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U.U.D.M. Project Report 2016:28

Contact Homology of Legendrian Knots in Five-Dimensional Circle Bundles

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Examensarbete i matematik, 30 hp
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Juni 2016

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CONTACT HOMOLOGY OF LEGENDRIAN KNOTS IN FIVE-DIMENSIONAL CIRCLE BUNDLES

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Abstract. In this paper we compute the Legendrian contact homology of the two-dimensional Legendrian unknot Λ in $\mathbb{R}^4 \times S^1$ equipped with its standard contact structure, by perturbing the contact form by a Morse function. We write down an explicit formula for the differential of the Legendrian contact homology algebra and show that the homology is generated by one Reeb chord in degree 2, which is the same as the homology of this knot in $\mathbb{R}^5$.

Considering the same knot Λ (with respect to a Darboux chart) in the circle bundle over $\mathbb{C}\mathrm{P}^2$ with total space S^5, we will make Λ describe a perturbation of Λ which gives that the Legendrian contact homology of Λ is the same as in $\mathbb{R}^4 \times S^1$.

Contents

1. Introduction

A *contact manifold* (M, ξ) of dimension $2n+1$ is a manifold of dimension $2n+1$ equipped with a completely non-integrable plane field ξ. If ξ is given as the kernel of a 1-form α, ξ is completely non-integrable if and only if $\alpha \wedge (d\alpha)^n \neq 0$. A submanifold $\Lambda^n \subset M$ is a *Legendrian submanifold* if Λ is everywhere tangent to ξ. In the case $n = 1$, an embedded S^1 that is everywhere tangent to ξ is called a *Legendrian knot*. A *Legendrian isotopy* of two Legendrian submanifolds Λ_0 and Λ_1 is an isotopy taking Λ_0 to Λ_1 through Legendrian submanifolds Λ_t. Assuming that $\xi = \ker \alpha$ with a 1-form α, we may define the *Reeb vector field* R_α as the unique vector field statisfying $\alpha(R_\alpha) = 1$ and $d\alpha(R_\alpha, -) = 0$. A segment of a flow line of the Reeb vector field with start and endpoints on a given Legendrian submanifold Λ is called a *Reeb chord*.

Date: June 20, 2016.

A general question in contact topology is to classify Legendrian submanifolds of a given contact manifold up to Legendrian isotopy. One attempt at answering this question is *Legendrian contact homology*, which is an invariant of Legendrian submanifolds up to Legendrian isotopy. Legendrian contact homology fits into the larger framework of symplectic field theory [8]. Legendrian contact homology was first introduced independently by Chekanov [3] and Eliashberg-Givental-Hofer [8]. Chekanov [3] successfully described the invariant for Legendrian knots combinatorially in $(\mathbb{R}^3, \xi_{\text{std}})$, which then later was generalized by Ekholm-Etnyre-Sullivan [4] to a geometrical description in $(\mathbb{R}^{2n+1}, \xi_{\text{std}})$ for embedded Legendrian submanifolds. Ekholm-Etnyre-Sullivan also computed Legendrian contact homology for Legendrian submanifolds in $(P \times \mathbb{R}, dz - \theta)$, where (P, θ) is an exact symplectic manifold [6]. From the work of Sabloff [19] we have a combinatorial description of Legendrian contact homology in circle bundles over Riemannian surfaces. In this thesis we use similar techniques to that of Sabloff to explicitly compute the Legendrian contact homology of the Legendrian unknot in $\mathbb{R}^4 \times S^1$.

1.1. **The Legendrian contact homology algebra.** In a contact 3-manifold, (M, α) Legendrian contact homology assigns a differential graded algebra (DGA) $(\mathcal{A}, \partial)$ to a Legendrian knot $\Lambda \subset (M, \alpha)$. The algebra is freely generated by Reeb chords $a_1, \ldots, a_n$, whose grading is given by the *Conley-Zehnder index* (see section 3.1 and appendix A). The differential is computed by studying moduli spaces of punctured pseudoholomorphic disks $u\colon (D, \partial D) \longrightarrow (M \times \mathbb{R}, \Lambda \times \mathbb{R})$ in the symplectization $(M \times \mathbb{R}, d(e^t \alpha))$ of (M, α). The moduli spaces consists of disks whose punctures $\{w_1, \ldots, w_k\} \subset \partial D$ are each one asymptotic to a Reeb chord in the symplectization. This means that, approaching $(s, \pm\infty)$ in $M \times \mathbb{R}$, each puncture w_i parametrize a Reeb chord of $\Lambda \subset M \times \{\pm\infty\}$. We may denote this moduli space by $\mathcal{M}(a_{i_1}, \ldots, a_{i_k})$, where $\{a_{i_1}, \ldots, a_{i_k}\}$ is the set of Reeb chords that appear as asymptotes of the pseudoholomorphic disks in this moduli space. The moduli spaces modulo reparametrization that consists of a finite number of isolated points contribute to the differential, and such pseudoholomorphic disks are called rigid.

The homology of $(\mathcal{A}, \partial)$ will be an Legendrian isotopy invariant, but is generally hard to compute. More generally, the *stable tame isomorphism class* of $(\mathcal{A}, \partial)$ (see section 3.4) is an invariant. There are also other more computable constructions related to $(\mathcal{A}, \partial)$ that are Legendrian isotopy invariants of the underlying knot, such as the linearized homology [3] and the characteristic algebra [17].

Following Sabloff [19], we consider a Riemannian surface F and E, which is a circle bundle over F. We equip E with a strictly negative curvature 2-form. Then there is a unique tight contact structure α, that is transverse to the S^1 fibers (see section 4). Letting $\Lambda \subset E$ be a Legendrian knot the Reeb vector field points in the S^1 fiber direction. In addition to the usual Reeb chords corresponding to the double points of $\pi(\Lambda)$, there is an infinite family of Reeb chords at every point $p \in \Lambda$. These Reeb chords are the ones that start at p, wind around the fiber $k \geq 1$ times and end at p again.

By an abstract perturbation $\alpha_\varepsilon = \alpha + \varepsilon f$, where f is a Morse function with one critical point on each edge of $\pi(\Lambda)$ we break this degeneracy. To each double point x_i we then have two families of Reeb chords $\{a_i^k\}_{k=0}^\infty$ and $\{b_i^k\}_{k=0}^\infty$, and to each critical point e_j we have either a family $\{c_j^k\}_{k=1}^\infty$ or a family $\{d_j^k\}_{k=1}^\infty$ of Reeb chords depending on whether the critical point is a maximum or a minimum.

1.2. **Main result and outline of the paper.** Consider $(\mathbb{R}^4 \times S^1, d\theta - \sum_{i=1}^2 y_i dx_i)$. We then consider the Legendrian unknot Λ, whose front projection is shown in fig. 1.

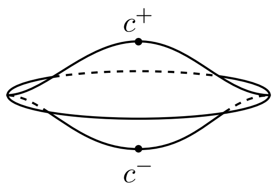

FIGURE 1. Front projection of the Legendrian unknot Λ.

We perturb the contact form with a Morse function that has one maximum and one minimum on Λ. Since this knot only has one double point in the Lagrangian projection, which corresponds to the points $c^\pm$ in the front projection as indicated in fig. 1, we have four families of Reeb chords, which we denote by

$\{\vec{q}^k\}_{k=0}^\infty$, $\{\overleftarrow{q}^k\}_{k=1}^\infty$, $\{q_\pm^k\}_{k=1}^\infty$. Then we consider the associative, unital, graded algebra $\mathcal{A}$ over $\mathbb{Z}_2$ that is generated by the formal power series

$$\boldsymbol{\vec{q}} = \sum_{k=0}^\infty \vec{q}^k T^k, \qquad \boldsymbol{\overleftarrow{q}} = \sum_{k=1}^\infty \overleftarrow{q}^k T^k, \qquad \boldsymbol{q}_\pm = \sum_{k=1}^\infty q_\pm^k T^k\,,$$

where $\boldsymbol{\vec{q}}$ and $\boldsymbol{\overleftarrow{q}}$ correspond to the only double point of the Lagrangian projection of Λ, and where $\boldsymbol{q}_\pm$ corresponds to the maximum and minimum respectively, of the perturbing Morse function. The grading of the generators is given by

$$|\vec{q}^k| = 2k+2, \qquad |\overleftarrow{q}^k| = 2k-2, \qquad |q_\pm^k| = 2k \pm 1\,.$$

The differential is split up as $\partial = \partial_{\text{hol}} + \partial_{\text{MB}}$. In this example we have that $\partial_{\text{hol}} = 0$ and that ∂_{MB} is given on generators as follows.

$$\begin{aligned} \partial_{\text{MB}}\boldsymbol{\vec{q}} &= \boldsymbol{q}_-\boldsymbol{\vec{q}} + \boldsymbol{\vec{q}}\boldsymbol{q}_- + \boldsymbol{q}_+ \\ \partial_{\text{MB}}\boldsymbol{\overleftarrow{q}} &= \boldsymbol{q}_-\boldsymbol{\overleftarrow{q}} + \boldsymbol{\overleftarrow{q}}\boldsymbol{q}_- \\ \partial_{\text{MB}}\boldsymbol{q}_+ &= \boldsymbol{q}_-\boldsymbol{q}_+ + \boldsymbol{q}_+\boldsymbol{q}_- + \boldsymbol{\vec{q}}\boldsymbol{\overleftarrow{q}} + \boldsymbol{\overleftarrow{q}}\boldsymbol{\vec{q}} \\ \partial_{\text{MB}}\boldsymbol{q}_- &= \boldsymbol{q}_-\boldsymbol{q}_- + \boldsymbol{\overleftarrow{q}}\,. \end{aligned}$$

This differential is computed via an explicit perturbation scheme as in [19, section 5]. The homology is then easily computed to be

$$CH_*(\Lambda) = \langle \vec{q}^0 \rangle\,.$$

That is, the homology is generated by one generator with grading 2. This is precisely the same result as in $\mathbb{R}^5$, which is what we expect.

First, section 2 is devoted to background notions regarding contact geometry, Legendrian knots and their classical invariants. In section 3 we discuss in broad terms the geometrical definition of the Legendrian contact homology DGA, and the combinatorial description in $(\mathbb{R}^3, \xi_{\text{std}})$ following [10]. In section 4 we follow [19] to describe how to compute Legendrian contact homology in three-dimensional circle bundles. In section 5 we explicitly compute the Legendrian contact homology of the Legendrian unknot in $\mathbb{R}^4 \times S^1$ and also consider the same knot in the circle bundle over $\mathbb{C}\mathrm{P}^2$ with S^5 as total space.

Acknowledgments. I would like to thank my supervisor Thomas Kragh for always taking time to discuss questions and thoughts of mine. I would also like to thank Tobias Ekholm for stimulating discussions.

2. Contact geometry and Legendrian knots

We recall basic definitions and notions of contact geometry. In section 2.1 we prove the Gray stability theorem which is a key ingredient in proving Darboux theorem which tells us that there are no local invariants of contact manifolds. The presentation roughly follows [9, 12].

In section 2.3 we define Legendrian knots and discuss the Lagrangian and front projection of Legendrian knots. To lift these projections to a Legendrian knot, the projections need to have certain features. This makes it possible to study Legendrian knots via their projections. In section 2.4 we compute the Thurston-Bennequin invariant and the rotation number in terms of the Lagrangian and front projections. Section 2.3 and section 2.4 follows [10].

A pair (M, ξ) of a $(2n+1)$-manifold M and a completely non-integrable plane field ξ, is called a *contact manifold*. ξ is called a contact structure and if we locally have $\xi = \ker \alpha$, then α is a 1-form such that

$$\alpha \wedge (d\alpha)^n \neq 0\,, \tag{2.1}$$

where $(d\alpha)^n$ is understood as the n-fold wedge product of $d\alpha$ with itself. Such 1-form α is called a *contact form* and we may sometimes write (M, α), if $\xi = \ker \alpha$ globally. If $\alpha \wedge (d\alpha)^n > 0$ we say that α is a *positive* contact form, and if $\alpha \wedge (d\alpha)^n < 0$ we say that α is a *negative* contact form. Note that (2.1) automatically implies that M has to be orientable, and that $\alpha \wedge (d\alpha)^n$ provides us with a volume form. In $\mathbb{R}^{2n+1}$ with coordinates (x_i, y_i, z), $i \in \{1, \ldots, n\}$, the standard contact structure is given as the kernel of the 1-form

$$\alpha = dz - \sum_{i=1}^n y_i dx_i\,.$$

If $\xi = \ker\alpha$ is a contact structure, any other 1-form $g\alpha$, for a nowhere vanishing function $g\colon M \longrightarrow \mathbb{R}$ also defines the same completely non-integrable plane field, and

$$(g\alpha) \wedge (d(g\alpha))^n = g\alpha \wedge (gd\alpha + dg \wedge \alpha)^n = g^{2n+1}\alpha \wedge (d\alpha)^n \neq 0\,.$$

The last equality holds by expanding the n-fold wedge product and by seeing that it is only one term that survives.

One can relate contact manifolds by *contactomorphisms*. A contactomorphism of two contact manifolds $(M, \xi = \ker\alpha)$ and $(N, \zeta = \ker\beta)$ is a diffeomorphism $f\colon M \longrightarrow N$ such that $df(\xi) = \zeta$, or equivalently $f^*\beta = g\alpha$, for some nowhere vanishing $g\colon M \longrightarrow \mathbb{R}$.

Example 2.1. In cylindrical coordinates (r, θ, z) on $\mathbb{R}^3$, $\xi_{\text{std}} = \ker(dz + r^2 d\theta)$ and $\xi_{\text{ot}} = \ker(\cos(r)dz + r\sin(r)d\theta)$ are contact structures.

The previous example provides an example of two non-contactomorphic contact structures on $\mathbb{R}^3$ [1]. The index of ξ_{ot} is short for *overtwisted*. A contact structure is called overtwisted if there is an embedded disk D which is transverse to ξ_{ot} at all points except at the boundary, where it is tangent to ξ_{ot}. A contact structure that is not overtwisted is called *tight*. It is called overtwisted, because the plane field of such a contact structure twists too much in some sense. Going out radially by $r = \frac{\pi}{2}$ in $(\mathbb{R}^3, \xi_{\text{ot}})$ makes the plane field twist a full turn, and so the planes makes an infinite number of full turns as r gets big. However, in $(\mathbb{R}^3, \xi_{\text{std}})$ the planes never do a full twist. As $r \to \infty$, the plane field approaches vertical planes.

There is a special vector field R_α called the *Reeb vector field*, which is defined uniquely by

$$\begin{cases} d\alpha(R_\alpha, -) = 0 \\ \alpha(R_\alpha) = 1 \end{cases}.$$

From the definition, we see that R_α is transverse to the contact structure, since $R_\alpha \notin \ker\alpha$. In the standard contact $\mathbb{R}^{2n+1}$, it is easy to see that $R_\alpha = \partial_z$. By the following result, it follows easily from the definition that the Reeb vector field preserves the contact structure.

Lemma 2.2. *Let $\varphi_t\colon M \longrightarrow M$ be the flow of the Reeb vector field, then $\varphi_t^*\alpha = \alpha$.*

Proof. It suffices to prove that $\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t^*\alpha = 0$, because $\varphi_0^*\alpha = \alpha$ by definition. By Cartan's formula $\mathcal{L}_X\omega = d\iota_X\omega + \iota_X d\omega$ we immediately have

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\varphi_t^*\alpha = \varphi_t^*(\mathcal{L}_{R_\alpha}\alpha) = \varphi_t^*(d\iota_{R_\alpha}\alpha + \iota_{R_\alpha}d\alpha) = \varphi_t^*(d(1) + 0) = 0\,.$$

□

A submanifold $N \subset M$ is a *contact submanifold* with contact structure ξ' if $\xi' = TN \cap \xi|_N$. A submanifold $N \subset M$ is called *isotropic* if $T_pN \subset \xi_p$ for each $p \in N$.

Proposition 2.3. *If $N \subset (M, \xi)$ is an isotropic submanifold and* $\dim M = 2n+1$, *then* $\dim N \leq n$.

Proof. Let α be a local contact form of ξ and let $\iota\colon N \longrightarrow M$ be the inclusion. The condition that N is isotropic can briefly be described as $\iota^*\alpha = 0$, and hence $\iota^*d\alpha = 0$. It is furthermore a fact that pointwise $\left(\xi_p, \omega := d\alpha|_{\xi_p}\right)$ is a symplectic vector space. Thus since $T_pN \subset \xi_p$, we have that T_pN is an isotropic subspace of ξ_p, seen as a symplectic vector space. Pointwise we therefore have $T_pN \subset T_pN^\omega$, so $\dim T_pN \leq \dim T_pN^\omega$ and so

$$\dim T_pN + \dim T_pN^\omega = \dim \xi_p = 2n \;\Rightarrow\; \dim T_pN \leq n\,.$$

□

An isotropic submanifold of maximal dimension is called *Legendrian*.

2.1. **Gray stability and Darboux theorem.** Locally, all contact manifolds look like $(\mathbb{R}^{2n+1}, \xi_{\text{std}})$ by Darboux theorem so it is not possible to find local invariants of contact manifolds.

The elementary proof is essentially pointwise linear algebra, while the more modern and the now standard proof uses Gray stability and Moser's trick. Recall that the Lie derivative on vector fields is defined as $\mathcal{L}_X Y = [X, Y]$. By Cartan's formula we can extend the definition to k-forms ω as $\mathcal{L}_X\omega = \iota_X d\omega + d(\iota_X\omega)$. Using this definition we have the identities

$$\begin{cases} \mathcal{L}_X(d\omega) = d(\mathcal{L}_X\omega) \\ \mathcal{L}_X(\omega\wedge\eta) = \mathcal{L}_X\omega\wedge\eta + \omega\wedge\mathcal{L}_X\eta \end{cases},$$

for differential forms ω and η.

Lemma 2.4 (Lemma 2.2.1 in [12])**.** *Let $\{\omega_t\}_{t\in[0,1]}$ be a smooth family of k-forms on a manifold M and let ψ_t be a smooth isotopy on M. Define a family of vector fields X_t,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\psi_t = X_t(\psi_t)\,,$$

that is, so that ψ_t is the flow of X_t. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\psi_t^*\omega_t\bigg|_{t=t_0} = \psi_{t_0}^*\left(\dot\omega_t|_{t=t_0} + \mathcal{L}_{X_0}\omega_{t_0}\right)\,.$$

Proof. We first show the formula for functions. Then, given that it holds for differential forms ω, ω' we also show that it holds for $d\omega$ and $\omega\wedge\omega'$. For a time-independent k-form ω we have $\frac{\mathrm{d}}{\mathrm{d}t}\psi_t^*\omega = \psi_t^*(\mathcal{L}_{X_t}\omega)$, so plugging in $t = t_0$ we see that the formula holds for time -independent k-forms.

(1) For a smooth real valued function $f\in\mathcal{C}^\infty(M)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(\psi_t^*f)(p) = \frac{\mathrm{d}}{\mathrm{d}t}\,f(\psi_t(p)) = \frac{\mathrm{d}}{\mathrm{d}t}\,\psi_t^*(X_tf)(p) = \psi_t^*(\mathcal{L}_{X_t}f)(p)\,.$$

(2) If it holds for differentials ω, ω' then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,(\psi_t^*(\alpha\wedge\beta)) &= \frac{\mathrm{d}}{\mathrm{d}t}\,(\psi_t^*\alpha\wedge\psi_t^*\beta) = \frac{\mathrm{d}}{\mathrm{d}t}\,\psi_t^*\alpha\wedge\beta + \alpha\wedge\frac{\mathrm{d}}{\mathrm{d}t}\,\psi_t^*\beta \\ &= \psi_t^*(\mathcal{L}_{X_t}\alpha)\wedge\psi_t^*\beta + (\psi_t^*\alpha)\wedge\psi_t^*(\mathcal{L}_{X_t}\beta) \\ &= \psi_t^*(\mathcal{L}_{X_t}(\alpha\wedge\beta))\,.t\end{aligned}$$

(3) If the formula holds for ω, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(\psi_t^*d\alpha) = \frac{\mathrm{d}}{\mathrm{d}t}\,(d(\psi_t^*\alpha)) = d\left(\psi_t^*\mathcal{L}_{X_t}\alpha\right) = \psi_t^*\mathcal{L}_{X_t}d\alpha\,.$$

Then for the time-dependent smooth family of forms ω_t we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,(\psi_t^*\omega_t) &= \lim_{h\to0}\frac{\psi_{t+h}^*\omega_{t+h} - \psi_t^*\omega_t}{h} = \lim_{h\to0}\psi_{t+h}^*\frac{\omega_{t+h}-\omega_t}{h} + \lim_{h\to0}\frac{\psi_{t+h}^*\omega_t - \psi_t^*\omega_t}{h} \\ &= \psi_t^*(\dot\omega_t) + \dot\psi_t^*\omega_t = \psi_t^*(\dot\omega_t + \mathcal{L}_{X_t}\omega_t)\,.\end{aligned}$$

□

This lemma along with the *Moser trick*, is used in the proof of the Gray stability theorem which is the following.

Theorem 2.5 (Gray stability, Theorem 2.2.2 in [12])**.** *Let ξ_t, $t\in[0,1]$ be a smooth family of contact structures on a closed manifold M. Then there is an isotopy $\{\psi_t\}_{t\in[0,1]}$ of M such that*

$$d\psi_t(\xi_0) = \xi_t,\ ,$$

for each $t\in[0,1]$.

Proof. The Moser trick is precisely to assume that the isotopy ψ_t is the flow of a vector field X_t. Equivalently, we may prove that for some smooth family of smooth positive functions $g_t\colon M\longrightarrow\mathbb{R}_+$ we have $\psi_t^*\alpha_t = g_t\alpha_0$. By differentiating this equation we can use lemma 2.4 to get

$$\psi_t^*(\dot\alpha_t + \mathcal{L}_{X_t}\alpha_t) = \dot g_t\alpha_0 = \frac{\dot g_t}{g_t}\psi_t^*\alpha_t\,.$$

Writing $\mu_t \coloneqq \frac{\mathrm{d}}{\mathrm{d}t}(\log g_t)\psi_t^{-1}$, and using Cartans formula in the left hand side gives the equation

$$\psi_t^*(\dot{\alpha}_t + d(\alpha_t(X_t)) + i_{X_t}d\alpha_t) = \psi_t^*(\mu_t\alpha_t)\,.$$

If we pick $X_t \in \xi_t$ we get

$$\dot{\alpha}_t = i_{X_t}d\alpha_t = \mu_t\alpha_t\,.$$

Plugging in the Reeb field R_{α_t} we get

$$\dot{\alpha}_t(R_{\alpha_t}) = \mu_t\,,$$

since $\alpha_t(R_{\alpha_t}) = 1$. So if we define $\mu_t \coloneqq \dot{\alpha}_t(R_{\alpha_t})$, then since $d\alpha_t|_{\xi_t}$, and since $R_{\alpha_t} \in \ker(\mu_t\alpha_t - \dot{\alpha}_t)$. We choose $X_t \in \xi_t$, such that $d\psi_t(\xi_0) = \xi_t$, $t \in [0,1]$. □

Theorem 2.6 (Darboux theorem, Theorem 2.5.1 in [12])**.** *Let (M,α) be a contact $(2n+1)$-manifold. For each $p \in M$ there is an open neighborhood $p \in U \subset M$, and coordinates (x_i, y_i, z) in U such that*

$$\alpha|_U = dz - \sum_{i=1}^{n} y_i dx_i\,.$$

Remark 2.7. Darboux theorem as stated above is equivalent with saying that there are open neighborhoods $p \in U \subset M$ and $0 \in V \subset \mathbb{R}^3$ so that with $\xi \coloneqq \ker\alpha$ there is a contactomorphism

$$\varphi\colon\ (M,\xi) \longrightarrow (\mathbb{R}^3, \xi_{\mathrm{std}})\,,$$

with $\varphi(p) = 0$.

2.2. **Basic notions in symplectic geometry.** A pair (M,ω) of a $2n$-manifold M and a closed non-degenerate 2-form ω is called a *symplectic manifold.* This means that $d\omega = 0$ and ω^n is a nowhere vanishing top form. Pointwise, $\omega_p\colon\ T_pM \times T_pM \longrightarrow \mathbb{R}$ is a skew-symmetric non-degenerate bilinear map. On $\mathbb{R}^{2n}$ with coordinates (x_i, y_i), $i \in \{1,\ldots,n\}$ the standard symplectic form is

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i\,.$$

Two symplectic manifolds (M,ω), (N,υ) are said to be *symplectomorphic* if there is a diffeomorphism $\varphi\colon\ M \longrightarrow N$ such that $f^*\upsilon = \omega$. A submanifold $N \subset M$ is called *isotropic* if $\iota^*\omega = 0$, where $\iota\colon\ N \longrightarrow M$ is the inclusion map. If $\dim N = \frac{1}{2}\dim M$, N is called a *Lagrangian submanifold.*

A symplectic manifold (M,ω) is called *exact* if there is a 1-form so that $\omega = d\lambda$. This 1-form is called a *Liouville form.*

2.3. **Legendrian knots.** This subsection follows the presentation given in [10].

We let $(M, \xi = \ker\alpha)$ be a 3-dimensional closed contact manifold. A *Legendrian knot* is an embedding $K\colon\ S^1 \longrightarrow M$ such that $T_pK \subset \xi_p$ for every $p \in K$. We will often refer to the image of K as the Legendrian knot, so we let $\mathit{\Lambda} = \operatorname{im} K$. We will restrict our attention to $(\mathbb{R}^3, \xi_{\mathrm{std}} = \ker(dz - ydx))$ since it is easier to study Legendrian knots in terms of the *front projection* and the *Lagrangian projection.*

We let $\mathit{\Lambda}$ be parametrized by an (at least) C^1-immersion

$$\begin{aligned}\varphi\colon\ S^1 &\longrightarrow \mathbb{R}^3\\ \theta &\mapsto (x(\theta), y(\theta), z(\theta))\,.\end{aligned}$$

The fact that $\mathit{\Lambda}$ is Legendrian means precisely that

$$T_{\varphi(\theta)}\mathit{\Lambda} = \operatorname{span}\Big\{(x'(\theta), y'(\theta), z'(\theta))\Big\} \subset \ker(dz - ydx) = \xi_{\varphi(\theta)}\,,$$

or equivalently

$$z'(\theta) - y(\theta)x'(\theta) = 0\,. \tag{2.2}$$

2.3.A. *Front projection.* The front projection is

$$\pi_F\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$
$$(x,y,z) \mapsto (x,z)\,.$$

The image of Λ, $\pi_F(\Lambda)$ is called the *front diagram* of Λ, and the parametrization of the front diagram is $\varphi_F(\theta) = (x(\theta), z(\theta))$. Since it is often easier to study knots using knot diagrams, it is useful to work out how front diagrams of Legendrian knots look like. Assuming that the parametrization φ_F is an immersion, we have $x'(\theta) \neq 0$ which in turn give that front diagrams do not have any vertical tangencies. Instead front diagrams have cusps. From (2.2) we can recover $y(\theta)$ by $y(\theta) = \frac{z'(\theta)}{x'(\theta)}$. A final note to further characterize front diagrams is that with the standard orientation of $\mathbb{R}^3$, the y direction is into the page. Hence, keeping $x(\theta)$ fixed at a double point of a front diagram, and moving from the overcrossing to the undercrossing, $y(\theta)$ gets bigger and thus also the slope $z'(\theta)$.

Thus the following characterize front diagrams of Legendrian knots

(1) Front diagrams have no vertical tangencies
(2) Any front diagram may be parametrized by a map that is an immersion except at a finite number of isolated points, at which there is still a well-defined tangent line of $\Lambda \subset \mathbb{R}^3$. Such points are called *generalized cusps*.
(3) The slope of the overcrossing is smaller than the undercrossing.

In fact, if we assume that we have a diagram satisfying the above conditions with a parametrization $f(\theta) := (x(\theta), z(\theta))$ and recover the y-coordinate via $y(\theta) = \frac{z'(\theta)}{x'(\theta)}$, then $\varphi(\theta) := (x(\theta), y(\theta), z(\theta))$ is a C^1-immersion which parametrizes a Legendrian knot in $(\mathbb{R}^3, \xi_{\text{std}})$.

FIGURE 2. The figure shows a Legendrian unknot to the left and a Legendrian right trefoil to the right.

2.3.B. *Lagrangian projection.* We then consider the Lagrangian projection

$$\pi_L\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$
$$(x,y,z) \mapsto (x,y)\,.$$

We call $\pi_L(\Lambda)$ the *Lagrangian diagram* of Λ, and the parametrization of the Lagrangian diagram is $\varphi_L(\theta) = (x(\theta), y(\theta))$. This map is always an immersion, because otherwise we would have that there is some $\theta \in S^1$ so that $x'(\theta) = y'(\theta) = 0$ which would imply $z'(\theta) \neq 0$, since φ is an immersion. But then $T_{\varphi(\theta)}\Lambda = \text{span}\,\{\partial_z\} \not\subset \xi_{\text{std}}$, and we have a contradiction.

From a Lagrangian diagram we can recover the z-coordinate by integrating the equation $z' = yx'$. If we choose z_0 then

$$z(\theta) = z_0 + \int_0^\theta y(\psi)x'(\psi)d\psi\,. \tag{2.3}$$

Working with Lagrangian projections is slightly more tricky than working with front projections. One obstruction is the following. Suppose that $z(\theta)$ is defined by (2.3). In order for $z(\theta)$ to be well-defined we need to have $z(0) = z(2\pi)$ (we parametrize the knot by an angle $[0, 2\pi]$), which is equivalent with having $\int_0^{2\pi} y(\psi)x'(\psi)d\psi = 0$. This is not true for all immersions, and it is not the only obstruction for when an immersion lifts to a Legendrian knot.

Proposition 2.8. *An immersion* $g\colon S^1 \longrightarrow \mathbb{R}^2$ *lifts to a Legendrian knot in* $(\mathbb{R}^3, \xi_{std})$ *if*

(1) $\int_0^{2\pi} y(\psi)x'(\psi)d\psi = 0$, *and*
(2) $\int_{\theta_0}^{\theta_1} y(\psi)x'(\psi)d\psi \neq 0$ *for every* $\theta_0 \neq \theta_1$ *with* $g(\theta_0) = g(\theta_1)$.

These conditions are not as easy to interpret diagrammatically as the corresponding conditions for the front projections. However, the following theorem should make it more clear how to construct Lagrangian diagrams, based on front diagrams.

Theorem 2.9 (Proposition 2.2 in [17]). *Given the front diagram of a Legendrian knot Λ one can obtain a Lagrangian projection by altering the cusps in the front diagram as in fig. 3.*

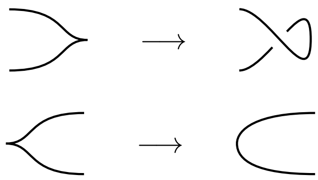

FIGURE 3. The figure shows how to transform a front diagram to a Lagrangian diagram.

Proof. See [17, prp 2.2]. □

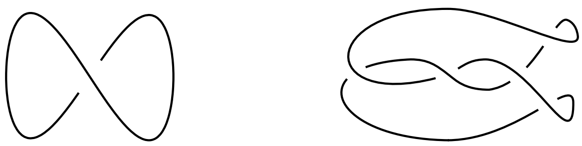

FIGURE 4. The figure shows a Legendrian unknot on the left and a Legendrian right trefoil on the right.

2.4. **Classical invariants of Legendrian knots.** Like in classical knot theory, the main question in Legendrian knot theory is to determine whether two Legendrian knots are considered to be equivalent. We want to classify Legendrian knots up to Legendrian isotopy, which is a smooth map

$$H\colon\ S^1 \times I \longrightarrow \mathbb{R}^3\,,$$

such that $K_i = H(s,i)$ are two given Legendrian knots for $i \in \{0,1\}$ and so that $K_t = H(s,t)$ is a Legendrian knot for each $t \in I = [0,1]$.

In classical knot theory one can determine whether two knots are (ambient) isotopic by checking whether their knot diagrams are related by a series of *Reidemeister moves*. For front and Lagrangian diagrams of Legendrian knots there are similar results.

Theorem 2.10 (Theorem 2.6 in [10] and Theorem B in [20]). *Two front diagrams represent Legendrian isotopic Legendrian knots if and only if they are related by regular homotopy, and a sequence of Legendrian Reidemeister moves shown in fig. 5.*

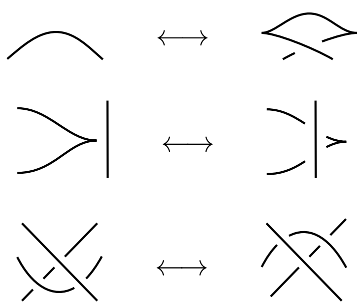

FIGURE 5. Three Legendrian Reidemeister moves of a front diagram.

Proof. See for example [20, section 3]. □

Remark 2.11. It can also be shown that if two Lagrangian diagrams represent Legendrian isoptopic Legendrian knots, then the Lagrangian diagrams are related by a sequence of the Reidemeister moves shown in fig. 6.

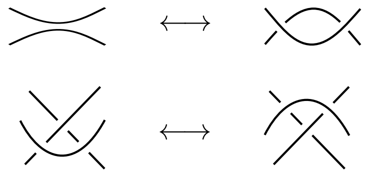

FIGURE 6. Two Legendrian Reidemeister moves of a Lagrangian diagram.

As in proposition 2.8, any immersion of S^1 into $\mathbb{R}^2$ with the appropriate area conditions will lift to a Legendrian knots. So in order to perform the Reidemeister moves shown in fig. 6 one has to make sure it is possible to maintain the area conditions. For a more precise formulation of this, see [16, thm. 4.1].

For Legendrian knots, there are three invariants that are referred to as classical invariants. We will assume that we have a Legendrian knot Λ which is oriented. The first classical invariant is the topological type of the knot when we simply forget about the contact structure. The two others are the *Thurston-Bennequin invariant* $\operatorname{tb}(\Lambda)$ and the *rotation number* $\operatorname{r}(\Lambda)$. The Thurston-Bennequin invariant roughly measures how much the contact structure twists around Λ, and the rotation number is intuitively the winding number of the knot. The rotation number is only defined for oriented and homologically trivial knots. That is knots that is the boundary of a 2-chain.

2.4.A. *Thurston-Bennequin invariant.* We may define the Thurston-Bennequin invariant as follows. Let v be a non-zero vector field along Λ, transverse to ξ. Define Λ' as a copy of Λ which has been slightly pushed in the direction of v. Then we can define $\operatorname{tb}(\Lambda) = \operatorname{lk}(\Lambda, \Lambda')$.

This defintion is not very convenient to work with in computations, so we translate this into the front and Lagrangian diagrams. $v = \partial_z$ is a vector field transverse to ξ. So we consider $\pi_F(\Lambda)$ and shift a copy of Λ slightly in this direction to obtain $\pi_F(\Lambda')$. The Thurston-Bennequin number can then be computed via the front diagram of a Legendrian knot as

$$\operatorname{tb}(\Lambda) = \operatorname{wr}(\pi_F(\Lambda)) - \frac{1}{2}\left(\#\text{ of cusps in } \pi_F(\Lambda)\right) .$$

In the Lagrangian projection we see that

$$\operatorname{tb}(\Lambda) = \operatorname{wr}(\pi_L(\Lambda)) ,$$

where wr is the writhe of the knot.

2.4.B. *Rotation number.* The Legendrian knot Λ is homologically trivial in $(\mathbb{R}^3, \xi)$, and so we may choose a Seifert surface Σ, which is a surface so that $\partial\Sigma = \Lambda$. The restriction $\xi|_\Sigma$ is a trivial plane bundle, which restricts to the boundary with a trivialization so that $\xi|_\Lambda = \Lambda \times \mathbb{R}^2$. If v is a non-zero vector field tangent to Λ in the direction according to the orientation of Λ, we may think of v as a path of non-zero vectors in the chosen trivialization of $\xi|_\Lambda = \Lambda \times \mathbb{R}^2$. This path has a winding number, which we call the rotation number. The rotation number is dependent on the orientation of Λ.

Since $\xi = \operatorname{span}\{\partial_y, \partial_z + y\partial_x\}$, $w := \partial_y$ is a non-zero section of ξ and thus w can be used to trivialize $\xi|_\Lambda$ independent of finding a Seifert surface of Λ. The winding number in this trivialization is the signed number of times v points in the same direction as w. In the front diagram, v points in the $\pm w$ direction at cusps. If D is the number of down-cusps (that is cusps that are traveled downwards when traversing Λ along to its orientation) and U is the number of up-cusps, then

$$\operatorname{r}(\Lambda) = \frac{1}{2}(D - U) .$$

In the Lagrangian projection, it is simply the winding number

$$\operatorname{r}(\Lambda) = \operatorname{winding}(\pi_L(\Lambda)) .$$

Both the Thruston-Bennequin invariant and the rotation numbers are Legendrian isotopy invariants. That is to say, two Legendrian isotopic Legendrian knots have the same tb and r. In general Legendrian isotopy classes are “finer” than toplogical types of knots in the sense that there exist non-Legendrian isotopic knots of the same topological type. In fact there are an infinite number of Legendrian unknots that are of the same topological type. The front diagrams of three of them is shown in fig. 7. Adding more cusps gives a Legendrian unknot with $\mathrm{tb} = -n$ for any $n \in \mathbb{Z}_+$.

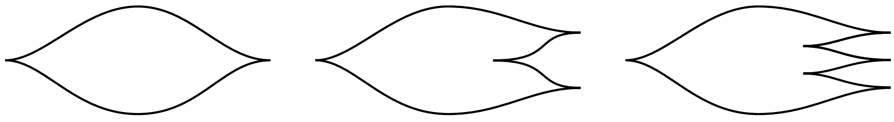

FIGURE 7. Non-Legendrian isotopic Legendrian unknots with $\mathrm{tb} = -1$, $\mathrm{tb} = -2$ and $\mathrm{tb} = -3$ respectively.

The classical invariants are enough to study some classes of knots. Namely the Legendrian unknots [7], torus knots and the figure eight knots [11].

3. LEGENDRIAN CONTACT HOMOLOGY IN A CONTACT 3-MANIFOLD

The classical invariants of Legendrian knots are not sufficient to classify Legendrian knots, as there exist non-Legendrian isotopic Legendrian knots with the same Thurston-Bennequin invariant and rotation number [3, section 4]. An attempt at defining a finer Legendrian isotopy invariant of Legendrian knots is Legendrian contact homology. In this section we will define the Legendrian contact homology DGA geometrically and consider the combinatorial decsription in $(\mathbb{R}^3, \xi_{\mathrm{std}})$ following [10].

Throughout this section we consider a contact 3-manifold (M, ξ) with contact form α. We denote by R_α the Reeb vector field, and let $\varphi_\alpha^t \colon M \longrightarrow M$ be the time t-Reeb flow. We assume that $\Lambda \subset M$ is a Legendrian knot with a finite number of Reeb chords. A segment of a Reeb flow line with endpoints on Λ is called a *Reeb chord.*

The *symplectization* of (M, α) is the symplectic manifold $(M \times \mathbb{R}, \omega := d(e^t\alpha))$, where t is the coordinate of the $\mathbb{R}$-factor. For the symplectization we fix a vertically invariant almost complex structure J, compatible with ω that is invariant along the t-coordinate. More precisely, J is an automorphism $J \colon T(M \times \mathbb{R}) \longrightarrow T(M \times \mathbb{R})$, with $J^2 = -\mathrm{id}$ and such that

$$\begin{cases} J(\partial_t) = R_\alpha \\ J(\xi) = \xi \end{cases} .$$

That J is compatible with ω means that

$$\begin{cases} \omega(-, J-) > 0 \\ \omega(J-, J-) = \omega(-,-) \end{cases} .$$

3.1. **The algebra.** We let $\mathcal{A}$ be the associative, unital graded algebra over $\mathbb{Z}_2$ which is freely generated by the Reeb chords of Λ. To define the grading, we need to pick a capping path. For a generating Reeb chord a parametrized by $[0,1]$, a path $\gamma_a \colon [0,1] \longrightarrow \Lambda$ so that $\gamma_a(0) = a(1)$ and $\gamma_a(1) = a(0)$ is called a *capping path.* We then trivialize ξ over a surface F_α with $\partial F_\alpha = a(t) \cup \gamma_a(t)$. A path of Lagrangian subspaces, E along $a \cup \gamma_a$ is then defined as

$$\begin{aligned} E|_{\gamma_a(t)} &= T_{\gamma_a(t)}\Lambda \\ E|_{a(t)} &= d\varphi_\alpha^t \cdot T_{a(0)}\Lambda\,, \end{aligned}$$

At each point on $a \cup \gamma_t$, the fiber of E is a Lagrangian submanifold of $\mathbb{R}^2$. We may look at it as a (non-closed) path Γ of Lagrangian submanifolds in a fixed symplectic vector space. We will make the path of Lagrangian submanifolds closed by considering the path $\lambda(V_0, V_1)(\tau) := e^{\tau I} V_1$ for $\tau \in \{0, \frac{\pi}{2}\}$, where I is a complex structure in $T_{a(1)}\Lambda$ such that $I(V_0) = V_1$. In this case we let $V_0 = d\varphi_\alpha^1(T_{a(0)}\Lambda)$ and $V_1 = T_{a(1)}\Lambda$. The path $\lambda(V_0, V_1)$ is the path which rotates V_0 to V_1 in the positive direction. We then consider the concatenation $\Gamma \star \lambda(V_0, V_1)$, which is a closed loop of Lagrangian submanifolds.

As such a path, it has a Maslov index $\mu(\Gamma \star \lambda(V_0, V_1))$ (see appendix A). We call this the *Conley-Zehnder index* of the Reeb chord $a(t)$, denoted $\mathrm{CZ}_{\gamma_a}(a)$. We then define the grading by

$$|a| := \mathrm{CZ}_{\gamma_a}(a) - 1\,.$$

The grading is extended to all of $\mathcal{A}$ as usual by $|ab| = |a| + |b|$. The grading is dependent on the capping path chosen. We note that since Λ is 1-dimensional, there are exactly two choices of capping paths. If γ_a and $\tilde{\gamma}_a$ are two cappings paths then the difference of the Conley-Zehnder indices with respect to these capping paths is

$$\mathrm{CZ}_{\gamma_a}(a) - \mathrm{CZ}_{\tilde{\gamma}_a}(a) = \mu(\gamma_a \star (-\tilde{\gamma}_a))\,.$$

$-\tilde{\gamma}_a$ is the path $\tilde{\gamma}_a$ traversed in the opposite direction. The image of $\gamma_a \star (-\tilde{\gamma}_a)$ is precisely our Legendrian knot Λ, and thus the grading $|a|$ is well-defined up to the Maslov number $m(\Lambda)$ which is the Maslov index of the closed loop

$$\begin{aligned} \pi\colon\ [0,1] &\longrightarrow \mathcal{L}(n) \\ t &\mapsto T_{p(t)}\Lambda\,, \end{aligned}$$

if $p\colon\ [0,1] \longrightarrow \Lambda$ is a parametrization of Λ and where $\mathcal{L}(n)$ is the Lagrangian Grassmannian. It is worth noting that the Maslov number $m(\Lambda)$ is equal to double the rotation number of the knot, which also will be evident in section 3.3.A.

With this, $\mathcal{A}$ has a well-defined $\mathbb{Z}_{m(\Lambda)}$-grading.

3.2. **The differential.** We want to define a map $\partial\colon\ \mathcal{A} \longrightarrow \mathcal{A}$ of degree -1, that is $|\partial a| = |a| - 1$, so that $\partial^2 = 0$ and that satisfies the Leibniz rule $\partial(ab) = \partial a b + (-1)^{|a|} a \partial b$. This means that $(\mathcal{A}, \partial)$ is turned into a DGA. The idea is to define this differential so that the homology of $(\mathcal{A}, \partial)$ is an invariant of Legendrian knots.

We will consider a punctured 2-disk $D_* := D^2 \setminus \{z, w_1, \ldots, w_n\} \subset \mathbb{C}$, with the marked points $\{z, w_1, \ldots, w_n\}$ lying in ∂D^2. We denote by j the restriction of the standard complex structure of $\mathbb{C}$ to D_*. Then we consider a map

$$f\colon\ (D_*, j) \longrightarrow (M \times \mathbb{R}, J)\,,$$

that satisfies $f(\partial D_*) \subset L \times \mathbb{R}$ and is J-holomorphic. That is $\overline{\partial} f = \frac{1}{2}(df + J \circ df \circ j) = 0$. The differential ∂ will be defined as a count of the number of such maps with some added conditions.

We want the map f to tend asympotically to Reeb chords in the symplectization $M \times \mathbb{R}$ close to a puncture. A neighborhood in D_* of a puncture x is conformally equivalent to $(0, \infty) \times [0, 1]$, so choosing coordinates in such a neigborhood (s, t) and letting $f = (f_M, f_{\mathbb{R}})$, we say that f *tends asymptotically to a Reeb chord* $a(t)$ at $\pm\infty$ near a puncture x if

$$\begin{cases} \lim_{s \to \infty} f_M(s,t) = a(t) \\ \lim_{s \to \infty} f_{\mathbb{R}}(s,t) = \pm\infty \end{cases}.$$

We may then define a suitable moduli space of J-holomorphic disks associated to a set of Reeb chords a, $\boldsymbol{b} = (b_1, \ldots, b_n)$

$$\mathcal{M}(a, \boldsymbol{b}) = \{f\colon\ (D_*, \partial D_*) \longrightarrow (M \times \mathbb{R}, \Lambda \times \mathbb{R}) \mid \text{(1)-(3) below holds}\} / \sim\,,$$

where

1. f has finite energy, $\int_{D_*} f^* d\alpha < \infty$,
2. f tends asymptotically to $a(t)$ at $+\infty$ near the puncture z, and
3. f tends asymptotically to $b_i(t)$ at $-\infty$ near the puncture w_i for $i \in \{1, \ldots, n\}$.

We say that $f \sim g$ if there is a biholomorphism $h\colon\ D_* \longrightarrow D_*$ such that $f = g \circ h$. Moreover, since the complex structure J is vertically invariant, there is an $\mathbb{R}$-action $t \mapsto t + \Delta t$ which we also will mod out, and thus consider $\mathcal{M}(a, \boldsymbol{b})/\mathbb{R}$. The differential is defined as

$$\partial a = \sum_{\substack{b_1 \cdots b_n \\ \dim \mathcal{M}(a, \boldsymbol{b})/\mathbb{R} = 0}} \#_2(\mathcal{M}(a, \boldsymbol{b})/\mathbb{R}) b_1 \cdots b_n\,,$$

where $\#_2$ is the modulo 2-count. We extend ∂ to all of $\mathcal{A}$ by the Leibniz rule $\partial(ab) = \partial a b + a \partial b$. The pseudoholomorphic disks in $\mathcal{M}(a, \boldsymbol{b})$ so that $\dim \mathcal{M}(a, \boldsymbol{b})/\mathbb{R} = 0$ are called *rigid*.

The local structure of moduli space is studied via functional analysis and more precisely the implicit function theorem. Near a J-holomorphic disk f we consider the linearization of the $\overline{\partial}$ operator, $D_f\overline{\partial}$, and assuming a nice enough Sobolev setup, the linearization is surjective and Fredholm. The grading that is given by the Conley-Zehnder index in the geometric picture, is then related by the Fredholm index of this linearization. This gives among other results that $\mathcal{M}(a, \boldsymbol{b}) = \overline{\partial}^{-1}(0)$ is a manifold. We only count maps which belongs to moduli spaces associated to collections of Reeb chords $a, b_1, \dots, b_n$ such that $\dim \mathcal{M}(a, \boldsymbol{b})/\mathbb{R} = 0$, so it only contains a finite number of isolated points.

3.3. **Combinatorial description.** The geometric definition is not very computable, since the non-linear Cauchy-Riemann equation $\overline{\partial} f = 0$ is hard to solve. For knots in $(\mathbb{R}^3, \xi_{\text{std}})$, Legendrian contact homology can be defined combinatorially [3] so that the definition coincides with the geometric description [8]. The combinatorial definition makes the Legendrian contact homology computable. This section follows the structure of [10].

We will consider the Lagrangian projection $\pi_L(\Lambda)$ of a Legendrian knot $\Lambda \subset \mathbb{R}^3$, and we assume without loss of generality that all the self-intersections of $\pi_L(\Lambda)$ are orthogonal. Recalling that the Reeb vector field is $R_\alpha = \partial_z$, we see that double points of $\pi_L(\Lambda)$ correspond to Reeb chords of Λ.

3.3.A. *The algebra.* If $\mathcal{C} := \{c_1, \dots, c_n\}$ is the set of double points of $\pi_L(\Lambda)$, we define the algebra $\mathcal{A} = \mathbb{Z}_2\langle c_1, \dots, c_n\rangle$ to be the associative, unital, graded algebra over $\mathbb{Z}_2$ freely generated by double points of $\pi_L(\Lambda)$. We make the choice to always consider Reeb chords that start on the undercrossing (the point with smaller z coordinate when $\pi_L(\Lambda)$ is lifted to a Legendrian knot in $\mathbb{R}^3$) and end on the overcrossing (the point with larger z coordinate) of a double point of $\pi_L(\Lambda)$.

Each double point c will obtain a grading as follows. First let $c^\pm$ denote the over- and undercrossing respectively. Then pick a capping path $\gamma_c \colon [0, 1] \longrightarrow \Lambda$ so that $\gamma_c(0) = c^+$, $\gamma_c(1) = c^-$. We only consider capping paths that are homotopic to an injective path in Λ, so there are only two choices of capping paths. With respect to the standard trivialization of $\mathbb{R}^2$, this path has a fractional rotation number $r(\gamma_c)$ which must be an odd multiple of 4, since self-intersections of $\pi_L(\Lambda)$ are orthogonal by assumption. We then define the grading of c as

$$|c| := 2r(\gamma_c) - \frac{1}{2},$$

and extend it to all of $\mathcal{A}$ by $|ab| = |a| + |b|$. Note that this is dependent on the choice of capping path, since if we pick the other possible capping path $\tilde{\gamma}_c$ we have $r(\gamma_c) - r(\tilde{\gamma}_c) = 2r(\Lambda)$. Thus the grading is only well-defined modulo $2r(\Lambda)$. So $\mathcal{A}$ is an algebra over $\mathbb{Z}_2$ with grading in $\mathbb{Z}_{2r(\Lambda)}$.

3.3.B. *The differential.* First we will decorate each double point in $\pi_L(\Lambda)$ with Reeb signs as in fig. 8.

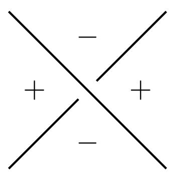

FIGURE 8. Reeb signs associated with a crossing.

We let P_{k+1} be a $(k+1)$-sided convex polygon with vertices $v_0, \dots, v_k$ labeled counterclockwise. We then let $a, b_1, \dots, b_k \in \mathcal{C}$ be generators of $\mathcal{A}$. We will count the number of such immersed P_{k+1} convex polygons with a *positive* corner at a, and *negative* corners at b_i. Let $\boldsymbol{b} = (b_1, \dots, b_k)$ and define

$$\mathcal{M}(a, \boldsymbol{b}) := \Big\{u\colon (P_{k+1}, \partial P_{k+1}) \longrightarrow (\mathbb{R}^2, \pi_L(\Lambda)) \mid \text{(1)-(3) below holds}\Big\}/\text{reparametrization},$$

where

(1) u is an immersion
(2) $u(v_0) = a$ and near a, the image of a small neighborhood of v_0 covers a quadrant labeled $+$ (we say that u has a positive corner at a).
(3) $u(v_i) = b_i$, $i \in \{1, \dots, k\}$ and near b_i, the image of a small neighborhood of v_i covers a quadrant labeled $-$ (we say that u has a negative corner at b_i).

Then define

$$\partial a := \sum_{b_1 \cdots b_k} \#_2 \mathcal{M}(a, \boldsymbol{b}) b_1 \cdots b_k \,,$$

where $\#_2$ is the modulo 2-count. We extend ∂ to all of $\mathcal{A}$ by the Leibniz rule $\partial(ab) = \partial a b + a \partial b$.

Example 3.1. We consider the left handed trefoil with $\mathrm{tb} = -6$, $r = 1$ with double points labeled as in fig. 9. The set of double points, $\{a_1, \ldots, a_6\}$ generate the algebra over $\mathbb{Z}_2$, with grading in $\mathbb{Z}_2$.

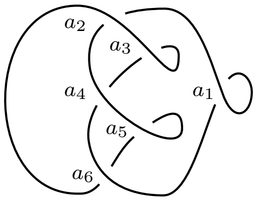

FIGURE 9. The Lagrangian projection of the left handed trefoil.

The gradings are

$$|a_i| = 1, \, \forall i \in \{1, \ldots, 6\} \,,$$

and the differential is

$$\begin{aligned} \partial a_1 &= 1 + a_4 a_3 + a_5 a_4 + a_4 a_4 \\ \partial a_2 &= \partial a_4 = \partial a_6 = 0 \\ \partial a_3 &= \partial a_5 = 1 \,. \end{aligned}$$

The polygons contributing to the three terms $a_4 a_3$, $a_5 a_4$ and $a_4 a_4$ in ∂a_1 are shown in fig. 10.

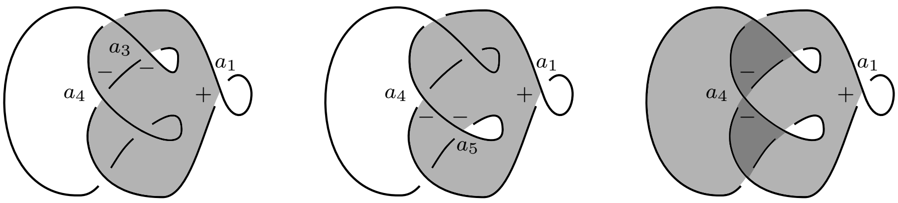

FIGURE 10. The three immersed convex polygons contributing to the non-trivial terms $a_4 a_3$, $a_4 a_5$ and $a_4 a_4$ in ∂a_1.

3.4. **Invariance under Legendrian isotopy and stable tame isomorphisms.** The goal is ultimately to consider the homology of the DGA $(\mathcal{A}, \partial)$ and show that it is invariant under Legendrian isotopies of the underlying Legendrian knot Λ, but there are in fact stronger results, saying that the *stable tame isomorphism* class of $(\mathcal{A}\langle c_1, \ldots, c_n \rangle, \partial)$ is an invariant. From hereon, we may write $\mathcal{A}$ instead of $\mathcal{A}\langle c_1, \ldots, c_n \rangle$, but we emphasize that our constructions are dependent on the free generating set $\{c_1, \ldots, c_n\}$.

Fix $j \in \{1, \ldots, n\}$. An algebra automorphism $\varphi_j \colon \mathcal{A} \longrightarrow \mathcal{A}$ defined by

$$\varphi_j(c_i) = \begin{cases} c_i, & i \neq j \\ \pm c_j + u, & i = j \end{cases} \,,$$

for $u \in \mathcal{A}\langle c_1, \ldots, \hat{c}_j, \ldots, c_n \rangle$, is called *elementary*. A composition of elementary automorphisms is called a *tame* automorphism of $\mathcal{A}$. A *tame isomorphism* of two algebras $\mathcal{A}\langle c_1, \ldots, c_n \rangle$ and $\tilde{\mathcal{A}}$ is an identification of generators $c_i \leftrightarrow \tilde{c}_i$ followed by a tame automorphism. All automorphisms are assumed to respect the grading of the algebras.

An *index j stabilization* of the DGA $(\mathcal{A}\langle c_1,\dots,c_n\rangle,\partial)$ is the DGA $(\tilde{\mathcal{A}}\langle c_1,\dots,c_n,a,b\rangle,\tilde{\partial})$ such that

$$\begin{cases}\tilde{\partial}(c_i)=\partial(c_i), & \forall i\in\{1,\dots,n\}\\ \tilde{\partial}b=a \\ \tilde{\partial}a=0\end{cases},$$

where $|a|=|b|-1=j$. We may use the notation $(\tilde{\mathcal{A}},\tilde{\partial})=S(\mathcal{A},\partial)$, and we let $S^m(\mathcal{A},\partial)$ be the m-fold stabilization of $(\mathcal{A},\partial)$, that is applying m stabilizations to $(\mathcal{A},\partial)$.

Two DGAs $(\mathcal{A},\partial)$ and $(\tilde{\mathcal{A}},\tilde{\partial})$ are *stably tame isomorphic* if there exist stabilizations S^k and $\overline{S}^l$ and a tame isomorphism

$$\psi\colon\ S^k(\mathcal{A},\partial)\longrightarrow\overline{S}^l(\tilde{\mathcal{A}},\tilde{\partial}),$$

such that $\psi\circ\partial=\tilde{\partial}\circ\psi$.

Theorem 3.2 (Chekanov [3]). *The DGA $(\mathcal{A},\partial)$ associated to a Legendrian knot Λ changes by stable tame isomorphisms under Legendrian isotopy. Moreover the homology of $(\mathcal{A},\partial)$*

$$CH_*(\Lambda)=\ker\partial/\operatorname{im}\partial,$$

is invariant under Legendrian isotopy of Λ.

4. Legendrian contact homology in a circle bundle

In this section, we will define the Legendrian contact homology DGA in a circle bundle, while following [19]. We will both discuss the geometric situation and how it differs from the situation discussed in section 3, and the combinatorial description. To this end, we first consider a Hermitian line bundle $\mathcal{E}\xrightarrow{\pi}F$ over a closed oriented Riemann surface F. Let $E\xrightarrow{\pi}F$ be the unit circle bundle with connection 1-form α associated to a Hermitian connection D. We then consider the metric $\langle -,-\rangle$ on E, which is induced by the Hermitian structure, which the connection D is compatible with, by definition. If we choose a frame of sections $\{e_i\}$ with $\langle e_i,e_j\rangle=\delta_{ij}$ such that $D=d+\alpha$ in this trivialization, then we can use compatability with $\langle -,-\rangle$ to get

$$d\langle\xi,\eta\rangle=\langle D\xi,\eta\rangle+\langle\xi,D\eta\rangle\ \Leftrightarrow\ d(\xi_i\eta_i)e_ie_i=d(\xi_i)\eta_ie_ie_i+\xi_id(\eta_i)e_ie_i+\alpha\xi_i\eta_ie_ie_i+\xi_i\overline{\alpha}\eta_ie_ie_i\,,$$

that is to say $\alpha+\overline{\alpha}=0$, which means that we can write $D=d+i\alpha$ and assume that our connection 1-form α takes real values. The curvature 2-form $\Omega\in\Omega^2(F,\mathbb{C})$ of this connection in a suitable trivialization is such that $\pi^*\Omega=id\alpha$. Assuming that $\Omega<0$, is the same as to say that $id\alpha<0$ in $\Omega^1(F,i\mathbb{R})$. We then see that

$$\alpha\wedge d\alpha=-i(\alpha\wedge id\alpha)>0\,,$$

so (E,α) forms a contact manifold with positive contact form. The unit circle bundle $E\xrightarrow{\pi}F$ equipped with such α is called a *contact circle bundle*. The Euler number is then strictly negative

$$e(E)=c_1(\mathcal{E})=\frac{1}{2\pi}\int_F\Omega<0\,.$$

It is sufficient to consider non-degenerate curvature forms in order for α to be a contact form, but we want the Euler number of E to be negative, since results from Giroux and Honda [14, 13] ensures the existence of a unique tight contact structure that is transverse to the fibers when $e(E)<0$. The results can be summarized in the following two theorems.

Theorem 4.1 (Theorem 3.8 (3) in [14]). *Let $E\longrightarrow F$ be a circle bundle over a closed orientable Riemann surface, with genus $g\geq 1$ and $e(E)\leq 2g-2$. Then E supports a unique transverse contact structure that is tight.*

Theorem 4.2 (Proposition 3.2 in [13]). *Let $E\longrightarrow S^2$ be a circle bundle and suppose that $e(E)<0$. Then E supports a unique transverse contact structure*

These two results combined removes the ambiguity that Legendrian contact homology depends on the contact structure.

4.1. **Morse-Bott perturbations and the geometric situation.** Let (E, α) be a contact circle bundle, with its unique transverse contact structure and let $\Lambda \subset E$ be a Legendrian knot with projection $\pi(\Lambda)$ in F. The first thing we note is that for a transverse α, the Reeb vector field points in the fiber direction, and there are two types of Reeb chords on Λ. The first type is the *long* Reeb chord. At any point $p \in \Lambda$ there is a Reeb chord that start and end at p and winds around the fiber k times. The second type is the *short* Reeb chord, which starts and ends at different points of Λ that corresponds to a double point in $\pi(\Lambda)$ and dont wind around the fiber.

So at any point that is not a double point of $\pi(\Lambda)$ there is a family of long Reeb chords, parametrized by how many times they wind around the fiber. At double points there is a family of Reeb chords which are compositions of a short and a long Reeb chord. The Reeb chords are therefore not isolated but they rather come in a continuous family of families parametrized by Λ, and hence we are in a Morse-Bott-type situation.

To describe the perturbation, first think of the projection $\pi(\Lambda)$ as a 4-valent graphs with vertices being precisely the double points. Then we will consider a Morse function $g\colon F \longrightarrow \mathbb{R}$ such that $g|_{\pi(\Lambda)}$ has exactly one critical point at each edge. When following the knot around, the types of the critical points will alternate between maxima and minima. We let j be the S^1-invariant lift of a complex structure on F. Then we consider the S^1-invariant lift $\hat{g}\colon E \longrightarrow \mathbb{R}$ of g. We then perturb α to get the contact form

$$\alpha_\varepsilon = \alpha + \varepsilon \hat{g} \alpha \,,$$

where $\varepsilon > 0$ is small. The resulting perturbed Reeb vector field is

$$R_{\alpha_\varepsilon} = R_\alpha + j\varepsilon \nabla \hat{g} \,.$$

At each critical point for g, we have $\nabla \hat{g} = 0$, and thus there is a family of long Reeb chords at each critical point for g. Moreover, we still have a family of Reeb chords at the double points of $\pi(\Lambda)$, since $\hat{g}$ is invariant along the S^1-fibers. The perturbation thus breaks our degeneracy of having a S^1-family of families of Reeb chords.

4.2. **Combinatorial description.** In this subsection we describe Legendrian contact homology in circle bundles $E \xrightarrow{\pi} F$ over a Riemann surface combinatorially, and our presentation follows the one given in [19].

The combinatorial description relies on perturbating the contact form α on E by a function which is invariant under the free S^1-action along the fibers. This introduces additional decorations in the diagram $\pi(\Lambda)$.

The knot diagram $\pi(\Lambda)$ consists of the following

(1) A choice of short Reeb chord at each double point of $\pi(\Lambda)$, and
(2) An integer vector $\boldsymbol{n}$ with one component for each connected component of $F \setminus \pi(\Lambda)$.

Since the lift of $\pi(\Lambda)$ to E is along a circle, it is ambiguous which strand goes over or under the other in $\pi(\Lambda)$, and hence there is not a canonical choice of a short Reeb chord, as in section 3.3. Also, depending on these choices, it determines how much the knot winds around in the fiber direction which is recorded in the integer vector $\boldsymbol{n}$.

Let $\{x_1, \dots, x_n\}$ be the set of double points in $\pi(\Lambda)$ and let Σ be an oriented surface with non-empty boundary and let $\{z_1, \dots, z_m\}$ be marked points on $\partial\Sigma$. Let $f\colon \Sigma \longrightarrow F$ be an orientation preserving embedding on the interior of Σ. Assume that f extends smoothly to $\partial\Sigma$ away from $\{z_1, \dots, z_m\}$ and that $f(z_j) = x_{i_j}$ for each $j \in \{1, \dots, m\}$. Use the notation $\partial\Sigma_* \coloneqq \partial\Sigma \setminus \{z_1, \dots, z_m\}$

With respect to the orientation of the quadrant that is covered by $f(\Sigma)$, We let the strands of $\pi(\Lambda)$ be the *incoming* and *outgoing* strand respectively, as in fig. 11.

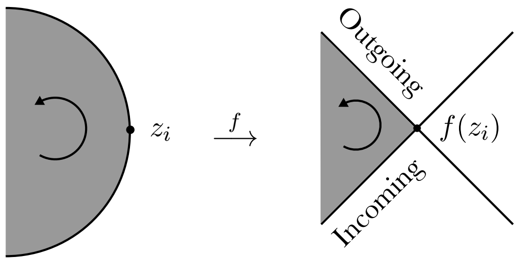

FIGURE 11. The figure shows how to mark the double points, depending on the choice of short Reeb chord.

If the choice of short Reeb chord is the one going from the incoming strand to the outgoing strand, we decorate the quadrant with a $+$, else we decorate it with a $-$ (or leave it undecorated, to reduce clutter). Conversely, the decorated diagram indicate which short Reeb chord is chosen.

Let R_i be the (chosen) short Reeb chord at the double point $f(z_i)$ of $\pi(\Lambda)$. Let $l(R_i)$ be the length of R_i, and assume without loss of generality that $l(R_i) \in (0, 2\pi)$. Define ε_i as

$$\varepsilon_i := \begin{cases} 1, & \text{if } R_i \text{ flows from the incoming to the outgoing strand,} \\ -1, & \text{otherwise} \end{cases} .$$

Then we define the *defect* of the embedding f with respect to the short Reeb chords $\{R_1, \ldots, R_m\}$ as the integer

$$n(f; R_1, \ldots, R_m) := \frac{1}{2\pi} \left(\int_\Sigma f^* \Omega + \sum_{j=1}^m \varepsilon_j l(R_j) \right) .$$

We extend n linearly to formal chains of embeddings.

Here is a geometric description of the defect. Assume that $\partial\Sigma$ is connected. Lift the component of $\partial\Sigma_*$ that lies between z_1 and z_2 to a Legendrian curve in f^*E. Start to lift the next component of $\partial\Sigma_*$, at a length $\varepsilon_2 l(R_2)$ away from the first. The curve obtained in this way, together with the lifted Reeb chords is a closed curve in f^*E, and the defect is the winding number around the fiber of this curve. This geometric description justifies the fact that n is an integer.

Lemma 4.3. *Assume that $m \leq n$. Then we have*

$$n(f; R_1, \ldots, R_m) \leq n - 1 .$$

Proof. We have $\Omega < 0$ by assumption, so $\int_\Sigma f^*\Omega < 0$. Also $\varepsilon_j \leq 1$, $l(R_j) < 2\pi$ and $m \leq n$. So

$$n(f; R_1, \ldots, R_m) = \frac{1}{2\pi} \left(\int_\Sigma f^* \Omega + \sum_{j=1}^m e_j l(R_j) \right) \leq \frac{1}{2\pi} \sum_{j=1}^n l(R_j) < \frac{1}{2\pi} \sum_{j=1}^n 2\pi = n .$$

□

The integer vector $\boldsymbol{n}$ has one component n_i for each connected component Σ_i of $F \setminus \pi(\Lambda)$. If we let Σ_i be a connected component of $F \setminus \pi(\Lambda)$ so that the boundary of Σ_i contains m double points, the short Reeb chords of which are $\{R_1, \ldots, R_m\}$. Then we define

$$n_i := n(\Sigma_i; R_1, \ldots, R_m) . \tag{4.1}$$

This defect is the defect of an embedding f as above so that $\operatorname{im} f = \Sigma_i$. The signs ε_i are positive if Σ_i covers a quadrant of a double point, decorated with a $+$.

Proposition 4.4. *Let $\boldsymbol{n} = (n_1, \ldots, n_k)$ be the integer vector of defects of the connected components of $F \setminus \pi(\Lambda)$. Then*

$$\sum_{i=1}^k n_i = e(E) . \tag{4.2}$$

Proof. Summing the n_i over each connected component of $F \setminus \pi(\Lambda)$, each term $e_j l(R_j)$ will cancel out, since there are exactly two positive and two negative quadrants around each double point of $\pi(\Lambda)$. Then the result follows almost immediately from the Gauss-Bonnet theorem. Letting f_i be a parametrizatrion of the i-th component of $F \setminus \pi(\Lambda)$ we have

$$\sum_{i=1}^{k} n_i = \frac{1}{2\pi}\left(\sum_{i=1}^{k}\int_{\Sigma} f_i^* \Omega\right) = \frac{1}{2\pi}\sum_{i=1}^{n}\int_{\Sigma_i}\Omega = \frac{1}{2\pi}\int_F \Omega = e(E)\,.$$

□

A diagram $\pi(\Lambda)$ (which is equipped with the choice of Reeb chord at each double point and with the integer vector $\boldsymbol{n}$) lifts to a Legendrian knot Λ in E if each component of $\boldsymbol{n}$ satisfies (4.1) and (4.2). A more precise formulation can be found in [19, prp 2.6].

4.2.A. *The algebra and grading.* To define the algebra we will introduce a few more generators, than in the case we encountered in section 3.3. Let $\{x_1, \ldots, x_n\}$ be the set of double points of $\pi(\Lambda)$. To each double point x_i we associate two countable sets $\{a_i^k\}_{k=0}^{\infty}$, $\{b_i^k\}_{k=0}^{\infty}$. Thinking of $\pi(\Lambda)$ as a 4-valent graph, we enumerate the edges by $\{e_i\}_{i=1}^{2n}$ by picking an arbitrary edge and calling it e_1, then we go along the knot and call the next edge e_2 and continue until all edges have been enumerated by some e_i. At the interior of each edge e_i we pick a point p_i. To the edge points p_{2i-1} we associate a countable set $\{c_i^k\}_{k=1}^{\infty}$, and to the edge points p_{2i} we associate a countable set $\{d_i^k\}_{k=1}^{\infty}$.

We say that the double points $\{x_i\}_{i=1}^{n}$ and edge points $\{p_i\}_{i=1}^{2n}$ are *special points* of $\pi(\Lambda)$. At each double point, we label the quadrants according to fig. 12.

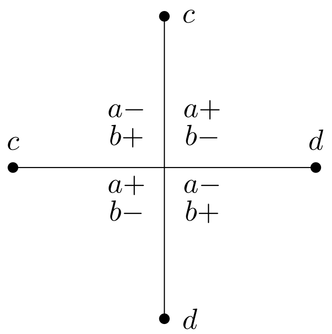

FIGURE 12. Reeb signs of generators a_i and b_i.

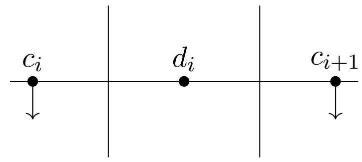

FIGURE 13. Marked edge points with a transverse direction chosen at the c labels.

Furthermore, we choose a direction at each edge point marked with c_i, transverse to $\pi(\Lambda)$ as in fig. 13.

Let $\mathcal{A}_\Lambda$ be the unital associative algebra over $\mathbb{Z}_2$, freely generated by the letters a_i^k, b_i^k, $c_i^{k'}$, $d_i^{k'}$ for $i \in \{1, \ldots, n\}$, $k \in \mathbb{N}$, $k' \in \mathbb{Z}_+$.

The generators a_i^0, b_i^0 correspond to the two possible choices of short Reeb chord, at a given double point x_i. The generators a_i^k, b_i^k for $k \geq 1$, correspond to Reeb chords which are the concatenation of a short Reeb chord and a long Reeb chord that winds around the fiber k times. The length of the concatenation is what one would expect, namely

$$l(a_i^k) = l(a_i^0) + 2\pi k$$
$$l(b_i^k) = l(b_i^0) + 2\pi k\,.$$

Since the families c_i^k and d_i^k correspond to Reeb chords that start and end at the same point, their lengths are integer multiples of 2π,

$$l(c_i^k) = l(d_i^k) = 2\pi k\,.$$

Instead of working directly with $\mathcal{A}_\Lambda$ we will consider $\mathcal{A}_\Lambda[\![T]\!]$ of formal power series that is generated by

$$\boldsymbol{a}_i = \sum_{k=0}^{\infty} a_i^k T^k, \qquad \boldsymbol{b}_i = \sum_{k=0}^{\infty} b_i^k T^k$$
$$\boldsymbol{c}_i = \sum_{k=1}^{\infty} c_i^k T^k \qquad \boldsymbol{d}_i = \sum_{k=1}^{\infty} d_i^k T^k .$$

To describe the grading, we will first describe the grading for all the a_i^k by considering a capping path in Λ. This is similar to how we define the grading in section 3.3, but we have to take the holonomy (the winding around the fiber) of the capping path into account. The grading of the other generators will be in terms of the grading for a_i^k (or independent of it).

Define a capping path γ_i for a_i at a double point x_i to be one of the two paths that start at x_i, run along $\pi(\Lambda)$ until it first hits x_i again. To a capping path we will associate two quantities. Its *holonomy* and its rotation number. Assume without loss of generality [1] that γ_i (concatenated with the Reeb chord that corresponds to x_i) bounds a (disjoint union of) connected component of $F \smallsetminus \pi(\Lambda)$ that is decorated with a a_i+ label (if it was decorated with a b_i+ label, the construction is carried out analogously). Let $\{\Sigma_1, \dots, \Sigma_{n+2}\}$ be the connected components of $F \smallsetminus \pi(\Lambda)$. Then consider a chain of embedded surfaces $\Sigma^{(i)} = \sum_{k=1}^{n+2} c_k^{(i)} \Sigma_k$ such that $\partial \Sigma^{(i)} = \gamma_i$. This chain is called a *capping surface* of γ_i and may be constructed via the Seifert circle algorithm. Define the holonomy of γ_i to be the defect

$$k_i := -n(\Sigma^{(i)}; a_i) = -\sum_{k=1}^{n+2} c_k n(\Sigma_k) ,$$

Where $n(\Sigma_k)$ is the defect for the connected component Σ_k of $F \smallsetminus \pi(\Lambda)$. Assume γ_i lies in an embedded disk[2] $\gamma_i \subset D \subset F$, it is possible to describe the grading combinatorially. If this is the case, we may pick a trivialization of TD, so that the fractional rotation number of γ_i may be computed (we first perturb the diagram $\pi(\Lambda)$ so that double points are orthogonal). We denote this number by $r_D(\gamma_i)$. In general we do not have $\Sigma^{(i)} \subset D$ in which case, we can not say that $\Sigma^{(i)}$ gets an induced trivialization from D. We have to correct for this, and we define the *rotation number* of γ_i as

$$r(\gamma_i) = r_D(\gamma_i) + \chi(F)(p \cdot \Sigma^{(i)}) ,$$

where $p \in F \smallsetminus (\pi(\Lambda) \cup D)$ is any point. To describe the grading of a_i^k, we let $\mu_E := -\frac{\chi(F)}{e(E)}$ be the Maslov index of a fiber, and we need to include a term, taking the k number of windings around the fiber into account in addition to the term involving the rotation number. This holonomy term is proportional to $2\mu_E$. The term involving the rotation number of γ_i is denoted $\left| a_i^{-n(\Sigma^{(i)};a_i)} \right|$ and is defined as

$$\left| a_i^{-n(\Sigma^{(i)};a_i)} \right| = 2r(\gamma_i) - \frac{1}{2} .$$

Then for any $k \in \mathbb{N}$ we define

$$\begin{aligned} |a_i^k| &= \left| a_i^{-n(\Sigma^{(i)};a_i)} \right| + 2\mu_E(k + n(\Sigma^{(i)}; a_i)) \\ |b_i^k| &= 2\mu_E(2k+1) - 1 - |a_i^k| \\ |c_i^k| &= 2k\mu_E \\ |d_i^k| &= 2k\mu_E - 1 . \end{aligned}$$

As in the case with $(\mathbb{R}^3, \xi_{\text{std}})$, the grading is only well-defined up to the choice of capping path. If $\pi(\Lambda)$ is contractible in F, the difference of two capping paths is a path that winds around the knot an integer number of times.

[1]If not, we may suppose that $\gamma_i \cup R_i$ is homologous to $-k \cdot [\text{fiber}] \in H_1(E)$, where R_i is one of the short Reeb chord corresponding to the double point x_i. Adding k circuits around the fiber to R_i, to yield R_i^k, we get that $\gamma_i \cup R_i^k$ is null-homologous in E, and we may carry out the construction with this Reeb chord instead.

[2]Actually it suffices to assume that γ_i is null-homotopic.

Proposition 4.5. *If $\pi(\Lambda)$ is contractible in F, the grading is well-defined modulo $2r(\Lambda) + 2\mu_E n(\Lambda)$ where $r(\Lambda)$ is the rotation number and $n(\Lambda)$ is the holonomy of the knot.*

Proposition 4.6. *Let γ be a capping path associated with a double point a of $\pi(\Lambda)$. Then $|a^k|$ is independent of the choice of capping surface Σ, and $r(\gamma)$ is independent of the point p.*

Proof. Let $\Sigma = \sum_{k=1}^{n+2} c_k \Sigma_k$ be the capping surface, where Σ_i are disjoint embedded surfaces which exhaust $F \setminus \pi(\Lambda)$. Any other capping surface must be of the form $\widetilde{\Sigma} = \sum_{k=1}^{n+2}(c_k + l)\Sigma_k$, for $l \in \mathbb{Z}$ since difference $|c_i - c_j|$ must be constant.

That $r(\gamma_i)$ does not depend on the choice of p amounts to showing that $c_i = c_j$ for regions Σ_i and Σ_j whose common boundary is not a point and does not belong to the image of γ. But this is clear since if Σ_i and Σ_j has common boundary α, which is a segment of $\pi(\Lambda)$ not belonging to the image of γ, then c_i and c_j must be so that the orientation along α cancels, and hence $c_i = c_j$.

To show that $|a^k|$ does not depend on the choice of capping surface, we want to show that $|\tilde{a}^k| - |a^k| = 0$, where $|\tilde{a}^k|$ is the grading associated to the capping surface $\widetilde{\Sigma}$. We have that

$$p \cdot \widetilde{\Sigma} = p \cdot \Sigma + l\,,$$

and

$$n(\widetilde{\Sigma}; a) = \sum_{k=1}^{n+2}(c_k + l)n(\Sigma_k) = n(\Sigma; a) + l\sum_{k=1}^{n+2} n(\Sigma_k)\,.$$

By proposition 4.4 we thus obtain $n(\widetilde{\Sigma}; a) = n(\Sigma; a) + le(E)$. Therefore

$$|\tilde{a}^k| - |a^k| = 2\chi(F)(p\cdot\widetilde{\Sigma} - p\cdot\Sigma) + 2\mu_E(n(\widetilde{\Sigma}; a) - n(\Sigma; a)) = 2\chi(F)l - 2\chi(F)l = 0\,.$$

□

4.2.B. *The differential.* Since we are in a Morse-Bott-type situation with a long Reeb chord at every point of Λ, the differential is split up into the *external* and the *internal* part, $\partial = \partial_{\text{ext}} + \partial_{\text{int}}$. The external differential comes from counting embedded marked disks (corresponding to punctured pseudoholomorphic disks in the symplectization $\mathcal{E} \setminus \{\text{zero section}\}$ in the geometric picture), and the internal differential comes from counting "Morse flow lines" within the Morse-Bott critical submanifolds, of which the generators c_i and d_i correspond to maxima and minima respectively.

Let D be a disk with non-empty boundary, and let $\{z, w_1, \dots, w_m\}$ be marked points (labeled counterclockwise around the boundary) on ∂D. D_* will denote the disk with the same interior as D and $\partial D_* = \partial D \setminus \{z_1, \dots, z_m\}$. Consider labels

$$\alpha \in \{x_1, \dots, x_n, p_2, p_4, \dots, p_{2n}\}$$
$$\boldsymbol{\beta} = (\beta_1, \dots, \beta_m) \subset \{x_1, \dots, x_n, p_1, p_3, \dots, p_{2n-1}\}^m\ ,$$

that is α is either a double point or an edge point with label d_j and each β_i for $i \in \{1, \dots, m\}$, is either a double point or an edge point with label c_j. We then define the space of *embedded marked disks*

$$\mathcal{M}(\alpha; \boldsymbol{\beta}) = \{f\colon\ D \longrightarrow \pi(\Lambda) \text{ orientation preserving embedding} \mid \text{ (1)-(4) below holds}\}\ ,$$

where

1. $f(\partial D) \subset \pi(\Lambda)$, and f is smooth on D_*. Moreover $f|_{\partial D_*}$ is an embedding.
2. $f(z) = \alpha$ and $f(w_j) = \beta_j$ for $j \in \{1, \dots, m\}$.
3. Near each marked point z_k such that $f(z_k) = \beta_k$ is a double point, the image of f covers exactly one quadrant of the double point.
   If α is a double point, then f covers one of the two quadrants labeled with $\alpha+$. If β_j is a double point, then f covers one of the two quadrants labeled with β_j-.
4. If an edge point with label c_k in $\pi(\Lambda)$ is covered by $f|_{\partial D_*}$, then c_k is the image of some z_j.

Two embeddings $f, g \in \mathcal{M}(\alpha; \boldsymbol{\beta})$ are equivalent if there is a smooth automorphism φ, so that $f = g \circ \varphi$.

Let $\alpha, \beta_1, \dots, \beta_m$ be labels as above. Then Sabloff computes the exterior differential to be

$$\partial_{\text{ext}}\boldsymbol{x} = \sum_{\boldsymbol{\beta}} \sum_{f\in\mathcal{M}(\alpha;\boldsymbol{\beta})} \tilde{\boldsymbol{y}}_1 \cdots \tilde{\boldsymbol{y}}_m T^{-n(f;\alpha,\boldsymbol{\beta})}\ ,$$

where

$$\tilde{\boldsymbol{y}} = \begin{cases} 1+\boldsymbol{y}, & y = c_j \text{ and the transversal direction at } c_j \text{ points into } \operatorname{im} f \\ (1+\boldsymbol{y})^{-1}, & y = c_j \text{ and the transversal direction at } c_j \text{ points out of } \operatorname{im} f \\ \boldsymbol{y}, & \text{otherwise} \end{cases}.$$

Remark 4.7. The notation for the defect $n(f;\alpha,\beta_1,\dots,\beta_m)$ in the formula for ∂_{ext} above, is a slight abuse of notation, since the labels $\alpha,\beta_1,\dots,\beta_m$ may be an edge point. In any case, the defect is easily computed by summing up the components of $\boldsymbol{n}$ in the connected components of $F\smallsetminus\pi(\Lambda)$ that is covered by $\operatorname{im} f$.

The internal differential is defined on generators as

$$\begin{aligned} \partial_{\text{int}}\boldsymbol{a} &= \boldsymbol{a}\boldsymbol{d}' + \boldsymbol{d}\boldsymbol{a} \\ \partial_{\text{int}}\boldsymbol{b} &= \boldsymbol{b}\boldsymbol{d} + \boldsymbol{d}'\boldsymbol{b} + \boldsymbol{b}\boldsymbol{a}\boldsymbol{b}T \\ \partial_{\text{int}}\boldsymbol{c} &= (1+\boldsymbol{c})(\boldsymbol{d}_1' + \boldsymbol{b}_1\boldsymbol{a}_1T) + (\boldsymbol{d}_2' + \boldsymbol{b}_2\boldsymbol{a}_2T)(1+\boldsymbol{c}) \\ \partial_{\text{int}}\boldsymbol{d} &= \boldsymbol{d}\boldsymbol{d}\,, \end{aligned}$$

when labels are as in fig. 14.

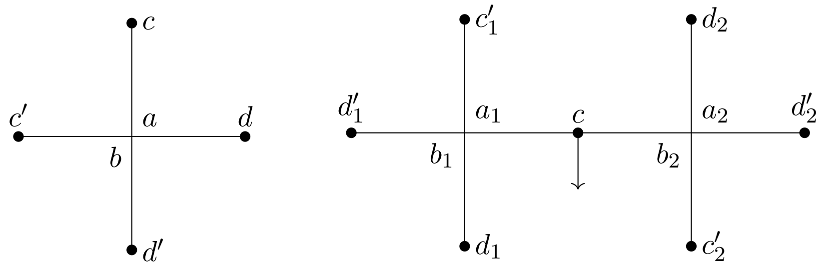

FIGURE 14. The configuration of labels used in the definition of the interior differential ∂_{int}.

5. Legendrian contact homology in five-dimensional circle bundles

In this section we provide an explicit calculation of the Legendrian contact homology of the Legendrian unknot in $\mathbb{R}^4\times S^1$. We will use a similar perturbation scheme to explicitly compute the differential, similar to what is found in [19, section 5].

5.1. Construction and example in $\mathbb{R}^4\times S^1$.

Let Λ be an embedded Legendrian knot in $\mathbb{R}^4\times S^1 \xrightarrow{\pi} \mathbb{R}^4$ with the standard contact form

$$\alpha = d\theta - \sum_{i=1}^{2} y_i dx_i\,,$$

using coordinates (x_1,y_1,x_2,y_2,θ). In fact $(\mathbb{R}^4,\omega)$ is an exact symplectic manifold with the standard Liouville form $\lambda=\sum_{i=1}^2 y_i dx_i$ on $\mathbb{R}^4$. The Reeb field is precisely ∂_θ in this setting, and hence double points of the projection $\pi(\Lambda)$ corresponds to Reeb chords of Λ. The projection $\pi(\Lambda)$ is a Lagrangian immersion, and we will assume that Λ is generic in the sense that $\pi(\Lambda)$ has finitely many isolated double points. Sometimes one says that Λ is *chord generic* in this situation.

Conversely, from the Lagrangian immersion $\pi(\Lambda)$ we may recover the Legendrian embedding. Namely, pick $p\in\pi(\Lambda)$ and choose any θ coordinate for p. The coordinate θ' along the S^1-factor at any other point $p'\in\pi(\Lambda)$ is then determined by

$$\theta' = \int_\gamma \lambda\,,$$

where γ is any path joining p and p'. The Lagrangian immersion lifts to a Legendrian embedding if the above integral is independent of γ, that is, if the pullback of λ via $\pi|_\Lambda$ is an exact 1-form. In fact if we pick any loop $\tilde{\gamma}$ in $\pi(\Lambda)$ then it suffices to assume $\int_{\tilde{\gamma}} \lambda \in \mathbb{Z}$, in order for $\pi(\Lambda)$ to lift to a Legendrian embedding. The front projection

$$\Pi\colon\ \mathbb{R}^4 \times S^1 \longrightarrow \mathbb{R}^2 \times S^1$$
$$(x_1, y_1, x_2, y_2, \theta) \mapsto (x_1, x_2, \theta)\,,$$

is easier to work with, since the target is a three-dimensional space. Front projections of higher dimensional Legendrian knots will contain other types of singularities than just cusp singularities, but any map into $\mathbb{R}^2 \times S^1$ that has the appropriate singularities lifts to an embedding in $\mathbb{R}^4 \times S^1$. Hence, in order to work in the front projection we need to identify double points of the Lagrangian projection in the front projection.

Proposition 5.1. *Let $c_1, c_2 \in \Pi(\Lambda)$ be any two non-singular points in the front projection. Then the following are equivalent*

(1) c_1 and c_2 correspond to a double point in the Lagrangian projection $\pi(\Lambda)$
(2) $T_{c_1}\Pi(\Lambda) = T_{c_2}\Pi(\Lambda)$ and the x_i-coordinates of c_1 and c_2 are the same.

Sketch of proof. If we pick two non-singular points $c_1, c_2 \in \Pi(\Lambda)$ that has the same x_i-coordinates and the same tangent spaces, the lift of c_1 and c_2 to $\Lambda \subset \mathbb{R}^4 \times S^1$ is given by $(y_1, y_2) = \left(\dfrac{\partial\theta}{\partial x_1}, \dfrac{\partial\theta}{\partial x_2}\right)$, since Λ is Legendrian. Since they have the same tangent spaces, they also project down to the same point through the Lagrangian projection and hence they correspond to a double point of $\pi(\Lambda)$.

Conversely, a double point $c \in \pi(\Lambda)$ lifts (up to translation along the S^1-fiber) to two points in $\Lambda \subset \mathbb{R}^4 \times S^1$ that has the same x_i- and y_i-coordinates, but different θ-coordinates. Through the front projection, they thus project down to two points $c_1, c_2 \in \Pi(\Lambda)$ that has the same tangent spaces since Λ is Legendrian. □

Example 5.2. We consider the Legendrian sphere Λ, whose front projection is shown in fig. 15. In the Lagrangian projection, there is only one double point c, which corresponds to the points $c^{\pm}$ in the front projection shown in fig. 15 and which has two families of Reeb chords assigned to it.

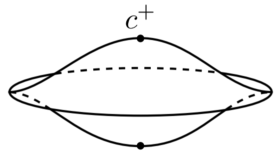

FIGURE 15. Front projection in $\mathbb{R}^2 \times S^1$ of a Legendrian sphere.

We may perturb the contact form with the Morse function $f(x_1, x_2, y_1, y_2) = \varepsilon x_1$. To break the S^2-family of Reeb chords coming from the fact that the Reeb vector field points in the S^1-fiber direction.

Lemma 5.3. *The only Reeb chords of Λ are the four families corresponding to the double point c of the Lagrangian projection, and the two critical points of f.*

Sketch of proof. The function $f(x_1, x_2, y_1, y_2) = \varepsilon x_1$ for some small $\varepsilon > 0$ is a Morse function with exactly two critical points when restricted to $\pi(\Lambda)$. Using this Morse function, the perturbed Reeb vector field is

$$R_\varepsilon = \partial_z + i\nabla f = \partial_z + (0, 0, \varepsilon, 0, 0)\,.$$

To find the new Reeb chords, we may perturb the knot by pushing it in the direction of $(0, 0, \varepsilon, 0, 0)$. Namely if $p \in \Lambda$ is arbitrary, then the corresponding point on the perturbed Λ, denoted by $\tilde{\Lambda}$, is given by $\tilde{p} = p + (0, 0, \varepsilon, 0, 0)$. Let $f^{\pm}(x_1, x_2, \theta) = \pm(1 - x_1^2 - x_2^2)^{\frac{3}{2}}$. Then the graph of the two functions $f^{\pm}$ together make up a model for $\Pi(\Lambda)$. Adding ε to the y_1-coordinate of Λ is equivalent to slightly tilt $\Pi(\Lambda)$ since $y_1 = \frac{d\theta}{dx_1}$. Namely, $\Pi(\tilde{\Lambda})$ may be described by the graphs of the functions

$$\tilde{f}^{\pm}(x_1, x_2, \theta) = \pm(1 - x_1^2 - x_2^2)^{\frac{3}{2}} + \varepsilon x_1\,.$$

As discussed in proposition 5.1, we will look for points on the graph of f^- and on the graph of $\tilde{f}^+$ with the same x_i-coordinates and the same gradients (and hence tangent spaces). We will thus solve for (x_1, x_2) in the equation $\nabla f^- = \nabla \tilde{f}^+ = \nabla f^+ + (\varepsilon, 0)$. This equation leads to the system

$$\begin{cases} 6x_1\sqrt{1-x_1^2-x_2^2} = \varepsilon \\ 6x_2\sqrt{1-x_1^2-x_2^2} = 0 \end{cases} .$$

The second equation gives $x_2 = 0$, since otherwise the first equation can not be solved. Solving for x_1 finally gives

$$x_1 = \pm\sqrt{\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\varepsilon^2}{6}}} .$$

So for small enough $\varepsilon > 0$, we break the S^1-family of Reeb chords, and obtain only 4. Two of which lie near the double point of $\pi(\Lambda)$, and the other two lie close to the generalized cusps of $\Pi(\Lambda)$. The latter two points correspond to the maximum and minimum of f when restricted to $\pi(\Lambda)$. The Reeb chords that correspond to these four points in the front projection, are the lowest energy ones, $\vec{q}^{\,0}$, $\overleftarrow{q}^{1}$ and $q^1_\pm$ (see notation below). To examine the ones with higher energy, we make ε even smaller, and we can carry out the same process to find four more Reeb chords. Making ε arbitrarily small makes it possible to find arbitrarily many Reeb chords, coming in precisely these four families. □

The perturbing function $f = \varepsilon x_1$ has exactly one maximum and one minimum on $\pi(\Lambda)$ which lifts to the two points $m_\pm$ on Λ. Then we have that away from the double points the long Reeb chords at each point $p \in \Lambda$ will correspond to two families of generators at two points $m_\pm$.

To the double point, we assign two families of generators $\{\vec{q}^{\,k}\}_{k=0}^\infty$ and $\{\overleftarrow{q}^{k}\}_{k=1}^\infty$. The elements $\vec{q}^{\,0}$ and $\overleftarrow{q}^{1}$ correspond to the two different short Reeb chord at the double point c, that pass through a fixed point in the S^1-fiber. At $m_\pm$ we have two more generators $\{q^k_\pm\}_{k=1}^\infty$. Arranging the generators into formal power series, we let $\mathcal{A}$ be the associative unital algebra over $\mathbb{Z}_2$ generated by the power series

$$\vec{\boldsymbol{q}} = \sum_{k=0}^\infty \vec{q}^{\,k} T^k, \qquad \overleftarrow{\boldsymbol{q}} = \sum_{k=0}^\infty \overleftarrow{q}^{k} T^k, \qquad \boldsymbol{q}_\pm = \sum_{k=1}^\infty q^k_\pm T^k .$$

The grading of these generators is given by the Conley-Zehnder index, and we may compute the Conley-Zehnder by the following lemma

Lemma 5.4. *Let $c_\pm \in \Pi(\Lambda)$ be two points which correspond to a Reeb chord $\vec{q}^{\,k}$ of Λ in the Lagrangian projection as described above. Let $U_\pm \subset \Lambda$ be a neighborhood around $c_\pm$ and let $f_\pm$ be a function with graph $U_\pm$. Also let γ be a capping path, starting at c_+ and ending at c_- (which is generic in the sense that it intersects cusp edges of $\Pi(\Lambda)$ transversally and meets no other singularity).*

Then the Conley-Zehnder index of the Reeb chord c is given by the formula

$$\mathrm{CZ}(c) = \mathrm{ind}_{c_+}(f_+ - f_-) + D(\gamma) - U(\gamma) + 2k ,$$

where ind denotes the Morse index, D is the number of down-cusps of γ (a down-cusp is a point on a cusp edge that γ passes through and for which the θ-coordinate locally decreases when traversed), and U the number of up-cusps of γ.

Sketch of proof. The case $k = 0$ is proven in [5, lemma 3.4]. If $k > 0$, and if we consider $\mathbb{R}^4 \times S^1$ to be the contact boundary of $\mathbb{R}^4 \times D^2$, then a trivialization of $T(\mathbb{R}^4 \times S^1)$ is induced by the canonical trivialization on $T(\mathbb{R}^4 \times D^2)$. In this induced trivialization, the Maslov index of the path $\Gamma \star \lambda(V_0, V_1)$ as described in section 3.1, has an additional term which is proportional to the Maslov number of the S^1 fibers and hence the term $2k$ appears in the Conley-Zehnder index. □

Remark 5.5. If we were to choose the "trivial" trivialization for $T(\mathbb{R}^4 \times S^1)$, then the term $2k$ will not appear in the grading.

The grading is then given by $|T| = 0$ and

$$\begin{aligned} \left|\vec{q}^{\,k}\right| &= \mathrm{CZ}(c) - 1 = 2k + 2 \\ \left|\overleftarrow{q}^{\,k}\right| &= 1 - \mathrm{CZ}(c) = 2k - 2 \\ \left|q_{\pm}^{k}\right| &= 2k \pm 1\,. \end{aligned}$$

As in [19] and similar to [2, section 8], the differential is split up as $\partial = \partial_{\text{hol}} + \partial_{\text{MB}}$. ∂_{hol} is the part coming from punctured pseudoholomorphic disks that only has punctures asymptotic in the symplectization to Reeb chords corresponding to double points. The rest of the differential is caputured in ∂_{MB}. Some disks captured in ∂_{MB} can be thought of as Morse flow lines inside Morse-Bott submanifolds, but perhaps a better picture is to compare it to counting *cascades* in Morse-Bott homology [15, p. 12].

Since punctured pseudoholomorphic disks $u\colon (D^2, \partial D^2) \longrightarrow (\mathbb{R}^4 \times \mathbb{C}^*, \Lambda \times \mathbb{R})$ projects down via Π to pseudoholomorphic disks $\tilde{u}\colon (D^2, \partial D^2) \longrightarrow (\mathbb{R}^2 \times S^1, \Pi(\Lambda))$, it is easy to see that any disk (modulo reparametrization by a biholomorphism) with boundary on $\Pi(\Lambda)$ that is asymptotic to the Reeb chord at the double point $c \in \pi(\Lambda)$, in the symplectization come as a part of a 1-parameter family in $\Pi(\Lambda)$. Hence there are no rigid punctured pseudoholomorphic disks, whereas $\partial_{\text{hol}} = 0$.

The only contribution to the differential thus come from the Morse-Bott part of the differential. We will count these disks using the explicit perturbation scheme with $f = \varepsilon x_1$ considered as above, which also is similar to the schemes used in [19, section 5]. To this end, we first consider the lift of the front projection of Λ to $\mathbb{R}^3$. The lift is a $\mathbb{Z}$-family of copies of Λ as indicated in fig. 16.

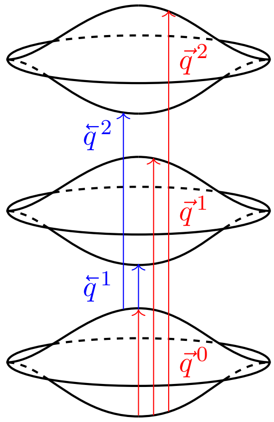

FIGURE 16. Some components of the lift of the front projection of Λ to $\mathbb{R}^3$, with generators indicated in the figure.

We may assume that the rigid disks lie in the slice $x_2 = 0$, because otherwise one can show that if a disk is not contained in the slice $x_2 = 0$, they can not be rigid. The Lagrangian projection after perturbing and projecting is shown in fig. 17.

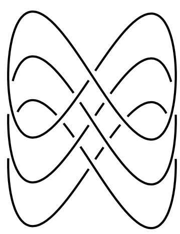

FIGURE 17. A projection of the Lagrangian projection (in $\mathbb{R}^2$) of Λ after perturbing the contact form with a Morse function f.

In order to locate the disks, we will draw a more schematic picture which will depict the same situation as in fig. 17, as also seen in [19, Section 5].

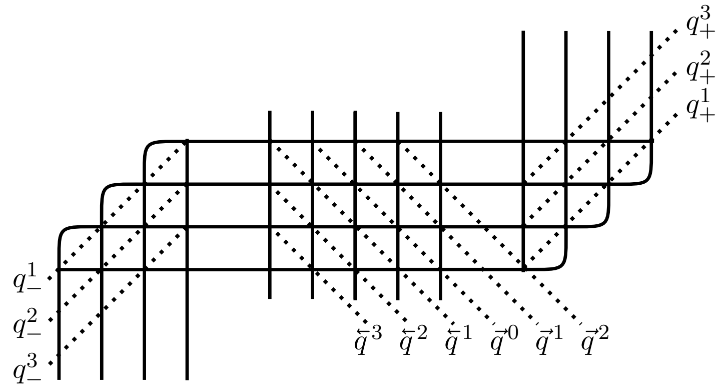

FIGURE 18. Each vertex along each dotted line represents a Reeb chord of Λ.

Using fig. 18, we may compute ∂_{MB}. Two disks that can be found in this diagram is shown in fig. 19.

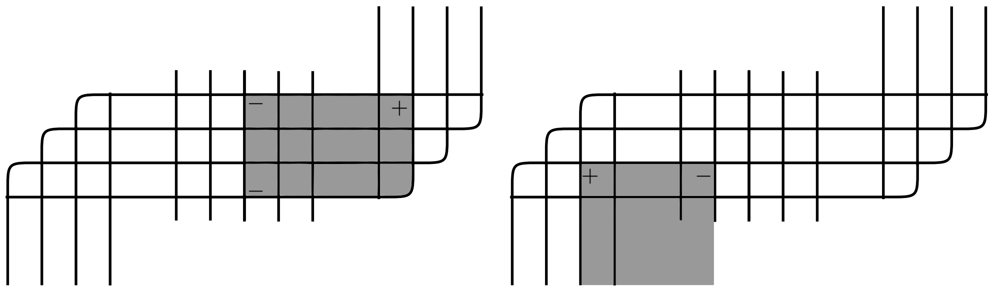

FIGURE 19. The two disks giving the differentials $\partial_{\mathrm{MB}} q^3_+ = \vec{q}^1 \overleftarrow{q}^2$ and $\partial_{\mathrm{MB}} q^2_- = \overleftarrow{q}^2$

The complete differential is given on generators as

$$\begin{aligned} \partial\vec{\boldsymbol{q}} &= \boldsymbol{q}_-\vec{\boldsymbol{q}} + \vec{\boldsymbol{q}}\boldsymbol{q}_- + \boldsymbol{q}_+ \\ \partial\overleftarrow{\boldsymbol{q}} &= \boldsymbol{q}_-\overleftarrow{\boldsymbol{q}} + \overleftarrow{\boldsymbol{q}}\boldsymbol{q}_- \\ \partial\boldsymbol{q}_+ &= \boldsymbol{q}_-\boldsymbol{q}_+ + \boldsymbol{q}_+\boldsymbol{q}_- + \vec{\boldsymbol{q}}\overleftarrow{\boldsymbol{q}} + \overleftarrow{\boldsymbol{q}}\vec{\boldsymbol{q}} \\ \partial\boldsymbol{q}_- &= \boldsymbol{q}_-\boldsymbol{q}_- + \overleftarrow{\boldsymbol{q}}\,. \end{aligned}$$

We see that the only contribution to $\ker\partial$ is given by

$$\begin{cases} \partial\vec{q}^0 = 0 \\ \partial\overleftarrow{q}^1 = 0 \end{cases},$$

but $\overleftarrow{q}^1$ appears in the image since $\partial q^1_- = \overleftarrow{q}^1$ and hence the homology is only generated by $\vec{q}^0$, which has degree 2

$$CH_*(\Lambda) = \langle \vec{q}^0 \rangle\,.$$

Example 5.6. If we now consider the Legendrian unknot as in example 5.2, but we make it large enough so that it forms a link with itself, if we consider the lift to $\mathbb{R}^5$. We call this knot Λ'. The front projection of the lift of Λ' to $\mathbb{R}^5$ is viewed in fig. 20.

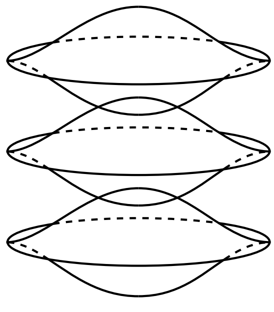

FIGURE 20. The front projection of the lift of Λ' to $\mathbb{R}^5$.

Similar to the above example we have $\partial_{\text{hol}} = 0$, and we may compute the Morse-Bott part of the differential in a similar way. We decompose $\partial = \partial_{\text{MB}} = \partial_1 + \partial_2$, where

$$\begin{cases} \partial_1 \vec{\boldsymbol{q}} = \vec{\boldsymbol{q}} \boldsymbol{q}_- + \boldsymbol{q}_- \vec{\boldsymbol{q}} + \boldsymbol{q}_+ \\ \partial_1 \overleftarrow{\boldsymbol{q}} = 0 \\ \partial_1 \boldsymbol{q}_- = \boldsymbol{q}_- \boldsymbol{q}_- \\ \partial_1 \boldsymbol{q}_+ = \boldsymbol{q}_+ \boldsymbol{q}_- + \boldsymbol{q}_- \boldsymbol{q}_+ \end{cases},$$

and

$$\begin{cases} \partial_2 \vec{q}^{\,k} = \overleftarrow{q}^1 q_-^{k+1} + q_-^{k+1} \overleftarrow{q}^1, & k \geq 0 \\ \partial_2 \overleftarrow{q}^k = \sum_{\substack{l+s=k \\ l \geq 2}} \overleftarrow{q}^l q_-^s + q_-^s \overleftarrow{q}^l, & k \geq 3 \\ \partial_2 \boldsymbol{q}_- = 0 \\ \partial_2 \boldsymbol{q}_+ = 0 \end{cases}.$$

From this, we get that the homology is

$$CH_*(\Lambda) = \langle \overleftarrow{q}^1, q_-^1 \rangle / \langle q_-^1 q_-^1, q_-^1 \overleftarrow{q}^1 + \overleftarrow{q}^1 q_-^1 \rangle\,.$$

5.1.A. *Another type of perturbation of the contact form.* One can compute the Legendrian contact homology of Λ as in example 5.2, by using a Morse-Bott perturbation as described in section 4.1. We first shrink Λ and make it small so that

$$l(\vec{q}^{\,0}) = \varepsilon$$
$$l(\overleftarrow{q}^1) = 2\pi - \varepsilon\,,$$

for some small $\varepsilon > 0$. Then we perturb the contact form by a function f on $\mathbb{R}^4 \times S^1$ so that $i\nabla f$ is transverse to the S^1-fibers. As in section 4.1, the perturbed Reeb flow is $\widetilde{R} = R + i\nabla f$. The idea is to choose ε small, and an appropriate f so that $\vec{q}^{\,0}$ is the only Reeb chord of $\widetilde{R}$. The Reeb chords $\vec{q}^{\,k+1}$ and $\overleftarrow{q}^k$ for $k \geq 0$ will all have length which is greater or equal to $2\pi - \varepsilon$ and will vanish, since those Reeb chords will get pushed off from Λ far enough so they are not Reeb chords anymore. We may for example choose $f = x$ where (x, y, z, w, θ) are coordinates on $\mathbb{R}^4 \times S^1$. Then $i\nabla f = (0, 0, 0, 1, 0)$ with the standard complex structure i on $\mathbb{R}^4$. Letting ε be small enough, all Reeb chords except for $\vec{q}^{\,0}$ will get pushed off far enough, and therefore the Legendrian contact homology algebra of the Legendrian unknot in $\mathbb{R}^4 \times S^1$ is the same as in $\mathbb{R}^5$ with the only generator being the short Reeb chord $\vec{q}^{\,0}$.

5.2. **Example in a circle bundle over** $\mathbb{C}\mathrm{P}^2$**.** On $S^5 \subset \mathbb{C}^3$ there is a free action given by

$$\rho\colon\ S^5 \times S^1 \longrightarrow S^5$$
$$(z_1, z_2, z_3, e^{i\theta}) \mapsto \left(e^{i\theta} z_1, e^{i\theta} z_2, e^{i\theta} z_3\right)\,.$$

The quotient S^5/S^1 gives $\mathbb{C}\mathrm{P}^2$ via this action. Hence we have a circle bundle over $\mathbb{C}\mathrm{P}^2$ with total space $S^5 \cong \mathbb{R}^5 \cup \{\infty\}$ and Lagrangian projection $\pi\colon\ S^5 \longrightarrow \mathbb{C}\mathrm{P}^2$. We consider the standard contact structure on S^5

$$\alpha = \iota^* \left(\frac{1}{2} \sum_{i=1}^{3} x_i dy_i - y_i dx_i \right) ,$$

where $\iota\colon\ S^5 \longrightarrow \mathbb{R}^6$ is the inclusion, and $(x_i, y_i)_{i=1}^3$ are coordinates in $\mathbb{R}^6$. The Reeb vector field is

$$R_\alpha = \frac{1}{2} \sum_{i=1}^{3} x_i \partial_{y_i} - y_i \partial_{x_i} .$$

Example 5.7. We let Λ be a Legendrian knot, which is small enough so that it is contained in a Darboux chart, in which Λ is the Legendrian 2-unknot in S^5 as in example 5.2. We may then perturb Λ so that it is small enough for $\pi(\Lambda)$ to be contained in the neighborhood

$$U = \left\{ [z_1, z_2, 1] \in \mathbb{C}\mathrm{P}^2 \mid (z_1, z_2) \in \mathbb{C} \right\} \subset \mathbb{C}\mathrm{P}^2 .$$

If necessary, we may shrink Λ so that it is contained in an even smaller neighborhood $V \subset U$, which lifts to $V \times S^1 \subset \mathbb{C}\mathrm{P}^2 \times S^1$ in S^5. Then we may consider the front projection

$$\Pi\colon\ U \longrightarrow \widetilde{U} ,$$

where $\widetilde{U} = \left\{ [x_1, x_2, 1] \in \mathbb{R}\mathrm{P}^2 \mid (x_1, x_2) \in \mathbb{R}^2 \right\} \subset U$. Again if necessary, we shrink $\Pi(\Lambda)$ to $\widetilde{V} \subset \widetilde{U}$ so that it lifts to $\widetilde{V} \times S^1 \subset \mathbb{R}\mathrm{P}^2 \times S^1$ in S^5. The front projection in $\mathbb{R}\mathrm{P}^2 \times S^1$ of the perturbed Λ can be made so it looks like the Legendrian sphere in fig. 15. This motivates the fact that Legendrian contact homology of the 2-unknot in the fibration $S^5 \xrightarrow{\pi} \mathbb{C}\mathrm{P}^2$ is the same as in $\mathbb{R}^4 \times S^1$.

Appendix A. Maslov index of a path of Lagrangian submanifolds

For this section we follow [18]. We let $(\mathbb{R}^{2n}, \omega)$ be a symplectic vector space. To any Lagrangian subspace $V \subset \mathbb{R}^{2n}$ we may define $\Sigma_k(V)$ as the set of Lagrangian subspaces of $\mathbb{R}^{2n}$ which intersect V in a k-dimensional subspace of $\mathbb{R}^{2n}$. $\Sigma_k(V)$ is a connected subspace of the Lagrangian Grassmannian $\mathcal{L}(n)$. *The Maslov cycle* associated to V is

$$\Sigma(V) := \overline{\Sigma_1(V)} = \bigcup_{k=1}^{n} \Sigma_k(V) .$$

can pick a Lagrangian complement W, $\mathbb{R}^{2n} = V \oplus W$. Any Lagrangian subspace transverse to W may be viewed as the graph of a quadratic form $A\colon\ V \longrightarrow W$. In fact we have a canonical isomorphism

$$\begin{aligned} T_V \mathcal{L}(n) &\longrightarrow S^2(V) \\ (V, \hat{V}) &\mapsto Q = Q(V, \hat{V}) , \end{aligned}$$

where $S^2(V)$ is the set of quadratic forms on V. The quadratic form Q is defined as follows.

Let $\Gamma\colon\ [0,1] \longrightarrow \mathcal{L}(n)$ be a smooth curve of Lagrangian subspaces with $\Gamma(0) = V$ and $\dot{\Gamma}(0) = \hat{V}$. If W is a fixed Lagrangian complement of V and for $v \in V$ define the path $w(t) \in W$ such that $v + w(t) \in \Gamma(t)$ for small t. Then

$$Q(V, \hat{V}) = \frac{\mathrm{d}}{\mathrm{d}t} \omega(v, w(t)) \Big|_{t=0} .$$

Proposition A.1. *Q is independent of the Lagrangian complement W chosen.*

Proof. We choose coordinates so that $\Gamma(0) = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{2n}$. Then any Lagrangian complement of $\Gamma(0)$ is the graph of a symmetric matrix $B \in \mathbb{R}^{n \times n}$,

$$W = \left\{ (By, y) \mid y \in \mathbb{R}^n \right\} .$$

For small t, the Lagrangian subspace $\Gamma(t)$ is the graph of a symmetric matrix $A(t) \in \mathbb{R}^{n \times n}$,

$$\Gamma(t) = \left\{ (x, A(t)x) \mid x \in \mathbb{R}^n \right\} .$$

In these coordinates, write $v = (x, 0)$, $w(t) = (By(t), y(t))$ so that $v + w(t) = (x + By(t), y(t)) \in \Gamma(t)$. So for some $z \in \mathbb{R}^n$, we have $y(t) = A(t)z$ and $x + By(t) = z$, so $y(t) = A(t)(x + By(t))$. So

$$Q(\Gamma(0), \dot{\Gamma}(0)) = \frac{\mathrm{d}}{\mathrm{d}t}\, \omega(v, w(t))\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \langle x, y(t)\rangle\Big|_{t=0} = \langle x, \dot{y}(0)\rangle \ . \tag{A.1}$$

From $y(t) = A(t)(x + By(t))$ we see $y(0) = 0$ and $\dot{y}(0) = \dot{A}(0)x$, so from (A.1) we get

$$Q(\Gamma(0), \dot{\Gamma}(0)) = \left\langle x, \dot{A}(0)x \right\rangle \ ,$$

which is independent of B and hence the Lagrangian complement W. □

Now, the Maslov index of a closed loop of Lagrangian subspaces is the intersection number of the loop with the Maslov cycle. More precisely, if $\Gamma\colon [0,1] \longrightarrow \mathcal{L}(n)$ is a loop of Lagrangian subspaces and $V \subset \mathbb{R}^{2n}$ we consider the quadratic form

$$g(\Gamma(t), V, t) := Q(\Gamma(t'), \dot{\Gamma}(t'))|_{\Gamma(t') \in \Sigma(V)} = \frac{\mathrm{d}}{\mathrm{d}t}\, \omega(v, w(t))\Big|_{t=t'} \ ,$$

for t near t', where v and $w(t)$ as as above for a Lagrangian complement W. The Maslov index is then the sum of signatures of the quadratic form $g(\Gamma(t), V, t)$, over such t so that $\Gamma(t) \in \Sigma(V)$, that is $\Gamma(t)$ intersects V non-trivially. Namely

$$\mu(\Gamma) := \sum_{t \in [0,1]\,:\, \Gamma(t) \in \Sigma(V)} \mathrm{sign}\left(g(\Gamma(t), V, t)\right) \ .$$

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