



Classification of plethories in characteristic zero

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Abstract

In this thesis we classify plethories over fields of characteristic zero, thus answering a question of Borger-Wieland and Bergman. All plethories over characteristic zero fields are linear, in the sense that they are free plethories on a bialgebra. For the proof we need some facts from the theory of ring schemes where we extend previously known results. We also classify plethories with trivial Verschiebung over a perfect field of non-zero characteristic and indicate future work.

Sammanfattning

I denna avhandling klassificerar vi s.k. plethories över kroppar av karakteristisk noll och svarar därmed på en fråga formulerad av Borger-Wieland och Bergman. Alla plethories över en kropp av karakteristisk 0 är linjära, i det avseende att de är fria konstruktioner på ett bialgebra. För att bevisa detta behöver vi några resultat från teorin om ringscheman där vi utvidgar tidigare kända satser. Vi klassificerar även plethories med trivial Verschiebung över en perfekt kropp av nollskild karakteristisk och indikerar hur vi tror framtida forskning på området skulle kunna te sig.

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1 Introduction

In this thesis we will study plethories and the aim of this introduction is to give an informal exposition to plethories and how they relate to different areas of mathematics. We will also give some background on the theory of group schemes, which is used heavily in the article we present in this thesis. In this introduction, we will sometimes define the objects of interest in less generality than in the aforementioned article. Our hope is that a reader, by reading this expository introduction, will gain some intuition and see some motivating examples without the theory laid out in full.

1.1 Witt vectors and Frobenius lifts

Let k be any ring. Given an algebra A over k one can construct its ring of p -typical Witt vectors

$$W_p(A).$$

The ring of p -typical Witt vectors of A is an amazingly rich algebraic structure and understanding its properties is of vital importance to many different areas of mathematics. The object $W_p(A)$ (and more generally, the functor W_p) satisfies a variety of universal properties but we shall focus on one property in particular, as explained in [1]. Let us now suppose that the k -algebra A has no p -torsion for the sake of simplicity (so k is necessarily p -torsion free as well).

Definition 1.1. Let A be a p -torsion free k -algebra and let $f : A \rightarrow A$ be an endomorphism of A . We say that f is a Frobenius lift if the induced map

$$\bar{f} : A/pA \rightarrow A/pA,$$

coincides with the p th power map, i.e if $\bar{f}(a) = a^p$ for all $a \in A/pA$.

Example 1.2. Consider \mathbb{Z} . In this case, given the prime p , a Frobenius lift is given by

$$f(a) = a.$$

Example 1.3. Let us now work with the ring $\mathbb{Z}[x]$. We will here give an example of a more interesting example of a Frobenius lift than the obvious one, taken from Clauwens [7]. Let us define the Frobenius lift

$$f_p : \mathbb{Z}[x] \rightarrow \mathbb{Z}[x]$$

as

$$f_p(a) = a \text{ for } a \in \mathbb{Z} \text{ and } f_p(x) = T_p(x)$$

where $T_p(x)$ is the p th Chebyshev polynomial, defined inductively by

$$T_0(x) = 2, \quad T_1(x) = x \text{ and } T_{n+1}(x) = xT_n - T_{n-1}.$$

f_p is then a Frobenius lift and the definition works for all primes p . If we want to find commuting Frobenius lifts for all primes p on $\mathbb{Z}[x]$, there is, up to isomorphism, only two choices.

Remark 1.1.1. Let A be a p -torsion free ring. Call a function (of sets) $\delta_p : A \rightarrow A$ a p -derivation if

$$\begin{aligned} \delta_p(a + b) &= \delta_p(a) + \delta_p(b) + (a^p + b^p - (a + b)^p)/p, \\ \delta_p(ab) &= \delta_p(a)b^p + a^p\delta_p(b) + p\delta_p(a)\delta_p(b) \end{aligned}$$

and $\delta(1) = 0$. One easily shows that there is a bijection between Frobenius lifts and p -derivations. Indeed, given a Frobenius lift f set $\delta_p(a) = (a - a^p)/p$ and given a p -derivation δ_p define a Frobenius lift by $f(a) = a^p + p\delta_p(a)$. For more on p -derivations see [5].

Denote by $\Lambda_p - \text{Alg}_k$ the category which has as objects the p -torsion free k -algebras A together with a Frobenius lift f , and which has as morphisms the maps

$$g : A \rightarrow B$$

commuting with the Frobenius lifts of A and B . We call an object of $\Lambda_p - \text{Alg}_k$ a Λ k -algebra. There is an evident forgetful functor F from

$$\Lambda_p - \text{Alg}_k$$

to the category of p -torsion free k -algebras and one can show that F has a right adjoint W_p , called the p -typical Witt functor. The universal property of $W_p(A)$, at least for p -torsion free k -algebras A is now particularly easy to describe: $W_p(A)$ is the terminal p -torsion free algebra equipped with a Frobenius lift

$$F_p : W_p(A) \rightarrow W_p(A)$$

and a map

$$W_p(A) \rightarrow A$$

of rings (so this map is the counit of the adjunction). One can extend this definition to all k -algebras as is done for example in [1]. So, in a very precise way one can state that the ring of p -typical Witt vectors gives us a best possible Frobenius lift on a p -torsion free algebra A . Despite this very natural universal property, the standard construction of the ring of Witt vectors is quite complicated and not as conceptual as one would want. It has been long known that the ring-valued Witt vector functor is representable (see for example the section by Bergman in [13]) by an affine ring scheme, i.e that there is a k -algebra S_p such that

$$\mathrm{Alg}_k(S_p, -) : \mathrm{Alg}_k \rightarrow \mathrm{Set}$$

factors through the category of rings and that

$$\mathrm{Alg}_k(S_p, A) \cong W_p(A).$$

The ring S_p is the ring of p -typical symmetric functions, for more on symmetric functions see [11]. Viewing W_p as a representable functor is a step in the right direction, but if one wants to construct certain natural maps involving Witt vectors, such as the Artin-Hasse exponential or the ghost maps, one must look more closely at S_p and by doing this one is faced with manipulations involving symmetric functions. In [2] Borger-Wieland gave a conceptual definition of the ring of Witt vectors, which avoids this "formulaic approach", using the theory of plethories which identifies $\Lambda_p - \mathrm{Alg}_k$ as a category of P -rings for some plethory P .

1.2 Affine ring schemes and plethories

Let us say that an affine ring scheme is a ring A together with a lift of the covariant functor

$$\mathrm{Spec} A(-) = \mathrm{Alg}_k(A, -) : \mathrm{Alg}_k \rightarrow \mathrm{Set}$$

to rings. We say that A is an affine k -algebra scheme if the lift of

$$\mathrm{Alg}_k(A, -) : \mathrm{Alg}_k \rightarrow \mathrm{Set}$$

to rings actually takes values in k -algebras. It is possible to compose two different k -algebra schemes, A and B as

$$\mathrm{Alg}_k(A, -) \circ \mathrm{Alg}_k(B, -) = \mathrm{Alg}_k(A, \mathrm{Alg}_k(B, -))$$

and one can show that this functor is representable by an object $A \odot_k B$ [2]. This gives us a monoidal structure on the category of affine k -algebra schemes with unit $k[e]$ (see the following example) and we define a plethory P as a comonoid in this category. We say that a k -algebra A is a P -ring if

$$\mathrm{Alg}_k(A, -) : \mathrm{Alg}_k \rightarrow \mathrm{Set}$$

has the structure of a coalgebra over the comonad $\mathrm{Alg}_k(P, -)$. This means that P has a natural action on A .

Example 1.4. For any ring k the affine line $k[e]$ is a plethory. Indeed, we note that the functor

$$\mathrm{Alg}_k(k[e], -)$$

has a lift to k -algebras and that it is naturally isomorphic to the identity functor

$$\mathrm{Id} : \mathrm{Alg}_k \rightarrow \mathrm{Alg}_k.$$

It is a comonad in the monoidal structure on representable endofunctors since the identity functor is obviously idempotent, so that

$$\mathrm{Alg}_k(k[e], \mathrm{Alg}_k(k[e], -)) \cong \mathrm{Alg}_k(k[e], -).$$

Example 1.5. Let G be a group (or a monoid) and consider the group algebra $k[G]$. One can then endow $S(k[G]) = P$ with the structure of a plethory (see [2], example 2.7). A P -ring is then precisely a ring with an action of the group G .

We will here explain why S_p is a plethory and also give a way of defining it in a conceptual manner, away from the symmetric functions approach. Let us consider the \mathbb{F}_p -plethory $\mathbb{F}_p[e]$. Let F be the Frobenius endomorphism and consider the free plethory

$$\mathbb{Z}\langle F \rangle = \mathbb{Z}[e, F, F \circ F, \dots].$$

Here, $\mathbb{Z}\langle F \rangle$ is the free plethory on the monoid ring $\mathbb{Z}[\mathbb{N}]$. A $\mathbb{Z}\langle F \rangle$ -ring is then precisely a ring A together with an endomorphism $\psi : A \rightarrow A$. There is a natural map of plethories

$$\mathbb{Z}\langle F \rangle \rightarrow \mathbb{F}_p[e],$$

given by $e \mapsto e$ and $F, F \circ F, \dots \mapsto 0$, that is surjective as a map of algebras. Let us say that a p -torsion-free ring A is a Frobenius-deformation of a $\mathbb{F}_p[e]$ -ring if the action of $\mathbb{Z}\langle F \rangle$ on A/pA factors through the action of $\mathbb{F}_p[e]$ on A/pA . Since an action of $\mathbb{Z}\langle F \rangle$ on A is equivalent to giving an endomorphism

$$\psi : A \rightarrow A,$$

the requirement is the same as saying that ψ reduces modulo p to the action of the Frobenius

$$a \mapsto a^p$$

on A/pA . One can now show that there is a \mathbb{Z} -plethory P' , which one calls the amplification of $\mathbb{Z}\langle F \rangle$ along $\mathbb{F}_p[e]$ such that Frobenius-deformations of $\mathbb{F}_p[e]$ -rings are precisely p -torsion free P' -rings. Note that P' is by its defining universal property unique up to unique isomorphism. The amplification construction is very natural and is akin to a blow-up (see Borger-Wieland [2]). In [2] it is then shown that P' , the amplification of $\mathbb{Z}\langle F \rangle$ along $\mathbb{F}_p[e]$ is isomorphic to S_p as plethories. This is equivalent to showing that for a p -torsion free ring A an action of S_p is the same as giving a Frobenius lift. The way we defined the plethory S easily generalizes to other Dedekind domains where p is replaced by any non-zero prime ideal.

1.3 Classification of plethories

As we sketched in the previous section, the functor taking A to its ring of Witt vectors is governed by the plethory $P' = S_p$. One of the main reasons for studying plethories is to understand what kind of things can act on rings. One particularly interesting example, in our opinion, is given by S_p . A fundamental question one might ask is whether one can classify plethories over a given ring k . We don't expect to be able to answer this for all rings but we would hope to some day understand plethories over \mathbb{Z} . In this thesis, we start a classification by classifying plethories over any field k of characteristic 0 and we show that they are "linear", which means that all plethories come from bialgebras in a sense made precise in the main part of this thesis. In particular, this is true for \mathbb{Q} . We saw that one could construct a particularly interesting example of a plethory P by looking at the amplification of a \mathbb{Z} -plethory along a \mathbb{F}_p -plethory, and we thus believe that to understand the situation fully over \mathbb{Z} one should first classify plethories over \mathbb{F}_p (or more generally, perfect fields). Here, we achieve a classification for plethories such that the Verschiebung is zero but further classification is needed.

1.4 Group schemes

Let k be a field. Let us consider Sch_k , the category of schemes over k .

Definition 1.6. A (commutative) group scheme G is a scheme G such that the representable functor

$$\text{Sch}_k(-, G) : \text{Sch}_k \rightarrow \text{Set}$$

has a lift to a functor with values in the category of (abelian) groups.

Let us mention that not all group schemes are affine: some fundamental examples of non-affine group schemes are abelian varieties over k . In the main article we will in particular be concerned with unipotent group schemes. For ease of exposition, we will from now on assume that G is also of finite type. Recall that a matrix A over a field k is unipotent if

$$P_A(t) = (t - 1)^m$$

for some $m > 0$ where $P_A(t)$ is the characteristic polynomial of A .

Any affine group scheme of finite type embeds faithfully into the affine group scheme GL_n , [9] [12] for some $n > 0$. For a ring R , $GL_n(R)$ consists of the invertible matrices with entries in R . There is a subgroup scheme \mathbb{U}_n of GL_n , which to R assigns $\mathbb{U}_n(R)$, the upper triangular matrices with entries in R .

Definition 1.7. Let G be a group scheme. We say that G is a unipotent group scheme if we can find a faithful embedding

$$r : G \rightarrow GL_n$$

such that

$$r(G) \subset \mathbb{U}_n.$$

There is another class of affine group schemes which are very natural to consider. Recall that a matrix M over a field k is semi-simple if the minimal polynomial of M has no square factors. This is equivalent to saying that if V is the vector space M acts on, we have a direct sum decomposition

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$$

where each V_i is stable under the action of M and such that there are no non-zero proper subspaces of V_i stable under the action of M . Given a matrix M over a perfect field k we always have the Jordan-Chevalley decomposition which says that we can write any matrix M as a product

$$M = M_u M_s$$

where M_u is unipotent and M_s is semisimple and they commute with each other. It is then natural to ask whether one can get a similar decomposition for G an affine group scheme of finite type. It turns out that if G is commutative, this is possible. We will first need to ask what a natural generalization of semisimple matrices is in the category of group schemes. Let us note that a matrix M is semisimple iff it is diagonalizable after base change to some possibly larger field. This motivates the following definition:

Definition 1.8. Let G be an affine group scheme. We say that G is of multiplicative type if the base change $G_{k^{\text{sep}}}$ is such that any representation

$$r : G_{k^{\text{sep}}} \rightarrow GL_n$$

is diagonalizable, i.e. a sum of one-dimensional representations.

Remark 1.4.1. There is also a class of group schemes called semisimple group schemes. We will not concern ourselves with them in this thesis, but just note their existence to avoid confusing multiplicative group schemes with semisimple group schemes.

We are now in a position to generalize the Jordan-Chevalley decomposition. For a proof, see [12], [9].

Theorem 1.9. *Let G be an affine commutative group scheme of finite type over a perfect field k . Then there is a canonical decomposition*

$$G = G_u \times G_m$$

where G_u is a unipotent group scheme and G_m is a multiplicative group scheme.

This structure theorem is a very powerful tool for understanding affine commutative group schemes G of finite type. A next step would thus be to gain a better understanding of the non-affine commutative group schemes. As previously noted, we have that the proper group varieties (i.e. abelian varieties) yield examples of non-affine group schemes. Essentially, in [6] Chevalley (see also [8] for a modern proof) shows that to understand connected group varieties, one important aspect is to understand abelian varieties and affine group varieties. More precisely, we have:

Theorem 1.10. *Let G be a connected group variety over a perfect field k . There is then a unique short exact sequence*

$$0 \rightarrow N \rightarrow G \rightarrow A \rightarrow 0$$

such that A is an abelian variety and N is a connected affine normal subgroup variety.

This theorem gives us in particular that there is a smallest connected affine group variety such that G/N is a variety (and in general, non-affine). There is a "dual" decomposition theorem in the sense that there is a largest affine quotient. We will make this precise after some definitions.

Definition 1.11. Let G be a group scheme of finite type over k . We say that G is anti-affine if $\mathcal{O}_G(G) = k$.

Anti-affine groups have been studied in great detail by Brion in [4]. Abelian varieties are in particular anti-affine groups and over a perfect field k all anti-affine group schemes are "semi-abelian" varieties (i.e. an extension of an abelian variety by a torus).

Theorem 1.12 (Brion [3] Theorem 1). *Let G be a group scheme of finite type over a field k . Then there is an exact sequence*

$$0 \rightarrow G^{\text{ant}} \rightarrow G \rightarrow G/G^{\text{ant}} \rightarrow 0$$

such that G^{ant} is anti-affine and G/G^{ant} is affine.

This theorem (which we also mention in the article included in this thesis) is used to show that any ring scheme of finite type over a field k is affine.

2 Summary of results

We study plethories as defined by Borger-Wieland [2] and Tall-Wraith [14]. We show that over a field k of characteristic zero all plethories are linear, meaning that they are the free plethory on a bialgebra. To prove this result, we use the theory of group schemes and ring schemes in some detail, and generalize some results first shown by Greenberg [10] on ring schemes. We then achieve our classification results by first classifying $k-k$ -birings for a field k of characteristic zero, which leads to the result that all plethories over k are linear. We also study the classification problem for perfect fields k of characteristic $p > 0$ and show that for plethories with trivial Verschiebung all plethories are quotients of linear plethories. We also include some new pathological examples of plethories which show what a future classification theorem must take into account.

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Part II



Classification of plethories in characteristic zero

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CLASSIFICATION OF PLETHORIES IN CHARACTERISTIC ZERO

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ABSTRACT. We classify plethories over fields of characteristic zero, thus answering a question of Borger-Wieland and Bergman. All plethories over characteristic zero fields are linear, in the sense that they are free plethories on a bialgebra. For the proof we need some facts from the theory of ring schemes where we extend previously known results. We also classify plethories with trivial Verschiebung over a perfect field of non-zero characteristic and indicate future work.

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1. INTRODUCTION

Plethories, first introduced by Tall-Wraith [13], and then studied by Borger-Wieland [3], are precisely the objects which act on k -algebras, for k a commutative ring. There are many fundamental questions regarding plethories which remain unanswered. One such question is, given a ring k , whether one can classify plethories over k , in this paper we will take a first step towards a classification.

For some motivation, let us start by looking at the category of modules Mod_k over a commutative ring k . If we consider the category of representable functors $\text{Mod}_k \rightarrow \text{Mod}_k$, there is a monoidal structure given by composition of functors. Then one defines a k -algebra R as a k -module R such that the representable endofunctor $\text{Mod}_k(R, -) : \text{Mod}_k \rightarrow \text{Mod}_k$ has

a comonoid structure with respect to composition of functors. Heuristically, this says that a k -algebra is precisely the kind of object which knows how to act on k -modules. This can be extended to a non-linear setting, so that instead of looking at k -modules we look at k -algebras Alg_k and consider representable endofunctors $\text{Alg}_k \rightarrow \text{Alg}_k$. A comonoid with respect to composition of functors is then called a plethory and analogously, a plethory is what knows how to act on k -algebras. One particular important example of a plethory is the \mathbb{Z} -algebra Λ which consist of the ring of symmetric functions in infinitely many variables with a certain biring structure. The functor $\text{Alg}_k(\Lambda, -) : \text{Alg}_k \rightarrow \text{Alg}_k$ represents the functor taking a ring R to its ring of Witt vectors. Using plethories one gets a very conceptual view of Witt vectors and in [2] James Borger develops the geometry of Witt vectors using the plethystic perspective.

Let now k be a field. If we let \mathcal{P}_k denote the category of plethories over k , there is a forgetful functor

$$F : \mathcal{P}_k \rightarrow \text{Bialg}_k$$

into the category of cocommutative counital bialgebras over k . This functor has a left adjoint $S(-) : \text{Bialg}_k \rightarrow \mathcal{P}$ and we say that a plethory P is linear if $P \cong S(Q)$ for some cocommutative, counital bialgebra Q . Heuristically, a plethory P is linear if every action of P on an algebra A comes from an action of a bialgebra on A . The main theorem of this paper is:

Theorem 1.1. Let k be a field of characteristic zero. Then any k -plethory is linear.

This answers a question of Bergman-Hausknecht [1, p.336] and Borger-Wieland [3] in the positive. The theorem is proved by studying the category of affine ring schemes. We there have the following results, extending those of Greenberg [8] to arbitrary fields and not necessarily reduced schemes:

Theorem 1.2. Let k be a field. Then any connected ring scheme of finite type is unipotent.

Theorem 1.3. Let P be a connected ring scheme of finite type over k . Then P is affine.

For the case of characteristic $p > 0$ our classification results on plethories are not as complete and further work is needed to have a complete classification. To explain our classification results here we need some definitions. Let F_k be the Frobenius homomorphism of k and $k\langle F \rangle$ be the non-commutative ring which as underlying set is $k[F]$ and has multiplication given by

$$F^i F^j = F^{i+j}$$

and

$$Fa = F_k(a)F.$$

We define Bialg_k^p to be the category of cocommutative, counital bialgebras over k which also are modules over $k\langle F \rangle$. Once again, for a plethory over a perfect field k of char $k > 0$ there is a forgetful functor

$$\mathcal{P}_k \rightarrow \text{Bialg}_k^p$$

which has a left adjoint $S^{[p]}$. Call a plethory P p -linear if $P \cong S^{[p]}(Q)$ for some $Q \in \text{Bialg}_k^p$. We have then the following classification result:

Theorem 1.4. Let k be a perfect field of characteristic $p > 0$. Assume that P is a plethory over k such that the Verschiebung $V_P = 0$. Then P is p -linear.

The structure of this paper is as follows. In section 2 we study ring schemes and prove some results which we will need for our classification theorem. The main theorems of this section that are needed for later purposes are Theorem 2.6 and Theorem 2.7. In section 3 we introduce plethories and $k-k$ -birings and provide some examples. This section contains no new results and gives just a brief introduction to the relevant objects as defined in Borger-Wieland [3]. In section 4 we prove that all plethories over a field k of characteristic zero is linear using the results from section 2. We also show that any $k-k$ -biring is connected. In section 5 we prove some initial classification results regarding plethories in characteristic $p > 0$.

NOTATION AND CONVENTIONS

Ring	category of rings.
$\text{BR}_{k,k}$	category of $k-k$ -birings.
\mathcal{P}_k	category of k -plethories.
Bialg_k	category of cocommutative k -bialgebras.
Bialg_k^p	category of cocommutative k - p -bialgebras.
\odot	composition product of $k-k$ -birings, Def. 3.2.
\mathcal{R}	generic name for a ring scheme.
Alg_k	category of commutative algebras over the ring k .
$\Delta_A^+, \Delta_A^\times$	coaddition resp. comultiplication map for a biring A .
$\epsilon_A^+, \epsilon_A^\times$	counit for coaddition resp. comultiplication for a biring A .
β_A	co- k -algebra structure on a $k-k$ -biring A .
$\Delta_2^+, \Delta_2^\times$	abbreviation for the composite $(1 \otimes \Delta^+) \circ \Delta^+$ resp. $(1 \otimes \Delta^\times) \circ \Delta^\times$.
P	primitive elements functor
\mathcal{O}_X	structure sheaf of a scheme X .
Sch_k	category of k -schemes for k a commutative ring.
\mathbb{G}_a	the affine line viewed as a group scheme, see Ex. 3.1

- \mathbb{G}_m the multiplicative group scheme, after Def. 2.5.
- μ_p the p -th root of unity group scheme, Ex. 5.2
- α_p see Ex. 3.3
- $\pi_0(G)$ the group scheme of connected components of a group scheme G over the field k , Def. 4.2
- S free plethory functor on a cocommutative bialgebra. Def. 4.1
- $S^{[p]}$ free plethory functor on a cocommutative p -bialgebra, after Def. 5.1.
- G° the identity component of a group scheme G .
- F_G, V_G the Frobenius resp. Verschiebung morphism of a group scheme G over a perfect field of characteristic $p > 0$.
- \mathbb{F}_q the constant group scheme on \mathbb{F}_q , Ex. 3.3.
- $k\langle F \rangle$ the twisted polynomial algebra.

For us, all rings are commutative and unital. We will use Swedler notation for coaddition Δ^+ and Δ^\times , so that $\Delta^+(x) = \sum_i x_i^{(1)} \otimes x_i^{(2)}$ and $\Delta^\times(x) = \sum_i x_i^{[1]} \otimes x_i^{[2]}$ if $x \in A$ where A is a biring. For concepts from the theory of group schemes not introduced properly here, we refer to [11] or [6].

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2. RING SCHEMES

Let k be a commutative ring. Recall that \mathcal{R} is a ring scheme over k if \mathcal{R} is a scheme and the functor

$$\mathrm{Sch}_k(-, \mathcal{R}) : \mathrm{Sch}_k \rightarrow \mathrm{Set}$$

has a lift to a functor

$$\mathrm{Sch}_k(-, \mathcal{R}) : \mathrm{Sch}_k \rightarrow \mathrm{Ring}.$$

We say that a functor is a k -algebra scheme if we can lift it to a functor taking values in k -algebras. We will mostly be concerned with affine ring schemes. Ring schemes were studied by Greenberg in [8] and he showed that for connected, reduced ring schemes of finite type over an algebraically closed field k , the underlying scheme is always affine. Further, he shows that the underlying group variety is always unipotent. We improve on these results by showing that any connected ring schemes of finite type over an arbitrary field is affine, and that the underlying group scheme is always unipotent. From now on, in this section, k is always a field.

Definition 2.1. A scheme X is anti-affine if $\mathcal{O}_X(X) = k$. We say that a group scheme is anti-affine if its underlying scheme is anti-affine.

For example, abelian varieties are all anti-affine group schemes. An anti-affine group scheme has the property that any morphism from it into an affine group scheme is trivial. Anti-affine groups are very important for the structure of group schemes as the following theorem shows:

Theorem 2.2 (Brion [4] Theorem 1). If G is a group scheme of finite type over a field k there is an exact sequence of group schemes

$$0 \rightarrow G^{\text{ant}} \rightarrow G \rightarrow G/G^{\text{ant}} \rightarrow 0$$

such that G^{ant} is anti-affine and G/G^{ant} is affine.

We will now want to show that all connected finite type ring schemes are affine, i.e that in the above exact sequence $G^{\text{ant}} = \text{Spec } k$. For this, we will need the following lemma.

Lemma 2.1. *Let X, Y, Z be schemes with X quasi-compact and anti-affine and Y locally noetherian and irreducible. Suppose that $f : X \times Y \rightarrow Z$ is a morphism such that there exist k -rational points $x_0 \in X$, $y_0 \in Y$ such that $f(x, y_0) = f(x_0, y_0)$ for all x . Then $f(x, y) = f(x_0, y)$ for all x, y .*

Proof. see [4] lemma 3.3.3 .

□

Theorem 2.3. Let \mathcal{R} a connected ring scheme of finite type over k . Then \mathcal{R} is affine.

Proof. We know that by Theorem 2.2 that \mathcal{R} sits in the middle of an extension of an affine group scheme by an anti-affine group. Let

$$0 \rightarrow \mathcal{R}^{\text{ant}} \rightarrow \mathcal{R} \rightarrow \mathcal{R}^{\text{aff}} \rightarrow 0$$

be the corresponding extension where \mathcal{R}^{ant} is anti-affine and \mathcal{R}^{aff} the affine quotient. Note that \mathcal{R}^{aff} is a ring scheme so that \mathcal{R}^{ant} defines an ideal scheme in \mathcal{R} , i.e for all rings S over k , $\mathcal{R}^{\text{ant}}(S)$ is an ideal of $\mathcal{R}(S)$. Now, we will apply the above lemma with $Y = \mathcal{R}$ and $X = Z = \mathcal{R}^{\text{ant}}$. Taking $x_0 = e_{\mathcal{R}^{\text{ant}}}$ and $y_0 = e_{\mathcal{R}}$ we have that $m(x, y_0) = m(x_0, y_0)$ is identically equal to zero. Thus, we have that $m(x, y) = m(x_0, y)$ is identically zero. But, letting $1_{\mathcal{R}}$ be the rational point corresponding to the multiplicative identity of $\mathcal{R}(k)$ we have that $m(1_{\mathcal{R}}, y)$ is zero. But multiplication by 1 is always injective, and thus, \mathcal{R}^{ant} is trivial and \mathcal{R}^{aff} is affine.

□

We don't know if the condition for \mathcal{R} to be of finite type is necessary in the above theorem. Let us recall the following definition from the theory of algebraic groups.

Definition 2.4. Let G be a group scheme over k . We say that G is unipotent if it is affine and if every non-zero closed subgroup H of G admits a non-zero homomorphism $H \rightarrow \mathbb{G}_a$.

The data of a homomorphism $G \rightarrow \mathbb{G}_a$ is the same as specifying an element $x \in A_G$ in the underlying Hopf algebra of G that satisfies $\Delta_G(x) = x \otimes 1 + 1 \otimes x$, i.e specifying a primitive element. If $G = \text{Spec } A_G$ is an affine group scheme and A_G the Hopf algebra associated to G , then saying that G is unipotent is the same as saying that it is coconnected (or conilpotent). The following definition will be useful for the proof of Theorem 2.6.

Definition 2.5. Let G be an affine group scheme over a field. We say that G is multiplicative if every homomorphism $G \rightarrow \mathbb{G}_a$ is zero.

An example of a multiplicative group is $\mathbb{G}_m = \text{Spec } k[x, x^{-1}]$. There can in general be no homomorphism from a multiplicative group into a unipotent group and no morphisms from a unipotent group to a multiplicative group (for a proof, see [11] Corollary 15.19-15.20).

The following theorem was shown for reduced ring varieties over an algebraically closed fields by Greenberg, but the results carry over for perfect fields without any modification. We improve on this by carrying through the proof when \mathcal{R} is not necessarily reduced and over any field k . Further, the theorem can be extended to ring schemes not necessarily of finite type if the ring scheme is already known to be affine.

Theorem 2.6. Over a field k , all connected ring schemes \mathcal{R} of finite type are unipotent.

Proof. By the previous theorem we know they are affine. We know that \mathcal{R} contains a greatest multiplicative subgroup \mathcal{R}_m that has the property that for all endomorphisms α of \mathcal{R}_S , (where \mathcal{R}_S is the base change of \mathcal{R} to S) for S a k -algebra, that $\alpha((\mathcal{R}_m)_S) \subset (\mathcal{R}_m)_S$ ([11], Theorem 17.16). Thus, since any $x \in \mathcal{R}(S)$ defines an endomorphism of \mathcal{R}_S (as a group scheme) through multiplication by x , we have that \mathcal{R}_m is an ideal of \mathcal{R} . It is known that any action of a connected algebraic group on a multiplicative group must be trivial, i.e for G connected and H multiplicative, a map $G \rightarrow \text{Aut}(H, H)$ must have image the identity. We will need the following, which says that any map $G \rightarrow \text{End}(H, H)$ where G is any connected group scheme and H is multiplicative is trivial. This is basically just deduced, *mutatis mutandis*, from the proof of [11] Theorem 14.28. So, we see that 0 and 1 defines the same endomorphisms on the ideal scheme \mathcal{R}_m . But this is only the case if $\mathcal{R}_m = 0$. The theorem thus follows. \square

To extend this to all connected ring schemes, we need the following:

Theorem 2.7. Let k be a field and \mathcal{R} be an affine ring scheme over k . Then \mathcal{R} is a filtered limit of its finite type ring schemes.

Proof. The following proof is inspired by the analogue theorem for Hopf algebras over a field, as occurs in for example Milne [11] Proposition 11.32 . Write $\mathcal{R} = \text{Spec } A_{\mathcal{R}}$. We know that $A_{\mathcal{R}}$ is a bialgebra and we see that we can reduce to proving that any $a \in A_{\mathcal{R}}$ is contained in a sub-bialgebra of finite type. Let $\Delta^+ : A_{\mathcal{R}} \rightarrow A_{\mathcal{R}} \otimes A_{\mathcal{R}}$ be the coaddition giving the additive group structure on \mathcal{R} and $\Delta^\times : A_{\mathcal{R}} \rightarrow A_{\mathcal{R}} \otimes A_{\mathcal{R}}$ the comultiplication defining the multiplication on \mathcal{R} . Consider

$$\Delta_2^+(a) = \sum_{i,j} c_i \otimes x_{ij} \otimes d_j$$

with c_i and d_j linearly independent. Now, by the fundamental theorem of coalgebras, we know that if we take X to be the subspace of $A_{\mathcal{R}}$ generated by $\{x_{ij}\}$, then this is a subcoalgebra, i.e that $\Delta^+(x_{ij}) \subset X \otimes X$. Now, for each x_{ij} in this system, consider

$$\Delta_2^\times(x_{ij}) = \sum_{k,l} e_i \otimes y_{kl} \otimes f_l$$

with e_i and f_l linearly independent. With the same arguments, one sees that for the subspace Y generated by $\{y_{kl}\}$ we have $\Delta^\times(y_{kl}) \subset Y \otimes Y$. Let now Z be subalgebra generated by the finite-dimensional subspace spanned by $\{x_{ij}, y_{kl}\}$. We claim that Z actually is closed under both the operation Δ^+ and Δ^\times . It is clear that

$$\Delta^\times(x_{ij}) \subset Z \otimes Z$$

and the same holds for coaddition. It is also easy to verify that $\Delta^\times(y_{kl}) \subset Z \otimes Z$. We will now prove that $\Delta^+(y_{kl}) \subset Z \otimes Z$ and for this, consider the following diagram which is easily verified if we reverse all arrows and think of it in terms of rings.

$$\begin{array}{ccc}
 A_{\mathcal{R}} & \xrightarrow{\Delta^\times} & A_{\mathcal{R}} \otimes A_{\mathcal{R}} \\
 \downarrow \Delta^+ & & \parallel \\
 A_{\mathcal{R}} \otimes A_{\mathcal{R}} & & A_{\mathcal{R}} \otimes A_{\mathcal{R}} \\
 \downarrow \Delta^\times \otimes \Delta^\times & & \parallel \\
 A_{\mathcal{R}} \otimes A_{\mathcal{R}} \otimes A_{\mathcal{R}} \otimes A_{\mathcal{R}} & & A_{\mathcal{R}} \otimes A_{\mathcal{R}} \\
 \downarrow 1 \otimes T \otimes 1 & & \parallel \\
 A_{\mathcal{R}} \otimes A_{\mathcal{R}} \otimes A_{\mathcal{R}} \otimes A_{\mathcal{R}} & & A_{\mathcal{R}} \otimes A_{\mathcal{R}} \\
 \downarrow \Delta^\times \otimes 1 \otimes 1 \otimes \Delta^\times & & \parallel \\
 A_{\mathcal{R}} \otimes (A_{\mathcal{R}} \otimes A_{\mathcal{R}}) \otimes (A_{\mathcal{R}} \otimes A_{\mathcal{R}}) & & A_{\mathcal{R}} \otimes A_{\mathcal{R}} \\
 \downarrow 1 \otimes T \otimes T \otimes 1 & & \downarrow \Delta^\times \otimes 1 \\
 (A_{\mathcal{R}} \otimes A_{\mathcal{R}}) \otimes A_{\mathcal{R}} \otimes A_{\mathcal{R}} \otimes (A_{\mathcal{R}} \otimes A_{\mathcal{R}}) & & (A_{\mathcal{R}} \otimes A_{\mathcal{R}}) \otimes A_{\mathcal{R}} \\
 \downarrow M \otimes 1 \otimes 1 \otimes M & & \downarrow 1 \otimes \Delta^+ \otimes 1 \\
 A_{\mathcal{R}} \otimes A_{\mathcal{R}} \otimes A_{\mathcal{R}} \otimes A_{\mathcal{R}} & \xlongequal{\quad} & A_{\mathcal{R}} \otimes A_{\mathcal{R}} \otimes A_{\mathcal{R}} \otimes A_{\mathcal{R}}
 \end{array}$$

What the opposite is saying, is just relating different ways of forming

$$abd + acd$$

for a, b, c, d in a ring. So this says, that

$$(1 \times \Delta^+ \times 1)(\Delta_2^\times(x_{ij})) = \sum_{k,l} e_k \otimes \Delta^+(y_{kl}) \otimes f_l \in Z \otimes Z \otimes Z \otimes Z.$$

Now, since e_k are independent, this means that

$$\sum_l \Delta^+(y_{kl}) \otimes f_l \in Z \otimes Z$$

and by linear independence of each f_l this means that

$$\Delta^+(y_{kl}) \in Z.$$

Now, let W be the sub-algebra generated by $Z \cup S(Z)$ where $S : A_{\mathcal{R}} \rightarrow A_{\mathcal{R}}$ is the antipode. It is easily verified that

$$\Delta^+ \circ S = (S \otimes S) \circ \Delta^+$$

and that

$$\Delta^\times(S(Z)) \subset W$$

follows from the identity

$$\Delta^\times \circ S = (1 \otimes S) \circ \Delta^\times.$$

We thus see that W is a bialgebra and we are done. \square

Corollary 2.2. *Any affine connected ring scheme over a field is unipotent.*

Proof. Indeed, we know that we can write $P = \varprojlim P_i$ where P_i ranges over ring schemes of finite type. Now, unipotence is stable under inverse limits and this immediately gives that P is unipotent. \square

3. PLETHORIES AND k - k -BIRINGS.

Let k be an arbitrary commutative ring. In this section we will recall the definition of a plethory as defined in [3].

Definition 3.1. A biring A is a coring object in the category of k -algebras. Explicitly, A is a k -algebra together with maps

$$\Delta^+ : A \rightarrow A \otimes_k A,$$

$$\Delta^\times : A \rightarrow A \otimes_k A,$$

$$S : A \rightarrow A,$$

$$\epsilon^+ : A \rightarrow k$$

and $\epsilon^\times : A \rightarrow k$ such that:

- The triple $(\Delta^+, \epsilon^+, S)$ defines a cocommutative Hopf algebra structure on A with S the antipode and ϵ^+ the counit.

- Δ^\times is cocommutative coassociative and codistributes over Δ^+ and $\epsilon^\times : S \rightarrow k$ is a counit for Δ^\times .

We say that A is a $k - k$ -biring if, in addition to the above data, it has a map

$$\beta : k \rightarrow \text{Ring}_k(A, k)$$

of rings, where we endow $\text{Ring}_k(A, k)$ with the ring structure induced from the coring structure on A .

Equivalently, a $k - k$ -biring A is just an affine scheme such that the functor $\text{Ring}_k(-, A)$ has a lift to k -algebras, i.e it is an affine k -algebra scheme.

Example 3.1. Let us note that $\mathbb{A}_k^1 = \text{Spec } k[e]$ is a k -algebra scheme which we will call \mathbb{G}_a . \mathbb{G}_a will represent the identity functor $\text{Ring}_k \rightarrow \text{Ring}_k$. Indeed, the coaddition and comultiplication is given by $\Delta^+(e) = e \otimes 1 + 1 \otimes e$, $\Delta^\times(e) = e \otimes e$, the additive resp. multiplicative counit by $\epsilon^+(e) = 0, \epsilon^\times(e) = 1$ the antipode by $S(e) = -e$ and the co- k -linear structure by $\beta(c)(e) = c$ for all $c \in k$.

Example 3.2. Consider $\mathbb{Z}[e, x]$. On e , we define all the structure maps as in the previous example. We then define

$$\Delta^+(x) = x \otimes 1 + 1 \otimes x,$$

$$\Delta^\times(x) = x \otimes e + e \otimes x$$

and $\epsilon^\times(x) = \epsilon^+(x) = 0, S(x) = -x$. This \mathbb{Z} -ring scheme represents the functor taking a ring R to $R[\epsilon]/(\epsilon^2)$, the ring of dual numbers over that ring.

Example 3.3. Let $k = \mathbb{F}_q$ be a finite field of characteristic p and consider

$$\alpha_p = \text{Spec } k[e]/(e^p)$$

as a group scheme where the group structure is induced from $\text{Spec } k[e]$. Define a multiplication

$$\alpha_p \times \alpha_p \rightarrow \alpha_p$$

by saying that $xy = 0$ for any $x, y \in \alpha_p(R)$ for R a k -algebra. Consider now the constant group scheme

$$\underline{\mathbb{F}}_q = \coprod_{x \in k} k.$$

Then we can define a structure of a ring scheme on $\alpha_p \times \underline{\mathbb{F}}_q$ by defining the multiplication to be $(x, y)(z, w) = (xz, xw + yz + yw)$ for $(x, y), (z, w) \in \alpha_p \times \underline{\mathbb{F}}_q(R)$. This is a non-reduced ring scheme.

A famous example is also that the functor taking a ring R to $W(R)$, its ring of big Witt vectors, is also representable by a ring scheme.

Let us note that we can form the category of $k - k$ -birings, with morphisms between objects those morphism of k -algebras respecting the

biring structure. We let $BR_{k,k}$ be the category of k - k -birings. Let us recall the following definition from [3].

Definition 3.2. Let A be a k - k -biring. Then the functor

$$\text{Ring}_k(A, -) : \text{Alg}_k \rightarrow \text{Alg}_k$$

has a left adjoint,

$$A \odot_k - : \text{Alg}_k \rightarrow \text{Alg}_k.$$

Explicitly, for a k -algebra B , $A \odot B$ is the k -algebra generated by all symbols $a \odot b$ subject to the conditions that:

- $\forall a, a' \in A, r \in R, aa' \odot r = (a \odot r)(a' \odot r).$
- $\forall a, a' \in A, r \in R, (a + a') \odot r = (a \odot r) + (a' \odot r).$
- $\forall c \in k, c \odot r = c.$
- $\forall a \in A, r, r' \in R, a \odot (r + r') = \sum_i (a_i^{(1)} \odot r)(a_i^{(2)} \odot r').$
- $\forall a \in A, r, r' \in R, a \odot rr' = \sum_i (a_i^{[1]} \odot r)(a_i^{[2]} \odot r').$
- $\forall a \in A, c \in k, a \odot c = \beta(c)(a).$

It is easy to see that $(\otimes_i A_i) \odot R \cong \otimes_i (A_i \odot R)$ and that $A \odot (\otimes_i R_i) \cong \otimes_i (A \odot R_i)$. If further, R is a k - k -bialgebra, we note that $A \odot R$ is a k - k -bialgebra. Indeed, we have that $\text{Ring}_k(A \odot R, S) \cong \text{Ring}_k(R, \text{Ring}_k(A, S))$ and since the latter set has a ring structure, so does the former. One then verifies that \odot_k gives a monoidal structure to $BR_{k,k}$. The unit of this monoidal structure is $k[e]$. $BR_{k,k}$ is a monoidal category, but it is not symmetric. Now, the Yoneda embedding sets up an equivalence of categories between the category of representable endofunctors $\text{Alg}_k \rightarrow \text{Alg}_k$ and $BR_{k,k}$ and under this equivalence, \odot corresponds to \circ , composition of representable endofunctors as given in the introduction. Denote the category of representable endofunctors $\text{Alg}_k \rightarrow \text{Alg}_k$ by $\text{Alg}_k^{\text{end}}$.

Definition 3.3. A k -plethory is a comonoid in $\text{Alg}_k^{\text{end}}$ where the monoidal structure is composition of endofunctors. Explicitly, on the level of representing objects, a k -plethory P is a monoid in $BR_{k,k}$. This means that P is a biring together with an associative map of birings $P \odot P \rightarrow P$ and a unit $k[e] \rightarrow P$.

Remark 3.4. For a plethory P one can define an action of P on a k -ring R to be a map $\circ : P \odot R \rightarrow R$ such that $(p_1 \odot p_2) \circ r = p_1 \odot (p_2 \circ r)$ and $e \circ p = p, \forall p_1, p_2 \in P, r \in R$.

Example 3.5. If k is a finite ring, then k^k , the set of functions $k \rightarrow k$ is a plethory where \circ is given by composition of functions.

4. CLASSIFICATION OF PLETHORIES OVER A FIELD OF CHARACTERISTIC ZERO.

In this section we will prove that all plethories over a field of characteristic zero are linear. This question was asked by Bergman-Hausknecht

[1] and Borger-Wieland [3]. To understand what it means for a plethory to be linear, we will introduce some terminology.

Definition 4.1. Let A be a cocommutative bialgebra (not necessarily commutative) over k with comultiplication Δ . Then there is a free k -plethory on A over k . The underlying algebra structure is $S(A)$, the symmetric algebra on A and the coaddition

$$\Delta^+ : S(A) \rightarrow S(A) \otimes S(A)$$

is induced from the map

$$A \rightarrow S(A) \otimes S(A)$$

sending a to $a \otimes 1 + 1 \otimes a$. The comultiplication Δ^* is induced from Δ . The plethysm

$$\circ : S(A) \odot S(A) \rightarrow S(A)$$

is given by

$$S(A) \odot S(A) \cong S(A \otimes A) \xrightarrow{S(m)} S(A)$$

where m is the multiplication on A . Among the pairs consisting of a plethory P and a morphism of bialgebras $f : A \rightarrow P$ the pair $S(A)$ and $j : A \rightarrow S(A)$ is initial with this property.

Call a plethory P linear if $P \cong S(A)$ for some bialgebra A . The reason for calling it linear is that if $P \cong S(A)$ for some bialgebra A then

$$\text{Ring}_k(-, S(A)) = \text{Mod}_k(-, A).$$

Let us note now that by Theorem 2.6, any connected reduced ring scheme of finite type is unipotent. Over \mathbb{Q} (or more generally any field of characteristic zero) all group schemes are reduced by a theorem of Cartier. We say that a group scheme G is étale if G is a finite scheme and geometrically reduced. This is equivalent to asking for the underlying Hopf algebra A_G to be an étale algebra. Let us recall the following definition from the theory of group schemes (see for example [6], II, §5, Proposition 1.8, or [11] Definition 9.4)

Definition 4.2. Let G be a group scheme of finite type over k . Let A_G be the underlying Hopf algebra of G and consider the largest étale k -subalgebra $\pi_0(A_G)$ of A_G . $\pi_0(A_G)$ then has a Hopf algebra structure induced from the one on A_G and we let $\pi_0(G) = \text{Spec } \pi_0(A_G)$ be the group scheme associated to this Hopf algebra.

Note that there is a canonical map $G \rightarrow \pi_0(G)$. It is easy to see that if $\pi_0(G) = \text{Spec } k$, then G is geometrically connected since in that case A_G has no nontrivial idempotents.

Lemma 4.1. *Any k -algebra scheme \mathcal{R} of finite type over any infinite field k is geometrically connected.*

Proof. Consider the connected-étale exact sequence

$$0 \rightarrow \mathcal{R}^\circ \rightarrow \mathcal{R} \rightarrow \pi_0(\mathcal{R}) \rightarrow 0$$

of group schemes where \mathcal{R}° is the identity component of \mathcal{R} . We will first show that $\pi_0(\mathcal{R})$ has a natural k -algebra scheme structure. Indeed, for this it is enough to show that \mathcal{R}° is a k -ideal scheme in \mathcal{R} . Let us start by proving that $m(\mathcal{R}^\circ \times \mathcal{R}) \subset \mathcal{R}^\circ$. We know that the multiplication

$$m : \mathcal{R}^\circ \times \mathcal{R} \rightarrow \mathcal{R}$$

takes the additive identity $e \in \mathcal{R}(k)$ to itself, i.e $m(e, x) = e$ for any $x \in \mathcal{R}(k)$. Further, the k -algebra structure on \mathcal{R}° is induced from the k -algebra structure on \mathcal{R} . This clearly implies that \mathcal{R}° is a k -ideal scheme. Thus, the quotient $\mathcal{R}/\mathcal{R}^\circ \cong \pi_0(\mathcal{R})$ is a k -algebra scheme. Let us see that $\pi_0(\mathcal{R})$ is isomorphic to $\text{Spec } k$. One knows that the underlying algebra of $\pi_0(\mathcal{R})$ is a product of finite separable k -extensions. We consider $\text{Sch}_k(\pi_0(\mathcal{R}), \pi_0(\mathcal{R}))$, this is a k -algebra (since $\pi_0(\mathcal{R})$ is a ring scheme). Because the underlying algebra of $\pi_0(\mathcal{R})$ is a finite product of finite separable field extensions, $\text{Sch}_k(\pi_0(\mathcal{R}), \pi_0(\mathcal{R}))$ is a finite set. However, for a finite set to have a k -algebra structure it must just contain one element, i.e it has to be the zero ring. This implies that $\pi_0(\mathcal{R}) = \text{Spec } k$ so \mathcal{R} is geometrically connected. \square

Now, let us consider a Hopf algebra H denote the primitive elements of H by $P(H)$. We say that a Hopf algebra is primitively generated if $P(H)$ generates H as an algebra. Over characteristic zero all unipotent affine group schemes of finite type are primitively generated. We then have the classical Milnor-Moore theorem (for a proof, see [12])

Theorem 4.3. For any commutative connected affine unipotent group scheme of finite type H over a field of characteristic zero, the canonical map

$$\text{Spec } H \rightarrow \text{Spec } S(P(H))$$

is an isomorphism of group schemes. In particular, the underlying scheme is affine space.

Remark 4.2. Let us note that we can view $P(H)$ as a Lie algebra with trivial commutator. Then the construction $S(P(H))$ is the same as the universal enveloping Lie algebra of $P(H)$.

In [3] it is shown that if Q is a plethory over a field k , then $P(Q)$ is a cocommutative k -bialgebra. Briefly, the multiplication in $P(Q)$ is given by the plethysm \circ and the maps

$$\Delta^\times : Q \rightarrow Q \otimes Q,$$

$\epsilon^\times : Q \rightarrow k$ induces a comultiplication respectively a counit on $P(Q)$ making it a cocommutative counital bialgebra.

Theorem 4.4. Let Q be a plethory over a field of characteristic zero k . Then Q is linear, i.e

$$Q \cong S(P(Q))$$

where $S(P(Q))$ has the plethory structure as given in Definition 3.1.

Proof. Suppose that Q is a plethory over k . $P(Q)$ naturally has a bialgebra structure as explained above. Given this, we can form the free plethory on $P(Q)$, $S(P(Q))$. We always have a natural map

$$v : S(P(Q)) \rightarrow Q$$

of Hopf algebras, and this is bijective by Milnor-Moore. Thus to show that any plethory is linear, it suffices to show that this is actually a morphism of plethories. But this is clear: the pair $S(P(Q))$ and

$$j : P(Q) \rightarrow S(P(Q))$$

is initial in the category of pairs consisting of a plethory P and a morphism $f : P(Q) \rightarrow P$ of bialgebras. It is immediate that the canonical map v_p is induced by this universal property, when we note that there clearly is a map $P(Q) \rightarrow Q$ of bialgebras. We will of course need to show that v is an isomorphism in the category of plethories. This follows easily from the fact that v is an isomorphism of affine schemes and thus has an inverse in the category of affine schemes. What remains to be checked is that this inverse is a morphism of plethories, but this is immediate since v is. \square

5. SOME CLASSIFICATION RESULTS IN CHARACTERISTIC $p > 0$.

In this section we will start a classification for plethories over a perfect field k of characteristic p . Our classification results here only apply to a certain class of plethories. We state future research directions, as well as give some "pathological" examples which a complete classification must take into account. For any scheme X over k with structure map $f : X \rightarrow \text{Spec } k$ we let G^p be the pullback of f along $F : \text{Spec } k \rightarrow \text{Spec } k$, the Frobenius.

Let us briefly recall that for perfect fields k , group schemes over k have two especially important maps, the (relative) Frobenius

$$F_G : G \rightarrow G^p \cong G$$

and the Verschiebung

$$V_G : G \cong G^p \rightarrow G.$$

These satisfy the property that $F_G V_G = V_G F_G = p$. A ring scheme \mathcal{R} is called elementary unipotent if $V_{\mathcal{R}} = 0$, i.e the Verschiebung is zero. Call a plethory Q weakly linear if there is a map of plethories $f : P \rightarrow Q$ where P is a linear plethory (as defined in the previous section) such that f when viewed as a map of algebras is surjective. This will, in particular, imply that Q is primitively generated and is a quotient of P

by a $P - P$ -ideal as defined in [3]. Not all plethories over a perfect field k are primitively generated, as the following example shows (built on an example from [10], Remark 1.6.2).

Example 5.1. Let G be the group scheme

$$\mathbb{G}_a \times_f \alpha_p$$

which as a scheme, is just $\mathbb{G}_a \times \alpha_p$. We let the the group structure be given by

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 + h_2 + f(g_1, g_2))$$

for

$$g_1, g_2, h_1, h_2 \in \mathbb{G}_a(R) \times \alpha_p(R)$$

where $f(x, y) = ((x + y)^p - x^p - y^p)/p$. This is a p -torsion group scheme but is not elementary unipotent. One can define a non-unital ring scheme structure on G be definining the multiplication to be trivial and then, when k is finite, i.e $k \cong \mathbb{F}_q$ "unitalize" this by taking the direct product with

$$\underline{\mathbb{F}}_q = \coprod_{a \in \mathbb{F}_q} \mathbb{F}_q$$

to get a ring scheme, as we did in Example 3.3. The underlying group scheme of this ring scheme is clearly not elementary unipotent, since the Verschiebung acts on each factor separately. In the case where $k = \mathbb{F}_p$ this is a $k - k$ -biring, and taking the free plethory on this $k - k$ -biring (see [3] 2.1) will then give us a plethory with its underlying group scheme not elementary unipotent.

Another feature which differs from the case over a field of characteristic zero is that there are plethories which have a non-trivial multiplicative subgroups. This stems from the fact that there are ring schemes with non-trivial multiplicative subgroups.

Example 5.2. Consider $\mu_p = \text{Spec } k[x, x^{-1}]/(x - 1)^p$ with comultiplication $\Delta : x \rightarrow x \otimes x$ and counit $\epsilon(x) = 1$. This is an example of a multiplicative group scheme which is p -torsion and we can as before define a trivial multiplication on μ_p , making it a non-unital ring scheme. We can then as previously stated, for finite fields, unitalize it to get a ring scheme by taking the direct product with

$$\underline{\mathbb{F}}_q$$

and if $k = \mathbb{F}_p$ we can form the free plethory to get a plethory Q with a non-trivial multiplicative subgroup. The fact that it has a non-trivial multiplicative subgroup comes from , for example, the fact that there is a non-zero homomorphism of group schemes $\mu_p \rightarrow Q$.

These two examples are rather artificial, but they show that plethories behave wildly different in characteristic $p > 0$ than in characteristic 0. We know that for any group scheme G over a perfect field k of characteristic

$p > 0$, the group $P(G)$ of primitive elements has a natural action of the Frobenius, taking $x \in P(G)$ to x^p . In fact, $P(G)$ becomes a module over a certain ring. As we previously stated, $P(G) = \text{Hom}(G, \mathbb{G}_a)$. We thus have that $P(G)$ is naturally a module over the endomorphism ring $\text{End}(\mathbb{G}_a, \mathbb{G}_a)$.

Definition 5.1. Let $k\langle F \rangle$ be the non-commutative polynomial ring over k in one variable F with multiplication given by, for $a \in k$ $aF = F_k(a)a$ where F_k is the Frobenius endomorphism of k .

It is a quick calculation to show that $\text{End}(\mathbb{G}_a, \mathbb{G}_a) \cong k\langle F \rangle$. We now see that $P(G)$ is a module over $k\langle F \rangle$. Let us denote the category of modules over $k\langle F \rangle$ by $\text{Mod}_{k\langle F \rangle}$. Given a $k\langle F \rangle$ -module M one can construct an elementary unipotent group scheme $S^{[p]}(M)$ as follows (for details we refer the reader to [11]). Form $S(M)$, the symmetric algebra on M , with its obvious Hopf algebra structure and consider the map $j : M \rightarrow S(M)$. We then quotient out by the ideal generated by the elements

$$j(Fx) - j(x)^p$$

to get $S^{[p]}(M)$. One notes that for any commutative algebraic group G one always has a map $G \rightarrow S^{[p]}(P(G))$. We have the following classical theorem (see [6] IV, §3, Proposition 6.6).

Theorem 5.2. Let G be an affine group scheme. The following are equivalent:

- (i) The Verschiebung V_G is zero.
- (ii) G is a closed subgroup of \mathbb{G}_a^r for some r .
- (iii) The canonical homomorphism $G \rightarrow S^{[p]}(P(G))$ is an isomorphism.

Remark 5.3. † What we call $S^{[p]}(P(Q))$ is the same as the enveloping p -algebra (also called the restricted universal enveloping algebra) on the p -Lie algebra $P(Q)$ where $P(Q)$ has trivial commutator.

Lemma 5.4. When Q is a plethory, then $S^{[p]}(P(Q))$ has the structure of a plethory.

Proof. We know that $P(Q)$ has a $k\langle F \rangle$ module structure where the action of F is just taking the p th power. Further, $S^{[p]}(P(Q))$ is the quotient of $S(P(Q))$, which we know is a plethory, by the ideal J generated by $j(x)^p - j(x^p)$, where $j : P(Q) \rightarrow S(P(Q))$ is the inclusion in degree 1. It now suffices to show that this is a $Q - Q$ -ideal (see [3] 6.1) for $U^{[p]}(P(Q))$ to be a plethory. This is equivalent to showing that for a generating set S of J that

$$\Delta_Q^+(S) \subset Q \otimes J + J \otimes Q,$$

$$\Delta_Q^\times(X) \subset Q \otimes J + J \otimes Q,$$

and

$$\beta_Q(c)(S) = 0$$

$\forall c \in k$ and that

$$P(Q) \odot X \odot Q \subset J.$$

The first is immediate, since taking S to be the set of all $j(x)^p - j(x^p)$, we have

$$\Delta^+(j(x)^p) - \Delta^+(j(x^p)) = \Delta^+(j(x))^p - (j(x^p) \otimes 1 + 1 \otimes j(x^p))$$

which is equal to

$$j(x)^p \otimes 1 + 1 \otimes j(x)^p - j(x^p) \otimes 1 - 1 \otimes j(x^p) \subset J \otimes Q + Q \otimes J.$$

Further,

$$\Delta^\times(j(x)^p) - \Delta^\times(j(x^p)) = \sum_i j(x_i^{[1]})^p \otimes j(x_i^{[2]})^p - \sum_i j((x_i^{[1]})^p) \otimes j((x_i^{[2]})^p)$$

and this is equal to

$$\sum_i j(x_i^{[1]})^p \otimes (j(x_i^{[2]})^p - j((x_i^{[2]})^p)) + \sum_i ((j((x_i^{[1]})^p) - j(x_i^{[1]})^p) \otimes j((x_i^{[2]})^p))$$

but this is in $J \otimes P + P \otimes J$. We also need to show that $\beta_Q(c)(S) = 0$, this is clear. The last containment is similarly easy to verify. \square

Theorem 5.3. When Q is a plethory over a perfect field such that $V_Q = 0$, then $S^{[p]}(P(Q)) \cong Q$. We then say that Q is a p -linear plethory.

Proof. All one has to verify is that the canonical map $f : Q \rightarrow S^{[p]}(P(Q))$ is a map of plethories. But this is obvious since this map is just the composition of the two plethory maps $Q \rightarrow S(P(Q))$ and $S(P(Q)) \rightarrow S^{[p]}(P(Q))$. \square

Remark 5.5. We have seen that plethories need not be elementary unipotent and not purely unipotent either (i.e it can have a non-trivial multiplicative subgroup). Let us note that there can be no non-trivial finite plethories over an infinite perfect field k . Indeed, from what we have seen all plethories Q are connected over an infinite field. By classical Dieudonné theory we can then decompose Q as $Q = Q^{\text{loc,red}} \times Q^{\text{loc,loc}}$. This would imply that the Frobenius is nilpotent, but this can never happen: the Frobenius is always a map of ring schemes.

It seems to us that to classify plethories over a perfect field one should establish an extension of ordinary Dieudonné theory to account for ring schemes, which has been done to some extent by Hedayatzadeh in [9] and for Hopf rings by Goerss [7] and Buchstaber-Lazarev [5]. Note that Hedayatzadeh work with finite / profinite group schemes and with local group schemes, which limits their applications to ring schemes since we have seen that there are no non-trivial finite connected ring schemes over a perfect field.

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