Supergravity and Kaluza-Klein dimensional reductions

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Abstract

This is a 15 credit project in basic supergravity. We start with the supersymmetry algebra to formulate the relation between fermions and bosons. This project also contains Kaluza-Klein dimensional reduction on a circle. We then continue with supergravity theory where we show that it is invariant under supersymmetry transformations, it contains both $D = 4, \mathcal{N} = 1$ and $D = 11, \mathcal{N} = 1$ supergravity theories. We also do a toroidal compactification of the eleven-dimensional supergravity.
1 Introduction

The aim of this thesis is to study supergravity from its four-dimensional formulation to its eleven-dimensional formulation. We also consider compactifications of higher dimensional supergravity theories which results in extended supergravity theories in four-dimensions. Supergravity emerges from combining supersymmetry and general relativity.

General relativity is a theory that describes gravity on large scales. The predictions of general relativity has been confirmed by experiments such as gravitational lensing and time dilation just to mention a few. However general relativity fails to describe gravity on smaller scales where the energies are much higher than current energy scales tested at the LHC.

The Standard model took shape in the late 1960’s and early 1970’s and was confirmed by the existence of quarks. The Standard model is a theory that describes the electromagnetic the weak and strong nuclear interactions. The interactions between these forces are described by the Lie group $SU(3) \otimes SU(2) \otimes U(1)$ where $SU(3)$ describes the strong interactions and $SU(2) \otimes U(1)$ describes the electromagnetic and weak interactions. Each interaction would have its own force carrier. The Weinberg-Salam model of electroweak interactions predicted four massless bosons, however a process of symmetry breaking gave rise to the mass of three of these particles, the $W^\pm$ and $Z^0$. These particles are carriers of the weak force, the last particle that remains massless is the photon which is the force carrier for the electromagnetic force.

We know that there are four forces in nature and the Standard model neglects gravitational force because its coupling constant is much smaller compared to the coupling constants of the three other forces. The Standard model does not have a symmetry that relates matter with the forces of nature. This led to the development of supersymmetry in the early 1970’s.

Supersymmetry is an extended Poincaré algebra that relates bosons with fermions. In other words since the fermions have a mass and all the force carriers are bosons, we now have a relation between matter and forces. However unbroken supersymmetry requires an equal mass for the boson-fermions pairs and this is not something we have observed in experiments. So if supersymmetry exist in nature it must appear as a broken symmetry. Each particle in nature must have a superpartner with a different spin. For example the gravitino which is a spin-3/2 particle has a superpartner called the graviton with spin-2. Graviton in theory is the particle that describes the gravitational force. We now see why supersymmetry is an important tool in the development of a theory that describes the three forces in the Standard model and gravity force. With supersymmetry one could now develop a theory that unifies all the forces in nature. One theory in particular that describes gravity in the framework of supersymmetry is called supergravity.

Supergravity was developed in 1976 by D.Z Freedman, Sergio Ferrara and Peter Van Nieuwenhuizen. They found an invariant Lagrangian describing the gravitino field which coincides with the gravitational gauge symmetry. In 1978 E.Cremmer, B.Julia and J.Scherk discovered the Lagrangian for eleven dimensional supergravity. The eleven dimensional theory is a crucial discovery which is the largest structure for a consistent supergravity theory. Eleven dimensions is the largest structure where one has a consistent theory containing a graviton, however there exist theories that contain higher dimensions which have interactions containing spin 5/2 particles. From the eleven-dimensional theory one can obtain lower-dimensional theories by doing dimensional reductions, one then finds ten-dimensional theories known as Type IIA and Type IIB and these theories are related to superstring theories with the same name. Supergravity appears as low-energy limit of superstring theory. The maximally extended supergravity theories
start from the eleven-dimensional Lagrangian, one obtains maximally extended theories by doing dimensional reductions, for example Type $IIA$ is a maximally extended supergravity theory since it contains two supercharges in ten dimensions.

The outline of this thesis goes as follows. In section 2 we begin with constructing Supersymmetry algebra and its representations. In section 3 we discuss Clifford algebra which plays an important role in Supergravity because it contains spinors. In section 4 we go through some differential geometry, in particular the Cartan formalism and differential forms. In section 5 we consider the $(D + 1)$ dimensional Einstein-Hilbert action and perform a dimensional reduction on a circle this is known as the Kaluza-Klein reduction. We obtain a $D$-dimensional Lagrangian that contains gravity, electromagnetic fields and scalar fields.

In section 6 we go through the Rarita-Schwinger equation and discuss degree’s of freedom for the gravitino field. Section 7 contains first and second order formulations of general relativity. In section 8 we consider the four-dimensional Supergravity theory and show the invariance of its Lagrangian. Section 9 contains eleven-dimensional Supergravity theory where we construct a Lagrangian that is invariant. We also perform dimensional reductions on the eleven-dimensional theory using Kaluza-Klein method to obtain lower dimensional theories.
2 Supersymmetry algebra

Supersymmetry is a symmetry that relates two classes of particles, fermions and bosons. Fermions have half-integer spin meanwhile bosons have integer spin. When one constructs a supersymmetry the fermions obtain integer spin and the bosons obtain half-integer spin, each particles has its own superpartner. So for example a graviton has spin-2 meanwhile its superpartner gravitino has spin-3/2.

The key references in this section will be [1], [2] and [3].

2.1 Poincaré symmetry

The Poincaré group \( ISO(3,1) \) is a extension of the Lorentz group. The Poincaré group is a 10-dimensional group which contains four translations, three rotations and three boosts. The Poincaré group corresponds to basic symmetries of special relativity, it acts on the spacetime coordinate \( x^\mu \) as

\[
x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu .
\]

(2.1)

Lorentz transformation leaves the metric invariant

\[
A^T \eta A = \eta
\]

(2.1)

where \( P \) is the generator of translations and \( M \) is the generator of Lorentz transformations. The corresponding Lorentz group \( SO(3,1) \) is homeomorphism to \( SU(2) \), where the following \( SU(2) \) generators correspond to \( J_i \) of rotations and \( K_i \) of Lorentz boosts defined as

\[
J_i = \frac{1}{2} \varepsilon_{ijk} M_{jk} , \quad K_i = M_{0i}
\]

and their linear combinations

\[
A_i = \frac{1}{2} (J_i + iK_i) , \quad B_i = \frac{1}{2} (J_i - iK_i) .
\]

The linear combinations satisfy \( SU(2) \) commutation relations such as

\[
[J_i, J_j] = i \varepsilon_{ijk} J_k \quad [J_i, K_j] = i \varepsilon_{ijk} K_k \quad [K_i, K_j] = -i \varepsilon_{ijk} K_k \quad [A_i, A_j] = 0 \quad [B_i, B_j] = 0 \quad [A_i, B_j] = 0 .
\]

(2.5) \quad (2.6) \quad (2.7) \quad (2.8) \quad (2.9) \quad (2.10)

We can interpret \( \vec{J} = \vec{A} + \vec{B} \) as physical spin. There is a homeomorphism \( SO(3,1) \cong SL(2,\mathbb{C}) \) which gives rise to spinor representations. We write down a \( 2 \times 2 \) hermitian matrix that can be parametrized as

\[
\vec{x} = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} .
\]

(2.11)
This will lead to an explicit description of the \((\frac{1}{2}, 0)\) and \((0, \frac{1}{2})\) representation and this is the central treatment for fermions in quantum field theory. \(^1\)

The determinant of equation (2.11) gives us \(x^2 = -x^\mu \eta_{\mu \nu} x^\nu\) which is negative of the Minkowski norm of the four vector \(x^\mu\). This gives rise to a relation between the linear space of the hermitian matrix and four-dimensional Minkowski space. The transformation \(x^\mu \rightarrow x^\mu \Lambda\) under \(SO(3,1)\) leaves the square of the four vector invariant, but what happens to the determinant if we consider a linear mapping \(^2\).

Let \(N\) be a matrix of \(SL(2, \mathbb{C})\) and consider the linear mapping
\[
\vec{x} \rightarrow N \vec{x} N^\dagger
\]
the four-vectors are linearly related \(i.e.\ x^\mu \rightarrow x^\mu \Lambda\) and from the \(2 \times 2\) matrix we can obtain the explicit form of the matrix \(\Lambda\). The linear transformation gives us that the determinant is preserved, and the Minkowski norm is invariant. This means that the matrix \(\Lambda\) must be a Lorentz transformation. Since \(SO(3,1) \cong SU(2) \otimes SU(2)\) is valid then the topology of \(SL(2, \mathbb{C})\) is connected due to the fact that \(\Lambda\) is a Lorentz transformation of \(SO(3,1)\) and the relation \(SO(3,1) \cong SL(2, \mathbb{C})\). The spinor representation will play an important role in the future chapters.

### 2.2 Representation of spinors

Let us consider the spinor of representations of \(SL(2, \mathbb{C})\). The corresponding spinors are called Weyl spinors. The \((\frac{1}{2}, 0)\) is given by the irreducible representation of the chirality \(2_L \equiv (\psi_{\alpha}, \alpha = 1, 2)\) and the spinor representation \((0, \frac{1}{2})\) is given by \(2_R = (\chi_{\dot{\alpha}}, \dot{\alpha} = 1, 2)\). The first representation gives us the following Weyl spinor
\[
\psi'_\alpha = N_{\alpha \beta} \psi_\beta, \quad \alpha, \beta = 1, 2
\]
this is the left-handed Weyl spinor. The other representation is given by
\[
\bar{\chi}'_{\dot{\alpha}} = N^{*}_{\dot{\alpha} \dot{\beta}} \bar{\chi}_{\dot{\beta}}, \quad \dot{\alpha}, \dot{\beta} = 1, 2
\]
this is the right-hand Weyl spinor. These spinors are the representation of the basic representation of \(SL(2, \mathbb{C})\) and the Lorentz group. From the previous chapter we can now see the relation between the Lorentz group and \(SL(2, \mathbb{C})\). Introducing another spinor \(\varepsilon^{\alpha \beta}\) which will be a contravariant representation of \(SL(2, \mathbb{C})\) defined as
\[
\varepsilon^{\alpha \beta} = \varepsilon^{\dot{\alpha} \dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\varepsilon_{\alpha \beta} = -\varepsilon_{\dot{\alpha} \dot{\beta}}
\]
hence we see that the spinor \(\varepsilon\) raises and lowers indices. Our representations of the Weyl spinors then becomes
\[
\psi^{\alpha} = \varepsilon^{\alpha \beta} \psi_\beta, \quad \bar{\chi}^{\dot{\alpha}} = \varepsilon^{\dot{\alpha} \dot{\beta}} \bar{\chi}_{\dot{\beta}}.
\]
When we have mixed indices of both \(SO(3,1)\) and \(SL(2, \mathbb{C})\) the transformations of the components \(x^\mu\) should look the same as we have seen in section 2.1. We saw that the determinant is invariant under transformation of \(SO(3,1)\) or a transformation via the matrix \(\vec{x} = x_\mu \sigma^\mu\), hence
\[
(x^\mu \sigma^\mu)^{\alpha \dot{\alpha}} \rightarrow N_{\alpha \beta} (x_\mu \sigma^\mu)^{\beta \dot{\gamma}} N^{*}_{\alpha \dot{\gamma}} = \Lambda_{\mu \nu} x_\mu \sigma^\nu
\]

\(^1\)The reason why we can find spinor representations is that we have written our \(2 \times 2\) matrix with Pauli matrices such that \(\vec{x} = x_\mu \sigma^\mu\)

\(^2\)Since we know that we have a relation between Minkowski space and a linear space of the hermitian matrix, it is valid to consider a linear mapping of the hermitian matrix.
the right transformation rule is then given by

\[(\sigma^\mu)^{\alpha\delta} = N^\alpha_\beta (\sigma^\nu)^{\beta\gamma} (\Lambda^{-1})^\mu_\nu N^{\gamma\delta}_\alpha\]

we can also introduce the matrices \(\bar{\sigma}^\mu\) defined as

\[(\bar{\sigma}^\nu)^{\alpha\dot{\alpha}} = \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} (\sigma^\mu)^{\beta\dot{\beta}}\]

2.3 Generators of \(SL(2,\mathbb{C})\)

We can define the tensors \(\sigma^{\mu\nu}, \bar{\sigma}^{\mu\nu}\) as antisymmetric products of \(\sigma\) matrices

\[(\sigma^{\mu\nu})^{\alpha\beta} = \frac{i}{4} (\sigma^{\mu\nu} - \sigma^{\nu\mu})^{\alpha\beta}\]

\[(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}\dot{\beta}} = \frac{i}{4} (\bar{\sigma}^{\mu\nu} - \bar{\sigma}^{\nu\mu})^{\dot{\alpha}\dot{\beta}}\]

which satisfy the Lorentz algebra

\[[\sigma^{\mu\nu}, \sigma^{\lambda\rho}] = i(\eta^{\mu\rho} \sigma^{\nu\lambda} + \eta^{\nu\lambda} \sigma^{\mu\rho} - \eta^{\mu\lambda} \sigma^{\nu\rho} - \eta^{\nu\rho} \sigma^{\mu\lambda})\]

The finite Lorentz transformation is then represented as

\[\psi_\alpha \rightarrow \exp \left(-\frac{i}{2} \omega_{\mu\nu} \sigma^{\mu\nu}\right)^{\beta}_{\alpha} \psi_\beta\]

which is the left handed spinor, the right hand spinor is defined as

\[\bar{\chi}^{\dot{\alpha}} \rightarrow \exp \left(-\frac{i}{2} \omega_{\mu\nu} \bar{\sigma}^{\mu\nu}\right)^{\dot{\alpha}}_{\dot{\beta}} \bar{\chi}^{\dot{\beta}}\]

We can obtain some useful identities concerning \(\sigma^{\mu}\) and \(\sigma^{\mu\nu}\), defined as

\[\sigma^{\mu\nu} = \frac{1}{2i} \varepsilon^{\mu\nu\rho\sigma} \sigma_{\rho\sigma}\]

\[\bar{\sigma}^{\mu\nu} = \frac{1}{2} i \varepsilon^{\mu\nu\rho\sigma} \bar{\sigma}_{\rho\sigma}\]

these identities are known as the \textit{self duality} and \textit{anti self duality}. These identities are important when we discuss Fierz identities which rewrite bilinear of the product of two spinors as a linear combination of products of the bilinear of the individual spinors. We will use Fierz identities when we consider Clifford algebra.

2.4 Super-Poincaré algebra

In this section we will consider the Super-Poincaré group which is an extension of the Poincaré group but with a new generator known as the spinor supercharge \(Q^{\dot{\alpha}}_\alpha\), where \(\alpha\) is the spacetime spinor index and \(A = 1, \ldots, N\) labels the supercharges. A superalgebra contains two classes of elements, even and odd. Let us introduce the concept of \textit{Graded algebras}.

Let \(O_a\) be a operator of the Lie algebra then

\[O_a O_b - (-1)^{\eta_a \eta_b} O_b O_a = i C^{\epsilon}{}_{\alpha\beta} O_\epsilon\]

where \(\eta_a\) takes the value

\[\eta_a = \begin{cases} 0 & : O_a \text{ bosonic generator} \\ 1 & : O_a \text{ fermionic generator} \end{cases}\]
This structure relations include both commutators and anti commutators in the pattern $[B, B] = B$, $[B, F] = F$ and $\{F, F\} = B$. We know the commutation relations in the Poincaré algebra, we now need to find the commutation relations between the supercharge and the Poincaré generators.

The following commutation relations with the supercharge generator are defined as

\[ [Q_\alpha, M^{\mu\nu}] = (\sigma^{\mu\nu})_{\alpha\beta} Q_\beta \]  
\[ [Q_\alpha, P^\mu] = 0 \]  
\[ \{Q_\alpha, Q_\beta\} = 0 \]  
\[ \{Q_\alpha, \bar{Q}_\beta\} = 2(\sigma^\mu)_{\alpha\beta} P_\mu \]

When $\mathcal{N} = 1$ then we have a simple SUSY, when $\mathcal{N} > 1$ then we have an extended SUSY. Let us consider $\mathcal{N} = 1$ and the commutator (2.27). We consider the $\mathcal{N} = 1$ SUSY representation, in any on-shell supermultiplet the number $n_B$ of bosons should be equal to the number $n_F$ of fermions.

**Proof.** Consider the fermion operator $(-1)^F = (-1)^F$ defined via

\[ (-)^F |B\rangle = |B\rangle, \quad (-)^F |F\rangle = -|F\rangle \]

The new operator $(-)^F$ anticommutes with $Q_\alpha$, since

\[ (-)^F Q_\alpha |F\rangle = (-)^F |B\rangle = |B\rangle = Q_\alpha |F\rangle = -Q_\alpha (-)^F |F\rangle \rightarrow \{-(-)^F, Q_\alpha\} = 0 \quad (*) \]

Next, consider the trace

\[ \text{Tr} \left\{ (-)^F \{Q_\alpha, \bar{Q}_\beta\} \right\} = \text{Tr} \left\{ (-)^F Q_\alpha \bar{Q}_\beta + (-)^F \bar{Q}_\beta Q_\alpha \right\} = \text{Tr} \left\{ -Q_\alpha (-)^F \bar{Q}_\beta + Q_\alpha (-)^F \bar{Q}_\beta \right\} = 0 \]

It can also be evaluated using \( \{Q_\alpha, \bar{Q}_\beta\} = 2(\sigma^\mu)_{\alpha\beta} P_\mu \)

\[ \text{Tr} \left\{ (-)^F \{Q_\alpha, \bar{Q}_\beta\} \right\} = \text{Tr} \left\{ (-)^F 2(\sigma^\mu)_{\alpha\beta} P_\mu \right\} = 2(\sigma^\mu)_{\alpha\beta} p_\mu \text{Tr}\{(-)^F\} = 0 \]

where $P_\mu$ is replaced by the eigenvalues $p_\mu$ for the specific state. The conclusion is

\[ 0 = \text{Tr}\{(-)^F\} = \sum_{\text{bosons}} \langle B|(-)^F|B\rangle + \sum_{\text{fermions}} \langle F|(-)^F|F\rangle = \sum_{\text{bosons}} \langle B|B\rangle - \sum_{\text{fermions}} \langle F|F\rangle = n_B - n_F \]

Each supermultiplet contains both fermion and boson states which are known as the superpartners of each other. This proof also shows that there are equal number of bosonic and fermionic degrees of freedom. This statement is only valid when the Super-Poincaré algebra holds. It is also valid when we have auxiliary fields that closes the algebra off-shell i.e. when off-shell degree’s of freedom disappear on-shell. If we consider the massless representation the eigenvalues are $p_\mu = (E, 0, 0, E)$, the algebra is

\[ \{Q_\alpha, \bar{Q}_\beta\} = 2(\sigma^\mu)_{\alpha\beta} p_\mu = 4E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \]

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3 A supermultiplet is a representation of a SUSY algebra

4 However there exist some examples where one also has off-shell supermultiplet that have equal amount of bosonic and fermionic degree’s of freedom. Off-shell equality holds for some extended SUSY and higher dimensional theories.
letting $\alpha$ take the value of 1 or 2 gives us the following commutation relations

$$\{Q_2, \bar{Q}_2\} = 0 \quad (2.28)$$
$$\{Q_1, \bar{Q}_1\} = 4E. \quad (2.29)$$

We can define the creation- and annihilation operators $a$ and $a^\dagger$ as

$$a = \frac{Q_1}{2\sqrt{E}}, \quad a^\dagger = \frac{\bar{Q}_1}{2\sqrt{E}} \quad (2.30)$$

we can see that the annihilation operator is in the representation $(0, \frac{1}{2})$ and has the helicity $\lambda = -1/2$ and the creation operator is in the representation $(\frac{1}{2}, 0)$ and has the helicity $\lambda = 1/2$. We can build the representation by using a vacuum state of minimum helicity, let’s call it $|\Omega\rangle$, if we let the annihilation operator act on this state we get zero. Letting the creation operator act on the vacuum state we obtain that the whole multiplet consisting of

$$|\Omega\rangle = |p^\mu, \lambda\rangle, \quad a^\dagger|\Omega\rangle = |p^\mu, \lambda + 1/2\rangle$$

so for example we have that the graviton has helicity $\lambda = 2$ and the superpartner gravitino has the helicity $\lambda = 3/2$.

## 3 Clifford algebra

In section 1.2 we only considered Weyl spinors. We will now introduce Clifford algebra which uses gamma matrices. The Clifford algebra satisfies the anticommutation relation

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}1 \quad (3.1)$$

these matrices are the generating elements of the Clifford algebra. The key references of this section are [2] and [4].

### 3.1 The generating $\gamma$-matrices

Let is discuss the Clifford algebra associated with the Lorentz group in $D$ dimensions. We can construct a Euclidean $\gamma$-matrices which satisfy (3.1) with Minkowski metric $\eta_{\mu\nu}$. The representation of the Clifford algebra can be written in terms of $\sigma$ matrices which are hermitian with square equal to 1 and cyclic

$$\gamma_1 = \sigma_1 \otimes 1 \otimes 1 \otimes \ldots$$
$$\gamma_2 = \sigma_2 \otimes 1 \otimes 1 \otimes \ldots$$
$$\gamma_3 = \sigma_3 \otimes \sigma_1 \otimes 1 \otimes \ldots$$
$$\gamma_4 = \sigma_3 \otimes \sigma_2 \otimes 1 \otimes \ldots$$
$$\gamma_5 = \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes \ldots$$
$$\ldots = \ldots$$

there are two representations of this algebra mainly an even representation and an odd representation. Let us assume that $D = 2m$ is even then the dimension of the representation is $2^{D/2}$, this means that we need $m$ factors to construct the $\gamma$-matrices. For odd representations we have $D = 2m + 1$ which gives us the same dimension of the representation. This is due to the fact that we need $\gamma^{2m+1}$ matrices to construct the algebra but since we only keep the factor of $m$ and delete a $\sigma_1$ matrices which does not increase the dimension. So the general construction of the algebra gives a
representation of the dimension $2^D/2$. We can construct the Lorentzian $\gamma$-matrices by picking any matrix from the Euclidean construction and multiplying it by $i$ and label it $\gamma_0$ for the time-like direction. The hermiticity property of the Lorentzian $\gamma$ are defined as

$$\gamma^\mu_1 = \gamma_0 \gamma^\mu \gamma_0 .$$

(3.2)

To preserve the Clifford algebra we introduce the definition of conjugacy

$$\gamma^\mu = U^{-1} \gamma^\mu U .$$

(3.3)

We only consider hermitian representation in which (3.2) holds, then the matrix $U$ has to be unitary. Given two equivalent representations the transformation matrix $U$ is unique.

### 3.2 $\gamma$-matrix manipulation

We need to define some $\gamma$-matrix manipulation in order to later use it for fermion spin calculations. Clifford algebra are needed to explore the physical properties of these fields. These manipulations are valid in odd and even dimensions $D$. Consider the index contraction such as

$$\gamma^{\mu\nu} \gamma_{\nu} = (D - 1) \gamma^\mu .$$

(3.4)

In general we obtain

$$\gamma^{\mu_1 \ldots \mu_r \nu_1 \ldots \nu_s} \gamma_{\nu_3 \ldots \nu_D} = \frac{(D - r)!}{(D - r - s)!} \gamma^{\mu_1 \ldots \mu_r} .$$

(3.5)

The general order reversal symmetry is defined as

$$\gamma^{\mu_1 \ldots \mu_r} = (-)^{r(r-1)/2} \gamma^{\mu_r \ldots \mu_1}$$

(3.6)

the sign factor $(-)^{r(r-1)/2}$ is negative for $r = 2, 3 \mod 4$. Another useful property is the contraction which is defined as

$$\gamma^{\mu_1 \mu_2 \nu_1 \ldots \nu_D} \epsilon^{\nu_1 \ldots \nu_D} = D(D - 1) \epsilon^{\mu_2 \nu_3 \ldots \nu_D} \gamma_{\nu_3 \ldots \nu_D}$$

(3.7)

$\gamma$-matrix without index contraction in the simplest case is defined as

$$\gamma^{\mu \nu} \gamma^\nu = \gamma^{\mu \nu} + \eta^{\mu \nu} .$$

(3.8)

This follows directly from the definitions: the antisymmetric part of the product is defined to be $\gamma^{\mu \nu}$ and the symmetric part is $\eta^{\mu \nu}$. In general one writes the totally antisymmetric Clifford matrix that contains all the indices and then add terms of possible index pairings. Another example is

$$\gamma^{\mu \nu \rho} \gamma_{\pi \sigma} = \gamma^{\mu \nu \rho} \gamma_{\pi \sigma} + 6 \gamma^{[\mu \nu} \delta^{\rho]}_\sigma + 6 \gamma^{[\mu \delta \nu]} \delta^{\rho]}_\sigma .$$

(3.9)

### 3.3 Symmetries of $\gamma$-matrices

The Clifford algebra of $2^m \times 2^m$ matrices for both even and odd representations, one can distinguish the antisymmetric and the symmetric matrices with a symmetry property called charge conjugation matrix. There exist a unitary matrix $C$ such that each matrix $C \gamma^A$ is either antisymmetric or symmetric. Symmetry only depends on the rank $r$ of the matrix $\gamma^A$, so we can write

$$(C \gamma^{(r)})^T = -t_r C \gamma^{(r)} , \quad t_r = \pm 1$$

(3.10)
where \( \gamma^A \) is the basis of the Clifford algebra defined as
\[
\gamma^A = 1, \gamma^\mu, \gamma^{\mu_1\mu_2}, \ldots, \gamma^{\mu_1\ldots\mu_D}.
\]

For rank \( r = 0 \) and \( 1 \) we obtain
\[
C^T = -t_0 C, \quad \gamma^\mu T = t_0 t_1 C \gamma^\mu C^{-1}
\] (3.11)

Given two possibilities in the representation for even dimensions we can construct the Clifford algebra with Pauli matrices
\[
C_+ = \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \ldots, \quad t_0 t_1 = 1
\] (3.12)
\[
C_- = \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \ldots, \quad t_0 t_1 = -1
\] (3.13)

for odd dimensions only one of the two can be used. To preserve the algebra the charge conjugation matrix transforms as
\[
C' = U^T C U.
\] (3.14)

The charge conjugation needs to be unitary \( C^\dagger = C^{-1} \) in any representation.

### 3.4 Fierz rearrangement

In this section we study the importance of the completeness of the Clifford algebra basis \( \gamma^A \). Fierz rearrangement properties are frequently used in supergravity, these properties involve changing the pairing of spinors in product of spinor bilinears.

Let us derive the basic Fierz identity. Using spinor indices defined as
\[
M_{\alpha\mu} M^{\mu\beta} M_{\gamma\nu} M^{\nu\delta} = \delta^\beta_\alpha \delta^\gamma_\delta.
\] (3.15)

We can expand the matrix \( M \) in the complete basis \( \gamma^A \) using our spinor indices we obtain
\[
\delta^\alpha_\beta \delta^\gamma_\delta = \frac{1}{2m} \sum_A (m_A)_\alpha^\beta (\gamma^A)_\gamma^\delta
\] (3.16)

the coefficients are \( m_A = 2^{-m} \delta^\alpha_\beta \delta^\gamma_\delta (\gamma^A)_\beta^\gamma \). Therefore we obtain the basic rearrangement lemma
\[
\delta^\alpha_\beta \delta^\gamma_\delta = \frac{1}{2m} \sum_A (\gamma^A)_\alpha^\beta (\gamma^A)_\gamma^\delta.
\] (3.17)

The Fierz rearrangement is valid for any set of four anticommuting spinor fields. The basic Fierz identity (3.17) gives us
\[
(\bar{\lambda}_1 \lambda_2)(\bar{\lambda}_3 \lambda_4) = -\frac{1}{2m} \sum_A (\bar{\lambda}_1 \gamma^A \lambda_4)(\bar{\lambda}_3 \gamma^A \lambda_2)
\] (3.18)

Useful Weyl spinor identities, which involves manipulating \( \sigma \) matrices interacting with two spinors, the symmetry properties are gives as
\[
\psi \sigma^\mu \bar{\chi} = -\bar{\chi} \sigma^\mu \psi
\] (3.19)
\[
\psi \sigma^\mu \sigma^\nu \bar{\chi} = \chi \sigma^\nu \sigma^\mu \psi
\] (3.20)
\[
\psi \sigma^{\mu\nu} \bar{\chi} = -\chi \sigma^{\mu\nu} \psi.
\] (3.21)

These manipulations will be useful later on.
Let us define the Majorana conjugate of any spinor $\lambda$ using its transpose and the charge conjugation matrix
\[
\bar{\lambda} = \lambda^T C
\] (3.22)
This equation is useful in SUSY and supergravity in which symmetry properties of $\gamma$-matrices and of spinor bilinears are important and these properties are determined by $C$. Using the definition of Majorana conjugate and equation (3.10) we obtain
\[
\bar{\lambda} \gamma_{\mu_1 \ldots \mu_r} \chi = t_r \bar{\chi} \gamma_{\mu_1 \ldots \mu_r} \lambda
\] (3.23)
the minus sign obtained by changing the Grassmann valued spinor components. The symmetry property is valid for Dirac spinors, but its main application is with Majorana spinors. Therefore we call it Majorana flip relations.

### 3.5 Majorana spinors

We consider the reality constraint and write down the following equations
\[
\psi = \psi^C = B^{-1} \psi^*, \quad \text{i.e.} \quad \psi^* = B \psi
\] (3.24)
the constraint is compatible with Lorentz symmetry, the equation (3.24) is the defining condition for Majorana spinors. In order to have Majorana spinors we consider when $D = 4$, using that $C^T = -t_1 C$ and equation (3.12). These equations and the condition indeed satisfies the equation (3.24). There are representations of $\gamma$-matrices that are real and may be called really real representations. Here is a real representation for $D = 4$

\[
\gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2 \otimes 1
\] (3.25)
\[
\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3 \otimes 1
\] (3.26)
\[
\gamma_2 = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix} = \sigma_1 \otimes \sigma_1
\] (3.27)
\[
\gamma_3 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} = \sigma_1 \otimes \sigma_3
\] (3.28)

The physics of Majorana spinors is the same, in any Clifford algebra the complex conjugate can be replaced with the charge conjugation. For example the complex conjugate of $\bar{\chi} \gamma_{\mu_1 \ldots \mu_r} \psi$ where $\chi$ and $\psi$ are Majorana, is computed in the following way
\[
(\bar{\chi} \gamma_{\mu_1 \ldots \mu_r} \psi)^* = (\bar{\chi} \gamma_{\mu_1 \ldots \mu_r} \psi)^C = \bar{\chi} (\gamma_{\mu_1 \ldots \mu_r})^C \psi = \bar{\chi} \gamma_{\mu_1 \ldots \mu_r} \psi
\]
where we have used $\psi^C = \psi$ and $\bar{\psi}^C = \bar{\psi}$. 

10
4 Differential geometry

In this section we will discuss differential geometry, where we will formulate the Cartan formalism also known as vierbein, spin connection and p-forms mainly. In supergravity fermions couple to gravity. The key references in this section will be [4] and [5].

4.1 Metric on manifold

A metric or inner product on a real vector space \( V \) is a bilinear map from \( V \otimes V \rightarrow \mathbb{R} \). The inner product of two vectors \( u \) and \( v \in V \) must satisfy the following properties:

- bilinearity, \((u, c_1 v_1 + c_2 v_2) = c_1 (u, v_1) + c_2 (u, v_2) \) and \(( c_1 v_1 + c_2 v_2, u) = c_1 (v_1, u) + c_2 (v_2, u) \)
- non-degeneracy, if \((u, v) = 0 \forall v \in V \) then \( u = 0 \)
- symmetry \((u, v) = (v, u) \).

The metric on a manifold is smooth assignment of an inner product map on each \( T_p(M) \otimes T_p(M) \rightarrow \mathbb{R} \). In local coordinates the metric is specified by a covariant rank-2 symmetric tensor field \( g_{\mu\nu} \) and the inner product of two contravariant vectors \( U^\mu(x) \) and \( V^\nu(x) \) is \( g_{\mu\nu} U^\mu(x) V^\nu(x) \) which is a scalar field. We can summarize the properties of a metric by the line element

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{4.1}\]

non-degeneracy means that \( \det(g_{\mu\nu}) \neq 0 \) so the inverse metric \( g^{\mu\nu} \) exist as a rank-2 symmetric contravariant tensor which satisfy

\[
g^{\mu\rho} g_{\rho\nu} = g_{\nu\rho} g^{\rho\mu} = \delta^\mu_\nu. \tag{4.2}\]

We can use this relation to raise and lower indices e.g \( V^\mu(x) = g_{\mu\nu} V^\nu(x) \) and \( \omega^\mu(x) = g^{\mu\nu} \omega_\nu(x) \). In gravity the metric has the following signature \(- + + \ldots +\). The metric tensor \( g_{\mu\nu} \) may be diagonalized by an orthogonal transformation \((O^{-1})^a_\mu = O^a_\mu \), and

\[
g_{\mu\nu} = O^a_\mu D_{ab} O^b_\nu \tag{4.3}\]

with positive eigenvalues \( \lambda^a \) in \( D_{ab} = \text{diag}(\lambda^0, \lambda^1, \ldots, \lambda^{D-1}) \)

4.2 Cartan formalism

Let us define an important auxiliary quantity

\[
e^a_\mu(x) = \sqrt{\lambda^a(x) O^a_\mu(x)} . \tag{4.4}\]

In four dimensions this quantity is known as the tetrad or vierbien. In general dimensions it is called vielbein but when we discuss gravity we prefer the term frame field. We can write the metric as

\[
g_{\mu\nu} = e^a_\mu \eta_{ab} e^b_\nu \tag{4.5}\]

where \( \eta_{ab} = \text{diag}(-1, 1, \ldots, 1) \) is the metric of flat \( D \)-dimensional Minkowski space-time. Given a \( x \)-dependent matrix \( \Lambda^a_b(x) \) which leaves \( \eta_{ab} \) invariant, which allows us to construct the equation (3.5) with a Lorentz transformation i.e

\[
e^a_\mu(x) = \Lambda^{-1a}_b(x) e^b_\mu(x) . \tag{4.6}\]
all frame fields related by a local Lorentz transformation are viewed as equivalent. Local Lorentz transformations in curved spacetime differ from global Lorentz transformation if Minkowski space. We require that the frame field $e^a_{\mu}$ transform as a covariant vector under coordinate transformations

$$e^\prime_a(x) = \frac{\partial x}{\partial x^\prime} e^a_{\mu}(x).$$

(4.7)

The vielbein $e^a_{\mu}$ has an inverse frame field $e^\mu_a$ which satisfy $e^a_{\mu}e^\nu_b = \delta^a_b$ and $e^\mu_a e^\nu_a = \delta^\mu_{\nu}$. The vielbein transforms under a local transformation, hence

$$e^\mu_a = g^\mu_{\nu} \eta_{ab} e^b_{\nu}, \quad e^\mu_a g_{\mu\nu} e^\nu_b = \eta_{ab}.$$

(4.8)

This term shows that the inverse frame field can be used relate a general metric to the Minkowski metric. The second relation of (4.8) indicates that $e^\mu_a$ form an orthonormal set of $T_p(M)$. Since it is non-degeneracy the $\det e^\mu_a \neq 0$ we have a basis of each tangent space. Any contravariant vector field has a unique expansion in the new basis, i.e $V^\mu = V^a e^a_{\mu}$ with $V^a = V^\mu e^a_{\mu}$. The $V^a$ are the frame components of the original vector field $V_\mu$. The vector fields transform under a Lorentz transformation, i.e $V^a = \Lambda_{ab} V^b$. We can then use $e^\mu_a$ and $e^\prime_a$ to transform vector and tensor fields back and forth between a coordinate basis with indices $\mu, \nu, \ldots$ and a local Lorentz basis with indices $a, b, \ldots$ in the metric $\eta_{ab}$. We can do an exercise to show how all these transformations work:

$$U^\mu V_\mu = g_{\mu\nu} U^\mu V^\nu = e^a_{\mu} \eta_{ab} e^b_{\nu} U^\mu V^\nu = \eta_{ab} U^\alpha V^b = U^\alpha V^a$$

where the second term is obtain from $V_\mu = g_{\mu\nu} V^\nu$, the third term we use equation (3.8) and the last term we use the same relation as the second term. We have now constructed a new local invariance, we have enlarge the symmetry of GR to be general coordinate transformations and Lorentz transformations. We need frame fields to describe fermions in general relativity.

We can also use the frame field $e^a_{\mu}$ to define the new basis $\Lambda^p(M)$ of differential forms. The local Lorentz basis of 1-form is

$$e^a \equiv e^a_{\mu} dx^\mu.$$

(4.9)

For 2-forms the basis consist of the wedge product $e^a \wedge e^b$ and so on. Local frames are useful when we consider fermions coupled to gravity, because spinors transform under Lorentz transformations.

### 4.3 Connections and covariant derivatives

Contravariant derivative on a manifold is a rule to differentiate a tensor of type $(p, q)$ producing a tensor $(p, q + 1)$. We need to introduce the affine connection $\Gamma^p_{\mu\nu}$. On vector fields the covariant derivative is defined as

$$\nabla_\mu V^\rho = \partial_\mu V^\rho + \Gamma^\rho_{\mu\nu} V^\nu$$

(4.10)

$$\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\rho_{\mu\nu} V_\rho.$$

(4.11)

In supergravity we will use the frame field $e^a_{\mu}$. Gravitational theories with fermions is antisymmetric in Lorentz indices $\omega^{ab} = \omega^{ab}_{\mu} dx^\mu$. The components $\omega^{ab}_{\mu}$ are called the spin connections because they describe the spinors on the manifold.

Given the 1-forms $e^a$ we examine the 2-form

$$de^a = \frac{1}{2} (\partial_\mu e^a_{\nu} - \partial_\nu e^a_{\mu}) dx^\mu \wedge dx^\nu$$

(4.12)
the antisymmetric components transforms as (0, 2) tensor under coordinate transformation. Using the Lorentz transformation (4.6) we obtain the following equation

\[ de'^a = \Lambda^{-1}a_b de^b + d\Lambda^{-1}a_b \wedge e^b \]  

(4.13)

the second term spoils the vector transformation so we need to add an extra term that absorbs the term involving \( d\Lambda^{-1}a_b \). We introduce the spin connection which is a 2-form, giving us

\[ de^a + \omega^a\wedge e^b \equiv T^a . \]  

(4.14)

If \( \omega^a\) is defined to transform under Lorentz transformation as

\[ \omega'{}^a_b = \Lambda^{-1}a_c d\Lambda^c_b + \Lambda^{-1}a_c \omega^d\Lambda^d_b \]  

(4.15)

then \( T^a \) transforms as a vector,

\[ T'^a_a = \Lambda^{-1}a_b T^b_a . \]  

We know that any tensor transforms as a vector

\[ \psi' = \exp \left( -\frac{1}{4} \gamma^a_a \right) \psi \]  

(4.16)

determines the covariant derivative

\[ D_\mu \psi = \left( \partial_\mu + \frac{1}{4} \omega_\mu a \gamma^a \right) \psi \]  

(4.17)

this will be very useful when we do calculations in supergravity.

Let us transform Lorentz covariant of the vector and tensor frame fields to the coordinate basis where they become covariant derivatives with respect to general coordinate transformations. We can write out the spin connection in terms of the affine connection \( \Gamma_{\mu\nu}^\rho \). The quantity \( \nabla_\mu V^\nu = e^\nu_a D_\mu V^a \) is the transform to coordinate basis of a frame field and covariant vector field. We can show that this quantity can take the form of equation (4.10) and (4.11)

\[ \nabla_\mu V^\nu = e^\nu_a D_\mu V^a \]  

(4.18)

where we have used that \( D_\mu V^\nu = \partial_\mu V^\nu + \omega_\mu a \gamma^a V^\nu \) and \( e^\nu_a e^a_\nu = \delta^\nu_\nu \). We can show that \( \nabla_\mu V^\nu = e^\nu_a D_\mu V^a \) in the same way

\[ \nabla_\mu V^\nu = e^\nu_a D_\mu V^a = e^\nu_a D_\mu (e^a\gamma^\nu) 
= \partial_\mu V^\nu + e^\nu_a (\partial_\mu e^a\gamma^\nu + \omega_\mu a \gamma^a e^b \gamma^b \gamma^\nu) 
= \partial_\mu V^\nu + e^\nu_a (\partial_\mu e^a\gamma^\nu + \omega_\mu a \gamma^a e^b \gamma^b \gamma^\nu) \]  

(4.19)

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These two examples can be written down in a general Lorentz tensor of type \((p,q)\). The transformation to coordinate basis is given by
\[
\nabla_\mu T^\rho_{\nu_1...\nu_q} \equiv e^\rho_{a_1} \cdots e^\rho_{a_p} e^{b_1}_{\nu_1} \cdots e^{b_q}_{\nu_q} D_\mu T^a_1...a_p b_1...b_q (4.20)
\]
define the tensor of type \((p,q+1)\) with the properties of the covariant derivative. The affine connection relates to the spin connection by
\[
\Gamma^\rho_{\mu\nu} = e^\rho_{a_1} (\partial_\mu e^{a_1}_\nu + \omega^a_{\mu\nu} \frac{e^b_a}{e^b_1} - e^a_{\mu\nu} \frac{\Gamma^\rho_{\mu\nu}}{\nabla g_{\mu\nu}}) (4.21)
\]
we can rewrite this as
\[
\partial_\mu e^{a}_\nu + \omega^a_{\mu\nu} \frac{e^b_a}{e^b_1} - e^a_{\mu\nu} \frac{\Gamma^\rho_{\mu\nu}}{\nabla g_{\mu\nu}} = 0 . (4.22)
\]
This property is called vielbein postulate. When we considered a connection without spin connection we had an restriction mainly the that the connection is metric compatible i.e \(\nabla g_{\mu\nu} = 0\). We have a similar restriction on the connection above,
\[
\nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^\sigma_{\mu\nu} g_{\sigma\rho} - \Gamma^\sigma_{\mu\rho} g_{\nu\sigma} = 0 (4.23)
\]
the metric postulate means that the tensor is covariantly constant. The \(\Gamma^\rho_{\mu\nu}\) is the Christoffel symbols defined as
\[
\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) (4.24)
\]
this is the torsion-free connection. When the torsion is present the affine connection is not symmetric rather
\[
\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu} = T^\rho_{\mu\nu} (4.25)
\]
Using equation (4.23) we can calculate the nonvanishing torsion, we start with
\[
0 = \nabla_a g_{bc} + \nabla_b g_{ca} - \nabla_c g_{ab} (4.26)
\]
using the metric compatibility property we obtain the following equation
\[
\partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab} = \Gamma^d a b c d + \Gamma^d a c b d + \Gamma^d b a c d - \Gamma^d c a b d - \Gamma^d b c d a = 2 \Gamma^d a b c d + 2 \Gamma^d a c b d + 2 \Gamma^d b a c d + 2 \Gamma^d b c a d (4.27)
\]
this connection is not unique. If the torsion vanishes we get back the affine connection.

We can now consider curvature, recall that we defined the curvature as
\[
R^a_{bcd} = \partial_b \Gamma^a_{cd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ce} \Gamma^e_{db} - \Gamma^a_{de} \Gamma^e_{cb} (4.28)
\]
we also have another connection which will have curvature too
\[
(\nabla_c \nabla_d - \nabla_d \nabla_c) V^\mu = \partial_c \partial_d V^\mu + (\partial_c \omega^d_{\mu\sigma} V^\sigma + \omega^d_{\mu\sigma} \partial_c V^\sigma - \Gamma^\sigma_{\mu\nu}(\partial_d V^\nu + \omega^\nu_{\mu\sigma} V^\sigma)) + \omega^\mu_{\nu\lambda}(\partial_d V^\lambda + \omega^\lambda_{\nu\lambda} V^\sigma) - (c \leftrightarrow d) (4.29)
\]
we would like this to be the spin curvature and a torsion term. We look at the terms involving \(V\) and no derivatives of \(V\), and identify
\[
R^a_{\mu\nu} = \partial_b \omega^a_{\mu\nu} - \partial_d \omega^a_{\mu\nu} + \omega^b_{\mu\lambda} \omega^a_{\lambda\sigma} - \omega^a_{\mu\lambda} \omega^b_{\lambda\sigma} (4.30)
\]
as the curvature of the spin connection. The remaining terms are

\[
(-\Gamma^c_{d} e^c + \Gamma^c_{d} e^c) (\partial_{e} V^\mu + \omega^e_{\sigma} V^\sigma) = -T^c_{d} \nabla_{e} V^\mu
\] (4.31)

Because of the antisymmetry in \(d\), it is also a \((0,2)\) tensor under general coordinate transformation, called the curvature tensor. Thus we can define the curvature 2-form

\[
R^\mu_{\sigma} = \frac{1}{2} R^\mu_{\sigma a b} dx^a \wedge dx^b
\] (4.32)

using equation (4.30) we then see that the curvature 2-form is related to connection 1-form by

\[
d\omega^\mu_{\sigma} + \omega^\mu_{\nu} \wedge \omega^\nu_{\sigma} = R^\mu_{\sigma}
\] (4.33)

this is known as the second Cartan structure equation. We can derive the Bianchi identities for the curvature tensor, but first we will introduce the basis of 1-forms

\[
E^\mu = e^\mu_a dx^a.
\] (4.34)

We can define the torsion 2-form

\[
T^\lambda = e^\lambda_a \frac{1}{2} T^a_{bc} dx^b \wedge dx^c.
\] (4.35)

We can use these p-forms to obtain the Cartan first and second structure equations in a much faster way. We consider the first structure equation where we will use the equation (4.34), rewriting the first structure equation and doing the calculation yields us

\[
dE^\mu + \omega^\mu_{\nu} \wedge E^\nu = d(e^\mu_a dx^a) + \omega^\mu_{\nu} dx^a \wedge e^\nu_b dx^b
\]

\[
= (\partial_{b} e^\mu_a) dx^b \wedge dx^a + \omega^\mu_{\nu} e^\nu_b dx^a \wedge dx^b
\]

\[
= \frac{1}{2} (\partial_{b} e^\mu_a - \partial_{a} e^\mu_b + \omega^\mu_{\nu} e^\nu_b - \omega^\mu_{\nu} e^\nu_a) dx^a \wedge dx^b
\]

\[
= \frac{1}{2} (\Gamma^c_{b} e^\mu_c - \omega^\mu_{\sigma} e^\sigma_b - \Gamma^c_{b} e^\mu_a + \omega^\mu_{\nu} e^\nu_a + \omega^\mu_{\nu} e^\nu_b - \omega^\mu_{\nu} e^\nu_a) dx^a \wedge dx^b
\]

\[
= \frac{1}{2} T^a_{b c} e^\mu_a dx^a \wedge dx^b = T^\mu
\]

this is Cartan’s first structure equation, where we have used \(\partial_{b} e^\mu_a = \Gamma^c_{b} e^\mu_c - \omega^\mu_{\sigma} e^\sigma_b\) and contracted the remaining terms which disappears when doing so. We can do the same with the curvature

\[
d\omega^\mu_{\sigma} + \omega^\mu_{\nu} \wedge \omega^\nu_{\sigma} = d(\omega^\mu_{\sigma a} dx^a) + \omega^\mu_{\nu a} dx^a \wedge \omega^\nu_{\sigma}
\]

\[
= d\omega^\mu_{\sigma a} dx^a + \omega^\mu_{\nu a} dx^a \wedge \omega^\nu_{\sigma b} dx^b
\]

\[
= \frac{1}{2} (\partial_{b} \omega^\mu_{\sigma a} - \partial_{a} \omega^\mu_{\sigma b} + \omega^\mu_{\nu a} \omega^\nu_{\sigma b} - \omega^\mu_{\nu b} \omega^\nu_{\sigma a}) dx^a \wedge dx^b
\]

\[
= \frac{1}{2} R^\mu_{\sigma a b} dx^a \wedge dx^b = R^\mu_{\sigma}
\] (4.36)

We can now find the Bianchi identities for Cartan’s first and second structure equation. Taking \(d\) of Cartan’s first structure equation gives us

\[
dT^\mu = d(dE^\mu + \omega^\mu_{\nu} \wedge E^\nu)
\]

\[
= d\omega^\mu_{\nu} \wedge E^\mu - \omega^\mu_{\nu} \wedge dE^\nu
\]

\[
= (R^\mu_{\sigma} - \omega^\mu_{\rho} \wedge \omega^\rho_{\sigma}) \wedge E^\nu - \omega^\mu_{\nu} \wedge (T^\nu - \omega^\nu_{\sigma} \wedge E^\sigma)
\]

\[
= R^\mu_{\sigma} \wedge E^\nu - \omega^\mu_{\rho} \wedge \omega^\rho_{\sigma} \wedge E^\nu - \omega^\mu_{\nu} \wedge T^\nu + \omega^\mu_{\nu} \wedge \omega^\nu_{\sigma} \wedge E^\sigma
\]

\[
= R^\mu_{\sigma} \wedge E^\nu - \omega^\mu_{\nu} \wedge T^\mu
\] (4.37)
and taking the $d$ of Cartan’s second structure equation gives us

$$d R^\lambda_\mu = d(d\omega^\lambda_\mu + \omega^\lambda_\rho \wedge \omega^\rho_\sigma)$$
$$= d\omega^\lambda_\rho \wedge \omega^\rho_\sigma - \omega^\lambda_\rho \wedge d\omega^\rho_\sigma$$
$$= (R^\mu_\sigma - \omega^\mu_\rho \wedge \omega^\rho_\sigma) \wedge \omega^\rho_\sigma - \omega^\lambda_\rho \wedge (R^\rho_\sigma - \omega^\rho_\delta \wedge \omega^\delta_\sigma)$$
$$= R^\mu_\sigma \wedge \omega^\rho_\sigma - \omega^\lambda_\rho \wedge R^\rho_\sigma . \tag{4.38}$$

Using the following relation $R^a_{\mu\nu\rho} = R^{eb}_\mu a e^b_\rho$, we obtain

$$R^a_{\mu\nu\rho} + R^a_{\nu\rho\mu} + R^a_{\rho\mu\nu} = -D_\mu T^a_{\nu\rho} - D_\nu T^a_{\rho\mu} - D_\rho T^a_{\mu\nu} \tag{4.39}$$
$$D_\mu R^a_{\nu\rho} + D_\nu R^a_{\rho\mu} + D_\rho R^a_{\mu\nu} = 0 . \tag{4.40}$$

The derivatives $D_\mu$ in the equation above are Lorentz covariant derivatives and contain only the spin connection. The first relation is the first Bianchi identity for the curvature tensor, but we include a torsion tensor. The second relation is known as the Bianchi identity for the curvature. The commutator of Lorentz covariant derivatives leads to Ricci identities. We define them as

$$[D_\mu, D_\nu] \Phi = \frac{1}{2} R_{\mu\nu ab} M^{ab} \Phi \tag{4.41}$$
$$[D_\mu, D_\nu] V^a = R^a_{\mu\nu b} V^b \tag{4.42}$$
$$[D_\mu, D_\nu] \Psi = \frac{1}{4} R_{\mu\nu ab} \gamma^{ab} \Psi \tag{4.43}$$

where $\Psi$ is the spinor fields, $\Phi$ is a field transforming in a representation of proper Lorentz group with the generator $M^{ab}$. 

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4.4 Einstein-Hilbert action in frame and curvature forms

Let us show that the Einstein-Hilbert action for GR can be written as the integral of a volume form involving the frame and curvature.

\[
\frac{1}{(D-2)!} \int \varepsilon_{abc1...c_{D-2}} e^{c1} \wedge ... \wedge e^{c_{D-2}} \wedge R^{ab}
\]
\[
= \frac{1}{(D-2)!} \int \varepsilon_{abc1...c_{D-2}} e^{c1} \wedge ... \wedge e^{c_{D-2}} \wedge \frac{1}{2} R^{ab}_{cd} e^c \wedge e^d
\]
\[
= \frac{1}{2(D-2)!} \int \varepsilon_{e1...e_{D-2}a} \varepsilon_{c1...c_{D-2}b} R^{ab}_{cd} dV
\]
\[
= \frac{1}{2(D-2)!} \int (D-2)! \delta_{abc1...c_{D-2}} R^{ab}_{cd} dV
\]
\[
= - \int \delta_a \delta_b c R^{ab}_{cd} dV = - \int d^D x \sqrt{-\det g} R
\]

where we have used the equation (4.32) and volume form defined as

\[
dV = d^D x \sqrt{-\det g}
\]

and

\[
e^{a1} \wedge ... \wedge e^{aq} \wedge e^{b1} \wedge ... \wedge e^{bp} = -\varepsilon^{a1...aq b1...bp} dV.
\]

This form of the action reveals how the connection with torsion is realised in the physical setting of gravity coupled to fermions.
5 Kaluza-Klein theory

In this section we will consider Kaluza-Klein theory which will be very useful when we do dimensional reductions. In this chapter we will go through dimensional reduction of Einstein-Hilbert action (EH) in $D + 1$ dimensions. The dimensional reduction of E-H action will yield us a D-dimensional Einstein field equations, the Maxwell equations for electromagnetic fields and an wave equation for scalar fields. The key references in this section are [1] and [6]

5.1 Kaluza-Klein reduction on $S^1$

Let us assume we start from Einstein gravity in ($D + 1$) dimensions described by the Einstein-Hilbert Lagrangian

$$L = \sqrt{-\hat{g}} \hat{R}$$

where $\hat{R}$ is the Ricci scalar and $\hat{g}$ denotes the determinant of the metric tensor. We now want to reduce the action to $D$ dimensions by doing so we need to compactify one of the coordinates on a circle $S^1$ of radius $L$. We can expand all the components of $D + 1$-dimensional metric as a Fourier series

$$\hat{g}_{MN} = \sum_n g_{NM}(x)e^{inz/L}$$

If we do this then we obtain infinite numbers of fields in $D$ dimensions by labelled the Fourier mode number $n$. So if we consider a scalar field in 5 dimensions defined by the action

$$S_{5D} = \int d^5x \hat{\phi} \partial_M \hat{\phi}$$

we set the extra dimensions $x^4 = z$ defining a circle with the radius $L$ with $z = z + 2\pi L$, the spacetime is now $M_4 \times S^1$. Using the Fourier expansion where we expand around $\hat{\phi}$, the coefficients describe infinite amount of D-dimensional scalar fields, which satisfy the equation of motion

$$\Box \phi_n - \frac{n^2}{L^2} \phi_n = 0.$$ (5.4)

This means that we have infinite amount of Klein-Gordon equations for massive D-dimensional fields and that each $\phi_n$ is a D-dimensional particle with mass

$$m_n^2 = \frac{n^2}{L^2}.$$ (5.5)

We assumed that the circle is very small, which tells us that the particles should be around the Planck mass. However since the modes are way beyond intergalactic scales, we can not observed these types of particles and we can thus neglect them.

Going back to the EH action, the Kaluza-Klein ansatz will be $\hat{g}_{MN}(x, z)$ where the metric tensor will not depend on the $z$ coordinate. The index $M$ runs over ($D + 1$) values of the higher dimension which splits into $D$ values of lower dimension, i.e takes the value associated with the compactification. Let us define the ($D + 1$)-dimensional metric in terms of $D$-dimensional fields $g_{\mu\nu}, A_\mu$ and $\phi$

$$ds^2_{D+1} = e^{2\alpha \phi} ds_D^2 + e^{2\beta \phi} (dz + A)^2.$$ (5.5)

This ansatz is independent of $z$ thus we have express the components of the higher dimensional metric $\hat{g}_{MN}$ which are given by the lower dimensional fields

$$\hat{g}_{MN} = \begin{pmatrix} g_{\mu\nu} + e^{2\beta \phi} A_\mu A_\nu & e^{2\beta \phi} A_\mu \\ e^{2\beta \phi} A_\mu & e^{2\beta \phi} \end{pmatrix}$$ (5.6)
To calculate the Ricci tensor for the Einstein-Hilbert Lagrangian we must choose a vielbein basis defined as
\[ \hat{e}^a = e^{\alpha \phi} e^a, \quad \hat{e}^z = e^{\beta \phi} (dz + A) \ . \] (5.7)

We can now calculate the one-forms, where we set the torsion tensor to zero.
\[ d\hat{e}^a = \alpha d\phi e^{\alpha \phi} \wedge e^a - e^{\alpha \phi} de^a \]
and
\[ d\hat{e}^z = \beta d\phi \wedge e^{\beta \phi} (dz + A) + e^{\beta \phi} dA \]

and the other one-form can be calculated in the same way
\[ d\hat{e}^z = \beta d\phi \wedge e^{\beta \phi} (dz + A) + e^{\beta \phi} dA \]

where we have used that \( F = dA \) defined as
\[ F = \frac{1}{2} F_{ab} e^a \wedge e^b = \frac{1}{2} (\partial_a A_b - \partial_b A_a) e^a \wedge e^b . \] (5.10)

We can now obtain the spin connection by using Cartan’s first structure equation (3.14),
\[ \tilde{\omega}^{ab} = \omega^{ab} + \alpha e^{-\alpha \phi} (\partial^a \phi e^b - \partial^b \phi e^a) - \frac{1}{2} F^{ab} e^{(\beta - 2\alpha) \phi} \hat{e}^z \]
and
\[ \tilde{\omega}^{az} = -\beta e^{-\alpha \phi} \partial^a \phi \hat{e}^z - \frac{1}{2} F^{ab} e^{(\beta - 2\alpha) \phi} \hat{e}^b \] (5.11)

Now that we have found the spin connection we can obtain the two-form by using Cartan’s second structure equation. Before we calculate the curvature, we want to choose the values of \( \alpha \) and \( \beta \) such that the dimensionally-reduced Lagrangian is defined as \( \mathcal{L} = \sqrt{-g} R \). If we do not fix the constants our Lagrangian takes the form \( \mathcal{L} = e^{(\beta - (D - 2)\alpha) \phi} \sqrt{-g} R \), we now see that we have to set \( \beta = -(D - 2)\alpha \) to achieve the Lagrangian we defined at the beginning. This is called the Einstein frame and it comes from our metric ansatz where we have Weyl rescaling \( e^{2\alpha \phi} \) and \( e^{2\beta \phi} \) which results in modified Ricci tensors. We also want to ensure that the scalar field \( \phi \) acquires a kinetic term with a canonical normalisation \( \frac{1}{2} (\partial \phi)^2 \) in the Lagrangian. Therefore we choose the constant to be
\[ \alpha^2 = \frac{1}{2(D - 1)(D - 2)}, \quad \beta = -(D - 2)\alpha \] (5.13)

The Ricci tensor are
\[ \hat{R}_{ab} = e^{-2\alpha \phi} \left( R_{ab} - \frac{1}{2} \partial_a \phi \partial_b \phi - \alpha \eta_{ab} \Box \phi \right) - \frac{1}{2} e^{-2\alpha \phi} F_a F_b \]
(5.14)
\[ \hat{R}_{az} = \frac{1}{2} e^{(D - 3)\alpha \phi} \partial^a \phi F_b \]
(5.15)
\[ \hat{R}_{zz} = (D - 2) e^{-2\alpha \phi} \Box \phi + \frac{1}{4} e^{-2\alpha \phi} F^2 . \]
(5.16)

We can obtain the Ricci scalar \( \hat{R} = \eta^{AB} \hat{R}_{AB} = \eta^{ab} \hat{R}_{ab} + \hat{R}_{zz} , \)
\[ \hat{R} = \eta^{ab} \left( e^{-2\alpha \phi} \left( R_{ab} - \frac{1}{2} \partial_a \phi \partial_b \phi - \alpha \eta_{ab} \Box \phi \right) - \frac{1}{2} e^{-2\alpha \phi} F_a F_b \right) \\
+ (D - 2) e^{-2\alpha \phi} \Box \phi + \frac{1}{4} e^{-2\alpha \phi} F^2 \]
(5.17)
We can now determine the determinant of the metric tensor,
\[
\sqrt{-\hat{g}} = e^{(\beta + D\alpha)\phi} \sqrt{-g} = e^{2\alpha\phi} \sqrt{-\hat{g}}
\]
(5.18)
the Einstein-Hilbert Lagrangian now takes the form
\[
\mathcal{L} = \sqrt{-g} \left( R - \frac{1}{2} (\nabla\phi)^2 - \frac{1}{4} e^{-2(D-1)\alpha\phi} F^2 \right).
\]
(5.19)
Doing the integral over the \( z \) coordinate now yields us the following action
\[
S_{EH} = \frac{1}{16\pi G_N^{D+1}} \int d^{D+1} \sqrt{-g} \hat{R} = \frac{2\pi L}{16\pi G_N^{D+1}} \int d^D x \sqrt{-g} \left( R - \frac{1}{2} (\nabla\phi)^2 - \frac{1}{4} e^{-2(D-1)\alpha\phi} F^2 \right).
\]
(5.20)
the scalar field \( \phi \) is called a dilaton. We can work out the field equations for this Lagrangian that tells us more about the fields. If one were to set the scalar field to zero we would obtain the Einstein-Maxwell Lagrangian in \( D \) dimensions, but this is not allowed due to the equations of motion.
\[
\begin{align*}
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} &= \frac{1}{2} \left( \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} (\partial \phi)^2 g_{\mu\nu} \right) + \frac{1}{2} e^{-2(D-1)\alpha\phi} \left( F^2_{\mu\nu} - \frac{1}{4} F^2 g_{\mu\nu} \right) \\
\nabla^\mu \left( e^{-2(D-1)\alpha\phi} F_{\mu\nu} \right) &= 0 \\
\Box \phi &= - \frac{1}{2} (D - 1) \alpha e^{-2(D-1)\alpha\phi} F^2.
\end{align*}
\]
(5.21)
the reason we have dropped the \( \Box \phi \) term in the Lagrangian is because it is a total derivative in \( \mathcal{L} \) which doesn’t contribute to the field equations. From the last equation in (5.21) we see that we cannot set \( \phi = 0 \) because it involves a \( F^2 \) term on the right-hand side. This means that the lower dimensional fields prevent the truncation of the scalar [6]. We neglected the massive modes in our Fourier expansion but assume we kept the massive modes, would we then be able to set the equation of motion of the massive field to zero. When we do a dimensional reduction on a circle our fields must transform covariantly under lower-dimensional \( U(1) \) gauge symmetry and under diffeomorphism of the lower-dimensional spacetime. Thus any mixture of massless modes with towers of massive modes will be avoided. Lower-dimensional solutions must be solutions to higher-dimensional theories which guarantees consistent truncation. So in other words when the Kaluza-Klein ansatz satisfies the equations of motion we have consistent truncation.
6 Free Rarita-Schwinger equation

Let us now introduce a free spin-3/2 field. We only consider fields that don’t interact and we consider them separately. In particular we consider \( \Psi_\mu(x) \) as a free field, the gauge transformation is

\[
\Psi_\mu(x) \rightarrow \Psi_\mu(x) + \partial_\mu \varepsilon x .
\]  

(6.1)

We will also assume that \( \Psi_\mu \) and \( \varepsilon \) are complex spinors with \( 2^{[D/2]} \) spinor components for spacetime dimension \( D \). The gauge field \( \Psi_\mu(x) \) should have anti-symmetric derivative of the gauge potential \( \partial_\mu \Psi_\nu - \partial_\nu \Psi_\mu \) which is invariant under gauge transformation. We now consider a action that is Lorentz invariant, first order in derivatives and invariant under gauge transformations (6.1) and conjugate transformation of \( \bar\Psi_\mu \). We also demand that the action is hermitian so that the equations of motion for \( \bar\Psi_\mu \) is Dirac conjugate of \( \Psi_\mu \). We obtain the following action

\[
S = - \int d^D x \bar\Psi_\mu \gamma^{\mu\nu\rho} \partial_\nu \Psi_\rho .
\]  

(6.2)

we see that the action contains a third rank of Clifford algebra \( \gamma^{\mu\nu\rho} \) which satisfies all these properties. Using the variational principle on the action we obtain the Euler-Lagrange equation given as

\[
\gamma^{\mu\nu\rho} \partial_\nu \Psi_\rho = 0 .
\]  

(6.3)

Let us consider the on-shell degree’s of freedom for the Rarita-Schwinger field. We need to fix the gauge so we impose the constraint

\[
\gamma^i \Psi_i = 0 .
\]  

(6.4)

We rewrite the components of (6.3) as \( \nu \rightarrow 0 \) and \( \nu \rightarrow i \) which gives us

\[
\gamma^{ij} \partial_i \Psi_j = 0
\]

\[
-\gamma^{0} \gamma^{ij} (\partial_0 \Psi_j - \partial_j \Psi_0) + \gamma^{ijk} \partial_j \Psi_k = 0 .
\]  

(6.5)

Using \( \gamma^{ij} = \gamma^i \gamma^j - \delta^{ij} \) and the gauge condition we obtain

\[
\partial^i \Psi_i = 0 .
\]  

(6.6)

Using the same gamma-matrix manipulation on the second equation in (6.5) and multiplying it with \( \gamma_i \) as well as using the gauge condition, we obtain that the two first terms are zero due to \( \Psi_0 = 0 \). The second equation then takes the form

\[
\gamma^0 \partial_0 \Psi_0 + \gamma^i \partial_j \Psi_i = \gamma^\mu \partial_\mu \Psi_i = 0 .
\]  

(6.7)

We have now shown that the components of \( \Psi_i \) satisfy a Dirac equation. Hence the classical degree’s of freedom after imposing the gauge condition and (6.6) is given by \((D - 3)^{[D/2]}\).

The spinor field \( \Psi \) is a representation of \( SO(D - 2) \) which would result in on-shell degree’s of freedom given by \((D - 2)^{[D/2]}\). However when one subtracts the \( \gamma \)-trace of the vector-spinor representation, we end up with a irreducible representation that contains \((D - 3)^{[D/2]}\) components.

We can rewrite the equation (6.3) using the \( \gamma \)-matrix relation \( \gamma_\mu \gamma^{\mu\nu} = (D - 2)\gamma^{\nu\rho} \), which implies that the equation \( \gamma^\nu \partial_\nu \Psi_\rho = 0 \) is zero in spacetime dimension \( D > 2 \). We also use that \( \gamma^{\mu\nu} = \gamma^\mu \gamma^\nu - 2\eta^{[\mu} \gamma^{\nu]} \). Using this we obtain the following equation

\[
\gamma^\mu (\partial_\mu \Psi_\nu - \partial_\nu \Psi_\mu ) = 0 .
\]  

(6.8)
This equation is equivalent to the equation of motion above by applying $\gamma^\nu$ and obtain $\gamma^\nu \partial_\rho \Psi_\rho = 0$. We can also apply $\partial_\rho$ to obtain the following equation of motion defined as

$$\partial_\rho (\partial_\rho \Psi_\rho) = 0 .$$

(6.9)

If we now take SUSY into account we know that the spin-3/2 particle should have a superpartner. If the spin-3/2 particle from the Rarita-Schwinger equation represents a gravitino then the superpartner should be a graviton. The graviton is expected to be massless, but the gravitino is expected to have mass.

### 6.1 Dimensional reduction to the massless Rarita-Schwinger field

Let us apply dimensional reduction the Rarita-Schwinger field in $D + 1$ dimensions with $D = 2m$. We assume that the field $\Psi_\mu(x,y)$ is periodic in $y$ so that the Fourier series involves modes with half-integer $k$. This yields us only massive modes since $k \neq 0$ occurs.

Let us derive the wave equation of a massive gravitino in $D$-dimensional Minkowski space. We impose a gauge condition $\Psi_{Dk}(x) = 0$ thus eliminating the field component $\Psi_{D}(x,y)$. Writing out the wave equation using $\gamma^D = \gamma^* \gamma^D$ where $\mu = D$ and $\mu \neq D - 1$:

$$\gamma^\nu \partial_\nu \Psi_{Dk} = 0$$

(6.10)

$$\gamma^\mu \partial_\nu \Psi_{Dk} - \frac{k}{L} \gamma^\nu \gamma^\mu \Psi_{Dk} = 0 .$$

(6.11)

We can obtain the equation of motion for a massive particle from (6.11) by applying the chiral transformation $\Psi = e^{-i\pi \gamma^* / 4} \Psi'$. We can rewrite the chiral transformation as a phase factor $(\cos(2\beta) + i \gamma^* \sin(2\beta)) \Psi$ then we see that the only contribution is the sinus term and the Fourier series mode $e^{iky/L} \Psi$ gives rise to the massive modes since $k \neq 0$, therefore our equation of motion takes the form

$$(\gamma^\mu \partial_\nu - m \gamma^\nu) \Psi_\mu = 0$$

(6.12)

where $m = k/L$ which is the mass of the vector fields. One can further investigate the constraints on the equation of motion by finding $\gamma^\mu \Psi_\nu = 0$ and letting $\mu = 0$ which yields us the following constraint

$$(\gamma^{ij} \partial_i - m \gamma^j) \Psi_j = 0 .$$

(6.13)

Gathering the information about the constraints of the equation of motion gives us information about degrees of freedom.

$$\gamma^\mu \Psi_\mu = 0$$

(6.14)

$$\gamma^{ij} \partial_i - m \gamma^j) \Psi_j = 0 .$$

(6.15)

$$\partial_\mu (\bar{\Psi} \gamma^\mu \Psi) = 0 .$$

(6.16)

The initial data are the values $t = 0$ of the $\Psi_\mu$ restricted by the first two equations above. The complex scalar field $\Psi_\mu$ with $D \times 2^{D/2}$ degrees of freedom contains $(D - 2) \times 2^{D/2}$ independent classical degrees of freedom and thus $\frac{1}{2} (D - 2) \times 2^{D/2}$ on-shell physical states [4].

We can obtain a more general action defined as

$$S = - \int d^D x \bar{\Psi} (\gamma^\mu \partial_\nu \Psi - m \gamma^\mu \Psi - m' \eta^\mu \Psi)$$

(6.17)

thus from the dimensional reduction we obtained that the mass term for massive gravitino should be $m \bar{\Psi} \gamma^\mu \Psi$ and the extra term in the action is a Lorentz invariant term which has the dimension of mass.
The first and second order formulation of general relativity

In this chapter we will discuss the first and second order formulation of GR, which will be important when we consider supergravity for $\mathcal{N} = 1$. The second order formulation in which the metric tensor or the frame field describes gravity. If fermions are involved we must use the frame field.

In first order a.k.a Palatini formalism one starts with an action in which $e^a_\mu$ and $\omega_{\mu ab}$ are independent variables and the Euler-Lagrange equations are first order in derivatives. When we couple gravity to spinor fields the $\omega_{\mu ab}$ field equation contains terms bilinear in the spinors and the solution is $\omega_{\mu ab} = \omega_{\mu ab}(e) + K_{\mu ab}$ with a contorsion tensor determined as a bilinear expression in the spinor fields.

7.1 Second order formalism for gravity and fermions

We consider the massless Dirac field $\Psi(x)$, the action is defined as

$$ S = S_2 + S_{1/2} = \int d^Dx \left( \frac{1}{2\kappa^2} e^\mu_a e^\nu_b R_{\mu\nu}^{ab}(\omega) - \frac{1}{2} \bar{\Psi} \gamma^\mu \nabla_\mu \Psi + \frac{1}{2} \bar{\Psi} \gamma^\mu \nabla_\mu \Psi \right). $$

(7.1)

We can obtain the relation

$$ \frac{\delta S_2(e, \omega)}{\delta \omega_{\mu ab}} \bigg|_{\omega = \omega(e)} = 0 $$

(7.2)

by using that the variation of the curvature is given by

$$ \delta R_{\mu\nu}^{ab} = D_\mu \delta \omega_{\nu ab} - D_\nu \delta \omega_{\mu ab} $$

(7.3)

thus making $\delta R_{\mu\nu}^{ab} \propto D_\mu e^a_\nu - D_\nu e^a_\mu$. Using Cartan’s first structure equation we have that the spin connection can be expressed as a unique torsion free spin connection defined as

$$ \omega_{\mu ab}(e) = 2 e^\nu_a [\partial_\mu e^b_\nu] - e^\nu_a e^b_\nu e^{\sigma}_c \partial_\nu e^c_\sigma. $$

(7.4)

The covariant derivatives defined from previous chapters:

$$ \nabla_\mu \Psi = D_\mu \Psi = (\partial_\mu + \frac{1}{4} \omega_{\mu}^{ab} \gamma_{ab}) \Psi $$

(7.5)

$$ \bar{\Psi} \nabla_\mu = \bar{\Psi} D_\mu = \bar{\Psi} (\partial_\mu - \frac{1}{4} \omega_{\mu}^{ab} \gamma_{ab}). $$

(7.6)

Frame fields are used to transform frame vector indices to a coordinate basis, e.g $\gamma^\mu = e^\mu_a \gamma^a = g^{\mu\nu} \gamma_\nu$. Thus $\gamma^\mu$ transforms as a covariant vector under coordinate transformation. The covariant derivative of $\gamma^\mu$ is therefore

$$ \nabla_\mu \gamma_\nu = \partial_\mu \gamma_\nu + \frac{1}{4} \omega_{\mu}^{ab} [\gamma_{ab}, \gamma_\nu] - \Gamma^\rho_{\mu\nu} \gamma_\rho. $$

(7.7)

The spin connection appears with the commutator as required for a spinor. We can obtain the following result

$$ \nabla_\mu \gamma_\nu = \partial_\mu \gamma_\nu + \frac{1}{4} \omega_{\mu}^{ab} [\gamma_{ab}, \gamma_\nu] - \Gamma^\rho_{\mu\nu} \gamma_\rho = 0 $$

(7.8)

which tells us that covariant derivatives commute with multiplication by $\gamma$-matrices. One obtains this by using the vielbein postulate (4.22). This relation holds for any affine connection the complete derivation can be seen in the appendix appendix C. For example the Dirac field $\nabla_\mu (\gamma_\nu \bar{\Psi}) = \gamma_\nu \nabla_\mu \bar{\Psi}$, and from the (7.1) one can find the equation of motion for the massless covariant Dirac equation

$$ \gamma^\mu \nabla_\mu \Psi = 0. $$

(7.9)
Let us now take the variation of the action defined in (7.1) with respect to $e^{a\mu}$, the goal is to find the Einstein field equations for a conserved symmetric fermionic stress tensor $T_{\mu\nu}$. The variation of the action is

$$\delta S = \int d^Dx \left(\frac{1}{\kappa^2} \left( e^{b\nu} R_{\mu\nu ab} - \frac{1}{2} \epsilon_{a\mu} R \right) \delta e^{a\mu} - \frac{1}{2} \Psi \gamma^\alpha \nabla_\mu \Psi \delta e^{a\mu} - \frac{1}{8} \Psi \{\gamma^\mu, \gamma^{ab}\} \Psi \delta \omega_{\mu ab} \right)$$

(7.10)

we have used (7.2), and we have dropped the terms that are satisfied by the equation of motion (7.9) in the fermionic part of the Lagrangian. Let us now consider the variation of the spin connection in the last part of the action above, when we integrate by parts we will obtain a stress tensor. The variation of the spin connection is defined as

$$e^b_{\rho} \epsilon^a_{\nu} \delta \omega_{\mu}^{ab} = D_{[\mu} \delta^{a}_{\rho]} e^{a}_{\nu} - D_{[\nu} \delta^{a}_{\rho]} e^{a}_{\mu} + (D_{[\rho} \delta^{a}_{\mu]} e^{a}_{\nu})$$

(7.11)

The anti commutator is equal to a third rank Clifford matrix $e^a_{\nu} \gamma^{cab}$. Using the variation of the spin connection above we see that it is totally anti symmetric in $\mu, \nu$ and $\rho$, thus we are able to write the last term

$$-\frac{1}{4} \bar{\Psi} \gamma^{\mu\rho\nu} \Psi e^{a\nu} e^{b\rho} \delta \omega_{\mu ab} = -\frac{1}{4} \bar{\Psi} \gamma^{\mu\rho\nu} \Psi e^{b\rho} \nabla_\mu \epsilon_{a\nu} \delta e^{a\mu}$$

(7.12)

One obtains this relation by doing some $\gamma$-matrix manipulations using $\frac{1}{2} \{\gamma^a, \gamma^b\}$ and inserting this into the anti-commutation relation above, after doing some calculations we obtain that this is $\frac{1}{4} \bar{\Psi} \gamma^{(a\nu)} \gamma^{b\rho} \Psi \delta \omega_{\mu ab}$ using that the gamma matrices are constant matrices i.e. $\gamma^\mu = \epsilon^a_{\nu} \gamma^a$ and using (7.11) one obtains the result in (7.12). Integrating by parts, and using the Dirac equation (7.9) and $\gamma$-matrix manipulations we obtain

$$-\frac{1}{4} \epsilon_{a\mu} \delta^{a}_{\nu} \Psi (\gamma^\nu \nabla_\rho \rho - \gamma^\rho \nabla_\nu) \Psi = \frac{1}{4} \Psi (\gamma^\alpha \nabla_\mu - \gamma^\mu \nabla_\alpha) \Psi \delta e^{a\mu}$$

(7.13)

Thus we arrived at the final expression defined as

$$\delta S = \int d^Dx \left(\frac{1}{\kappa^2} \left( e^{b\nu} R_{\mu\nu ab} - \frac{1}{2} \epsilon_{a\mu} R \right) \delta e^{a\mu} - \frac{1}{4} \Psi (\gamma^\alpha \nabla_\mu - \gamma^\mu \nabla_\alpha) \Psi \delta e^{a\mu} \right).$$

(7.14)

since $\delta e^{a\mu}$ is arbitrary we can set it to zero thus we obtain the Einstein equation of the form

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{\kappa^2}{4} \bar{\Psi} \left(\gamma^\mu \nabla_\nu + \gamma^\nu \nabla_\mu\right) \Psi$$

(7.15)

The Einstein equations are consistent only if the matter stress tensor is covariantly conserved $T_{\mu\nu} = 0$, and symmetric $T_{\mu\nu} = T_{\nu\mu}$. This follows from the requirements that the matter action is invariant under general coordinate transformation and local Lorentz transformations.

Let us show that the stress tensor is Lorentz invariant, we consider the action

$$S = \int d^Dx L(\psi, \omega(\psi))$$

(7.16)

where $\psi$ is the spinor field, thje independent fields transform as

$$\delta \epsilon^{a\mu} = -\lambda^b_\alpha e_{b\mu}, \quad \delta \Psi = -\frac{1}{4} \lambda^{ab} \gamma^{ab\nu} \Psi, \quad \delta \psi_\mu = -\frac{1}{4} \lambda_{ab} \gamma^{ab\mu} \psi_\mu$$

(7.17)

The $\psi_\mu$ field is the gravitino field that appears in the Rarita-Schwinger equation and $\lambda_{ab}$ is the infinitesimal Lorentz transformation. We assume that the action is invariant so the variation is given by

$$\delta S = \int d^Dx \left( \frac{\delta S}{\delta \epsilon^{a\mu}} \delta \epsilon^{a\mu} + \frac{\delta S}{\delta \psi} \delta \psi \right) = 0$$

(7.18)
From the first term we obtain

\[ T^\alpha_\mu = \frac{1}{e} \frac{\delta S}{\delta a_\mu} \]  

(7.19)

Thus if the equation of motion (7.9) is satisfied then the equation above becomes

\[ \delta S = \int d^D x e T^{\mu
u} e^a_\nu e^b_\mu \lambda_{ab}(x) = 0 \]  

(7.20)

since \( \lambda_{ab}(x) \) is antisymmetric the stress tensor must be symmetric. Symmetry is guaranteed only if the fermion equation of motion is satisfied.

### 7.2 First order formalism for gravity and fermions

The field equation for \( \omega_{\mu ab} \) contains terms bilinear in the spinors giving a connection with torsion

\[ \omega_{\mu ab} = \omega_{\mu ab}(g) + K_{\mu ab} \].

We will use the action (7.1) and the covariant derivative (7.6) for the first order action. In first order formalism the frame field and the spin connection \( \omega_{\mu ab} \) are independent variables and that the curvature tensor \( R_{\mu
u}^{ab} \) can be constructed from this. The field equation for the spin connection gives a connection with the torsion

\[ \omega_{\mu ab} = \omega_{\mu ab}(e) + K_{\mu ab}(\phi) \]  

(7.21)

The variation of the gravitational action is

\[ \delta S_2 = \frac{1}{2\kappa^2} \int d^D x e e^\mu_a e^\nu_b \left( D_\mu \delta \omega_{\nu ab} - D_\nu \delta \omega_{\mu ab} \right) \]  

(7.22)

\[ = \int d^D x \sqrt{-g} e^\mu_a e^\nu_b D_\mu \delta \omega_{\nu ab} \]  

(7.23)

The Lorentz covariant derivatives can be integrated by parts. We have to include the action of \( D_\mu \) on each of the three factors in \( \sqrt{-g} e^\mu_a e^\nu_b \). Using the relation \( \partial_\mu \sqrt{-g} = \sqrt{-g} \Gamma_{\rho}^\mu_{\rho\nu}(g) \), we obtain

\[ \delta S_2 = -\int d^D x \sqrt{-g} e^\mu_a e^\nu_b \left( (\Gamma^\rho_{\mu\nu}(g)) e^\mu_a + D_\mu e^\nu_b \right) \delta \omega_{\nu ab} \]  

(7.24)

The last term in the action can be written as

\[ (\Gamma^\rho_{\mu\nu}(g) - \Gamma^\rho_{\nu\mu}) e^\nu_b + \nabla_\mu e^\mu_a \]  

(7.25)

The last term is a total covariant derivative of \( e^\mu_a \) which vanishes as a consequence of the vielbein postulate. Using that \( \Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}(g) - K_{\mu
u}^\rho \) and the contorsion tensor

\[ K_{[\mu|\nu]} = -\frac{1}{2} (T_{[\mu\nu]\rho} - T_{[\nu\mu]\rho} + T_{[\rho\mu]\nu}) \]  

(7.26)

thus we can rewrite the equation (7.25) as

\[ (\Gamma^\rho_{\mu\nu}(g) - \Gamma^\rho_{\nu\mu}) e^\nu_a = T_{\mu\nu}^\rho \]  

(7.27)

The last term in the action can be written as

\[ D_\mu e^\nu_b = -e^\nu_b (D_\mu e^\nu_c) e^\nu_{\mu} \]  

(7.28)

The integrand of the second term is then

\[ -e^\mu_a e^\nu_b D_\mu e^\nu_c \delta \omega_{\nu ab} = -\frac{1}{2} e^\mu_a e^\nu_b (D_\mu e^\nu_{\rho}) e^\nu_c \delta \omega_{\nu ab} \]  

(7.29)

\[ = -\frac{1}{2} T_{\mu\nu}^\rho \delta \omega_{\nu ab} \]
this follows from equation (4.12). Combining all the information above we arrive at the final expression of the variation of $S_2$ defined as

$$\delta S_2 = \frac{1}{2} \int d^D x \sqrt{-g} \left( T_{ab}^{\nu} - T_{ap}^{\nu} e_{b}^{\nu} + T_{bp}^{\nu} e_{a}^{\nu} \right) \delta \omega_{ab}^{\nu \cdot \nu}. \quad (7.30)$$

Let us look at the connection variation of the spinor action using equation (7.6), we obtain

$$\delta S_{1/2} = -\frac{1}{4} \int d^D x \ e \bar{\Psi}_{\gamma}^{ab} \gamma_{\nu}^{ab} \Psi \delta \omega_{\nu}^{ab}. \quad (7.31)$$

We can now find a equation for the torsion tensor, by considering $\delta \omega_{\nu}^{ab}$ to be arbitrary and thus we obtain

$$T_{ab}^{\nu} - T_{ap}^{\nu} e_{b}^{\nu} + T_{bp}^{\nu} e_{a}^{\nu} = \frac{1}{2} \kappa^2 \bar{\Psi}_{\gamma}^{ab} \gamma^{\nu} \Psi. \quad (7.32)$$

We have seen that the spin connection in first order formalism has a connection with the torsion tensor $\omega_{\mu ab} = \omega_{\mu ab}(e) + K_{\mu ab}(\phi)$, thus we see that the spin connection $\omega_{\mu ab}$ is a function of the other fields. The contorsion tensor appears when we couple gravity to fermions.
8 4D Supergravity

We have now arrived to supergravity theory, which is a locally invariant supersymmetry theory. This means that the action is invariant for SUSY transformations where \( \varepsilon(x) \) is a function of the spacetime coordinates. A supergravity theory is nonlinear which is an interacting field theory that contains gauge and gravity multiplets and other multiplets describing scalar fields. The gauge multiplet consist of a frame field \( e^a_{\mu}(x) \) describing the graviton and \( N \) of vector-spinor fields \( \Psi^i_{\mu}(x) \) where \( i = 1, \ldots, N \). In this section we will focus on \( D = 4 \) with \( N = 1 \).

8.1 Supergravity in second order formalism

We will not start with the second order formalism of supergravity. We have the Einstein-Hilbert Lagrangian which should be interpreted as the kinetic term for gravity and we should add a kinetic term for the spin \( 3/2 \) field which is known as the Rarita-Schwinger Lagrangian. The action should be of form

\[
S = S_2 + S_{3/2},
\]

hence the supergravity action is defined as

\[
S_{SG} = \frac{1}{2\kappa^2} \int d^D x \left( e e^{\mu\nu} R_{\mu\nu} - e \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho \right) \tag{8.1}
\]

where \( e \) stands for the determinant of \( e^a_{\mu} \), the gravitino covariant derivative is given by

\[
D_\nu \psi_\rho = \partial_\nu \psi_\rho + \frac{1}{4} \omega_{\nu ab} \gamma^{ab} \psi_\rho. \tag{8.2}
\]

The supersymmetry transformations laws of the vielbein and the gravitino are given by

\[
\delta e^a_{\mu} = \frac{1}{2} \bar{\varepsilon} \gamma^a \psi_\mu \tag{8.3}
\]

\[
\delta \psi_\mu = D_\mu \varepsilon(x). \tag{8.4}
\]

The gravitino is the gauge field of local supersymmetry, and one can interpret (8.3) as a bosonic vielbein that transforms into its superpartner. We have to check that the transformations satisfy the supersymmetry algebra. The commutator of two supersymmetry transformations should give rise to a suitable combination of local supersymmetry transformations. Thus the commutator of two infinitesimal supersymmetry transformations yields space-time dependent vector field \( \bar{\varepsilon} \gamma^\mu \varepsilon \). Consider the commutator of two supersymmetry transformations on the vielbein is

\[
[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] e^a_{\mu} = 2 \delta_{\varepsilon_1} (\varepsilon_2 \gamma^a \psi_\mu) - \delta_{\varepsilon_2} (\varepsilon_1 \gamma^a \psi_\mu)
\]

\[
= \varepsilon_1 \gamma^a D_{\mu} \varepsilon_2 - \varepsilon_2 \gamma^a D_{\mu} \varepsilon_1
\]

\[
= D_{\mu} (\bar{\varepsilon}_2 \gamma^a \varepsilon_1). \tag{8.5}
\]

We define the space-time vector field \( \xi^\mu = \varepsilon_2 \gamma^a \varepsilon_1 \), which yields us

\[
[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] = \xi^\mu D_{\mu} e^a_{\nu} + e^a_{\nu} \partial_{\mu} \xi^\nu. \tag{8.6}
\]

We have used that \( D_{\mu} \) is the covariant derivative which acts as a space-time derivative on \( \xi^\nu \). For now we set our torsion tensor to zero thus we have a relation from the vielbein postulate that allows us to write

\[
D_{\mu} e^a_{\nu} = 0. \tag{8.7}
\]

Considering this property and the equation (8.6), we can write the covariant derivative as

\[
[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] e^a_{\mu} = \xi^\nu \partial_{\nu} e^a_{\mu} + e^a_{\nu} \partial_{\mu} \xi^\nu + \xi^\nu \omega_{\nu b} e^b_{\mu}. \tag{8.8}
\]
the first and second term are infinitesimal action of a diffeomorphism on the frame field and the last term as a infinitesimal internal Lorentz transformation $\delta \Lambda e^a_\mu$ with $\Lambda^{ab} = \xi^\nu \omega^\nu_{ab}$. We have shown that the commutator of two supersymmetry transformations can be written as

$$[\delta_1, \delta_2] e^a_\mu = \delta_+ + \delta_- \tag{8.9}$$

which is a combination of local symmetries: diffeomorphisms and Lorentz transformations. So the space-time dependent vector field $\xi^\mu$ is a element of the group of local diffeomorphism on space-time. We have to treat the space-time metric as a dynamical object because it is coordinate invariant.

Let us go back to our action we want to show that the action (8.1) is invariant under supersymmetry transformations. We start with the gravitational part of the action

$$\delta S_2 = \frac{1}{2\kappa^2} \int d^D x \ e^{a_\mu} e^{b_\nu} R_{\mu \nu a b} \tag{8.10}$$

we will follow the same procedure as we did in section (7.1), where we take the curvature tensor to depend on the spin connection $\omega$, hence $R_{\mu \nu a b}(\omega)$. Thus the variation of the gravitational action under (8.3) is

$$\delta S_2 = \frac{1}{2\kappa^2} \int d^D x \ e (2\int(\delta e^{a_\mu}) e^{b_\nu} + (\delta e) e^{a_\mu} e^{b_\nu}) R_{\mu \nu a b} + \epsilon^{a_\mu} e^{b_\nu} \delta R_{\mu \nu a b} \tag{8.11}$$

Due to symmetry $R_{\mu \nu a b} = R_{\nu \mu a b}$ gives the factor of 2 in the first line, the second term vanishes because of the variation on the action where we set the variations on the boundary to zero.

Let us now consider the gravitino variation. We want to partial integrate so we vary $\delta\bar{\psi}_\mu$ and multiply with 2. We then obtain

$$\delta S_{3/2} = -\frac{1}{2\kappa^2} \int d^D x \ e \bar{\epsilon} D_\nu \gamma^{\mu \rho} D_\nu \psi_\rho \tag{8.12}$$

we integrated by parts and used (7.8) to obtain the second line and we replaced the $\nabla_\mu$ by $D_\mu$ because of antisymmetry and no absence of torsion. The last expression can be obtained by using the Ricci identity (4.43). We have to evaluate the $\gamma^{\mu \rho} \gamma^{ab}$, we contract the Riemann tensor with the gamma matrices and using the relation (3.9), we obtain

$$R_{\mu \nu a b} \gamma^{\mu \rho} \gamma^{ab} = \gamma^{\mu \rho} R_{\mu \nu a b} + 6 R_{\mu \nu} \gamma^{\rho} R_{b}^{[a} \gamma^{b]} + 6 \gamma^{[\mu R_{\mu \nu \rho]} + 4 \gamma^{[\mu R_{\mu \nu \rho]} + 4 \gamma^{[\mu R_{\mu \nu \rho]} + 2 \gamma^{[\mu R_{\mu \nu \rho]} + 2 \gamma^{\rho} R_{\mu \nu \rho}] \tag{8.13}$$

The first and second term vanishes from the Bianchi identity (4.39) with a zero torsion and the third term vanishes because of symmetry clash between the symmetric Ricci tensor $R_{ab} = R_{\mu \nu \rho}^{\mu \nu \rho}$ and the anti symmetry of $\gamma^{\rho pb}$. We have now obtain the variation on the gravitino field, given as

$$\delta S_{3/2} = \frac{1}{2\kappa^2} \int d^D x \ e \bar{\epsilon} (R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R) (\bar{\epsilon} \gamma^{\mu} \psi^{\nu}) \tag{8.14}$$

We see that $\delta S_2 + \delta S_{3/2} = 0$ and thus confirms local supersymmetry to linear order in $\psi_\mu$ for any spacetime dimension.
8.2 Supergravity in first order formalism

In the previous section we only considered linear order terms but it becomes more complicated when we go beyond linear order, due to complication in establishing local supersymmetry for supergravity theory. Let us now consider the spin connection $\omega_{\mu ab}$ as an independent variable. Considering the action (8.1), we will use the connection variation (7.30) of $S_2$. We are left with the spin variation on the spin-$3/2$ field, we obtain the following variation

$$\delta S_{3/2} = -\frac{1}{8\kappa^2} \int d^Dx \ e \left( \bar{\psi}_\mu \gamma^{\mu \rho} \gamma_{\rho \sigma} \psi_\sigma \right) \delta \omega_{\nu}^{\sigma}. \quad (8.15)$$

The rank 3 terms from (3.9) vanish because of antisymmetry in the gravitino indices $\mu, \rho$, therefore we obtain

$$\bar{\psi}_\mu \gamma^{\mu \rho} \gamma_{\rho \sigma} \psi_\sigma = \bar{\psi}_\mu \left( \gamma^{\mu \rho} \gamma_{\rho \sigma} \psi_\sigma + 6 \gamma^{[\mu} \psi_{e^{\nu]} \gamma_{\rho] a}} \psi_\rho \right). \quad (8.16)$$

Using the equation (7.30) we can solve $S_2 + S_{3/2} = 0$, we obtain the following solution to the torsion

$$T_{\mu \nu} = \frac{1}{2} \bar{\psi}_a \gamma^\nu \psi_b + \frac{1}{4} \bar{\psi}_\mu \gamma^{\mu \rho} \gamma_{\nu} \psi_\rho. \quad (8.17)$$

The first formalism determines a connection with torsion in supergravity. The fifth rank term in the torsion vanishes in $D = 4$, and we obtain the same torsion as in section (7.2). We will see that writing the spin connection as $\omega_{\mu ab} = \omega_{\mu ab}(e) + K_{\mu ab}$ will leave the supergravity action invariant under the transformation rules (8.3) and (8.4).

We can substitute the torsion above in the action (8.1) to obtain the second order action of supergravity.

$$S = \frac{1}{2\kappa^2} \int d^4x \ e \left( R(e) - \bar{\psi}_\mu \gamma^{\mu \rho} D_\rho \psi_\mu + \mathcal{L}_{SG,torsion} \right) \quad (8.18)$$

where

$$\mathcal{L}_{SG,torsion} = -\frac{1}{16} \left( \left( \bar{\psi}_\rho \gamma^\mu \psi^\nu \right) \left( \bar{\psi}_\mu \gamma_\rho \psi_\nu + 2 \bar{\psi}_\rho \gamma_\nu \psi_\mu \right) - 4 \left( \bar{\psi}_\mu \gamma_\cdot \psi \right) \left( \bar{\psi} \cdot \psi \right) \right) \quad (8.19)$$

8.3 1.5 order formalism

The second order action (8.18) is invariant under supersymmetry transformations, with the connection

$$\omega_{\mu ab} = \omega_{\mu ab}(e) + K_{\mu ab}, \quad (8.20)$$

$$K_{\mu \nu \rho} = -\frac{1}{4} (\bar{\psi}_\mu \gamma_\rho \psi_\nu - \bar{\psi}_\nu \gamma_\rho \psi_\mu + \bar{\psi}_\rho \gamma_\mu \psi_\nu) \quad (8.21)$$

that includes the gravitino torsion. For higher order terms using first and second order formalism becomes very complicated to solve, because of the variations on different orders in the gravitino field. In the first order the spin connection is an independent variable this approach becomes complicated when we couple matter multiplets to supergravity. The simplest approach to supergravity is between the first and second order called the 1.5 formalism, we can apply this formalism in any dimensions.

We are working in the second order formalism since there are only two independent fields $\psi_\mu$ and $e^{\mu}_{\nu}$. Let us consider a functional of three variables $S[e, \omega, \psi]$, we use the chain rule to calculate its variation in the second order formalism

$$\delta S[e, \omega(e) + K, \psi] = \int d^Dx \left( \frac{\delta S}{\delta e} \delta e + \frac{\delta S}{\delta \omega} \delta (\omega(e) + K) + \frac{\delta S}{\delta \psi} \delta \psi \right). \quad (8.22)$$
We can neglect all the $\delta \omega$ variations, because in 1.5 order formalism we don’t need to calculate the algebraic field equation $\delta S/\delta \omega = 0$.

We can summarize this

- Use the first order form of the action $S[e, \omega, \psi]$ and the transformation rules $\delta e$ and $\delta \psi$ with an unspecified connection
- Ignore the connection variation and calculate

$$
\delta S = \int d^Dx \left( \frac{\delta S}{\delta e} \delta e + \frac{\delta S}{\delta \psi} \delta \psi \right)
$$

(8.23)

### 8.4 Local supersymmetry of $N = 1$, $D = 4$ supergravity

We will now use the 1.5 order formalism to show that supergravity in $D = 4$ and $N = 1$ is invariant under the transformations rules (8.13) and (8.20) and (8.21). To simplify the gravitino action we introduce the highest rank Clifford element $\gamma_* = i\gamma_0\gamma_1\gamma_2\gamma_3$. We can express the third rank Clifford matrices as

$$
\gamma^{abc} = -i\epsilon^{abcd} \gamma_* \gamma_d, \quad \gamma^{\mu
u\rho} = -i\epsilon^{\mu
u\rho\sigma} \gamma_* \gamma_* \gamma_*
$$

(8.24)

The first relation holds for local frames and the second is needed for the coordinate basis. We can now rewrite the gravitino action using these relations and we obtain

$$
S_{3/2} = \frac{i}{2\kappa^2} \int d^4x \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_* \gamma_* D_\nu \psi_\rho .
$$

(8.25)

The advantage of this form is that the frame field variation is needed only in $\gamma_*$ instead of $e\gamma^{\mu\nu\rho}$. Since we are working in 1.5 order formalism we can ignore the variation $\delta \omega$ of $R^{\mu\nu\rho\sigma}$, so we consider the following terms

$$
\delta S = \delta S_2 + \delta S_{3/2,e} + \delta S_{3/2,\psi} + \delta S_{3/2,\bar{\psi}}
$$

(8.26)

where the first term is the variation of the gravity and the second term is the variation of frame field in $S_{3/2}$, meanwhile the third and forth term are variations of $\psi$ and $\bar{\psi}$.

The variation of the gravity was obtain from (8.11)

$$
\delta S_2 = \frac{1}{2\kappa^2} \int d^Dx \varepsilon \left( R_{\mu\nu}(\omega) - \frac{1}{2} g_{\mu\nu} R \right) (-\bar{\varepsilon} \gamma^\mu \psi_\mu) .
$$

(8.27)

and the second term in (8.26) is obtained by the variation of the frame field which is obtained in the same fashion as the second order formalism

$$
\delta S_{3/2,e} = \frac{i}{4\kappa^2} \int d^4x \varepsilon^{\mu\nu\rho\sigma} (\bar{\psi}_\mu \gamma_* \gamma_* D_\nu \psi_\rho) .
$$

(8.28)

The variation of the $S_{3/2,\psi}$ is given by

$$
\delta S_{3/2,\psi} = \frac{i}{2\kappa^2} \int d^4x \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_* \gamma_* D_\nu \psi_\rho
$$

$$
= \frac{i}{16\kappa^2} \int d^4x \bar{\psi}_\mu \varepsilon^{\mu\nu\rho\sigma} \gamma_* \gamma_* \gamma_* \gamma_* R_{\nu\rho\mu\sigma}(\omega) \varepsilon
$$

(8.29)

the last line is obtained from the equation (4.43), and we were allowed to shift the derivative $D_\nu$ due to partial integration. Let us now write down the variation of $S_{3/2,\bar{\psi}}$, where we will use the relation (3.23) with $t_3 = 1$. We obtain

$$
\delta S_{3/2,\bar{\psi}} = \frac{i}{2\kappa^2} \int d^4x \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \bar{D}_\nu \gamma_* \gamma_* D_\mu \varepsilon
$$

(8.30)
we can now use the covariant derivative given in previous chapters

\[ \bar{\psi}_\mu D_\nu = \partial_\nu \bar{\psi}_\rho - \frac{1}{4} \bar{\psi}_\rho \omega_{\nu ab} \gamma^{ab} \]  

(8.31)

if we then partial integrate we can use the same property as above which gives us the right to shift derivatives and we now obtain the following variation

\[ \delta S_{3/2, \tilde{\psi}} = \frac{-i}{2\kappa^2} \int d^4 x \varepsilon^{\mu \nu \rho \sigma} \bar{\psi}_\rho \gamma^*_b \left( (D_\nu \gamma_a) D_\mu \varepsilon + \gamma_a D_\nu D_\mu \varepsilon \right) \]

\[ = \frac{-i}{2\kappa^2} \int d^4 x \varepsilon^{\mu \nu \rho \sigma} \bar{\psi}_\rho \gamma^*_b \left( \frac{1}{2} T_{\nu \sigma} \gamma_a D_\mu \varepsilon - \frac{1}{8} \gamma_\sigma \gamma^{ab} R_{\mu \nu ab} (\omega) \varepsilon \right) \]  

(8.32)

The torsion term comes from when we add the Christoffel symbols and use antisymmetry in \( \nu \sigma \), this relation comes from (4.25). We see that the Riemann term \( R(\omega) \) is equal in both variations on the gravitino fields thus we can add them and we obtain the following variation

\[ \delta S_{3/2, \tilde{\psi}} + \delta S_{3/2, \tilde{\psi}} = \frac{-i}{2\kappa^2} \int d^4 x \varepsilon^{\mu \nu \rho \sigma} \bar{\psi}_\rho \gamma^*_b \left( \frac{1}{2} T_{\nu \sigma} \gamma_a D_\mu \varepsilon - \frac{1}{4} \gamma_\sigma \gamma^{ab} R_{\mu \nu ab} (\omega) \varepsilon \right) \]  

(8.33)

Using the following \( \gamma \)-matrix manipulation

\[ \gamma_\sigma \gamma_{ab} = \gamma_{\sigma ab} + 2 \epsilon_{\sigma \rho a b} \gamma^c = i \epsilon^d_{\sigma \rho a b} \gamma^c \gamma^+ + 2 \epsilon_{\sigma \rho a b} \]  

(8.34)

where we use the relation from (8.24), using this relation and plugging it back into the action (8.33) and keeping in mind that \( \bar{\psi}_\mu \gamma^+ \varepsilon = -\bar{\varepsilon} \gamma^+ \bar{\psi}_\mu \). We consider the parts separated, let us start with the first part and we see that we have a contraction of two Levi-Civita symbols, using \( \epsilon_{\mu_1 \cdots \mu_\rho \nu_1 \cdots \nu_\rho} \epsilon^{\mu_1 \cdots \mu_\rho \nu_1 \cdots \nu_\rho} = -p! \eta^{\rho_1 \cdots \rho_p \nu_1 \cdots \nu_p} \). The contraction of two Levi-Civita with the Riemann tensor \( R(\omega) \) becomes

\[ \varepsilon^{\mu \nu \rho \sigma} \varepsilon_{\alpha \beta \gamma \delta} \gamma^{ab} R_{\mu \nu ab} = -2 \epsilon \left( \epsilon^\alpha_{\mu a} \epsilon^\beta_{\nu \rho} \epsilon^\gamma_{\delta b} + \epsilon^\alpha_{\mu b} \epsilon^\beta_{\nu \rho} \epsilon^\gamma_{\delta a} + \epsilon^\alpha_{\mu c} \epsilon^\beta_{\nu \rho} \epsilon^\gamma_{\delta e} \right) R_{\mu \nu ab} (\omega) \]

\[ = 4 \epsilon \left( R_{\mu \nu} (\omega) - \frac{1}{2} \epsilon^\mu_\rho R(\omega) \right) \]  

(8.35)

Using this and plugging it back in to the action (8.27), with the majorana spinor condition above we see that these terms cancel out. We see that we have obtained the same result as in the second order formalism of supergravity. Which tells us that we have a linear local supersymmetry. The second term in (8.34) to the integrand in (8.32) involves the factor

\[ \varepsilon^{\mu \nu \rho \sigma} \omega_{\mu \nu ab} (\omega) = -\varepsilon^{\mu \nu \rho \sigma} D_\nu T_{\sigma \rho b} \]  

(8.36)

where we have used the Bianchi identity (4.39), where the derivative \( D_\nu \) is a Lorentz covariant derivative and contain only the spin connection acting on the index \( b \). We are left with

\[ \delta S_2 + \delta S_{3/2, \tilde{\psi}} = \frac{-i}{4\kappa^2} \int d^4 x \varepsilon^{\mu \nu \rho \sigma} \bar{\psi}_\rho \gamma^*_b \left( T_{\nu \sigma} \gamma_a D_\mu \varepsilon + (D_\nu T_{\sigma \rho a}) \varepsilon \right) \]  

(8.37)

We now have the variation of the frame field for the \( S_{3/2} \) action,

\[ \delta S_{3/2, e} = \frac{i}{4\kappa^2} \int d^4 x \varepsilon^{\mu \nu \rho \sigma} \left( \bar{\varepsilon} \gamma^a \psi_\rho \right) \left( \bar{\psi} \gamma^*_a D_\nu \psi_\rho \right) \]  

(8.38)

We need to reorder the spinors using Fierz rearrangement, using

\[ (\gamma^a)_\alpha^\beta (\gamma_\mu)_\gamma^\delta = \frac{1}{2m} \sum_A v_A (\Gamma_A)_\alpha^\beta (\Gamma^A)_\gamma^\delta \]  

(8.39)
the expansion coefficient \( v_A = (-)^{r_A}(D - 2r_A) \) where \( r_A \) is the tensor rank of the Clifford basis element \( \Gamma_A \), thus for even dimensions \( D = 2m \) we have that \( 0 \leq r_A \leq D \). Applying this to the integrand above we obtain

\[
(\bar{\epsilon} \gamma^a \psi_a)(\bar{\psi} \gamma^a \gamma_a D_\nu \psi_\rho) = -\frac{1}{4} \sum_A (-)^{r_A}(4 - 2r_A)(\bar{\psi} \gamma^a \gamma_a D_\nu \psi_\rho)(\bar{\psi} \gamma^a \psi_\sigma)
\]

\[
= \frac{1}{2}(\bar{\epsilon} \alpha \beta \gamma^a \gamma_a \psi_\mu)(\bar{\psi} \gamma^a \psi_\sigma)
\]

\[
= (\bar{\epsilon} \gamma^a \gamma_a D_\nu \psi_\rho) T_{\mu \nu \rho \sigma} \tag{8.40}
\]

Due to antisymmetry in \( \mu \sigma \) of the action \( S_{3/2,e} \) with the contraction of the Levi-Civita symbols, allows us to write the last line using the \( D = 4 \) torsion tensor defined as (8.17). We insert this into (8.38) and reorder the \( (\bar{\epsilon} \ldots D \psi) \) bilinear and exchange the indices \( \mu \rho \) to obtain

\[
\delta S_{3/2,e} = \frac{-i}{4\kappa^2} \int d^4 x \bar{\epsilon} \gamma^a \gamma_a D_\nu \psi_\rho \gamma_\sigma \psi_\rho T_{\mu \nu \rho \sigma} \bar{\psi} \gamma_\sigma \psi_\rho \bar{\psi} \gamma_a \psi_\rho \tag{8.41}
\]

We have now expressed all the variations in terms of the torsion tensor and adding them together we obtain

\[
\delta S = \frac{-i}{4\kappa^2} \int d^4 x \bar{\epsilon} \gamma^a \gamma_a \left( T_{\mu \sigma} \bar{\psi} \gamma^a \gamma_a D_\nu \psi_\rho + \bar{\psi} \gamma_a D_\nu \gamma_a \psi_\rho \right) + D_\nu T_{\mu \rho \sigma} \bar{\psi} \gamma^a \psi_\rho \gamma_\sigma \psi_\rho \tag{8.42}
\]

We have shown that the action is invariant up to a total derivative. We also see that the spin connection among the three terms in the second line cancel, thus we have showed that the \( \mathcal{N} = 1, D = 4 \) supergravity theory is locally supersymmetric.
9 Toroidal compactification

We have seen Kaluza-Klein reductions on a circle, in this chapter we will consider the toroidal compactification of the eleven-dimensional supergravity Lagrangian. We will reduce the 11D supergravity Lagrangian down to 4D. Performing dimensional reductions will yield us e.g Type IIA and Type IIB theories which are low-energy theories of superstring theory with the same name. The main references in this chapter will be [6], [10] and [11].

9.1 D=11 supergravity

We will now consider supergravity in 11 dimensions, before we construct the Lagrangian and the transformation rules we must first investigate the field content for D = 11 supergravity.

The field content of D = 11 theory, consists of elfbein $e^\mu_a$, a Majorana spin $3/2$ $\psi_\mu$ and a antisymmetric gauge tensor with three indices $A_{\mu \nu \rho}$, this was first formulated by [8]. To obtain the set of fields we simply calculate the number of states. The formulas for counting the states are defined as

$$e^\mu_a = \frac{d(d-3)}{2}, \quad \psi_\mu = \frac{(d-3)^2(d/2)}{2}, \quad A_{\mu_1...\mu_p} = \left(\frac{d-2}{p}\right).$$ (9.1)

We see that we obtain 44 degrees of freedom for the vierbein and we have 84 degrees of freedom. This comes from the fact that we need equal amounts of boson and fermion states.

We know that doing a dimensional reduction on $T^7$ generates the D = 4, $\mathcal{N} = 8$ supergravity theory, which contains one graviton $g_{\mu \nu}$, seven vectors $g_{\mu i}$ and $g_{ij}$ contains 28 scalars. The Majorana field decomposes into 8 spin 3/2 fields, and 56 1/2 fields.

We can now begin to construct the 11-dimensional Lagrangian and its transformation rules. We begin with the 3-form potential $A_{\mu \nu \rho}$ this field must be coordinate invariant and local Lorentz invariant, it must also transform under gauge transformation involving a parameter $\theta_{\nu \rho}$. The theory involves a gauge invariant 4-form field strength $F_{\mu \nu \rho \sigma}$. The equations are

$$\delta A_{\mu \nu \rho} = \partial_\mu \theta_{\nu \rho} + \partial_\nu \theta_{\rho \mu} + \partial_\rho \theta_{\mu \nu}$$ (9.2)

$$F_{\mu \nu \rho \sigma} = \partial_\mu A_{\nu \rho \sigma} - \partial_\nu A_{\rho \sigma \mu} + \partial_\rho A_{\sigma \mu \nu} - \partial_\sigma A_{\mu \nu \rho}$$ (9.3)

$$\partial_{[\tau} F_{\mu \nu \rho \sigma]} = 0$$ (9.4)

the last equations follows from the Bianchi identity, this follows from the fact that letting a covariant derivative act on $F = dA$ gives us zero when written in differential form.

We will also use Majorana flip relation, we have the same properties as in four dimensions

$$\bar{\chi}^\gamma^\mu_1...^\mu_r \lambda = t_r \chi^\gamma^\mu_1...^\mu_r \chi, \quad t_0 = t_3 = 1, \quad t_1 = t_2 = -1, \quad t_{r+4} = t_r.$$ (9.5)

We know that the 11-dimensional supergravity theory must have graviton and gravitino terms like we had in the D = 4 theory, we also have to include the 3-form potential.
$A_{\mu\nu\rho}$ plus several terms which we need to find by evaluating the gravitino field and the potential. We begin with the action defined as

$$S = \frac{1}{2\kappa^2} \int d^{11}x \left( e^{\mu\nu} \epsilon^{b\rho} R_{\mu\nu\alpha\beta} - \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho - \frac{1}{24} F^{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma} + \ldots \right)$$

(9.6)

where $\ldots$ are the terms we need to find. Let us evaluate the gravitino and 3-form potential now using the second order formalism where we have a torsion-free spin connection $\omega_{\mu\nu\rho}$. The transformation rules are given by

$$\delta e^a_{\mu} = \frac{1}{2} \bar{\epsilon} \gamma^a \psi_\mu$$

(9.7)

$$\delta \psi_\mu = D_\mu \epsilon + (a \gamma^{\alpha\beta\gamma\delta}_\mu + b \gamma^{\beta\gamma\delta}_\mu) F_{\alpha\beta\gamma\delta} \epsilon$$

(9.8)

$$\delta A_{\mu\nu\rho} = -\frac{1}{3} \bar{c} \epsilon (\gamma_{\mu\nu} \psi_\rho + \gamma_{\nu\rho} \psi_\mu + \gamma_{\rho\mu} \psi_\nu) .$$

(9.9)

We have expressed the transformations rules that contain numerical constants $a, b, c$, which we have to determine. The action for a free theory where we don’t have any interactions, is given by

$$S_0 = \frac{1}{2} \int d^{11}x \left( -\bar{\psi}_\mu \gamma^{\mu\nu\rho} \partial_\nu \psi_\rho - \frac{1}{24} F^{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma} \right)$$

(9.10)

Let us now take the variation of this action, which is needed to determine the numerical constants. We vary the action w.r.t $\bar{\psi}_\mu$ and $A_{\mu\nu\rho}$

$$\delta S_0 = \int d^{11}x \bar{\epsilon} \left( (a \gamma^{\alpha\beta\gamma\delta}_\mu - b \gamma^{\beta\gamma\delta}_\mu) F_{\alpha\beta\gamma\delta} \gamma^{\mu\nu\rho} \partial_\nu \psi_\rho - \frac{1}{6} \bar{c} \gamma_{\mu\nu\rho} \psi_\sigma \partial F^{\mu\nu\rho\sigma} \right)$$

$$= \int d^{11}x \bar{\epsilon} \left( (a \gamma^{\alpha\beta\gamma\delta}_\mu - b \gamma^{\beta\gamma\delta}_\mu) F_{\alpha\beta\gamma\delta} \gamma^{\mu\nu\rho} \psi_\rho - \frac{1}{6} \bar{c} \gamma_{\mu\nu\rho} \psi_\sigma \partial F_{\mu\nu\rho\sigma} \right)$$

(9.11)

where we have used

$$\delta \bar{\psi}_\mu = \bar{\epsilon} (a \gamma^{\alpha\beta\gamma\delta}_\mu - b \gamma^{\beta\gamma\delta}_\mu) F_{\alpha\beta\gamma\delta} \gamma^{\mu\nu\rho} .$$

(9.12)

The factor of 2 disappears because we vary the fields twice hence we can combine them and only vary one field, the variation of the field strength tensor was obtained by using (9.9). We now see that we need to reduce the products of the $\gamma$ matrices, to sums over rank 6, rank 4 and rank 2 elements of $\Gamma^A$ of the Clifford algebra. We start with the first term in (9.11)

$$\gamma^{\alpha\beta\gamma\delta}_\mu \gamma^{\mu\nu\rho} F_{\alpha\beta\gamma\delta} = (D - 6) \gamma^{\alpha\beta\gamma\delta}_\mu \gamma^{\mu\nu\rho} F_{\alpha\beta\gamma\delta} + 8(D - 5) \gamma^{\alpha\beta\gamma\delta}_\mu |\nu F_\rho|_{\alpha\beta\gamma}$$

$$- 12(D - 4) \gamma^{\alpha\beta\gamma\delta}_\mu F_{\alpha\beta\gamma\delta}$$

(9.13)

we can obtain this by using $\gamma$-matrix manipulations. In the first term we see that $\mu$ can run over all $D$ values except $\alpha\beta\gamma\delta$ and $\nu\rho$ which gives us $(D - 6)$ factor in front of the first term. We can use the same logic for the other terms. In the second term we see that $\nu$ can run over all $D$ values except the five indices. We can also choose four indices from the first factor and two indices from the second term which gives us the factor $8$. We also have one contraction from the $\delta$-index thus combining all this we obtain the second term. The third term contains a contraction from the $\gamma$-index and the same logic applied we obtain the last term.

Let us now take the second term

$$\gamma^{\beta\gamma\delta}_\mu \gamma^{\mu\nu\rho} F_{\mu\beta\gamma\delta} = -\gamma^{\nu\rho\alpha\beta\gamma\delta}_\mu F_{\alpha\beta\gamma\delta} - 6 \gamma^{\alpha\beta\gamma\delta}_\mu |\nu F_\rho|_{\alpha\beta\gamma} + 6 \gamma^{\alpha\beta}_\mu F_{\alpha \nu \rho}.$$ 

(9.14)
This relation holds directly from the definition of the Clifford algebra (3.1), we have seen this relation in (3.8) and (3.9). We first write the totally antisymmetric Clifford matrix that contains all the indices and then add possible pairings. We also use the same logic as above hence the constants in front of the second and third term.

We can now use these relations above in the variational action (9.11) to solve the numerical constants. By doing so we see that the rank 6 elements disappear due to the Bianchi identity (9.4) we are left with the rank 4 elements and the rank 2 elements of the Clifford algebra. We can summarize this as

\[ 8a(D - 5) + 6b = 0 \]  \hspace{1cm} (9.15)
\[ 12a(D - 4) + 6b = \frac{1}{6}c \]  \hspace{1cm} (9.16)

we solve for \( D = 11 \) and we obtain that \( a = \frac{c}{216} \) and \( b = -8a \). The transformation rules are then given as

\[ \delta \psi_\mu = \partial_\mu \varepsilon + \frac{c}{216} \left( \gamma^{\alpha\beta\gamma\delta}_\mu - 8\gamma^{\beta\gamma\delta}_\mu \right) F_{\alpha\beta\gamma\delta} \varepsilon \]  \hspace{1cm} (9.17)
\[ \delta A_{\mu\nu\rho} = -c \bar{\varepsilon}_2 \gamma_{[\mu\nu\rho]} \cdot \]  \hspace{1cm} (9.18)

We now have to solve the \( c \) constant, we do so by examining the commutator of two SUSY transformations and require that they agree with the local supergravity. Let us write down the SUSY transformation for the gauge potential, defined as

\[ [\delta_1, \delta_2] A_{\mu\nu\rho} = -\frac{1}{216} c^2 \bar{\varepsilon}_2 \gamma_{[\mu\nu} \left( \gamma^{\alpha\beta\gamma\delta}_{\rho]} - 8\gamma^{\beta\gamma\delta}_{\rho]} \right) \varepsilon_1 F_{\alpha\beta\gamma\delta} - \text{ (1 }\leftrightarrow 2) \]  \hspace{1cm} (9.19)

We know that each contraction in these terms contains a pair of indices so the contributions to the Clifford algebra will be of rank 1, 3, 5, 7. We can do the same \( \gamma \)-matrix manipulations as above, but we can simplify this by observing that the parameters \( \varepsilon_1 \) and \( \varepsilon_2 \) are antisymmetric and we only obtain contributions from the rank 1 and rank 5 terms. We also see that the first term does not have any rank 1 terms because we would need to contract three pairs of indices to obtain a non-vanishing rank 1 term, and the antisymmetrization do not allows this. We can see this by doing the \( \gamma \)-matrix manipulation where the contraction is between the antisymmetric indices hence why we are not allowed to do this. We can see that the rank 3 terms vanish when we consider the antisymmetry of the parameters \( \varepsilon_1 \) and \( \varepsilon_2 \), using the relations in (3.11) and a very useful table in [2]

<table>
<thead>
<tr>
<th>d( mod 8)</th>
<th>S</th>
<th>A</th>
<th>( \varepsilon )</th>
<th>( \eta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.3</td>
<td>2,1</td>
<td>-1</td>
<td>+1</td>
</tr>
<tr>
<td></td>
<td>0.1</td>
<td>2,3</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>0.1</td>
<td>2,3</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>1.0</td>
<td>3,2</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td></td>
<td>1.2</td>
<td>3,0</td>
<td>+1</td>
<td>+1</td>
</tr>
<tr>
<td>3</td>
<td>1.2</td>
<td>0,3</td>
<td>+1</td>
<td>+1</td>
</tr>
</tbody>
</table>

Table 1: The \( S \) and \( A \) stands for symmetric and antisymmetric, for a detailed description of the table we refer to [2]

note that \( \varepsilon = \pm t_0 \) and \( \eta = \pm t_0 t_1 \). We can see that when we do the calculation for rank 3 elements they all cancel with each other.

The second term is already given in (3.9). We know from the definition that the SUSY algebra contain a spacetime translation which involves a rank 1 bilinear \( \bar{\varepsilon}_1 \gamma^\sigma \varepsilon_2 \)
from the second term in (9.19). We then obtain the commutator

$$[\delta_1, \delta_2] A_{\mu\nu\rho} = -\frac{4}{9} c^2 \varepsilon_1 \gamma^\sigma \varepsilon_2 F_{\sigma\mu\nu\rho}$$  \hspace{1cm} (9.20)$$

SUSY requires that the rank 5 elements vanish in the term (9.19) which they do.

Let us interpret the SUSY commutation relation by looking at the field strength defined in (9.3). The first term $\partial_\sigma A_{\mu\nu\rho}$ is the space-time translation from the SUSY algebra, the remaining terms just add up to the 3-form potential. The gauge field is proportional gauge transformation $\theta_{\mu\nu} = -\varepsilon_1 \gamma^\sigma \varepsilon_2 A_{\mu\nu\rho}$. These gauge transformations can be found when one considers the super Yang-Mills theory and also in local supersymmetry for supergravity theories.

We can now find the $c$ constant, from above we see that $c^2 = 9/8$. We choose the positive root and obtain $c = 3\sqrt{2}/2$.

We can find the supercurrent in (9.11) by considering the spinor parameter $\varepsilon$ depending on $x^\mu$, taking the variation of the action and performing partial integration. We find that the variation contains a term that is proportional to $D_\nu \varepsilon$ whose coefficients are the supercurrent

$$J^\nu = \sqrt{\frac{2}{96}} (\gamma^{\alpha\beta\gamma\delta\nu\rho} F_{\alpha\beta\gamma\delta} + 12 \gamma^{\alpha\beta} F_{\alpha\beta}^\nu \psi^\rho)$$  \hspace{1cm} (9.21)$$

This is a new term to the 11-dimensional supergravity. We can now extend the results from the free system to an interacting system. We consider a general frame field $e^a_{\mu}(x)$ and consider the general $\varepsilon(x)$, we then have the transformation rules

$$\delta e^a_{\mu} = \frac{1}{2} \varepsilon^{a\nu} \psi_{\nu}$$  \hspace{1cm} (9.22)$$

$$\delta \psi_{\mu} = D_\mu \varepsilon + \sqrt{\frac{2}{96}} (\gamma^{\alpha\beta\gamma\delta\nu\rho} F_{\alpha\beta\gamma\delta} + 12 \gamma^{\alpha\beta} F_{\alpha\beta}^\nu \psi^\rho)$$  \hspace{1cm} (9.23)$$

$$\delta A_{\mu\nu\rho} = -\frac{3\sqrt{2}}{4} \varepsilon_{[\mu\nu\psi^\rho]}$$  \hspace{1cm} (9.24)$$

and the action

$$S = \frac{1}{2\kappa^2} \int d^{11} x \left( e^{\mu\nu} e^{\rho\sigma} R_{\mu\nu\rho\sigma} - \bar{\psi} \gamma^{\mu\nu \rho} D_\sigma \psi - \frac{1}{24} F^{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma} \right.$$

$$- \sqrt{\frac{2}{96}} \bar{\psi}_{\nu} (\gamma^{\alpha\beta\gamma\delta\nu\rho} F_{\alpha\beta\gamma\delta} + 12 \gamma^{\alpha\beta} F_{\alpha\beta}^\nu \psi^\rho + \ldots) \left. \right).$$  \hspace{1cm} (9.25)$$

We still need to figure out the rest of the Lagrangian. We have seen that the first and second term cancel in previous section and that $\bar{\varepsilon} F_{\alpha\beta\gamma\delta} \psi_{\rho}$ also cancel.

We are now left with the variation of the field strength $\bar{\varepsilon} F^2 \psi$ and the variation of $\bar{\psi} F \psi$ term. Writing the variational terms explicitly,

$$\delta L_{FF} = \frac{1}{48} \left( 4 \bar{\varepsilon} \gamma^{\mu} \psi^\nu - \frac{1}{2} g^{\mu\nu} \bar{\varepsilon} \gamma \cdot \psi \right) F^{\rho\sigma\tau}_{\mu} F_{\nu\rho\sigma\tau}$$

$$\delta L_{\bar{\psi} F \psi} = \frac{1}{96 \times 144} \bar{\varepsilon} \left( (\gamma^{\alpha\beta\gamma\delta\nu\rho} F_{\alpha\beta\gamma\delta} + 12 \gamma^{\alpha\beta} F_{\alpha\beta}^\nu \psi^\rho) \right. \left. \times (\gamma^{\alpha\beta\gamma\delta\nu\rho} F_{\alpha\beta\gamma\delta} + 12 \gamma^{\alpha\beta} F_{\alpha\beta}^\nu \psi^\rho) \right).$$  \hspace{1cm} (9.26)$$

We have now used the 1.5 order formalism and taking the linear parts in the $\delta S$, the terms of the form $\bar{\varepsilon} \psi$ in the variation vanish as it did in $D = 4$ supergravity. The products of $\gamma$-matrices contain rank 9, 7, 5, 3, 1 terms. For a detailed description on how to solve these $\gamma$-matrices we refer to [9]. The rank 1 terms in (9.26) cancel between
each other and several rank 3, 5 and 7 terms cancel with the term $\bar{\psi}F\psi$. We have the rank 9 terms, which we can obtain from the $\gamma$-matrix products

$$
\gamma^{\alpha\beta\gamma\delta}\gamma^{\alpha\beta\gamma\delta\rho} = (D - 9)\gamma^{\alpha\beta\gamma\delta}\alpha\beta\gamma\delta + \ldots = 2\gamma^{\alpha\beta\gamma\delta}\alpha\beta\gamma\delta\rho + \ldots
$$

(9.27)

$$
\bar{\gamma}^{\alpha\beta\gamma\delta}\gamma^{\alpha\beta\gamma\delta}\alpha\beta\gamma\delta\rho = -\gamma^{\alpha\beta\gamma\delta}\alpha\beta\gamma\delta\rho + \ldots
$$

(9.28)

where the dots are the lower ranked terms which we ignore. Combining these results we are left with

$$
\delta L_{FF} + \delta L_{\bar{\psi}F\psi} = -\frac{1}{16 \times 144} \bar{\gamma}^{\alpha\beta\gamma\delta}\alpha\beta\gamma\delta\rho \bar{F}_{\alpha\beta\gamma\delta} = \frac{1}{2 \sqrt{2}} \bar{\gamma}^{\alpha\beta\gamma\delta}\alpha\beta\gamma\delta\rho \bar{F}_{\alpha\beta\gamma\delta}
$$

(9.29)

this is the term that survives from the variational terms. In order to make the Lagrangian invariant the terms must cancel, hence we need Clifford algebra in odd spacetime dimensions $D = 2m + 1$. For $D = 11$ the generating matrices are $32 \times 32$, using

$$
\gamma_{\pm\mu} = (\gamma^{0}, \gamma^{1}, \ldots, \gamma^{(2m - 1)}, \gamma^{2m}) = \pm \gamma_{\mu}
$$

(9.30)

we obtain that the rank 9 Clifford element is related to rank 2 by this equation. Thus the Clifford algebra is given as

$$
\gamma^{\alpha\beta\gamma\delta}\alpha\beta\gamma\delta\rho = -\frac{1}{2} \epsilon^{\alpha\beta\gamma\delta}\alpha\beta\gamma\delta\rho \epsilon_{\gamma\mu}. \quad (9.31)
$$

Rewriting the equation (9.29) we get

$$
e (\delta L_{FF} + \delta L_{\bar{\psi}F\psi}) = \frac{1}{32 \times 144} \epsilon^{\alpha\beta\gamma\delta}\alpha\beta\gamma\delta\rho \epsilon_{\gamma\mu}
\bar{F}_{\alpha\beta\gamma\delta} = \frac{1}{3 \sqrt{2}} \epsilon^{\alpha\beta\gamma\delta}\alpha\beta\gamma\delta\rho \epsilon_{\gamma\mu}
\bar{F}_{\alpha\beta\gamma\delta}
$$

(9.32)

We are left with the following variation, called the Chern-Simons term for the 3-form potential

$$
S_{C-S} = \int A_{3} \wedge F_{4} \wedge F_{4}
$$

(9.33)

there is a $U(1)$ gauge symmetry where the gravitino is charged and for which the 3-form potential is the gauge field. We also note that $F_{4} = dA_{3}$ is the $U(1)$ field strength 4-form. The Chern-Simons term is gauge invariant up to a total derivative. The action transform as

$$
\int A_{3} \wedge F_{4} \wedge F_{4} = \int A_{3} \wedge F_{4} \wedge F_{4} + \int dA_{2} \wedge F_{4} \wedge F_{4}
$$

$$
= \int A_{3} \wedge F_{4} \wedge F_{4} + \int d(A_{2} \wedge F_{4} \wedge F_{4})
$$

(9.34)

the last term vanishes due to the Bianchi identity $dF_{4} = 0$, and we have used the gauge transformation $A_{3} \rightarrow A_{3} + dA_{2}$. This variation does not produce any further variations since there are no frame fields in the expression. A typical property of the Chern-Simons action always guarantees gauge invariance.

We have now obtained all the transformations rules. We can thus write the 11-dimensional Lagrangian as

$$
S = \frac{1}{2 \kappa^{2}} \int d^{11}x \left( e^{\mu\nu} e^{ab} R_{\mu
u ab} - \bar{\psi}_{\mu} \gamma^{\mu\nu\rho} D_{\nu} \left( \omega + \hat{\omega} \right) \bar{\psi}_{\rho} - \frac{1}{24} F_{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma}
$$

$$
- \frac{\sqrt{2}}{192} \bar{\psi}_{\mu} (\gamma^{\alpha\beta\gamma\delta\rho} + 12 \gamma^{\alpha\beta} g^{\gamma\delta\rho}) \bar{\psi}_{\rho} (F_{\alpha\beta\gamma\delta} + \hat{F}_{\alpha\beta\gamma\delta})
$$

$$
- \frac{2 \sqrt{2}}{(144)^{2}} \epsilon^{\alpha\beta\gamma\delta\rho} \epsilon_{\alpha\beta\gamma\delta\rho} F_{\alpha\beta\gamma\delta} F_{\alpha\beta\gamma\delta} A_{\mu\rho}
\right).
$$

(9.35)
We replaced the spin-connection field by $\omega$ with $(\omega + \tilde{\omega})/2$ in the covariant derivative of the gravitino kinetic term and we also replaced the field-strength with $(F_{\alpha\beta\gamma\delta} + \tilde{F}_{\alpha\beta\gamma\delta})/2$, these substitutions ensure that the field equations are supercovariant. The hatted connection and field strength are given by

$$\tilde{\omega}_{\mu ab} = \omega_{\mu ab} + K_{\mu ab}, \quad (9.36)$$

$$\tilde{\omega}_{\mu ab} = \omega_{\mu ab} - \frac{1}{4} (\tilde{\psi}_\mu \gamma^a \psi_a - \tilde{\psi}_a \gamma^\mu \psi_b + \tilde{\psi}_b \gamma^\mu \psi_a) + K_{\mu ab}, \quad (9.37)$$

$$K_{\mu ab} = - \frac{1}{4} (\tilde{\psi}_\mu \gamma^b \psi_a - \tilde{\psi}_a \gamma^\mu \psi_b + \tilde{\psi}_b \gamma^\mu \psi_a) + \frac{1}{8} \tilde{\psi}_\nu \gamma^{\nu \rho} \mu ab \psi_\rho, \quad (9.38)$$

$$\tilde{F}_{\mu \nu \rho \sigma} = 4 \partial_{[\mu} A_{\nu \rho \sigma]} + \frac{3}{2} \sqrt{2} \tilde{\psi}_{[\mu} \gamma_{\nu \rho \sigma]} \psi] \quad (9.39)$$

The action is invariant under the transformations defined in (9.22 - 9.24)

### 9.2 11-Dimensional toroidal compactification

Before we start with the dimensional reduction on the eleven-dimensional supergravity Lagrangian we first have to investigate the reductions of antisymmetric tensor field strengths. We will consider the reduction from $(D+1)$ to $D$ dimensions, we assume that we have an $n$-index field strength in higher dimension which we denote $\hat{F}_n$. Let us rewrite the field strength in terms of the gauge potential $\hat{A}_{n-1}$ so that $\hat{F}_n = d\hat{A}_{n-1}$. Thus after reduction we obtain two dimensional reduced potentials, namely $(n-1)$ which lies in the $D$-dimensional spacetime and the $(n-2)$ which lies is in the $S^1$ direction. We can express this as

$$\hat{A}_{n-1}(x, z) = A_{n-1}(x) + A_{n-2}(x) \wedge dz \quad (9.40)$$

the field strength can be expressed as

$$\hat{F}_n = F_n + F_{n-1} \wedge (dz + A_1) \quad (9.41)$$

where $A_1$ is the metric reduction. Thus we obtain that the $D$-dimensional field strength are given by

$$F_n = dA_{n-1} - dA_{n-2} \wedge A_1, \quad F_{n-1} = dA_{n-2} \quad (9.42)$$

we can then conclude that the field strengths after dimensional reduction will mean different things. For each reduction we do we obtain one field strength of same rank as the one we reduce from and the other field strength has $(n-1)$ rank.

The m-th reduction for the gauge-potential can now be written as

$$\hat{A}_{n-1} = A_{n-1} + A_{(n-2)a} \wedge dz^a + \frac{1}{2!} A_{(n-3)ab} \wedge dz^a \wedge dz^b$$

$$= \sum_{p=0}^{m,n} \frac{1}{p!} A_{(n-1-p)a_1...a_p} \wedge dz^{a_1} \wedge ... \wedge dz^{a_p} \quad (9.43)$$

and the field strength can be written as

$$F_n = \sum_{p=0}^{m,n} \frac{1}{p!} F_{(n-p)a_1...a_p} \wedge dz^{a_1} \wedge ... \wedge dz^{a_p} \quad (9.44)$$

We have now established the dimensional reduction of field strengths and gauge-potentials.

Let us now begin with finding the $D$-dimension Lagrangian for the toroidal compactification, we consider the $(D+1)$-dimensional Lagrangian of the form where we follow

$$L = \hat{e}R - \frac{1}{2} \hat{e}(\partial \phi)^2 - \frac{1}{2n!} \hat{e}^{a\hat{a}\hat{b}} \hat{F}_{n}^a \quad (9.45)$$
doing a dimensional reduction on this Lagrangian gives us the following Lagrangian

\[
L = eR - \frac{1}{2}e^2(\partial \phi)^2 - \frac{1}{4}ee^{-2(D-1)\phi}F^2 - \frac{1}{2n!}ee^{-2(n-1)\phi+\hat{\alpha}\phi}F^2_n - \frac{1}{2(n-1)!}ee^{2(D-n)\phi+\hat{\alpha}\phi}F^2_{n-1}
\]  

(9.46)

where we have the 2-form defined as \( F = d\mathcal{A} \), so for each time we reduce the Lagrangian we obtain a new two-form field strength from the metric, and one dilaton scalar so there will be \((D - 11)\) dilaton scalars. The dilaton factor of the field strength in (9.45) takes the general form of \( e^{\hat{\alpha}D+1\hat{\phi}D+1} \) where \( \hat{\phi}_{D+1} = (\phi_1, \phi_2, \ldots, \phi_{10-D}) \). We can see that there is no dilaton factor in the eleven-dimensional supergravity Lagrangian so we set \( \hat{\alpha}1 = 0 \), we can obtain the lower dimensional vectors \( \vec{a} \) by the following recursion formula

\[
\vec{a}_{D} = \left( \frac{2}{(D-1)(D-2)} \right)^{x} \begin{cases} -(n-1), & \text{for } \hat{F}_n \rightarrow \hat{F}_n \\ (D-n), & \text{for } \hat{F}_n \rightarrow \hat{F}_{n-1} \\ -1, & \text{for } F \end{cases}
\]

(9.47)

So for each reduction we obtain n-form field strength from the n-form in \((D + 1)\) dimension, an \((n - 1)\)-form from the n-form field strength and the 2-form from the metric. We note that the 2-form appears in the \( D \) dimensional Lagrangian for the first time since there is no dilaton vectors in the eleven-dimensional supergravity theory we set \( \vec{a}_{D+1} = 0 \), from this formula we can obtain all the factors for the dilaton vectors that appear when we do a dimensional reduction. One can easily calculate the prefactors using this formula.

Let us denote the dilaton vector for the 4-form as \( F_{MNij} \), the 3-forms as \( F_{MNP} \), and the 2-forms as \( F_{MNij} \) 1-forms as \( F_{M} \) by \( \vec{a}, \vec{a}_i, \vec{a}_{ij}, \vec{a}_{ijk} \) where \( i \) labels the internal (11-D) indices in \( D \) dimensions. We also obtain 2-forms \( F_{MN} \) and 1-forms \( F_{M} \) with \( i < j \) which comes from the dimensional reduction of the vielbein. We denote the vielbein dilaton scalars with \( \vec{b}_i \) and \( \vec{b}_{ij} \), we summarize this with this table:

<table>
<thead>
<tr>
<th>4-form:</th>
<th>( \vec{a} = -\vec{g} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3-forms:</td>
<td>( \vec{a}_i = \vec{f}_i - \vec{g} )</td>
</tr>
<tr>
<td>2-forms:</td>
<td>( \vec{a}_{ij} = \vec{f}_i + \vec{f}_j - \vec{g} )</td>
</tr>
<tr>
<td>1-forms:</td>
<td>( \vec{a}_{ijk} = \vec{f}_i + \vec{f}_j + \vec{f}_k - \vec{g} )</td>
</tr>
</tbody>
</table>

Table 2: Dilaton vector

where the vectors \( \vec{g} \) and \( \vec{f} \) have \((11 - D)\) components in \( D \) dimensions and are given by

\[
\vec{g} = 3(s_1, s_2, \ldots, s_{11-D})
\]

(9.48)

\[
\vec{f}_i = (0, 0, \ldots, 0, (10 - 1)s_i, s_{i+1}, \ldots, s_{11-D})
\]

(9.49)

where \( s_i = \sqrt{2/((10 - i)(9 - i))} \). We can now obtain the Lagrangian for the toroidal compactification of the eleven-dimensional supergravity theory.

We write down the compact form of the eleven-dimensional Lagrangian, given as

\[
L = eR - \frac{1}{48}e\hat{F}_4^2 + \frac{1}{6}\hat{F}_4 \wedge \hat{A}
\]

(9.50)

the Chern-Simons term needs to be treated separately, which we will come back to. We can apply the rules above to the kinetic term where we consider a modification to the
equation (9.44), where we now include the dilaton vectors and order them as following
\(i < j < k\), which follows directly from the Table 2. The field strength dimensional
reduction equation takes the general form

\[
\frac{1}{n!} F_n^2 = \sum_{p=0}^{n-1} \sum_{a_1 < \ldots < a_p} \frac{1}{(n-p)!} (F_{(n-p)a_1 \ldots a_p})^2 e^{-\frac{n-1}{3} \phi + \tilde{f}_{a_1} + \ldots + \tilde{f}_{a_p}} \phi
\]  

(9.51)
since we have a 4-form in our Lagrangian the dimensional reduction of the kinetic term
using the equation above and use the dilaton scalar notations in the table

\[
\frac{1}{24} F_4^2 = \frac{1}{24} e^{\tilde{\phi}} F_4^2 + \frac{1}{6} \sum_i e^{\tilde{\phi}} (F_{3i}^i)^2 + \frac{1}{2} \sum_{i<j} e^{\tilde{\phi}} (F_{2ij}^j)^2 + \sum_{i<j<k} e^{\tilde{\phi}} (F_{1ijk}^k)^2 \]  

(9.52)
and the 2-forms coming from the vielbein reduction are

\[
\frac{1}{24} F_4^2 = \frac{1}{2} \sum_i e^{\tilde{\phi}} (F_{2i}^i)^2 + \sum_{i<j} e^{\tilde{\phi}} (F_{1ij}^j)^2 . \]  

(9.53)
The complete Lagrangian for the toroidal compactification of the eleven-dimensional
supergravity theory is gives as

\[
\mathcal{L} = eR - \frac{1}{2} e (\partial \phi)^2 - \frac{1}{48} e e^{\tilde{\phi}} F_4^2 - \frac{1}{12} \sum_i e^{\tilde{\phi}} (F_{3i}^i)^2 - \frac{1}{4} \sum_{i<j} e^{\tilde{\phi}} (F_{2ij}^j)^2 \]  

\[
- \frac{1}{2} \sum_{i<j<k} e^{\tilde{\phi}} (F_{1ijk}^k)^2 - \frac{1}{4} \sum_i e^{\tilde{\phi}} (F_{2i}^i)^2 - \frac{1}{2} \sum_{i<j} e^{\tilde{\phi}} (F_{1ij}^j)^2 + \mathcal{L}_{FFA} . \]  

(9.54)
We have now written out the toroidal compactified Lagrangian, let us now consider the
Chern-Simons term. We begin with writing out the 3-form gauge potential and the
4-form field strengths associated with the Chern-Simons term

\[
\hat{F}_4 = \hat{F}_4 + \hat{F}_4^i \wedge dz^i - \frac{1}{2} \hat{F}_{2ij} \wedge dz^i \wedge dz^j - \frac{1}{6} \hat{F}_{1ijk} \wedge dz^i \wedge dz^j \wedge dz^k \]  

(9.55)
\[
\hat{F}_4 = \bar{F}_4 + \hat{F}_4^i \wedge dz^i - \frac{1}{2} \bar{F}_{2im} \wedge dz^m \wedge dz^l - \frac{1}{6} \bar{F}_{1i} \wedge dz^i \wedge dz^m \wedge dz^n \]  

(9.56)
\[
\hat{A}_3 = A_3 + A_2^p \wedge dz^p - \frac{1}{2} A_1^{pq} \wedge dz^p \wedge dz^q - \frac{1}{6} A_0^{pqr} \wedge dz^p \wedge dz^q \wedge dz^r . \]  

(9.57)
Note that we have \(\hat{F}_4 \wedge \hat{F}_4 \wedge \hat{A}_3\), thus giving us 64 terms in total. The complete expression
is written out in the appendix. We also note that the amount of indices the terms have
will tell us what dimension the Lagrangian will have after the compactification on a
torus. Before we go on let us define a important property of differential forms, namely

\[
d(\omega^p \wedge \omega^q) = d\omega^p \wedge \omega^q + (-1)^p \omega^p \wedge d\omega^q \]  

(9.58)
When we do our calculations we set the right-hand side to zero this is due to having
a eleven-dimensional theory. When we let a differential operator act on a form we
increase its degree by one, which would result in a having p-forms of higher degree than
our theory where our Lagrangian is written out as an 11-form. From a more technical
point of view we see that letting a differential operator act on \(\hat{F}_4 \wedge \hat{F}_4 \wedge \hat{A}_3\) will result
in having Bianchi identities \(d\hat{F}_4 = 0\), hence why we set the right-hand side to zero.
Let us begin with the simplest case where we have terms with 1 index, we write out
the Lagrangian as

\[
\mathcal{L}_{FFA}^{n=1} = \frac{1}{6} \int \left( \hat{F}_4 \wedge \hat{F}_4 \wedge A_2^i \wedge dz^i + \hat{F}_4 \wedge \hat{F}_4 \wedge dz^i \wedge A_3 + \hat{F}_4 \wedge dz^i \wedge \hat{F}_4 \wedge A_3 \right) . \]  

(9.59)
The next step is to partial integrate the terms. Let us consider the second term where we now perform a partial integration and use the differential form property above

\[\int \left( \tilde{F}_4 \wedge \tilde{F}_3^i \wedge dz^i \wedge A_3 \right) = \int \left( dA_2^i \wedge \tilde{F}_4 \wedge A_3 \right) \wedge dz^i \]

\[= \int \left( A_2^i \wedge d(\tilde{F}_4 \wedge A_3) \right) \wedge dz^i \]

\[= \int \left( A_2^i \wedge (d\tilde{F}_4 \wedge A_3 + \tilde{F}_4 \wedge dA_3) \right) \wedge dz^i \]

\[= \int \left( A_2^i \wedge \tilde{F}_4 \wedge \tilde{F}_4 \right) \wedge dz^i \]

\[= \int \left( \tilde{F}_4 \wedge \tilde{F}_4 \wedge A_2 \right) \varepsilon^i \]

where we have used the differential form property in the first line, in the third line we see that the first term we obtain is zero due to the Bianchi identity. We are now left with something that has form as the first term. Applying the same procedure to the last term should give us the same terms as above, one could easily see from the similarities of the terms \(^6\). We can now gather all the terms and obtain the following Lagrangian

\[\mathcal{L}_{FFA}^{\text{red}} = \frac{1}{6} \int \left( \tilde{F}_4 \wedge \tilde{F}_4 \wedge A_2 + \tilde{F}_4 \wedge \tilde{F}_4 \wedge A_2 + \tilde{F}_4 \wedge \tilde{F}_4 \wedge A_2 \right) \varepsilon^i \]

\[= \frac{1}{6} \int 3(\tilde{F}_4 \wedge \tilde{F}_4 \wedge A_2)\varepsilon^i = \frac{1}{2} \int (\tilde{F}_4 \wedge \tilde{F}_4 \wedge A_2)\varepsilon^i .\]

We have now found the Chern-Simons term for \(D=10\) Lagrangian. Note that we now have a Lagrangian of 10-form, we can thus conclude that each time we do a dimensional reduction we integrate over the coordinate in the toroidal direction which results in having one dimension less. We refer to the appendix for the rest of the dimensional reductions. Let us write down all the Lagrangian one obtains after dimensional reduction:

\[
\begin{align*}
D = 10 : & \quad \frac{1}{2} \tilde{F}_4 \wedge \tilde{F}_4 \wedge A_2 , \\
D = 9 : & \quad \left( \frac{1}{4} \tilde{F}_4 \wedge \tilde{F}_4 \wedge A_0^{ijkl} - \frac{1}{2} \tilde{F}_3^i \wedge \tilde{F}_3^j \wedge A_3 \right) \varepsilon^{ij} , \\
D = 8 : & \quad \left( \frac{1}{12} \tilde{F}_4 \wedge \tilde{F}_4 \wedge A_0^{ijkl} - \frac{1}{6} \tilde{F}_3^i \wedge \tilde{F}_3^j \wedge A_2^k + \frac{1}{2} \tilde{F}_3^i \wedge \tilde{F}_2^j \wedge A_3 \right) \varepsilon^{ijk} , \\
D = 7 : & \quad \left( \frac{1}{6} \tilde{F}_4 \wedge \tilde{F}_3^i \wedge A_0^{ijkl} - \frac{1}{4} \tilde{F}_3^i \wedge \tilde{F}_3^j \wedge A_2^k + \frac{1}{8} \tilde{F}_2^i \wedge \tilde{F}_2^j \wedge A_3 \right) \varepsilon^{ijkl} , \\
D = 6 : & \quad \left( \frac{1}{12} \tilde{F}_4 \wedge \tilde{F}_2^ij \wedge A_0^{klm} - \frac{1}{12} \tilde{F}_3^i \wedge \tilde{F}_3^j \wedge A_0^{klm} + \frac{1}{8} \tilde{F}_2^i \wedge \tilde{F}_2^j \wedge A_2^m \right) \varepsilon^{ijklm} , \\
D = 5 : & \quad \left( \frac{1}{12} \tilde{F}_4 \wedge \tilde{F}_2^ij \wedge A_0^{klm} + \frac{1}{48} \tilde{F}_2^i \wedge \tilde{F}_2^j \wedge A_2^k \right) \varepsilon^{ijklm} , \\
D = 4 : & \quad \left( \frac{1}{48} \tilde{F}_2^i \wedge \tilde{F}_2^j \wedge A_0^{mnp} - \frac{1}{144} \tilde{F}_2^i \wedge \tilde{F}_1^j \wedge A_2^k \right) \varepsilon^{ijklmnp} , \\
D = 3 : & \quad - \frac{1}{144} \tilde{F}_1^i \wedge \tilde{F}_1^j \wedge A_0^{mnp} \varepsilon^{ijklmnp} , \\
D = 2 : & \quad - \frac{1}{1296} \tilde{F}_1^i \wedge \tilde{F}_1^j \wedge A_0^{mnpq} \varepsilon^{ijklmnpq} .
\end{align*}
\]

These are all the reduced Lagrangian we can obtain from the \(D = 11\) Chern-Simons term.

\(^6\)One could choose to partial integrate the first term and obtain something similar to \(\tilde{F}_4 \wedge \tilde{F}_3 \wedge A_3\) but we chose to partial integrate the other two terms to obtain the maximally ranked field-strengths in our Lagrangian.
These reductions are not from the unmodified field strengths, i.e. the field strength reduction \((9.55 - 9.57)\) are not expressed in the vielbein basis. However if one were to consider dualisations as \([10]\) one would need modified field strengths which are expressed in vielbein basis. Let us re express the toroidal coordinates \(dz^i\) in terms of 

\[
h^i = dz^j + A_{0}^{ij}dz^j.
\]  

(9.62)

We obtain this from the dimensional reduction of the metric ansatz

\[
ds^2 = e^{k\tilde{g}}ds^2_{D} + \sum_i e^{2\tilde{g}_i}(h^i)^2
\]

(9.63)

where \(\gamma_i = \frac{1}{6}\tilde{g} - \frac{1}{2}\tilde{f}_i\). We have considered the field strengths in the Chern-Simons term to be associated with the gauge potentials e.g \(F^4 = dA_3\) and so one, in general one would also obtain non-linear Kaluza-Klein modifications. If we express the field strength reductions in terms of \((9.62)\), we obtain the following modified field strengths

\[
F_{ij} = (\gamma^{ij}F_{ij} - (\gamma_{kl}F_{kl} \wedge A^i_k \wedge A^j_l + \frac{1}{6}\gamma^{ij}\gamma^{km}\gamma^{ln}F_{ijkl} \wedge A^j_k \wedge A^i_k \wedge A^m_l)
\]

(9.64)

these modified field strengths appear in the kinetic term of the Lagrangian \((9.54)\). The modified field strengths that come from the vielbein are given by

\[
F^i_j = F^i_j - \gamma^{jk}F^j_k \wedge A^i_1
\]

(9.65)

Notice that we do not have any 0-form potentials, this is because \(A_{ij}^0\) are defined only for \(i < j\), in other words \(A_{ij}^0 = 0\) for \(i \geq j\). This can be seen when we to dimensional reduction from the Kaluza-Klein vielbein ansatz by taking the \(i\)’th step of the reduction process. We obtain the following relation

\[
dz^i = \gamma^{ij}(h^j - A^j_i)
\]

(9.66)

one then sees that the 0-form potentials are eliminated\(^7\), notice that \(\gamma^{ij} = 0\) when \(i > j\) and \(\gamma^{ij} = 1\) for \(i = j\).

If one were to consider brane’s we would need to have these modified Chern-Simons terms in the kinetic term.

For example in \([11]\) the authors attempt to find p-brane solutions in maximal supergravity, where they find the equations of motion of the supergravity Lagrangian and impose additional constraints on the Chern-Simons modifications of the field strengths. These constraints gives rise to which field strengths can be dualised in certain dimensions.

\(^7\) A detailed mathematical description can be found in \([11]\)
10 Conclusion

In this thesis we have seen how Supergravity theories are invariant under the local SUSY transformations where the spinor parameters $\varepsilon(x)$ are arbitrary functions of spacetime coordinates. Showing that the Lagrangians are invariant tells us that Supergravity is consistent as a classical theory of graviton and gravitino.

In section 5 we saw how one obtained a theory containing gravity, electromagnetic fields and scalar fields from a $(D + 1)$-dimensional Einstein-Hilbert action by using Kaluza-Klein theory. This method turns out to be very important when considering Supergravity theories. When we performed a dimensional reductions of the eleven-dimensional Supergravity theory on a 7-torus, we obtained the degrees of freedom of $D = 4, \, N = 8$ theory. The theory contains one graviton $g_{\mu\nu}$, seven vectors $g_{\mu i}$ and 28 scalars $g_{ij}$. The Majorana field is decomposed into eight gravitinos and 56 spin-1/2 fields this is obtained from calculating the states for the Rarita-Schwinger field. We know from Supersymmetry that a theory should contain equal amount of bosonic and fermionic states on-shell degrees of freedom, so we are missing bosons to account for the fermions. These bosons are obtained from the 3-form gauge potential $A_{\mu ij}$ which supplies 21 scalars. The form components $A_{ijk}$ contain 35 scalars and the components $A_{\mu i}$ give additional seven scalars. This completes the field content of the $D = 4, \, N = 8$ theory.

We also saw that we obtained ten-dimensional Supergravity theories, Type IIA and Type IIB which are low-energy limits of superstring theories of the same name.

Type IIB and and $SO(6)$-gauged $D = 5$ supergravities have important applications in AdS/CFT-correspondence. The modified field strengths appear in the dimensional reduction of the Chern-Simons term in the eleven-dimensional supergravity. The modified field strengths are needed to guarantee gauge invariance in the kinetic term. When one considers external sources such as brane worlds one needs the modified field strengths. With the modified field strengths one can consider dualisations of higher dimensional theories to lower dimensional theories. The eleven-dimensional Supergravity theory combined with $M2$ and $M5$-brane solutions is the basis for M-theory, which is the theory that contains all the various string theories.

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References


A Riemann tensor for Einstein-Hilbert action in D dimensions

The derivatives of the ON-Basis are defined as

\[
dE^i = \beta e^{-\beta \phi} \partial_j E^j \wedge E^i - \hat{\omega}^i_j \wedge E^j \\
dE^5 = \alpha e^{-\beta \phi} \partial_j \phi E^j \wedge E^5 + \frac{1}{2} e^{(\alpha-2\beta)\phi} F_{ij} E^i \wedge E^j.
\]  

(A.1) \hspace{1cm} (A.2)

The spin connection can be obtained by using Cartan’s first structure equation and we obtain

\[
\omega^5_i = \alpha \partial_i \phi e^{-\beta \phi} E^5 + \frac{1}{2} e^{(\alpha-2\beta)\phi} F_{ij} E^j
\]  

(A.3)

\[
\omega^j_i = \hat{\omega}^i_j - \beta e^{-\beta \phi} (\partial_i \phi E_j - \partial_j \phi E_i) - \frac{1}{2} e^{(\alpha-2\beta)\phi} F_{ij} E^5
\]  

(A.4)

Cartan’s second structure equation is defined as

\[
R^{\mu}_{\nu \sigma} = d\omega^{\mu}_{\nu \sigma} + \omega^{\mu \nu} \wedge \omega^{\nu \sigma}
\]  

(A.5)

Let us now find the Riemann tensor $R^5_{i}$, we start with the first component $d\omega^5_{\mu \sigma}$

\[
d\omega^5_i = d(\alpha \partial_i \phi e^{-\beta \phi} E^5 + \frac{1}{2} e^{(\alpha-2\beta)\phi} F_{ij} E^j)
\]

\[
= \alpha \partial_j \partial_i \phi e^{-\beta \phi} E^j \wedge E^5 - \alpha \beta \partial_i \phi e^{-\beta \phi} d\phi E^5 + \alpha \partial_i \phi e^{-\beta \phi} dE^5
\]

\[
+ \frac{1}{2} (dF_{ij} e^{(\alpha-\beta)\phi} E^j + (\alpha - \beta) F_{ij} \partial_k \phi e^{(\alpha-\beta)\phi} E^k + e^{(\alpha-\beta)\phi} F_{ij} dE^j)
\]

\[
= \alpha \partial_j \partial_i \phi e^{-\beta \phi} E^j \wedge E^5 - \alpha \beta \partial_i \phi e^{-\beta \phi} \partial_j \phi E^j \wedge E^5
\]

\[
+ \alpha \beta \partial_i \phi e^{-\beta \phi} (\partial \alpha \beta \phi \partial_j \phi E^j \wedge E^5 + \frac{1}{2} e^{(\alpha-\beta)\phi} F_{ij} E^i \wedge E^j)
\]

\[
+ \frac{1}{2} \partial_k \phi F_{ij} e^{(\alpha-\beta)\phi} E^k \wedge E^j + \frac{1}{2} (\alpha - \beta) F_{ij} \partial_k \phi e^{(\alpha-\beta)\phi} E^k \wedge E^j
\]

\[
+ \frac{1}{2} e^{(\alpha-\beta)\phi} F_{ij} (\beta e^{-\beta \phi} \partial_k E^k \wedge E^j - \hat{\omega}^j_k \wedge E^k)
\]

\[
= \alpha \partial_j \partial_i \phi e^{-\beta \phi} E^j \wedge E^5 - \alpha \beta \partial_i \phi e^{-\beta \phi} \partial_j \phi E^j \wedge E^5
\]

\[
+ \alpha \beta \partial_i \phi \partial_j \phi e^{-\beta \phi} E^j \wedge E^5 + \frac{1}{2} \alpha \partial_i \phi F_{ij} e^{(\alpha-\beta)\phi} E^j \wedge E^j
\]

\[
+ \frac{1}{2} \partial_k \phi F_{ij} e^{(\alpha-\beta)\phi} E^k \wedge E^j + \frac{1}{2} (\alpha - \beta) F_{ij} \partial_k \phi e^{(\alpha-\beta)\phi} E^k \wedge E^j
\]

\[
+ \frac{1}{2} e^{(\alpha-\beta)\phi} F_{ij} (\beta e^{-\beta \phi} \partial_k E^k \wedge E^j - \hat{\omega}^j_k \wedge E^k)
\]

Now we consider the second part of the equation above and we obtain

\[
\omega^5_j \wedge \omega^j_i = \alpha \partial_j \phi e^{-\beta \phi} E^5 \wedge \hat{\omega}^j_i - \alpha \beta \partial_j \phi \partial_i \phi e^{-2\beta \phi} E^5 \wedge E^j
\]

\[
+ \alpha \beta \partial_j \phi \partial_i \phi e^{-2\beta \phi} E^5 \wedge E^j + \frac{1}{2} e^{(\alpha-2\beta)\phi} F_{kj} E^k \wedge \hat{\omega}^j_i
\]

\[
+ \frac{1}{2} \beta e^{(\alpha-3\beta)\phi} \partial_k \phi F_{ij} E^k \wedge E^j - \frac{1}{4} e^{2(\alpha-2\beta)\phi} F^{k} F_{ij} F^k E^j \wedge E^5
\]

\[
+ \frac{1}{2} \beta \partial_i \phi \partial_j \phi e^{(\alpha-3\beta)\phi} E^k \wedge E^j
\]
combining all the equations now we obtain the following Riemann tensor

\[ R^i_j = \alpha \partial_j \partial_i \phi e^{-2\beta \phi} E^j \wedge E^5 - \beta e^{-2\phi} \partial_j \phi \partial_i \phi e^{-\beta \phi} E^j \wedge E^5 + \alpha \partial_i \partial_j \phi e^{-2\beta \phi} E^j \wedge E^5 + \alpha^2 \partial_i \partial_j \phi e^{-2\beta \phi} E^j \wedge E^5 \\
+ \frac{1}{2} \alpha \partial_i \phi F_{ij} e^{(\alpha-\beta \phi)} E^i \wedge E^j + \frac{1}{2} \beta \partial_i \phi \partial_j \phi e^{(\alpha-\beta \phi)} E^k \wedge E^i \\
+ \frac{1}{2} \partial_k F_{ij} e^{(\alpha-\beta \phi)} E^k \wedge E^j + \frac{1}{2} (\alpha - 2\beta) F_{ij} \partial_k \phi e^{(\alpha-\beta \phi)} E^k \wedge E^j \\
+ \frac{1}{2} \beta \partial_i \phi \partial_j \phi e^{(\alpha-\beta \phi)} E^k \wedge E^j - e^{(\alpha-\beta \phi)} F_{ij} \hat{\omega}^i_j \wedge E^k \\
+ \alpha \partial_j \phi e^{-\beta \phi} E^5 \wedge \hat{\omega}_j + \alpha \beta \partial_j \phi \partial_i \phi e^{-2\beta \phi} E^5 \wedge E^i \\
- \alpha \beta \partial_j \phi \partial_i \phi e^{-2\beta \phi} E^5 \wedge E^j + \frac{1}{2} e^{(\alpha-\beta \phi)} F_{ij} E^k \wedge \hat{\omega}^i_j \\
+ \frac{1}{2} \beta e^{(\alpha-\beta \phi)} \partial_k \phi F_{ij} E^k \wedge E^j - \frac{1}{4} e^{(\alpha-\beta \phi)} F_{ij} E^k \wedge E^5 \wedge E^5 \\
\]

simplify this by gather all the wedge products under same brackets we obtain

\[ R^5_i = e^{-2\beta \phi} \left( \alpha (\alpha - 2\beta) \partial_i \partial_j \phi + \alpha \partial_j \partial_i \phi + \eta_j \alpha \beta \partial_k \phi \partial^k \phi - \frac{1}{4} e^{2(\alpha-\beta \phi)} F_{kj} F^{kj} \right) E^j \wedge E^5 \\
+ e^{(\alpha-\beta \phi)} \left( \frac{1}{2} (\alpha - \beta) \partial_i \phi F_{kj} - \frac{1}{2} (\alpha - \beta) \partial_j \phi F_{ki} - \frac{1}{2} \partial_k F_{ij} + \frac{1}{2} \eta_k F_{ij} \phi \right) E^k \wedge E^j \\
- \frac{1}{2} e^{(\alpha-\beta \phi)} \left( F_{ij} \hat{\omega}^i_j \wedge E^k + F_{jk} \hat{\omega}^j_i \wedge E^k \right) + \alpha \partial_j \phi e^{-\beta \phi} E^5 \wedge \hat{\omega}_j \\
\]

We have another curvature tensor to calculate namely \( R^i_j \) we follow the same procedure as above

\[ R^i_j = d\omega^i_j + \omega^i_{\nu} \wedge \omega^\nu_j \]

We start with the first component which yields us

\[ d\omega^j_i = \beta e^{-2\beta \phi} \partial_i \partial_j \phi E^k \wedge E^j - \beta e^{-2\beta \phi} \partial_j \partial_k \phi E^k \wedge E^i + \beta^2 e^{-2\beta \phi} \partial_j \phi \partial_i \phi E^k \wedge E^j - \beta e^{-2\beta \phi} \partial_i \partial_j \phi E^k \wedge E^i + \beta e^{-2\beta \phi} \partial_j \partial_i \phi E^k \wedge E^j \\
- \beta^2 e^{-2\beta \phi} \partial_i \phi \partial_j \phi E^k \wedge E^j + \beta^2 e^{-2\beta \phi} \partial_j \phi \partial_i \phi E^k \wedge E^j - \frac{1}{2} \partial_k \phi F_{ij} e^{(\alpha-\beta \phi)} E^k \wedge E^5 \\
- \frac{1}{2} (\alpha - 2\beta) \partial_k \phi F_{ij} e^{(\alpha-\beta \phi)} E^k \wedge E^5 + \frac{1}{2} e^{(\alpha-\beta \phi)} \alpha \partial_j \phi F_{ij} E^j \wedge E^5 - \frac{1}{4} F_{kj} F_{ij} e^{2(\alpha-\beta \phi)} E^j \wedge E^j \]

the second term

\[ \omega^i_{\nu} \wedge \omega^\nu_j = \omega^i_5 \wedge \omega^5_j + \omega^i_k \wedge \omega^k_j \]

Which yields us the following expression without going through the entire calculation

\[ \omega^i_{\nu} \wedge \omega^\nu_j = \frac{1}{2} \alpha \partial_i \phi F_{jk} e^{(\alpha-\beta \phi)} E^5 \wedge E^k + \frac{1}{2} F_{ij} \alpha \partial_j \phi e^{(\alpha-\beta \phi)} E^j \wedge E^5 \\
+ \frac{1}{4} F_{ij} F_{kj} e^{2(\alpha-2\beta \phi)} E^j \wedge E^5 + \omega^i_k \wedge \omega^k_j - \beta \partial^k \phi e^{-\beta \phi} \hat{\omega}_j^i \wedge E^j \\
+ \beta \partial_j \phi e^{-\beta \phi} \hat{\omega}_j^i \wedge E^k + \beta^2 \partial^i \phi \partial^j \phi e^{-2\beta \phi} E_k \wedge E_j - \beta^2 \partial^i \phi \partial_j \phi e^{-2\beta \phi} E_k \wedge E^k \\
- \beta^2 \partial_i \phi \partial^j \phi e^{-2\beta \phi} E^k \wedge E_j + \beta^2 \partial_i \phi \partial^j \phi e^{-2\beta \phi} E^k \wedge E^k \\
+ \frac{1}{2} \beta \partial_j \phi F_{kj} e^{(\alpha-3\beta \phi)} E^k \wedge E^5 - \frac{1}{2} \beta \partial_j \phi F_{kj} e^{(\alpha-3\phi)} E^j \wedge E^5 \\
- \frac{1}{2} F_{ij} e^{(\alpha-2\beta \phi)} E^5 \wedge \hat{\omega}_j^i + \frac{1}{2} \beta \partial^k \phi F_{kj} e^{(\alpha-3\beta \phi)} E^5 \wedge E^j - \frac{1}{2} \beta \partial_j \phi F_{kj} E^5 \wedge \hat{\omega}_j^i \]

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Gathering all the terms we obtain the following Riemann tensor

\[ R_{ij} = r_{ij} + E_z \wedge E_k e^{(\alpha - 3\beta)\phi} \left( (\alpha - \beta) F^i_j \partial_k \phi + \frac{1}{2} \partial_k F^i_j - \frac{1}{2} (\alpha - \beta)(\partial^i \phi F_{jk} - F^i_k \partial_j \phi) \right. \\
+ \frac{1}{2} \beta \left( \partial_n \phi F^m_{ij} \delta^l_k + F^i_l \partial^j \phi \eta_{lk} \right) + E^k \wedge E^l e^{-2\beta \phi} \left( \beta \left( \partial_j \partial_k \phi \delta^l_1 - \partial_k \partial^j \phi \eta_{lj} \right) \\
- \frac{1}{4} e^{2(\alpha - 2\beta)\phi} \left( F^i_j F_{kl} + F^i_{[k} \eta_{lj]} \right) + \beta^2 \left( \delta^i \phi \partial_j \phi \delta^j_1 - \partial_j \partial^i \phi \phi \delta^j_1 \right) \\
+ \beta e^{-\beta \phi} \left( \partial_k \phi E^i \wedge \hat{\omega}^k_j - \delta^k \phi \hat{\omega}^i_k \wedge E_j \right) - \frac{1}{2} \beta e^{(\alpha - 2\beta)\phi} \left( F^i_j \hat{\omega}^k_i \wedge E^5 - F^i_k \hat{\omega}^j_k \wedge E^5 \right) \]

**B Equations of motion for Einstein-Hilbert action in D dimensions**

The derivation for the equations of motion for Einstein-Hilbert action in \( D \) dimensions where the action is given as

\[ S_{EH} = \kappa^{-1} \int d^D x \sqrt{-g} \left( R - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{4} e^{-2(D-1)\alpha \phi} F^2 \right) \]

we vary this action with respect to the metric \( g^{\mu\nu} \)

\[ \delta S_{EH} = \kappa^{-1} \int d^D x \delta \left( \sqrt{-g} R - \frac{1}{2} \sqrt{-g} (\nabla \phi)^2 - \frac{1}{4} \sqrt{-g} e^{-2(D-1)\alpha \phi} F^2 \right) \]

\[ = \kappa^{-1} \int d^D x \left( \sqrt{-g} \delta R_{\mu\nu} g^{\mu\nu} + \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} + R \delta \sqrt{-g} \right. \\
- \frac{1}{2} \sqrt{-g} (\partial_\mu \delta \phi \partial^\mu \phi + \partial_\mu \phi \partial^\mu \delta \phi) - \frac{1}{2} \sqrt{-g} (\nabla \phi)^2 \delta g^{\mu\nu} \\
- \frac{1}{2} (\partial^\phi)^2 \sqrt{-g} - \frac{1}{4} \sqrt{-g} g^{\mu\nu} e^{-2(D-1)\alpha \phi} \left( \delta F_{\mu\nu} F^{\mu\nu} + F_{\mu\nu} \delta F^{\mu\nu} \right) \\
- \frac{1}{4} \sqrt{-g} F_{\mu\nu}^2 e^{-2(D-1)\alpha \phi} \delta g^{\mu\nu} + \frac{1}{4} F^2 e^{-2(D-1)\alpha \phi} \delta \sqrt{-g} \\
+ \frac{1}{2} \sqrt{-g} (D - 1) \alpha \delta \phi e^{-2(D-1)\alpha \phi} F^2 \right) \]

we set \( \partial_\mu \delta \phi \partial^\mu \phi = \partial_\mu \phi \partial^\mu \phi \) and \( \delta F_{\mu\nu} F^{\mu\nu} = F_{\mu\nu} \delta F^{\mu\nu} \) and all the boundary terms are set to zero. Using the relation

\[ \delta \sqrt{-g} = - \frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \]

and integrating by parts we obtain
\[ \delta S_{EH} = \kappa^{-1} \int d^D x \left( \sqrt{-g} R_{\mu \nu} \delta g^{\mu \nu} - \frac{1}{2} \sqrt{-g} R g_{\mu \nu} \delta g^{\mu \nu} - \sqrt{-g} \partial_{\mu} \delta \phi \partial^{\nu} \phi \right. \\
\left. - \frac{1}{2} \sqrt{-g} \partial_{\mu} \phi \partial^{\nu} \phi \delta g^{\mu \nu} + \frac{1}{4} \sqrt{-g} (\partial \phi)^2 g_{\mu \nu} \delta g^{\mu \nu} \right. \\
\left. - \frac{1}{2} \sqrt{-g} g^{\mu \nu} F_{\mu \nu} \delta \nabla^\alpha A_\nu e^{-2(D-1)\alpha \phi} - \frac{1}{4} \sqrt{-g} F_{\mu \nu}^2 e^{-2(D-1)\alpha \phi} \delta g^{\mu \nu} \\
\left. - \frac{1}{8} \sqrt{-g} g_{\mu \nu} F^2 e^{-2(D-1)\alpha \phi} \delta g^{\mu \nu} + \frac{1}{2} \sqrt{-g} (D - 1) \alpha \delta \phi e^{-2(D-1)\alpha \phi} F^2 \right) \\
= \kappa^{-1} \int d^D x \sqrt{-g} \left( \delta g^{\mu \nu} \left( R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} - \frac{1}{2} (\partial_{\mu} \phi \partial^{\nu} \phi + \frac{1}{2} (\partial \phi)^2 g_{\mu \nu} \right) \\
- \frac{1}{4} e^{-2(D-1)\alpha \phi} (F_{\mu \nu}^2 - \frac{1}{4} F^{\mu \nu} g_{\mu \nu}) \right) \\
- \delta A_\nu \nabla^\mu \left( F_{\mu \nu} e^{-2(D-1)\alpha \phi} \right) \\
- \delta \phi \left( \partial^\mu \partial^\nu \phi - \frac{1}{2} (D - 1) \alpha e^{-2(D-1)\alpha \phi} F^2 \right). \]

The equations of motion can be read of as

\[ R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} = \frac{1}{2} \left( \partial_{\mu} \phi \partial_{\nu} \phi - \frac{1}{2} (\partial \phi)^2 g_{\mu \nu} \right) + \frac{1}{2} e^{-2(D-1)\alpha \phi} \left( F_{\mu \nu}^2 - \frac{1}{4} F^{\mu \nu} g_{\mu \nu} \right) \]

\[ \nabla^\mu \left( e^{-2(D-1)\alpha \phi} F_{\mu \nu} \right) = 0 \]

\[ \Box \phi = - \frac{1}{2} (D - 1) \alpha e^{-2(D-1)\alpha \phi} F^2 \]

(B.3)

C Covariant derivative of $\gamma_{\nu}$

\[ \nabla_\mu \gamma_{\nu} = \partial_\mu \gamma_{\nu} + \frac{1}{4} \omega_\mu^{\ k} [\gamma_{ab}, \gamma_{\nu}] - \Gamma^p_{\mu \nu} \gamma_p \]

\[ = \partial_\mu \gamma_{\nu} + \frac{1}{4} \omega_\mu^{\ k} [\gamma_{\alpha \beta}, \gamma_{\nu}] - \Gamma^p_{\mu \nu} \gamma_p \]

\[ = \partial_\mu \gamma_{\nu} + \frac{1}{8} \omega_\mu^{\ ab} (\gamma_{\alpha \beta} \gamma_{\nu} - (a \leftrightarrow b)) - \Gamma^p_{\mu \nu} \gamma_p \]

\[ = \partial_\mu \gamma_{\nu} + \frac{1}{8} \omega_\mu^{\ ab} (\gamma_{\alpha \beta} \gamma_{\nu} - \gamma_{\nu} \gamma_{\alpha \beta}) - (a \leftrightarrow b) - \Gamma^p_{\mu \nu} \gamma_p \]

\[ = \partial_\mu \gamma_{\nu} + \frac{1}{8} \omega_\mu^{\ ab} (\gamma_{\alpha \beta} \gamma_{\nu} - \gamma_{\nu} \gamma_{\alpha \beta} - 2 \eta_{\mu \nu} \gamma^b - (a \leftrightarrow b)) - \Gamma^p_{\mu \nu} \gamma_p \]

\[ = \partial_\mu \gamma_{\nu} + \frac{1}{8} \omega_\mu^{\ ab} (2 \gamma^a \eta_{\mu \nu} - 2 \gamma^b \eta_{\mu \nu} - (a \leftrightarrow b)) - \Gamma^p_{\mu \nu} \gamma_p \]

\[ = \partial_\mu \gamma_{\nu} + \omega_\mu^{\ ab} \gamma^c \eta_{\nu} - \Gamma^p_{\mu \nu} \gamma_p \]

\[ = \gamma^c (\partial_\mu \epsilon_{ab} + \omega_\mu^{\ ab} e^b - \Gamma^p_{\mu \nu} \epsilon_{ab}) = 0 \]

D Variation of the spin connection

We are interested in finding the variation of the spin connection which will be useful when doing calculations in the second order formalism of gravity and fermions. We start by varying the first Cartan structure equation

\[ d\delta e^a + \omega^a_b \wedge \delta e^b + \delta \omega^a_b \wedge e^b = 0 \]
Let us multiply with $\eta_{\alpha\epsilon\rho}^\lambda$ and do a cyclic permutation of $\mu, \nu$ and $\rho$ with the sign $+++$

$$\eta_{\alpha\epsilon\rho}^\lambda \left( D_{\mu\nu}^\alpha \dot{e}_{\rho}^\epsilon \cdot e_{\mu j} \cdot e_{\nu}^\epsilon \right) - \eta_{\alpha\epsilon\mu}^\lambda \left( D_{\nu\rho}^\alpha \dot{e}_{\nu}^\epsilon \cdot e_{\nu}^\epsilon \right) + \eta_{\alpha\epsilon\rho}^\lambda \left( D_{\rho\mu}^\alpha \dot{e}_{\rho}^\epsilon \cdot e_{\mu j} \cdot e_{\nu}^\epsilon \right) = 0$$

writing out the explicit terms of the spin connection we obtain

$$\eta_{\alpha\epsilon\rho}^\lambda \left( D_{\mu\nu}^\alpha \dot{e}_{\rho}^\epsilon \cdot e_{\mu j} \cdot e_{\nu}^\epsilon - \frac{1}{2} \delta \omega_{\mu}^\epsilon \cdot e_{\nu}^\epsilon \right) + \frac{1}{32} \delta \omega_{\mu}^\epsilon \cdot e_{\nu}^\epsilon$$

we see that certain terms vanish and we have contracted the metric so all the $c$ are replaced with $a$ and we obtain

$$e_{\mu}^a \cdot \omega_{\mu}^b \cdot e_{\nu}^c = D_{\mu}^a \cdot \omega_{\mu}^b \cdot e_{\nu}^c = D_{\mu}^a \cdot \omega_{\mu}^b \cdot e_{\nu}^c + D_{\mu}^a \cdot \omega_{\mu}^b \cdot e_{\nu}^c$$

### E Chern-Simons dimensional reduction

$$\hat{F}_4 \wedge \hat{F}_4 \wedge \hat{A}_3 = \left( \hat{F}_4 + \hat{F}_4^2 \wedge dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4 - \frac{1}{6} \hat{F}_4^2 \wedge dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4 \right)$$

$$\wedge \left( \hat{F}_4 + \hat{F}_4^2 \wedge dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4 - \frac{1}{6} \hat{F}_4^2 \wedge dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4 \right)$$

$$\wedge \left( \hat{A}_3 + A_0 \wedge dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4 - \frac{1}{6} \hat{A}_3 \wedge dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4 \right)$$

(E.1)

In order to obtain the dimensional reductions from eleven-dimensional supergravity to four-dimensional supergravity, we need to expand the expression above and we obtain a very messy expression. In order to avoid having a expression that is to long we will break it into 4 parts, for the simplicity.

$$\hat{F}_4 \wedge \hat{F}_4 \wedge \hat{A}_3 = \hat{F}_4 \wedge \hat{F}_4 \wedge A_3 + \hat{F}_4 \wedge \hat{F}_4 \wedge A_3^2 \wedge dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4 \wedge dz^5 \wedge dz^6$$

$$- \frac{1}{12} \hat{F}_4 \wedge \hat{F}_4 \wedge A_0 \wedge dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4 \wedge dz^5 \wedge dz^6$$

$$+ \hat{F}_4 \wedge \hat{F}_4 \wedge A_2^2 \wedge dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4 \wedge dz^5 \wedge dz^6$$

$$- \frac{1}{6} \hat{F}_4 \wedge \hat{F}_4 \wedge A_0 \wedge dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4 \wedge dz^5 \wedge dz^6$$

$$+ \frac{1}{2} \hat{F}_4 \wedge \hat{F}_4 \wedge A_2 \wedge dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4 \wedge dz^5 \wedge dz^6$$

$$+ \frac{1}{12} \hat{F}_4 \wedge \hat{F}_4 \wedge dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4 \wedge dz^5 \wedge dz^6$$

$$- \frac{1}{6} \hat{F}_4 \wedge \hat{F}_4 \wedge dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4 \wedge dz^5 \wedge dz^6$$

$$+ \frac{1}{12} \hat{F}_4 \wedge \hat{F}_4 \wedge dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4 \wedge dz^5 \wedge dz^6$$

$$+ \frac{1}{36} \hat{F}_4 \wedge \hat{F}_4 \wedge dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4 \wedge dz^5 \wedge dz^6$$

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this is the expression for the first extensions that contains 16 terms, we see that we should in total have 64 terms in total. The next extensions are expressed as

\[
\hat{F}_4 \wedge \hat{F}_4 \wedge A_3 = \frac{1}{2} \hat{F}_2^{ij} \wedge d z^i \wedge \hat{F}_4 \wedge A_3 + \frac{1}{2} \hat{F}_2^{ij} \wedge d z^j \wedge \hat{F}_4 \wedge A_3 - \frac{1}{2} \hat{F}_2^{ij} \wedge d z^i \wedge \hat{F}_4 \wedge A_3 - \frac{1}{2} \hat{F}_2^{ij} \wedge d z^j \wedge \hat{F}_4 \wedge A_3
\]

\[
\hat{F}_4 \wedge \hat{F}_4 \wedge A_3 = \frac{1}{2} \hat{F}_2^{ij} \wedge d z^i \wedge \hat{F}_4 \wedge A_3 + \frac{1}{2} \hat{F}_2^{ij} \wedge d z^j \wedge \hat{F}_4 \wedge A_3 - \frac{1}{2} \hat{F}_2^{ij} \wedge d z^i \wedge \hat{F}_4 \wedge A_3 - \frac{1}{2} \hat{F}_2^{ij} \wedge d z^j \wedge \hat{F}_4 \wedge A_3
\]
\[ \tilde{F}_4 \wedge \tilde{F}_4 \wedge A_3 = -\frac{1}{6} \tilde{F}_{1}^{ijk} \wedge dz^i \wedge dz^j \wedge dz^k \wedge \tilde{F}_4 \wedge A_3 - \frac{1}{6} \tilde{F}_1^{ijk} \wedge dz^i \wedge dz^j \wedge dz^k \wedge \tilde{F}_4 \wedge A_2^{i} \wedge dz^i \\
+ \frac{1}{12} \tilde{F}_1^{ijk} \wedge dz^i \wedge dz^j \wedge dz^k \wedge \tilde{F}_4 \wedge A_1^{ij} \wedge dz^i \wedge dz^j \\
+ \frac{1}{36} \tilde{F}_1^{ijk} \wedge dz^i \wedge dz^j \wedge dz^k \wedge \tilde{F}_4 \wedge A_0^{ijk} \wedge dz^i \wedge dz^j \wedge dz^k \\
- \frac{1}{6} \tilde{F}_1^{ijk} \wedge dz^i \wedge dz^j \wedge dz^k \wedge \tilde{F}_3^i \wedge dz^l \wedge A_3 - \frac{1}{6} \tilde{F}_1^{ijk} \wedge dz^i \wedge dz^j \wedge dz^k \wedge \tilde{F}_3^i \wedge dz^l \wedge A_2^i \wedge dz^i \\
+ \frac{1}{12} \tilde{F}_1^{ijk} \wedge dz^i \wedge dz^j \wedge dz^k \wedge \tilde{F}_3^i \wedge dz^l \wedge A_1^{ij} \wedge dz^i \wedge dz^j \\
+ \frac{1}{36} \tilde{F}_1^{ijk} \wedge dz^i \wedge dz^j \wedge dz^k \wedge \tilde{F}_3^i \wedge dz^l \wedge A_0^{ijk} \wedge dz^i \wedge dz^j \wedge dz^k \\
+ \frac{1}{12} \tilde{F}_1^{ijk} \wedge dz^i \wedge dz^j \wedge dz^k \wedge \tilde{F}_3^i \wedge dz^l \wedge A_2^i \wedge dz^i \\
+ \frac{1}{12} \tilde{F}_1^{ijk} \wedge dz^i \wedge dz^j \wedge dz^k \wedge \tilde{F}_3^i \wedge dz^l \wedge A_1^{ij} \wedge dz^i \wedge dz^j \\
- \frac{1}{24} \tilde{F}_1^{ijk} \wedge dz^i \wedge dz^j \wedge dz^k \wedge \tilde{F}_3^i \wedge dz^l \wedge A_1^{ij} \wedge dz^i \wedge dz^j \\
- \frac{1}{72} \tilde{F}_1^{ijk} \wedge dz^i \wedge dz^j \wedge dz^k \wedge \tilde{F}_3^i \wedge dz^l \wedge A_0^{ijk} \wedge dz^i \wedge dz^j \wedge dz^k \\
+ \frac{1}{36} \tilde{F}_1^{ijk} \wedge dz^i \wedge dz^j \wedge dz^k \wedge \tilde{F}_3^{lm} \wedge dz^l \wedge dz^m \wedge A_3 \\
+ \frac{1}{36} \tilde{F}_1^{ijk} \wedge dz^i \wedge dz^j \wedge dz^k \wedge \tilde{F}_3^{lm} \wedge dz^l \wedge dz^m \wedge A_2^i \wedge dz^i \\
- \frac{1}{72} \tilde{F}_1^{ijk} \wedge dz^i \wedge dz^j \wedge dz^k \wedge \tilde{F}_3^{lm} \wedge dz^l \wedge dz^m \wedge A_1^{ij} \wedge dz^i \wedge dz^j \\
- \frac{1}{72} \tilde{F}_1^{ijk} \wedge dz^i \wedge dz^j \wedge dz^k \wedge \tilde{F}_3^{lm} \wedge dz^l \wedge dz^m \wedge A_0^{ijk} \wedge dz^i \wedge dz^j \wedge dz^k \\
- \frac{1}{216} \tilde{F}_1^{ijk} \wedge dz^i \wedge dz^j \wedge dz^k \wedge \tilde{F}_3^{lm} \wedge dz^l \wedge dz^m \wedge A_2^{i} \wedge dz^i \wedge dz^j \wedge dz^k .
\]

Continue with terms with 2 indices
\[
\mathcal{L}_{\tilde{F}_4A_3} = \int \left( -\frac{1}{2} \tilde{F}_4 \wedge \tilde{F}_4 \wedge A_1^{ij} \wedge dz^i \wedge dz^j + \tilde{F}_4 \wedge \tilde{F}_3 \wedge A_2^i \wedge dz^i \wedge dz^j + \tilde{F}_4 \wedge \tilde{F}_3 \wedge A_1^{ij} \wedge dz^i \wedge dz^j + \tilde{F}_4 \wedge A_3 \wedge A_2^i \wedge dz^i \wedge dz^j \right) \quad \text{(E.2)}
\]

rearranging the terms using the property of changing sign each time we run a p-form through another q-form, we obtain
\[
\mathcal{L}_{\tilde{F}_4A_3} = \int \left( -\frac{1}{2} \tilde{F}_4 \wedge \tilde{F}_4 \wedge A_1^{ij} \wedge dz^i \wedge dz^j - \tilde{F}_4 \wedge \tilde{F}_3 \wedge A_2^i \wedge dz^i \wedge dz^j \right) \quad \text{(E.3)}
\]

Note that the factors in front of the various terms can help us see what type of term we should obtain after integration. Consider the first term we see that we have 4 \& 4 \& 1 \ i.e we have four “leg” in the first and second field strength and one “leg” in the gauge potential. So we should perform partial integration on the terms with the same factor
in front but different setup of legs in each term.

\[-\tilde{F}_4 \wedge \tilde{F}_3 \wedge A_2^i \wedge dz^i \wedge dz^j = -dA_3 \wedge A_2^i \wedge \tilde{F}_3^j \wedge dz^i \wedge dz^j \]

\[= -A_3 \wedge d(\tilde{F}_3^j \wedge A_2^i) \wedge dz^i \wedge dz^j \]

\[= -A_3 \wedge \tilde{F}_3^j \wedge \tilde{F}_3^j \wedge dz^i \wedge dz^j \]

\[= -\tilde{F}_3^j \wedge \tilde{F}_3^j \wedge A_3 \wedge dz^i \wedge dz^j \]

we see that after partial integration we obtain a term that has the same form as the fifth term in the integral, doing a partial integration on the term that has a 1/2 factor in front gives us

\[-\frac{1}{2} \tilde{F}_4 \wedge \tilde{F}_2^{ij} \wedge A_3 \wedge dz^i \wedge dz^j = -\frac{1}{2} \tilde{F}_4 \wedge dA_1^{ij} \wedge A_3 \wedge dz^i \wedge dz^j \]

\[= +\frac{1}{2} A_1^{ij} \wedge d(\tilde{F}_4 \wedge A_3) \wedge dz^i \wedge dz^j \]

\[= -\frac{1}{2} A_1^{ij} \wedge \tilde{F}_4 \wedge \tilde{F}_4 \wedge dz^i \wedge dz^j \]

\[= -\frac{1}{2} \tilde{F}_4 \wedge \tilde{F}_4 \wedge A_1^{ij} \wedge dz^i \wedge dz^j . \]

We have now performed partial integration on two terms with different setup of legs, the same procedure applies to the other terms and we obtain

\[\mathcal{L}_{FFA}^{n=2} = \frac{1}{6} \int \left( -\frac{3}{2} \tilde{F}_4 \wedge \tilde{F}_4 \wedge A_1^{ij} - 3\tilde{F}_3^i \wedge \tilde{F}_3^j \wedge A_3 \right) dz^i \wedge dz^j \]

\[= \int \left( -\frac{1}{4} \tilde{F}_4 \wedge \tilde{F}_4 \wedge A_{ij} - \frac{1}{2} \tilde{F}_3^i \wedge \tilde{F}_3^j \wedge A_3 \right) \varepsilon^{ij} \] (E.4)

thus we have found the \( D = 9 \) C-S term. We can continue with this procedure and obtain dimensional reductions all the way down to \( D = 2 \). Let us continue with 3 indices which should yield us a \( D = 8 \) C-S term. Gathering all the terms now with 3 indices gives us the following Lagrangian

\[\mathcal{L}_{FFA}^{n=3} = \frac{1}{6} \int \left( -\frac{1}{6} \tilde{F}_4 \wedge \tilde{F}_4 \wedge A_1^{ijk} \wedge dz^i \wedge dz^j \wedge dz^k - \frac{1}{2} \tilde{F}_4 \wedge \tilde{F}_3^i \wedge dz^i \wedge A_1^{jk} \wedge dz^j \wedge dz^k \right. \]

\[-\frac{1}{2} \tilde{F}_2 \wedge \tilde{F}_2^{ij} \wedge dz^i \wedge dz^j \wedge A_2^k \wedge dz^k \right. \]

\[\left. -\frac{1}{2} \tilde{F}_3 \wedge \tilde{F}_3^i \wedge dz^i \wedge A_1^{ij} \wedge dz^j \wedge dz^k \right. \]

\[-\frac{1}{2} \tilde{F}_2 \wedge \tilde{F}_2^{ij} \wedge dz^i \wedge dz^j \wedge A_3 \right. \]

\[-\frac{1}{2} \tilde{F}_3 \wedge \tilde{F}_3^i \wedge dz^i \wedge A_1^{ij} \wedge dz^j \wedge dz^k \left. \right) \widetilde{\mathcal{L}} = \frac{1}{6} \int \left( -\frac{1}{4} \tilde{F}_4 \wedge \tilde{F}_4 \wedge A_{ijk} - \frac{1}{2} \tilde{F}_3^i \wedge \tilde{F}_3^j \wedge A_3 \right) \varepsilon^{ijk} \] (E.5)

We start by noticing that we have three terms with 1/6-term in front, these terms should yield us the same expression. Let us assume we want something of the form like the

\(^8\)Hope this is understandable, if we think in this way we can save some time on the calculations
first term \( i.e \ 4 \times 4 \times 0 \times 1 \times 1 \times 1 \). The second term with the same factor is expressed as

\[
-\frac{1}{6} \tilde{F}_4 \wedge \tilde{F}_4 \wedge dz^j \wedge dz^j \wedge dz^k \wedge A_3 = \frac{1}{6} \tilde{F}_4 \wedge \tilde{F}_4 \wedge A_3 \wedge dz^j \wedge dz^j \wedge dz^k
\]

\[
= \frac{1}{6} \tilde{F}_4 \wedge dA_0 \wedge A_3 \wedge dz^i \wedge dz^j \wedge dz^k
\]

\[
= -\frac{1}{6} A_0 \wedge d(\tilde{F}_4 \wedge A_3) \wedge dz^i \wedge dz^j \wedge dz^k
\]

\[
= -\frac{1}{6} A_0 \wedge \tilde{F}_4 \wedge A_3 \wedge dz^j \wedge dz^j \wedge dz^k
\]

\[
= -\frac{1}{6} \tilde{F}_4 \wedge \tilde{F}_4 \wedge A_0 \wedge dz^j \wedge dz^j \wedge dz^k
\]

we see that this term has the same form as the first term mainly \( 4 \times 4 \times 0 \times 1 \times 1 \times 1 \), we also obtain the same expression for the last term with the factor \( 1/6 \) one can also show this but we see that the term resembles the term above. Let us now look on the terms with the factors \( 1/2 \) and see what expression we obtain after partial integration

\[
-\frac{1}{2} \tilde{F}_4 \wedge \tilde{F}_4 \wedge dz^i \wedge A_1 \wedge dz^j \wedge dz^k = \frac{1}{2} \tilde{F}_4 \wedge \tilde{F}_4 \wedge A_1 \wedge dz^i \wedge dz^j \wedge dz^k
\]

\[
= \frac{1}{2} A_3 \wedge \tilde{F}_4 \wedge A_1 \wedge dz^j \wedge dz^j \wedge dz^k
\]

\[
= \frac{1}{2} A_3 \wedge d(\tilde{F}_4 \wedge A_1) \wedge dz^i \wedge dz^j \wedge dz^k
\]

\[
= \frac{1}{2} A_3 \wedge \tilde{F}_4 \wedge A_1 \wedge dz^j \wedge dz^j \wedge dz^k
\]

\[
= \frac{1}{2} \tilde{F}_4 \wedge \tilde{F}_4 \wedge A_3 \wedge dz^i \wedge dz^j \wedge dz^k
\]

we can see that we obtain the same expression as the other \( 1/2 \) factor terms. Gathering all of them together we obtain the following integral

\[
\mathcal{L}_{FFA}^{n=3} = \frac{1}{6} \int \left( -\frac{1}{2} \tilde{F}_4 \wedge \tilde{F}_4 \wedge A_0 \wedge dz^j - \frac{1}{2} \tilde{F}_4 \wedge \tilde{F}_4 \wedge A_1 \wedge dz^j + 3 \tilde{F}_4 \wedge \tilde{F}_4 \wedge A_3 \wedge dz^k \right) \epsilon^{ijk}
\]

\[
= \int \left( -\frac{1}{12} \tilde{F}_4 \wedge \tilde{F}_4 \wedge A_0 \wedge dz^j - \frac{1}{6} \tilde{F}_4 \wedge \tilde{F}_4 \wedge A_1 \wedge dz^j + \frac{1}{2} \tilde{F}_4 \wedge \tilde{F}_4 \wedge A_3 \wedge dz^k \right) \epsilon^{ijk} \ . \tag{E.6}
\]

We thus see that we have 8-forms which yields us a 8-dimensional Chern-Simons term.
Let us continue now with terms that have four index. The following integral becomes

\[ \mathcal{L}_{FFA}^{n=4} = \int \left( -\frac{1}{6} \tilde{F}_4 \wedge \tilde{F}_3 \wedge dz^i \wedge A_0^{ijkl} \wedge dz^j \wedge dz^k \wedge dz^l \\
+ \frac{1}{4} \tilde{F}_4 \wedge \tilde{F}_2 \wedge dz^i \wedge A_1^{kl} \wedge dz^j \wedge dz^k \wedge dz^l \\
- \frac{1}{6} \tilde{F}_4 \wedge \tilde{F}_1^{ijk} \wedge dz^i \wedge A_2^j \wedge dz^j \wedge dz^k \wedge dz^l \\
- \frac{1}{6} \tilde{F}_3^j \wedge dz^i \wedge \tilde{F}_4 \wedge A_0^{ijkl} \wedge dz^j \wedge dz^k \wedge dz^l \\
- \frac{1}{2} \tilde{F}_3^j \wedge dz^i \wedge \tilde{F}_3^j \wedge A_1^{kl} \wedge dz^j \wedge dz^k \wedge dz^l \\
- \frac{1}{2} \tilde{F}_3^j \wedge dz^i \wedge \tilde{F}_3^{jkl} \wedge dz^j \wedge dz^k \wedge dz^l \wedge A_3 \\
+ \frac{1}{4} \tilde{F}_2^{ij} \wedge dz^i \wedge \tilde{F}_4 \wedge A_1^{kl} \wedge dz^j \wedge dz^l \\
- \frac{1}{2} \tilde{F}_2^{ij} \wedge dz^i \wedge \tilde{F}_2^k \wedge A_2^l \wedge dz^j \wedge dz^l \\
+ \frac{1}{4} \tilde{F}_2^{ij} \wedge dz^i \wedge \tilde{F}_2^{kl} \wedge dz^j \wedge dz^k \wedge dz^l \wedge A_3 \\
- \frac{1}{6} \tilde{F}_1^{ijk} \wedge dz^i \wedge \tilde{F}_4 \wedge A_2^j \wedge dz^j \wedge dz^l \\
- \frac{1}{6} \tilde{F}_1^{ijk} \wedge dz^i \wedge \tilde{F}_3^j \wedge A_3^l \wedge dz^j \wedge dz^k \wedge dz^l \wedge A_3 \right). \tag{E.7}

We see that the terms with the factor 1/6 should resemble each other and so on. Like always we choose a term and perform a partial integration.

\[ \frac{1}{6} \tilde{F}_4 \wedge \tilde{F}_1^{ijk} \wedge dz^i \wedge dz^j \wedge dz^k \wedge A_3^l \wedge dz^l = \frac{1}{6} \tilde{F}_4 \wedge \tilde{F}_1^{ijk} \wedge A_2^j \wedge dz^j \wedge dz^k \wedge dz^l \wedge dz^l \\
= \frac{1}{6} \tilde{F}_4 \wedge \tilde{F}_1^{ijk} \wedge A_2^j \wedge dz^j \wedge dz^k \wedge dz^l \wedge dz^l \\
= \frac{1}{6} \tilde{F}_4 \wedge \tilde{F}_1^j \wedge A_2^l \wedge dz^j \wedge dz^k \wedge dz^l \\
= \frac{1}{6} \tilde{F}_4 \wedge \tilde{F}_1^j \wedge A_2^l \wedge dz^j \wedge dz^k \wedge dz^l \\
= \frac{1}{6} \tilde{F}_4 \wedge \tilde{F}_1^j \wedge A_2^l \wedge dz^j \wedge dz^k \wedge dz^l . \]

We thus see that this configuration becomes the first term in our integral above, after partial integration.
We continue with the $1/2$ factor terms and see what we obtain after partial integration

\[-\frac{1}{2} \tilde{F}_3^i \wedge dz^i \wedge \tilde{F}_2^{jk} \wedge dz^j \wedge dz^k \wedge A_2^l \wedge dz^l = -\frac{1}{2} \tilde{F}_3^i \wedge \tilde{F}_2^{jk} \wedge A_2^i \wedge dz^i \wedge dz^j \wedge dz^k \wedge dz^l \]
\[= \frac{1}{2} dA_1^{ij} \wedge \tilde{F}_3^{kl} \wedge A_2^l \wedge dz^i \wedge dz^j \wedge dz^k \wedge dz^l \]
\[= \frac{1}{2} A_1^{ij} \wedge d(\tilde{F}_3^{kl} \wedge A_2^l) \wedge dz^i \wedge dz^j \wedge dz^k \wedge dz^l \]
\[= \frac{1}{2} A_1^{ij} \wedge \tilde{F}_3^{kl} \wedge A_2^l \wedge dz^i \wedge dz^j \wedge dz^k \wedge dz^l \]
\[= \frac{1}{2} \tilde{F}_3^i \wedge \tilde{F}_3^{kl} \wedge A_1^l \wedge dz^i \wedge dz^j \wedge dz^k \wedge dz^l \]
\[= 1/2 \tilde{F}_3^i \wedge \tilde{F}_3^{jk} \wedge A_1^l \wedge dz^i \wedge dz^j \wedge dz^k \wedge dz^l \].

The terms left now are the terms with the factor $1/4$, doing a partial integration on for example the term

\[\frac{1}{4} \tilde{F}_4 \wedge \tilde{F}_2^{jk} \wedge dz^i \wedge dz^j \wedge A^{kl} \wedge dz^k \wedge dz^l = \frac{1}{4} dA_3 \wedge \tilde{F}_2^{ij} \wedge A_1^{kl} \wedge dz^i \wedge dz^j \wedge dz^k \wedge dz^l \]
\[= -\frac{1}{4} A_3 \wedge d(\tilde{F}_2^{ij} \wedge A_1^{kl}) \wedge dz^i \wedge dz^j \wedge dz^k \wedge dz^l \]
\[= -\frac{1}{4} A_3 \wedge \tilde{F}_2^{ij} \wedge \tilde{F}_2^{kl} \wedge dz^i \wedge dz^j \wedge dz^k \wedge dz^l \]
\[= -\frac{1}{4} \tilde{F}_2^{ij} \wedge \tilde{F}_2^{kl} \wedge A_3 \wedge dz^i \wedge dz^j \wedge dz^k \wedge dz^l \].

We see that this term is the same as the other $1/4$ factor terms, we can now add them together and obtain the following integral

\[L_{FFA}^{n=1} = \int \left( \frac{1}{6} \tilde{F}_4 \wedge F_3^i \wedge A_0^{kl} - \frac{1}{4} \tilde{F}_3^i \wedge \tilde{F}_2^{jk} \wedge A_2^l + \frac{1}{8} \tilde{F}_2^{ij} \wedge \tilde{F}_2^{kl} \wedge A_3 \right) \varepsilon^{ijkl} \quad (E.8)\]
Let us continue with 5 indices now, collecting all the term with 5 indices gives us the following integral

\[ L^{n=5}_{F, A} = \frac{1}{6} \int \left( + \frac{1}{12} \tilde{F}_4 \wedge \tilde{F}^{ijk} \wedge dz^i \wedge dz^j \wedge dz^k \wedge dz^l \wedge dz^m \right. \\
+ \frac{1}{12} \tilde{F}^{4} \wedge \tilde{F}^{ij} \wedge dz^i \wedge dz^j \wedge dz^k \wedge A_0^{klm} \wedge dz^l \wedge dz^m \\
- \frac{1}{6} \tilde{F}^{4} \wedge dz^i \wedge \tilde{F}^{ij} \wedge dz^j \wedge A_0^{klm} \wedge dz^k \wedge dz^l \wedge dz^m \\
- \frac{1}{6} A_0^{ijklm} \wedge dA_0^{ijklm} \\
- \frac{1}{6} \tilde{F}^{4} \wedge dz^i \wedge \tilde{F}^{ij} \wedge dz^j \wedge \tilde{F}^{kl} \wedge dz^k \wedge dz^l \wedge dz^m \\
+ \frac{1}{12} \tilde{F}^{ij} \wedge dz^i \wedge dz^j \wedge \tilde{F}^{4} \wedge A_0^{klm} \wedge dz^k \wedge dz^l \wedge A_0^m \wedge dz^m \\
- \frac{1}{12} \tilde{F}^{4} \wedge dz^i \wedge dz^j \wedge \tilde{F}^{klm} \wedge dz^k \wedge dz^l \wedge \tilde{F}^{m} \wedge dz^m \wedge A_3 \\
- \frac{1}{12} \tilde{F}^{4} \wedge dz^i \wedge \tilde{F}^{ijklm} \wedge dz^j \wedge A_3 \\
+ \frac{1}{12} \tilde{F}^{4} \wedge dz^i \wedge dz^j \wedge dz^k \wedge \tilde{F}^{ijklm} \wedge dz^l \wedge dz^m \wedge A_3 \\
+ \frac{1}{12} \tilde{F}^{4} \wedge dz^i \wedge dz^j \wedge dz^k \wedge \tilde{F}^{ijklm} \wedge dz^l \wedge dz^m \wedge A_3 \right). \\

Let us calculate the partial integration for one of the 1/12 terms and see what we obtain

\[ \frac{1}{12} \tilde{F}^{4} \wedge \tilde{F}^{ijkl} \wedge A_1^{lm} \wedge dz^i \wedge dz^j \wedge dz^k \wedge dz^l \wedge dz^m = \frac{1}{12} dA_0^{ijkl} \wedge \tilde{F}^{4} \wedge A_1^{lm} \wedge dz^{ijklm} \\
= \frac{1}{12} \tilde{F}^{4} \wedge d(\tilde{F}^{4} \wedge A_1^{lm}) \wedge dz^{ijklm} \\
= \frac{1}{12} \tilde{F}^{4} \wedge \tilde{F}^{ijkl} \wedge A_0^{lm} \wedge dz^{ijklm} \\
= \frac{1}{12} \tilde{F}^{4} \wedge \tilde{F}^{ijkl} \wedge A_1^{lm} \wedge dz^{ijklm} \]

we introduced a short-hand notation to save space i.e., \( dz^i \wedge dz^j \wedge dz^k \wedge dz^l \wedge dz^m = dz^{ijklm} \). We continue with the factor 1/6 term and obtain the following

\[ - \frac{1}{6} \tilde{F}^{ijkl} \wedge dz^i \wedge dz^j \wedge dz^k \wedge \tilde{F}^{4} \wedge dz^l \wedge A_0^m \wedge dz^m = \frac{1}{6} \tilde{F}^{ijkl} \wedge \tilde{F}^{4} \wedge A_0^m \wedge dz^{ijklm} \\
= \frac{1}{6} dA_0^{ijkl} \wedge \tilde{F}^{4} \wedge A_0^m \wedge dz^{ijklm} \\
= - \frac{1}{6} A_0^{ijkl} \wedge d(\tilde{F}^{4} \wedge A_0^m) \wedge dz^{ijklm} \\
= - \frac{1}{6} \tilde{F}^{4} \wedge \tilde{F}^{ijkl} \wedge A_0^m \wedge dz^{ijklm} \\
= - \frac{1}{6} \tilde{F}^{4} \wedge \tilde{F}^{ijkl} \wedge A_0^m \wedge dz^{ijklm}. \]

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and finally the last term gives us

\[
\frac{1}{4} \tilde{F}_3^i \wedge dz^i \wedge \tilde{F}_2^{jk} \wedge dz^j \wedge dz^k \wedge A_{1m} \wedge dz^l \wedge dz^m = \frac{1}{4} \tilde{F}_3^i \wedge \tilde{F}_2^{jk} \wedge A_{1m} \wedge dz^{ijklm}
\]

\[
= \frac{1}{4} dA_0^i \wedge \tilde{F}_2^{jk} \wedge A_{1m} \wedge dz^{ijklm}
\]

\[
= \frac{1}{4} A_0^i \wedge d(\tilde{F}_2^{jk} \wedge A_{1m}) \wedge dz^{ijklm}
\]

\[
= \frac{1}{4} A_0^i \wedge \tilde{F}_2^{jk} \wedge \tilde{F}_{lm} \wedge dz^{ijklm}
\]

\[
= \frac{1}{4} \tilde{F}_2^{ij} \wedge \tilde{F}_{kl} \wedge A_{2}^m \wedge dz^{ijklm}
\]

we have now done all the partial integrations we need, collecting the results gives us

\[
\mathcal{L}^{n=5}_{FFA} = \frac{1}{6} \int \left( \frac{1}{2} \tilde{F}_4 \wedge \tilde{F}_2^2 \wedge A_{klm} - \frac{1}{2} \tilde{F}_3^i \wedge \tilde{F}_3^j \wedge A_{0,klm} + \frac{3}{4} \tilde{F}_2^{ij} \wedge \tilde{F}_{2}^{kl} \wedge A_{2}^m \right) \varepsilon^{ijklm}
\]

\[
= \int \left( \frac{1}{12} \tilde{F}_4 \wedge \tilde{F}_2^2 \wedge A_{klm} - \frac{1}{12} \tilde{F}_3^i \wedge \tilde{F}_3^j \wedge A_{0,klm} + \frac{1}{8} \tilde{F}_2^{ij} \wedge \tilde{F}_{2}^{kl} \wedge A_{2}^m \right) \varepsilon^{ijklm}
\]