The Whitney embedding theorem

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Abstract
A fundamental theorem in differential geometry is proven in this essay. It is the embedding theorem due to Hassler Whitney, which shows that the ever so general and useful topological spaces called manifolds, can all be regarded as subspaces of some Euclidean space. The version of the proof given in this essay is very similar to the original from 1944. Modern definitions are used, however, and many illustrations have been made, wherever it helps the understanding.

Sammanfattning
En fundamental sats i differentialgeometri bevisas i denna uppsats. Det är inbäddningssatsen av Hassler Witney, som visar att de ofantligt generella och användbara topologiska rummen kallade mångfalder, kan ses som delrum av något Euklidiskt rum. Versionen av beviset given i denna uppsats är väldigt lik originalet från 1944. Moderna definitioner används dock, och många illustrationer har gjorts, varhelst det hjälper förståelsen.
**CONTENTS**

List of Figures 1
1. Introduction 3
2. Preliminaries 5
  2.1. Manifolds 5
  2.2. Immersions and embeddings 7
  2.3. Bundles and polyhedra 8
3. Whitney’s first embedding theorem 13
4. Whitney’s second embedding theorem 27
5. Acknowledgements 45
References 47
List of Figures

2.1 Example of a topological space that is not a differentiable manifold 6
2.2 A mapping on manifolds and the corresponding mapping on Euclidean spaces. 7
2.3 Example of what is not the image of an immersion. 8
2.4 A 2-simplex with an illustration of how it is generated by its points. 11
2.5 2-dimensional polyhedron in \( \mathbb{R}^3 \) consisting of seven faces. 12
3.1 Dependency diagram for the path to the proof of Whitney’s first embedding theorem. 13
3.2 Illustration for proof of Lemma 3.11 18
3.3 Illustration for proof of Lemma 3.12. 19
4.1 Dependency diagram for the path to the proof of Whitney’s second embedding theorem. 27
4.2 Mappings of Theorem 4.5. 30
4.3 A completely regular immersion with one self-intersection. 32
4.4 A completely regular immersion considered in Lemma 4.12, Lemma 4.13, and Theorem 4.14, as well as a construction used for the Whitney trick. 36
4.5 Defining vector fields necessary for the Whitney trick. 38
4.6 Modification of neighbourhoods, part of the Whitney trick. 39
4.7 Introducing a pair of regular self-intersections to a 1-dimensional manifold. 41
4.8 Undoing intersections. The purpose of the Whitney trick. 42
1. Introduction

As every one knows an $n$-dimensional differentiable manifold is defined to be a set $M$ together with a differentiable structure $\mathcal{D}$ such that the induced topology is Hausdorff, second countable and such that $M$ is locally Euclidean of dimension $n$. Some, for example [9], assume only paracompactness instead of the stronger assumption of second countability. We can not proceed in this manner since we want to apply Sard’s theorem. It should also be emphasized that not every set admits a differentiable structure (see e.g. [8] or [3, 6]). This crucial point is something that most textbooks in differential geometry forget to mention. For convenience we shall assume that our manifolds are $C^\infty$-smooth, even if the proofs go through for $C^1$-smooth manifolds.

In [14], Scholz argue that the concept of manifolds can be traced back to Lagrange and his “Mécanique analytique” published in 1788, other interpret it differently. We shall not here entangle in this fight. Instead, we refer the interested reader to [2, 4, 7, 10, 21] and the references therein. On the other hand, most agree on that it was Hermann Weyl who in “Idee der Riemannfläche” published in 1913 introduced for the first time what we today know as real manifolds.

The modern definition of manifolds is completely intrinsic, so we can not a priori view a manifold globally as a subset of $\mathbb{R}^N$ for some $N$. In other words, we can not assume that the manifold can be embedded in $\mathbb{R}^N$. This is one of the most crucial points of our definition. The main star of this essay is the American mathematician Hassler Whitney who in 1936 proved that any $n$-dimensional differentiable manifold can be embedded in $\mathbb{R}^{2n+1}$ ([18]). The proof of this theorem is included in most introductory textbooks on differential geometry (see e.g. [9, 17]), and it has been considerably simplified over time. But the original work of Whitney is still to this day very well written, and therefore we shall here follow his proof as presented in [1].

In 1944, eight years after the first paper, Whitney proved in [19] the main aim of this essay. He prove the following:

**Theorem 4.15.** Every $n$-dimensional differentiable manifold can be embedded in $\mathbb{R}^{2n}$.

This theorem is sharp as it is shown by the example that the 2-dimensional projective plane $\mathbb{R}P^2$ can be embedded in $\mathbb{R}^4$, but not in $\mathbb{R}^3$. It is hard to grasp the importance of Whitney’s embedding theorem and its proof. One thing that might indicate this is that Field Medal laureates Simon Donaldson, Michael Freedman, Grigori Perelman, and Stephen Smale have all, directly or indirectly, been inspired by Theorem 4.15 and its proof. Another thing is that it yields the foundation of a whole field of mathematics known as surgery theory. Finally, to quote the French legend Jean Alexandre Eugène Dieudonné in his book [2] (page 60) on Theorem 4.15:

“...the proof is long and difficult . . .”

To this day, seventy years after publication, the proof of Theorem 4.15 has not been noticeably simplified. The complete proof of our focal point is rarely found in the literature. Except for the original articles by Whitney it can be found in [1], a Japanese translation of Whitney’s original work that in 1993 was translated back
into English. Another rare exposition was written by Prasolov; the case \( \mathbb{R}^{2n+1} \) can be found in [12], and the generalization can be found in [13]. In this essay we follow [1], even though to grasp detail we have consulted [13] and [19].

One might wonder what the secret is behind the proof of Theorem 4.15. It is what is known today as the “Whitney Trick”, it even got its own Wikipedia page. In this essay it is divided into Lemma 4.12, Lemma 4.13 and Theorem 4.14, and it is only valid for dimension 5 or higher. Therefore, in the proof of Theorem 4.15 for the lower dimensions we shall need to use the classification theorems for lower dimensional manifolds.

The overview of this thesis is as follows. In §2, we give some necessary background to differential geometry and fibre bundles. For further information we refer to [9, 15, 16, 17]. In §3, we give the first proof of the first part of the embedding theorem \( \mathbb{R}^{2n+1} \), and we end this essay in §4 by proving the general case (Theorem 4.15).

Before starting the proof of the all so mighty Whitney’s embedding theorem, and its trick, it should be pointed out that some depth of detail is ignored. The main thing omitted is in Lemma 4.11 when we refer to [16] for the details of extending a section over \( |K'| \) to \( |K| \) in the \((m-i)\)-dimensional sphere bundle over \(|K|\) (see §2.3 for definition).
2. Preliminaries

Important or reoccurring notions and concepts of differential geometry are presented in the three subsections that follow. Concepts specific to certain lemmas and theorems are defined when necessary.

2.1. Manifolds.

**Definition 2.1.** A set \( M \) is locally Euclidean of dimension \( n \) if every point \( p \in M \) has a neighbourhood \( U \) such that there is a bijection \( \phi \) from \( U \) into an open subset of \( \mathbb{R}^n \). We call the pair \((U, \phi : U \rightarrow \mathbb{R}^n)\) a chart, \( U \) a coordinate neighbourhood, and \( \phi \) a coordinate map on \( U \).

**Definition 2.2.** Two charts \((U \subset \mathbb{R}^n, \phi : U \rightarrow \mathbb{R}^n), (V \subset \mathbb{R}^n, \psi : V \rightarrow \mathbb{R}^n)\) are \( C^\infty \)-compatible if the two maps
\[
\phi \circ \psi^{-1} : \psi(U \cap V) \rightarrow \phi(U \cap V),
\]
\[
\psi \circ \phi^{-1} : \phi(U \cap V) \rightarrow \psi(U \cap V),
\]
are both \( C^\infty \)-smooth as maps \( \mathbb{R}^n \rightarrow \mathbb{R}^n \).

**Definition 2.3.** An atlas on a set \( M \) is a collection \( \mathcal{S} = \{(U_\alpha, \phi_\alpha : U_\alpha \rightarrow \mathbb{R}^n)\} \) that satisfy the following:

(i) The coordinate neighbourhoods cover \( M \), i.e.
\[
M = \bigcup_\alpha U_\alpha.
\]

(ii) Every image of the form \( \phi_\alpha(U_\alpha \cap U_\beta) \) is open in \( \mathbb{R}^n \).

(iii) They are pairwise \( C^\infty \)-compatible.

The neighbourhoods of the charts from an atlas \( \mathcal{S} \) on a set \( M \) constitute a basis for a topology on \( M \), called the topology induced by the atlas. Thus, the pair \((M, \mathcal{S})\) is a locally Euclidean topological space. In [9] (Proposition 1.32, p. 14) one can find the requirements necessary, on an atlas \( \mathcal{S} \), for the induced topology to be Hausdorff and second countable. For us, the important part is that there exist such requirements.

**Definition 2.4.** Two atlases on a set \( M \) are said to be equivalent if their union is also an atlas on \( M \). An equivalence class of atlases on \( M \) is called a differentiable structure on \( M \). The pair \((M, \mathcal{D})\) of a set \( M \) and a differentiable structure \( \mathcal{D} \) on \( M \) is called a differentiable manifold if the topology induced by \( \mathcal{D} \) makes \((M, \mathcal{D})\) a Hausdorff, locally Euclidean, second countable topological space. We will often write \( M \) instead of \((M, \mathcal{D})\), when the differentiable structure is understood.

**Remark.** Manifolds do not have to be connected. Manifolds are said to be compact if every component is compact. They are said to be non-compact no component is compact. Until the very last part of Whitney’s second embedding theorem we will only consider compact manifolds.

Let us now give some examples of differentiable manifolds, as well as sets that cannot be made into differentiable manifolds.
Example 2.5. Euclidean space $\mathbb{R}^n$ with differentiable structure represented by $\{(\mathbb{R}^n, \text{id})\}$, is a differentiable manifold, and so is every open $U \subset \mathbb{R}^n$, with representatives $\{(U, \text{id}|_U)\}$. Here, $\text{id} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map.

Example 2.6. The cross given by

$\{(x, 0) \mid x \in (-1, 1)\} \cup \{(0, y) \mid y \in (-1, 1)\} \subset \mathbb{R}^2$,

is not a differentiable manifold (see Figure 2.1). The reason is that there is no bijection between a neighbourhood of the intersection and an open subset of any Euclidean space (see e.g. [17]).

Figure 2.1. Example of a topological space that cannot be made locally Euclidean and hence is not a differentiable manifold.

An alternative definition of differentiable manifold is as a set $M$ together with a so called maximal atlas. Given any atlas on $M$, a maximal atlas is formed as the union of all equivalent atlases, which is the set of all compatible charts. This is precisely what the differentiable structure consists of. Thus, given an atlas on $M$, we obtain the same differentiable manifold whether we give it the corresponding maximal atlas or the corresponding differentiable structure.

Given a set $M$, there is no guarantee that all charts on $M$ will be compatible. The existence of incompatible charts implies the existence of different equivalence classes of charts. This, in turn, means that $M$ may have more than one possible differentiable structure. For example, the unit sphere $S^{31}$ has over 16 million different differentiable structures (see [6]).

Of course, there is no guarantee that there exists a differentiable structure for an arbitrary set $M$. Even if $M$ is a Hausdorff, locally Euclidean, second countable topological space from the outset, its topology may not be the same as that induced by a differentiable structure. In this essay, however, we will simply assume that this problem does not occur.

We shall now introduce the concept of submanifold. There are various definitions in the literature but for this essay, there is only Definition 2.7.

Definition 2.7. Let $(M, \mathscr{D})$ be an $n$-dimensional manifold, and let $A$ be a subset of $M$. Regard $\mathbb{R}^k$, $0 \leq k \leq n$, as a subspace of $\mathbb{R}^n$,

$\mathbb{R}^k = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_{k+1}, \ldots, x_n = 0\}$.

Now assume that we can choose a representative $\mathscr{D} = \{(V_j, \varphi_j)\}$ of $\mathscr{D}$ such that for each nonempty $V_j \cap A$, the restriction

$\varphi_j|_{V_j \cap A} : V_j \cap A \to \mathbb{R}^k \subset \mathbb{R}^n$
is a homeomorphism into an open subset of $\mathbb{R}^k$ and the chart $(V_j \cap A, \varphi_j|_{V_j \cap A})$ is said to be adapted to $A$. Thus, we have an atlas $\mathcal{A}_A = \{(V_j \cap A, \varphi_j|_{V_j \cap A})\}$ representing a differentiable structure $\mathcal{D}_A$. We say that $(A, \mathcal{D}_A)$ is a submanifold of $(M, \mathcal{D})$ of dimension $k$.

2.2. Immersions and embeddings.
In this section we define immersions and embeddings and all that is necessary for those definitions. We shall also give some examples of sets that can, and that cannot, be the images of immersions or embeddings.

**Definition 2.8.** Let $(M_1, \mathcal{D}_1)$ and $(M_2, \mathcal{D}_2)$ be differentiable manifolds. A mapping $f : M_1 \to M_2$ is smooth at a point $x \in M_1$ if there exists charts $(U_1, h_1)$, from some atlas in $\mathcal{D}_1$, and $(U_2, h_2)$, from some atlas in $\mathcal{D}_2$, satisfying the following:

(i) The point $x$ belongs to $U_1$.

(ii) The set $f(U_1)$ is a subset of $U_2$.

(iii) The map $h_2 \circ f \circ h_1^{-1} : h_1(U_1 \cap f^{-1}(U_2)) \to h_2(U_2)$ is smooth (see Figure 2.2).

If $f$ is smooth at every point $x \in M_1$, then we say that $f$ is smooth.

**Definition 2.9.** A bijective mapping $f$ of manifolds is a diffeomorphism if both $f$ and $f^{-1}$ are smooth.

**Definition 2.10.** Let $M_1$ and $M_2$ be $n$-dimensional manifolds and $f : M_1 \to M_2$ a smooth map. For $x$ in $M_1$ choose charts $(U_1, h_1)$ and $(U_2, h_2)$ about $x$ and $f(x)$. We define the rank of $f$ at $x$ to be the rank of the Jacobian matrix of the map

$$h_2 \circ f \circ h_1^{-1} : h_1(U_1 \cap f^{-1}(U_2)) \to h_2(U_2)$$

at $h_1(x)$. 

---

**Figure 2.2.** A mapping $f$ of $n$-dimensional manifolds $M_1$ and $M_2$. The point $h_1(x) \in U_1 \subset M_1$ is mapped to $h_2(f(x))$ by the composition $h_2 \circ f \circ h_1^{-1} : h_1(U_1) \to \mathbb{R}^n$. 

Remark. The definition is independent of the choice of charts and it is sometimes possible to choose identity maps as $h_1$ and $h_2$ so that the rank of $f$ becomes the rank of its own Jacobian matrix.

**Definition 2.11.** A smooth map of manifolds $f : M_1 \to M_2$ is an *immersion* if the rank of $f$ at each point of $M_1$ is the same as the dimension of $M_1$. If $f$ is both an immersion and a homeomorphism, then it is an *embedding*. If there exists an embedding of $M_1$ into $M_2$, then we say that $M_1$ is *embedded* in $M_2$.

**Example 2.12.** The map $f : \mathbb{R}^2 \to S^1 \times \mathbb{R} \to \mathbb{R}^3$ defined by

$$f(u, v) = (v \cos u, v \sin u, v),$$

is smooth but not an embedding, since it intersects itself at $(u, v) = (u, 0)$ (see Figure 2.3). Examining $f$ more closely, as follows, we see that it is not even an immersion. Since $f : \mathbb{R}^2 \to \mathbb{R}^3$ is a map of $\mathbb{R}^2$, we can choose identity maps as the coordinate maps in the definition of rank (Definition 2.10). Thus, its rank is the rank of its Jacobian matrix

$$Df(u, v) = \begin{bmatrix}
\cos u & -v \sin u \\
\sin u & v \cos u \\
1 & 0
\end{bmatrix},$$

which has rank $1 < 2$ whenever $v = 0$.

**Example 2.13.** The mapping $\alpha : \mathbb{R} \to \mathbb{R}^2$ defined by,

$$y = x - \frac{x}{(1 + x^2)}, \quad z = \frac{1}{(1 + x^2)}$$

with graph in Figure 4.3 on p. 32, is an immersion but not an embedding since it is not injective, passing through $(0, 1/2)$ twice. We shall get back to this example in (4.2).

2.3. **Bundles and polyhedra.**

This section consists of definitions necessary for Lemma 4.11, which will in turn be used in Lemma 4.12 and Lemma 4.13. For a full description of how the lemmas and theorems of §4 depend upon one another, see Figure 4.1.

First we shall introduce the concept of *topological transformation groups*. We then need the theory of topological groups. We refer to [5] for the necessary background.
**Definition 2.14.** Let $G$ be a topological group, and let $Y$ be a topological space. Assume there is a continuous map $\eta : G \times Y \to Y$ satisfying the following:

(i) For the identity element $e$ of $G$, $\eta(e, y) = y$.

(ii) For all $g, g' \in G$ and $y \in Y$, $\eta(gg', y) = \eta(g, \eta(g', y))$.

Then we say that $G$ is a topological transformation group of $Y$, with respect to $\eta$, and that $G$ acts or operates on $Y$.

**Example 2.15.** The general linear group of dimension $n$ is the set of all non-singular $n \times n$ matrices $\text{GL}(n, \mathbb{R}) = \{ A \in \mathbb{R}^{nn} \mid \det(A) \neq 0 \}$. The orthogonal group of dimension $n$ is the set of all orthogonal $n \times n$ matrices $\text{O}(n) = \{ A \in \mathbb{R}^{nn} \mid A^T A = I \} \subset \text{GL}(n, \mathbb{R})$. Both of these are topological transformation groups of $\mathbb{R}^n$, their identity element is the identity matrix, $e = I$, and the continuous map $\eta$ corresponds to matrix multiplication from the left.

**Definition 2.16.** A coordinate bundle $\mathcal{B} = \{ B, p, X, Y, G \}$ is a collection of topological spaces and continuous maps with structures satisfying the following:

(1) The members $B$ and $X$ are topological spaces, $B$ is the total space and $X$ is the base space. The member $p : B \to X$ is a surjective continuous map called the projection map of $\mathcal{B}$.

(2) The member $Y$ is also a topological space; $Y$ is the fiber of $\mathcal{B}$. The member $G$ is a topological transformation group called the structural group of $\mathcal{B}$.

(3) The base $X$ has an open covering $\{ V_j \}_{j \in J}$, and for each $j \in J$ there is a homeomorphism $\phi_j : V_j \times Y \to p^{-1}(V_j)$; the $V_j$’s are coordinate neighbourhoods and the $\phi_j$’s are coordinate functions.

(4) The coordinate functions satisfy the following:

(i) $p \circ \phi_j(x, y) = x$, $x \in V_j$, $y \in Y$, $j \in J$.

(ii) The map $\phi_{j,x} : Y \to p^{-1}(x)$ defined by $\phi_{j,x}(y) = \phi_j(x, y)$, $y \in Y$ gives a homeomorphism of $Y$,

$$\phi_{j,x}^{-1} \circ \phi_{i,x} : Y \to Y$$

for $x \in V_i \cap V_j$, which agrees with the action of an element $g_{ji}(x)$ of $G$.

(iii) Define a map $g_{ji} : V_i \cap V_j \to G$ by

$$g_{ji}(x) = \phi_{j,x}^{-1} \circ \phi_{i,x}.$$

Then $g_{ji}$ is continuous; we say that the $g_{ji}$’s are coordinate transformation or transition functions of $\mathcal{B}$.

We write $Y_x$ for $p^{-1}(x)$; $Y_x$ is the fiber over $x$. 
Definition 2.17. For any two maps \( p : E \rightarrow M \) and \( p' : E' \rightarrow M \) with the same target space \( M \), a map \( \phi : E \rightarrow E' \) is said to be fiber-preserving if \( \phi(p^{-1}(x)) \subset p'^{-1}(x) \) for all \( x \in M \).

Definition 2.18. A surjective smooth map \( p : B \rightarrow X \) of manifolds is said to be locally trivial of rank \( r \) if

(i) each fiber \( p^{-1}(x) \) has the structure of a vector space of dimension \( r \);

(ii) for each \( x \in X \), there are an open neighbourhood \( U \) of \( x \) and a fiber-preserving diffeomorphism \( \phi : p^{-1}(U) \rightarrow U \times \mathbb{R}^r \) such that for every \( x \in U \) the restriction

\[
\phi|_{p^{-1}(x)} : p^{-1}(x) \rightarrow \{x\} \times \mathbb{R}^r
\]

is a vector space isomorphism.

Definition 2.19. A bundle \( \mathcal{B} = \{B, p, X, Y, G\} \) in which \( Y = \mathbb{R}^n \) and \( G = \text{GL}(n, \mathbb{R}) \) acts on \( Y \) by the usual linear transformations is called an \( n \) dimensional vector bundle. If \( p : B \rightarrow X \) is locally trivial of rank \( r \) then \( \mathcal{B} \) is said to be a vector bundle of rank \( r \).

Definition 2.20. Given a manifold \( M \), let \( p_1 : M \times \mathbb{R}^r \rightarrow M \) be the projection to the first factor, i.e. \( p_1(m, x) = m \), where \( m \in M \) and \( x \in \mathbb{R}^r \). Then \( M \times \mathbb{R}^r \rightarrow M \) is a vector bundle of rank \( r \), called the product bundle of rank \( r \) over \( M \). The vector space structure on the fiber \( p_1^{-1}(x) = \{(x, v) \mid v \in \mathbb{R}^r\} \) is given by

\[
(x, u) + (x, v) = (x, u + v), \quad k \cdot (x, v) = (x, kv) \quad \text{for } b \in \mathbb{R}.
\]

Definition 2.21. A vector bundle of rank \( r \) over a manifold \( M \) isomorphic over \( M \) to the product bundle \( M \times \mathbb{R}^r \) is called a trivial bundle.

The remainder of this section consists of the definition of polyhedra and all that is necessary for that definition. The importance of polyhedra in this essay comes from the fact that all differentiable manifolds are homeomorphic to a polyhedron. In other words, every differentiable manifold is triangulable.

Definition 2.22. Let \( P_0, \ldots, P_n \) be affinely independent points in \( \mathbb{R}^m \), \( m \geq n \), which means that \( P_0P_1, \ldots, P_0P_n \) are linearly independent. We call the set

\[
|P_0P_1 \ldots P_n| = \left\{ X \in \mathbb{R}^m \mid \begin{array}{c}
\overrightarrow{OX} = \lambda_0\overrightarrow{OP_0} + \cdots + \lambda_n\overrightarrow{OP_n}, \\
\lambda_0 + \cdots + \lambda_n = 1,
\end{array} \right\}
\]

an \( n \)-simplex. The \( n \) is the dimension of the simplex (see Figure 2.4).

Definition 2.23. With a subset \( \{i_k \mid 0 \leq i_k \leq n\}_{k=0}^q \) of an index set \( \{i\}_{i=0}^n \), any set of \( q+1 \) points \( P_{i_0}, P_{i_1}, \ldots, P_{i_q} \), among the vertices \( P_0, P_1, \ldots, P_n \) of an \( n \)-simplex, \( \sigma = |P_0P_1 \ldots P_n| \), are again linearly independent; hence, they define a \( q \)-simplex

\[
\tau = |P_{i_0}P_{i_1} \ldots P_{i_q}|,
\]

called a \( q \)-face of \( \sigma \). If \( \tau \) is a face of \( \sigma \), then we write

\[
\tau \prec \sigma \quad \text{or} \quad \sigma \succ \tau.
\]
\[ \lambda_0 = 1/5, \lambda_1 = 3/5, \lambda_2 = 1/5 \]

**Figure 2.4.** A 2-simplex with an illustration of how it is generated by its points. Faces of 2-dimensional simplices are either a point, a line or the simplex itself.

**Definition 2.24.** A finite set \( K \) of simplices is called a *simplicial complex* if it satisfies the following:

(i) If \( \sigma \in K \) and \( \sigma \succ \tau \), then \( \tau \in K \).

(ii) If \( \sigma, \tau \in K \) and \( \sigma \cap \tau \neq \emptyset \), then \( \sigma \cap \tau \prec \sigma \) and \( \sigma \cap \tau \prec \tau \).

The *dimension* of a simplicial complex \( K \) is the maximum value among the dimensions of the simplices belonging to \( K \), and is denoted by \( \dim K \).

A subcollection of a simplicial complex, containing all faces of its own elements, is called a *simplicial subcomplex* or just a *subcomplex*.

**Definition 2.25.** Let \( K \) be a simplicial complex of dimension \( n \). The union of all simplices belonging to \( K \) is a *polyhedron* of \( K \) denoted by \( |K| \),

\[ |K| = \bigcup_{\sigma \in K} \sigma \subset \mathbb{R}^n. \]
Simplices $\sigma$ are polyhedra $|K|$ of simplicial complexes $K$ consisting of all faces $\tau \prec \sigma$. In this sense, polyhedra are generalizations of simplices, see Figure 2.5.
3. Whitney’s first embedding theorem

We have included a diagram below (Figure 3.1) which gives an overview of this section and how we will reach the proof of Theorem 3.22.

![Diagram](https://via.placeholder.com/150)

**Figure 3.1.** Dependency diagram for the path to the proof of Whitney’s first embedding theorem.

We follow the diagram in Figure 3.1 by proving the three theorems as soon as we can, and by the order in which they are numbered.

**Definition 3.1.** An $n$-cube $C^n(x, r)$ is the Cartesian product of $n$ open intervals $(a, b)$ centered at $x$ and with length $2r$, i.e. such that
\[
\frac{a + b}{2} = x, \quad \|a - b\|_\mathbb{R}^n = 2r.
\]
When $x = 0$ we will omit it, writing $C^n(r)$ for $C^n(0, r)$.

Some measure theory is necessary before we can proceed. To be more specific, it is the concept of Lebesgue measure zero that is central to Lemma 3.5 and its corollaries.

**Definition 3.2.** A *cover* of a topological space $S$ is a collection of sets $\{U_\alpha\}_{\alpha \in A}$ such that $S = \bigcup_{\alpha \in A} U_\alpha$. If all the $U_\alpha$’s are open, it is an *open cover*. A *refinement* of a cover of a topological space is another cover of the same space such that every set in the refinement is a subset of at least one of the sets in the original cover. A cover of a topological space is said to be *locally finite* if every point has a neighbourhood that intersects only a finite number of sets from the cover.

**Definition 3.3.** A topological space $X$ is said to be *paracompact* if every open cover of $X$ has a refinement which is locally finite and open.
Definition 3.4. A subset $A$ of $\mathbb{R}^n$ is said to have measure zero if for an arbitrary $\varepsilon > 0$ there exists an open cover of $A$,

$$A \subset \bigcup_{i=1}^{\infty} C^n(x_i, r_i),$$

such that

$$\sum_{i=1}^{\infty} r_i^n < \varepsilon.$$ 

Lemma 3.5. Let $f : U \rightarrow \mathbb{R}^n$ be a smooth map, with $U$ an open subset of $\mathbb{R}^n$. If $A$ is an open subset of $U$ with measure zero, then $f(A)$ too has measure zero.

Proof. We take an $n$-cube $C$ with $\overline{C} \subset U$ and notice that $A \cap C$ must also have measure zero. Then, for an arbitrary $\varepsilon > 0$, there exists an open covering of $A \cap C$,

$$A \cap C \subset \bigcup_{i=1}^{\infty} C^n(x_i, r_i),$$

such that

$$\sum_{i=1}^{\infty} r_i^n < \varepsilon.$$ 

Furthermore, if

$$b = \max_{x \in \overline{C}, i, j} \left| \left( \frac{\partial f_i}{\partial x_j} \right)_x \right|$$

then by applying the mean value theorem in several variables to each component of $f$, we get

$$\|f(x) - f(y)\|_{\mathbb{R}^n} \leq \sqrt{b^2 + \cdots + b^2} \|x - y\|_{\mathbb{R}^n} = \sqrt{n^2 \text{ terms}} \|x - y\|_{\mathbb{R}^n}, \quad x, y \in \overline{C}.$$

With this we have found that

$$f(C^n(x_i, r_i)) \subset C^n(f(x_i), bn r_i).$$

Thus

$$f(A \cap C) \subset f \left( \bigcup_{i=1}^{\infty} C^n(x_i, r_i) \right) = \bigcup_{i=1}^{\infty} f(C^n(x_i, r_i)) \subset$$

$$\subset \bigcup_{i=1}^{\infty} C^n(f(x_i), bn r_i).$$

This means that $f(A \cap C)$ has measure zero, since

$$\sum_{i=1}^{\infty} 2^n b^n n^n r_i^n = 2^n b^n n^n \sum_{i=1}^{\infty} r_i^n < 2^n b^n n^n \varepsilon.$$
Finally, $A$ can be covered with a countable number of $n$-cubes such as $C$, so the measure of $f(A)$ must be zero. \hfill \Box

**Corollary 3.6.** With $n < p$ let $U$ be an open subset of $\mathbb{R}^n$, and let $f : U \to \mathbb{R}^p$ be a smooth map. Then $f(U)$ has measure zero.

**Proof.** We may think of $U \times \mathbb{R}^{p-n}$ as an open subset of $\mathbb{R}^p$. Hence, we define $g : U \times \mathbb{R}^{p-n} \to \mathbb{R}^p$ by $g = f \circ p_1$, i.e.

$$g : U \times \mathbb{R}^{p-n} \overset{p_1}{\to} U \overset{f}{\to} \mathbb{R}^p,$$

where $p_1$ is the projection into the first component. As a composition of $C^\infty$ maps, $g$ too, is a $C^\infty$ map. But $U \cong U \times \{0\} \subset U \times \mathbb{R}^{p-n}$ has measure zero, and so $f(U) = g(U \times \{0\})$ has measure zero in $\mathbb{R}^p$. \hfill \Box

In Definition 3.4 we defined the concept of measure zero for a subset of Euclidean space, but we must also define it for manifolds.

**Definition 3.7.** Let $M$ be an $n$-dimensional manifold with associated differentiable structure $\mathcal{D}$, and let $A$ be a subset of $M$. We say that $A$ has **measure zero** if $\varphi_A(U_A \cap A) \subset \mathbb{R}^n$ has measure zero for an arbitrary chart $(U_A, \varphi_A)$ of $\mathcal{D}$.

**Corollary 3.8.** Let $V$ and $M$ be manifolds of dimensions $n$ and $m$, with $n < m$. Let $f : V \to M$ be a smooth map. Then $f(V)$ has measure zero in $M$.

**Lemma 3.9.** Let $M(p, n; \mathbb{R})$ be the set of real $(p \times n)$-matrices. Since $M(p, n; \mathbb{R})$ is bijective to $\mathbb{R}^{pn}$, we give the usual topology of $\mathbb{R}^{pn}$ to $M(p, n; \mathbb{R})$; thus $M(p, n; \mathbb{R})$ becomes a differentiable manifold. Let $M(p, n; k) \subset M(p, n; \mathbb{R})$ be the set of all $(p \times n)$-matrices of rank $k$. If $k \leq \min(p, n)$, then $M(p, n; k)$ is a $(p + n - k)$-dimensional submanifold of $M(p, n; \mathbb{R})$.

**Proof.** Let $E_0$ be an element of $M(p, n; k)$. Without loss of generality we may assume that $E_0 = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}$ with $A_0 \in \text{GL}(k, \mathbb{R})$. Then there exists an $\varepsilon > 0$ such that $\det(A) \neq 0$ if $A$ is an element of $\text{GL}(n, \mathbb{R})$ such that the absolute value of each entry of $A - A_0$ is less than $\varepsilon$.

Now let $U \subset M(p, n; \mathbb{R})$ be the set consisting of all $(p \times n)$-matrices of the form $E = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A$ is a $(k \times k)$-matrix such that the absolute value of each entry of $A - A_0$ is less than $\varepsilon$.

Then we have $E \in M(p, n; k)$ if and only if $D = CA^{-1}B$, because the rank of

$$\begin{pmatrix} I_k & 0 \\ X & I_{p-k} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ XA + C & XB + D \end{pmatrix}$$

is equal to the rank of $E$ for an arbitrary $((p - k) \times k)$-matrix $X$. Now setting $X = -CA^{-1}$ we see that the above matrix becomes

$$\begin{pmatrix} A & B \\ 0 & -CA^{-1}B + D \end{pmatrix}.$$
The rank of this matrix is $k$ if $D = CA^{-1}B$. The converse also holds, since if $-CA^{-1}B + D \neq 0$, then the rank of the above matrix becomes greater than $k$.

Let $W$ be an open subset of $\mathbb{R}^m$, where

$$m = pn - \dim(D) = pn - (p - k)(n - k) = k(p + n - k),$$

$$W = \left\{ \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \in M(p, n; \mathbb{R}) \left| \begin{array}{l} \text{The absolute value of each entry of } A - A_0 \text{ is less than } \varepsilon. \end{array} \right. \right\}. $$

Then the correspondence

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \mapsto \begin{bmatrix} A & B \\ C & CA^{-1}B \end{bmatrix}$$

defines a diffeomorphism between $W$ and the neighbourhood $U \cap M(p, n; k)$ of $E_0$ in $M(p, n; k)$. Therefore, $M(p, n; k)$ is a $k(p + n - k)$-submanifold of $M(p, n; \mathbb{R})$. □

**Theorem 3.10.** If $f$ is a smooth map of an open subset $U$ of $\mathbb{R}^n$ into $\mathbb{R}^p$, with $p \geq 2n$, then for an arbitrary $\varepsilon > 0$, there exists a $(p \times n)$-matrix $A = [a_{ij}]$ satisfying the following:

(i) $|a_{ij}| < \varepsilon$.

(ii) The map $g : U \rightarrow \mathbb{R}^p$ defined by $g(x) = f(x) + Ax$ is an immersion.

**Proof.** We must show that the rank of

$$Dg(x) = Df(x) + A$$

is $n$ at every $x \in U$. We do this by showing that the set

$$\{A = B - Df(x) | \text{rank } B < n\}$$

has measure zero in $M(p, n; n)$. To show this we will use Corollary 3.8. Consider the maps $F_k : M(p, n; k) \times U \rightarrow M(p, n; \mathbb{R})$ defined by

$$F_k(Q, x) = Q - Df(x).$$

Assume that $k \leq n - 1$. By Lemma 3.9 we have that the dimension of $M(p, n; k) \times U$ is at most

$$k(p + n - k) + n = (n - 1)(p + n - (n - 1)) + n = (2n - p) + pn - 1$$

since $k(p + n - k) + n$ is monotonically increasing with respect to $k$. Furthermore,

$$(2n - p) + pn - 1 \leq pm = \dim(M(p, n, \mathbb{R})),$$

since $p \geq 2n$. Now, by Corollary 3.8 , the measure of $F_k(M(p, n; k) \times U)$ must be zero in $M(p, n, \mathbb{R})$. Throughout all of this, no assumptions had to be made about the size of the entries of $A$. Thus, there is a matrix $A$ which satisfies (i) and which is not contained in any of the $F_k$’s, $k = 0, 1, \ldots, n - 1$, i.e. which does not have rank $0, 1, \ldots, n - 1$. This is the desired matrix $A$. □
In Lemma 3.12 a certain atlas is defined and proven to exist, which will be used in Lemma 3.18 and Lemma 3.21, Theorem 3.17 and Theorem 4.5. One of the properties of this atlas is that the neighbourhoods constitute a locally finite refinement of an arbitrary open covering of the underlying manifold. Thus, the underlying manifold must be paracompact. In the following, we use \( \text{int} \ A \) to denote the interior of a set \( A \).

**Lemma 3.11.** A locally compact topological space \( X \) with a countable basis is paracompact.

**Proof.** First, take a countable basis \( \{ U_i \} \) of \( X \) with \( U_i \) being compact. We proceed to construct a sequence of compact sets \( \{ A_i \} \) such that

\[
X = \bigcup_{i=1}^{\infty} A_i,
\]

and

\[
A_i \subseteq \text{int} \ A_{i+1}.
\]

The \( A_i \)'s obtain these properties if we set \( A_1 = U_1 \) and

\[
A_{i+1} = (U_1 \cup \cdots \cup U_k) \cup U_{i+1},
\]

where \( k \) is the smallest natural number such that \( A_i \subseteq U_1 \cup \cdots \cup U_k \).

Now, we take an arbitrary open covering \( W = \{ W_j \} \) of \( X \) and show that it has a locally finite refinement, as follows. Since the sets \( A_{i+1} \setminus \text{int} \ A_i \) are compact they can be covered by finitely many of the \( W_j \)'s. Therefore, there exist a finite number of sets \( V_i \), such that

1. \( A_{i+1} \setminus \text{int} \ A_i \subseteq \bigcup_{r=1}^{s} V_i \),
2. \( V_i \subseteq W_j \) for some \( j \),
3. \( V_i \subseteq \text{int} (A_{i+2}) \setminus A_{i-1} \).

(See Figure 3.2.) Setting \( P_i = \{ V_i, \ldots, V_s \} \) and \( \mathcal{P} = P_0 \cup P_1 \cup \ldots \), we see that \( \mathcal{P} \) is a locally finite refinement of the arbitrary covering \( W \) of \( X \). Thus, \( X \) is paracompact. \( \square \)

Our manifolds, which have a countable basis, are locally compact, hence paracompact, and the atlas mentioned previously can be shown to exist.

**Lemma 3.12.** Let \( \{ U_\alpha \} \) be an open covering of an \( n \)-dimensional manifold \( M \). Then \( M \) has an atlas \( \{(V_j, h_j)\} \) satisfying the following:

1. \( \{(V_j, h_j)\} \) is denumerable (countably infinite).
2. \( \{V_j\} \) is a locally finite refinement of \( \{U_\alpha\} \).
3. \( h_j(V_j) = C^n(3) \).
4. \( M = \bigcup_j W_j \), where \( W_j = h_j^{-1}(C^n(1)) \).
Figure 3.2. Some of the $V_3$, which cover $A_4 \setminus \text{int} A_3$ and are contained in $\text{int}(A_5) \setminus A_2$.

Proof. (i), (ii) and (iii) are immediately satisfied as we construct a locally finite refinement $\{V_j\}$ of $\{U_\alpha\}$ such that $h_j(V_j) = C^n(3)$. For condition (iv) we first select a sequence of compact sets as in the proof of Lemma 3.11, $A_1, A_2, \ldots$, such that

$$\mathcal{M} = \bigcup_{i=1}^\infty A_i,$$

and

$$A_i \subset \text{int} A_{i+1}.$$

With the additional requirement that

$$A_{i+1} \setminus \text{int} A_i \subset \bigcup_j h_j^{-1}(C^n(1))$$

including $A_1 \subset \bigcup_j h_j^{-1}(C^n(1))$,

the last condition (iv) is satisfied, as follows. What we want is $\bigcup_j h_j^{-1}(C^n(1)) = M = \bigcup_i A_i$ and since $A_i \subset \text{int} A_{i+1}$ we have

$$\bigcup_i A_{i+1} \setminus \text{int} A_i = \bigcup_i A_i,$$

see Figure 3.3, so

$$M = \bigcup_i A_i = \bigcup_i A_{i+1} \setminus \text{int} A_i \subset \bigcup_j h_j^{-1}(C^n(1)).$$

But

$$\bigcup_j h_j^{-1}(C^n(1)) \subset \bigcup_j h_j^{-1}(C^n(3)) = \bigcup_j V_j = M.$$
so

\[ M \subset \bigcup_{j} h^{-1}_j(C^n(1)) \subset M \]

i.e.

\[ M = \bigcup_{j} h^{-1}_j(C^n(1)) = \bigcup_{j} W_j. \]

\[ \square \]

If we can prove there is a mapping with some desirable properties on just a
neighbourhood of some point in a manifold, then we can extend it to obtain a map
with the same properties on the entire manifold using what is known as a cut-off
function. We simply multiply the mapping with a function that is 1 where the
mapping is defined and has compact support containing that domain. In this essay
the following special case of such a function will suffice.

**Lemma 3.13.** There exists a smooth function \( \varphi : \mathbb{R}^n \to \mathbb{R} \) satisfying the following:

(i) \( \varphi(x) = 1, \ x \in C^n(1) \).

(ii) \( 0 < \varphi(x) < 1, \ x \in C^n(2) \setminus C^n(1) \).

(iii) \( \varphi(x) = 0, \ x \in \mathbb{R}^n \setminus C^n(2) \).

We call \( \varphi \) a cut-off function.

**Proof.** Define a function \( \lambda : \mathbb{R} \to \mathbb{R} \) by

\[ \lambda(x) = \begin{cases} e^{-1/x}, & x > 0, \\ 0, & x \leq 0. \end{cases} \]

and set

\[ \psi(x) = \frac{\lambda(2 + x)\lambda(2 - x)}{\lambda(2 + x)\lambda(2 - x) + \lambda(x - 1) + \lambda(-x - 1)}. \]

Now define \( \varphi \) by

\[ \varphi(x_1, \ldots, x_n) = \prod_{i=1}^{n} \psi(x_i). \]

\[ \square \]
**Definition 3.14.** Let \( f_0 : X \to Y \) and \( f_1 : X \to Y \) be smooth maps. A homotopy from \( f_0 \) to \( f_1 \) is a smooth map \( F : X \times [0, 1] \to Y \) such that
\[
F(x, 0) = f_0(x), \\
F(x, 1) = f_1(x)
\]
for all \( x \). If there exists such a smooth homotopy, then we say that \( f_0 \) is a deformation of \( f_1 \) or that \( f_0 \) is homotopic to \( f_1 \) and write \( f_0 \simeq f_1 \). If \( A \subset X \) is a closed subset and if \( F(a, t) = f_0(a) = f_1(a) \) for all \( a \in A \) and all \( t \in [0, 1] \) then we say that \( f_0 \) is homotopic to \( f_1 \) relative to \( A \) and we write \( f_0 \simeq f_1 \) (rel \( A \)). We sometimes write \( f_t(x) = F(x, t) \) and call the \( \{f_t\} \) a homotopy.

**Definition 3.15.** If the elements \( f_t \) of a homotopy \( \{f_t\} \) are immersions for all \( t \in [0, 1] \) then we say that it is a regular homotopy and that \( f_0 \) is regularly homotopic to \( f_1 \) and write \( f_0 \simeq_{\text{reg}} f_1 \). To be regularly homotopic relative to some closed subset \( A \subset X \) is defined in the obvious way.

** Remark.** In some texts, regular is the same as full rank. Then immersions are regular maps.

We denote the set of all strictly positive real numbers by \( \mathbb{R}_+ \).

**Definition 3.16.** Let \( X \) be a topological space and \((Y, d)\) be a metric space with metric \( d \), and let \( \delta : X \to \mathbb{R}_+ \) be a continuous function. We say that \( g : X \to Y \) is a \( \delta \)-approximation of \( f : X \to Y \) if
\[
d(f(x), g(x)) < \delta(x) \quad \text{for all} \ x \in X.
\]

Now, given a smooth map \( f : M \to \mathbb{R}^p \) (\( p \geq 2n \), \( n \) as the dimension of \( M \), we prove the existence of an immersion as a \( \delta \)-approximation of \( f \). Later we will prove the existence of an embedding, given an immersion \( M \to \mathbb{R}^p \) (\( p \geq 2n + 1 \)). Thus, given a smooth map \( M \to \mathbb{R}^p \) (\( p \geq 2n + 1 \)) we prove the existence of an embedding \( M \to \mathbb{R}^p \) (\( p \geq 2n + 1 \)), i.e. Whitney’s first embedding theorem, which in turn is used in the proof of his second.

**Theorem 3.17.** Let \( M \) be an \( n \)-dimensional manifold, and let \( f : M \to \mathbb{R}^p \) be a smooth map, with \( p \geq 2n \). There exists an immersion \( g : M \to \mathbb{R}^p \) which is a \( \delta \)-approximation of \( f \). In addition, if the rank of \( f \) is \( n \) on a closed subset \( N \) of \( M \), then we may choose \( g \) such that \( g \) is homotopic to \( f \) relative to \( N \).

**Proof.** Assume the rank of \( f \) is \( n \) on \( N \), that is, the rank of \( f \) is \( n \) on some open neighbourhood \( U \) of \( N \). The family \( \{U, M \setminus N\} \) is an open cover of \( M \). We select an atlas \( \mathcal{A}_0 = \{(V_j, h_j)\} \) according to Lemma 3.12, which is a locally finite refinement of \( \{U, M \setminus N\} \) with \( h_i(V_i) = C^n(3) \) and \( W_i = h_i^{-1}(C^n(1)) \). We next set \( U_i = h_i^{-1}(C^n(2)) \) and re-index the \( \{(V_i, h_i)\} \) so that
\[
i \leq 0 \quad \text{if and only if} \quad V_i \subset U, \\
i > 0 \quad \text{if and only if} \quad V_i \subset M \setminus N.
\]
Since \( \overline{U}_i \) is compact, we can set
\[
\varepsilon_i = \min_{x \in \overline{U}_i} \delta(x).
\]
Now we construct the desired $g$ by induction.

**Induction basis.** Set $f_0 = f$; then the rank of $f_0$ is $n$ on $U$, and so it is $n$ on $\bigcup_{j \leq 0} W_j$ since $W_j \subset V_j \subset U$ for $j \leq 0$.

**Induction hypothesis.** Assume next that $f_{k-1} : M \to \mathbb{R}^p$ is a smooth map having rank $n$ on $N_{k-1} = \bigcup_{j < k} W_j$.

**Induction step.** We now construct as a $\delta/2^k$-approximation of $f_{k-1}$, a smooth map $f_k : M \to \mathbb{R}^p$ whose rank is $n$ on $N_k$. Consider the map $f_{k-1} \circ h_k^{-1} : C^n(3) \to \mathbb{R}^p$ and the cut-off function $\varphi : \mathbb{R}^n \to \mathbb{R}$ of Lemma 3.13. Choose a $(p \times n)$-matrix $A$ in such a way that if we define $F_A : C^n(3) \to \mathbb{R}^p$ by

$$F_A(x) = f_{k-1} \circ h_k^{-1}(x) + \varphi(x)Ax,$$

the following conditions (i), (ii), and (iii) are met.

(i) $F_A$ is of rank $n$ on $K = h_k(N_{k-1} \cap U_k)$. Since $h_k^{-1}(K) = N_{k-1} \cap U_k \subset N_{k-1}$, the rank of $f_{k-1} \circ h_k^{-1}$ is $n$ on $K$ by assumption. The Jacobian matrix of $F_A$ is

$$DF_A(x) = D(f_{k-1} \circ h_k^{-1}(x)) + D(\varphi(x))Ax + \varphi(x)A.$$

The map

$$\Phi : K \times M(p,n;\mathbb{R}) \to M(p,n;\mathbb{R})$$

which assigns $DF_A(x)$ to $(x,A)$ is continuous and the subset $M(p,n;n)$ consisting of matrices of rank $n$ is open in $M(p,n;\mathbb{R})$. But since

$$\Phi(K,0_{pn}) = \{D(f_{k-1} \circ h_k^{-1}(x))|x \in K\} \subset M(p,n;n)$$

we have $\Phi(K,A) \subset M(p,n;n)$ for a small enough $A$, by which we mean that the largest entry of $A$ should be small enough.

(ii) The desired $A$ should also be sufficiently small that

$$\|Ax\|_{\mathbb{R}^n} \leq \frac{\varepsilon_k}{2^k}, \quad x \in C^n(3).$$

(iii) Finally, by Theorem 3.10 we may take $A$ so small that $f_{k-1} \circ h_k^{-1}(x) + Ax$ has rank $n$ on $C^n(2)$.

Having chosen the desired $A$ as above, we define for each $k$ a map $f_k : M \to \mathbb{R}^p$ by

$$f_k(x) = \begin{cases} f_{k-1}(x) + \varphi(h_k(x))Ah_k(x) & \text{if } x \in V_k, \\ f_{k-1}(x) & \text{if } x \in M \setminus U_k. \end{cases}$$

The map $f_k$ is well defined, i.e. for a point $x \in V_k \cap (M \setminus U_k) = V_k \setminus U_k$, we have

$$f_{k-1}(x) + \varphi(h_k(x))Ah_k(x) = f_{k-1}(x).$$

The rank of $f_k$ is by (i) $n$ on $N_{k-1}$ and by (iii) $n$ on $U_k$. Therefore, the rank of $f_k$ is $n$ on $N_k = \bigcup_{j < k+1} W_j$. Furthermore, by (ii) the map $f_k$ is a $\delta/2^k$-approximation of $f_{k-1}$. 

21
Define \( g : M \to \mathbb{R}^p \) by \( g(x) = \lim_{k \to \infty} f_k(x) \). This means the following. Recall that

\[ M = \bigcup_i W_i, \]

and

\[ N_0 \subset N_1 \subset N_2 \subset \ldots, \quad N_{k-1} = \bigcup_{j<k} W_j, \]

so

\[ M = \bigcup_k N_k. \]

For an arbitrary point \( x \) of \( M \), since the \( \{V_i\} \) is locally finite, there exists a neighbourhood \( U(x) \) of \( x \) such that \( U(x) \cap V_j \neq \emptyset \) only for a finite number of \( j \)'s, the largest among which we call \( k \); then

\[ g(x) = f_k(x) = f_{k+1}(x) = \ldots, \]

by the definition of the \( f_k \), (3.1).

Clearly, the map \( g \) is smooth and of rank \( n \) on \( \bigcup_k N_k = M \). Moreover, \( g \) is a \( \delta \)-approximation of \( f \). The construction of \( f_k \) implies immediately that \( f_k \) is homotopic to \( f_{k-1} \) while \( N \) is kept fixed. Hence, \( f \) is homotopic to \( g \) with \( N \) fixed. \( \square \)

Proving the existence of an embedding, given an immersion, has been divided into the three lemmas Lemma 3.18, Lemma 3.20 and Lemma 3.21, that remain before Whitney’s first embedding theorem, Theorem 3.22. First we prove the existence of an injective immersion as an approximation of a given immersion \( M \to \mathbb{R}^p \) \((p \geq 2n+1)\), which is Lemma 3.18. Then, Lemma 3.20 tells us what requirements an injective immersion has to fulfill in order to be an embedding. Lastly, in Lemma 3.21, we show that our injective immersion actually fulfills these requirements.

**Lemma 3.18.** Let \( f : M \to \mathbb{R}^p \) be an immersion of an \( n \)-dimensional manifold \( M \), with \( 2n < p \). There exists an injective immersion \( g \) which is a \( \delta \)-approximation of \( f \). Furthermore, if \( f \) is injective in an open neighbourhood \( U \) of a closed subset \( N \) of \( M \), we can choose \( g \) such that \( g \) is regularly homotopic to \( f \) relative to \( N \).

**Proof.** On one hand, because \( f \) is an immersion, there exists an open covering \( \{U_\alpha\} \) of \( M \) such that \( f|_{U_\alpha} \) is an embedding for each \( \alpha \). On the other hand, the set \( \{U, M\setminus N\} \) is another covering of \( M \). Together these coverings form a third covering with respect to which we consider an atlas \( \{(V_j,h_j)\} \) as given in Lemma 3.12. Furthermore, using the cut-off function \( \varphi : \mathbb{R}^n \to \mathbb{R} \) of Lemma 3.13, we define \( \varphi_j : M \to \mathbb{R} \) by

\[
\varphi_j(y) = \begin{cases} \varphi \circ h_j(y) & \text{if } y \in V_j, \\ 0 & \text{if } y \notin V_j. \end{cases}
\]  

(3.2)
Then it follows immediately that \( \varphi_j \) is a smooth function. For the sake of convenience we take
\[ V_i \subset U \quad \text{if and only if} \quad j \leq 0. \]

Now we define the desired \( g \) by induction.

**Induction basis.** First put \( f_0 = f \).

**Induction hypothesis.** Assuming next that we have an immersion \( f_{k-1} : M \to \mathbb{R}^p \) which is injective on \( \bigcup_{j<k} V_j \), we define \( f_k : M \to \mathbb{R}^p \) by
\[ f_k(x) = f_{k-1}(x) + \varphi_k(x)b_k, \] (3.3)
where \( b_k \) is a point of \( \mathbb{R}^p \), which we select as follows:

(i) \( b_k \) is small enough to make \( f_k \) an immersion.

(ii) \( b_k \) is small enough for \( f_k \) to be a \( \delta/2^k \)-approximation of \( f_{k-1} \).

(iii) Let \( N \) be a \( 2n \)-dimensional subset of \( M \times M \),
\[ N = \{(x,y) \in M \times M | \varphi_k(x) \neq \varphi_k(y)\}. \]
Evidently \( N \) is open in \( M \times M \).

**Induction step.** Now consider the following smooth map \( \Phi : N \to \mathbb{R}^p \),
\[ \Phi(x,y) = \frac{- (f_{k-1}(x) - f_{k-1}(y))}{\varphi_k(x) - \varphi_k(y)}. \]
By assumption \( 2n < p \), and so the measure of \( \Phi(N) \) in \( \mathbb{R}^p \) is zero. Hence, we want \( b_k \) to be outside \( \Phi(N) \). If \( b_k \) is small enough to satisfy the conditions (i) and (ii) it will work for the condition (iii). It is by Theorem 3.10 that condition (i) can be satisfied.

The map \( f_k \) satisfies
\[ f_k(x) - f_k(y) = 0 \quad \text{if and only if} \quad \begin{cases} 
\varphi_k(x) - \varphi_k(y) = 0, \\
 f_{k-1}(x) - f_{k-1}(y) = 0.
\end{cases} \] (3.4)
The ‘if part’ follows directly from the definition (3.3) of the \( f_k \),
\[ f_k(x) - f_k(y) = (f_{k-1}(x) - f_{k-1}(y)) + (\varphi_k(x) - \varphi_k(y))b_k = 0. \]
For the converse, assume that
\[ 0 = f_k(x) - f_k(y) = (f_{k-1}(x) - f_{k-1}(y)) + (\varphi_k(x) - \varphi_k(y))b_k. \]
Then if \( \varphi_k(x) - \varphi_k(y) \neq 0 \) we have
\[ b_k = \frac{-(f_{k-1}(x) - f_{k-1}(y))}{\varphi_k(x) - \varphi_k(y)} \in \Phi(N), \]
a contradiction. Hence, we must have \( \varphi_k(x) - \varphi_k(y) = 0 \). Therefore, we also get
\[ f_{k-1}(x) - f_{k-1}(y) = 0. \]
Define \( g : M \to \mathbb{R}^p \) by
\[ g(x) = \lim_{k \to \infty} f_k(x). \]
This means the following. We have

\[ \varphi_k(x) = \begin{cases} \varphi \circ h_k(x) & \text{if } x \in V_k, \\ 0 & \text{if } x \in V_k^c, \end{cases} \]

and, therefore,

\[ f_k(x) = \begin{cases} f_{k-1}(x) + \varphi \circ h_k(x), & \text{if } x \in V_k, \\ f_{k-1}(x), & \text{if } x \in V_k^c. \end{cases} \]

Since \( \{V_j\} \) is locally finite, any given \( x \in M \) lies only in a finite number of \( V_j \)'s. For all other \( k, x \in V_k^c \) so that \( f_{k-1}(x) = f_k(x) \). Assume, therefore, that \( k_0 = k_0(x) \) is the largest \( k \) such that \( x \in V_k \). Then \( f_k(x) = f_{k_0(x)}(x) \) for all \( k \geq k_0(x) \) so that \( g(x) = f_{k_0(x)}(x) \).

The definition of \( g(x) \) readily implies that \( g \) is a smooth map which is an immersion as well. Since \( k > 0 \) we also have that \( g|_N = f|_N \) so that \( V_k \cap N = \emptyset \). It remains to show that \( g \) is injective. Assume now that \( g(x) = g(x_0) \) and \( x \neq x_0 \). Then from (3.4) we have

\[ f_{k-1}(x) = f_{k-1}(x_0), \quad \varphi_k(x) = \varphi_k(x_0) \quad \text{for all integers } k > 0. \]

From the former equation we get \( f(x) = f(x_0) \). Therefore, \( x \) and \( x_0 \) do not belong to the same \( V_j \). But from the latter equation we see that if \( x \in V_k \) for \( k > 0 \), then \( x_0 \) must also be in \( V_k \), which cannot happen; hence, both \( x \) and \( x_0 \) must be in \( U \) (the \( V_j \) were re-indexed this way). This is a contradiction as \( f \) is injective on \( U \).

Finally, a close look at the definition of \( f_k \) reveals that \( f_k \) and \( f_{k-1} \) are regularly homotopic relative to \( N \), and hence the same is true for \( f \) and \( g \).

\[ \square \]

**Definition 3.19.** Let \( M \) be a manifold and let \( f : M \to \mathbb{R}^p \) be a continuous map. We say that the set

\[ L(f) = \left\{ y \in \mathbb{R}^p \mid y = \lim_{n \to \infty} f(x_n) \text{ for some sequence } \{x_n\}_{n=1}^\infty \subset M \text{ which has no limit point in } M. \right\} \]

is the limit set of \( f \).

**Lemma 3.20.** If \( f : M \to \mathbb{R}^n \) is an injective immersion of the \( n \)-dimensional manifold \( M \) then

(i) \( f(M) \) is a closed subset of \( \mathbb{R}^n \) if and only if \( L(f) \subset f(M) \), and

(ii) \( f \) is an embedding if and only if \( L(f) \cap f(M) = \emptyset \).

**Proof.** Let \( \{x_n\}_{n=1}^\infty \) be a sequence in \( M \), that does not converge in \( M \). If \( f(M) \) is closed then \( \lim_{n \to \infty} f(x_n) \in f(M) \).

Now, assume that \( L(f) \subset f(M) \) and assume that \( y \in f(M) \) and define a sequence \( \{x_n\}_{n=1}^\infty \) in \( M \) such that \( f(x_n) \in C^n(y, 1/n) \). Then

\[ f(\lim_{n \to \infty} x_n) = \lim_{n \to \infty} f(x_n) = y. \]
If \( \lim_{n \to \infty} x_n \in f(M) \) then \( y \in f(M) \). Otherwise, \( y \in L(f) \subset f(M) \). Thus, \( f(M) = f(M) \).

Now for (ii), assume that \( \{x_n\}_{n=1}^{\infty} \) is a sequence in \( M \) without limit point in \( M \). It is true for all immersions \( f \) of \( M \) that \( L(f) \cap f(M) = \emptyset \). To see this, assume that \( L(f) \cap f(M) \neq \emptyset \) and that \( y \in L(f) \cap f(M) \). Then
\[
y = \lim_{n \to \infty} f(x_n),
\]
because \( y \in L(f) \). Furthermore, since \( f \) is continuous, we have
\[
f(\lim_{n \to \infty} x_n) = \lim_{n \to \infty} f(x_n).
\]
Therefore
\[
f(\lim_{n \to \infty} x_n) = y
\]
but \( y \in f(M) \) so this would imply that \( \lim_{n \to \infty} x_n \in M \), which is a contradiction.

Lastly, for the injective immersion \( f \) to be an embedding we only need its inverse to be continuous. So, assume that its inverse \( f^{-1} : f(M) \to M \) is not continuous. Then there exists a point \( x \in M \) such that \( f(x) \in L(f) \). But this contradicts \( f(M) \cap L(f) = \emptyset \).

\( \square \)

Remark. Note that for injective immersions \( f : M \to \mathbb{R}^n \) to be embeddings with closed images \( f(M) \subset \mathbb{R}^n \), we need both \( L(f) \subset f(M) \) and \( L(f) \cap f(M) = \emptyset \), which is only possible if \( L(f) = \emptyset \). Hence, Lemma 3.21.

**Lemma 3.21.** Let \( M \) be an \( n \)-dimensional manifold. Then there exists a smooth map \( f : M \to \mathbb{R}^p \) with \( L(f) = \emptyset \).

**Proof.** Let \( \{(V_j, h_j)\} \) be the atlas of Lemma 3.12 for \( M \), and let \( \varphi \) be the smooth function of Lemma 3.13. For each \( j > 0 \) let \( \varphi_j : M \to \mathbb{R} \) be the smooth function given in the proof of Lemma 3.18, equation (3.2). Set
\[
f(x) = \sum_{j>0} j \varphi_j(x).
\]
The right-hand side of this equation makes sense because the \( \{V_j\} \) is locally finite. It follows that \( f \) is continuous.

We want to show that \( L(f) = \emptyset \). Let \( \{x_n\}_{n=1}^{\infty} \) be a sequence in \( M \), which has no limit point. Then for any integer \( m > 0 \) there exists an integer \( n > 0 \) with \( x_n \notin W_1 \cup \cdots \cup W_m \), and so \( x_n \in W_j \) for some \( j > m \). Therefore, \( f(x_n) > m \) and the sequence \( \{f(x_n)\}_{n=1}^{\infty} \) has no limit point. \( \square \)

Observe that we have only showed the existence of a smooth map with empty limit set, and not an injective immersion with empty limit set. It is for this reason alone that the immersion of Theorem 3.17 and the injective immersion of Lemma 3.18 are approximations of the given smooth map and immersion, respectively. That way, the injective immersion is an approximation of the smooth map and so has empty limit set too. To see this, take a smooth function \( g : M \to \mathbb{R}^{2n+1} \) with \( L(g) = \emptyset \) and an injective immersion \( f : M \to \mathbb{R}^{2n+1} \) that is a \( \delta \)-approximation.
of \( g \). Assume that \( \{x_k\}_{k=1}^\infty \) is a sequence in \( M \) without limit point in \( M \). Assume that \( y = \lim_{k \to \infty} f(x_k) \) exists. There is an \( \varepsilon > 0 \) and a \( k \geq N \) such that
\[
\|g(x_k) - y\|_{\mathbb{R}^{2n+1}} > \varepsilon
\]
for all \( N \geq 1 \). Then, we can choose the continuous map \( \delta : M \to \mathbb{R}_+ \) as a constant map \( \delta = \mu < \varepsilon \), which would mean that \( \lim_{k \to \infty} f(x_k) \neq y \).

We now have all that is required for the proof of the first embedding theorem.

**Theorem 3.22** (Whitney’s first embedding theorem). Every \( n \)-dimensional differentiable manifold can be embedded in \( \mathbb{R}^{2n+1} \) as a closed subset.

**Proof.** Let \( M \) be an \( n \)-dimensional manifold. Let \( f : M \to \mathbb{R}^{2n+1} \) be a smooth map with \( L(f) = \emptyset \). Let \( \delta : M \to \mathbb{R}_+ \) be the constant map \( \delta(x) = \varepsilon > 0 \). Then by Theorem 3.17 there exists an immersion \( g : M \to \mathbb{R}^{2n+1} \) which is a \( \delta \)-approximation of \( f \). Furthermore, by Lemma 3.18 there exists an injective immersion \( h : M \to \mathbb{R}^{2n+1} \) which is a \( \delta \)-approximation of \( g \). By the argument proceeding the proof, there is \( \varepsilon > 0 \) so small that \( L(h) = \emptyset \). Hence, Lemma 3.20 implies that \( h \) is an embedding, and so \( h(M) \) is a closed subset of \( \mathbb{R}^{2n+1} \).

\[ \square \]
4. Whitney’s second embedding theorem

In this section we prove the main theorem of this essay, Whitney’s second embedding theorem, Theorem 4.15. Just like for the first embedding theorem, we have made a diagram illustrating the role of each lemma and theorem prior to Theorem 4.15 (Figure 4.1).

Theorem 4.15: The second embedding theorem.

Theorem 3.17 Theorem 4.14

Theorem 4.5 Theorem 4.15: The second embedding theorem.

Lemma 4.12 Lemma 4.13

Figure 4.1. Dependency diagram for the path to the proof of Whitney’s second embedding theorem.

We begin by defining tangent vector, tangent space and tangent bundle.

**Definition 4.1.** For an \( n \)-dimensional manifold \( M \), a **tangent vector** \( v_p \) at \( p \in M \) is a linear map \( v_p : C^\infty(M) \to \mathbb{R} \) with the property that for \( f, g \in C^\infty(M) \),

\[
v_p(fg) = g(p)v_p(f) + f(p)v_p(g).
\]

The tangent space \( T_pM \) at \( p \in M \) is the set of all tangent vectors \( v_p \) at \( p \in M \). The **tangent bundle** of \( M \) is the disjoint union of all tangent spaces:

\[
TM = \bigcup_{p \in M} T_pM.
\]

**Definition 4.2.** Let \( M \) and \( V \) be manifolds of dimensions \( n \) and \( p \), and let \( f : M \to V \) be a smooth map. A point \( y \) in \( V \) is a **regular value** of \( f \) if the rank of \( f \) at each point \( x \) in \( f^{-1}(y) \) is \( p \); otherwise, \( y \) is a **critical value**.
**Definition 4.3.** Let $M$ be an $n$-dimensional manifold and let $f : M \to \mathbb{R}^{2n}$ be an immersion. We say that $f$ is *completely regular* if it satisfies the following:

(i) $f$ has no triple points, i.e. no points

$$f(p_1) = f(p_2) = f(p_3),$$

with

$$p_1 \neq p_2 \neq p_3.$$  

(ii) For $p_1$ and $p_2$, with $p_1 \neq p_2$ and $f(p_1) = f(p_2) = q$,

$$(df)_{p_1}(T_{p_1}M) \oplus (df)_{p_2}(T_{p_2}M) = T_q\mathbb{R}^{2n},$$  \hspace{1cm} (4.1)

where the differentials are defined as

$$(df)_p(v) = Df(p) \cdot v, \quad v \in M.$$

We say that $f$ *intersects transversely* at $q$ or that $q$ is a regular self-intersection of $f$, if $f$ satisfies (ii).

We shall immediately rewrite (4.1) into a form better suited for explicit calculations, by which we shall prove the transversality of a certain parametrization (4.3). The direct sum of a finite number of vector spaces coincides with the Cartesian product. Thus,

$$(df)_{p_1}(T_{p_1}M) \oplus (df)_{p_2}(T_{p_2}M) =$$

$$(Df(p_1) \cdot v_1, Df(p_2) \cdot v_2) \mid v_1 \in T_{p_1}M, v_2 \in T_{p_2}M =$$

$$= \left\{ \begin{bmatrix} Df(p_1) \ Df(p_2) \end{bmatrix} (v_1, v_2)^T \mid (v_1, v_2) \in T_{p_1}M \times T_{p_2}M \right\}.$$

Since $T_q\mathbb{R}^{2n} = \mathbb{R}^{2n}$ for all $q \in \mathbb{R}^{2n}$, equation (4.1) becomes

$$\left\{ \begin{bmatrix} Df(p_1) \ Df(p_2) \end{bmatrix} v \mid v \in T_{p_1}M \times T_{p_2}M \right\} = \mathbb{R}^{2n}.$$

**Remark.** With these rewritings we see that the dimension of the image must be twice that of the domain for there to exist regular self-intersections: For the column vectors of the above matrix to span $\mathbb{R}^{2n}$ there must be $2n$ of them ($n$ for each Jacobian matrix) and they must have $2n$ components. The only other way for a mapping to be completely regular is for it to have no self-intersections.

We prove the second embedding theorem by first proving the existence of a completely regular immersion of an $n$-dimensional manifold $M$ to $\mathbb{R}^{2n}$, Theorem 4.5. In Theorem 4.9, we show how to modify a completely regular immersion so as to introduce new regular self-intersections. With this we can ensure that there are just as many regular self-intersections of *positive type* as of *negative type* (see Definition 4.8). Then, with Theorem 4.14, we can remove pairs of regular self-intersections of opposite types. Thus, we will obtain a completely regular immersion.
with no self-intersections, i.e. an injective immersion which is the same as an embedding \( M \to \mathbb{R}^{2n} \) when \( M \) is compact.

To prove the existence of a completely regular immersion we shall use Sard’s theorem which we state here as Theorem 4.4. A proof can be found in [11].

**Theorem 4.4.** Let \( f : U \to \mathbb{R}^p \) be a smooth map, defined on an open set \( U \subset \mathbb{R}^n \), and let

\[
C = \{ x \in U \mid \text{rank } df_x < p \}.
\]

Then the image \( f(C) \) has measure zero in \( \mathbb{R}^p \).

**Theorem 4.5.** Let \( M \) be an \( n \)-dimensional manifold, and let \( f : M \to \mathbb{R}^{2n} \) be a smooth map. There exists a completely regular immersion \( g : M \to \mathbb{R}^{2n} \) which is a \( \delta \)-approximation of \( f \). Furthermore, if \( f \) is an immersion that is completely regular on some open neighbourhood \( U \) of a compact set \( N \), then \( g \) may be chosen to be regularly homotopic to \( f \) relative to \( N \).

**Proof.** By Theorem 3.17 there exists an immersion \( \bar{f} \) which is a \( \delta/2 \)-approximation of \( f \) with \( \bar{f}|_N = f|_N \). The map \( \bar{f} \) is completely regular on some open neighbourhood of \( N \). Now, select an atlas of \( \mathbb{R}^{2n} \), \( \mathcal{S}' = \{(C^{2n}(x_i, 1), \psi_i) \mid i = 1, 2, \ldots\} \), where \( \psi_i : C^{2n}(x_i, 1) \to \mathbb{C}^{2n}(1) \). The family

\[
\mathcal{U} = \{ U, (M \setminus N) \cap \bar{f}^{-1}(C^{2n}(x_i, 1)) \mid i = 1, 2, \ldots \}
\]

is an open covering of \( M \). We take an atlas \( \{(V_j, h_j)\} \) of \( M \) given by Lemma 3.12 with respect to \( \mathcal{U} \), such that for each \( V_j \) the following conditions hold.

(i) \( \bar{f}|_{V_j} : V_j \to \mathbb{R}^{2n} \) is an embedding.

(ii) For some \( \lambda_j \) such that \( f(V_j) \subset C^{2n}(x_{\lambda_j}, 1) \), there exists a diffeomorphism

\[
\varphi_j : C^{2n}(1) \to \mathbb{R}^{2n}
\]

with

\[
\varphi_j \circ \psi_{\lambda_j} \circ \bar{f}(V_j) \subset C^{2n}(1) \cap \mathbb{R}^n.
\]

(We think of \( \mathbb{R}^n \) as \( \mathbb{R}^n = \{(x_1, \ldots, x_n, 0, \ldots, 0) \in \mathbb{R}^{2n} \} \subset \mathbb{R}^{2n} \).) (See Figure 4.2.)

Re-indexing the \( \{(V_i, h_i)\}'s \), as we did in the proof of Theorem 3.17, we assume

\[
i \leq 0 \quad \text{if and only if } V_i \subset U,
\]

\[
i > 0 \quad \text{if and only if } V_i \subset M \setminus N.
\]

We write \( \varphi'_j = \varphi_j \circ \psi_{\lambda_j} \).

We shall construct \( g \) inductively.

**Induction basis.** Set \( g_0 = \bar{f} \).

**Induction hypothesis.** Assume that we have defined \( g_j : M \to \mathbb{R}^{2n} \) with \( g_j |_N = \bar{f}|_N \).

(a) Replacing the \( \varphi_j \) we may assume that \( g_j \) satisfies the conditions (i) and (ii) in place of \( \bar{f} \).
\[ \varphi_j \circ \psi_{\lambda_j} \circ \tilde{f}(V_j) \subset C^{2n}(1) \cap \mathbb{R}^n \]

**Figure 4.2.** Mappings of Theorem 4.5.

(b) We assume that if \( N_j = N \cup_{i \leq j} W_i \), then for any point \( p \) of \( g_j(N_j) \) the set \( (g_j|_{N_j})^{-1}(p) \) contains at most two points, since \( g|_N \) is completely regular, and in case it contains two points \( g_j|_{N_j} \) intersects transversely at \( p \).

**Induction step.** Now we construct \( g_{j+1} : M \to \mathbb{R}^{2n} \). Consider the map

\[ \varphi'_{j+1} \circ g_j \circ (h_{j+1})^{-1} : C^n(3) \to C^{2n}(1). \]

Define a projection \( \pi : \mathbb{R}^{2n} \to \mathbb{R}^n \) by \( \pi(x_1, \ldots, x_{2n}) = (x_{n+1}, \ldots, x_{2n}) \). Then by the choice of \( \varphi'_{j+1} \) we have

\[ \varphi'_{j+1} \circ g_j \circ (h_{j+1})^{-1}(C^n(3)) \subset \pi^{-1}(0, \ldots, 0) = \{(x_1, \ldots, x_n, 0, \ldots, 0) \in \mathbb{R}^{2n}\}. \]

And by Sard’s theorem the set of critical values of the smooth map

\[ \pi \circ \varphi'_{j+1} \circ g_j \circ (U \cup_{i \leq j} W_i) \cap g_j^{-1}(U_{\lambda_j}) \to C^n(1) \subset \mathbb{R}^n \]

has measure zero. Hence, we may select a point \( c_q \in \mathbb{R}^n \) satisfying the following:

1. \( c_q \) is sufficiently close to \((0, \ldots, 0)\).
2. \( c_q \) is a regular value of \( \pi \circ \varphi'_{j+1} \circ g_j \).
3. If the points \( p_1 \) and \( p_2, p_1 \neq p_2 \), satisfy \( g_j(p_1) = g_j(p_2) \in C^{2n}(x_{\lambda_j}, 1) \), then

\[ \varphi'_{j+1} \circ g_j(p_1) = \varphi'_{j+1} \circ g_j(p_2) \notin \pi^{-1}(c_q). \]
Using the \( c_q \), we define a smooth map \( g_{j+1} : M \to \mathbb{R}^{2n} \) by

\[
g_{j+1}(x) = \begin{cases} 
g_j(x) & \text{if } x \in M \setminus h^{-1}_{j+1}(C^n(2)), \\
(\varphi'_{j+1})^{-1}\{\varphi'_{j+1} \circ g_j(x) + c_q\varphi(h_{j+1}(x))\} & \text{if } x \in V_j,
\end{cases}
\]

where \( \varphi \) is our cut-off function. Then the conditions (1), (2), (3) for \( c_q \) guarantee that \( g_{j+1} \) is an immersion and that \( g_{j+1} \) is a \( \delta/2^{j+1} \)-approximation of \( g_j \). Furthermore setting \( N'_{j+1} = N_j' \cup \overline{W}_j \) and noting that \( N'_j \) is compact, we see that for any point \( p \) of \( g_{j+1}(N'_j) \), the set \( (g_{j+1}\mid_{N'_j})^{-1}(p) \) contains at most two points, and if it contains two points, then \( g_{j+1} \) intersects transversely at \( p \). It is also routine that \( g_{j+1} \mid_{N} = \bar{f} \mid_{N} \). In addition, by readjusting the \( \varphi_j \) we see that \( g_{j+1} \) replacing \( \bar{f} \) satisfies the conditions (i) and (ii).

Finally, by examining the definition of \( g_{j+1} \) carefully we see that \( g_{j+1} \) and \( g_j \) are regularly homotopic relative to \( N \).

Defining a map \( g : M \to \mathbb{R}^{2n} \) by

\[
g(x) = \lim_{i \to \infty} g_i(x),
\]

we have what we wanted. \( \square \)

In order to introduce additional regular self-intersections to a given mapping \( M \to \mathbb{R}^{2n} \) we need to construct a model of an immersion with a single regular self-intersection. The map \( \alpha : \mathbb{R} \to \mathbb{R}^2 \) defined by

\[
y = x - \frac{x}{(1 + x^2)}, \quad z = \frac{1}{1 + x^2}
\]

is a completely regular immersion, with one regular self-intersection at \((0, 1/2)\), see Figure 4.3. Furthermore, there exists \( r > 0 \) so large that \( \alpha \) is the identity map outside the unity disk \( D(0, r) \),

\[
y = x - \frac{x}{(1 + x^2)} \quad \xrightarrow{|x| \to \infty} \quad x,
\]

\[
z = \frac{1}{1 + x^2} \quad \xrightarrow{|x| \to \infty} \quad 0
\]

so that, for large enough \( x \), we have approximately

\[
x \mapsto (x, 0).
\]

For \( n \geq 2 \), \( \alpha : \mathbb{R}^n \to \mathbb{R}^{2n} \) is defined as follows,

\[
\alpha(x_1, \ldots, x_n) = (y_1, \ldots, y_{2n}), \quad (x_1, \ldots, x_n) \in \mathbb{R}^n,
\]

with

\[
y_1 = x_1 - \frac{2x_1}{u}, \quad y_i = x_i, \quad i = 2, \ldots, n,
\]

and

\[
y_{n+1} = \frac{1}{u}, \quad y_{n+i} = \frac{x_1x_i}{u}, \quad i = 2, \ldots, n.
\]
Figure 4.3. The completely regular immersion $\alpha : \mathbb{R}^1 \to \mathbb{R}^2$ defined by (4.2) with a single (regular) self-intersection at $(0, 1/2)$.

where

$$u = (1 + x_1^2) \cdots (1 + x_n^2).$$

The rank of a mapping is independent of the choice of coordinate maps for the charts in the definition of rank. Since we are mapping $\mathbb{R}^n$ into $\mathbb{R}^{2n}$ we can choose the identity maps as coordinate maps. With this, the rank of $\alpha$ is simply the rank of its Jacobian matrix $(D\alpha)(x)$ which is

$$\begin{bmatrix}
1 & \frac{-2x_1}{u(1+x_1^2)} & \cdots & \frac{4x_1}{u(1+x_n^2)} \\
0 & 1 & 0 & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\frac{-2x_n}{u(1+x_1^2)} & \frac{-2x_2}{u(1+x_2^2)} & \cdots & 1 \\
\frac{x_1(1-x_1^2)}{u(1+x_1^2)} & \frac{x_1(1-x_1^2)}{u(1+x_2^2)} & \cdots & \frac{-2x_1x_2}{u(1+x_n^2)} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{x_n(1-x_1^2)}{u(1+x_n^2)} & \frac{-2x_1x_n}{u(1+x_2^2)} & \cdots & \frac{x_1(1-x_n^2)}{u(1+x_n^2)}
\end{bmatrix}.$$ 

Not all elements of the first column are zero. If $x_1 \neq 0$ then $-2x_1/[u(1+x_1^2)] \neq 0$. If $x_1 = 0$ then the first element is $1 - 2/u$. If this element is zero then $u = 2$ which means that some of the $x_i$’s, $i = 2, \ldots, n$, must be nonzero. Then the corresponding term $x_i(1-x_i^2)/[u(1+x_i^2)] = x_i/u$ is nonzero. Thus, with the 1’s in the other columns, there are $n$ linearly independent nonzero columns, i.e. the rank of $D(\alpha: \mathbb{R}^n \to \mathbb{R}^{2n})(x)$ is $n$ and $\alpha$ is an immersion.

We now want to show that $\alpha$ has exactly one (regular) self-intersection, i.e. a double point which intersects transversely. Let

$$x = (x_1, \ldots, x_n), \quad x' = (x'_1, \ldots, x'_n),$$

$$u' = (1 + (x'_1)^2) \cdots (1 + (x'_n)^2),$$

$$\alpha(x) = (y_1, \ldots, y_{2n}), \quad \alpha(x') = (y'_1, \ldots, y'_{2n}).$$

At a double point $\alpha(x) = \alpha(x')$ we have $x \neq x'$. First we observe that $x_i = x'_i$ for $i = 2, \ldots, n$ by the definition of $\alpha$. That $x \neq x'$ then implies that $x_1 \neq x'_1$. Secondly,
since \( y_{n+1} = y'_{n+1} \) we must have \( u = u' \). Combining this with the first observation we have that \( 1 + x_1^2 = 1 + (x_1')^2 \), which is only possible if \( x_1 = -x_1' \neq 0 \). With this, \( y_{n+i} = y'_{n+i} \) can only be true if \( x_i = 0, \; i = 2, \ldots, n \). Therefore, \( u = 1 + x_1^2 \) and \( y_1 = y'_1 \) tells us that

\[
x_1 - \frac{2x_1}{u} = -x_1 + \frac{2x_1}{u}
\]

so

\[
1 - \frac{2}{u} = -1 + \frac{2}{u}
\]

thus

\[
u = 2 = 1 + x_1^2\]

or

\[
x_1 = \pm 1.
\]

The only double point turns out to be

\[
y = \alpha(\pm 1, 0, \ldots, 0).
\]

Set \( x_+ = (1, 0, \ldots, 0) \) and \( x_- = (-1, 0, \ldots, 0) \). For \( \alpha \) to intersect transversely we need, by the discussion preceding the definition 4.3,

\[
\begin{cases}
D\alpha(x_+)D\alpha(x_-)v \\
v \in T_{x_+}\mathbb{R}^n \times T_{x_-}\mathbb{R}^n
\end{cases} = \{ \text{since} \; T_x\mathbb{R}^m = \mathbb{R}^m, \; x \in \mathbb{R}^m \} = \mathbb{R}^{2n}.
\]

This is equivalent to the matrix \([Df(p_1) Df(p_2)]\) being invertible,

\[
[D\alpha(x_+) D\alpha(x_-)] = \begin{bmatrix}
1 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 & 0 & 1 & \ldots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 1 \\
-1/2 & 0 & \ldots & 0 & 1/2 & 0 & \ldots & 0 \\
0 & 1/2 & \ldots & 0 & 0 & -1/2 & \ldots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1/2 & 0 & 0 & \ldots & -1/2
\end{bmatrix}
\]

which we see is true since the columns are all linearly independent.

We shall now classify all regular self-intersections. To do that we first need some definitions regarding the notion of orientation.

**Definition 4.6.** Consider an \( n \)-dimensional manifold \((M, \mathcal{D})\). Let \( \mathcal{D} = [\mathcal{S}], \mathcal{S} = \{(V_j, \varphi_j)\} \). For \( x \in V_i \cap V_j \) let \( a_{ji}(x) \) be the Jacobian matrix of \( \varphi_j \circ \varphi_i^{-1} \) at \( \varphi_i(x) \):

\[
a_{ji}(x) = D(\varphi_j \circ \varphi_i^{-1})(\varphi_i(x)), \; x \in V_i \cap V_j.
\]
Then it is easy to see that
\[ a_{kj}(x) \cdot a_{ji}(x) = a_{ki}(x), \quad x \in V_i \cap V_j \cap V_k. \]

If we set \( k = i \), it follows that \( a_{ji}(x) \) has an inverse. Hence \( a_{ji}(x) \in \text{GL}(n, \mathbb{R}) \) and we have a continuous map
\[ a_{ij} : V_i \cap V_j \to \text{GL}(n, \mathbb{R}). \]

An atlas \( \mathcal{S} = \{(V_j, \varphi_j)\} \) is oriented if for all \( i, j \) and all \( x \in V_i \cap V_j \), the determinant \( |a_{ij}(x)| \) is positive.

A manifold is orientable if and only if it has an orientable atlas (see [17]). We take this to be the definition of orientable manifold. For the special case of Euclidean spaces, which are all orientable, we also need the following.

**Definition 4.7.** Two ordered set of \( n \) vectors, \( B \) and \( B' \), that are bases for \( \mathbb{R}^n \), are said to be equivalent if their transition matrix has positive determinant. This partitions the set of all bases for \( \mathbb{R}^n \) into two equivalence classes. If the transition matrix from a basis \( B \) of \( \mathbb{R}^n \) to the standard basis has positive determinant, then \( B \) is said to define the positive orientation on \( \mathbb{R}^n \). Otherwise it is said to define the negative orientation on \( \mathbb{R}^n \).

**Definition 4.8.** Let \( f : M \to \mathbb{R}^{2n} \) be a completely regular immersion of an \( n \)-dimensional manifold \( M \).

(i) The case where \( M \) is orientable and \( n \) is even. Choose an orientation in \( M \). Assume that \( f(p) = f(q), \ p \neq q \). Let \( u_1, \ldots, u_n \in T_pM \) and \( v_1, \ldots, v_n \in T_qM \) be ordered sets of linearly independent vectors, which define the orientations of \( T_pM \) and \( T_qM \). We say that the self-intersection at \( f(p) \) has positive type or negative type depending on whether the ordered set of vectors
\[ (df)_p(u_1), \ldots, (df)_p(u_n), (df)_q(v_1), \ldots, (df)_q(v_n) \in T_{f(p)}\mathbb{R}^{2n} \]
defines the positive orientation or the negative orientation in \( \mathbb{R}^{2n} \), and we define the intersection number of this self-intersection to be \(+1\) or \(-1\) accordingly. The intersection number \( I_f \) of \( f \) is the sum of intersection numbers of self-intersections of \( f \): \( I_f \in \mathbb{Z} \).

(ii) The case where \( M \) is non-orientable or \( n \) is odd. In this case we define the intersection number \( I_f \in \mathbb{Z} \), in the same way as above.

**Theorem 4.9.** Let \( M \) be an \( n \) dimensional compact manifold.

(i) If \( M \) is orientable and \( n \) is even, then for an arbitrary integer \( m \) there exists a completely regular immersion \( f : M \to \mathbb{R}^{2n} \) with \( I_f = m \).

(ii) If \( M \) is non-orientable or \( n \) is odd, then for any \( m \in \mathbb{Z}_2 \), there exists a completely regular immersion \( f : M \to \mathbb{R}^{2n} \) with \( I_f = m \).

**Proof.** By Theorem 4.5 there exists a completely regular immersion \( f_0 \) of \( M \) in \( \mathbb{R}^{2n} \). Take a point \( x_0 \) of \( M \) and some neighbourhood \( U \) of \( x_0 \) to replace \( f_0 \) by the map \( \beta \) with exactly one self-intersection defined in (4.3) or by the composite of \( \beta \) with the map \( r : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \) defined by \( r(x_1, \ldots, x_{2n}) = (-x_1, x_2, \ldots, x_{2n}) \). This can be
Lemma 4.12. Let \( \Lambda \) be a coordinate bundle. By a section of \( \Lambda \) we mean a continuous map \( s : X \to B \) with \( p \circ s = \text{id}_X \), the identity on \( X \). A smooth map \( s \) which is a section of a smooth coordinate bundle is called a smooth section.

**Proof.** We give only part of the proof and refer to Steenrod [16] for the rest together with the necessary background in cohomology theory.

Fix a point \( p \) of \( K \) and let \( \mathbb{R}^m_p \) denote the fiber over \( p \). Then the desired \( \zeta_i(p) \) may be a unit vector in the orthogonal complement of the \((i-1)\)-subspace spanned by \( \zeta_1(p), \ldots, \zeta_{i-1}(p) \), i.e., we want \( \zeta_i(p) \in S^{m-i}_p \subset \mathbb{R}^{m-i}_p \) such that for each \( p \in K \), \( \zeta_1(p), \ldots, \zeta_{i-1}(p) \) form an orthonormal system. Assume that (i) \( \zeta_1, \ldots, \zeta_{i-1} \) are sections over \( K \) such that for each \( p \in K \), \( \zeta_1(p), \ldots, \zeta_{i-1}(p) \) form an orthonormal system, and (ii) \( \zeta_i \) is a section over \( K' \) such that for each point \( p \in K' \), \( \zeta_1(p), \ldots, \zeta_{i-1}(p), \zeta_i(p) \) form an orthonormal system. Then if \( \dim K \leq m - i \), we can extend \( \zeta_i \) over \( K \) in such a way that the \( \zeta_1(p), \ldots, \zeta_{i-1}(p), \zeta_i(p) \) is an orthonormal system. 

For Lemma 4.12 and Lemma 4.13, consider a completely regular immersion \( f : M \to \mathbb{R}^n \) (see Figure 4.4 but keep in mind that the Whitney Trick is really only valid for manifolds \( M \) of dimension 3 or higher). Assume \( f \) has two regular self-intersection points \( q \) and \( q' \) of opposite sign,

\[
\begin{align*}
  f(p_1) &= f(p_2) = q, \quad p_1 \neq p_2; \\
  f(p'_1) &= f(p'_2) = q', \quad p'_1 \neq p'_2.
\end{align*}
\]

Let \( C_1 \) and \( C_2 \) be disjoint curves in \( M \): \( C_i \) connecting \( p_i \) and \( p'_i \) without passing through any other point where \( f \) has a self-intersection, \( i = 1, 2 \). Let \( B_i \) be the curve \( f(C_i) \) connecting \( q \) to \( q' \). Then \( B = B_1 \cup B_2 \) is a simple closed curve in \( f(M) \).

**Lemma 4.11.** Let \( \Lambda = \{B, p, \mathbb{R}^m, O(m)\} \) be an \( m \)-vector bundle over a polyhedron \( K \). Let \( K' \) be a subcomplex of \( K \). Assume that (i) \( \zeta_1, \ldots, \zeta_{i-1} \) are sections over \( K' \) such that for each \( p \in K' \), \( \zeta_1(p), \ldots, \zeta_{i-1}(p) \) form an orthonormal system, and (ii) \( \zeta_i \) is a section over \( K' \) such that for each point \( p \in K' \), \( \zeta_1(p), \ldots, \zeta_{i-1}(p), \zeta_i(p) \) form an orthonormal system. Then if \( \dim K \leq m - i \), we can extend \( \zeta_i \) over \( K \) in such a way that the \( \zeta_1(p), \ldots, \zeta_{i-1}(p), \zeta_i(p) \) is an orthonormal system.

\( \square \)

\( ^1 \)A coordinate bundle with fiber \( S^{m-i} \) and group \( O(m-i+1) \).
$f(M^n)$

$\psi(\tau'') = \sigma^2$

$\tau''$

Figure 4.4. (Upper two, with the mapping $f$.) The completely regular immersion $f : M^n \rightarrow \mathbb{R}^{2n}$ considered in Lemma 4.12, Lemma 4.13, and Theorem 4.14. (Lower two, with the mapping $\psi$.) Construction in $\mathbb{R}^2$ for the Whitney trick

(See Figure 4.4.) Let $A = A_1 \cup A_2$. Let $\tau'$ be a small neighbourhood of $A$ in $\mathbb{R}^2$ and let $\tau''$ be the region enclosed by $A$. Lastly, set $\tau = \tau' \cup \tau''$. With this there is an embedding $\psi : \tau \rightarrow \mathbb{R}^{2n}$ satisfying the following:

(I) $\psi(r) = q$, $\psi(r') = q'$ and $\psi(A_i) = B_i$, $i = 1, 2$,

(II) $\psi(\tau) \cap f(M) = B$,

(III) $T_q^*\psi(\tau) \not\subset T_q^*f(M)$, where $q^* \in B$.

Proof. We prove the theorem for an embedding $\psi : \tau' \rightarrow \mathbb{R}^{2n}$ to begin with.
Change $f$ slightly (by so little that it is still an embedding) at $p_i$ in $M_i$ so that $f$ maps a neighbourhood of $p_i$ in $M_i$ surjectively on the exponential image of some neighbourhood of $q$ in

$$T^n_i := T_qf(M_i).$$

Near $q$ we assume that $f$ maps a neighbourhood of $C_i$ in $M_i$ onto the exponential image of some neighbourhood of a line in $T^n_i$, given by its intersection with

$$T^2 := T_1f(M_1) \times T_qf(M_2).$$

Call the lines $T^1_i$, i.e.

$$T^1_i := T^2 \cap T_qf(M).$$

Now we define the linear mapping $\phi: \bar{V} \rightarrow T^2$ which maps a closed neighbourhood $\bar{V}$ of $r$ in $\tau'$ to a closed neighbourhood of $q$ in $T^2$. Then the image of the restriction $\phi|_{\bar{V} \cap A}$ is a closed neighbourhood of $q$ in $T^1_i$. In a similar way we define the linear mapping $\phi'$ of a closed neighbourhood $\bar{V}'$ of $r'$. Now we parametrize $A_i$,

$$r_i : [0, 1] \rightarrow A_i \in \mathbb{R}^2,$$

so that $\phi(r_i(t)) = q_i(t)$, wherever $\phi$ is defined, meaning that these $q_i(t)$ are points on the part of $C_i$ mapped onto the exponential image of $T^1_i$. Below we will, among other things, extend $\bar{V}$ and $\bar{V}'$ over all of $A$ in order for $\phi$ to fulfill the rest of (I) together with $\phi'$.

Let $u_i(t), t \in [0, 1]$, be the vector field of tangents to $A_i$ in $\tau'$, i.e.

$$u_i(t) \in T(\tau)|_{A_i},$$

which satisfies the following conditions.

(i) $\exp(u_i(t)) \subset \tau, i = 1, 2.$ \hspace{1cm} (4.4)

(ii)

$$\begin{cases}
  u_1(0) \in T_\tau A_2, & u_1(0) \text{ is pointed forward along } A_2, \\
  u_1(1) \in T_\tau A_2, & u_1(1) \text{ is pointed backward along } A_2,
\end{cases} \hspace{1cm} (4.5)$$

(iii) $u_i(t) \in T_{r_i(t)} \tau, u_i(t) \not\in T_{r_i(t)} A_i, i = 1, 2.$

(iv)

$$\begin{cases}
  u_1(t) \text{ rotates counterclockwise as } t \text{ goes from } 0 \text{ to } 1; \\
  u_2(t) \text{ rotates clockwise as } t \text{ goes from } 0 \text{ to } 1.
\end{cases} \hspace{1cm} (4.6)$$

(See Figure 4.5.)
Currently, $\phi(V)$ lies in $T^2$ and contains part of $T_1^1$ which constitutes part of $f(C_i) = B_i$. Further away from $q$, $B_i$ does not, necessarily, coincide with $T_1^1$. Since we want $\phi$ to satisfy (I), we extend $V$ in such a way that even if $q_i(t) \notin T_1^1$ we still have $q_i(t) \in \phi(V)$. This is done as follows. Let $R_i(t)$ be a line segment collinear with $u_i(t)$ and centered at $r_i(t)$. Assume that part of the boundaries of both $\tilde{V}$ and $\tilde{V}'$ are formed by the $R_i(t)$’s, whose length $\rho$ is such that no $R_1(t)$ touches an $R_2(t)$ if $r_1(t), r_2(t) \notin \tilde{V} \cup \tilde{V}'$. (See Figure 4.6.)

Now define the vector field $v_i(t) = (d\phi)_{r_i(t)}(u_i(t))$ for $r_i(t) \in [\tilde{V} \cup \tilde{V}'] \cup A_i$. We have that $v_i(t)$ is in $f(\tilde{V} \cap \tilde{V}')$ and by Lemma 4.11 we can extend it to $B_i$ without making it touch $f(M)$ at $q_i(t)$.

For linear maps, the Jacobian matrix is constant and equal to the coefficient matrix. Thus, on $\tilde{V} \cup \tilde{V}'$ where $\phi$ is linear, we have

$$v_i(t) = (d\phi)_{r_i(t)}(u_i(t)) = D\phi(r_i(t)) \cdot u_i(t) = \phi(u_i(t)).$$

Therefore, for $r_i(t) \in \tilde{V} \cup \tilde{V}'$, we have

$$\phi(r_i(t) + \alpha u_i(t)) = q_i(t) + \alpha v_i(t), \quad r_i(t) \in \tilde{V} \cup \tilde{V}', |\alpha| \leq \rho.$$

We can choose closed intervals $I_i(t) \subset [-\rho, \rho]$ such that

$$\{r_i(t) + \alpha u_i(t) | t \in [0, 1], i = 1, 2, \alpha \in I_i(t)\} = \tilde{T}',$$

and hence extend $\phi$ to the closed neighbourhood $\tilde{T}'$ of $A$.

Now, we can extend $\phi$ continuously over $\tau$ as $\psi'$, which is a smooth map by a smooth approximation. We find the embedding $\psi$ as a $\delta$-approximation of $\psi'$ by Theorem 3.22, since $2n \geq 5$. The embedding $\psi$ satisfies (II), and since $2 + n < 2n$ it also satisfies (III).

This last condition is important as a property of the 2-cell $\sigma^2 = \psi(\tau)$, which we will need in Theorem 4.14. We will also need a certain set of sections on the tangent bundle $T\mathbb{R}^{2n}$ restricted to $\sigma^2$. This is taken care of in Lemma 4.13.

**Lemma 4.13.** Assume $q$ and $q'$ are self-intersections of $f$ of opposite types. Then there exist sections $w_1, \ldots, w_{2n}$ in $T(\mathbb{R}^{2n})_{|\sigma^2}$, the restriction of the tangent bundle $T(\mathbb{R}^{2n})$ of $\mathbb{R}^{2n}$ to $\sigma^2$, satisfying the following:

1. For each $q^* \in \sigma^2$, $w_1(q^*), \ldots, w_{2n}(q^*)$ are linearly independent.
Proof. Since $\psi$ is an embedding, $w_1(q^*)$ and $w_2(q^*)$ are linearly independent. Set

$$V_1^{n-1} = \{e_3, \ldots, e_{n+1}\}, \quad V_2^{n-1} = \{e_{n+2}, \ldots, e_{2n}\},$$

where $\{\ldots\}$ denotes the subspace of $\mathbb{R}^{2n}$ spanned by $\{\ldots\}$. For each point $q^*$ of $B_1$ put

$$V_1^{n-1}(q^*) = \{v \in T_{q^*}f(M_1) \mid v \perp T_{q^*}B_1\}.$$ 

Setting $\mathcal{V}_1 = \bigcup_{q^* \in B_1} V_1^{n-1}(q^*)$, we obtain an $(n-1)$-dimensional vector bundle $\mathcal{V}_1 = \{\mathcal{V}_1, \pi_1, B_1\}$ over $B_1$. As the base space $B_1$ is contractible, the bundle $\mathcal{V}_1$ is trivial. Hence, there exist linearly independent vector fields, say, $w_1, w_3, \ldots, w_{n+1}$, in $\mathcal{V}_1$, which define an orientation of $f(M_1)$ at each point $q^*$ of $B_1$.

Similarly for $q^* \in B_2$ setting

$$V_2^{n-1}(q^*) = \{v \in T_{q^*}f(M_2) \mid v \perp T_{q^*}B_2\},$$

we obtain an $(n-1)$-dimensional vector bundle $\mathcal{V}_2 = \{\mathcal{V}_2, \pi_2, B_2\}$, where $\mathcal{V}_2 = \bigcup_{q^* \in B_2} V_2^{n-1}(q^*)$. Again there exist linearly independent vector fields $w_2, w_3, \ldots, w_{2n}$ in $\mathcal{V}_2$, which define an orientation of $f(M_2)$ at each point $q^*$ of $B_2$. Here we assume $w_2(q^*)$ to be an element of $T_{q^*}B_2$, which points in the positive direction along $B_2$.

Let $\mathcal{V} = \{\mathcal{V}_1, \mathcal{V}_2\}$ be a $2n$-dimensional trivial bundle. The restriction $\mathcal{V}|_{B_2}$ of $\mathcal{V}$ over $B_2$ has $n+1$ linearly independent sections $w_1, w_2, w_{n+2}, \ldots, w_{2n}$. Furthermore, if $q^* = q$ or $q^* = q'$, the $w_1(q^*), \ldots, w_{2n}(q^*)$ are linearly independent. By Lemma 4.11 we can extend the $w_1, \ldots, w_n$ over $B_2$ and for each $q^* \in B_1$...
the $2n - 1$ vectors
\[ w_1(q^*), w_2(q^*), w_3(q^*), \ldots, w_n(q^*), w_{n+2}(q^*), \ldots, w_{2n}(q^*) \]
are linearly independent.

Recall that the self-intersections $q$ and $q'$ have opposite types. By suitable choices of $w_1, w_3, \ldots, w_{n+1}$ in $B_1$ and $w_2, w_{n+2}, \ldots, w_{2n}$ in $B_2$ we may assume that for $q^* = q$ or $q^* = q'$, the vectors
\[ w_1(q^*), w_3(q^*), \ldots, w_n(q^*), w_{n+2}(q^*), \ldots, w_{2n}(q^*) \]
define the orientation opposite from the preassigned orientation in $\mathbb{R}^{2n}$.

Now we can deform $w_2(q)$ and $w_2(q')$ to $w_2'(q)$ and $w_2'(q')$ keeping them in $T(\sigma^2)$ and out of $T(B_1)$; thus, these vectors remain linearly independent of the above vectors. Hence, the vectors
\[ w_1(q^*), w_2(q^*), \ldots, w_{2n}(q^*) \]
define the positive orientation of $\mathbb{R}^{2n}$ at $q^* = q$ or $q^* = q'$. Therefore, we can extend the section $w_{n+1}$ over $B_2$ while keeping its linear independence with the other vectors.

Using Lemma 4.11 again we extend the sections $w_3, \ldots, w_{n+1}$ over $\sigma^2$ so that $w_1, \ldots, w_{n+1}$ are linearly independent at each point of $\sigma^2$.

Finally, we extend the sections $w_{n+2}, \ldots, w_{2n}$ defined over $B_2$ to $\sigma^2$ while keeping their linear independence. This is possible since we may think of $B_2$ as a smooth deformation retract of $\sigma^2$. Hence, we have shown the lemma is valid. □

As a last step before proving Whitney's second embedding theorem, we now show how to "add" and "remove" pairs of regular self-intersections. Thus we prove the existence of completely regular immersions with one pair more or less than a given completely regular immersion $f$, of Figure 4.4.

The following theorem states that it is, under certain circumstances, possible to both add and remove pairs of regular singularities. For the second embedding theorem, we only need to be able to remove pairs of regular singularities. Therefore, only a sketch is made of how to add them.

**Theorem 4.14.** Let $f: M \to \mathbb{R}^{2n}$ be an immersion, where $M$ is an $n$-dimensional manifold and $n \geq 3$. Then there exists a regular homotopy $\{f_t|t \in [0, 1]\}$ of $f$ satisfying the following:

1. $f_0 = f$.
2. $f_1$ is a completely regular immersion.
3. The number of self-intersections of $f_1$ is two more than that of $f$.

If (i) $M$ is orientable, $n$ is even, and the number of self-intersections of $f$ is greater than $|I_f|$, or (ii) $M$ is non-orientable or $n$ is odd, and $f$ has at least one self-intersection, then there exists a regular homotopy $\{f_t\}$ of $f$ such that the number of self-intersections of $f_1$ is two less than that of $f = f_0$. 

40
Figure 4.7. Introducing a pair of regular self-intersections to a 1-dimensional manifold by pulling it through itself. An intermediate step is shown, where the point $p$ has just been pulled away from the $x_1$-axis.

Proof. Notice first that by Theorem 4.5 we may assume that $f$ is a completely regular immersion. Take a sufficiently small $n$-disk $D^n$ in $f(M)$, $D^n \subset f(M) \subset \mathbb{R}^{2n}$, and choose distinct points $p$ and $q$ in $D^n$.

(a) We try to grasp a whole picture with $n = 1$. We take an embedding of $D^1$ in $\mathbb{R}^2$ as shown in Figure 4.7 and make two self-intersections by moving the disk around by a regular homotopy, say, $q = 0$, $p \in D^1 \subset \mathbb{R}^1 \subset \mathbb{R}^2$.

(b) Assume $D^n$ is embedded in $\mathbb{R}^n \subset \mathbb{R}^{2n}$. Let $q = 0$ and $p$ be points of $D^n$ with $p \neq q$. Assume $p$ sits on the $x_1$-axis. Pull up $p$ to position some neighbourhood of $x$ parallel to the $(x_1, x_{n+2}, \ldots, x_{2n})$-plane. Next we move this neighbourhood in the $(x_1, x_{n+2}, \ldots, x_{2n})$-plane through the origin 0 (at this point the disk crosses itself) making sure not to create any other intersections. The only intersections are on the $x_1$-axis; we have created two self intersections. We show how to eliminate two self-intersections in case (ii) where either $M$ is non-orientable or $n$ is odd, and how to eliminate two self-intersections of distinct types in case (i) where $M$ is orientable and $n$ is even, in each case through a regular homotopy.

Take a two-cell $\sigma^2$ with boundary $B$ such that $f(M) \cap \sigma^2 = B$ (Lemma 4.12).

Next, deform $f$ through $\sigma^2$ in a neighbourhood of $C_2$ in $M$ so that the new image of $C_2$ will not meet $B_1$. In this way we will remove the two self-intersections.

Now, for the case of removing pair of regular singularities. Consider $\tau \subset \mathbb{R}^2 \subset \mathbb{R}^{2n}$. For each point $r = (a_1, a_2, \ldots, a_{2n})$ of $\mathbb{R}^{2n}$ set $r^* = (a_1, a_2, 0, \ldots, 0)$ and

$$\psi(r) = \psi \left( r^* + \sum_{i=3}^{2n} a_i e_i \right) = \psi(r^*) + \sum_{i=3}^{2n} a_i w_i(\psi(r^*)).$$
For each point $q^*$ of $\sigma^2$, the vectors $w_1(q^*), \ldots, w_{2n}(q^*)$ are linearly independent and they are smooth in $q^*$. Therefore, $\psi$ when considered as a map from a neighbourhood of $\sigma^2$ to $\mathbb{R}^{2n}$ has the nonzero Jacobian matrix at each point in its domain. Hence, we have the inverse $\psi^{-1}$. Setting

$$N_1 = \psi^{-1}(f(M_1)), \quad N_2 = \psi^{-1}(f(M_2)),$$

we see that $N_1$ and $N_2$ are contained in a neighbourhood $U$ of $\tau$ in $\mathbb{R}^{2n}$. If we can deform $N_2$ in $U$ so $N_2$ does not intersect $N_1$ (i.e., there exists an isotopy $\{i_t : t \in [0, 1]\}$ of the inclusion map $i : N_2 \to U$ such that $i_0 = i$, $i_1(N_2) \cap N_1 = \emptyset$), the family $\{\psi \circ i_t\}$ defines a deformation of $f$ and the number of self-intersections of $f$ decreases by two. Hence, in this case the proof will be complete. Set

$$\pi(x_1, \ldots, x_{2n}) = (x_1, 0, x_3, \ldots, x_{2n}).$$

By the conditions (i), (ii), and (iii) of Lemma 4.11 and the definition of $\psi(r)$ as given above for $r^* \in A_1$, $T_r, N_1$ is in the $(x_1, x_3, \ldots, x_{n+1})$-plane. Hence $T_r, \pi(N_1)$ is also in the $(x_1, x_3, \ldots, x_{n+1})$-plane. Similarly $T_r, \pi(N_2)$ is in the $(x_1, x_{n+2}, \ldots, x_{2n})$-plane. Hence, $\pi(N_1) \cap \pi(N_2)$ is on the $x_1$-axis.

Let $\mu(x_1)$ be a smooth function whose graph $x_2 = \mu(x_1)$ is the union of the set $A_2$ and the $x_1$-axis minus the set $A_1$ smoothed out at the points $r$ and $r'$ (Figure 4.8). Take $\varepsilon > 0$ such that the interior of $N_2$ consists of points whose distances from the $(x_1, x_2)$-plane are each less than $\varepsilon$. Consider a smooth function $\nu : \mathbb{R} \to \mathbb{R}$ as follows:

$$|\nu(\lambda)| \leq 1, \quad \nu(0) = 1,$$

$$\nu(\lambda) = 0 \quad \text{if } |\lambda| \geq \varepsilon^2.$$

Now define a map $\theta_t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by

$$\theta_t(r) = r - t\nu(x_3^2 + \cdots + x_{2n}^2)\mu(x_1)e_2, \quad r = (x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n}.$$

By the definition of $\nu$, $\theta_t$ is the identity map outside $N_2$. Evidently the $\{\theta_t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}\}$ is a regular homotopy with $\theta_0 = 1$. When $t = 1$, $\theta_1$ maps the portion of $N_2$ on $A_2$ to the halfplane $x_1 < 0$. But $\pi(\theta_1(N_2)) = \pi(N_2)$ and $\pi(N_1) \cap \pi(N_2)$ is on the $x_1$-axis, and so $N_1 \cap \theta_1(N_2)$ must be on the $x_1$-axis. However, $\theta_1(N_2)$ does not intersect the $x_1$-axis, and so $\theta_1(N_2)$ must be empty.
Furthermore, since \( \psi(\tau) \cap f(M) = B \) no new self-intersection will arise if we take \( \varepsilon \) sufficiently small. This concludes the proof for the case that \( M \) is orientable and \( n \) is even.

In other cases we can also remove pairs of self-intersections by regular homotopies in just about the same way as above. To do this we only need to show that \( C_1 \) and \( C_2 \) can be chosen so that \( q \) and \( q' \) have opposite types.

**The case \( M \) is not orientable.** If we can take \( C_1 \) and \( C_2 \) as above and \( q \) and \( q' \) are of opposite types we follow the previous argument. If \( q \) and \( q' \) is of the same type, we choose a curve \( C'_2 \) from \( p_2 \) to \( p'_2 \) so that \( C_2 \cup C'_2 \) reverses the orientation in \( M \). Then \( q \) and \( q' \) are of opposite type with respect to \( C_1 \) and \( C_2 \).

**The case \( n \) is odd and \( M \) is orientable.** Assume \( q \) and \( q' \) have the same type for \( C_1 \) and \( C_2 \). Then choose a curve \( C'_1 \) from \( p_1 \) to \( p'_2 \) and a curve \( C'_2 \) from \( p_2 \) to \( p'_1 \) as follows. The curve \( C'_i \) agrees with \( C_i \) near the starting point and with \( C_j, i \neq j \), near the endpoint. The neighbourhood \( M'_i \) of \( C'_i \) is chosen in such a way that \( M_i \) agrees with \( M'_i \) near the point \( p_i \) and with \( M'_j \) near \( p'_j \), \( j \neq i \). Orient \( M_i \) and \( M'_i \) with the preassigned orientations near \( p_i \) and \( p'_j \). Then \( q \) and \( q' \) have the opposing orientations with respect to \( (M'_1, M'_2) \). \( \square \)

Immersions and embeddings of non-compact manifolds are proper, in addition to what the original Definition 2.11 specifies. Continuous maps are proper if the inverse image of every compact set is compact. With this, we finally have everything we need for the proof of Whitney’s second embedding theorem.

**Theorem 4.15** (Whitney’s second embedding theorem). Every \( n \)-dimensional differentiable manifold can be embedded in \( \mathbb{R}^{2n} \).

**Proof.** Let \( M \) be an \( n \)-dimensional manifold. The theorem is routine for \( n = 1 \) as \( M \) is a finite union of \( S^1 \).

Case \( n = 2 \). We can embed the sphere \( S^2 \), the projective space \( \mathbb{R}P^2 \) and the Klein bottle \( K^2 \) in \( \mathbb{R}^4 \). By the classification theorem for closed surfaces, \( M \) is a connected sum of a finite number of copies of the above surfaces. Hence, we can embed \( M \) in \( \mathbb{R}^4 \) (cf. any elementary text covering the classification of surfaces, e.g. [20]).

Let \( n \geq 3 \). By Theorem 4.9 there exists a completely regular immersion \( f_0 : M \to \mathbb{R}^{2n} \) with \( I_{f_0} = 0 \). By Theorem 4.14 we can remove all self-intersections. Thus, we have obtained an embedding \( f : M \to \mathbb{R}^{2n} \) as an injective immersion on a compact set.

Now assume that \( M \) is not compact and let \( (p_i, q_i) \) denote the double points of \( f_0 \). For each \( p_i \), take a curve \( \gamma_i \) from \( p_i \) to infinity. Now define neighbourhoods \( U_i \), for each curve \( \gamma_i \), such that \( U_i \) intersects no other \( U_j \) or contains any other \( p_j \). The points of the \( U_i \)'s may be written as \( (p, r, s) \) with \( p \in S^{n-2}_0, 0 \leq r \leq 1 \) and \( 0 \leq s < 1 \), such that \( r = 0 \) and \( 2\delta \leq s < 1 \) represents the curve while \( r = 1 \) or \( s = 0 \) gives the boundary of \( U_i \). Next, define the deformation

\[
\phi_t(p, r, s) = (p, r, [1 - (1 - \delta)(1 - r)]t)\, |\, s, \]


which leaves the boundary of $U_i$ fixed. Replacing $\phi_t$ by a smooth deformation for each $i$ gives a mapping $f$ with $f(M) \subset f_0(M) \setminus \cup_i \gamma_i$. In this way we remove all self-intersections, and $f$ is still proper; Hence, an embedding. □
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