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Functions of bounded variation

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Introduction

In this paper we investigate the functions of bounded variations. We study basic properties of these functions and solve some problems.

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1 Monotone functions

The properties of monotone functions will be useful to us because later we shall see that some of them can be extended directly to the functions of bounded variation. First a definition:

Definition 1.0.1 Let $f : [a, b] \rightarrow \mathbb{R}$ be a function. Then f is said to be

- **increasing** on $[a, b]$ if for every $x, y \in [a, b]$ $x < y \Rightarrow f(x) \leq f(y)$
- **decreasing** on $[a, b]$ if for every $x, y \in [a, b]$ $x < y \Rightarrow f(x) \geq f(y)$
- **monotone** if f is either increasing or decreasing on $[a, b]$

If the interval $[a, b]$ can be divided into a finite number of intervals such that f is monotone on each of them then f is said to be *piecewise monotone* on $[a, b]$. The following theorem is given in [2], we refer to it for a proof.

Theorem 1.0.2 [2, p. 95]. Let $f : [a, b] \rightarrow \mathbb{R}$ be increasing on $[a, b]$ and suppose that $c \in (a, b)$. Then $f(c + 0)$ and $f(c - 0)$ ¹ exists and

$$\sup\{f(x) : x < c\} = f(c - 0) \leq f(c) \leq f(c + 0) = \inf\{f(x) : x > c\}$$

If f is decreasing the theorem above still holds, with opposite inequalities of course. Thus we can state that if f is monotone and $c \in (a, b)$ then both $f(c + 0)$ and $f(c - 0)$ exists. Therefore the following definition makes sense:

Definition 1.0.3 Let $f : [a, b] \rightarrow \mathbb{R}$ be monotone on $[a, b]$ and let $c \in [a, b]$.

(i) If $c \in [a, b)$ we define **the right-hand jump** of f at c to be

$$\sigma_c^+ = f(c + 0) - f(c)$$

(ii) If $c \in (a, b]$ we define **the left-hand jump** of f at c to be

$$\sigma_c^- = f(c) - f(c - 0)$$

¹We denote the right-hand and left-hand limits

$$\lim_{h \rightarrow 0} f(c + h) = f(c + 0) \quad \lim_{h \rightarrow 0} f(c - h) = f(c - 0)$$

where h tends to 0 from the positive side.

(iii) If $c \in [a, b]$ we define **the jump** of f at c to be

$$\sigma_c = \begin{cases} \sigma_c^+ + \sigma_c^- & \text{if } c \in (a, b) \\ \sigma_c^+ & \text{if } c = a \\ \sigma_c^- & \text{if } c = b \end{cases}$$

We now make the following simple observation: if $f : [a, b] \rightarrow \mathbb{R}$ is monotone on $[a, b]$, then f is continuous at $c \in [a, b]$ if and only if $\sigma_c = 0$. Here, the necessity is obvious and the sufficiency follows at once from theorem 1.0.2.

Of course, a monotone function needn't be continuous. However we can now prove that if $f : [a, b] \rightarrow \mathbb{R}$ is monotone then it can't be "too" discontinuous.

Theorem 1.0.4 [2, p. 96]. *Let $f : [a, b] \rightarrow \mathbb{R}$ be monotone on $[a, b]$ and let D be the set of all points of discontinuity of f . Then D is at most countable.*

Proof: Suppose that f is increasing on $[a, b]$. A point $c \in [a, b]$ is a point of discontinuity of f if and only if $\sigma_c \neq 0$. Since f is increasing on $[a, b]$ clearly $\sigma_c \geq 0$ for every $c \in [a, b]$ and hence

$$D = \{x \in [a, b] : \sigma_x > 0\}$$

For any given $n \in \mathbb{N}$ take points x_1, x_2, \dots, x_n satisfying $a \leq x_1 < x_2 < \dots < x_n \leq b$. For $0 \leq j \leq n$ take points t_j such that

$$a = t_0 \leq x_1 < t_1 < x_2 < t_2 < \dots < t_{n-1} < x_n \leq t_n = b$$

Then, since f is increasing we have $\sigma_{x_j} \leq f(t_j) - f(t_{j-1})$ for $1 \leq j \leq n$ and it follows that

$$\begin{aligned} \sigma_{x_1} + \sigma_{x_2} + \dots + \sigma_{x_n} &\leq \sum_{j=1}^n (f(t_j) - f(t_{j-1})) \\ &= f(b) - f(a) \end{aligned}$$

So if $D_k = \{x : \sigma_x \geq (f(b) - f(a))/k\}$ then D_k can have at most k elements. Furthermore, $D = \bigcup_{k=1}^{\infty} D_k$ and since every D_k is finite D is at most countable.

□

2 Functions of bounded variation

2.1 General properties

Let $f : [a, b] \rightarrow \mathbb{R}$ be a function and $\Pi = \{x_0, x_1, \dots, x_n\}$ a partition of $[a, b]$. We denote $V_\Pi(f) = \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|$ and set

$$V_a^b(f) = \sup_{\Pi} V_\Pi(f)$$

where the supremum is taken over all partitions of $[a, b]$. We clearly have $0 \leq V_a^b(f) \leq \infty$. The quantity $V_a^b(f)$ is called the *total variation of f over $[a, b]$* .

Definition 2.1.1 *A function $f : [a, b] \rightarrow \mathbb{R}$ is said to be of **bounded variation** on $[a, b]$ if $V_a^b(f)$ is finite. If f is of bounded variation on $[a, b]$ we write $f \in V[a, b]$.*

Occasionally we shall say that a function is of bounded variation, leaving out the specification of interval, when the interval in question is clear.

We shall state and prove some important properties of functions of bounded variation and their total variation but first we need a theorem concerning refinements of partitions.

Theorem 2.1.2 *Let $f : [a, b] \rightarrow \mathbb{R}$ be a function and Π any partition of $[a, b]$. If Π' is any refinement of Π then $V_\Pi(f) \leq V_{\Pi'}(f)$.*

Proof: Since any refinement of Π can be obtained by adding points to Π one at a time it's enough to prove the theorem in the case when we add just one point. Take $\Pi = \{x_0, x_1, \dots, x_n\}$ and add the point c to Π and denote the result Π' . Assume that $x_j < c < x_{j+1}$ for some $0 \leq j \leq n-1$, then the triangle inequality gives that

$$\begin{aligned} |f(x_{j+1}) - f(x_j)| &= |f(x_{j+1}) - f(c) + f(c) - f(x_j)| \\ &\leq |f(x_{j+1}) - f(c)| + |f(c) - f(x_j)| \end{aligned}$$

and hence

$$\begin{aligned} V_\Pi(f) &= \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| \\ &= |f(x_{j+1}) - f(x_j)| + \sum_{k=0, k \neq j}^{n-1} |f(x_{k+1}) - f(x_k)| \\ &\leq |f(x_{j+1}) - f(c)| + |f(c) - f(x_j)| + \sum_{k=0, k \neq j}^{n-1} |f(x_{k+1}) - f(x_k)| \\ &= V_{\Pi'}(f) \end{aligned}$$

□

The theorem above assures us that adding points to a partition Π will only make the sum $V_{\Pi}(f)$ larger or perhaps leave it unchanged, a useful fact that we shall use in the proof of the following theorem, given in [1].

Theorem 2.1.3 [1, p. 120]. *Let $f : [a, b] \rightarrow \mathbb{R}$ and c an arbitrary point in (a, b) . Then $f \in V[a, b]$ if and only if $f \in V[a, c]$ and $f \in V[c, b]$. Furthermore, if $f \in V[a, b]$ then*

$$V_a^b(f) = V_a^c(f) + V_c^b(f)$$

Proof: Assume that $f \in V[a, b]$. We will show that $f \in V[a, c]$, the proof is similar to prove that $f \in V[c, b]$. Take an arbitrary partition Π of $[a, c]$ and add the point b to Π and denote the result Π' , which is a partition of $[a, b]$. We then have

$$\begin{aligned} V_{\Pi'}(f) &= V_{\Pi}(f) + |f(b) - f(c)| \leq V_a^b(f) \\ \Leftrightarrow V_{\Pi}(f) &\leq V_a^b(f) - |f(b) - f(c)| \end{aligned}$$

Since $V_a^b(f)$ is finite, the sums $V_{\Pi}(f)$ are bounded above and thus $\sup_{\Pi} V_{\Pi}(f)$ is finite, that is $f \in V[a, c]$.

Now assume that $f \in V[a, c]$ and $f \in V[c, b]$. Let Π be any partition of $[a, b]$. Add the point c to Π and denote the result Π_1 . Then $\Pi_1 = \Pi' \cup \Pi''$ where Π' is a partition of $[a, c]$ and Π'' is a partition of $[c, b]$. Then we have

$$V_{\Pi}(f) \leq V_{\Pi_1}(f) = V_{\Pi'}(f) + V_{\Pi''}(f) \leq V_a^c(f) + V_c^b(f)$$

and since both $V_a^c(f)$ and $V_c^b(f)$ are finite the sums $V_{\Pi}(f)$ are bounded above and thus $f \in V[a, b]$ and $V_a^b(f) \leq V_a^c(f) + V_c^b(f)$.

Now we take any two partitions Π' and Π'' of $[a, c]$ and $[c, b]$ respectively and let Π be the union of Π' and Π'' , then Π is a partition of $[a, b]$. We have

$$V_{\Pi'}(f) + V_{\Pi''}(f) = V_{\Pi}(f) \leq V_a^b(f)$$

and thus $V_{\Pi'}(f) \leq V_a^b(f) - V_{\Pi''}(f)$. For any fixed partition Π'' of $[c, b]$ the number $V_a^b(f) - V_{\Pi''}(f)$ is an upper bound for the sums $V_{\Pi'}(f)$ and therefore $V_a^c(f) \leq V_a^b(f) - V_{\Pi''}(f)$. This is equivalent to $V_{\Pi''}(f) \leq V_a^b(f) - V_a^c(f)$ and thus $V_a^b(f) - V_a^c(f)$ is an upper bound for the sums $V_{\Pi''}(f)$ and therefore $V_c^b(f) \leq V_a^b(f) - V_a^c(f)$ whence $V_a^c(f) + V_c^b(f) \leq V_a^b(f)$. But then we must have $V_a^b(f) = V_a^c(f) + V_c^b(f)$.

□

The following two theorems are given in [1] and are straightforward to prove and therefore their proofs are omitted.

Theorem 2.1.4 [1, p. 120]. *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be of bounded variation on $[a, b]$. Then $(f + g) \in V[a, b]$ and $V_a^b(f + g) \leq V_a^b(f) + V_a^b(g)$.*

Theorem 2.1.5 [1, p. 120]. *Let $f : [a, b] \rightarrow \mathbb{R}$ be of bounded variation on $[a, b]$. Then $cf \in V[a, b]$ for any $c \in \mathbb{R}$ and $V_a^b(cf) = |c|V_a^b(f)$*

Theorem 2.1.6 [1, p. 119]. *If $f : [a, b] \rightarrow \mathbb{R}$ is monotone on $[a, b]$ then $f \in V[a, b]$ and $V_a^b(f) = |f(b) - f(a)|$*

Proof: We will give the proof in the case when f is increasing, it is similar when f is decreasing. Let f be increasing on $[a, b]$, then $|f(b) - f(a)| = f(b) - f(a)$. Take an arbitrary partition $\Pi = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$. Since f is increasing we have $|f(x_{k+1}) - f(x_k)| = f(x_{k+1}) - f(x_k)$ and hence

$$V_{\Pi}(f) = \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| = \sum_{k=0}^{n-1} (f(x_{k+1}) - f(x_k)) = f(b) - f(a)$$

Since the sum $V_{\Pi}(f)$ is independent of the partition Π we conclude that

$$V_a^b(f) = f(b) - f(a)$$

□

Combining the theorem above with theorem 2.1.3 we see that any piecewise monotone function defined on a compact interval is of bounded variation. However, the converse is certainly not true. Indeed, there exists functions of bounded variation that aren't monotone on any subinterval.

Even so, functions of bounded variation can be characterized in terms of monotone functions, as the following theorem due to Jordan shows.

Theorem 2.1.7 (Jordan's theorem) [1, p. 121]. *Let $f : [a, b] \rightarrow \mathbb{R}$, f is of bounded variation if and only if f is the difference of two increasing functions.*

Proof: Assume that $f \in V[a, b]$ and let $v(x) = V_a^x(f)$, $x \in (a, b]$ and $v(a) = 0$. Then clearly $f(x) = v(x) - [v(x) - f(x)]$. We will show $v(x)$ and $v(x) - f(x)$ are increasing. For any $x_1 < x_2$ we have

$$v(x_2) - v(x_1) = V_{x_1}^{x_2}(f) \geq 0 \Leftrightarrow v(x_2) \geq v(x_1)$$

so $v(x)$ is increasing. Furthermore,

$$\begin{aligned} f(x_2) - f(x_1) &\leq |f(x_2) - f(x_1)| \leq V_{x_1}^{x_2}(f) = v(x_2) - v(x_1) \\ &\Leftrightarrow v(x_1) - f(x_1) \leq v(x_2) - f(x_2) \end{aligned}$$

and thus $v(x) - f(x)$ is increasing.

Conversely, suppose that $f(x) = g(x) - h(x)$ with g and h increasing. Since h is increasing $-h$ is decreasing and thus $f(x) = g(x) + (-h(x))$ is the sum of two monotone functions so theorem 2.1.6 together with theorem 2.1.4 gives that $f \in V[a, b]$

□

Since any function of bounded variation can be written as the sum of two monotone functions many of the properties of monotone functions are inherited by functions of bounded variation. If $f \in V[a, b]$, then

- The limits $f(c + 0)$ and $f(c - 0)$ exists for any $c \in (a, b)$.
- The set of points where f is discontinuous is at most countable.

We shall return to these facts later on.

The following two problems demonstrate how one can use the theorems given above in computations.

Problem 2.1.8 Represent $f(x) = \cos^2 x$, $0 \leq x \leq 2\pi$ as a difference of two increasing functions.

Solution: The proof of Jordans theorem shows that the functions $v(x) - f(x)$ and $v(x)$ will do, so the problem is to determine $v(x)$. Divide $[0, 2\pi]$ into four subintervals $I_1 = [0, \frac{\pi}{2}]$, $I_2 = [\frac{\pi}{2}, \pi]$, $I_3 = [\pi, \frac{3\pi}{2}]$ and $I_4 = [\frac{3\pi}{2}, 2\pi]$.

The function $f(x)$ decreases from 0 to 1 on I_1 and I_3 and increases from 0 to 1 on I_2 and I_4 , so the total variation of f over any of these subintervals is 1. To determine $V_0^x(f)$ we need to study separate cases depending on which interval x lies in. To demonstrate the principle, assume that $x \in I_3$, then

$$\begin{aligned} V_0^x(f) &= V_0^{\frac{\pi}{2}}(f) + V_{\frac{\pi}{2}}^{\pi}(f) + |f(x) - f(\pi)| \\ &= 1 + 1 + |\cos^2 x - 1| = 3 - \cos^2 x \end{aligned}$$

Similar calculations for the other subintervals gives that

$$v(x) = \begin{cases} 1 - \cos^2 x & 0 \leq x \leq \frac{\pi}{2} \\ 1 + \cos^2 x & \frac{\pi}{2} \leq x \leq \pi \\ 3 - \cos^2 x & \pi \leq x \leq \frac{3\pi}{2} \\ 3 + \cos^2 x & \frac{3\pi}{2} \leq x \leq 2\pi \end{cases}$$

□

Problem 2.1.9 Represent the function

$$f(x) = \begin{cases} -x^2 & 0 \leq x < 1 \\ 0 & x = 1 \\ 1 & 1 < x \leq 2 \end{cases}$$

as a difference of two increasing functions.

Solution: As above, we determine $V_0^x(f)$. On $[0, 1)$ the function $f(x)$ is decreasing so if $x \in [0, 1)$ then

$$V_0^x(f) = |-x^2 - 0| = x^2$$

To determine $V_0^1(f)$, let $\Pi = \{x_0, \dots, x_n\}$ be any partition of $[0, 1]$ and consider $V_\Pi(f)$. We have

$$\begin{aligned} V_\Pi(f) &= \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| \\ &= \sum_{k=0}^{n-2} |f(x_{k+1}) - f(x_k)| + |f(1) - f(x_{n-1})| \\ &= \sum_{k=0}^{n-2} (x_{k+1}^2 - x_k^2) + x_{n-1}^2 = 2x_{n-1}^2 \end{aligned}$$

By taking the point x_{n-1} "close enough" to 1, $V_\Pi(f)$ can be made arbitrary close to but less than 2, and thus $V_0^1(f) = 2$. Finally, if $x \in (1, 2]$ we have

$$V_0^x(f) = V_0^1(f) + V_1^x(f) = 2 + V_1^x(f)$$

Let $\Pi = \{x_0, \dots, x_n\}$ be any partition of $[1, x]$ and consider $V_\Pi(f)$

$$\begin{aligned} V_\Pi(f) &= \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| \\ &= |f(x_1) - f(x_0)| + \sum_{k=1}^{n-1} |f(x_{k+1}) - f(x_k)| \\ &= |1 - 0| = 1 \end{aligned}$$

Clearly $V_{\Pi}(f)$ is independent of the partition Π , so $V_1^x(f) = 1$ and thus $V_0^x(f) = 3$, $x \in (1, 2]$. Hence

$$V_0^x(f) = \begin{cases} x^2 & 0 \leq x < 1 \\ 2 & x = 1 \\ 3 & 1 < x \leq 2 \end{cases}$$

Then $f(x) = V_0^x(f) - (V_0^x(f) - f(x))$ where

$$V_0^x(f) - f(x) = \begin{cases} 2x^2 & 0 \leq x < 1 \\ 2 & x = 1 \\ 2 & 1 < x \leq 2 \end{cases}$$

□

2.2 Positive and negative variation

For any $a \in \mathbb{R}$ set

$$a^+ = \max\{a, 0\} \quad \text{and} \quad a^- = \max\{-a, 0\}$$

We begin by noticing the following equalities:

$$a^+ + a^- = |a| \tag{1}$$

$$a^+ - a^- = a \tag{2}$$

Indeed, if $a > 0$ then $a^+ = a = |a|$ and $a^- = 0$ so $a^+ + a^- = |a|$ and $a^+ - a^- = a$. The case $a < 0$ is treated similarly.

Equations 1 and 2 gives that $a^+ = (a + |a|)/2$. Then we have

$$(\alpha + \beta)^+ = \frac{\alpha + \beta + |\alpha + \beta|}{2} \leq \frac{\alpha + \beta + |\alpha| + |\beta|}{2} = \alpha^+ + \beta^+$$

that is

$$(\alpha + \beta)^+ \leq \alpha^+ + \beta^+ \tag{3}$$

Let $f : [a, b] \rightarrow \mathbb{R}$ and $\Pi = \{x_0, x_1, \dots, x_n\}$ any partition of $[a, b]$. Denote

$$P_{\Pi}(f) = \sum_{k=0}^{n-1} [f(x_{k+1}) - f(x_k)]^+ \quad \text{and} \quad Q_{\Pi}(f) = \sum_{k=0}^{n-1} [f(x_{k+1}) - f(x_k)]^-$$

and set

$$P_a^b(f) = \sup_{\Pi} P_{\Pi}(f) \quad \text{and} \quad Q_a^b(f) = \sup_{\Pi} Q_{\Pi}(f)$$

$P_a^b(f)$ and $Q_a^b(f)$ will be referred to as the positive respectively negative variation of f on $[a, b]$. There is a connection between $P_a^b(f)$, $Q_a^b(f)$ and $V_a^b(f)$, as the following problem shows:

Problem 2.2.1 *If one of the magnitudes $P_a^b(f)$, $Q_a^b(f)$ and $V_a^b(f)$ is finite then so are the two others.*

Proof: For any partition Π of $[a, b]$ we have $P_{\Pi}(f) + Q_{\Pi}(f) = V_{\Pi}(f)$ and $P_{\Pi}(f) - Q_{\Pi}(f) = f(b) - f(a)$ according to equations 1 and 2. The equality $P_{\Pi}(f) - Q_{\Pi}(f) = f(b) - f(a)$ gives that

- (i) $P_{\Pi}(f) \leq Q_a^b(f) + f(b) - f(a)$
- (ii) $Q_{\Pi}(f) \leq P_a^b(f) + f(a) - f(b)$

Hence $P_a^b(f)$ is finite if and only if $Q_a^b(f)$ is finite.

Suppose that $V_a^b(f)$ is finite. For any partition Π of $[a, b]$ we have $P_\Pi(f) + Q_\Pi(f) = V_\Pi(f) \leq V_a^b(f)$ and since $P_\Pi(f) \geq 0$ and $Q_\Pi(f) \geq 0$ we also have $P_\Pi(f) \leq V_a^b(f)$ and $Q_\Pi(f) \leq V_a^b(f)$ whence it follows that $P_a^b(f)$ and $Q_a^b(f)$ is finite.

Suppose now that one of $P_a^b(f), Q_a^b(f)$ is finite, then the other one is finite as well. Then, for any partition Π

$$V_\Pi(f) = P_\Pi(f) + Q_\Pi(f) \leq P_a^b(f) + Q_a^b(f)$$

so $V_a^b(f)$ must be finite. □

Let f be a function of bounded variation. Then the additive property (theorem 2.1.3) holds also for the positive and negative variation of f , the problem below shows this for the positive variation.

Problem 2.2.2 *Let $f \in V[a, b]$ and $a < c < b$. Then*

$$P_a^b(f) = P_a^c(f) + P_c^b(f)$$

Proof: We will prove an analogue of the theorem 2.1.2 on refinements of partitions for the positive variation. Once this is done the proof is exactly as the proof of theorem 2.1.3.

As in theorem 2.1.2 we take an arbitrary partition $\Pi = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ and add one additional point c , where $x_j < c < x_{j+1}$ for some j , and denote the result Π' . By (3) we have

$$[f(x_{j+1}) - f(x_j)]^+ \leq [f(x_{j+1}) - f(c)]^+ + [f(c) - f(x_j)]^+$$

and thus

$$\begin{aligned} P_\Pi(f) &= \sum_{k=0}^{n-1} [f(x_{k+1}) - f(x_k)]^+ \\ &= [f(x_{j+1}) - f(x_j)]^+ + \sum_{k=0, k \neq j}^{n-1} [f(x_{k+1}) - f(x_k)]^+ \\ &\leq [f(x_{j+1}) - f(c)]^+ + [f(c) - f(x_j)]^+ + \sum_{k=0, k \neq j}^{n-1} [f(x_{k+1}) - f(x_k)]^+ \\ &= P_{\Pi'}(f) \end{aligned}$$

□

With the useful additivity property established we consider the following problem.

Problem 2.2.3 Find $P_a^x(f)$, $Q_a^x(f)$ and $V_a^x(f)$ if:

a) $f(x) = 4x^3 - 3x^4, -2 \leq x \leq 2$

b) $f(x) = x + 2[x], 0 \leq x \leq 3$

Solution: a) The derivative $f'(x) = 12x^2 - 12x^3 = 12x^2(1 - x)$ gives that $f(x)$ is increasing on $[-2, 1]$ and decreasing on $[1, 2]$. If $x \in [-2, 1]$ then $V_{-2}^x(f) = |f(x) - f(-2)| = 4x^3 - 3x^4 + 80$ since $f(x)$ is increasing. If $x \in (1, 2]$ then

$$\begin{aligned} V_{-2}^x(f) &= V_{-2}^1(f) + V_1^x(f) = |f(1) - f(-2)| + |f(x) - f(1)| \\ &= 81 + |4x^3 - 3x^4 - 1| = 81 + 1 - 4x^3 + 3x^4 \end{aligned}$$

Thus

$$V_{-2}^x(f) = \begin{cases} 4x^3 - 3x^4 + 80 & x \in [-2, 1] \\ 82 - 4x^3 + 3x^4 & x \in (1, 2] \end{cases}$$

Since f is increasing on $[-2, 1]$ then $[f(x) - f(y)]^+ = |f(x) - f(y)|$ and $[f(x) - f(y)]^- = 0$ for any $x, y \in [-2, 1]$. It follows that $P_{-2}^x(f) = V_{-2}^x(f)$ and $Q_{-2}^x(f) = 0$ when $x \in [-2, 1]$.

Since f is decreasing on $[1, 2]$ then $[f(x) - f(y)]^+ = 0$ and $[f(x) - f(y)]^- = |f(x) - f(y)|$ for any $x, y \in [1, 2]$. It follows that $P_1^x(f) = 0$ and $Q_1^x(f) = V_1^x(f)$ and then $P_{-2}^x(f) = P_{-2}^1(f) + P_1^x(f) = 81$ and $Q_{-2}^x(f) = Q_{-2}^1(f) + Q_1^x(f) = V_1^x(f) = 1 - 4x^3 + 3x^4$ for $x \in [1, 2]$.

Thus we have

$$P_{-2}^x(f) = \begin{cases} 4x^3 - 3x^4 + 80 & x \in [-2, 1] \\ 81 & x \in (1, 2] \end{cases}$$

$$Q_{-2}^x(f) = \begin{cases} 0 & x \in [-2, 1] \\ 1 - 4x^3 + 3x^4 & x \in (1, 2] \end{cases}$$

b) The function $f(x)$ is increasing on $[0, 3]$ so $V_0^x(f) = P_0^x(f)$ and $Q_0^x(f) = 0$ for any $x \in [0, 3]$. Furthermore

$$V_0^x(f) = |f(x) - f(0)| = x + 2[x]$$

Thus, $V_0^x(f) = P_0^x(f) = x + 2[x]$ and $Q_0^x(f) = 0$.

□

We shall finish this section with another characterization of functions of bounded variation.

Problem 2.2.4 *Let f be defined on $[a, b]$. Then $f \in V[a, b]$ if and only if there exists an increasing function φ on $[a, b]$ such that*

$$f(x'') - f(x') \leq \varphi(x'') - \varphi(x')$$

for any $a \leq x' < x'' \leq b$

Proof: Suppose that $f \in V[a, b]$. We take $\varphi(x) = V_a^x(f)$, this function is increasing and for any $a \leq x' < x'' \leq b$ we have

$$\varphi(x'') - \varphi(x') = V_{x'}^{x''}(f) \geq |f(x'') - f(x')| \geq f(x'') - f(x')$$

Conversely, suppose that there exist an increasing function φ on $[a, b]$ such that $f(x'') - f(x') \leq \varphi(x'') - \varphi(x')$ for any $a \leq x' < x'' \leq b$.

Since φ is increasing on $[a, b]$ we have $\varphi \in V[a, b]$ and $V_a^b(\varphi) = P_a^b(\varphi)$.

Furthermore, since $[f(x'') - f(x')]^+$ equals to either $f(x'') - f(x')$ or 0 we have

$$[f(x'') - f(x')]^+ \leq \varphi(x'') - \varphi(x')$$

for any $a \leq x' < x'' \leq b$. Therefore $P_\Pi(f) \leq V_a^b(\varphi)$ for any partition Π of $[a, b]$ whence

$$P_a^b(f) \leq V_a^b(\varphi)$$

Since $V_a^b(\varphi)$ is finite $P_a^b(f)$ must be finite as well. But then $V_a^b(f)$ is finite according to problem 2.2.1, that is $f \in V[a, b]$.

□

2.3 Conditions for bounded variation

We know that piecewise monotone functions and any function that can be expressed as the difference of two increasing functions are of bounded variation. In this section we shall give some additional conditions which will guarantee that a function is of bounded variation.

Definition 2.3.1 Let $f : [a, b] \rightarrow \mathbb{R}$, f is said to satisfy a **Lipschitz condition** if there exists a constant $M > 0$ such that for every $x, y \in [a, b]$ we have

$$|f(x) - f(y)| \leq M|x - y|$$

Theorem 2.3.2 [1, p. 119]. If $f : [a, b] \rightarrow \mathbb{R}$ satisfies a Lipschitz condition on $[a, b]$ with constant K , then $f \in V[a, b]$ and $V_a^b(f) \leq K(b - a)$.

Proof: Suppose that $|f(x) - f(y)| \leq K|x - y|$ for every $x, y \in [a, b]$. Take an arbitrary partition $\Pi = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$. Then

$$V_{\Pi}(f) = \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| \leq \sum_{k=0}^{n-1} K|x_{k+1} - x_k| = K(b - a)$$

Since Π was arbitrary the inequality above is valid for any partition, which means that the sums $V_{\Pi}(f)$ are bounded above by $K(b - a)$ whence it follows that $f \in V[a, b]$ and $V_a^b(f) \leq K(b - a)$.

□

Theorem 2.3.3 [1, p. 119]. If $f : [a, b] \rightarrow \mathbb{R}$ is differentiable on $[a, b]$ and if there exists $M > 0$ such that $|f'(x)| \leq M$ on $[a, b]$ then $f \in V[a, b]$ and $V_a^b(f) \leq M(b - a)$

Proof: For any $x, y \in [a, b]$ we have $f(x) - f(y) = f'(c)(x - y)$ for some c between x and y according to the Mean value theorem. Hence

$$|f(x) - f(y)| = |f'(c)||x - y| \leq M|x - y|$$

for any $x, y \in [a, b]$, that is f satisfies a Lipschitz condition on $[a, b]$ with constant M and thus Theorem 2.3.2 gives the result.

□

Problem 2.3.4 Prove that $f \in V[0, 1]$ if

$$f(x) = \begin{cases} x^{3/2} \cos(\pi/\sqrt{x}) & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}$$

Solution: For any $x \in (0, 1]$ the function $f(x)$ has the derivative

$$f'(x) = \frac{3}{2}x^{1/2} \cos(\pi/\sqrt{x}) + \frac{\pi}{2} \sin(\pi/\sqrt{x})$$

and in the point 0 the derivative

$$f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} x^{1/2} \cos(\pi/\sqrt{x}) = 0$$

Now, for any $x \in [0, 1]$ we have

$$|f'(x)| \leq \left| \frac{3}{2}x^{1/2} \cos(\pi/\sqrt{x}) \right| + \left| \frac{\pi}{2} \sin(\pi/\sqrt{x}) \right| \leq \frac{3}{2} + \frac{\pi}{2}$$

Since f has a bounded derivative on $[0, 1]$ we have that $f \in V[0, 1]$.

□

With the additional assumption that the derivative f' is continuous on $[a, b]$ we can give a formula for the total variation:

Theorem 2.3.5 *If f is continuously differentiable on $[a, b]$ then $f \in V[a, b]$ and the total variation is given by*

$$V_a^b(f) = \int_a^b |f'(x)| dx$$

Proof: Since f' is continuous then f' is bounded so $f \in V[a, b]$. Furthermore, the continuity of $|f'|$ implies that $|f'|$ is Riemann-integrable on $[a, b]$. Set $I = \int_a^b |f'(x)| dx$ and let $\epsilon > 0$ be arbitrary. Then there exists $\delta > 0$ such that for any partition \mathcal{P} with $d(\mathcal{P}) < \delta$ the Riemann-sum $S(|f'|, \mathcal{P})$, with arbitrary intermediate points, satisfy the following inequalities:

$$I - \epsilon < S(|f'|, \mathcal{P}) < I + \epsilon$$

So let Π be a partition of $[a, b]$ with $d(\Pi) < \delta$. According to the Mean Value theorem $|f(x_{k+1}) - f(x_k)| = |f'(t_k)|(x_{k+1} - x_k)$ for some $t_k \in (x_k, x_{k+1})$. Hence

$$V_{\Pi}(f) = \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| = \sum_{k=0}^{n-1} |f'(t_k)|(x_{k+1} - x_k)$$

The right hand side is a Riemann sum $S(|f'|, \Pi)$ and since $d(\Pi) < \delta$ we have $I - \epsilon < V_{\Pi}(f) < I + \epsilon \Rightarrow I - \epsilon < V_a^b(f)$ whence $I \leq V_a^b(f)$ since ϵ is arbitrary. Further, for any partition $\Pi = \{x_0, x_1, \dots, x_n\}$ we have

$$\begin{aligned} V_{\Pi}(f) &= \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| = \sum_{k=0}^{n-1} \left| \int_{x_k}^{x_{k+1}} f'(x) dx \right| \\ &\leq \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} |f'(x)| dx = \int_a^b |f'(x)| dx \end{aligned}$$

Since Π is an arbitrary partition it follows that $V_a^b(f) \leq I$. Then we must have $I = V_a^b(f)$. □

In the following problem we will establish a fact that will be very useful together with theorem 2.3.5.

Problem 2.3.6 *Let f be defined on $[a, b]$. If $f \in V[a, c]$ for any $a < c < b$ and if there exists a number M such that $V_a^c(f) \leq M$ for any $a < c < b$ then $f \in V[a, b]$.*

Proof: Let $\Pi = \{x_0, x_1, \dots, x_n\}$ be an arbitrary partition of $[a, b]$. Set $\Pi' = \{x_0, x_1, \dots, x_{n-1}\}$. Then Π' is a partition of $[a, x_{n-1}]$ and since $V_a^{x_{n-1}}(f) \leq M$ we have

$$\begin{aligned} V_{\Pi}(f) &= V_{\Pi'} + |f(b) - f(x_{n-1})| \\ &\leq V_a^{x_{n-1}}(f) + |f(b) - f(x_{n-1})| \\ &\leq M + |f(b) - f(x_{n-1})| \\ &= M + |f(b) + (f(a) - f(a)) - f(x_{n-1})| \\ &\leq M + |f(b) - f(a)| + |f(a) - f(x_{n-1})| \\ &\leq M + |f(b) - f(a)| + V_a^{x_{n-1}}(f) \\ &\leq 2M + |f(b) - f(a)| \end{aligned}$$

Thus $V_{\Pi}(f)$ is bounded above by $2M + |f(b) - f(a)|$ and therefore $f \in V[a, b]$. □

The problem below demonstrates how one can use theorem 2.3.5 and problem 2.3.6:

Problem 2.3.7 Prove that $f \in V[0, 1]$ if

$$f(x) = \begin{cases} x^{3/2} \cos(\pi/x) & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}$$

Solution: For any $x \in (0, 1]$ the function $f(x)$ has the derivative

$$f'(x) = \frac{3}{2}\sqrt{x} \cos(\pi/x) + \frac{\pi}{\sqrt{x}} \sin(\pi/x)$$

For any $0 < a < 1$ the function $|f'(x)|$ is continuous on $[a, 1]$ and thus $f \in V[a, 1]$ and the total variation of f over $[a, 1]$ is given by

$$V_a^1(f) = \int_a^1 |f'(x)| dx$$

Furthermore

$$|f'(x)| = \left| \frac{3}{2}\sqrt{x} \cos(\pi/x) + \frac{\pi}{\sqrt{x}} \sin(\pi/x) \right| \leq \frac{3}{2}\sqrt{x} + \frac{\pi}{\sqrt{x}}$$

so it follows that

$$V_a^1(f) = \int_a^1 |f'(x)| dx \leq \int_a^1 \left(\frac{3}{2}\sqrt{x} + \frac{\pi}{\sqrt{x}} \right) dx < 1 + 2\pi$$

that is $V_a^1(f) \leq 1 + 2\pi$ for any $a > 0$. Then $f \in V[0, 1]$ according to problem 2.3.6.

□

2.4 The function $v(x)$

In the proof of Jordans theorem we introduced the function of total variation, $v(x) = V_a^x(f)$. In this section we shall study some properties of this function, more specifically continuity and differentiability.

The following theorem is given in [1]:

Theorem 2.4.1 [1, p. 125]. *Let $f \in V[a, b]$, then $v(x)$ is continuous at a point c if and only if $f(x)$ is continuous at c .*

Instead of proving 2.4.1 we shall give a result that is a bit stronger:

Problem 2.4.2 *Let $f \in V[a, b]$, then for any $x \in (a, b)$ we have*

$$v(x+0) - v(x) = |f(x+0) - f(x)| \text{ and } v(x) - v(x-0) = |f(x) - f(x-0)|$$

Proof: We shall prove the first equality, the proof of the second one is similar.

Take a fixed but arbitrary $x_0 \in (a, b)$ and set $L = |f(x_0+0) - f(x_0)|$

Given $\epsilon > 0$, there exists a $\delta > 0$ such that $x_0 < x < x_0 + \delta \Rightarrow$

$$L - \frac{\epsilon}{2} < |f(x) - f(x_0)| < L + \frac{\epsilon}{2}$$

Let $\Pi = \{x_0, x_1, \dots, x_n\}$ be a partition of $[x_0, b]$ such that

$$\begin{aligned} (i) \quad & V_{x_0}^b(f) - \frac{\epsilon}{2} < V_{\Pi}(f) \\ (ii) \quad & x_1 < x_0 + \delta \end{aligned}$$

The point x_1 can be treated as an arbitrary point satisfying $x_0 < x_1 < x_0 + \delta$. Now set $\Pi' = \Pi \setminus \{x_0\}$, clearly Π' is a partition of $[x_1, b]$ and we have

$$V_{\Pi}(f) - V_{\Pi'}(f) = |f(x_1) - f(x_0)|$$

Furthermore,

$$\begin{aligned} V_{\Pi}(f) - V_{\Pi'}(f) &\geq (V_{x_0}^b(f) - \frac{\epsilon}{2}) - V_{\Pi'}(f) \\ &\geq V_{x_0}^b(f) - V_{x_1}^b(f) - \frac{\epsilon}{2} \\ &= V_{x_0}^{x_1}(f) - \frac{\epsilon}{2} \\ &= V_a^{x_1}(f) - V_a^{x_0}(f) - \frac{\epsilon}{2} \\ &= v(x_1) - v(x_0) - \frac{\epsilon}{2} \end{aligned}$$

Thus $v(x_1) - v(x_0) - \frac{\epsilon}{2} \leq |f(x_1) - f(x_0)|$ and since $x_1 < x_0 + \delta$ we have $|f(x_1) - f(x_0)| < L + \frac{\epsilon}{2}$ whence

$$v(x_1) - v(x_0) < L + \epsilon$$

But we also have that $|f(x_1) - f(x_0)| \leq V_{x_0}^{x_1}(f) = v(x_1) - v(x_0)$
 $\Rightarrow L - \frac{\epsilon}{2} < v(x_1) - v(x_0) \Rightarrow L - \epsilon < v(x_1) - v(x_0)$

Since we regard the point x_1 as an arbitrary point satisfying $x_0 < x_1 < x_0 + \delta$ we have

$$x_0 < x < x_0 + \delta \Rightarrow L - \epsilon < v(x) - v(x_0) < L + \epsilon$$

and hence $v(x_0 + 0) - v(x_0) = L$

□

We shall proceed to study how the differentiability of the function $v(x)$ is related to that of $f(x)$.

Problem 2.4.3 *If f has a continuous derivative on $[a, b]$, then the function $v(x)$ is differentiable and has a continuous derivative on $[a, b]$.*

Proof: First of all, since f' is continuous on $[a, b]$ then f' is bounded on $[a, b]$ and therefore $f \in V[a, b]$. Further, by theorem 2.3.5

$$v(x) = \int_a^x |f'(t)| dt$$

and according to the fundamental theorem of calculus $v'(x) = |f'(x)|$ which is continuous on $[a, b]$.

□

If $f \in V[a, b]$ then the existence of a derivative f' on $[a, b]$ is not sufficient to guarantee that v is differentiable on $[a, b]$, as we shall demonstrate in a problem. But first we need a lemma:

Lemma 2.4.4 *Set $\varphi(x) = \int_0^x |\sin t| dt$, then*

$$\lim_{x \rightarrow \infty} \frac{\varphi(x)}{x} = \frac{2}{\pi}$$

Proof: For any $n \in \mathbb{N}$

$$\varphi(n\pi) = \int_0^{n\pi} |\sin t| dt = \sum_{k=0}^{n-1} \int_{k\pi}^{(k+1)\pi} |\sin t| dt$$

To evaluate $\int_{k\pi}^{(k+1)\pi} |\sin t| dt$ for a given k , we make the change of variable $t = k\pi + u$, then $0 \leq u \leq \pi$ and since the function $|\sin|$ is periodic with a period π then

$$\int_{k\pi}^{(k+1)\pi} |\sin t| dt = \int_0^\pi \sin u du = 2$$

and thus $\varphi(n\pi) = 2n$.

Assume that $n\pi \leq x < (n+1)\pi$ for some $n \in \mathbb{N}$, then

$$\frac{2n}{(n+1)\pi} < \frac{2n}{x} = \frac{\varphi(n\pi)}{x} \leq \frac{\varphi(x)}{x}$$

since φ is an increasing function. On the other hand,

$$\frac{\varphi(x)}{x} \leq \frac{\varphi(x)}{n\pi} \leq \frac{\varphi((n+1)\pi)}{n\pi} = \frac{2(n+1)}{n\pi}$$

and therefore

$$\frac{2}{\pi} \cdot \frac{n}{n+1} < \frac{\varphi(x)}{x} \leq \frac{2}{\pi} \cdot \frac{n+1}{n}$$

Letting $x \rightarrow \infty$ gives the result. □

Problem 2.4.5 Let $f(x) = x^2 \cos(|x|^{-3/2})$ for $0 < |x| \leq 1$ and $f(0) = 0$. Then $f \in V[-1, 1]$ and $f(x)$ is differentiable on $[-1, 1]$ but $v(x)$ is not differentiable at the point $x = 0$.

Proof: If $x \in [-1, 1] \setminus \{0\}$ then the standard rules of differentiation applies, for example if $x \in (0, 1]$ then $f'(x) = 2x \cos(x^{-3/2}) + \frac{3}{2}x^{-1/2} \sin(x^{-3/2})$. Using the definition of the derivative we can easily show that $f'(0) = 0$.

By applying a similar argument as in problem 2.3.7 one shows that $f \in V[-1, 0]$ and $f \in V[0, 1]$ and thus $f \in V[-1, 1]$.

To show that $v(x)$ isn't differentiable in $x = 0$ it's sufficient to show that the limit $\lim_{x \rightarrow +0} (v(x) - v(0))/x$ doesn't exist. For any a such that $0 < a < x$ we have $v(x) - v(a) = \int_a^x |f'(t)| dt$ and since $v(x)$ is continuous at the point $x = 0$ it follows that $v(x) - v(0) = \int_0^x |f'(t)| dt$ where the integral is improper.

Thus we need to show that the limit

$$\lim_{x \rightarrow +0} \frac{1}{x} \int_0^x |2t \cos(t^{-3/2}) + \frac{3}{2}t^{-1/2} \sin(t^{-3/2})| dt$$

doesn't exist. It's easy to see that

$$\int_0^x |2t \cos(t^{-3/2}) + \frac{3}{2}t^{-1/2} \sin(t^{-3/2})| dt = \int_0^x |\frac{3}{2}t^{-1/2} \sin(t^{-3/2})| dt + \mathbf{O}(x^2)$$

and therefore we consider $\frac{1}{x} \int_0^x |t^{-1/2} \sin(t^{-3/2})| dt$.

Make the change of variable $u = t^{-3/2}$, then $t = u^{-2/3}$ and $dt = -\frac{2}{3}u^{-5/3}du$, the integral above becomes

$$\frac{1}{x} \int_{x^{-3/2}}^{\infty} \frac{2}{3} |\sin u| u^{-4/3} du$$

Set $y = x^{-3/2}$, then $y \rightarrow \infty$ as $x \rightarrow +0$ and therefore we shall show that

$$y^{2/3} \int_y^{\infty} |\sin u| u^{-4/3} du \rightarrow \infty \text{ as } y \rightarrow \infty$$

We shall integrate $\int_y^{\infty} |\sin u| u^{-4/3} du$ by using integration by parts. Set $\varphi(u) = \int_0^u |\sin t| dt$, then φ is a primitive of $|\sin|$ and integration by parts gives

$$\begin{aligned} \int_y^{\infty} |\sin u| u^{-4/3} du &= \varphi(u) u^{-4/3} \Big|_y^{\infty} + \frac{4}{3} \int_y^{\infty} \varphi(u) u^{-7/3} du \\ &= \frac{4}{3} \int_y^{\infty} \varphi(u) u^{-7/3} du - \varphi(y) y^{-4/3} \end{aligned}$$

and therefore $y^{2/3} \int_y^{\infty} |\sin u| u^{-4/3} du = y^{1/3} \left(\frac{4}{3} y^{1/3} \int_y^{\infty} \varphi(u) u^{-7/3} du - \frac{\varphi(y)}{y} \right)$.

According to lemma 2.4.4 $\lim_{y \rightarrow \infty} \frac{\varphi(y)}{y} = \frac{2}{\pi}$ and $\lim_{y \rightarrow \infty} y^{1/3} \int_y^{\infty} \varphi(u) u^{-7/3} du = \frac{6}{\pi}$ which is shown by using L'Hospital's rule:

$$\lim_{y \rightarrow \infty} \frac{(\int_y^{\infty} \varphi(u) u^{-7/3} du)'}{(y^{-1/3})'} = \lim_{y \rightarrow \infty} \frac{-\varphi(y) y^{-7/3}}{-\frac{1}{3} y^{-4/3}} = \lim_{y \rightarrow \infty} \frac{3\varphi(y)}{y} = \frac{6}{\pi}$$

by lemma 2.4.4. Thus $\lim_{y \rightarrow \infty} \left(\frac{4}{3} y^{1/3} \int_y^{\infty} \varphi(u) u^{-7/3} du - \frac{\varphi(y)}{y} \right)$ exists and therefore $y^{1/3} \left(\frac{4}{3} y^{1/3} \int_y^{\infty} \varphi(u) u^{-7/3} du - \frac{\varphi(y)}{y} \right) \rightarrow \infty$ as $y \rightarrow \infty$.

□

According to problem 2.4.3, if f' is continuous on $[a, b]$ then v is differentiable on $[a, b]$. The above problem demonstrates that only the existence of a derivative f' at a point is not sufficient to guarantee that v is differentiable. We shall show that if f' is continuous at a point, then v will be differentiable at the point. We cannot use the same approach as in problem 2.4.3 since we only assume that f' is continuous at one point. Indeed, f' might not even be Riemann-integrable.

Problem 2.4.6 *Let $f \in V[a, b]$ be differentiable on $[a, b]$. If f' is continuous at a point $x_0 \in [a, b]$ then $v(x)$ is differentiable at x_0 .*

Proof: Continuity of f' in x_0 implies that $|f'|$ is continuous in x_0 . Let $\epsilon > 0$ be given, there exists $\delta_1 > 0$ such that if $|x - x_0| < \delta_1$ then

$$|f'(x_0)| - \epsilon < |f'(x)| < |f'(x_0)| + \epsilon$$

Take $\delta = \frac{\delta_1}{2}$, for any $x \in [x_0 - \delta, x_0 + \delta]$ we have $|f'(x)| < |f'(x_0)| + \epsilon$ and since f has a bounded derivative on $[x_0 - \delta, x_0 + \delta]$ then f satisfies a Lipschitz condition on $[x_0 - \delta, x_0 + \delta]$ with constant $|f'(x_0)| + \epsilon$.

If $x_0 < x < x_0 + \delta$ then $v(x) - v(x_0) = V_{x_0}^x(f) \leq (|f'(x_0)| + \epsilon)(x - x_0)$ whence

$$\frac{v(x) - v(x_0)}{x - x_0} < |f'(x_0)| + \epsilon \quad (4)$$

In the same way one shows that the inequality above also holds for x such that $x_0 - \delta < x < x_0$. On the other hand, if $x > x_0$ we have

$$\frac{v(x) - v(x_0)}{x - x_0} \geq \frac{|f(x) - f(x_0)|}{|x - x_0|} = |f'(x_1)|$$

for some x_1 between x and x_0 according to the Mean value theorem. Since $|x_1 - x_0| < \delta$ we have $|f'(x_1)| > |f'(x_0)| - \epsilon$ and therefore

$$|f'(x_0)| - \epsilon < \frac{v(x) - v(x_0)}{x - x_0} \quad (5)$$

A similar inequality holds for $x < x_0$. Combining inequality 4 with inequality 5 the following implication follows:

$$0 < |x - x_0| < \delta \Rightarrow \left| \frac{v(x) - v(x_0)}{x - x_0} - |f'(x_0)| \right| < \epsilon$$

and hence v is differentiable in x_0 , with the derivative $|f'(x_0)|$.

□

We know now that continuity of f' in a point is sufficient for v to be differentiable in the point. However, it's not a necessary condition.

Problem 2.4.7 Let $f(x) = x^2 \cos(1/x)$ for $0 < |x| \leq 1$ and $f(0) = 0$. Then $f(x)$ and $v(x)$ are differentiable everywhere on $[-1, 1]$ but both f' and v' are discontinuous at the point $x = 0$.

Proof: Clearly f is differentiable with derivative $f'(x) = 2x \cos(1/x) + \sin(1/x)$ for $x \in [-1, 1] \setminus \{0\}$. We have $(f(x) - f(0))/x = x \cos(1/x) \rightarrow 0$ as $x \rightarrow 0$ and thus $f'(0) = 0$. However, since $\lim_{x \rightarrow 0} f'(x)$ doesn't exist f' is

not continuous at the point $x = 0$. It follows from problem 2.4.6 that v is differentiable and has the derivative $v'(x) = |f'(x)|$ for every $x \in [-1, 1] \setminus \{0\}$ since $f'(x)$ is continuous there.

Now we shall show that $\lim_{x \rightarrow 0} (v(x) - v(0))/x$ exists. Let $x > 0$, as in problem 2.4.5 we have $v(x) - v(0) = \int_0^x |f'(t)| dt$ and therefore the derivative in $x = 0$ is given by

$$\lim_{x \rightarrow +0} \frac{1}{x} \int_0^x |2t \cos \frac{1}{t} + \sin \frac{1}{t}| dt$$

Further,

$$\int_0^x |2t \cos \frac{1}{t} + \sin \frac{1}{t}| dt = \int_0^x |\sin \frac{1}{t}| dt + \mathbf{O}(x^2)$$

so it'll be sufficient to show the existence of $\lim_{x \rightarrow +0} \frac{1}{x} \int_0^x |\sin \frac{1}{t}| dt$.

Make the change of variable $u = \frac{1}{t}$, then $dt = -\frac{1}{u^2} du$ and we get

$$\lim_{x \rightarrow +0} \frac{1}{x} \int_{1/x}^{\infty} |\sin u| \frac{1}{u^2} du$$

Set $y = \frac{1}{x}$, then $y \rightarrow +\infty$ as $x \rightarrow +0$ which yields

$$\lim_{y \rightarrow \infty} y \int_y^{\infty} |\sin u| \frac{1}{u^2} du$$

Let $\varphi(u)$ be as in problem 2.4.5 and use integration by parts:

$$\begin{aligned} \int_y^{\infty} |\sin u| u^{-2} du &= \varphi(u) u^{-2} \Big|_y^{\infty} + 2 \int_y^{\infty} \varphi(u) u^{-3} du \\ &= 2 \int_y^{\infty} \varphi(u) u^{-3} du - \varphi(y) y^{-2} \end{aligned}$$

and thus $y \int_y^{\infty} |\sin u| u^{-2} du = 2y \int_y^{\infty} \varphi(u) u^{-3} du - \varphi(y) y^{-1}$

Write $y \int_y^{\infty} \varphi(u) u^{-3} du = \int_y^{\infty} \varphi(u) u^{-3} du / y^{-1}$ and apply L'Hospital's rule:

$$\lim_{y \rightarrow \infty} \frac{(\int_y^{\infty} \varphi(u) u^{-3} du)'}{(y^{-1})'} = \lim_{y \rightarrow \infty} \frac{-\varphi(y) y^{-3}}{-y^{-2}} = \lim_{y \rightarrow \infty} \frac{\varphi(y)}{y} = \frac{2}{\pi}$$

by lemma 2.4.4 and therefore $\lim_{y \rightarrow \infty} y \int_y^{\infty} \varphi(u) u^{-3} du = \frac{2}{\pi}$ according to L'Hospital's rule. Thus

$$\begin{aligned} \lim_{y \rightarrow \infty} y \int_y^{\infty} |\sin u| u^{-2} du &= \lim_{y \rightarrow \infty} (2y \int_y^{\infty} \varphi(u) u^{-3} du - \frac{\varphi(y)}{y}) \\ &= \frac{4}{\pi} - \frac{2}{\pi} = \frac{2}{\pi} \end{aligned}$$

Therefore $v'(0) = \frac{2}{\pi}$.

□

2.5 Two limits

In this section we shall consider continuous functions of bounded variation. If f is continuous then the total variation is given as a limit of the sums $V_{\Pi}(f)$. We shall introduce some notations:

Let $\Pi = \{x_0, x_1, \dots, x_n\}$ be any partition of $[a, b]$. Then we set

$$d(\Pi) = \max_{0 \leq k \leq n-1} (x_{k+1} - x_k)$$

Now let $f : [a, b] \rightarrow \mathbb{R}$ be continuous and $\Pi = \{x_0, x_1, \dots, x_n\}$ any partition of $[a, b]$. Let $M_k = \max_{x_k \leq x \leq x_{k+1}} f(x)$ and $m_k = \min_{x_k \leq x \leq x_{k+1}} f(x)$ and set

$$\Omega_{\Pi}(f) = \sum_{k=0}^{n-1} (M_k - m_k)$$

Problem 2.5.1 *If $f \in C[a, b]$ then both sums $V_{\Pi}(f)$ and $\Omega_{\Pi}(f)$ tend to $V_a^b(f)$ as $d(\Pi) \rightarrow 0$.*

Proof: We deal with $V_{\Pi}(f)$ first. We're going to show that for every $\epsilon > 0$ there exists a $\delta > 0$ so that if Π is any partition of $[a, b]$ with $d(\Pi) < \delta$ then $V_a^b(f) - \epsilon < V_{\Pi}(f)$.

Let $\epsilon > 0$ be given, then there exists a partition $\Pi_0 = \{y_0, y_1, \dots, y_{m+1}\}$ (the reason for this notation will be clear later) such that

$$V_a^b(f) - \frac{\epsilon}{2} < V_{\Pi_0}(f)$$

Since f is continuous on a compact interval $[a, b]$ f is uniformly continuous on $[a, b]$. Therefore there exists a $\delta_1 > 0$ such that

$$|x - y| < \delta_1 \Rightarrow |f(x) - f(y)| < \frac{\epsilon}{4m}$$

Now let $\Pi = \{x_0, x_1, \dots, x_n\}$ be any partition with $d(\Pi) < \delta$ where $\delta < \delta_1$ but also small enough to ensure that there is only one point $y_j \in \Pi_0$ in every interval $[x_k, x_{k+1}]$.

Take the partition Π' consisting of the points in Π together with the points in Π_0 . Then $V_{\Pi'}(f) \geq V_{\Pi_0}(f)$ since Π' is a refinement of Π_0 .

We will now compare the sums $V_{\Pi}(f)$ and $V_{\Pi'}(f)$:

$V_{\Pi}(f)$ consists of terms $|f(x_{k+1}) - f(x_k)|$ and they coincides with the terms of $V_{\Pi'}(f)$ except for those terms that corresponds to those intervals $[x_k, x_{k+1}]$ which contains a point y_j with $1 \leq j \leq m$. Here the term $|f(x_{k+1}) - f(x_k)|$ is exchanged for $|f(y_j) - f(x_k)| + |f(x_{k+1}) - f(y_j)|$.

Thus $V_{\Pi'}(f) - V_{\Pi}(f)$ consists of m terms on the form

$$|f(y_j) - f(x_k)| + |f(x_{k+1}) - f(y_j)| - |f(x_{k+1}) - f(x_k)|$$

and therefore

$$\begin{aligned} 0 &\leq |f(y_j) - f(x_k)| + |f(x_{k+1}) - f(y_j)| - |f(x_{k+1}) - f(x_k)| \\ &\leq |f(y_j) - f(x_k)| + |f(x_{k+1}) - f(y_j)| \\ &< \epsilon/4m + \epsilon/4m = \epsilon/2m \end{aligned}$$

where the last inequality is true because $|x_{k+1} - y_j|, |y_j - x_k| < \delta_1$.
Therefore

$$V_{\Pi'}(f) - V_{\Pi}(f) < m \cdot \epsilon/2m = \epsilon/2$$

and thus

$$V_a^b(f) - \epsilon/2 < V_{\Pi'}(f) < V_{\Pi}(f) + \epsilon/2 \Leftrightarrow V_a^b(f) - \epsilon < V_{\Pi}(f)$$

We conclude that $V_{\Pi}(f) \rightarrow V_a^b(f)$ when $d(\Pi) \rightarrow 0$

For the second part of the problem, let $\epsilon > 0$ be given, there exists a $\delta > 0$ such that $d(\Pi) < \delta \Rightarrow V_a^b(f) - \epsilon < V_{\Pi}(f)$. Take any partition Π with $d(\Pi) < \delta$. We notice that for any partition we have $V_{\Pi}(f) \leq \Omega_{\Pi}(f)$ and thus $d(\Pi) < \delta \Rightarrow V_a^b(f) - \epsilon < \Omega_{\Pi}(f)$

On the other hand, for any partition $\Pi = \{x_0, x_1, \dots, x_n\}$ let t_k, s_k be points in which f attains its maximum respectively minimum on $[x_k, x_{k+1}]$, that is $M_k = f(t_k)$ and $m_k = f(s_k)$. Now add t_k and s_k , $0 \leq k \leq n-1$, to the partition Π and denote the resulting partition Π' . The sum $V_{\Pi'}(f)$ will then contain terms $|f(t_k) - f(s_k)| = M_k - m_k$ so we have

$$V_a^b(f) \geq V_{\Pi'}(f) = \sum_{k=0}^{n-1} (M_k - m_k) + \text{additional positive terms}$$

$\Rightarrow V_a^b(f) \geq \Omega_{\Pi}(f)$ for any partition Π .

Hence, for every $\epsilon > 0$ there exists a $\delta > 0$ such that if Π is any partition with $d(\Pi) < \delta$ then $V_a^b(f) - \epsilon < \Omega_{\Pi}(f) \leq V_a^b(f)$, that is $\Omega_{\Pi}(f) \rightarrow V_a^b(f)$ as $d(\Pi) \rightarrow 0$.

□

The total variation of a continuous function of bounded variation is also given by another limit, as the following problem shows:

Problem 2.5.2 *Let $f \in V[a, b]$ and let f be continuous on $[a, b]$. Then*

$$V_a^b(f) = \lim_{h \rightarrow +0} \frac{1}{h} \int_a^{b-h} |f(x) - f(x+h)| dx$$

Proof: Since $|f(x) - f(x+h)| \leq v(x+h) - v(x)$ it follows that

$$\begin{aligned}
\frac{1}{h} \int_a^{b-h} |f(x) - f(x+h)| dx &\leq \frac{1}{h} \int_a^{b-h} [v(x+h) - v(x)] dx \\
&= \frac{1}{h} \int_a^{b-h} v(x+h) dx - \frac{1}{h} \int_a^{b-h} v(x) dx \\
&= \frac{1}{h} \int_{a+h}^b v(x) dx - \frac{1}{h} \int_a^{b-h} v(x) dx \\
&= \frac{1}{h} \int_{b-h}^b v(x) dx - \frac{1}{h} \int_a^{a+h} v(x) dx \\
&\leq v(b) = V_a^b(f)
\end{aligned}$$

The inequality above is valid for every h and thus

$$\limsup_{h \rightarrow +0} \frac{1}{h} \int_a^{b-h} |f(x) - f(x+h)| dx \leq V_a^b(f)$$

Now let $\Pi = \{x_0, x_1, \dots, x_n\}$ be any partition of $[a, b]$.

For any h small enough to ensure that $x_{n-1} < b-h$ and for $j \in \mathbb{N}$ such that $0 \leq j \leq n-2$ the following is true:

$$\begin{aligned}
\frac{1}{h} \int_{x_j}^{x_{j+1}} |f(x) - f(x+h)| dx &\geq \left| \frac{1}{h} \int_{x_j}^{x_{j+1}} f(x) - f(x+h) dx \right| \\
&= \left| \frac{1}{h} \int_{x_j}^{x_{j+1}} f(x) dx - \frac{1}{h} \int_{x_{j+h}}^{x_{j+1}+h} f(x) dx \right| \\
&= \left| \frac{1}{h} \int_{x_j}^{x_{j+h}} f(x) dx - \frac{1}{h} \int_{x_{j+1}}^{x_{j+1}+h} f(x) dx \right|
\end{aligned}$$

The right-hand side tends to $|f(x_j) - f(x_{j+1})|$ as $h \rightarrow +0$ according to the fundamental theorem of calculus. So, for $0 \leq j \leq n-2$

$$\liminf_{h \rightarrow +0} \frac{1}{h} \int_{x_j}^{x_{j+1}} |f(x) - f(x+h)| dx \geq |f(x_{j+1}) - f(x_j)|$$

In the same way one shows that

$$\liminf_{h \rightarrow +0} \frac{1}{h} \int_{x_{n-1}}^{b-h} |f(x) - f(x+h)| dx \geq |f(b) - f(x_{n-1})|$$

Summing these inequalities gives

$$\begin{aligned}
\liminf_{h \rightarrow +0} \frac{1}{h} \int_a^{b-h} |f(x) - f(x+h)| dx &\geq \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| \\
&= V_{\Pi}(f)
\end{aligned}$$

for any partition Π . It follows that

$$\liminf_{h \rightarrow +0} \frac{1}{h} \int_a^{b-h} |f(x) - f(x+h)| dx \geq V_a^b(f)$$

Thus we've obtained the following inequalities:

$$\begin{aligned} V_a^b(f) &\leq \liminf_{h \rightarrow +0} \frac{1}{h} \int_a^{b-h} |f(x) - f(x+h)| dx \\ &\leq \limsup_{h \rightarrow +0} \frac{1}{h} \int_a^{b-h} |f(x) - f(x+h)| dx \\ &\leq V_a^b(f) \end{aligned}$$

and therefore

$$\lim_{h \rightarrow +0} \frac{1}{h} \int_a^{b-h} |f(x) - f(x+h)| dx = V_a^b(f)$$

□

3 Jump functions

In this section we shall introduce the so called jump function of a function of bounded variation. Before we give a definition we shall establish some facts.

One of the consequences of Jordan's theorem is that if f is of bounded variation, then the set of points at which f is discontinuous is at most countable. Let $\{d_n\}_{n \geq 1}$ be the sequence of points of discontinuity of f and let σ_n^+ be the right-hand jump of f at d_n and σ_n^- the left-hand jump of f at d_n , that is

$$\sigma_n^+ = f(d_n + 0) - f(d_n), \text{ and } \sigma_n^- = f(d_n) - f(d_n - 0)$$

Problem 3.0.3 *If $f \in V[a, b]$ then*

$$\sum_{n=1}^{\infty} (|\sigma_n^+| + |\sigma_n^-|) \leq V_a^b(f)$$

Proof: Set

$$S_n = \sum_{k=1}^n (|\sigma_k^+| + |\sigma_k^-|)$$

It is sufficient to show that the sequence $\{S_n\}_{n \in \mathbb{N}}$ is bounded above by $V_a^b(f)$, that is $S_n \leq V_a^b(f)$ for every n .

Let $\{d_k\}_{k \in \mathbb{N}}$ be the sequence of all points of discontinuity of f in $[a, b]$. For the sake of simplicity assume that neither a nor b is a point of discontinuity. The problem that arises when a or b is a point of discontinuity is purely notational, the reasoning is the same.

Let $\epsilon > 0$ be given and take $n \in \mathbb{N}$. Consider the points d_k , $1 \leq k \leq n$, we may reorder these points so that $d_1 < d_2 < \dots < d_n$. For $1 \leq k \leq n$ let r_k, l_k be points such that

$$\begin{aligned} (i) \quad & r_{k-1} < l_k < d_k < r_k < l_{k+1} \quad , 2 \leq k \leq n-1 \\ (ii) \quad & |f(d_k) - f(l_k)| > |\sigma_k^-| - \frac{\epsilon}{2n} \\ (iii) \quad & |f(r_k) - f(d_k)| > |\sigma_k^+| - \frac{\epsilon}{2n} \end{aligned}$$

Now let Π be the partition of $[a, b]$ consisting of a, b and the points l_k, d_k, r_k for $1 \leq k \leq n$. Then we have

$$\begin{aligned} V_a^b(f) &\geq V_{\Pi}(f) \geq \sum_{k=1}^n (|f(d_k) - f(l_k)| + |f(r_k) - f(d_k)|) \\ &> \sum_{k=1}^n (|\sigma_k^-| - \frac{\epsilon}{2n} + |\sigma_k^+| - \frac{\epsilon}{2n}) = S_n - \epsilon \end{aligned}$$

Since ϵ is arbitrary $S_n \leq V_a^b(f)$ for every n .

□

Definition 3.0.4 Set $s(a) = 0$ and for $a < x \leq b$

$$s(x) = \sum_{a \leq d_k < x} \sigma_k^+ + \sum_{a < d_k \leq x} \sigma_k^-$$

The function $s(x)$ is called the **jump function** of f .

It follows from problem 3.0.3 that the series $\sum_{k=1}^{\infty} \sigma_k^+$ and $\sum_{k=1}^{\infty} \sigma_k^-$ are absolutely convergent and thus the subseries $\sum_{a \leq d_k < x} \sigma_k^+$ and $\sum_{a < d_k \leq x} \sigma_k^-$ are absolutely convergent. This is crucial in order for the jump function to be well-defined.

In the following problem some properties of the jump function is proved.

Problem 3.0.5 If s is the jump function of $f \in V[a, b]$ then:

- (1) the function s is continuous at every point $x \neq d_k$ and at every point d_k has the right-hand jump σ_k^+ and the left-hand jump σ_k^- .
- (2) $s \in V[a, b]$ and for $a \leq x \leq b$ we have

$$V_a^x(s) = \sum_{a \leq d_k < x} |\sigma_k^+| + \sum_{a < d_k \leq x} |\sigma_k^-|$$

- (3) the difference $f - s$ is a continuous function of bounded variation on $[a, b]$.
- (4) if f is increasing then $f - s$ also is increasing.

Proof: (1) Let $x_0 \neq d_k$, since f is continuous in x_0 the function of total variation $v(x)$ is continuous in x_0 . Then, given $\epsilon > 0$ there exists a $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |v(x) - v(x_0)| < \epsilon$. Take x such that $x_0 < x < x_0 + \delta$. Then we have:

$$\begin{aligned} |s(x) - s(x_0)| &= \left| \sum_{x_0 \leq d_k < x} \sigma_k^+ + \sum_{x_0 < d_k \leq x} \sigma_k^- \right| \\ &\leq \left| \sum_{x_0 \leq d_k < x} \sigma_k^+ \right| + \left| \sum_{x_0 < d_k \leq x} \sigma_k^- \right| \\ &\leq \sum_{x_0 \leq d_k < x} |\sigma_k^+| + \sum_{x_0 < d_k \leq x} |\sigma_k^-| \\ &\leq v(x) - v(x_0) \end{aligned}$$

where the last inequality is true because of problem 3.0.3. Furthermore, $v(x) - v(x_0) < \epsilon$ and thus $x_0 < x < x_0 + \delta \Rightarrow |s(x) - s(x_0)| < \epsilon$. The same inequality holds when $x_0 - \delta < x < x_0$ and thus s is continuous in x_0 .

For the second part, take a fixed point of discontinuity d_j . We shall show that $\lim_{x \rightarrow +d_j} s(x) - s(d_j) = \sigma_j^+$, one shows that $\lim_{x \rightarrow -d_j} s(d_j) - s(x) = \sigma_j^-$ in the same way.

For $x > d_j$ we set $\omega'(x) = \sum_{d_j \leq d_k < x} \sigma_k^+$ and $\omega''(x) = \sum_{d_j < d_k \leq x} \sigma_k^-$. Then

$$s(x) - s(d_j) = \omega'(x) + \omega''(x)$$

We shall show that $\omega'(x) \rightarrow \sigma_j^+$ and $\omega''(x) \rightarrow 0$ as $x \rightarrow +d_j$.

By problem 3.0.3 both series $\sum_{k=1}^{\infty} \sigma_k^+$ and $\sum_{k=1}^{\infty} \sigma_k^-$ are absolutely convergent and therefore we have that for a given $\epsilon > 0$ there exists natural numbers N_1, N_2 such that $\sum_{k \geq N_1} |\sigma_k^+| < \epsilon$ and $\sum_{k \geq N_2} |\sigma_k^-| < \epsilon$.

For $N = \max\{N_1, N_2\}$ there exists $\delta > 0$ such that if $k \leq N$ then the point d_k does not belong to $(d_j, d_j + \delta)$. Hence, if $d_j < x < d_j + \delta$ then

$$|\omega'(x) - \sigma_j^+| = \left| \sum_{d_j < d_k < x} \sigma_k^+ \right| \leq \sum_{d_j < d_k < x} |\sigma_k^+| \leq \sum_{k \geq N} |\sigma_k^+| < \epsilon$$

and

$$|\omega''(x)| = \left| \sum_{d_j < d_k \leq x} \sigma_k^- \right| \leq \sum_{d_j < d_k \leq x} |\sigma_k^-| \leq \sum_{k \geq N} |\sigma_k^-| < \epsilon$$

and we conclude that $\lim_{x \rightarrow +d_j} \omega'(x) = \sigma_j^+$ and $\lim_{x \rightarrow +d_j} \omega''(x) = 0$.

(2) Let $x \in (a, b]$, according to (1) the function $s(x)$ has the same discontinuities as $f(x)$ and also the same left and right-hand jumps as $f(x)$ at these points. Then problem 3.0.3 applied to the function $s(x)$ gives that

$$\sum_{a \leq d_k < x} |\sigma_k^+| + \sum_{a < d_k \leq x} |\sigma_k^-| \leq V_a^x(s)$$

We shall show the opposite inequality. Set

$$t(x) = \sum_{a \leq d_j < x} |\sigma_j^+| + \sum_{a < d_j \leq x} |\sigma_j^-|$$

and let $\Pi = \{x_0, x_1, \dots, x_n\}$ be any partition of $[a, x]$. For $0 \leq j \leq n-1$ we have

$$\begin{aligned} |s(x_{k+1}) - s(x_k)| &= \left| \sum_{x_k \leq d_j < x_{k+1}} \sigma_j^+ + \sum_{x_k < d_j \leq x_{k+1}} \sigma_j^- \right| \\ &\leq \sum_{x_k \leq d_j < x_{k+1}} |\sigma_j^+| + \sum_{x_k < d_j \leq x_{k+1}} |\sigma_j^-| \\ &= t(x_{k+1}) - t(x_k) \end{aligned}$$

and therefore

$$V_{\Pi}(s) = \sum_{k=0}^{n-1} |s(x_{k+1}) - s(x_k)| \leq \sum_{k=0}^{n-1} (t(x_{k+1}) - t(x_k)) = t(x)$$

It follows that $V_a^x(s) \leq t(x)$ and therefore

$$V_a^x(s) = \sum_{a \leq d_j < x} |\sigma_j^+| + \sum_{a < d_j \leq x} |\sigma_j^-|$$

(3) If $x \in [a, b]$ is a point of continuity for f then s is continuous in x as well, by (1), and thus $(f - s)$ is continuous in x . If x is a point of discontinuity for f , that is $x = d_k$ for some k , then

$$\begin{aligned} (f - s)(d_k + 0) &= f(d_k + 0) - s(d_k + 0) \\ &= f(d_k + 0) - s(d_k + 0) + s(d_k) - s(d_k) + f(d_k) - f(d_k) \\ &= f(d_k + 0) - f(d_k) - [s(d_k + 0) - s(d_k)] + f(d_k) - s(d_k) \\ &= \sigma_k^+ - \sigma_k^- + f(d_k) - s(d_k) \\ &= f(d_k) - s(d_k) \end{aligned}$$

One shows that $(f - s)(d_k - 0) = (f - s)(d_k)$ in the same manner, so $(f - s)$ is continuous in d_k .

(4) Let $x, y \in [a, b]$ and $y < x$. In the proof of (1) above we showed that

$$s(x) - s(y) \leq |s(x) - s(y)| \leq V_y^x(f)$$

Since f is increasing $V_y^x(f) = f(x) - f(y)$ and thus we have

$$s(x) - s(y) \leq f(x) - f(y)$$

which is equivalent with $f(y) - s(y) \leq f(x) - s(x)$ if $y < x$, hence $f - s$ is increasing.

□

We can give an alternative definition of the jump function that will be useful:

Problem 3.0.6 For those n such that $a < d_n < b$ set

$$\sigma_n(x) = \begin{cases} 0 & a \leq x < d_n \\ \sigma_n^- & x = d_n \\ \sigma_n^- + \sigma_n^+ & d_n < x \leq b \end{cases}$$

If $d_n = a$ for some n set

$$\sigma_n(x) = \begin{cases} \sigma_n^+ & a < x \leq b \\ 0 & x = a \end{cases}$$

Finally, if $d_n = b$ for some n set

$$\sigma_n(x) = \begin{cases} 0 & a \leq x < b \\ \sigma_n^- & x = b \end{cases}$$

Then

$$s(x) = \sum_{n=1}^{\infty} \sigma_n(x)$$

Proof: Set $s_1(x) = \sum_{n=1}^{\infty} \sigma_n(x)$, we shall prove that $s_1 = s$. Since $\sigma_n(a) = 0$ for every n clearly $s_1(a) = s(a)$. Take $x \in (a, b]$ fixed but arbitrary, for those $n \in \mathbb{N}$ such that $a < d_n < x$ we have $\sigma_n(x) = \sigma_n^+ + \sigma_n^-$ and for those $n \in \mathbb{N}$ such that $x < d_n$ we have $\sigma_n(x) = 0$. We get different cases depending on whether or not the points x and a are points of discontinuity.

We'll do the case when a is a point of discontinuity, that is $a = d_i$ for some i , and x isn't a point of discontinuity:

$$s_1(x) = \sigma_i(x) + \sum_{a < d_n < x} \sigma_n(x) = \sum_{a \leq d_n < x} \sigma_n^+ + \sum_{a < d_n < x} \sigma_n^- = s(x)$$

□

We shall conclude this section with two problems on how the total variation of the function $f - s$ relates to the total variation of f .

Problem 3.0.7 Let $f \in V[a, b]$ and let s be the jump function of f . Then

$$V_a^b(f) = V_a^b(f - s) + \sum_{n=1}^{\infty} (|\sigma_n^+| + |\sigma_n^-|)$$

Proof: We have

$$V_a^b(f) = V_a^b(f - s + s) \leq V_a^b(f - s) + V_a^b(s) = V_a^b(f - s) + \sum_{k=1}^{\infty} (|\sigma_k^+| + |\sigma_k^-|)$$

For the opposite inequality, set for every $n \in \mathbb{N}$

$$s_n(x) = \sum_{k=1}^n \sigma_k(x)$$

where $\sigma_k(x)$ is as defined in problem 3.0.6. For notational simplicity we assume that a and b are not points of discontinuity of f . We shall show that

$$\begin{aligned} (i) \quad & V_a^b(f - s_n) = V_a^b(f) - \sum_{k=1}^n (|\sigma_k^+| + |\sigma_k^-|) \\ (ii) \quad & V_a^b(s - s_n) = V_a^b(s) - \sum_{k=1}^n (|\sigma_k^+| + |\sigma_k^-|) \end{aligned}$$

In order to show (i), first consider the case when $n = 1$. We shall show that $V_a^b(f - s_1) = V_a^b(f) - (|\sigma_1^+| + |\sigma_1^-|)$.

Let h be an arbitrary number satisfying $0 < h < \min\{c - a, b - c\}$.

The function s_1 is constant on $[a, c - h]$ and $[c + h, b]$ and thus $V_a^{c-h}(f - s_1) = V_a^{c-h}(f)$ and $V_{c+h}^b(f - s_1) = V_{c+h}^b(f)$ and the following equality follows:

$$V_a^b(f) - V_a^b(f - s_1) = V_{c-h}^{c+h}(f) - V_{c-h}^{c+h}(f - s_1)$$

Let $h \rightarrow +0$, with $v(x) = V_a^x(f)$ problem 2.4.2 gives that $v(c+0) - v(c) = |\sigma^+|$ and $v(c) - v(c-0) = |\sigma^-|$ so therefore $v(c+0) - v(c-0) = |\sigma^+| + |\sigma^-|$. Further, since $f - s_1$ is continuous in the point c then $V_{c-h}^{c+h}(f - s_1) \rightarrow 0$ and thus we obtain $V_a^b(f) - V_a^b(f - s_1) = |\sigma_1^+| + |\sigma_1^-|$.

For the general case of (i), order the points of discontinuity d_1, \dots, d_n such that $d_1 < d_2 < \dots < d_n$ and divide $[a, b]$ into n compact, non-overlapping subintervals I_1, \dots, I_n such that d_k lies in the interior of $I_k = [x_{k-1}, x_k]$ for $1 \leq k \leq n$. Then applying the case $n = 1$ to each interval $[x_{k-1}, x_k]$ yields

$$V_{x_{k-1}}^{x_k}(f - s_n) = V_{x_{k-1}}^{x_k}(f) - (|\sigma_k^+| + |\sigma_k^-|) \quad \text{for } 1 \leq k \leq n$$

Summing these equalities gives (i).

Since the function s has the same points of discontinuity as f and the same left-hand and right-hand jumps at these points (ii) follows directly by applying (i) to the function s .

Now, we have

$$\begin{aligned} V_a^b(f - s) &= V_a^b(f - s_n + s_n - s) \\ &\leq V_a^b(f - s_n) + V_a^b(s - s_n) \\ &= V_a^b(f) - \sum_{k=1}^n (|\sigma_k^+| + |\sigma_k^-|) + V_a^b(s) - \sum_{k=1}^n (|\sigma_k^+| + |\sigma_k^-|) \\ &= V_a^b(f) - \sum_{k=1}^n (|\sigma_k^+| + |\sigma_k^-|) + \sum_{k=n+1}^{\infty} (|\sigma_k^+| + |\sigma_k^-|) \end{aligned}$$

Given any ϵ there exists natural numbers N_1 and N_2 such that if $n \geq N_1$ then

$$\sum_{k>n} (|\sigma_k^+| + |\sigma_k^-|) < \frac{\epsilon}{2}$$

and if $n \geq N_2$ then

$$\sum_{k=1}^n (|\sigma_k^+| + |\sigma_k^-|) \geq \sum_{k=1}^{\infty} (|\sigma_k^+| + |\sigma_k^-|) - \frac{\epsilon}{2}$$

Thus if $n \geq \max\{N_1, N_2\}$ then

$$\begin{aligned} V_a^b(f - s) &\leq V_a^b(f) - \sum_{k=1}^n (|\sigma_k^+| + |\sigma_k^-|) + \sum_{k=n+1}^{\infty} (|\sigma_k^+| + |\sigma_k^-|) \\ &< V_a^b(f) - \left[\sum_{k=1}^{\infty} (|\sigma_k^+| + |\sigma_k^-|) - \frac{\epsilon}{2} \right] + \frac{\epsilon}{2} \\ &= V_a^b(f) - \sum_{k=1}^{\infty} (|\sigma_k^+| + |\sigma_k^-|) + \epsilon \end{aligned}$$

whence it follows that $V_a^b(f - s) + \sum_{k=1}^{\infty} (|\sigma_k^+| + |\sigma_k^-|) \leq V_a^b(f)$ and thus we must have

$$V_a^b(f) = V_a^b(f - s) + \sum_{k=1}^{\infty} (|\sigma_k^+| + |\sigma_k^-|)$$

□

Problem 3.0.8 If $f \in V[a, b]$ and if a function φ is such that $f - \varphi$ is continuous on $[a, b]$ and

$$V_a^b(\varphi) = \sum_{n=1}^{\infty} (|\sigma_n^+| + |\sigma_n^-|)$$

then $\varphi - s = \text{constant}$

Proof: If we let $h = f - \varphi$ then $\varphi = f - h$ and since h is continuous on $[a, b]$ we see that if f is continuous in a point then so is φ . On the other hand $f = h + \varphi$ so if φ is continuous in a point then so is f .

Thus, for any $x_0 \in [a, b]$, $f(x)$ is continuous in x_0 if and only if $\varphi(x)$ is continuous in x_0 . It follows that f and φ have exactly the same sequence of points of discontinuity.

Furthermore, if d_k is any point of discontinuity of f we have, since $f - \varphi$ is continuous, that

$$\begin{aligned} 0 &= (f - \varphi)(d_k + 0) - (f - \varphi)(d_k) \\ &= f(d_k + 0) - f(d_k) - [\varphi(d_k + 0) - \varphi(d_k)] \\ &= \sigma_k^+ - [\varphi(d_k + 0) - \varphi(d_k)] \end{aligned}$$

That is, the function φ has the same right-hand jump as f at d_k . In the same way we can show that φ also has the same left-hand jump as f at d_k .

Then problem 3.0.7 gives that

$$V_a^b(\varphi) = V_a^b(\varphi - s_\varphi) + \sum_{k=1}^{\infty} (|\sigma_k^+| + |\sigma_k^-|)$$

where s_φ is the jump function of φ . However, f and φ have the same points of discontinuity and the same right-hand and left-hand jumps at these points so then $s_f = s_\varphi$. Thus

$$V_a^b(\varphi) = V_a^b(\varphi - s_f) + \sum_{k=1}^{\infty} (|\sigma_k^+| + |\sigma_k^-|)$$

Finally we're given that $V_a^b(\varphi) = \sum_{k=1}^{\infty} (|\sigma_k^+| + |\sigma_k^-|)$ and it follows that $V_a^b(\varphi - s_f) = 0 \Leftrightarrow \varphi - s_f = \text{constant}$

□

References

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