Experimental study of
the rotating-disk boundary-layer flow

by

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Rotating-disk flow has been investigated not only as a simple model of cross flow instability to compare with swept-wing flow but also for industrial flow applications with rotating configurations. However the exact nature of laminar-turbulent transition on the rotating-disk flow is still a major problem and further research is required for it to be fully understood, in particular, the laminar-turbulent transition process with absolute instability. In addition, the studies of the rotating-disk turbulent boundary-layer flow are inadequate to understand the physics of three-dimensional turbulent boundary-layer flow.

In present thesis, a rotating-rotating disk boundary-layer flow has been investigated experimentally using hot-wire anemometry. A glass disk with a flat surface has been prepared to achieve low disturbance rotating-disk environment. Azimuthal velocity measurements using a hot-wire probe have been taken for various conditions. To get a better insight into the laminar-turbulent transition region, a new way to describe the process is proposed using the probability density function (PDF) map of azimuthal fluctuation velocity.

The effect of the edge of the disk on the laminar-turbulent transition process has been investigated. The disturbance growth of azimuthal fluctuation velocity as a function of Reynolds number has a similar trend irrespective of the various edge conditions.

The behaviour of secondary instability and turbulent breakdown has been investigated. It has been found that the kinked azimuthal velocity associated with secondary instability just before turbulent breakdown became less apparent at a certain wall normal heights. Furthermore, the turbulent breakdown of the stationary mode seems not to be triggered by its amplitude, however, depend on the appearance of the travelling secondary instability.

Finally, the turbulent boundary layer on a rotating disk has been investigated. An azimuthal friction velocity has been directly measured from the azimuthal velocity profile in the viscous sub-layer. The turbulent statistics normalized by the inner and outer scales are presented.
Preface

This licentiate thesis within the area of fluid mechanics deals with boundary-layer flow on a rotating disk based on experimental work, including the instability, laminar-turbulent transition and fully developed turbulence. The thesis is divided into two parts. The first part contains an introduction, an overview, summary of the present study and an appendix giving the governing equations. The second part consists of four papers. One of them has already been published however it is presented in a different format here to align with the formatting of the thesis. In chapter 5 of the first part of the thesis, the authors' contributions to the papers are stated.

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Part I

Overview and summary
CHAPTER 1

Introduction

An incompressible boundary layer over a rotating disk without any imposed flow is discussed in this study. The aim of this experimental work is to investigate the laminar-turbulent transition process and turbulence on the rotating-disk boundary-layer flow.

The flow is purely driven by the disk rotation. The laminar boundary layer has a three-dimensional velocity profile with an inflection point in the radial velocity component. This boundary layer is known as the ‘von Kármán boundary layer’ and it belongs to a family of rotating boundary-layer flows, including the so-called Bödewadt, Ekman and von Kármán boundary layers (BEK boundary layers) that are exact solutions of the Navier-Stokes equations. The differences between these flows are characterized by the Rossby number $Ro$, which is written as

$$Ro = \frac{\Omega_f - \Omega_d}{\Omega_a}$$

with

$$\Omega_a = \left(\frac{\Omega_f + \Omega_d}{2}\right) + \left(\frac{(\Omega_f - \Omega_d)^2}{16} + (\Omega_f - \Omega_d)^2/2\right)^{1/2},$$

where $\Omega_f$ and $\Omega_d$ are the fluid angular velocity outside the boundary layer and the disk angular velocity, respectively. $Ro$ on this study is $-1$ as $\Omega_f = 0$ and therefore $\Omega_a = \Omega_d$. The flow is also characterized by the Reynolds number, which is the ratio of inertial forces to viscous forces. It is defined as

$$R = r^* \sqrt{\frac{\Omega^*}{\nu^*}},$$

where $r^*$ is the radius of the disk at the measurement position, $\Omega^*$ is the angular velocity of the disk, $\nu^*$ is the kinematic viscosity of the fluid and $^*$ denotes a dimensional quantity.

The radial velocity component with an inflection point satisfies Rayleigh’s inflection-point criterion which relates to the existence of an inviscidly unstable mode. The rotating-disk flow is therefore inviscidly unstable, namely it remains
unstable at infinite Reynolds number. Three-dimensional boundary layers that have an inviscid instability such as this are said to have ‘crossflow instability’.

The rotating-disk boundary-layer flow has been used as a model for the flow over a swept wing because of the similarity of the velocity profiles and because both flows are susceptible to crossflow instability since the work by Gregory et al. (1955). Furthermore there is an advantage in investigating the crossflow instability of a rotating-disk flow rather than swept-wing flow. Because the rotating-disk flow is independent of a pressure-gradient parameter or a variable sweep angle that are required for the boundary layer flow over a swept wing. However, the flow over the rotating disk has Coriolis effects in contrast to swept-wing boundary layers. Nevertheless, Lingwood (1995a) found ‘local absolute instability’, attributed by an inviscid mechanism in the rotating-disk boundary layer, linked to the onset of nonlinearity and transition and therefore the Coriolis and streamline curvature effects were shown not to be of primary importance to the laminar-turbulent transition mechanism. Lingwood (1997b) revealed that the flow over the swept wing could in certain circumstances be absolutely unstable in the chordwise direction but because the swept wing has no spanwise periodicity the laminar-turbulent transition could still be a convective process.

The exact nature of the laminar-turbulent transition process for the rotating-disk flow is still not well understood. In particular, to what extent the absolute instability is involved in the transition process. This study helps us to understand a nature of the absolutely-unstable crossflow instability. Flows driven by one or more rotating disks have constituted a major field of study in fluid mechanics since the last century. Many application areas, such as rotating machinery, viscometry, computer storage devices and crystal growth processes, require the study of rotating flows (Brady 1987). Thus, this study will help the understanding of flows in more complicated applications.

The thesis is organised as follows: Part I, chapter 2, will continue describing the basis of the work, including the aims of the study, application examples, previous authors’ work and the governing equations; chapter 3 describes the experimental set-up and measurement method including the calibrations. Part I ends with a summary of results and a list of publications as well as describing the author’s contribution to the papers in chapter 4 and 5, respectively. Part II contains four papers on various aspects of the rotating-disk flow.
CHAPTER 2

Rotating-disk flow studies

This chapter will introduce, first of all, the basic equations governing rotating-disk flow. Then an overview of the rotating-disk flow following previous authors’ studies since the derivation of the von Kármán (1921) similarity solution for an infinite disk rotating in otherwise quiescent fluid is described. Thus overview of recent studies of laminar-turbulent transition of the rotating-disk boundary-layer flow is discussed. Then application examples of the rotating-disk flow are introduced to specify the importance of this study for an industrial applications. Finally, the concepts of convective instability and absolute instability are discussed.

2.1. The governing equations

2.1. The governing equations

Figure 2.1. A sketch of the von Kármán boundary layer on a rotating-disk showing the mean velocity profiles (in a stationary laboratory frame).
The rotating-disk flow system is modelled in a cylindrical coordinate system as an infinite planar disk with a constant angular speed $\Omega^*$. The position vector is given as $r = (r^* \cos \theta, r^* \sin \theta, z^*)$. The instantaneous velocity vector is represented by $\mathbf{v} = (\hat{u}^*, \hat{v}^*, \hat{w}^*)$. The rotation vector is given by $\mathbf{\omega} = (0, 0, \Omega^*)$.

The continuity equation and Navier-Stokes equation (NSE) in a uniformly rotating co-ordinate system are written as

$$\nabla \cdot \mathbf{v} = 0, \quad (2.1)$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} + 2\mathbf{\omega} \times \mathbf{v} + \mathbf{\omega} \times (\mathbf{v} \times \mathbf{r}) = -\frac{1}{\rho^*} \nabla \tilde{p}^* + \nu^* \nabla^2 \mathbf{v}, \quad (2.2)$$

where $2\mathbf{\omega} \times \mathbf{v}$ is the Coriolis acceleration term and $\mathbf{\omega} \times (\mathbf{v} \times \mathbf{r})$ is a centrifugal acceleration term, $\tilde{p}^*$ is an instantaneous pressure and $\rho^*$ and $\nu^*$ are density and kinematic viscosity of a Newtonian fluid, respectively, and $\nabla$ and $\nabla^2$ are the gradient and Laplace operators respectively, in cylindrical coordinates. The continuity equation and the NSE can be decomposed radial, azimuthal and axial components and are written as

**Continuity equation:**

$$\frac{\partial \hat{u}^*}{\partial r^*} + \frac{1}{r^*} \frac{\partial \hat{v}^*}{\partial \theta} + \frac{\partial \hat{w}^*}{\partial z^*} + \frac{\tilde{u}^*}{r^*} = 0, \quad (2.3)$$

**Radial component of NSE:**

$$\frac{\partial \hat{u}^*}{\partial r^*} + \left( \tilde{u}^* \frac{\partial \hat{u}^*}{\partial r^*} + \tilde{v}^* \frac{\partial \hat{u}^*}{\partial \theta} + \tilde{w}^* \frac{\partial \hat{u}^*}{\partial z^*} \right) = -\frac{1}{\rho^*} \frac{\partial \tilde{p}^*}{\partial r^*} + \nu^* \left[ \frac{\partial^2 \hat{u}^*}{\partial r^*^2} + \frac{1}{r^*} \frac{\partial \hat{u}^*}{\partial \theta^2} + \frac{\partial^2 \hat{u}^*}{\partial z^*^2} \right] + \frac{1}{r^*} \frac{\partial \tilde{u}^*}{\partial \theta^2} - \tilde{u}^* - 2 \frac{\partial \hat{v}^*}{\partial \theta} , \quad (2.4)$$

**Azimuthal component of NSE:**

$$\frac{\partial \hat{v}^*}{\partial \theta} + \left( \tilde{u}^* \frac{\partial \hat{v}^*}{\partial r^*} + \tilde{v}^* \frac{\partial \hat{v}^*}{\partial \theta} + \tilde{w}^* \frac{\partial \hat{w}^*}{\partial z^*} \right) = -\frac{1}{\rho^* r^*} \frac{\partial \tilde{p}^*}{\partial \theta} + \nu^* \left[ \frac{\partial^2 \hat{v}^*}{\partial \theta^2} + \frac{1}{r^*} \frac{\partial \hat{v}^*}{\partial \theta^2} + \frac{\partial^2 \hat{v}^*}{\partial z^*^2} \right] + \frac{1}{r^*} \frac{\partial \tilde{v}^*}{\partial \theta^2} - \tilde{v}^* + 2 \frac{\partial \hat{u}^*}{\partial \theta} , \quad (2.5)$$

**Axial component of NSE:**

$$\frac{\partial \hat{w}^*}{\partial z^*} + \left( \tilde{u}^* \frac{\partial \hat{w}^*}{\partial r^*} + \tilde{v}^* \frac{\partial \hat{w}^*}{\partial \theta} + \tilde{w}^* \frac{\partial \hat{w}^*}{\partial z^*} \right) = -\frac{1}{\rho^*} \frac{\partial \tilde{p}^*}{\partial z^*} + \nu^* \left[ \frac{\partial^2 \hat{w}^*}{\partial r^*^2} + \frac{1}{r^*} \frac{\partial \hat{w}^*}{\partial \theta^2} + \frac{\partial^2 \hat{w}^*}{\partial z^*^2} \right] + \frac{1}{r^*} \frac{\partial \tilde{w}^*}{\partial \theta^2} \quad (2.6)$$
2.1. THE GOVERNING EQUATIONS

2.1.1. Mean velocity profile

The instantaneous velocity \( (\tilde{u}^*, \tilde{v}^*, \tilde{w}^*) \) and instantaneous pressure \( (\tilde{p}^*) \) are decomposed into mean (time-independent) and fluctuation (time-dependent) components, namely

\[
\begin{align*}
\tilde{u}^* &= U^* + u^*, \\
\tilde{v}^* &= V^* + v^*, \\
\tilde{w}^* &= W^* + w^*, \\
\tilde{p}^* &= P^* + p^*,
\end{align*}
\]

(2.7)

where \( U^*, V^*, W^* \) are the mean radial, azimuthal and axial velocities, \( P^* \) is the mean pressure, \( u^*, v^*, w^* \) are fluctuating velocities in the radial, azimuthal and axial directions, and \( p^* \) is the fluctuating pressure. This operation is called Reynolds decomposition. Kármán (1921) derived an exact axi-symmetric similarity solution of the Navier-Stokes equation for the (time-independent) base flow. Then velocity and pressure similarity variables are defined by

\[
U(z) = \frac{U^*}{r^*\Omega^*}, \quad V(z) = \frac{V^*}{r^*\Omega^*}, \quad W(z) = \frac{W^*}{(u^*/r^*)^{1/2}}, \quad P(z) = \frac{P^*}{\rho^*r^*\Omega^*},
\]

(2.8)

where \( U, V, W \) are nondimensional radial, azimuthal and axial mean velocity components, \( P \) is the nondimensional mean pressure. \( z \) is the wall normal position from the disk surface normalized by the characteristic length \( L^* = (u^*/r^*)^{1/2} \), namely written as

\[
z = z^*/L^*.
\]

(2.9)

The mean basic flow equations are derived from equations (2.3–2.6), with time-independence and axi-symmetry, which yields nonlinear ordinary differential equations written as:

\[
\begin{align*}
2U + W' &= 0, \\
U^2 - V^2 + U'W - U'' &= 0, \\
2UV + V'W - V'' &= 0, \\
P' + WW' - W'' &= 0
\end{align*}
\]

(2.10–2.13)

where the prime denotes differentiation with respect to \( z \). The boundary conditions on a rotating-disk flow are no-slip conditions at the wall, and no radial or azimuthal velocity at \( z = \infty \), so in the laboratory frame they become:

\[
\begin{align*}
U(0) &= 0, \quad V(0) = 1, \quad W(0) = 0, \\
U(\infty) &= 0, \quad V(\infty) = 0
\end{align*}
\]

(2.14)
The solutions of the differential equations in equations (2.10–2.13) are plotted in figure 2.2. And the flow direction between azimuthal and radial components is also plotted in figure 2.3.

**Figure 2.2.** Laminar mean velocity profiles $U$ (dash line), $V$ (solid line) and $W$ (chain line), respectively, in a stationary frame.

**Figure 2.3.** A laminar mean velocity angle profile of $U$ and $V$. 
2.1. THE GOVERNING EQUATIONS

2.1.2. Reynolds averaged equations

To derive the governing equations for turbulent flow over the rotating disk, Reynolds averaged continuity equation and Navier-Stokes equations (RANS) are derived from equations (2.3–2.6) with some assumptions. The detail of these derivations are described in Appendix A. To derive Reynolds averaged continuity equation, the decomposed velocity and pressure components given in equation (2.7) are substituted into equation (2.3), and time averages taken. Then derivaties with respect to the \( \theta \) direction are neglected due to the axisymmetry of the mean flow. Therefore, the Reynolds averaged continuity equation becomes

\[
\frac{\partial U^*}{\partial r^*} + \frac{\partial W^*}{\partial z^*} + \frac{U^*}{r^*} = 0.
\]  

(2.15)

The RANS equations are derived by substituting equation (2.7) into equation (2.4), equation (2.5) and equation (2.6) and taking time averages (denoted with an overscore). With the usual assumptions and assuming axisymmetry, the resulting RANS equations for the incompressible turbulent rotating-disk boundary-layer flow are given as

**Radial component:**

\[
U^* \frac{\partial U^*}{\partial r^*} + W^* \frac{\partial U^*}{\partial z^*} - \frac{V^*}{r} \frac{V^*}{r} - 2V^* \Omega^* = -\frac{1}{\rho^*} \frac{\partial P^*}{\partial r^*} + r^* \Omega^*^2 + \frac{1}{\rho^*} \frac{\partial}{\partial z^*} \left( \mu^* \frac{\partial U^*}{\partial z^*} - \rho^* u^* w^* \right),
\]  

(2.16)

**Azimuthal component:**

\[
U^* \frac{\partial V^*}{\partial r^*} + W^* \frac{\partial V^*}{\partial z^*} + \frac{U^* V^*}{r^*} + 2U^* \Omega^* = \frac{1}{\rho^*} \frac{\partial}{\partial z^*} \left( \mu^* \frac{\partial V^*}{\partial z^*} - \rho^* v^* w^* \right),
\]  

(2.17)

**Axial component:**

\[
\frac{\partial w^*}{\partial z^*} = -\frac{1}{\rho^*} \frac{\partial P^*}{\partial z^*},
\]  

(2.18)
2. ROTATING-DISK FLOW STUDIES

2.2. Overview of previous studies

2.2.1. Instabilities and laminar-turbulent transition process

The laminar boundary-layer flow on the rotating disk has a three-dimensional profile with an inflection point in the radial direction and the exact similarity solution, which corresponds to the von Kármán similarity solution (1921) for an infinite disk rotating in an otherwise quiescent fluid. The instability and transition on the rotating-disk flow has been investigated since Theodorsen & Regier (1944) and Smith (1947) who noticed fluctuations in the boundary layer on the rotating disk using a hot-wire probe. Local stability analysis\(^1\) (e.g. Lingwood 1995\(a\)) reveals that the flow can be inviscidly unstable and the critical Reynolds number \(R_{CC}\) for the onset of instability of the stationary mode, i.e. for disturbances that are fixed with respect to the rotating disk, is about \(R_{CC} = 290\). This inviscidly unstable mode is called Type I. There is another unstable mode which is called Type II and is due to viscous effect. However for Type II stationary disturbances have smaller growth rates. Travelling disturbances can also be unstable. Hussain \textit{et al.} (2011) showed growth rates of travelling disturbances with different frequencies. The maximum growth rates were observed for a mode travelling slowly with respect to the disk and growing at a rate larger than the stationary mode. Note also that disturbances travelling faster than the disk can have a critical Reynolds number for Type II that is lower than that for Type I. However, flow visualisation of the rotating-disk flow, e.g. by Kohama (1984), shows 31 or 32 stationary spiral vortices. The predominance of the stationary disturbances observed in experiments may be because of unavoidable roughness(es) on the surface that cause repeatable excitation of the stationary mode rather than travelling modes.

Early studies of laminar-turbulent transition of the rotating-disk flow had only considered convective instability (e.g. Huerre & Monkewitz 1990). Convective instability is desired, for example, the linear impulse response goes to zero at infinite time at the excited position, while the disturbance grows as it is convected downstream.

More recently the work done by Lingwood (1995\(a\), 1997\(a\)) using Briggs’ method (Briggs 1964) found that for certain travelling waves above \(R_{CA} = 507\) the flow becomes absolutely unstable.\(^2\) This is caused by a stable upstream-travelling mode, henceforth called Type III, coalescing with an unstable downstream travelling Type I mode. Here \(R_{CA}\) is the critical Reynolds number

---

\(^1\)In order to render the perturbation equations separable, variations in \(R\) with respect to \(r\) are neglected and this results in a ‘local’ stability analysis, which is often referred to as a ‘parallel-flow’ approximation (as it is for other boundary-layer flows) even though for the rotating-disk laminar boundary layer the boundary-layer thickness is in fact constant.

\(^2\)Absolute instability is where the linear impulse response goes to infinity at infinite time at the excited position.
2.2. OVERVIEW OF PREVIOUS STUDIES

for onset of local absolute instability. Furthermore, she revealed the absolute instability of the rotating-disk flow is an inviscid mechanism and suggested that the absolute instability triggers nonlinearity what is the start of the laminar-turbulent transition process. By introducing impulsive excitation to the rotating-disk boundary-layer flow, Lingwood (1996) confirmed experimentally the absolute instability (above about $R = 507$) by tracking the trajectory of the excited wavepacket. Furthermore, she showed the onset of nonlinearity appeared at Reynolds numbers above 502 and below 514, see Lingwood (1995b) and the development of the laminar-turbulent transition process from there, resulting in a fully turbulent flow at about $R = 600 - 650$. She performed these experiments with a ‘clean’ disk to minimize the excitation of stationary disturbances and the amplitude of the primary disturbance was small enough that the peak amplitude of disturbances were 3% of the local disk speed at $R = 500$ (smaller than Balachandar et al.’s (1992) threshold where the root-mean-square amplitude of the primary disturbance reaches about 9% at $R = 500$). Based on these results and because she did not see clear evidence of kinks in the timeseries indicative of secondary instabilities Lingwood (1996) stated “the stationary disturbances are sufficiently small, even close to the onset of transition, for the boundary layer stability to be governed by the mean velocity profiles rather than secondary instabilities”.

Davies & Carpenter (2003) performed direct numerical simulations solving the linearized Navier-Stokes equations and suggested that the convective behaviour eventually dominates even for strongly locally absolutely unstable regions and concluded that the absolute instability does not produce a linear amplified global mode. Othman & Corke (2006) performed experiments similar to Lingwood (1996) but using a low-amplitude and a high-amplitude initial pulse-jet excitation to create the wavepacket disturbances in the boundary-layer flow. Contrary to Lingwood (1996) the trailing edge of the wavepacket did not become fixed at $R_{CA}$ with the low amplitude initial disturbance, and that result agrees well with the linearized DNS of Davies & Carpenter (2003). On the other hand, the amplitude of the wavepacket with the high-amplitude initial disturbance compares better with Lingwood’s (1996) results although it is not certain that the trailing edge of the wavepacket becomes fixed at $R_{CA}$, which would be indicative of the absolute instability.

In contrast to Lingwood’s (1996) experimental observation, Kobayashi et al. (1980), Kohama (1984) and Wilkinson & Malik (1985) observed signs associated with secondary instability at the final stage of laminar-turbulent transition, namely just before the turbulent breakdown region. Kobayashi et al. (1980) who performed the flow visualization on the rotating-disk flow captured “a new striped flow pattern originating along the axis of a spiral vortex”. Kohama (1984) who also carried out the visualization study suggested “ring-like vortices which occur on the surfaces of each spiral vortices [sic]”.


2. ROTATING-DISK FLOW STUDIES

<table>
<thead>
<tr>
<th>Reynolds number</th>
<th>The description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$</td>
<td>Local Reynolds number.</td>
</tr>
<tr>
<td>$R_{CC}$</td>
<td>Critical Reynolds number of Type I instability for stationary mode.</td>
</tr>
<tr>
<td>$R_{CA}$</td>
<td>Critical Reynolds number of absolute instability.</td>
</tr>
<tr>
<td>$R_t$</td>
<td>Transition Reynolds number.</td>
</tr>
<tr>
<td>$R_{edge}$</td>
<td>Edge Reynolds number.</td>
</tr>
</tbody>
</table>

Table 1. Descriptions of various Reynolds numbers.

Furthermore both Kobayashi et al. (1980) and Wilkinson & Malik (1985) observed kinked velocity fluctuations just before the turbulent breakdown region. Then Wilkinson & Malik (1985) concluded that “stationary, secondary instabilities between the primary vortices were observed”. From a theoretical analysis Balachandar et al. (1992) suggested that the travelling secondary instability appears as a pair of counter-rotating vortices. Pier (2003) suggested that a nonlinear approach is required to explain the self-sustained behaviour of the rotating-disk flow. He suggested that the rotating-disk boundary layer has a primary nonlinear global mode fixed by the local absolute instability, found by Lingwood (1995a), which has itself a secondary absolute instability that triggers the transition to turbulence.

In addition to the studies described above, Healey (2010) suggested that the proximity of the edge of the disk to the transition region is important; an effect not captured by standard analyses, which assume an infinite disk radius. He recognized the scatter of experimentally-observed transition Reynolds number $R_t$ reported by some previous authors who performed hot-wire measurements as shown in table 2. He argued that, based on investigations of the linearized complex Ginzburg-Landau equation, the transition Reynolds number should depend on the Reynolds number at the edge of the disk, $R_{edge}$, where $R_{edge} = r_k^* \left( \frac{\Omega}{\nu} \right)^{1/2}$, $r_k^*$ is the actual radius of the disk, with the assumption that the transition to turbulence is related to the appearance of a steep-fronted nonlinear global mode. Figure 7(b) of Healey (2010) shows the variance of the experimentally-observed transition Reynolds number depending on the edge Reynolds number compared with his theoretical prediction.

As mentioned above, the exact nature of the laminar-turbulent transition process is not yet fully understood. In particular, the behaviour of the secondary instability and its relation to its primary absolute instability, and the effects of the edge Reynolds number and edge conditions of the disk should also be investigated.
2.2. OVERVIEW OF PREVIOUS STUDIES

<table>
<thead>
<tr>
<th>Authors</th>
<th>$R_t$</th>
<th>Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theodorsen &amp; Regier (1944)</td>
<td>557</td>
<td>Hot-wire</td>
</tr>
<tr>
<td>Gregory, Stuart &amp; Walker (1955)</td>
<td>533</td>
<td>Visual, China-clay</td>
</tr>
<tr>
<td>Cobb &amp; Saunders (1956)</td>
<td>490</td>
<td>Heat transfer</td>
</tr>
<tr>
<td>Gregory &amp; Walker (1960)</td>
<td>524</td>
<td>Pressure probe</td>
</tr>
<tr>
<td>Chin &amp; Litt (1972)</td>
<td>510</td>
<td>Mass transfer</td>
</tr>
<tr>
<td>Fedorov et al. (1976)</td>
<td>515</td>
<td>Visual, naphthalene</td>
</tr>
<tr>
<td>Clarkson, Chin &amp; Shacter (1980)</td>
<td>562</td>
<td>Visual, dye</td>
</tr>
<tr>
<td>Kobayashi, Kohama &amp; Takamadate (1980)</td>
<td>566</td>
<td>Hot-wire</td>
</tr>
<tr>
<td>Malik, Wilkinson &amp; Orszag (1981)</td>
<td>520</td>
<td>Hot-wire</td>
</tr>
<tr>
<td>Wilkinson &amp; Malik (1985)</td>
<td>550</td>
<td>Hot-wire</td>
</tr>
<tr>
<td>Lingwood (1996)</td>
<td>508</td>
<td>Hot-wire</td>
</tr>
<tr>
<td>Othman &amp; Corke (2006)</td>
<td>539</td>
<td>Hot-wire</td>
</tr>
</tbody>
</table>

Table 2. Experimental $R_t$ (differently defined) given in previous studies.

2.2.2. Turbulent boundary-layer flow

In contrast to the many studies of the laminar-turbulent transition process, as mentioned in the previous section, experimental results of turbulent boundary-layer flow on a rotating disk are still limited (Littell & Eaton 1994), despite many industrial applications (e.g. rotor-stator systems (Arco et al. 2005)). The turbulent boundary layer on a rotating disk is three-dimensional due to the crossflow driven by the centrifugal force and at the most skewed position in the boundary layer the mean crossflow velocity component reaches 11% of the local disk velocity (e.g. Littell & Eaton 1994). The early work on the turbulent rotating-disk flow was done by Goldstein (1935) who did a torque measurement. Theodorsen & Regier (1944) measured the azimuthal turbulent velocity profile using a hot-wire probe up to $R = 2646$. The velocity profile agreed well with the $1/7$ power law in the measurement range. Cham & Head (1969) performed turbulent radial and azimuthal velocity profile measurements using Pitot and entrainment measurement techniques and concluded that both of azimuthal and radial velocity profiles are well fitted by Thompson’s (1965) two-dimensional family and Mager’s (1952) cross flow expression, respectively. They also estimated the azimuthal local skin friction coefficient using a Clauser (1954) plot, resulting in “realistic” values. Erian & Tong (1971) performed experimental turbulent statistics measurements on a rotating-disk flow and concluded that “the eddy viscosity in the turbulent boundary layer generated by the disk rotation is substantially larger than that of the turbulent boundary layer over a flat plate”. Littell & Eaton’s (1994) azimuthal velocity profile normalized by an inner scale acquired by a conventional two-dimensional law
of the wall, shows that compared with the two-dimensional turbulent boundary layer the rotating-disk boundary layer lacks a wake component. They suggested that the reason for the absence of a wake region was not understood because for the two-dimensional turbulent boundary layer the wake would be missing if there was a streamwise favourable pressure gradient, but this could not explain the rotating-disk case where there is no azimuthal pressure gradient.

Inner variables (e.g. friction velocity) used in these previous are obtained by classical empirical methods. However, Nagib & Chauhan (2008) show that the von Kármán constant, $\kappa$, which is one of overlap coefficients for the logarithmic law in the turbulent boundary layer is no longer constant, but that it can change depending not only on Reynolds number but also on the flow systems (e.g. boundary layers, pipes and channels). To evaluate accurately the turbulent statistics normalized by the inner variables, measurement of these inner variables is required for the turbulent boundary layer on a rotating disk.

2.3. The application examples for a rotating-disk flow

The investigation of rotating-disk flow is useful not only for understanding crossflow instability as a fundamental element of three dimensional boundary layers but also for industrial flow applications with rotating configurations. Flows driven by rotating disks have constituted a major field of study in fluid mechanics since the twentieth century. The rotating-disk flow in this study is purely driven by a single rotating disk, however in more applied studies several geometrical configurations and flow conditions are taken into account.

Chemical vapour deposition (CVD) reactors often used in the semiconductor industry to deposite thin films of electrical and optical materials on substrates are one of the applications of the rotating-disk flow, see Hussain et al. (2011); Chen, K. Mortazavi (1986); Vanka et al. (2004). In the CVD reactor a disk-like rotor is mounted horizontally in the flow. A substrate placed on the disk-like supporter rotates to generate the homogeneous axial flow over the substrate to get rid of any non-uniformity of the incident flow (Hussain et al. 2011). Then the flow containing the reactive molecules attaches to the substrate and homogeneous thin film is deposited. Hussain et al. (2011) investigated the instability of the boundary layer over a rotating disk in an enforced axial flow to model the flow situation inside CVD reactors. They found the relative importance of the type II modes increases with axial flow.

The flow between rotating disks enclosed by a stationary sidewall has similar geometry to rotor-stator configurations, and is therefore geometry that is related to many industrial applications. The review paper about stability, transition and turbulence in rotating cavities written by Arco et al. (2005) describes this flow well showing that the flow instabilities with this configuration are strongly dependent on the aspect ratio of a diameter of the disk and height between two disks.
2.4. Convective instability and absolute instability

The concept of convective and absolute instability was introduced in the study of plasma physics by Briggs (1964).

Convective and absolute instability both relate to growth of disturbances in space and time and are distinguished by the different linear impulse response given at a certain spatial location. Figure 2.4 shows a sketch of impulse responses at nondimensional time $t = t_1$ for different instability conditions, where the linear impulse is introduced at a nondimensional position $x_s = 0$ and at $t = 0$. Figure 2.4(a) shows a stable condition; the introduced impulse decays in time and at $t = t_1$ the system reverts back to the initial condition. Figure 2.4(b) shows the behaviour of the linear impulse in a convective unstable region; the introduced impulse is exponentially amplified within the chain lines as the resulting wavepacket convects downstream. At infinite time the system reverts back to the initial condition at the location where initial impulse was introduced. Figure 2.4(c) shows the response in an absolutely unstable region; the introduced impulse is exponentially amplified within the chain lines at the introduced location. At a infinite time the system has a growing response at the location where the initial impulse was introduced.

These differences between convective instability and absolute instability shown in figure 2.4 can also be described mathematically. A general dispersion relation is given (Schmid & Henningson 2001) as

$$D(\alpha, \omega; R) = 0,$$

where $\alpha$ is the wave number of the streamwise direction and $\omega$ is the frequency, and both are in general complex for the necessary spatio-temporal analysis $R$ is the control parameter, namely Reynolds number here. Combining equation (2.19) with fluctuations $v(x, t)$ into the linear system satisfies the following condition:

$$D \left( -i \frac{\partial}{\partial x}, i \frac{\partial}{\partial x} \right) v(x, t) = 0,$$

where $x$ is the streamwise position, $i = \sqrt{-1}$ is the imaginary unit. To investigate the response to a linear impulse into the linear system, the Green's function $G(x, t)$ is introduced into equation (2.20) with the linear impulse introduced as a Kronecker delta functions $\delta(x)\delta(t)$ so that equation (2.20) satisfies

$$D \left( -i \frac{\partial}{\partial x}, i \frac{\partial}{\partial x} \right) G(x, t) = \delta(x)\delta(t).$$

Then the response $G(x, t)$ to the linear impulse $\delta(x)\delta(t)$ is defined as follows
to distinguish between linearly stable and linearly unstable and between convec-tively unstable and absolutely unstable responses. The system is linearly stable if

$$\lim_{t \to \infty} G(x, t) = 0 \quad \text{along all rays} \quad \frac{x}{t} = \text{constant.} \quad (2.22)$$

On the otherhand, the system is linearly unstable if

$$\lim_{t \to \infty} G(x, t) \to \infty \quad \text{along at least one ray} \quad \frac{x}{t} = \text{constant.} \quad (2.23)$$

If the system is linearly unstable, the convective instability and absolute instability are distinguished mathematically as follows. The system is convectively unstable if

![Figure 2.4](image)

**Figure 2.4.** The concept of a linear impulse response to distinguish between convective and absolute instability in the $x-t$ plane: (a) stable, (b) convectively unstable, (c) absolutely unstable. The linear impulse is introduced at $x = x_s$ at $t = 0$ for all cases.
2.4. CONVECTIVE INSTABILITY AND ABSOLUTE INSTABILITY

\[ \lim_{t \to \infty} G(x, t) = 0 \quad \text{along the ray} \quad \frac{x}{t} = 0. \quad (2.24) \]

The system is absolutely unstable if

\[ \lim_{t \to \infty} G(x, t) \to \infty \quad \text{along the ray} \quad \frac{x}{t} = 0. \quad (2.25) \]

These definitions are compatible with the sketches of the growth of the impulse in time and space shown in figure 2.4.
CHAPTER 3

Experimental methods

3.1. Experimental set-up of rotating-disk system

In this chapter, first of all, the experimental apparatus used in this study is described in some detail. Secondly, the measurement procedure of the fluid velocity by hot-wire anemometry is introduced and the typical calibration result is shown. Thirdly the measurement procedures of other relevant quantities are explained.

(a)  
(b)

Figure 3.1. (a) The experimental set-up of the rotating disk with plate edge condition. (b) The horizontal and inclined traverses and mounted hot-wire over the disk with open edge condition.
3.1. EXPERIMENTAL SET-UP OF ROTATING-DISK SYSTEM

3.1.1. Rotating apparatus

The rotating disk experimental set-up is a modified version of the one used by Lingwood (1996), see figure 3.1. In the present study a float glass is selected as a surface material of the rotating disk. The detail of the glass disk is stated in the following section. The glass disk is connected to the aluminum alloy disk constricted by 8 aluminum components. The iron disk with diameter of 270 mm is an adaptor to connect between aluminum alloy disk and a vertical shaft which is mounted vertically on a shaft driven by a d.c. servo-motor (Mavilor MS6). The d.c. servo-motor to operate the rotating disk is controlled by a main motor control inverter (Infranor SMVE 1510). These components of the rotating objects are sustained by an air-bearing spindle operated by pressurized air with 5.5 bar supplied by an air compressor through an air filter (HPC, DomnickHunter AO-0013G) and air dryer (KAESER KMM3 Compressed Air Dryer). Then this air bearing makes the operation with lower acoustic noise and vibration possible to the rotating system compared with a ball bearing. This system helps to bring in low initial noise environment into flow which is basically referred to an instability measurement. The rotating system is supported by a basement box made of steel that is filled with sandbags. The total weight of the system is approx 250 kg.

3.1.2. Glass disk

Lingwood (1996) used an aluminum alloy disk that was polished by a single-crystal diamond cutting tool as a surface of the rotating system. However the new float glass plate polished to get near optical quality is prepared for this experimental work instead of the aluminum alloy disk and it has an advantage that it is harder to make scratches on the surface than for metal materials. This selection is also proper in hot-wire anemometry point of view to limit heat conduction effect close to the disk wall that the surface absorbs the heat of hot-wire and the higher velocity speed than actual value is observed. The diameter of the glass disk $D^*$ is 474 mm to mount on the previous used aluminum alloy disk which has same diameter. At the edge of the disk the glass disk is ground down approximately 1.5 mm with a 45° angle. This is why the actual radius of this glass is $r^*_d = 233.5$ mm. The thickness of the glass is 24 mm.

The glass disk is required to have an as small imbalance and roughness as possible to avoid the possibility that enough large initial disturbance excited by them causes the bypass transition before the appearance of the absolute instability, namely the flow changes to turbulence in the completely convective route. Then the surface of the glass disk used in this study is polished and the imbalance measurement by mechanical test indicator is shown in figure 3.2. The figure 3.2 shows the azimuthal imbalance variation and the radial imbalance variation is neglected because it is hard to measure by the mechanical test indicator.
indicator. The maximum imbalance is observed at the edge of the disk and it is maximum 10 µm. The typical surface roughness is less than 1 µm.

To avoid the break of the glass disk during the operation, the maximum operation rotational speed is estimated by the following procedure. The relation between the failure stress $\sigma_f^*$ and the maximum angular velocity, $\Omega_{\text{max}}^*$, of the rotating disk (Ashby 2005; Lingwood 1995b) is given as

$$\frac{D^*}{2} \Omega_{\text{max}}^* = \left( \frac{8 \sigma_f^*}{S_f \rho_{\text{glass}} (3 + \nu_{\text{Po}})} \right)^{1/2},$$

(3.1)

where $\nu_{\text{Po}}$ is Poisson’s ratio, which has an approximately constant value of 1/3 for all solids and $S_f$ is an appropriate safety factor so that $S_f = 10$ is selected in this study. As the result, the maximum rotational speed of the glass disk is given using parameters in table 1 as $\Omega_{\text{max}}^* = 2553$ rpm.

3.1.3. Edge conditions

To investigate the effect on the transition process caused by the disk edge condition, different conditions have been considered. Figure 3.3 shows the three

![Figure 3.2](image)

**Figure 3.2.** The azimuthal imbalance measured by a mechanical test indicator up to $r^* = 230$ mm. The colour contour indicates the surface height variation $\Delta I$ from the reference position (-2 µm (Blue) $< \Delta I < +7$ µm (Red)) with 1 µm step.
3.1. EXPERIMENTAL SET-UP OF ROTATING-DISK SYSTEM

<table>
<thead>
<tr>
<th>σ_f [MPa]</th>
<th>ν</th>
<th>ρ_{glass} [kg/m^3]</th>
</tr>
</thead>
<tbody>
<tr>
<td>41</td>
<td>0.23</td>
<td>2.53 × 10^3</td>
</tr>
</tbody>
</table>

Table 1. Float glass parameters, where σ_f is the failure stress, ν is the Poisson’s ratio and ρ_{glass} is the density of the glass (Source: http://www.industrialglasstech.com/pdf/soda-lime-properties.pdf).

Figure 3.3. Three edge conditions.

edge conditions. Figure 3.3(a) shows the ‘open type’, which has no extended plate or cover, figure 3.3(b) shows the ‘ring type’, where there is a steel ring mounted below the surface of the disk covering eight aluminum clamps fixing the glass disk to the aluminum-alloy disk. These components on the vertical edge of the glass disk generate a disturbances field with eight oscillations per
rotation of the disk as measured in the laboratory frame of reference. The ring is mounted around them to eliminate their contribution to the flow disturbance field. The ring itself does not rotate and the horizontal gap between the ring and glass disk is less than 1 mm. The edge of the glass disk is still exposed in a similar way to the open-type edge condition because the top of the ring is located 11 mm vertically below the disk surface. The third case, figure 3.3(c) is called ‘plate type’, which consists of a non-rotational extended annular plate made of wood with an outside diameter of 900 mm mounted around the glass disk. This extended plate eliminates the effects of the eight aluminum fixing components and also reduces the effects of noise coming from the air bearing and DC-servo motor. The horizontal gap between the disk and plate is less than 1 mm and vertically the disk surface and plate are approximately flush. These differences of the edge conditions to the laminar-turbulent transition process on the rotating-disk flow are discussed in paper 2.

3.1.4. Traverse system
A traverse system with two axes is connected to a basement steel box through aluminum and steel beams (see figure 3.1(a)). One of the traverse moves in the horizontal (radial) direction, and the other traverse is mounted on the horizontal traverse at a 45° inclination not to disturb the axial flow which approaches the rotating-disk from above. The horizontal traverse and inclined traverse are operated by absolute encoders (AVAGO AEAS-7000 and Mitsu-toyo ID-C125B) and d.c. motors (micro motors E192.24.67 and RH158 510:1), respectively. These traverses achieved resolution of 5 µm for horizontal traverse and 3 µm for inclined traverse, respectively. They can operate all the way to the radius of the glass disk for the horizontal direction and beyond the boundary layer thickness of the rotating-disk flow for the vertical direction. At the edge of the inclined traverse a hot-wire holder directed so that the sensor is radial oriented and hence is predominately sensitive to the azimuthal velocity component. The traverse system is operated by the computer sampling signal with the software of LabVIEW8.6 through a controller board (National Instruments USB-6216).

3.2. Measurement techniques

3.2.1. Hot-wire anemometry
A hot-wire probe operational with a constant temperature anemometer (CTA) is used to measure the fluid velocity. The advantage of hot-wire anemometry compared to the other methods (e.g. Laser Doppler Velocimetry (LDV) or Particle Image Velocimetry (PIV)) is that it can sample flow velocity at a small localised position with high temporal resolution. In the present study, hot-wire probes with a single sensor made of platinum, with a diameter of 5 µm and 1 mm (laminar-turbulent transition measurement) and 1 µm and 0.3 mm
(turbulence measurement) are prepared, respectively, which are operated by CTA system (DANTEC StreamLine) with an overheat ratio ($\alpha_R$) of 0.8, where it is defined as

$$\alpha_R = \frac{R^*(T^*_{h}) - R^*(T^*_{ref})}{R^*(T^*_{ref})},$$

(3.2)

where $R^*(T^*_{h})$ is the resistance of the sensor at the operation temperature of $T^*_{h}$ and $R^*(T^*_{ref})$ is the resistance of the sensor at the reference ambient temperature of $T^*_{ref}$. The typical example of the probe used in laminar-turbulent transition measurement is shown in figure 3.4(a). The sensing element of the hot-wire is oriented in the radial direction (figure 3.4(b)), making it mainly sensitive to the azimuthal velocity. A low pass filter with 30 kHz (laminar-turbulent transition measurement) or 100 kHz turbulent measurement) is applied to the CTA circuit. The output voltage from the CTA is digitalized using a 16-bit A/D converter (National Instruments USB-6216) at a specific sampling rate and sampling time and recorded by the same computer on the software of LabVIEW8.6.

![Figure 3.4. Hot-wire setup. (a) The typical edge of the hot-wire probe used in laminar-turbulent transition measurement. (b) The hot-wire probe mounted on the traverse through an fixing adapter. The hot-wire probe is oriented to the azimuthal direction.](image)
3. EXPERIMENTAL METHODS

3.2.2. Hot-wire calibration

Here the hot-wire calibration method for instability and laminar-turbulent transition measurement is stated. The calibration for turbulent boundary-layer flow is discussed in detail in Paper 4.

The calibration of the hot-wire probe is generally performed in the free-stream with a reference velocity meter (e.g. Prandtl tube). However, this procedure can be problematic for the rotating-disk boundary layer measurement, because, first of all, the boundary layer of the rotating-disk flow does not have a free-stream region. Therefore, another calibration method is performed for this study which is comparing the laminar velocity profile. This calibration requires the absolute wall-normal position from the wall of the hot-wire sensitive area. Then the height of the hot-wire sensitive area is determined by the photograph taken with a precision gauge block with 1.000 mm thickness. The typical setup to determine the hot-wire position is shown in figure 3.5. The reference precision gauge block is put next to the hot-wire probe. The height of the hot-wire sensitive area and the gauge block is captured from the front by a microscopic lens (Nikon Micro-Nikkor AF 200mm f/4 D ED) and a camera (Canon EOS 7D) through a mirror located between its optical path. A typical photograph of the wall position determination is shown in figure 3.6. In Fig 3.6 1 pixel of the image is equivalent to 2.4 µm. By this method, the probe height from the wall is determined with an accuracy of 10 – 15 µm. This error is caused mainly by resolution of the micro lens and quality of the mirror in the path which makes the image blur.

In hot-wire anemometry CTA outputs in principle the correct voltage related to fluid velocity if the physical properties of flow field are constant during the operation. However, the measurements in this study observed maximum 1 °C temperature deviation basically depending on a rotational speed of a d.c. servo-motor. To compensate this temperature variance in hot-wire anemometry the output voltage is corrected by the following equation (e.g. Bruun 1995):

\[ E^* (T_{ref}) = E^* (T^*) \left(1 - \frac{T^* - T_{ref}^*}{\alpha_R / \alpha_{el}}\right)^{-1}, \]  

where \( E^* (T_{ref}) \) is a corrected output voltage from CTA, \( T_{ref}^* \) is a reference ambient temperature, namely an ambient temperature in hot-wire calibration, \( E^* (T^*) \) is the output voltage in the measurement, \( T^* \) is an ambient temperature in the measurement and \( \alpha_{el} \) is temperature coefficient of resistivity, which is \( \alpha_{el} = 0.0038 \, K^{-1} \) for platinum (Bruun 1995).

The hot-wire calibration is performed using a azimuthal laminar velocity profile by varying rotational speed, radial position and axial height. Figure 3.7 shows typical calibration results with laminar velocity profile. The calibrated
3.2. MEASUREMENT TECHNIQUES

Figure 3.5. The typical hot-wire calibration set-up. The hot-wire sensitive area with a precision gauge block put next to the hot-wire probe is captured by a digital camera with a micro lens through the mirror put in the optical path.

Figure 3.6. Photograph showing the hot-wire probe for an instability measurement during the wall position determination using a precision gauge block with 1.000 mm thickness. The above half-plane shows the real objects and the objects below half-plane are due to reflections on the glass surface.
data points are fitted by modified King’s law (for better accuracy at low velocities, see Johansson & Alfredsson 1982) given as

\[ V^* = k_1 (E^{*2} - E_0^{*2})^{1/n} + k_2 E^* - E_0^{*1/2}, \] (3.4)

where \( E^* \) and \( E_0^* \) are the mean anemometer output voltages at mean velocities \( V^* \) and zero, respectively, and \( k_{1,2} \) and \( n \) are constants to be determined by a linear least-squares fit of the calibration data. Figure 3.8 shows the deviation of the calibration data points from the fitted equation (3.4). The deviations are within ±1.5% except at low speed region (\( V^* \leq 0.5 \text{ m/sec} \)). The effect of a radial velocity component on the hot-wire reading depends on the rotational speed and axial position, see figure 2.3. However figure 3.8 shows that its effect is negligible because the deviations of calibration data points from theoretical laminar profile in different rotational speeds are small. The axial velocity component is also negligible.

3.2.3. Rotational speed of disk

The rotational speed of the disk \( \Omega^* \) is measured by photo-micro sensor (EE-SX 498). A brass disk with 30 slits at regular intervals in the azimuthal direction is mounted below the iron disk. The slits of the brass disk is sandwiched by the photo-micro sensor, then the sensor outputs the corresponding voltage (4 or 0 V) which depends whether the slit is located between the sensor or not. The frequency of the output voltage from the sensor is recorded with 80MHz sample clock timebase (National Instruments USB-6216) and converted to the rotational speed. This photo-micro sensor is able to measure the rotational speed up to 3000 rpm. The measured rotational speed of the disk is shown in figure 3.9. It shows that the disk rotates in steady rotational speed within ±1.5 rpm.

3.2.4. Ambient temperature and pressure

The ambient temperature is measured by a platinum resistance thermometer (PT100). The accuracy of this sensor is checked by a mercury thermometer with 0.01°C resolution shown in figure 3.10. The deviations of the PT100 used in present study from the mercury thermometer is ±0.15°C in the measurement range.

The kinematic viscosity \( \nu^* \) of the fluid is given as

\[ \nu^* = \frac{\mu^*}{\rho^*}, \] (3.5)

where \( \mu^* \) is the viscosity of fluid and \( \rho^* \) is the density. Here \( \mu^* \) is calculated
3.2. MEASUREMENT TECHNIQUES

Figure 3.7. Hot-wire calibration using the laminar velocity profile varying the rotational speed and the normal height. The symbols indicate \(\Omega^*=0\) rpm (\(\star\)), 300 rpm (\(\circ\)), 500 rpm (\(\Box\)), 600 rpm (\(\circlearrowleft\)), 700 rpm (\(\triangle\)), 770 rpm (\(\downarrow\)), 860 rpm (\(\diamondsuit\)). The solid line shows the modified King’s law fitting.

Figure 3.8. Deviations of calibration data points from the modified King’s law fitting \((V^{\star}_{fit})\). The symbols are the same as in figure 3.7.
using Sutherland law which is written as

$$\mu^* = \frac{1.4578 \times 10^{-6} \times T^{*3/2}}{T^* + 110.4},$$  \hspace{1cm} (3.6)

where $T^*$ is a atmospheric temperature measured by the PT100 in Kelvin. The $\rho^*$ for dry air is calculated using gas law which is written as

$$\rho^* = \frac{P_{atm}^*}{287.0 \times T^*},$$  \hspace{1cm} (3.7)

where $P_{atm}^*$ is an atmospheric pressure in Pascal unit measured by a precision barometer.

**Figure 3.9.** Deviations $\Delta \Omega^*$ rpm of the rotational speeds from the target rotational speeds in revolutions per minute. Target rotational speeds are 400 rpm (Blue), 700 rpm (Green), 1000 rpm (Red), 1500 rpm (Black), respectively.
Figure 3.10. The temperature difference between calibrated PT100 temperature ($T^*_c$) and a precision mercury thermometer temperature ($T_{Mar}$) with 0.01°C step.
Main contribution and conclusions

The following chapter summarizes the main contributions and conclusions from the papers constituting Part II of the thesis. For details on the results the reader is referred to the appended papers.

4.1. Laminar-turbulent transition process of the a rotating-disk boundary-layer flow.

- To investigate laminar-turbulent transition process of the rotating-disk flow azimuthal velocities are measured using a hot-wire probe which is calibrated using the laminar mean velocity profile. The measured mean azimuthal velocity profile at $R = 430 - 510$ corresponds well to its theoretical laminar profile except far away from the disk where the smallness of the azimuthal velocity makes the hot-wire measurements inaccurate. Lingwood (1995a) showed there is a local absolute above $R_{CA} = 507$, and suggested that this triggers nonlinearity behaviour. Then the present results show the onset of nonlinearity at $R = 510$ in the frequency spectrum as a harmonic of the primary vortices. The laminar-turbulent transition observed in the present study includes the effects not only of convective instability but also absolute instability.

- The growth of azimuthal fluctuation velocities captures its exponential growth up to $R = 580$. The slope of the exponential growth for $475 < R < 530$ corresponds approximately to the maximum spatial growth rate for stationary linear disturbances, see e.g. figure 6a in Hussain et al. (2011). The change in slope at around $R = 545$ could correspond to Viaud et al.’s so-called ‘secondary front’. If so, then the present results may represent the first experimental validation of Viaud et al.’s (2011) DNS results and Pier’s (2003) theoretical predictions of absolute instability of the primary global instability.

- To give a better understanding of the laminar-transition on the rotating-disk flow, probability density function (PDF) maps of azimuthal fluctuation velocity normalized by the local disk speed are presented. The PDF map measured at $z = 1.3$ over a range of Reynolds nubers dramatically
4.1. LAMINAR-TURBULENT TRANSITION

shows the change in distribution at $R = 550$ from exponential growth to a strongly skewed distribution. This change in PDF corresponds to the change of the slope of the disturbances growth. At around $R = 600$, the skewed PDF starts to disappear and the positive deviation of $v$ has its maximum. The almost symmetric PDF above $R = 650$ indicates that the flow has reached a fully-developed turbulent state. These characteristics are not obvious in the spectral distributions.

- The application of PDF maps to velocity-profile measurements reveals the structure normal to the wall. In particular, at $R = 570$ peaks in the PDF may be associated with a secondary instability.

- Effects that the edge of the disk may have on laminar-turbulent transition have been investigated. Healey (2010) suggested, using the linearized complex Ginzburg-Landau equation, that the transition Reynolds number for the rotating-disk flow can be affected by the Reynolds number at the edge of the disk. He compared his suggestion with previous experimental results, which seemed to confirm his hypothesis. In this study three different edge conditions are considered and the results do not show such a behaviour in the measurement range; the differences in transition Reynolds numbers stated by previous authors are explained by the different definitions of the transition Reynolds number used in each case, rather than the effect of the edge Reynolds number (or edge condition).

- The secondary instability and turbulent breakdown of the rotating-disk flow are investigated using hot-wire anemometry. The kinked azimuthal velocity fluctuations associated with secondary instability are observed in single-realization timeseries at the final stage of the laminar-turbulent transition. It is found that the appearance of kinked timeseries becomes less apparent at certain wall-normal heights. On the other hand, ensemble-averaged fluctuation-velocity timeseries do not seem to feature kinks, indicating that the secondary instability is a travelling wave.

- The turbulent breakdown of the stationary mode has been investigated as a function of Reynolds number. It is found that the exponential growth shown by ensemble-averaged velocity fluctuations saturates at $R = 550$, plateaus for $R = 580 - 585$, and is followed by the turbulent breakdown of the stationary mode beyond that Reynolds number. To investigate more details of the stationary mode, normalized peak amplitudes of each stationary vortex are plotted as a function of Reynolds number. Each stationary vortex grows exponentially but with different amplitudes up to $R = 550$, showing a convective behaviour. However
the turbulent breakdown of each stationary vortex seems to be independent of its amplitude, which suggests that the turbulent breakdown process is not due to its convective behaviour but due to the appearance of the travelling secondary instability observed at $R = 570$.

- Based on the almost constant transition Reynolds number reported in the literature, the amplitude independence of turbulent breakdown of stationary vortex and the appearance of the travelling secondary instability at the final stage of the laminar-turbulent transition, we hypothesise that the secondary instability could be triggered by the primary absolute instability (Lingwood 1995a) and could be absolute unstable itself as suggested by Pier (2003).

4.2. Turbulent boundary-layer flow on a rotating disk.

- The turbulent boundary layer on the rotating disk has been studied and compared with two-dimensional flat-plate turbulent boundary layer with respect to the mean-flow distributions as well as the higher moments.

- A methodology to determine the wall shear stress was developed, including a new idea to calibrate the hot-wire sensor against the laminar profile and extrapolating to higher velocities, a procedure that could be tested a posteriori.

- The results show that the statistics in the near-wall region are similar to the two-dimensional boundary layer, whereas the outer regions differ.

- The spectral map of the streamwise/azimuthal velocity show clear differences between the two cases, possibly because of the three-dimensional character of the rotating-disk boundary layer, which may change the inclination of the near wall structures with respect to the hot wire.
CHAPTER 5

Papers and authors’ contributions

Paper 1

*A new way to describe the transition characteristics of a rotating-disk boundary-layer flow*

The laminar-turbulent transition of the rotating-disk flow has been investigated. The original apparatus was borrowed from the University of Cambridge Department of Engineering and was modified and put into operation by SI. The experimental investigations were performed by SI under the supervision of HAL and RL, and the writing was jointly done by SI, HAL and RL. Part of these results have been presented at EUROMECH Colloquium 525 Instabilities and transition in three-dimensional flows with rotation, 21 – 23 June 2011, Lyon, France.

Paper 2

*An experimental study of edge effects on rotating-disk transition*

The effects of the finite radius of the disk on the laminar-turbulent transition of the rotating-disk flow have been investigated experimentally. The experimental investigations were performed by SI using the same facility used in Paper 1 under supervision of HAL and RL, and the writing was jointly done by SI, HAL and RL. Some of these results have been presented at the Annual Meeting of the American Physical Society’s Division of Fluid Dynamics, 20 – 22 November 2011, Baltimore, Maryland, USA.
5. PAPERS AND AUTHORS’ CONTRIBUTIONS

Paper 3

*Secondary instability and turbulent breakdown of the rotating-disk flow*
Shintaro Imayama (SI).

This paper focuses on the transition process and especially the role of secondary instability on the primary stationary vortices, through separating the stationary and travelling modes by comparison of ensemble-averaged and single-realization measurements of the fluctuating velocity.

Paper 4

*An experimental study of a rotating-disk turbulent boundary-layer flow*
Shintaro Imayama (SI), R. J. Lingwood (RL) & P. Henrik Alfredsson (HAL).

The turbulent boundary layer on the rotating disk flow has been investigated. The azimuthal friction velocity is determined using hot-wire measurement directly and turbulence statistics normalized by the inner scales are represented. The experiments were performed by SI using the same facility used in Paper 1 under the supervision of RL and HAL, and the writing was jointly done by SI, RL and HAL. Some of these results have been presented at EUROMECH Colloquium 525 Instabilities and transition in three-dimensional flows with rotation, 21 – 23 June 2011, Lyon, France.
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References


REFERENCES


Derivations of governing equations

A.1. Introduction

This appendix shows the derivation in detail of the governing equations for the incompressible rotating disk flow in cylindrical coordinates (Cham & Head 1969; Hsu 2002; Kundu & Cohen 2008). First general definitions of the cylindrical coordinates, the continuity equation and Navier-Stokes equations (NSE) are derived. Secondly taking a Reynolds decomposition and ensemble average, Reynolds Averaged Navier-Stokes equation (RANS) is derived. Finally making some assumptions about the magnitude of various terms the RANS for turbulent flow on the rotating-disk is derived. The dimensional mark ‘*’ used in the introduction parts and papers of this thesis is taken away in the appendix.

Figure A.1. A sketch of the von Kármán boundary layer on a rotating disk showing the mean velocity profiles (in a stationary laboratory frame), where $U_N = U/(r\Omega)$, $V_N = V/(r\Omega)$, $W_N = W/(\nu \Omega)^{1/2}$.
A.2. CYLINDRICAL FRAME SYSTEM IN A ROTATING-DISK FLOW

The system of incompressible rotating-disk flow is modeled as an infinite disk rotating at a constant speed $\Omega$ in cylindrical coordinate frame. Figure A.1 shows a sketch of the rotating-disk flow system. The position vector $\mathbf{r}$, velocity vector $\mathbf{v}$ and rotation vector $\mathbf{\omega}$ are given as

$$\mathbf{r} = \begin{pmatrix} r \cos \theta \\ r \sin \theta \\ z \end{pmatrix}, \quad (A.1)$$

$$\mathbf{v} = \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix}, \quad (A.2)$$

$$\mathbf{\omega} = \begin{pmatrix} 0 \\ 0 \\ \Omega \end{pmatrix}, \quad (A.3)$$

where $r$ is the radius, $\theta$ is the angle and $z$ is a wall normal coordinate, $\tilde{u}, \tilde{v}, \tilde{w}$ are radial, azimuthal and axial velocity components respectively, and $\Omega$ is the constant angular rotational speed of the disk.

To confirm the independence relation between the orthogonal coordinate system and the cylindrical frame system, a Jacobian matrix of positions in both coordinate systems can be written as

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \partial_x x & \partial_y x & \partial_z x \\ \partial_x y & \partial_y y & \partial_z y \\ \partial_x z & \partial_y z & \partial_z z \end{vmatrix}$$

$$= \begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r \left( \cos^2 \theta + \sin^2 \theta \right) = r \neq 0 \text{ (except } r = 0), \quad (A.4)$$

where $(x, y, z)$ is a position vector in the orthogonal coordinate system and $\partial_r, \partial_{\theta}, \partial_z$ are derivatives in the radial, azimuthal and axial directions, respectively. The relation of the position vector between the orthogonal coordinate system and the cylindrical coordinate system is written as

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z. \quad (A.5)$$

The Jacobian matrix is nonzero except $r = 0$, which indicates that the positions in orthogonal coordinate system and cylindrical coordinate system have one-on-one relations except at $r = 0$. 

To confirm the independence relation between the orthogonal coordinate system and the cylindrical frame system, a Jacobian matrix of positions in both coordinate systems can be written as
A.3. Continuity equation

The divergence operator $\nabla$ is in the cylindrical coordinate system written as

$$\nabla = \frac{1}{h_r h_\theta h_z} \left( \frac{\partial}{\partial r} (h_\theta h_z) \right) \left( \frac{\partial}{\partial \theta} (h_z h_r) \right) \left( \frac{\partial}{\partial z} (h_r h_\theta) \right)$$  \hspace{1cm} (A.6)

where $h_r$, $h_\theta$, $h_z$ are given as

$$h_r = \left| \frac{\partial r}{\partial r} \right| = (\cos^2 \theta + \sin^2 \theta)^{1/2} = 1,$$

$$h_\theta = \left| \frac{\partial r}{\partial \theta} \right| = [(-r \sin \theta)^2 + (r \cos \theta)^2]^{1/2} = r,$$  \hspace{1cm} (A.7)

$$h_z = \left| \frac{\partial r}{\partial z} \right| = 1.$$

Then the divergence of the velocity vector is

$$\nabla \cdot \mathbf{v} = \frac{1}{r} \frac{\partial (r \tilde{u})}{\partial r} + \frac{1}{r} \frac{\partial \tilde{v}}{\partial \theta} + \frac{\partial \tilde{w}}{\partial z}. \hspace{1cm} (A.8)$$

The continuity equation in incompressible flow is

$$\nabla \cdot \mathbf{v} = 0. \hspace{1cm} (A.9)$$

Then equation (A.2) and equation (A.6) are substituted into equation (A.9), giving

$$\frac{\partial \tilde{u}}{\partial r} + \frac{1}{r} \frac{\partial \tilde{v}}{\partial \theta} + \frac{\partial \tilde{w}}{\partial z} + \tilde{u} = 0. \hspace{1cm} (A.10)$$

A.4. Navier-Stokes equation (NSE)

The Navier-Stokes equation (NSE) in a uniformly rotating co-ordinate system is written as

$$\frac{\partial \mathbf{v}}{\partial t} + \left( \mathbf{v} \cdot \nabla \right) \mathbf{v} + 2 \omega \times \mathbf{v} + \omega \times (\omega \times \mathbf{r}) = -\frac{1}{\rho} \nabla p + \mu \nabla^2 \mathbf{v}, \hspace{1cm} (A.11)$$

where $\textcircled{1}$ is the material differential term, where $(\mathbf{v} \cdot \nabla)\mathbf{v}$ is defined as

$$(\mathbf{v} \cdot \nabla)\mathbf{v} = \left( \tilde{u} \frac{\partial}{\partial r} + \frac{\tilde{v}}{r} \frac{\partial}{\partial \theta} + \tilde{w} \frac{\partial}{\partial z} \right) \mathbf{v} + \frac{1}{r} \left( \begin{array}{c} \tilde{v} \tilde{v} \\ \tilde{u} \tilde{w} \\ 0 \end{array} \right), \hspace{1cm} (A.12)$$
the Coriolis force term \((\mathbf{2})\) is

\[
2\mathbf{\omega} \times \mathbf{v} = 2 \begin{pmatrix} 0 \\ 0 \\ \Omega \end{pmatrix} \times \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix} = \begin{pmatrix} -2\Omega \tilde{v} \\ 2\Omega \tilde{u} \\ 0 \end{pmatrix},
\]

(A.13)

the centrifugal force term \((\mathbf{3})\) is

\[
\mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}) = -\Omega^2 \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix},
\]

(A.14)

the pressure term \((\mathbf{4})\) is

\[
-\frac{1}{\rho} \nabla \tilde{p} = -\frac{1}{\rho} \left( \frac{\partial_p \tilde{p}}{\tilde{r}} \right),
\]

(A.15)

and viscous term \((\mathbf{5})\) is

\[
\nu \nabla^2 \mathbf{v} = \nu \left[ \begin{pmatrix} \nabla^2 \tilde{u} \\ \nabla^2 \tilde{v} \\ \nabla^2 \tilde{w} \end{pmatrix} + \frac{1}{r^2} \begin{pmatrix} -\tilde{u} - 2\partial_\tilde{v} \tilde{v} \\ -\tilde{v} + 2\partial_\tilde{u} \tilde{u} \\ 0 \end{pmatrix} \right],
\]

(A.16)

respectively, where \(\rho\) is a density, \(\tilde{p}\) is a pressure and \(\nu\) is the kinematic viscosity and the Laplace operator \(\nabla^2\) in a cylindrical coordinate system is defined as

\[
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \tilde{\theta}^2} + \frac{\partial^2}{\partial \tilde{z}^2} + \frac{1}{r} \frac{\partial}{\partial \tilde{z}}.
\]

(A.17)

Therefore the viscous term can be written

\[
\nu \nabla^2 \mathbf{v} = \nu \left[ \begin{pmatrix} \partial^2 \tilde{u} \\ \partial^2 \tilde{v} \\ \partial^2 \tilde{w} \end{pmatrix} + \frac{1}{r^2} \frac{\partial}{\partial r} \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix} + \frac{\partial}{\partial \tilde{\theta}} \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ 0 \end{pmatrix} + \frac{1}{r^2} \begin{pmatrix} -\tilde{u} - 2\partial_\tilde{\theta} \tilde{v} \\ -\tilde{v} + 2\partial_\tilde{\theta} \tilde{u} \\ 0 \end{pmatrix} \right],
\]

(A.18)
Then in a cylindrical coordinate system, the three components of the Navier-Stokes equations become

**Radial component:**

\[
\frac{\partial \tilde{u}}{\partial t} + \left( \frac{\partial \tilde{u}}{\partial r} + \frac{\tilde{v}}{r} \frac{\partial \tilde{u}}{\partial \theta} + \frac{\tilde{w}}{r} \frac{\partial \tilde{u}}{\partial z} \right) - \frac{\tilde{v}}{r} = \frac{1}{\rho} \frac{\partial \tilde{p}}{\partial r} - 2 \tilde{\varepsilon} \Omega - r \Omega^2 \]

(A.19)

**Azimuthal component:**

\[
\frac{\partial \tilde{v}}{\partial t} + \left( \frac{\partial \tilde{v}}{\partial r} + \frac{\tilde{v}}{r} \frac{\partial \tilde{v}}{\partial \theta} + \tilde{w} \frac{\partial \tilde{v}}{\partial z} \right) = \frac{1}{\rho} \frac{\partial \tilde{p}}{\partial \theta} + 2 \tilde{\Omega} \]

(A.20)

**Axial component:**

\[
\frac{\partial \tilde{w}}{\partial t} + \left( \frac{\partial \tilde{w}}{\partial r} + \frac{\tilde{v}}{r} \frac{\partial \tilde{w}}{\partial \theta} + \tilde{w} \frac{\partial \tilde{w}}{\partial z} \right) = \frac{1}{\rho} \frac{\partial \tilde{p}}{\partial z} \]

(A.21)

### A.5. Reynolds average

In this section the Reynolds averaged continuity equation and Navier-Stokes equations (RANS) are derived, respectively. The instantaneous velocity \((\tilde{u}, \tilde{v}, \tilde{w})\) and instantaneous pressure \((\tilde{p})\) are decomposed in their mean and fluctuations (time-dependence), through an operation called Reynolds decomposition, giving

\[
\tilde{u} = U + u, \quad \tilde{v} = V + v, \quad \tilde{w} = W + w, \quad \tilde{p} = P + p \]

(A.22)

where \(U, V, W, P\) are mean components of the radial, azimuthal and axial velocities and pressure, and \(u, v, w, p\) are fluctuation components of radial, azimuthal and axial velocities and pressure, respectively.
A.5. REYNOLDS AVERAGE

A.5.1. Reynolds averaged continuity equation

The continuity equation is given from equation (A.10) as

\[ \frac{\partial \tilde{u}}{\partial r} + \frac{1}{r} \frac{\partial \tilde{v}}{\partial \theta} + \frac{\partial \tilde{w}}{\partial z} + \frac{\tilde{u}}{r} = 0. \]  

(A.23)

The decomposed components (equation (A.22)) are substituted into equation (A.23), whereafter an ensemble averages is taken here (denoted with an overscore), resulting in

\[ \frac{\partial \bar{U}}{\partial r} + \frac{1}{r} \frac{\partial \bar{V}}{\partial \theta} + \frac{1}{r} \frac{\partial \bar{W}}{\partial \theta} + \frac{\partial \bar{U}}{\partial z} + \frac{\bar{U}}{r} + \frac{\bar{V}}{r} = 0. \]  

(A.24)

The derivation with respect to the \( \theta \) direction is neglected due to that the mean flow is independent of the azimuthal direction. Therefore, the Reynolds averaged continuity equation becomes

\[ \frac{\partial \bar{U}}{\partial r} + \frac{\partial \bar{W}}{\partial z} + \frac{\bar{U}}{r} = 0. \]  

(A.25)

A.5.2. The radial component of the RANS

The radial component of NSE the is given by equation (A.19) and decomposed components (equation (A.22)) are substituted into equation (A.19). Then the left-hand side (LHS) of equation (A.19) becomes

\[ \text{LHS}(A.19) = \frac{\partial \bar{U}}{\partial t} + \frac{\partial \bar{P}}{\partial r} + \frac{u \partial \bar{U}}{\partial r} + \frac{\partial \bar{u}}{\partial r} + \frac{\partial \bar{u}}{\partial r} + \frac{\partial \bar{u}}{\partial r} + \frac{\partial \bar{v}}{\partial \theta} + \frac{\partial \bar{v}}{\partial \theta} + \frac{\partial \bar{v}}{\partial \theta} + \frac{\partial \bar{v}}{\partial \theta} + \frac{\partial \bar{w}}{\partial z} + \frac{\partial \bar{w}}{\partial z} + \frac{\partial \bar{w}}{\partial z} + \frac{\partial \bar{w}}{\partial z} + \frac{V^2}{r} - \frac{\bar{V}^2}{r} - \frac{2V\bar{W}}{r} - 2\bar{V}\Omega - r\bar{\Omega}^2, \]

(A.26)

where \( A_r \) is
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\[ A_r = \frac{\partial uu}{\partial r} - u \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial w u}{\partial \theta} - \frac{u}{r} \frac{\partial u}{\partial \theta} + \frac{\partial u w}{\partial z} - \frac{u}{r} \frac{\partial w}{\partial z} \]
\[ = \frac{\partial uu}{\partial r} + \frac{1}{r} \frac{\partial w u}{\partial \theta} + \frac{\partial u w}{\partial z} - u \left( \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial w}{\partial \theta} + \frac{\partial w}{\partial z} \right) \]
\[ = \frac{\partial uu}{\partial r} + \frac{\partial w w}{\partial z} - \frac{u}{r} \]

(A.27)

The right-hand side (RHS) of equation (A.19) becomes

\[ \text{RHS(A.19)} = - \frac{1}{\rho} \frac{\partial P}{\partial r} - \frac{1}{\rho} \frac{\partial P}{\partial \theta} + \nu \left[ \frac{\partial^2 U}{\partial r^2} + \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r} \frac{\partial^2 U}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 U}{\partial \theta^2} \right] + \frac{\partial^2 U}{\partial z^2} \]
\[ + \frac{\partial^2 U}{\partial z^2} + \frac{1}{r} \frac{\partial U}{\partial r} + \frac{1}{r} \frac{\partial U}{\partial r} - \frac{1}{r^2} \frac{\partial U}{\partial \theta} + \frac{2}{r^2} \frac{\partial V}{\partial \theta} - \frac{1}{r^2} \frac{\partial U}{\partial \theta} \]
\[ = - \frac{1}{\rho} \frac{\partial P}{\partial r} + \nu \left[ \frac{\partial^2 U}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2} + \frac{1}{r} \frac{\partial U}{\partial r} - \frac{U}{r^2} - \frac{2}{r^2} \frac{\partial V}{\partial \theta} \right]. \]

(A.28)

Then the RANS of the radial component is given as

\[ \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial r} + \frac{V}{r} \frac{\partial U}{\partial \theta} + W \frac{\partial U}{\partial z} + \frac{\partial uu}{\partial r} + \frac{\partial uu}{\partial \theta} + \frac{\partial uu}{\partial z} \]
\[ = - \frac{1}{\rho} \frac{\partial P}{\partial r} + \nu \left[ \frac{\partial^2 U}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2} + \frac{1}{r} \frac{\partial U}{\partial r} - \frac{U}{r^2} - \frac{2}{r^2} \frac{\partial V}{\partial \theta} \right]. \]

(A.29)

A.5.3. The azimuthal component of the RANS

The azimuthal component of the NSE is given from equation (A.20) and decomposed components from equation (A.22) are substituted into equation (A.20). Then the LHS of equation (A.20) becomes
LHS(A.20) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial t} + u_0 \frac{\partial V}{\partial t} + \frac{\partial V}{\partial t} + u_0 \frac{\partial V}{\partial t}
+ \frac{V \partial V}{r \partial \theta} + \frac{\partial V}{\partial \theta} + \frac{V \partial V}{r \partial \theta} + \frac{\partial V}{\partial \theta}
+ \frac{W \partial V}{\partial z} + \frac{\partial V}{\partial z} + \frac{W \partial V}{\partial z} + \frac{\partial V}{\partial z}
+ \frac{UV}{r} + \frac{\partial V}{r} + \frac{\partial V}{r} + \frac{\partial V}{r} + 2U \Omega + 2\Omega

= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial r} + \frac{V \partial V}{r \partial \theta} + W \frac{\partial V}{\partial z} + \frac{\partial V}{\partial r} + \frac{\partial V}{r \partial \theta} + \frac{\partial V}{r \partial \theta} + \frac{\partial V}{r \partial \theta} + \frac{\partial V}{r \partial \theta} + \frac{\partial V}{r \partial \theta}
+ \frac{UV}{r} + \frac{\partial V}{r} + 2U \Omega,

where \( A_\theta \) is

\[ A_\theta = \frac{\partial \nu}{\partial r} - \frac{\nu \partial \nu}{r \partial \theta} - \frac{\nu \partial \nu}{r \partial \theta} + \frac{\partial \nu}{\partial \theta} - \frac{\partial \nu}{\partial \theta} - \frac{\partial \nu}{\partial \theta} - \frac{\partial \nu}{\partial \theta} - \frac{\partial \nu}{\partial \theta} \]

\[ = \frac{\partial \nu}{\partial r} + \frac{1}{r} \frac{\partial \nu}{\partial \theta} + \frac{\partial \nu}{\partial z} + \frac{\partial \nu}{\partial z} \]

The RHS of equation (A.20) becomes

\[ \text{RHS(A.20)} = - \frac{1}{\rho r} \frac{\partial P}{\partial \theta} - \frac{1}{\rho r} \frac{\partial \rho}{\partial \theta} + \nu \left[ \frac{\partial^2 V}{\partial r^2} + \frac{\partial^2 V}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 V}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 V}{\partial \theta^2} \right]
+ \frac{\partial^2 V}{\partial z^2} + \frac{\partial^2 V}{\partial z^2} + \frac{1}{r} \frac{\partial V}{\partial r} + \frac{1}{r} \frac{\partial V}{\partial r} - \frac{V}{r^2} - \frac{\partial V}{r^2} + \frac{2}{r^2} \frac{\partial V}{r^2} + \frac{2}{r^2} \frac{\partial V}{r^2} \]

\[ = - \frac{1}{\rho r} \frac{\partial P}{\partial \theta} + \nu \left[ \frac{\partial^2 V}{\partial r^2} + \frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} + \frac{1}{r^2} \frac{\partial V}{\partial r} - \frac{V}{r^2} + \frac{2}{r^2} \frac{\partial V}{r^2} \right] \]

(A.32)

Then the RANS of the azimuthal component is written as
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\[
\begin{align*}
\frac{\partial V}{\partial t} + U \frac{\partial V}{\partial r} + \frac{V}{r} \frac{\partial V}{\partial \theta} + W \frac{\partial V}{\partial z} + \frac{1}{r} \frac{\partial \nu}{\partial \theta} &+ \frac{1}{r^2} \frac{\partial \nu}{\partial \theta} \\
&+ \frac{UV}{r} + 2 \frac{\nu}{r} + 2U \Omega \\
= &- \frac{1}{\rho r} \frac{\partial P}{\partial \theta} + \nu \left[ \frac{\partial^2 V}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 V}{\partial \theta^2} + \frac{1}{r} \frac{\partial V}{\partial r} - \frac{V}{r^2} \right] .
\end{align*}
\]

(A.33)

A.5.4. The axial component of the RANS

The axial direction of the NSE is given from equation (A.21) and decomposed components from equation (A.22) are substituted into equation (A.21). Then the LHS of equation (A.21) is given as

\[
\text{LHS(A.21)} = \frac{\partial W}{\partial t} + \frac{\partial W}{\partial \theta} + \frac{U}{r} \frac{\partial W}{\partial r} + \frac{W}{r} \frac{\partial W}{\partial \theta} + \frac{1}{r} \frac{\partial w}{\partial \theta} + \frac{w}{r} \frac{\partial w}{\partial \theta} \\
+ \frac{V}{r^2} \frac{\partial W}{\partial \theta} + \frac{W}{r^2} \frac{\partial W}{\partial \theta} + \frac{1}{r^2} \frac{\partial W}{\partial \theta} + \frac{W}{r} \frac{\partial w}{\partial \theta} + \frac{w}{r} \frac{\partial w}{\partial \theta} \\
+ \frac{W}{r} \frac{\partial W}{\partial z} + \frac{w}{r} \frac{\partial W}{\partial z} + \frac{W}{r} \frac{\partial w}{\partial z} + \frac{w}{r} \frac{\partial w}{\partial z} \\
= \frac{\partial W}{\partial t} + U \frac{\partial W}{\partial r} + \frac{V}{r} \frac{\partial W}{\partial \theta} + W \frac{\partial W}{\partial z} + \frac{u}{\partial r} + \frac{v}{\partial \theta} + \frac{w}{\partial z} \\
\end{align*}
\]

(A.34)

where \( A_z \) is

\[
\begin{align*}
\frac{\partial w}{\partial r} - \frac{w}{\partial r} + \frac{1}{r} \frac{\partial w}{\partial \theta} - \frac{w}{\partial \theta} + \frac{\partial w}{\partial z} - \frac{w}{\partial z} = \frac{\partial w}{\partial r} + \frac{1}{r} \frac{\partial w}{\partial \theta} + \frac{\partial w}{\partial z} - \frac{w}{\partial z} \\
= \frac{\partial w}{\partial r} + \frac{1}{r} \frac{\partial w}{\partial \theta} + \frac{\partial w}{\partial z} - \frac{w}{\partial z} \\
= \frac{\partial w}{\partial r} + \frac{1}{r} \frac{\partial w}{\partial \theta} + \frac{\partial w}{\partial z} - \frac{w}{\partial z} .
\end{align*}
\]

(A.35)

The RHS of equation (A.21) becomes
\[ \text{RHS}(A.21) = -\frac{1}{\rho} \frac{\partial P}{\partial z} - \frac{1}{\rho} \frac{\partial \rho}{\partial z} + \nu \left[ \frac{\partial^2 W}{\partial r^2} + \frac{\partial^2 W}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial \rho}{\partial r} \right] + \frac{\partial^2 W}{\partial z^2} + \frac{1}{r} \frac{\partial W}{\partial r} + \frac{1}{r} \frac{\partial^2 \rho}{\partial \theta^2} \]

\[ = -\frac{1}{\rho} \frac{\partial P}{\partial z} + \nu \left[ \frac{\partial^2 W}{\partial r^2} + \frac{1}{r^2} \frac{\partial \rho}{\partial r} + \frac{\partial^2 W}{\partial z^2} + \frac{1}{r} \frac{\partial W}{\partial r} \right]. \quad (A.36) \]

Then the axial component of the RANS is written as

\[ \frac{\partial W}{\partial t} + U \frac{\partial W}{\partial r} + \frac{V}{r} \frac{\partial W}{\partial \theta} + W \frac{\partial W}{\partial z} + \frac{\partial^2 W}{\partial r^2} + \frac{1}{r} \frac{\partial W}{\partial r} + \frac{\partial^2 \rho}{\partial \theta^2} + \frac{1}{r} \frac{\partial W}{\partial r} \]

\[ = -\frac{1}{\rho} \frac{\partial P}{\partial z} + \nu \left[ \frac{\partial^2 W}{\partial r^2} + \frac{1}{r^2} \frac{\partial \rho}{\partial r} + \frac{\partial^2 W}{\partial z^2} + \frac{1}{r} \frac{\partial W}{\partial r} \right]. \quad (A.37) \]

A.5.5. RANS for incompressible turbulent rotating-disk flow

RANS for incompressible turbulent rotating-disk flow are derived applying the boundary-layer approximation and some other assumptions. The order of each term is estimated as \( r \sim L, z \sim \delta, U and V \sim V', W \sim (\delta/L)V', u, v and w \sim v', V \sim r\Omega, \) namely \( \Omega \sim V'/L \) and \( P \sim \rho V'^2, \) where \( L \) is characteristic length, \( \delta \) is boundary layer thickness, \( V' \) is characteristic mean velocity and \( v' \) is characteristic fluctuation velocity.

The assumptions for incompressible turbulent rotating-disk flow are:

- ASM1. Steady flow: \( \partial / \partial t = 0, \)
- ASM2. Axisymmetry: \( \partial / \partial \theta = 0, \)
- ASM3. Boundary layer approximation: \( L \gg \delta, \)
- ASM4. A fluid element is convected with a velocity \( O(V'). \) In the two time scale of convection \( \Delta t_{\text{convection}} \) and diffusion \( \Delta t_{\text{diffusion}} \) are defined as

\[ \Delta t_{\text{convection}} \sim L/V', \quad \Delta t_{\text{diffusion}} \sim \delta^2/\nu, \]

respectively. In the boundary layer both of \( \Delta t_{\text{convection}} \) and \( \Delta t_{\text{diffusion}} \) are assumed to be of similar size, such that

\[ \Delta t_{\text{convection}} \sim \Delta t_{\text{diffusion}}, \]

giving

\[ L/V' \sim \delta^2/\nu. \]
These assumptions are applied to equations (A.29), (A.33) and (A.37), resulting

**Radial component (equation (A.29)):**

\[
\frac{\partial V}{\partial t} + U \frac{\partial U}{\partial r} + V \frac{\partial V}{\partial \theta} + W \frac{\partial V}{\partial z} + \frac{\partial \mu}{\partial r} \frac{\partial \mu}{\partial \theta} + \frac{\partial \mu}{\partial z} + \frac{\partial \mu}{\partial t} + UV - \frac{V^2}{r} + \frac{2\Omega}{r} - 2V\Omega - \frac{r\Omega^2}{2} \\
= - \frac{1}{\rho} \frac{\partial P}{\partial r} + \nu \left[ \frac{\partial^2 V}{\partial r^2} + \frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} + \frac{\partial V}{\partial \theta} - \frac{2}{r^2} \frac{\partial V}{\partial \theta} \right] \\
\]  

(A.38)

**Azimuthal component (equation (A.33)):**

\[
\frac{\partial V}{\partial t} + U \frac{\partial V}{\partial r} + V \frac{\partial V}{\partial \theta} + W \frac{\partial V}{\partial z} + \frac{\partial \mu}{\partial r} \frac{\partial \mu}{\partial \theta} + \frac{\partial \mu}{\partial z} + \frac{\partial \mu}{\partial t} + UV + \frac{2\mu}{r} + 2U\Omega \\
= - \frac{1}{\rho} \frac{\partial P}{\partial \theta} + \nu \left[ \frac{\partial^2 V}{\partial r^2} + \frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} + \frac{\partial V}{\partial \theta} - \frac{2}{r^2} \frac{\partial V}{\partial \theta} \right] \\
\]  

(A.39)

**Axial component (equation (A.37)):**

\[
\frac{\partial W}{\partial t} + U \frac{\partial W}{\partial r} + V \frac{\partial W}{\partial \theta} + W \frac{\partial W}{\partial z} + \frac{\partial \mu}{\partial r} \frac{\partial \mu}{\partial \theta} + \frac{\partial \mu}{\partial z} + \frac{\partial \mu}{\partial t} + \frac{\partial \mu}{r} \\
= - \frac{1}{\rho} \frac{\partial P}{\partial z} + \nu \left[ \frac{\partial^2 W}{\partial r^2} + \frac{\partial^2 W}{\partial \theta^2} + \frac{\partial^2 W}{\partial z^2} + \frac{\partial W}{\partial \theta} \right] \\
\]  

(A.40)

The following terms in equations (A.38), (A.39) and (A.40) are negligible:
A.5. REYNOLDS AVERAGE

- 101, 201, 301 due to ASM1,
- 103, 106, 115, 119, 203, 206, 211, 213, 217, 303, 306, 311 due to ASM2,
- 105, 108, 110 due to ASM3 which are smaller \((\delta/L)\) than 107,
- 114, 117, 118 due to ASM3 which are much smaller \((\delta^2/L^2)\) than 116,
- 205, 209 due to ASM3 which are smaller \((\delta/L)\) than 207,
- 212, 215, 216 due to ASM3 which are much smaller \((\delta^2/L^2)\) than 214,
- 302, 304, 312 due to ASM3 which are much smaller \((\delta^2/L^2)\) than 309,
- 310, 313 due to 3 which are much smaller \((\delta^2/L^2)\) than 312.

In summary, Reynolds averaged continuity equation and RANS for incompressible turbulent rotating-disk boundary-layer flow are written as

\[
\frac{\partial U}{\partial r} + \frac{\partial W}{\partial z} + \frac{U}{r} = 0, \tag{A.41}
\]

The radial component of the RANS:

\[
U \frac{\partial U}{\partial r} + W \frac{\partial U}{\partial z} - \frac{V^2}{r} - 2V \Omega = -\frac{1}{\rho} \frac{\partial P}{\partial r} + r \Omega^2 + \frac{1}{\rho} \frac{\partial}{\partial z} \left( \mu \frac{\partial U}{\partial z} - \rho \omega \right), \tag{A.42}
\]

The azimuthal component of the RANS:

\[
U \frac{\partial V}{\partial r} + W \frac{\partial V}{\partial z} + \frac{UV}{r} + 2U \Omega = \frac{1}{\rho} \frac{\partial}{\partial z} \left( \mu \frac{\partial V}{\partial z} - \rho \omega \right), \tag{A.43}
\]

The axial component of the RANS:

\[
\frac{\partial \rho \omega}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial z}. \tag{A.44}
\]
The pressure and centrifugal force terms in equation (A.42) can be transformed as

\[-\frac{1}{\rho} \frac{\partial P}{\partial r} + r \Omega^2 = -\frac{1}{\rho} \left( \frac{\partial P}{\partial r} - \rho r \Omega^2 \right)\]
\[= -\frac{1}{\rho} \frac{\partial}{\partial r} \left( P - \frac{r^2}{2} \Omega^2 \right)\]
\[= -\frac{\partial}{\partial r} \left( \frac{P}{\rho} - \frac{1}{2} r^2 \Omega^2 \right).\]  
(A.45)

Equation (A.44) is integrated along the boundary layer, resulting

\[\rho \bar{w} = -P + P_W,\]  
(A.46)

where \(P_W\) is a wall static pressure. Thus the static pressure \(P\) in the turbulent boundary layer on the rotating disk is given as

\[P = P_W - \rho \bar{w}.\]  
(A.47)