Fractal Geometry, Graph and Tree Constructions

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Abstract

In the 18th and 19th centuries the branch of mathematics that would later be known as fractal geometry was developed. It was the ideas of Benoît Mandelbrot that made the area expand so rapidly as it has done recently, and since the publication of his works there have for fractals, and most commonly the estimation of the *fractal dimension*, been found uses in the most diverse applications. Fractal geometry has been used in information theory, economics, flow dynamics and image analysis, among many different areas.

This thesis covers the foundations of fractal geometry, and gives most of the fundamental definitions and theorems that are needed to understand the area. Concepts such as measure and dimension are explained thoroughly, especially for the Hausdorff dimension and the Box-counting dimension. An account of the graph-theoretic approach, which is a more general way to describe self-similar sets is given, as well as a tree-construction method that is shown to be equivalent to the graph-theoretic approach.

Fraktalgeometri, graf- och trädkonstruktioner

Sammanfattning

På 1800- och 1900-talen utvecklades det område som senare skulle komma att kallas fraktalgeometri. Det var Benoît Mandelbrots idéer som fick området att växa så mycket som det gjort de senaste decennierna och alltsedan hans arbeten publicerades har det för fraktaler, och då främst för skattningar av den *fraktala dimensionen*, funnits användningsområden i de mest skilda tillämpningar. Bl.a. har fraktalgeometri använts i informationsteori, ekonomi, flödesdynamik och bildanalys.

Detta examensarbete går igenom grunderna av området fraktalgeometri och förklarar de flesta av de grundläggande definitionerna och satserna som behövs för att förstå området. Saker såsom mått och dimension förklaras genomgående, speciellt för Hausdorffdimensionen och Lådräkningsdimensionen. Den generellare metoden att med grafteori beskriva självlika mängder förklaras och förklaras gör även en metod att med trädkonstruktioner beskriva mängderna. Dessa två metoder visar sig vara ekvivalenta.

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Chapter 1

Introduction

This report describes the master thesis project *"Fractal Geometry, Graph and Tree Constructions"* performed at the Department of Mathematics and Mathematical Statistics at Umeå University.

The history of fractal geometry is filled with work done on nowhere differentiable but everywhere continuous functions and curves, self-similar sets and sets with fractional dimension. The work and the results was initially seen as anomalies, and any suggestion that i.e. a non-differentiable curve might have some practical application was not at all taken seriously. This situation has been completely reversed today.

The first ideas concerning fractal sets came well over hundred years ago, at the end of the 19th century, but have only received practical use since the 1970's. The use of fractal and multifractal geometry today spans from physics, through economy, biology, medicine, to computer science; among many other areas.

More and more applications of fractal geometry are found. Some of the current applications are, among others, image compression and enhancement, computer graphics and special effects in movies, music generation and pattern classification.

The goal of this thesis is to give a recollection of the theory of fractal geometry and connect this theory to the theory of the much broader class of sets created by graph-directed constructions and tree constructions.

This thesis describes fractals geometry from the very most simple definitions of set theory and non-formal explanations of fractals, through the topological dimension and similarity dimension, to the much more general definitions of Hausdorff and Box-counting dimensions. For the theory to be understood, some general measure theory is described; starting with the Lebesgue measure to the more general notions. A brief account of how to estimate the fractal dimension, by using the Box-counting theorem, is given. Algorithms that are used to generate fractals are also explained.

In the later chapters, the classical theory of fractal geometry is broadened to constructions using graphs and trees. It turns out that the Iterated Function Systems and tree constructions are just special cases of the graph-directed approach.

1.1 Prerequisites

This thesis does not require any previous experience in fractal geometry. All theory will be built bottom-up, and all graduate students and alike should be able to follow the text. Some of the theory might anyway be new to some readers, in which case Chapter 4 probably should be read first. Readers without experience in reading mathematical proofs should still be able to follow this text without so much reduced comprehension. In most cases, the proofs are merely for completeness and sometimes they are omitted. The only mathematical prerequisites are basic set theory and a moderate competence in calculus; especially the notion of limits is important.

1.2 Thesis Outline

This thesis is organized as follows. Chapter 1 is this introduction. Chapter 2 states the problem at hand, i.e. describe the goals and purpose of this thesis. Chapter 3 gives a brief summary of the history of fractal geometry, but also describes some of the scientific and real world work that involve fractal geometry.

In Chapter 4 can be found some theory that might be unknown to the reader. The reader should browse this chapter before embarking the rest of the thesis, just to make sure to have the correct prerequisites. This chapter is a short and concise recollection of what is needed to understand the thesis.

Chapter 5 introduces the concept of a fractal and defines what a fractal is. The best, and most general definition there is will be stated, and an explanation of what a fractal is will be given. The concept of fractal dimension will be explained, and descriptions of a number of different (though, in some cases equivalent) definitions are included. A number of properties, which all *good* definitions of fractal dimension should possess will be stated. Finally, a description on how to estimate the fractal dimension numerically is given.

In Chapter 6 we describe how to generate fractals. Three methods, or algorithms, are given that are build on the Iterated Function Systems approach. They are *The Deterministic Algorithm* and two versions of the *Random Iteration Algorithm*. There is a dimension that goes together with the particular Iterated Function System that will be described here; this dimension is the general case of the similarity dimension that was explained in Chapter 5.

In Chapter 7 the graph-directed constructions are described. The connection to the classical fractal geometry, and Iterated Function Systems are established, and the general case is explained. How to find the Hausdorff dimension of the graph-directed constructions is also explained.

The recursiveness of fractals, and especially fractals that stem from Iterated Function Systems are easily interpreted as a recursion tree, and thus the entire fractal can be explained using a tree. It turns out that trees in fact are metric spaces in their own right, and therefore have e.g. Hausdorff dimensions. With the right metric and translation functions, the Hausdorff dimension of the tree is exactly that of the underlying set that the tree describes. Trees can also be described using what is called the *branching number* and *growth rate*. These numbers relate to both the Hausdorff and lower and upper Box-counting dimensions.

There are several arithmetic operations that can be applied to sets, and therefore to fractals sets. The results for set union is described in Chapter 9. It turns out that the properties of union for classical fractal sets is equivalent to graph-directed constructions and tree constructions.

In Chapter 10, the results and conclusions of this thesis are stated. Since this thesis work was mostly to summarize the equalities between the classical fractal geometry and the more recent results, the theoretical results are few. But some insights and conclusions are drawn. Also, the method described in Chapter 5 to estimate the fractal dimension is tested with a number of fractals generated by the methods described in Chapter 6. Also, the result that graph-union is done by matrix augmentation is tested.

The theorems and proofs in this thesis are collected from a vast amount of sources, but several of them are those of the author. When a proof is someone else's, the theorem is preceded by the proper citation, thus, when a citation is not given, the proof is done by the author. There are some cases, however, where no citation for e.g. a definition is given, but the statement is still not that of the author. These are the cases when a definition or theorem is considered too general or elementary to show, and there is no specific reference that states it or that it is ubiquitous in the literature. These cases are few, and shall hopefully not confuse the reader.

Chapter 2

Problem Description

In this chapter the problem statement, the guide to do this master's thesis, is given. Over the course of the work, the original plan was followed quite well, but since the specifications where rather loose, excavations have been done in some different directions.

2.1 Problem Statement

The theory of fractal geometry have grown for well over a hundred years by now. The classical theory is pretty well understood and therefore, quite naturally, new areas of fractal geometry have evolved. The most prominent being multifractal geometry and graph-directed constructions. A study of the first area is found in i.e. [Nil07] and [Löf07]. The latter area was introduced by Mauldin and William in [MW88], and is what this thesis is focused on.

2.2 Goals

The goal of this thesis is to correlate the theory of classical fractal geometry to the theory of graph-directed constructions.

A literature study of the classical fractal geometry as well as the theory of graph-directed constructions should be performed and documented.

The equivalence or differences between the two approaches should be evaluated and compared.

If time be, a study of tree constructions should be done as well, comparing the approach to that of the classical fractal geometry and the graph-directed constructions.

2.3 Purpose

The area of graph-directed constructions of fractals is likely to grow in the years to come, with more and more applications following. Therefore, a study of how the classical theory relate to the *new* theory is of great interest to the research area.

Chapter 3

Previous Work

Often when approaching a new area of science, we look at its history. There is always much to learn from history, and this is particularly true in mathematics, in which all (or at least most) new results are explained in terms of old ones.

This chapter is a summary of the history of fractal geometry. The main source for this summary is [Edg03], where some of the most influential articles since the 1880's are reprinted. The articles that made the most sense in this context, and that felt the most important for this thesis are mentioned here. But in addition to to those articles several other results are mentioned here that also seems to have made a strong impression on the fractal geometry community.

The study of the special class of sets, which nowadays is known as *Fractals*, begun already in the 19th century, and in the beginning of the 20th century, the interest in this area flourished and much literature was written on the subject. The interest subsided however, until it renaissanced in the 1970's, much thank to Benoît Mandelbrot's work, and the advancement of computers in science. Computers made it possible to draw these figures in a way that was never possible before. The Fractal dimension became one of the most popular tool with which these sets where described.

The field of mathematics blossomed in the end of the 17th century, when Isaac Newton and Gottfried Leibniz developed calculus. Many ideas came and went during the 18th century and by the 19th century, the mathematicians thought they had the area, by most part, figured out. But in 1872, Karl Weierstrass wrote an article where he proved that there are functions which are everywhere continuous, but nowhere differentiable, see [Wei72]. This was something completely new, the mathematical community had assumed that the derivative of a function could be undefined only at isolated points. Much research followed, and several *counterexamples* to the classical calculus was found.

In 1904, the Swedish mathematician Helge von Koch wrote an article about a continuous curve, constructed from very elementary geometry, that did not have a tangent at any of its points, see [vK04]. This curve is described in Example 5.1.2. Ernesto Cesàro immediately recognized this geometrical figure as being self-similar, and did much work on the theory of self-similar curves. The work of Cesàro was taken further in a 1938 paper by Paul Lévy, in which he introduces new, more general, self-similar curves.

The idea of using a measure to extend the notion of length was used by Georg Cantor, the father of set theory, in 1884 in [Can84], and Émile Borel in 1895 when he studied "pathological" real functions. Their ideas where extended by Henri Lebesgue in 1901 in [Leb01] by the Lebesgue integral.

The ideas of Lebesgue where developed further by Constantin Carathéodory in 1914, who adapted the theory to lengths in arbitrary spaces. This was later generalized further by Felix Hausdorff in 1919 in [Hau19] to extend to non-integral dimensions. This contribution is the foundation of the theory of fractional dimensions. The Hausdorff measure provides a natural way of measuring the s-dimensional volume of a set and the Hausdorff dimension is today generally considered the Fractal dimension of a set. Much of the early work with the Hausdorff dimension was done by Abram Besicovitch in the 1930's.

Karl Menger wrote two papers in a communication with the Amsterdam Academy of Sciences, in the beginning of the 20th century. In these papers, he introduced the ideas of a topological dimension. These ideas where new, but other authors had, independently of him, the same, or similar, ideas; e.g. Henri Lebesgue.

The definition of the Box-counting dimension dates back to the late 1920's, and is due to Georges Bouligand. It has become a very popular definition because of its ease of numerical computation, but also rigorous computation. In his 1928 paper, Bouligand defined several different variants of the new definition of dimension. In the same paper there can also be found theory of the dimension of a Cartesian product of two sets being the sum of the dimensions of the two sets.

In the 1940's and 1950's, several different authors proved results of arithmetic properties of fractal sets. In a 1946 paper, Patrick Moran proves results concerning the Hausdorff dimension of a Cartesian product of two sets. The general result was proven in a 1954 paper by J. M. Marstrand. In a 1954 paper, [Mar54], Marstrand also proved several results concerning projections of fractal sets. This was later generalized by Pertti Mattila in 1975 in [Mat75]. Authors like Falconer, Howroyd and others investigated this further. Theoretical results for the intersection of fractal sets was also introduced by J. M. Marstrand, and can be found in [Mar54]. But more work was done by Jean-Pierre Kahane and Pertti Mattila in the 1980's.

In the 1960's, Benoît Mandelbrot did work on self-similar sets, and in a 1967 paper, see [Man67], he describes the Similarity dimension. Mandelbrot formalizes the work of Lewis Fry Richardson, who noticed that the length of a coast line depends on the unit of measurement used. He also suggests that the theory of fractional dimensions and self-similarity could be used not only in mathematics, but in other branches of science as well. Mandelbrot coined the term *Fractal* in 1975, see [Man75]. Mandelbrot also says that self-similar objects seldom are found in nature, but that a statistical form of self-similarity is ubiquitous. He manifests in his 1982 book [Man82] the idea that fractal geometry is better at describing the nature than classical Euclidean geometry is.

Mandelbrot's book inspired literally thousands of new papers in science, engineering, social sciences, economics and other areas. The introduction of computers in science made it possible to do impressive colorful visualizations of the sets that earlier was only existing in theory and possibly on paper.

The study of self-similar sets became one of the main fields of study in fractal geometry in the 1980's and 1990's. The theory of self-similar sets was formalized by John Hutchinson in a 1981 paper, but popularized by Michael Barnsley in his 1988 book [Bar88]. The iterated function system approach, used to create self-similar sets, was extended to graph-directed constructs by Mauldin and Williams in a 1988 paper.

Measures have always been a fundamental tool in the study of geometrical fractal sets. But because there exists natural fractal measures in many constructions in fractal geometry, i.e. self-similar measures, fractal properties of measures received increased attention in the 1980's and 1990's. The idea leads to the notion of a dimension of a measure.

Multifractal analysis, which became one of the most popular topics in geometric measure theory in the 1990's, studies the local structure of measures and provides much more detailed information than the uni-dimensional notion that was popular earlier. The multifractal spectra was first explicitly defined by physicists Halsey et al. in 1986 in [HJK$^+$86].

Recent uses of fractal geometry and multifractal analysis is mainly in (medical) image analysis. In a 1989 paper, [KC89], Keller et al. used the Box-counting dimension and the concept of *lacunarity* to discriminate between different textures. They showed that the fractal dimension alone is not enough to classify natural textures. In a 2002 paper, [CMV02], Caron et al. used the multifractal spectrum for texture analysis and object detection in images with natural background. In a 2003 paper, [NSM03], Novianto et al. used the local fractal dimension in image segmentation and edge detection. In 2006, Stojić et al., [SRR06], used multifractal analysis for the segmentation of microcalcifications in digital mammograms. Their method successfully detected microcalcifications in all test cases.

Another area that recently has attracted a great deal of attention is analysis on fractals. This area studies dynamical aspects of fractals, such as how heat diffuse on a fractal and how a fractal vibrate. But other areas have been explored as well, such as Fourier or wavelet analysis on fractals.

The area of fractal geometry is young and prosperous, and we can safely conjecture that the area will continue to grow for many years to come, with new subareas, discoveries and applications likely to pop up all the time.

Chapter 4

Prerequisites

This chapter aims to give the reader a brief review of theory used in this thesis, and also, perhaps most importantly, to explain some of the less basic ideas (concerning both set theory and general mathematics) and notations used in this work. This review is by no means complete, but should give the reader a good enough summary to be able to understand the general topics in the text.

4.1 Set Theory

In almost every area of mathematics, the notion of sets, and ideas from set theory is used. A set is a well-defined collection of objects, where the objects are called members, or elements of the set. Well-defined in the preceding definition means that we are always able to determine whether an element is in a set or not [Gri90].

In this text, the theory we work with is mostly concerned with sets of points from the *n-dimensional Euclidean space*, $\mathbb{R}^n$.

We will use upper case letters to denote sets, and lower case letters to denote elements of the set. Sometimes we will use the coordinate form of the points in $\mathbb{R}^n$, and denote them $x = (x_1, \ldots, x_n)$, we then call them vectors. However, it is the *object* x that is the member of the set $\mathbb{R}^n$. For a set A, we will write $e \in A$ to say that the element e is a member if the set A (e is in A), and $e \notin A$ to say that e is *not* a member of A (e is not in A). Sometimes, to distinguish vectors from scalars (i.e. real or complex numbers), we denote vectors with a bold face font, i.e. as $\mathbf{x}$.

We will write $\{e : \textit{condition}\}$ to denote the set of all elements of some set fulfilling a given condition. E.g., the set of all even numbers is denoted:

$$\{2x : x \in \mathbb{Z}\}.$$

Remember that $\mathbb{Z}$ is the set of integers, $\mathbb{Q}$ is the set of rational numbers, $\mathbb{R}$ is the set of real numbers, and $\mathbb{C}$ is the set of complex numbers. We will use a superscript $^+$ to denote only the positive elements of a set, e.g. $\mathbb{Z}^+$ is the set of all positive integers.

Vector addition and scalar multiplication (multiplication with a scalar, not to be confused with the scalar product) are defined as usual, so that $x \pm y = (x_1 \pm y_1, \ldots, x_n \pm y_n)$ and $\lambda x = (\lambda x_1, \ldots, \lambda x_n)$, and also so that $A + B = \{x + y : x \in A \wedge y \in B\}$ and $\lambda A = \{\lambda x : x \in A\}$. Scalar product, or dot product, is defined as follows:

Definition 4.1.1: *Let $v = (v_1, \ldots, v_n)$ and $u = (u_1, \ldots, u_n)$ be vectors in coordinate form, then their dot product is*

$$v \cdot u = \sum_{i=1}^{n} v_i u_i. \tag{4.1}$$

We define *subsets* as follows [Gri90]:

Definition 4.1.2: *If A, B are sets, we say that A is a* subset *of B, and write $A \subseteq B$ or $B \supseteq A$, if every element in A is also an element of B. Also, if B contains at least one element that is not in A, we say that A is a* proper subset *of B, and write $A \subset B$, or $B \supset A$. Thus, if $A \subseteq B$, then*

$$\forall x (x \in A \Rightarrow x \in B). \tag{4.2}$$

If a set is empty, that is, it contains no elements, it is called the *null set*, or *empty set*. It is a unique set denoted by $\emptyset$ or $\{\}$.

We define *union*, *intersection* and *difference* of sets as follows:

Definition 4.1.3: *For two sets A, B, we define the following:*

a) *The* union *of A and B is:* $A \cup B = \{x : x \in A \vee x \in B\}$

b) *The* intersection *of A and B is:* $A \cap B = \{x : x \in A \wedge x \in B\}$

c) *The* difference *of A and B is:* $A \backslash B = A - B = \{x : x \in A \wedge x \notin B\}$

Two sets are called *disjoint* if $A \cap B = \emptyset$. Remember that $A \cup B = B \cup A$ and $A \cap B = B \cap A$, $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup A = A$ and $A \cap A = A$, $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$, and $A \cap \mathbb{R}^n = A$ and $A \cup \mathbb{R}^n = \mathbb{R}^n$. Proofs of these properties are omitted, but can be found in [Gri90]. The *complement* of a set, $A \subseteq B$, is the set $B \backslash A$.

The *Cartesian product* of two sets A and B is the set of all ordered pairs $\{(a, b) : a \in A \wedge b \in B\}$, and is denoted $A \times B$. If $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$, then $A \times B \subseteq \mathbb{R}^{n+m}$.

We can use the dot product to define the length of a vector [Roe03]:

Definition 4.1.4: *If $v \in \mathbb{R}^n$ is a vector, we define the length, $|v|$, of v as $\sqrt{v \cdot v}$, where we take the positive square root. If $x, y \in \mathbb{R}^n$ are vectors, or points in $\mathbb{R}^n$, then the distance between them is the length, $|x - y|$, of the vector $x - y$.*

In the above definition (and in the following), $|\cdot|$ denotes the Euclidean norm,

$$|x| = L_2(x) = \sqrt{\sum_{i=1}^{n} x_i^2}$$

for $x = (x_1, \ldots, x_n)$, but it could in general be any norm. A more general setting for *lengths* is in a *metric space* [Edg90]:

Definition 4.1.5: *A* metric space *is a set S together with a function $\rho : S \times S \to [0, \infty)$ satisfying:*

a) *$\rho(x, y) = 0$ if and only if $x = y$;*

b) *$\rho(x,y) = \rho(x,y)$;*

c) *$\rho(x,z) \leq \rho(x,y) + \rho(y,z)$.*

The number $\rho(x,y)$ is called the distance *between the points x and y, and the function itself is called a* metric *on the set S.*

A metric space may be written as a pair (S, ρ). For the Euclidean spaces, $\mathbb{R}^n$, the metric is the euclidean distance defined above, i.e. $\rho(x,y) = L_2(x-y)$.

The infimum of a set, denoted $\inf(A)$, is the greatest element that is smaller than, or equal to, all elements in A, i.e. the greatest m such that $m \leq x$ for all $x \in A$. If such an element does not exist, we define $\inf(A) = -\infty$. If $A = \emptyset$, then $\inf(A) = \infty$. The supremum of a set, denoted $\sup(A)$, is the smallest element that is greater than, or equal to, all elements in A, i.e. the least number m such that $x \leq m$ for all $x \in A$. If such an element does not exist, we define $\sup(A) = \infty$. If $A = \emptyset$, then $\sup(A) = -\infty$. Intuitively, the infimum and supremum can be thought of as the minimum and maximum of a set respectively, but need not be members of the set themselves, and they always exist.

The *diameter* of a non-empty set A is defined as $\mathrm{diam}(A) = |A| = \sup\{\rho(x,y) : x, y \in A\}$, with the convention that $|\emptyset| = 0$. A set is *bounded* if it has finite diameter, and *unbounded* otherwise. The distance between two non-empty sets A, B is defined as $\mathrm{dist}(A,B) = \inf\{\rho(x,y) : x \in A \wedge y \in B\}$. The *$\delta$-neighbourhood*, or *$\delta$-parallel body*, A_δ, of a set A is defined as $A_\delta = \{b : \inf_{a \in A} \rho(a,b) \leq \delta\}$, with $\delta > 0$. It is the set of points within distance δ of A [Fal97].

We define the *closed* and *open balls* of center x and radius r by

$$B_r(x) = \{y : \rho(x,y) \leq r\}$$

and

$$B_r^o(x) = \{y : \rho(x,y) < r\}$$

respectively. Thus, a closed ball contains its bounding sphere. If $a, b \in \mathbb{R}$ and $a < b$, we write $[a,b]$ for the closed interval $\{x : a \leq x \leq b\}$ and (a,b) for the open interval $\{x : a < x < b\}$. $[a,b)$ denotes the half-open interval $\{x : a \leq x < b\}$.

A set A is *open* if there for every $x \in A$ is some $\varepsilon > 0$ such that $B_\varepsilon^o(x) \subseteq A$, i.e. the distance between any point in A and the edge of A is always greater than zero.

If S is a space, and $A \subseteq S$. A point $x \in S$ is called a *limit point* of A if every open set containing x also contains at least one point $y \in A$ such that $x \neq y$. A set is *closed* if every limit point of the set is a point in the set. A set is said to be *compact* if it is both closed and bounded.

A set is called *clopen* if it is both closed and open. In any space S, the empty set and the entire space S are clopen.

The intersection of all closed sets containing a set A is called the *closure* of A, and is written $\overline{A}$. The closure of A is thought of as the smallest set containing A [Fal97].

A *cover* of a set A is a countable (or finite) collection of sets that cover A. I.e., if $\mathcal{C} = \{U_i\}$ is a collection of subsets of A, then $\mathcal{C}$ is a cover of A if

$$\bigcup_i U_i = A.$$

More generally, a *cover* of a set $A \subseteq X$ is a countable (or finite) collection of subsets $U_i \subseteq X$ such that

$$\bigcup_i U_i \supseteq A.$$

If $\mathcal{C}$ is a cover of A, then a *subcover* of $\mathcal{C}$ is a subset of $\mathcal{C}$ which still covers A.

A set is said to be finite, if there is a bijection between it and a set of the form $\{1, 2, \ldots, N\}$, where N is zero or a positive integer. I.e., if we can list the elements in the set by enumerating them. An infinite set is a set that is not finite.

An infinite set is said to be *countable*, if its elements can be listed in the form $x_1, x_2, \ldots$, where every element has its specific place in the list. Otherwise the set is said to be *uncountable*. Remember that $\mathbb{Z}$ and $\mathbb{Q}$ are countable, but that $\mathbb{R}$ is not. See [Fla98] for a good explanation of countability.

A space, S, is *connected* if the only simultaneously open and closed sets (*clopen* sets) are S and the empty set. Less formally we can say that S is connected if we can *move* continuously from any one point of the space to any other point of the space. A subset of a connected space S is a *connected set* if it is a connected space when seen as a subspace of S. A *connected component* of a space is a maximal connected subset, i.e. a connected subset to which we cannot add any point and keep it connected. A space is said to be *totally disconnected* if all connected components are singletons. A space is said to be *disconnected* if it is not connected.

There is a special class of sets that need to be mentioned. For that, we need the following definition [Edg90]:

Definition 4.1.6: *A collection $\mathscr{F}$ of subsets of a set F is called a* σ-algebra *on F if and only if:*

- ***a)*** *$\emptyset, F \in \mathscr{F}$;*
- ***b)*** *if $A \in \mathscr{F}$, then $F \backslash A \in \mathscr{F}$;*
- ***c)*** *if $A_1, A_2, \ldots \in \mathscr{F}$, then $\bigcup_{i\in\mathbb{N}} A_i \in \mathscr{F}$;*
- ***d)*** *if $A_1, A_2, \ldots \in \mathscr{F}$, then $\bigcap_{i\in\mathbb{N}} A_i \in \mathscr{F}$;*
- ***e)*** *if $A, B \in \mathscr{F}$, then $A \backslash B \in \mathscr{F}$.*

We now define this special class of sets as follows [Fal90, Edg90]:

Definition 4.1.7: *The class of* Borel sets *is the smallest collection of subsets of a metric space, (S, ρ), with the following properties:*

- ***a)*** *every open set and every closed set is a Borel set;*
- ***b)*** *all subsets in a σ-algebra generated by Borel sets are Borel sets. I.e. the union or intersection of every finite or countable collection of Borel sets is a Borel set, and the set difference between Borel sets is a Borel set.*

The above definition says that Borel sets are sets that can be constructed from open or closed sets by repeatedly taking countable unions and intersections. The Borel sets are measurable. In this text, almost all theory deals with Borel sets.

4.2 Calculus

Let S and T be two metric spaces and let $x \in S$. A function $h : S \to T$ is said to be *continuous* at x if and only if for every $\varepsilon > 0$, there is $\delta > 0$ such that

$$\rho(x, y) < \delta \implies \rho(h(x), h(y)) < \varepsilon.$$

A function $h : S \to T$ is a *homeomorphism* of S onto T if and only if it is bijective, and both h and h^{-1} are continuous. Two metric spaces are *homeomorphic* if and only if there is a homeomorphism of one onto the other.

A property of a space is called a *topological* property if and only if it is preserved by a homeomorphism.

4.3 Linear Algebra

Vector addition and *scalar multiplication* where defined above as $x \pm y = (x_1 \pm y_1, \ldots, x_n \pm y_n)$ and $\lambda x = (\lambda x_1, \ldots, \lambda x_n)$, respectively. A combination of these operations gives a *linear combination* $\lambda_1 x_1 + \lambda_2 x_2$.

Vectors are either column vectors,

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},$$

or row vectors,

$$y = [y_1, \ldots, y_n].$$

Sometimes, to save space, column vectors are denoted by their transpose as

$$x = [x_1, \ldots, x_n]^T.$$

A vector with only zeros is called the *zero vector* and is denoted by $\mathbf{0}$.

The dot product was defined in the previous section as:

Definition 4.3.1: *Let $v = (v_1, \ldots, v_n)$ and $u = (u_1, \ldots, u_n)$ be vectors in coordinate form, then their dot product is*

$$v \cdot u = \sum_{i=1}^{n} v_i u_i. \tag{4.3}$$

If the dot product of two vectors is zero, i.e. $v \cdot u = 0$, the vectors are called *perpendicular.*

We will say that vectors are linearly independent if they fulfill the following criteria [Str05]:

Definition 4.3.2: *A sequence of vectors, $v_1, \ldots, v_n$ are called* linearly independent *if the only linear combination that results in the zero vector is $0v_1 + \cdots + 0v_n$, thus the sequence of vectors are linearly independent if*

$$k_1 v_1 + k_2 v_2 + \cdots + k_n v_n = \mathbf{0}, \qquad k_i = 0 \text{ for } i = 1, \ldots, n. \tag{4.4}$$

A matrix is a rectangular table of elements, in which the elements are real or complex numbers, but could be any objects that allows for vector addition and scalar multiplication. A matrix, $\mathbf{A}$, of size $m \times n$, has structure

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \ldots & a_{m,n} \end{bmatrix}.$$

We may simplify this notation and write

$$\mathbf{A} := (a_{i,j})_{m \times n},$$

for the matrix with m rows and n columns. The elements of the matrix, $\mathbf{A}$, is identified by $\mathbf{A}_{ij}$, meaning the element in the ith row and jth column. A square matrix with ones on the diagonal, and zeros outside of the diagonal is called an *identity matrix.* A vector is a special case of a matrix, with only one row or one column.

A system of linear combinations, e.g.

$$\begin{gathered}\lambda_{1,1}x_1 + \ldots + \lambda_{1,n}x_n = b_1 \\ \vdots \\ \lambda_{m,1}x_1 + \ldots + \lambda_{m,n}x_n = b_m,\end{gathered}$$

is simplified by a *matrix equation* as

$$\mathbf{M}x = b,$$

in which $\mathbf{M}$ is a *coefficient matrix* with elements $\mathbf{M}_{ij} = \lambda_{i,j}$, x is the column vector $x = [x_1, \ldots, x_n]^T$ and b is the column vector $b = [b_1, \ldots, b_m]^T$.

And *eigenvector* to a matrix, $\mathbf{A}$, is a vector that does not change direction when multiplied by $\mathbf{A}$, but might be stretched or shrunk. I.e. $\mathbf{A}x = \lambda x$, where λ is the factor with which x is stretched or shrunk. The number λ is the *eigenvalue* corresponding to the eigenvector x. The number λ is an eigenvalue if and only if $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$.

A matrix is said to be non-negative if all elements of $\mathbf{A}$ are greater than zero, i.e. $a_{i,j} \geq 0$, and we denote this by $\mathbf{A} \geq 0$. A matrix is called positive if $a_{i,j} > 0$, denoted by $\mathbf{A} > 0$ and similarly $\mathbf{A} < 0$ if the matrix is negative. If all elements of the matrix are zero, it is denoted by $\mathbf{A} = 0$. These notions are valid for vectors also.

The *norm* of a matrix is a function $\|\cdot\| : A_{m\times n} \to \mathbb{R}$ that fulfills the following properties [Str05]:

Definition 4.3.3: *If* $\mathbf{A}$ *is an* $m \times n$ *matrix and* k *is a scalar, then*

a) $\|\mathbf{A}\| \geq 0$

b) $\|\mathbf{A}\| = 0$ *if and only if* $\mathbf{A} = 0$

c) $\|k\mathbf{A}\| = |k|\|\mathbf{A}\|$.

The norm of a matrix is a natural extension to the norm of a vector, which fulfills the same properties.

4.4 Graph Theory

A *graph* is a pair, $G = (V, E)$, of sets with $E \subseteq V \times V$. V is the set of *vertices* (or *nodes*), and E is the set of *edges* of the graph, G. A graph is called *directed* (or a *digraph*) if the edges between vertices have an implied direction, and *undirected* otherwise. In Figure 4.1 a) is an example of an undirected graph with vertex set $V = \{1, 2, 3, 4\}$, and edge set $E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}\}$, and in Figure 4.1 c) is an example of an directed graph with the same vertex set, but with edge set $E = \{(1, 2), (1, 3), (3, 2), (4, 3)\}$. In the undirected case, the edge set is unordered, and ordered in the directed case.

Normally, the vertices are painted as dots, and edges are painted as lines connecting the dots. The direction of an edge is normally indicated by an arrow. How this is done is irrelevant and merely a way to illustrate the connections between vertices.

The vertex set of a graph G is denoted $V(G)$, and the edge set of G is denoted $E(G)$. When the context is clear, we will, however, write $v \in G$ or $v \in V$ instead of $v \in V(G)$ for some vertex v in G, and $e \in G$ or $e \in E$ instead of $e \in E(G)$.

Two vertices, x and y, are *adjacent*, or *neighbours* if there is an edge joining them. Instead of writing $\{x, y\}$, or (x, y), for that edge, we will simplify the edge as xy. The set of all edges between vertex x and vertex y is denoted $E_{x,y}$. If the graph is directed, the vertex x is called *origin*, *source* or *initial vertex* of the edge, and y is called *terminal* or *terminating vertex*. A vertex can have an edge to itself, i.e. $\{x, x\}$ or (a, a), and is then called a *loop*. If all vertices of G are adjacent, and the graph is loop-free, then the graph is called *complete*.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. If $V_2 \subseteq V_1$ and $E_2 \subseteq E_1$, then G_2 is a *subgraph* of G_1, and G_1 is a *supergraph* of G_2. We write $G_2 \subseteq G_1$, i.e. G_1 contains G_2.

The set of neighbours of a vertex v in a graph G is denoted by $N_G(v)$, or $N(v)$ if the context is clear. The degree of a vertex, $d_G(v) = d(v)$ is the number of edges at v, i.e. the number of neighbours of v. In the directed case, we count *in* and *out degrees*, $id(v)$ and $od(v)$ respectively. A vertex of degree 0 is called *isolated*. If all vertices of a graph, G, have the same degree, say k, then the graph is *k-regular*, or just *regular*. The *average degree* of a graph G is

$$d(G) := \frac{1}{|V|} \sum_{v \in V} d(v). \tag{4.5}$$

A *path* in a graph is a non-empty graph $P = (V, E)$ such that

$$V = \{x_0, \ldots, x_k\} \quad E = \{x_0x_1, \ldots, x_{k-1}x_k\}, \tag{4.6}$$

with $x_i \neq x_j$ when $i \neq j$ for all $0 \leq i, j \leq k$. The number of edges of a path is its *length*. We may write $P = x_0x_1 \ldots x_k$, and call it a path from x_0 to x_k. If $P = x_0 \ldots x_k$ is a path, then $C = P + x_kx_0 = x_0x_1 \ldots x_{k-1}x_kx_0$ is called a *cycle*. The *length* of a cycle is its number of edges, and the cycle of length k is called a *k-cycle*. If the graph is directed, we call a path and a cycle a directed path and a directed cycle, respectively.

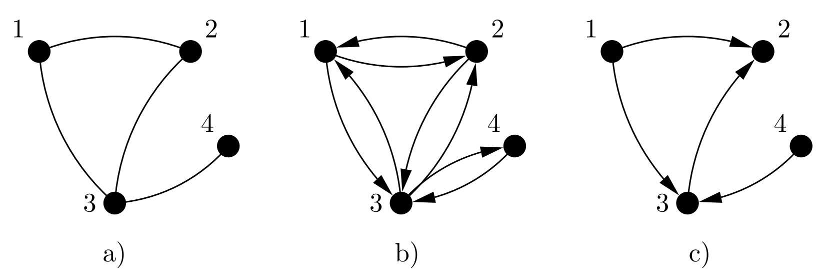


Figure 4.1: *a) A graph, G, with $V = \{1, 2, 3, 4\}$, and $E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}\}$. b) The same graph as in a). c) The graph G with edge set $E = \{(1, 2), (1, 3), (3, 2), (4, 3)\}$.*

A graph, $G = (V, E)$, is called a *multigraph* if for some $x, y \in V$, there are two or more edges in E, i.e. (x, y) or $\{x, y\}$ exists multiple times in E. If there are k edges between two vertices x, y we say that the edge $\{x, y\}$, or (x, y), has multiplicity k.

A graph, G, is called *connected* if for any two distinct vertices, $x, y \in V$, where $x \neq y$, there is a path joining x and y. A graph that is not connected is called *disconnected*. A directed graph G is called *strongly connected* if for all $x, y \in V$, where $x \neq y$, there is a path of directed edges from x to y.

A connected subgraph is called *maximal* if there are no other vertices and edges that can be added to the subgraph, and still leave it connected. A *connected component* is a maximal connected subgraph. A *strongly connected component* is a maximal connected subgraph that is strongly connected. The set of all strongly connected components of a graph G is denoted $SC(G)$.

The edges of a graph may have a *weight* assigned to them. This weight is normally a (positive) real number, but could be anything. The weight could be e.g. the cost to travel between the two vertices, or the length between them and so on. The weight of an edge is denoted $w(e)$ or $w(x, y)$, for $e \in E$ and $x, y \in V$. If there is no edge between two vertices, i.e. $\{x, y\} \notin E$ or $(x, y) \notin E$, the weight is set to some appropriately default value, normally 0, $-\infty$ or ∞. A graph with weights like this is denoted a *weighted graph*.

The adjacency matrix, $A := (a_{i,j})_{n \times n}$, of a graph G with n vertices is defined by

$$a_{ij} := \begin{cases} 1 & \text{if } v_i v_j \in E \\ 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

I.e. the adjacency matrix indicates whether there is a directed edge between vertices v_i and v_j. If the matrix is symmetric, i.e. $A = A^T$, then the matrix corresponds to an undirected graph, otherwise it corresponds to a directed graph. The adjacency matrix allows for loops, and the non-zero entries can indicate the number of edges between vertices, i.e. it can describe multigraphs. When the entries in the matrix are allowed to be other than 0 and 1, the entries can denote the weights between vertices.

4.4.1 Trees

An *acyclic* graph, i.e. a graph without any cycles, is called a *forest*. A connected forest is called a *tree*. The vertices of degree 1 in a tree are called *leaves*. Any two leaves, and in fact even any two vertices, of a tree are connected by a unique path. The vertices of a tree can always be enumerated, even if the tree is infinite. A connected graph with n vertices is a tree if and only if it has $n - 1$ edges. Figure 4.2 is an example of a tree.

If any edge is removed from a tree, the tree will be disconnected, and if any edge is added to the tree between vertices that are not adjacent will add a cycle to the tree.

It is often convenient to consider one vertex of the tree as special. That vertex is denoted the *root* of the tree, Λ. A tree with a root is called a *rooted tree*. A tree without root is called a *free tree*. With a root, there is a natural orientation on the tree; towards or away from the root.

If σ is a vertex, then $|\sigma|$ denote the number of edges on the unique path from Λ to σ. We write $\sigma \leq \tau$ if σ is on the unique path from Λ to τ; $\sigma < \tau$ if $\sigma \leq \tau$ and $\sigma \neq \tau$; $\sigma \to \tau$ if $\sigma \leq \tau$ and $|\sigma| = |\tau| - 1$. If $\sigma \to \tau$, σ is said to be the *parent* of τ, and τ is said to be the *child*, or *successor* of σ. If $\sigma \neq \Lambda$, then $\overleftarrow{\sigma}$ denotes the unique vertex such that $\overleftarrow{\sigma} \to \sigma$.

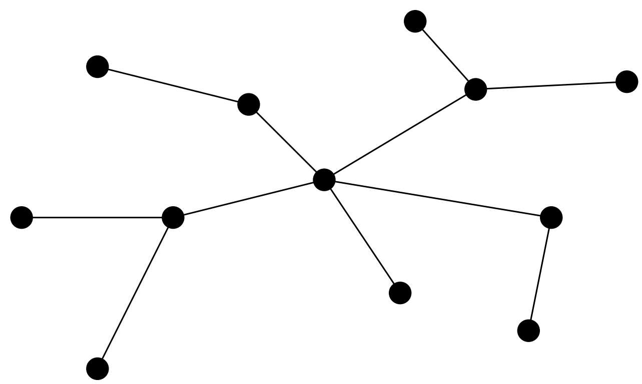


Figure 4.2: *An example of a tree*

Chapter 5

Fractal Geometry

When dealing with the class of geometrical objects called *Fractals*, the classical Euclidean Geometry is not enough to describe their complex nature. In the past decades, a new branch of geometry, called Fractal Geometry, have grown and received a great deal of attention from a variety of fields.

This chapter will describe the general theory of fractals and their geometry. First, we will see a short explanation of what a fractal really is. Then some various notions of dimension, something very important in Fractal Geometry, will be described, followed by a description of some methods to estimate the fractal dimension.

Some notions and theory in this chapter might be new to the reader, in which case we recommend reading Chapter 4 first.

5.1 What is a Fractal?

In his founding paper (see [Man75]) Benoît Mandelbrot coined the term *Fractal*, and described it as follows:

> *A [fractal is a] rough or fragmented geometric shape that can be subdivided in parts, each of which is (at least approximately) a reduced-size copy of the whole.*

The word is derived from the Latin word *fractus* meaning broken, and is a collective name for a diverse class of geometrical objects, or sets, holding most of, or all of the following properties [Fal90]:

i. The set has fine structure, it has details on arbitrary scales.

ii. The set is too irregular to be described with classical euclidean geometry, both locally and globally.

iii. The set has some form of self-similarity, this could be approximate or statistical self-similarity.

iv. The *Hausdorff dimension* of the set is strictly greater than its *Topological dimension.*

v. The set has a very simple definition, i.e. it can be defined recursively.

Property (iv) is Mandelbrot's original definition of a fractal, however, this property has been proven not to hold for all sets that should be considered fractal. In fact, each of the above properties have been proven not to hold for at least one fractal. Several attempts to give a pure Mathematical definition of fractals have been proposed, but all proven unsatisfactory. We will therefore, rather loosely, use the above properties when talking about fractals [Fal90].

Perhaps a couple of examples are required to get a better understanding of the geometrical objects we are talking about.

Example 5.1.1: *The Cantor set, see Figure 5.1, is created by removing the middle third segment of a unit line segment, call it E_0. We now have two line segments, each one third of the original line's length, call this set E_1, see Figure 5.1 b). We get E_2 by removing the middle third of the two line segments of E_1, see Figure 5.1 c). If we apply the rule (removing the middle third of the line segments) on E_{k-1} we obtain E_k, and when k tends to infinity, we get the Cantor set in Figure 5.1 d).*

We see that in the kth iteration, there is are 2^k disjoint intervals, each of length $(1/3)^k$. Thus, the total length of the Cantor set at iteration k is $(2/3)^k$. The limit of this is

$$\lim_{k\to\infty}\left(\frac{2}{3}\right)^k = 0,$$

and thus, the total length of the Cantor set is zero. This length is what we later will call the measure of the set. We note that the endpoints of the intervals are members of the set, and notice that in each iteration, there are 2^k new endpoints. Thus, when $k \to \infty$, the number of points in the cantor set tends to infinity.

Figure 5.1: *a) One, b) two, c) three, and d) several iterations of the Cantor set*

Example 5.1.2: *The von Koch curve, see Figure 5.2, is created as follows: Start with a unit line segment E_0. Remove the middle third of E_0 and replace it by two lines, each of the same length as the removed piece, call it E_1. We now obtained an equilateral triangle (with the base segment gone) as Figure 5.2 b) suggests. E_1 now has four lines of equal length, $\frac{1}{3}$ of that of E_0. We can now create E_2 by applying the same procedure as when we created E_1, and thus obtains the curve in Figure 5.2 c). Thus, applying the rules on E_{k-1}, we obtain E_k, and when k tends to infinity, we get the von Koch curve of Figure 5.2 d).*

We note that in each iteration, we have 4^k line segments of length 3^{-k}. Thus, the length of the von Koch curve tends to infinity as $k \to \infty$, i.e.

$$\lim_{k\to\infty}\left(\frac{4}{3}\right)^k = \infty.$$

The von Koch curve can be proven to be continuous, but without tangent at any of its points. Read more about the von Koch curve in [vK04].

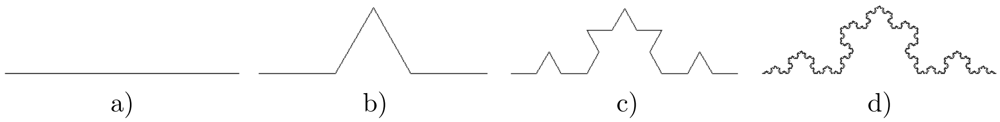


Figure 5.2: *a) One, b) two, c) three, and d) several iterations of the von Koch curve*

Example 5.1.3: *The Sierpinski triangle is created by replacing an equilateral triangle of unit size (Figure 5.3 a)), E_0, by three triangles of half its size, leaving the middle region empty, giving E_1, see Figure 5.3 b). E_2 is created by replacing each of the three triangles of E_1 by three half-sized triangles, leaving the middle region empty as before, see Figure 5.3 c). Thus, as in Example 5.1.2, applying the rules on E_{k-1}, we obtain E_k, and when k tends to infinity, we get the Sierpinski triangle of Figure 5.3 d).*

We see that the set E_k consists of 3^k triangles, each with side length 2^{-k}. Thus, the total area of the Sierpinski triangle is $3^k \cdot (2^{-k})^2 \cdot \sqrt{3}/4$, which tends to zero when $k \to \infty$, i.e.

$$\lim_{k\to\infty} \frac{\sqrt{3}}{4}\left(\frac{3}{4}\right)^k = 0.$$

Note that in each iteration, we always keep the line segments that constitute the boundary of the triangles from every earlier iteration, and we always get new line segments from the new triangles. Starting with three line segments, we get one new for each triangle of the kth iteration. Thus, in the kth iteration we have $3 + \sum_{k=1}^{\infty} 3^k$ line segments. This goes to ∞ as $k \to \infty$, which means that the length *of the Sierpinski triangle is infinite.*

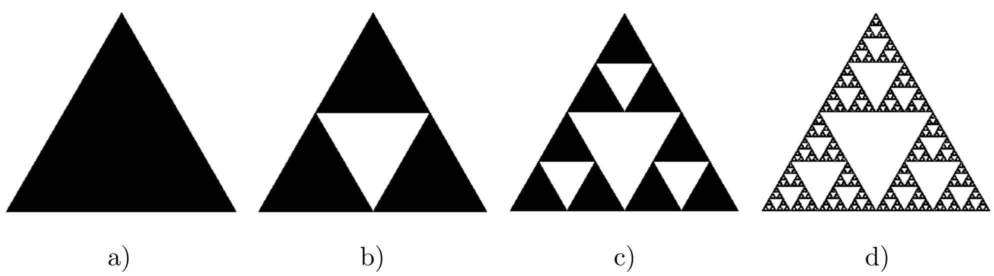


Figure 5.3: *a) One, b) two, c) three, and d) several iterations of the Sierpinski triangle, or Sierpinski casket*

Objects in nature often have fractal properties (i.e. a tree has a stem, on which each branch is a reduced size *copy* of the step), and therefore, fractals are used to better approximate objects in nature than classical euclidean geometry can do [Man82]. Natural objects with fractal properties could be trees, clouds, coast lines, mountains, and lightning bolts.

5.2 Fractal Dimension

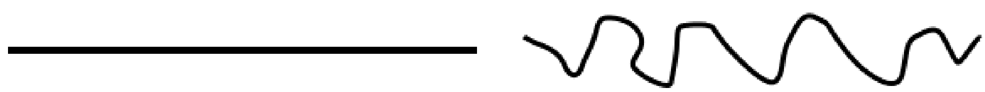
An important feature of fractal geometry is that it enables a characterization of irregularity at different scales that the classical Euclidean geometry does not allow for. As a result, many fractal features have been identified, among which the fractal dimension is one of the most important [JOJ95].

From the works of Euclid, see [Roe03], we know that the dimension of a point is considered 0, the dimension of a line is 1, the dimension of a square is 2, and that the dimension of a cube is 3. Roughly, we can say that the dimension of a set describes how much space the set fills [Fal90]. We will build the theory of fractal dimension from the basic Euclidean definition to the more mathematically exhaustive definitions of Hausdorff and Box-counting dimensions.

One might ask why there are several different definitions of dimension. This is simply because a certain definition might be useful for one purpose, but not for another. In many cases the definitions are equivalent, but when they are not, it is their particular properties that makes them more suitable for the task at hand.

5.2.1 Topological Dimension

The intuitive feeling of dimension that was mentioned in the beginning of this section is called the *Topological dimension.* Topology is the study of the geometrical properties of an object that remains unchanged when continuously transforming the object [Kay94]. Thus, the lines in Figure 5.4 both have topological dimension 1, since we could *stretch* them both to fit each other, and we know that a line has dimension 1.



a) b)

Figure 5.4: *a) A straight line, and b) a rugged line. Both a) and b) have Topological dimension 1.*

We need the following definitions [Edg90]:

Definition 5.2.1: *If $\mathcal{A}$ and $\mathcal{B}$ are two covers of a metric space, S, then $\mathcal{B}$ is a* refinement *of $\mathcal{A}$ if and only if for every $B \in \mathcal{B}$ there is $A \in \mathcal{A}$ with $B \subseteq A$. We say that $\mathcal{B}$* refines *$\mathcal{A}$. (E.g. a subcover of $\mathcal{A}$ is a refinement of $\mathcal{A}$.)*

Definition 5.2.2: *The* order *of a family $\mathcal{A}$ of sets is $\leq n$ if and only if any $n+2$ of the sets have empty intersection. It has order n if and only if it has order $\leq n$ but does not have order $\leq n-1$.*

The Topological dimension of a set is always an integer, and is 0 if the set is totally disconnected. The set should be considered zero-dimensional if it can be covered by disjoint sets. The Topological dimension of $\mathbb{R}^n$ is n (this can be proven, but we will not do that now.) Formally, the topological dimension is defined as [Edg90]:

Definition 5.2.3: *Let A be a set, and $n \geq -1$ be an integer. We say that A has* Topological dimension $\dim_{\mathrm{T}} A \leq n$ *if and only if every finite open cover of A has an* *open refinement of order $\leq n$. The Topological dimension is n if and only if the covering dimension if $\leq n$ but not $\leq n-1$. If the Topological dimension if $\leq n$ for no integer n, then we say that* $\dim_{\mathrm{T}} A = \infty$. *The only set with Topological dimension -1 is the empty set, i.e.* $\dim_{\mathrm{T}} \emptyset = -1$.

In Figure 5.5 you can see, more descriptively, how the topological dimension of the line in Figure 5.4 b) is found. This version of the Topological dimension (there are several different definitions of Topological dimension, which in general are equivalent) is called the *Covering dimension*.

5.2.2 Similarity Dimension

Imagine measuring the length of a seashore coastline, if we try to approximate the length of the coastline using a fixed sized ruler, we would find that the length increases as the length of the ruler decreases, because we are taking finer and finer details into account. The ruler is therefore inadequate to describe the complexity of a geographical curve [Man67].

The von Koch curve is a theoretical equivalence of a coastline, and as we saw in Example 5.1.2, the length of the von Koch curve tends to infinity as k tends to infinity (i.e. when we use a shorter ruler). But also, since the von Koch curve is created from finite line segments, we know that the curve must occupy zero area in the plane. Thus, we cannot use neither length, nor area to describe the *size* of the curve. See [vK04] for details.

Mandelbrot suggests in [Man67] that dimension should be considered a continuous quantity that ranges from 0 to infinity, and in particular that curves could have their ruggedness described by a real number between 1 and 2 that describes their *space-filling ability* [Kay94]. This is what we will investigate in this and the following sections.

A *transformation* $S : \mathbb{R}^n \to \mathbb{R}^n$ is called a *congruence* or *isometry* if it preserves distances (and thus also angles), i.e. $|T(x) - T(y)| = |x - y|$, for all $x, y \in \mathbb{R}^n$. The mapping is thus done without *deforming* the object [Roe03]. A *similar transformation* is a mapping $T : \mathbb{R}^n \to \mathbb{R}^n$ where there is a constant c such that $|T(x)-T(y)| = c|x-y|$, for all $x, y \in \mathbb{R}$ [Fal90].

Consider a unit line segment, which thus is 1-dimensional in the classical sense. If

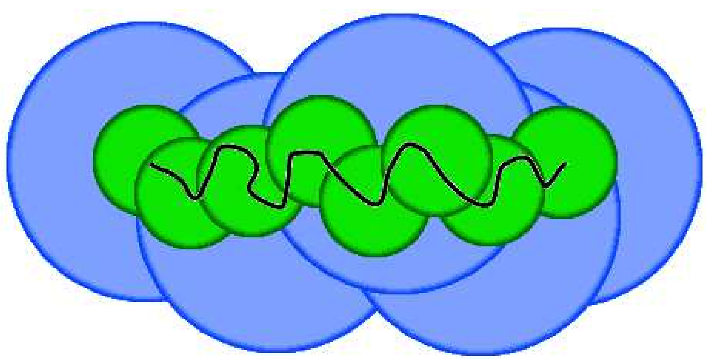


Figure 5.5: *A cover on the rugged line from Figure 5.4 b), and its refinement. Note that every point on the line is an element of at most two subsets of the refinement cover.*

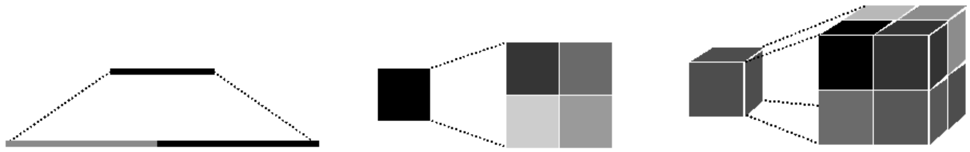


Figure 5.6: *a) A unit length line magnified two times will make the magnified object be made up of two identical unit line segments. b) A unit square magnified two times gives four cubes of unit size. c) A unit cube magnified two times gives eight unit cubes.*

we magnify the line segment twice, we will have two connected line segments both of the same length as the original one, see Figure 5.6 a). Now we look at a unit square in the plane, a 2-dimensional object. Magnify it twice (the side lengths, not the area) and we will have a similar square, made up of four connected unit squares, see Figure 5.6 b). A unit cube, 3-dimensional of course, magnified two times will result in a cube made up of eight cubes of unit size, see Figure 5.6 c).

Note that the number of copies of the original object when magnified two times is two to the power of the dimension. That is

$$m^D = N, \tag{5.1}$$

where m is the magnification, D is the dimension, and N is the number of copies of the original object when magnified m times. Now, if we solve for D in Equation 5.1 we obtain

$$D = \frac{\log N}{\log m}. \tag{5.2}$$

Of course, this is accurate for the objects in Figure 5.6, because $1 = \log 2/\log 2$, $2 = \log 4/\log 2$, $3 = \log 8/\log 2$, but what happens with the more complex objects of Example 5.1.2 and Example 5.1.3?

The von Koch curve gives, in E_{k+1}, four copies of E_k of size 1/3. Thus, we find D as the number

$$\frac{\log 4}{\log 1/\frac{1}{3}} = 1.2618\ldots.$$

It is *more* than 1-dimensional, but *less* than 2-dimensional. This agrees with the idea that neither length nor area can describe the curve, simple because it is *more* than a line (the length between any two points on the curve is infinite), but does not fill the plane either (it has zero area).

The Sierpinski triangle is in E_{k+1} made up of three copies of E_k of size 1/2, thus the dimension is

$$D = \frac{\log 3}{\log 1/\frac{1}{2}} = 1.5849\ldots.$$

All triangles are replaced by smaller triangles, and the area of each triangle tends to zero. Note that the dimension of the Sierpinski triangle is higher than that of the von Koch curve, we say that the Sierpinski triangle fills the plane more than the von Koch curve does.

The number obtained in the above way is called the *Similarity dimension* of a set, and might appear to be a suitable way to calculate the dimension. However, the Similarity dimension can only be calculated for a small class of *strictly self-similar sets*. In the following sections we will describe some other definitions of dimension that are much more general [Fal90].

5.2.3 Hausdorff Dimension

The similarity dimension is not general enough to describe all sets, and also, as we will see in Chapter 6, it can easily be fooled to give a too large value for the dimension.

The most general notion of dimension is the Hausdorff dimension, which is defined for all metric spaces. This dimension is the topic of the current section. To be able to understand the more general notions of dimension, such as the Hausdorff dimension, we first need to set the basics with some elementary Measure Theory.

The Notion of a Measure

Before delving into the mathematics of fractal dimensions, we need to briefly look at some notions of measure. We use measures in the definition of dimension, and in any case, measures are important in fractal geometry, in some form or another, and we need to set the basics here.

A measure is exactly what the intuitive feeling tells us it is; a way to give a numerical size to a set such that the sum of the sets in a collection of disjoint subsets have the same measure as the whole set (the union of the subsets). The numerical *size* of a set could e.g. be the mass distribution, or the electrical charge of the set. We define a measure, μ, on $\mathbb{R}^n$ as follows [Fal90]:

Definition 5.2.4: *The* measure μ *assigns a non-negative value, possibly* ∞*, to subsets of* $\mathbb{R}^n$ *such that:*

a) $\mu(\emptyset) = 0$; (5.3)

b) $\mu(A) \leq \mu(B)$ *if* $A \subseteq B$; (5.4)

c) *If* $A_1, A_2, \ldots$ *is a countable (or finite) sequence of sets then*

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i) \tag{5.5}$$

with equality in Equation 5.5, i.e.

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) \tag{5.6}$$

if the A_i *are disjoint Borel sets.*

This follows our intuitive feeling that an empty set has no *size* (a), that a smaller set has smaller *size* (b), and, as noted above, that the sum of the *sizes* of the pieces is the *size* of the whole, Equation 5.6.

If $A \supset B$, we can express $A = B \cup (A \backslash B)$, and thus, by Equation 5.6, if A and B are Borel sets, we have:

$$\mu(A \backslash B) = \mu(A) - \mu(B). \tag{5.7}$$

In fact, if X is a set, a set $E \subseteq X$ is called μ-measurable if and only if every set $A \subseteq X$ satisfies

$$\mu(A) = \mu(A \cap E) + \mu(A \backslash E). \tag{5.8}$$

A *probability measure* is a measure μ on a set A, such that $\mu(A) = 1$.

The *support* of a measure is the set on which the measure is concentrated. The support of a measure μ, written $\operatorname{spt}\mu$, is the smallest closed set with complement of measure zero. More formally we define the support of a set as [Fal90]:

Definition 5.2.5: *The support of measure, μ, is the smallest closed set, $F \subseteq \mathrm{R}^n$, such that*

$$\mu(\mathrm{R}^n \backslash X) = 0. \tag{5.9}$$

We say that μ is a measure on *a set A, if A contains the support of μ.*

If μ is a measure on a set A, the set is said to be *finite* or have *finite measure* if $\mu(A) < \infty$. The set A is said to be *σ-finite* or have *σ-finite measure* if there exists a countable sequence of sets $\{A_i\}$ such that $A = \bigcup_{i=1}^{\infty} A_i$ and $\mu(A_i) < \infty$.

The Lebesgue Measure

The *Lebesgue measure*, $\mathscr{L}^1$, extends the idea of a *length* to a large collection of subsets of $\mathbb{R}$. For open and closed intervals, we have $\mathscr{L}^1(a,b) = \mathscr{L}^1[a,b] = b - a$. If $A = \bigcup_i [a_i, b_i]$ is a finite or countable union of disjoint intervals we let $\mathscr{L}^1(A) = \sum(b_i - a_i)$ be the length of A, which leads to the formal definition of *Lebesgue measure* [Fal90]:

Definition 5.2.6: *The* Lebesgue measure*, $\mathscr{L}^1$ of an arbitrary set is:*

$$\mathscr{L}^1(A) = \inf \left\{ \sum_{i=1}^{\infty} (b_i - a_i) : A \subseteq \bigcup_{i=1}^{\infty} [a_i, b_i] \right\} \tag{5.10}$$

This measure follows our intuitive feeling of a length in $\mathbb{R}$, but extends also to areas in $\mathbb{R}^2$, volumes in $\mathbb{R}^3$, and to the volume of n-dimensional hypercubes in $\mathbb{R}^n$:

$$\mathrm{vol}^n(A) = \mathscr{L}^n(A) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n). \tag{5.11}$$

The n-dimensional Lebesgue measure, $\mathscr{L}^n$ may be thought of as an extension to the n-dimensional volume for a large collection of sets. By simply extending Definition 5.2.6, we obtain [Fal90]:

Definition 5.2.7: *The* Lebesgue measure *on $\mathbb{R}^n$ of an arbitrary set is:*

$$\mathscr{L}^n(A) = \inf \left\{ \sum_{i=1}^{\infty} \mathrm{vol}^n(A_i) : A \subseteq \bigcup_{i=1}^{\infty} A_i \right\} \tag{5.12}$$

We have the following proposition [Edg90]:

Proposition 5.2.8: *The Lebesgue measure fulfills the properties of Definition 5.2.4. That is, we have*

a) $\mathscr{L}^n(\emptyset) = 0$;

b) $\mathscr{L}^n(A) \leq \mathscr{L}^n(B)$ *if* $A \subseteq B$;

c) $\mathscr{L}^n(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mathscr{L}^n(A_i)$;

d) $\mathscr{L}^n(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathscr{L}^n(A_i)$ *if* A_i *are disjoint Lebesgue measurable sets.*

Proof. For a), note that $\emptyset$ is the subset of any arbitrarily small set, i.e. the subset of a set with Lebesgue measure zero. For b), any cover of B is also a cover of A. See [Edg90] for proofs of c) and d) for $\mathscr{L}^1$. □

The following example illustrates the use of the above rules, and how to use the Lebesgue measure on sets:

Example 5.2.9: *The Cantor dust has Lebesgue measure 0.*

Calculation. The Cantor set, C, is the limit of $C = \lim_{k\to\infty} E_k$, as described in Example 5.1.1. The set consists of 2^k disjoint intervals of length 3^{-k}. Thus, $\mathscr{L}^1(C) \leq 2^k 3^{-k}$ by Proposition 5.2.8 c). This limit is 0, so $\mathscr{L}^1(C) = 0$. I.e. the *length* of the Cantor set is 0, as we showed in Example 5.1.1. □

Results are sometimes said to hold for *almost all* subsets, or almost all angles, and so on. The meaning of this is the following:

Definition 5.2.10: *A property that holds for* almost all *members of any given set will mean all members of the set with the exception of a subset of measure zero.*

The following example illustrates the consequences of the above definition:

Example 5.2.11: *We can say that* almost all *real numbers are irrational with regard to the Lebesgue measure. The rational numbers are countable, i.e.* $\mathbb{Q} = \{x_1, x_2, \ldots\}$, *and thus we can use Proposition 5.2.8 c) and write* $\mu(\mathbb{Q}) \leq \sum_{i=1}^{\infty} \mu(\{x_i\}) = 0$, *since every* x_i *is a point. Thus* $\mu(\mathbb{Q}) = 0$.

Not all sets are Lebesgue measurable, but all *normal* sets, and all sets we are dealing with are. We have the following propositions [Edg90]:

Proposition 5.2.12: *A compact set* $K \subseteq \mathbb{R}^n$ *is Lebesgue measurable. An open set* $U \subseteq \mathbb{R}^n$ *is Lebesgue measurable.*

Proof. See the proof in [Edg90]. The proof is in $\mathbb{R}^1$, but extends to $\mathbb{R}^n$ trivially. □

Lemma 5.2.13: *Let* $A \subseteq \mathbb{R}^n$. *Then* A *is measurable if and only if, for every* $\epsilon > 0$, *there exists an open set* U *and a closed set* F *with* $F \subseteq A \subseteq U$ *and* $\mathscr{L}^n(U \backslash F) < \epsilon$.

Proof. Omitted, see [Edg90]. □

Proposition 5.2.14:

a) *Both* $\emptyset$ *and* $\mathbb{R}^n$ *are Lebesgue measurable.*

b) *If* $A \subseteq \mathbb{R}^n$ *is Lebesgue measurable, then so is its complement* $\mathbb{R}^n \backslash A$.

c) *If* A *and* B *are Lebesgue measurable, then so are* $A \cup B$, $A \cap B$ *and* $A \backslash B$.

d) *If A_n is Lebesgue measurable for $n \in \mathbb{N}$, then so are $\bigcup_{n\in\mathbb{N}} A_n$ and $\bigcap_{n\in\mathbb{N}} A_n$.*

Proof. For a), note that $\mathscr{L}^n(\emptyset) = 0$ and $\mathbb{R}^n \cap [-n, n] \times \cdots \times [-n, n]$ is measurable for all n.

For b), note that if $U \subseteq A \subseteq V$, then $\mathbb{R}^n \backslash V \subseteq \mathbb{R}^n \backslash A \subseteq \mathbb{R}^n \backslash U$ and $(\mathbb{R}^n \backslash U) \backslash (\mathbb{R}^n \backslash V) = V \backslash U$.

For the intersection in c), note that if $U_1 \subseteq A \subseteq V_1$ and $U_2 \subseteq B \subseteq V_2$, then $U_1 \cap U_2 \subseteq A \cap B \subseteq V_2 \cap V_2$ and $(V_1 \cap V_2) \backslash (U_1 \cap U_2) \subseteq (V_1 \backslash U_1) \cup (V_2 \backslash U_2)$. This shows that $A \cap B$ is measurable. Now, $A \cup B = \mathbb{R}^n \backslash ((\mathbb{R}^n \backslash A) \cap (\mathbb{R}^n \backslash B))$, so $A \cup B$ is measurable. And $A \backslash B = A \cap (\mathbb{R}^n \backslash B)$, so $A \backslash B$ is measurable.

Finally, for d), note that by c) we may find disjoint measurable sets B_n with the same union as A_n, so that Proposition 5.2.8 is applicable. The intersection follows by taking complements. □

This leads us to the following result [Edg90]:

Corollary 5.2.15: *Every Borel set in $\mathbb{R}^n$ is Lebesgue measurable.*

Proof. Open sets are measurable by Proposition 5.2.12. Countable (or finite) unions and intersections of open sets are measurable by Proposition 5.2.14. Hence, the Borel sets of $\mathbb{R}^n$ are Lebesgue measurable by Definition 4.1.7. □

The Hausdorff Measure

The *Hausdorff dimension* is the oldest, and probably the most important. As opposed to many of the other definitions of dimension, the Hausdorff dimension is defined for all sets, and is convenient for mathematicians since it is based on measures – The Hausdorff measure. We have the following definitions [Fal90, Edg90]:

Definition 5.2.16: *Let F be a subset of S, a metric space, and $s \in \mathbb{R}^+$, then for any $\delta > 0$ we let*

$$\mathscr{H}^s_\delta(F) = \inf\left\{ \sum_{i=1}^{\infty} (\operatorname{diam} U_i)^s : \{U_i\} \text{ is a } \delta\text{-cover of } F \right\}. \tag{5.13}$$

We try to minimize the sum of the s-powers of the diameters of the covers of F with diameter at most δ. When δ decreases, the class of possible covers of F is reduced, and $\mathscr{H}^s_\delta(F)$ increases, and thus approach a limit as $\delta \to 0$. We define [Fal90, Edg90]:

Definition 5.2.17: *Let*

$$\mathscr{H}^s(F) = \lim_{\delta \to 0} \mathscr{H}^s_\delta(F). \tag{5.14}$$

*This limit exists for all subsets of a metric space S, but is usually 0 or ∞. We call $\mathscr{H}^s(F)$ the s-*dimensional Hausdorff measure *of F.*

It can be shown that the definition above is invariant of the choice of metric when calculating the diameter in Equation 5.13 [The90].

$\mathscr{H}^s$ fulfills the properties of Definition 5.2.4. We have the following proposition:

Proposition 5.2.18: *The Hausdorff measure fulfills the properties of Definition 5.2.4. That is, we have*

a) $\mathscr{H}^s(\emptyset) = 0$;

b) $\mathscr{H}^s(A) \leq \mathscr{H}^s(B)$ *if* $A \subseteq B$;

c) $\mathscr{H}^s(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mathscr{H}^s(A_i)$;

d) $\mathscr{H}^s(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathscr{H}^s(A_i)$ *if* A_i *are disjoint Borel sets.*

Proof. For a), by the definition, $\mathscr{H}_\delta^s$ tends to zero when $\delta \to 0$ for any cover of $\emptyset$. For b), any cover of B is also a cover of A, so that for each positive δ, $\mathscr{H}_\delta^s(A) \leq \mathscr{H}_\delta^s(B)$, let $\delta \to 0$ and the proposition follows. The proofs of c) and d) are omitted. □

We have the following corollary

Corollary 5.2.19: *If a set* F *is countable or finite, then* $\mathscr{H}^s(F) = 0$*, for all* $s > 0$*.*

Proof. Follows from the definition and Proposition 5.2.18. □

We also have the following proposition [Edg98]:

Proposition 5.2.20: *The* 0*-dimensional Hausdorff measure is a* counting measure*. Every subset* $A \subseteq S$ *is measurable; if* A *is an infinite set, then* $\mathscr{H}^s(A) = \infty$*; if* n *is a non-negative integer and* A *is a set with* n *elements, then* $\mathscr{H}^0(A) = n$*.*

Proof. Follows from the definition. See details in e.g. [Edg98]. □

The Hausdorff measure generalizes to the intuitive idea of length, area, and volume. We have the following proposition [Edg90]:

Proposition 5.2.21: *In the metric space* $\mathbb{R}$*, the one-dimensional Hausdorff measure,* $\mathscr{H}^1$*, coincides with the Lebesgue measure,* $\mathscr{L}^1$*.*

Proof. Omitted, see [Edg90]. □

In the general case, for subsets of $\mathbb{R}^n$, the s-dimensional Hausdorff measure is just a constant multiple of the Lebesgue measure [Edg90, Fal90]:

Proposition 5.2.22: *If* F *is a Borel set of* $\mathbb{R}^n$*, then there exists positive constants such that*

$$a_n \mathscr{L}^n(F) \leq \mathscr{H}^n(F) \leq b_n \mathscr{L}^n(F), \tag{5.15}$$

and in particular, such that

$$\mathscr{H}^n(F) = c_n \mathscr{L}^n(F). \tag{5.16}$$

Proof. See [Edg90] for a proof in $\mathbb{R}^2$ of Equation 5.15, the proof generalizes to higher dimensions. Equation 5.16 follows from Equation 5.15. □

The scaling properties of *normal measures* applies to Hausdorff measures as well, and thus, on magnification by a factor λ the length of a curve is multiplied by λ, the area of a figure in the plane is multiplied by λ^2, the volume of an object in space is multiplied by λ^3, and in general, the s-dimensional Hausdorff measure scales with a factor λ^s. We have the following proposition [Fal90]:

Proposition 5.2.23 (Scaling property of the Hausdorff measure)**:** *If $F \subseteq \mathbb{R}^n$ and $\lambda > 0$ then*

$$\mathscr{H}^s(\lambda F) = \lambda^s \mathscr{H}^s(F). \tag{5.17}$$

Proof. If $\{U_i\}$ is a δ-cover of F then $\{\lambda U_i\}$ is a $\lambda\delta$-cover of λF. Hence

$$\mathscr{H}^s_{\lambda\delta}(\lambda F) \leq \sum |\lambda U_i|^s = \lambda^s \sum |U_i|^s \leq \lambda^s \mathscr{H}^s_\delta(F) \tag{5.18}$$

since this holds for any δ-cover $\{U_i\}$. Letting $\delta \to 0$ gives that $\mathscr{H}^s(\lambda F) \leq \lambda^s \mathscr{H}^s(F)$. Replacing λ by $1/\lambda$ and F by λF gives the opposite inequality required. □

There is a very nice proof of Pythagoras' theorem using the above proposition, which is demonstrated with the following example [Bar88]:

Example 5.2.24: *A right-angle triangle in $\mathbb{R}^2$ is made up of two smaller copies of itself (this need also be proven, but is done with some elementary geometry; see i.e. [Roe03]), as in Figure 5.7. Clearly both transformations are similarities. The scaling factors are b/c and a/c (see [Roe03] for a proof.) The Hausdorff measure of the triangle, T, is thus $\mathscr{H}^2(T) = (b/c)^2\mathscr{H}^2(T) + (a/c)^2\mathscr{H}^2(T)$ by Proposition 5.2.18 and Proposition 5.2.23. Multiplying by c^2 and dividing by $\mathscr{H}^2(T)$ (assuming $\mathscr{H}^2(T) > 0$) on both sides yields the famous*

$$a^2 + b^2 = c^2.$$

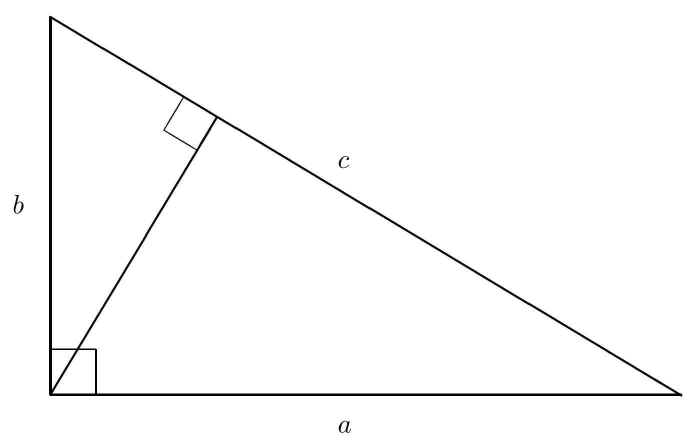


Figure 5.7: *A proof of Pythagoras' theorem using the Scaling property of the Hausdorff measure. The region bounded by a right-angle triangle is the union of two similar scaled copies of itself.*

How Measure Relate to Dimension

We begin with the following observation about the Hausdorff measure [Fal90]:

Proposition 5.2.25: *Studying Equation 5.13 we note that for any set F and $\delta < 1$, $\mathscr{H}^s_\delta$ is non-increasing with s, and thus $\mathscr{H}^s$ is non-increasing as well. If $t > s$ and $\{U_i\}$ is a δ-cover of F, we have*

$$\sum_i |U_i|^t \leq \delta^{t-s} \sum_i |U_i|^s. \tag{5.19}$$

Thus, taking the infimum of both sides, we get

$$\mathscr{H}^t_\delta(F) \leq \delta^{t-s}\mathscr{H}^s_\delta(F). \tag{5.20}$$

If we let $\delta \to 0$ *we see that if* $\mathscr{H}^s(F) < \infty$ *then* $\mathscr{H}^t(F) = 0$ *when* $t > s$*.*

Now, if we plot $\mathscr{H}^s(F)$ against s we see that there is a critical point at which $\mathscr{H}^s(F)$ *jumps* from ∞ to 0, see Figure 5.8. This critical value is known as the *Hausdorff dimension* (sometimes called the *Hausdorff-Besicovitch dimension*) of F, and is denoted $\dim_{\mathrm{H}} F$.

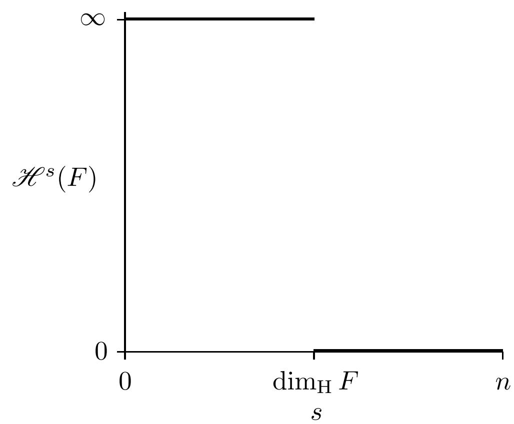


Figure 5.8: *A plot of* $\mathscr{H}^s(F)$ *against* s *for some set* F*. The Hausdorff dimension of* F *is the value* $s = \dim_{\mathrm{H}} F$ *at which the graph* jumps *from* ∞ *to* 0*.*

Formally, we define the Hausdorff dimension as follows [Fal90]:

Definition 5.2.26: *The value*

$$s = \inf\{s : \mathscr{H}^s(F) = 0\} = \sup\{s : \mathscr{H}^s(F) = \infty\} \tag{5.21}$$

such that

$$\mathscr{H}^s(F) = \begin{cases} \infty & \text{if } s < \dim_{\mathrm{H}} F \\ 0 & \text{if } s > \dim_{\mathrm{H}} F \end{cases} \tag{5.22}$$

is called the Hausdorff dimension *of the set* F*, and is denoted* $\dim_{\mathrm{H}} F$*. If* $s = \dim_{\mathrm{H}} F$*, then* $\mathscr{H}^s(F)$ *may be zero or infinite, or may satisfy*

$$0 < \mathscr{H}^s(F) < \infty.$$

This *jump* is easily understood by considering the dimension when measuring i.e. the length of lines, the area of squares and the volume of cubes. We can fit an infinite number of points on a line, but the *area* of a line is zero. The *length* of a square is infinite, i.e. we can fit an infinite number of lines (or a curve with infinite length) on the square, but the volume of a square is zero. A cube has infinite area in the sense that we can fit an infinite number of planes (or a plane curve with infinite area) in the cube. Thus, if we use a too small dimension when measuring a set, the measure is infinite, and if we use a too large dimension when measuring, the measure is zero.

The Hausdorff dimension relates to the Topological dimension by the following proposition [Edg90]:

Proposition 5.2.27: *Let S be a (possibly compact) metric space. Then* $\dim_{\mathrm{T}} S \leq \dim_{\mathrm{H}} S$.

Proof. Omitted, see [Edg90] and the references therein. □

The Hausdorff dimension fulfills the following properties:

Proposition 5.2.28: *Let $A, B \subset \mathbb{R}^n$*

a) *If $A \subseteq B$ then* $\dim_{\mathrm{H}} A \leq \dim_{\mathrm{H}} B$;

b) $\dim_{\mathrm{H}} A \leq n$;

c) *If A is a single point, then* $\dim_{\mathrm{H}} A = 0$.

d) *The Hausdorff dimension is finitely stable. I.e. let $A_i \subset \mathbb{R}^n$, for $i = 1, \ldots, k$, then*

$$\dim_{\mathrm{H}} \bigcup_{i=1}^{k} A_i = \max_{1 \leq i \leq k} \dim_{\mathrm{H}} A_i. \tag{5.23}$$

e) *The Hausdorff dimension is countably stable. I.e. let $A_i \subset \mathbb{R}^n$, for $i = 1, 2, \ldots$, then*

$$\dim_{\mathrm{H}} \bigcup_{i=1}^{\infty} A_i = \sup_{1 \leq i < \infty} \dim_{\mathrm{H}} A_i. \tag{5.24}$$

Proof. a) Follows from the monotonicity of the Hausdorff measure, Proposition 5.2.18. b) Holds by a) since $A \subset \mathbb{R}^n$ and $\dim_{\mathrm{H}} \mathbb{R}^n = n$. c) follows from the definition. A single point can be covered by a set with arbitrarily small diameter. d) and e) are proven in Section 9.1. □

Corollary 5.2.29: *If a set F is countable or finite, then* $\dim_{\mathrm{H}} F = 0$, *for all $s > 0$.*

Proof. The proof for finite sets follows from Proposition 5.2.20 but we state a full proof here for completeness. We note that for a single point set, F_0, we have

$$\mathscr{H}^0(F_0) = 1 \tag{5.25}$$

by the definition. Then we have, by Corollary 5.2.19 (or by Proposition 5.2.25) and the definition of Hausdorff dimension, that $\dim_{\mathrm{H}} F_0 = 0$. I.e., by Proposition 5.2.28 we have

$$\dim_{\mathrm{H}} F = \dim_{\mathrm{H}} \bigcup_{i=1}^{\infty} F_i = 0 \tag{5.26}$$

if F is countable, and

$$\dim_{\mathrm{H}} F = \dim_{\mathrm{H}} \bigcup_{i=1}^{n} F_i = 0 \tag{5.27}$$

if F is finite. □

We can also deduce the result of the following proposition:

Proposition 5.2.30: *Let $F \subset \mathbb{R}^n$. If F contains an open ball, then* $\dim_{\mathrm{H}} F = n$. *Also, since $\mathbb{R}^n$ contains an open ball,* $\dim_{\mathrm{H}} \mathbb{R}^n = n$.

Proof. The open ball has finite diameter, and therefore finite Lebesgue measure. We know from Proposition 5.2.22 that

$$a_n \mathscr{L}^n(F) \leq \mathscr{H}^n(F) \leq b_n \mathscr{L}^n(F),$$

and thus the n-dimensional Hausdorff measure of F is finite; hence F has Hausdorff dimension n. □

A consequence of Proposition 5.2.23 is the following [Edg90]:

Proposition 5.2.31: *Let $f(S) = rS$, where $r > 0$. Let s be a positive real number, and let $F \subseteq S$ be a Borel set. Then $\mathscr{H}^s(f(F)) = r^s \mathscr{H}^s(F)$, and thus $\dim_H f(F) = \dim_H F$.*

Proof. By Proposition 5.2.23 we know that $\mathscr{H}^s(f(F)) = r^s \mathscr{H}^s(F)$ and thus $\dim_H f(F) = \dim_H F$. □

The problem with the Hausdorff dimension is that it is difficult to calculate, or measure, in practice (it is not feasible to find the infimum, or the supremum, in Equation 5.21) [Fal90]. Because of this mathematicians felt the need for some other general definition of dimension, that could also easily be calculated in practice. The answer to this problem was the Box or Box-counting dimension, which we describe in the following section.

5.2.4 Box-Counting Dimension

The Box-counting dimension is one of the most common in practical use. This is mainly because it is easy to calculate mathematically and because it is easily estimated empirically.

We note that the number of line segments of length δ that are needed to cover a line of length l is l/δ, that the number of squares with side length δ that are needed to cover a square with area A is A/δ^2, and that the number of cubes with side length δ that are needed to cover a cube with volume V are V/δ^3. The dimension of the object we try to cover is obviously the power of the side length, δ. Now, can we generalize this to find the dimension of any set using this method?

Let the number of boxes of side length δ that we need to cover an object be N_δ. Following the discussion above, the number of boxes needed to cover the object is proportional to the box size [Fal90]:

$$N_\delta \sim \frac{C}{\delta^s}, \tag{5.28}$$

when $\delta \to 0$. Thus, for the constant C we have

$$\lim_{\delta \to 0} \frac{N_\delta}{\delta^{-s}} = C. \tag{5.29}$$

Taking the logarithm of both sides gives:

$$\lim_{\delta \to 0} (\log N_\delta + s \log \delta) = \log C. \tag{5.30}$$

We solve for s and get an expression for the dimension as

$$s = \lim_{\delta \to 0} \frac{\log N_\delta - \log C}{-\log \delta} = -\lim_{\delta \to 0} \frac{\log N_\delta}{\log \delta}. \tag{5.31}$$

We let F be a bounded non-empty subset of $\mathbb{R}^n$ and $N_\delta(F)$ be the smallest number of sets of diameter at most δ that covers F. Then we have the following definition [Fal97]:

Definition 5.2.32: *The* lower *and* upper Box-counting dimensions *of a set F are defined as*

$$\underline{\dim}_{\mathrm{B}} F = \liminf_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta} \tag{5.32}$$

and

$$\overline{\dim}_{\mathrm{B}} F = \limsup_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}, \tag{5.33}$$

respectively. If their values are equal, we refer to the common value as the Box-counting dimension *of F*

$$\dim_{\mathrm{B}} F = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}. \tag{5.34}$$

This says that the least number of sets of diameter δ that can cover F is of the order δ^{-s} where $s = \dim_{\mathrm{B}} F$. The dimension reflects how rapidly the irregularities of the object develop as $\delta \to 0$ [Fal90].

There are a number of equivalent definitions of the Box-counting dimension. The differences mainly concerns the shape of the *box* used to cover the set. However, the shape of the box is of no importance, and we can use both squares and circles, and their higher dimensional equivalences. In fact, we can even use general subsets of $\mathbb{R}^n$ with diameter δ. As a matter of fact, any decreasing sequence δ_k such that $\delta_{k+1} \geq c\delta_k$, for some $0 < c < 1$, with δ_k tending towards 0 will do equally well; see [Fal90] for details. In the limit, the shape will not matter [Fal97].

The following proposition establishes that we can use (hyper) cubes with side length δ to cover a set in $\mathbb{R}^n$ to calculate its Box-counting dimension, and that this method is equivalent to using sets with maximal diameter δ [Bar88]:

Proposition 5.2.33 (The Box-counting Theorem)**:** *Let $A \subset \mathbb{R}^n$. Cover A by closed square boxes of side length $\delta = 2^{-n}$. Let $N'_\delta(A)$ denote the number of boxes of side length $\delta = 2^{-n}$ which intersects A. Then*

$$\underline{\dim}_{\mathrm{B}} A = \liminf_{\delta \to 0} \frac{\log N'_\delta(A)}{-\log \delta} \tag{5.35}$$

and

$$\overline{\dim}_{\mathrm{B}} A = \limsup_{\delta \to 0} \frac{\log N'_\delta(A)}{-\log \delta}, \tag{5.36}$$

is the upper and lower box-counting dimension of A, and if their values are equal, we refer to the common value as the Box-counting dimension *of F*

$$\dim_{\mathrm{B}} A = \lim_{\delta \to 0} \frac{\log N'_\delta(A)}{-\log \delta}. \tag{5.37}$$

Proof. The proof is from [Fal90]. Since $N'_\delta(A)$ is the number of δ-cubes that intersect A, they are a collection of sets with diameter $\delta\sqrt{n}$ that cover A. Thus,

$$N_{\delta\sqrt{n}}(A) \leq N'_\delta(A).$$

For small enough δ, we have $\delta\sqrt{n} < 1$ and then

$$\frac{\log N_{\delta\sqrt{n}}(A)}{-\log \delta\sqrt{n}} \leq \frac{\log N'_\delta(A)}{-\log\sqrt{n} - \log\delta}$$

so taking limits as $\delta \to 0$

$$\underline{\dim}_{\mathrm{B}} A \leq \liminf_{\delta\to 0} \frac{\log N'_\delta(A)}{-\log\delta} \tag{5.38}$$

and

$$\overline{\dim}_{\mathrm{B}} A \leq \limsup_{\delta\to 0} \frac{\log N'_\delta(A)}{-\log\delta}. \tag{5.39}$$

On the other hand, any set of diameter at most δ is contained in 3^n mesh cubes of side δ. Thus

$$N'_\delta(A) \leq 3^n N_\delta(A),$$

and taking logarithms, dividing by $\log\delta$ and letting $\delta \to 0$ yields

$$\liminf_{\delta\to 0} \frac{\log N'_\delta(A)}{-\log\delta} \leq \underline{\dim}_{\mathrm{B}} A \tag{5.40}$$

and

$$\limsup_{\delta\to 0} \frac{\log N'_\delta(A)}{-\log\delta} \leq \overline{\dim}_{\mathrm{B}} A. \tag{5.41}$$

Thus, we can equally well let $N_\delta(A)$ be the number of δ-cubes that intersect A. □

The above proposition is widely used in practice for estimating the Box-counting dimension of a set. Put a δ-mesh on the set, and count the number of cubes that intersect the set for various small δ. The dimension is the logarithmic rate at which N_δ increases when $\delta \to 0$, and this may be estimated as the slope of a log-log plot of $\log N_\delta$ against $\log\delta$. We will explain this in more detail in Section 5.3.

There is a very nice connection between the Lebesgue measure and the Box-counting dimension, that the following proposition establishes [Fal90]:

Proposition 5.2.34: *If $F \subseteq \mathbb{R}^n$, then*

$$\underline{\dim}_{\mathrm{B}} F = n - \limsup_{\delta\to 0} \frac{\log \mathscr{L}^n(F_\delta)}{\log\delta} \tag{5.42}$$

and

$$\overline{\dim}_{\mathrm{B}} F = n - \liminf_{\delta\to 0} \frac{\log \mathscr{L}^n(F_\delta)}{\log\delta} \tag{5.43}$$

where F_δ is the δ-parallel body to F.

Proof. Omitted, but can be found in [Fal90]. □

The above proposition is the reason why the Box-counting dimensions sometimes is referred to as the *Minkowski dimension*.

The relationship between the Hausdorff dimension and the Box-counting dimension is established by the following proposition [Fal90]:

Proposition 5.2.35: *The following is true for* $F \subseteq \mathbb{R}^n$

$$\dim_{\mathrm{H}} F \leq \underline{\dim}_{\mathrm{B}} F \leq \overline{\dim}_{\mathrm{B}} F \tag{5.44}$$

Proof. If a set $F \subseteq \mathbb{R}^n$ can be covered by $N_\delta(F)$ sets of diameter δ, then, by Definition 5.2.17,

$$\mathscr{H}^s_\delta(F) \leq N_\delta(F)\delta^s.$$

If $1 < \mathscr{H}^s(F) = \lim_{\delta\to 0} \mathscr{H}^s_\delta(F)$ then taking logarithms gives $\log N_\delta(F) + s\log\delta > \log 1 = 0$ if δ is sufficiently small. Thus $s \leq \liminf_{\delta\to 0} \log N_\delta(F)/-\log\delta$ so the proposition follows for any $F \subseteq \mathbb{R}^n$ because of Definition 5.2.32. □

The above proposition does not in general have equality. The Hausdorff and Box-counting dimensions are only equal for *reasonably regular sets*, but there are several examples where the inequality is strict [Fal90].

We have the following proposition:

Proposition 5.2.36: *The Box-counting dimension of* $\mathbb{R}^n$ *is* n*, i.e.* $\dim_{\mathrm{B}} \mathbb{R}^n = n$*.*

Proof. We use Proposition 5.2.33 and note that for a hypercube, H, with side 1 in $\mathbb{R}^n$ we have $N_1(H) = 1$, $N_{2^{-1}}(H) = 2^n$, $N_{2^{-2}}(H) = 4^n$ and in general $N_{2^{-k}}(H) = 2^{kn}$. Thus, for a hypercube with side length 2^{-k} we have 2^{kn} hypercubes, i.e.

$$\dim_{\mathrm{B}} H = \lim_{k\to\infty} \frac{\log N_{2^{-k}}(H)}{-\log 2^{-k}} = \lim_{k\to\infty} \frac{\log 2^{kn}}{-\log 2^{-k}} = \lim_{k\to\infty} \frac{nk\log 2}{k\log 2} = n.$$

The above is in fact true for any hypercube in $\mathbb{R}^n$, i.e. a hypercube with side length l. Let $l \to \infty$ and the result follows. □

The above proposition is demonstrated with the following example [Bar88]:

Example 5.2.37: *Consider the unit square in the plane,* U_2*. It is easy to see that* $N_1(U_2) = 1$, $N_{2^{-1}}(U_2) = 4$, $N_{2^{-2}}(U_2) = 16$, $N_{2^{-3}}(U_2) = 64$ *and in general that* $N_{2^{-k}}(U_2) = 4^k$ *for* $k = 1, 2, 3, \ldots$*. By Proposition 5.2.36 we see that*

$$\dim_{\mathrm{B}} U_2 = \lim_{k\to\infty} \frac{\log N_{2^{-k}}(U_2)}{-\log 2^{-k}} = \lim_{k\to\infty} \frac{\log 4^k}{-\log 2^{-k}} = 2.$$

The Box-counting dimension of a unit cube in the plane is thus 2.

The Box-counting dimension fulfills the following properties:

Proposition 5.2.38: *Let* $A, B \subset \mathbb{R}^n$

- ***a)*** *If* $A \subseteq B$ *then* $\dim_{\mathrm{B}} A \leq \dim_{\mathrm{B}} B$*;*
- ***b)*** $\dim_{\mathrm{B}} A \leq n$*;*
- ***c)*** *If* A *is a single point, then* $\dim_{\mathrm{B}} A = 0$*.*
- ***d)*** *The upper Box-counting dimension is finitely stable. I.e. let* $A_i \subset \mathbb{R}^n$*, for* $i = 1, \ldots, k$*, then*

$$\overline{\dim}_{\mathrm{H}} \bigcup_{i=1}^{k} A_i = \max_{1\leq i\leq k} \overline{\dim}_{\mathrm{H}} A_i. \tag{5.45}$$

Proof. a) It is immediate that $N_\delta(A) \le N_\delta(B)$ for all $\delta > 0$. Thus, for all $0 < \delta < 1$, we have

$$0 \le \frac{\log N_\delta(A)}{-\log\delta} \le \frac{\log N_\delta(B)}{-\log\delta}.$$

Taking limits yields the result. b) This follows from a) and Proposition 5.2.36, since $A \subset \mathbb{R}^n$ and $\dim_{\mathrm{H}} \mathbb{R}^n = n$. c) For each $\delta > 0$ we have $N_\delta(A) = 1$. Since $\log 1 = 0$, it follows that $\dim_{\mathrm{B}} A = 0$. d) is proven in Section 9.1. □

Corollary 5.2.39: *The Box-counting dimension of a finite set is 0.*

Proof. This is immediate from Proposition 5.2.28 c) and d). But in fact it follows from the the definition of the Box-counting dimension as well. Let $A \subseteq S$ be the set in its metric space (S, ρ), let k be the number of elements in A and let $r = \min\{\rho(x, y) : x, y \in A\}$. Then, for any $\delta < r/2$, we have $N_\delta(A) = k$. Thus, the Box-counting dimension of A is

$$\dim_{\mathrm{B}} A = \lim_{\delta\to 0} \frac{\log k}{-\log\delta} = 0.$$

□

There is a problem with the Box-counting dimension, however, which at first might seem appealing, but has undesirable consequences [Fal90]:

Proposition 5.2.40: *Let $\overline{F}$ denote the closure of F. Then*

$$\underline{\dim}_{\mathrm{B}}\overline{F} = \underline{\dim}_{\mathrm{B}}F \tag{5.46}$$

and

$$\overline{\dim}_{\mathrm{B}}\overline{F} = \overline{\dim}_{\mathrm{B}}F. \tag{5.47}$$

Proof. Let $B_1, \dots, B_k$ be a finite collection of closed balls of radii δ. If the closed set $\cup_{i=1}^k B_i$ contains F, it also contains $\overline{F}$. Hence the smallest number of closed balls of radius δ that cover F is enough to cover the larger set $\overline{F}$. The result follows. □

Let F be the countable set of rational numbers between 0 and 1. Then $\overline{F}$ is the entire interval $[0, 1]$, so that $\underline{\dim}_{\mathrm{B}}F = \overline{\dim}_{\mathrm{B}}F = 1$. Thus, countable sets can have non-zero Box-counting dimension. The Box-counting dimension of each rational number is zero, but the countable union of these *points* has dimension 1.

5.2.5 Properties of Dimensions

The above definitions for the Hausdorff and upper and lower Box-counting dimensions have several important properties in common. Some of them are the following [Fal97]:

Monotonicity: If $E_1 \subseteq E_2$ then $\dim E_1 \le \dim E_2$.

Finite sets: If E is finite then $\dim E = 0$.

Open sets: If E is a (non-empty) open subset of $\mathbb{R}^n$ then $\dim E = n$.

Smooth manifolds: If E is a smooth m-dimensional manifold in $\mathbb{R}^n$ then $\dim E = m$.

Lipschitz mappings: If $f : E \to \mathbb{R}^m$ is Lipschitz then $\dim f(E) \le \dim E$.

Bi-Lipschitz invariance: If $f : E \to f(E)$ is bi-Lipschitz then $\dim f(E) = \dim E$.

Geometric invariance: If f is a similarity or affine transformation then $\dim f(E) = \dim E$.

We have already proven some of these in the sections above. The Hausdorff and the upper Box-dimensions are also, as we have seen, *finitely stable*, that is $\dim \cup_{i=1}^{k} E_i = \max_{1 \leq i \leq k} \dim E_i$, for any finite collection of sets $\{E_1, \ldots, E_k\}$. The lower Box-counting dimension is, however, not in general finitely stable. The Hausdorff dimension is also *countably stable*, meaning that $\dim_{\mathrm{H}} \cup_{i=1}^{\infty} E_i = \sup_{1 \leq i < \infty} \dim_{\mathrm{H}} E_i$. This is not true for the Box-counting dimension, however, as was described above.

Most definitions of dimension will take on values between the Hausdorff dimension and the upper Box-counting dimension. Thus, if $\dim_{\mathrm{H}} E = \overline{\dim}_{\mathrm{B}} E$ then all *normal* definitions of dimension will take on this value [Fal97].

5.3 Estimating the Fractal Dimension

The difficulty with implementing the Hausdorff dimension for numerical applications is the need for finding the infimum and supremum for all coverings, described in Equation 5.21. When we relax this requirement, and e.g. considers a fixed-size grid instead, we can numerically estimate the Box-counting dimension, as mentioned above, and thus find an upper bound for the Hausdorff dimension as Proposition 5.2.35 suggests. However, as will be explained in the following chapters, and according to Proposition 6.4.5, the Hausdorff and Box-counting dimensions are equal for most sets that are of interest to us [The90].

When numerically estimating the fractal dimension of a set, we are faced with an immediate problem. The number of points in a constructed set is inevitably finite, and thus the theoretic dimension of the sets under consideration is always zero (see Section 5.2.5). However, estimating blindly the dimension of the underlying set according to the definitions yields, in general, very good results.

Most definitions of dimension is based on the idea of a measurement at scale δ. For each δ, we look at properties of the set, but ignoring irregularities smaller than δ, and look at how these measurements change when $\delta \to 0$ [Fal90]. Remember from Section 5.2.4 that the number of *boxes* needed to cover a set is:

$$N_\delta(F) \sim \frac{C}{\delta^s},$$

which gives the dimension as

$$s = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}.$$

In [Man67], Mandelbrot says that geographers cannot be concerned with minute details when they measure the length of a coastline. Simply because below a certain level of detail, it is no longer the coastline that is being measured, but other details that affects the length of the curve. There is no clear cross-over either, so one simply have to choose a lower limit of geographically interesting features. A similar problem occurs with e.g. digital images, but here we have a strictly imposed lower limit of measurement granularity in the single pixel being the smallest measurable unit. Thus, we cannot let $\delta \to 0$, but have to stop when $\delta = 1$. This means we will not get the true value for the dimension this way. However, if we use successively smaller value of δ, say δ_k, and let

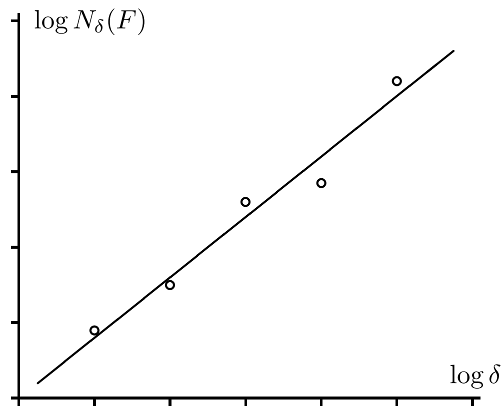


Figure 5.9: *Least-squares fit to the points of* $\log N_\delta(F)$ *against* $\log \delta$*.*

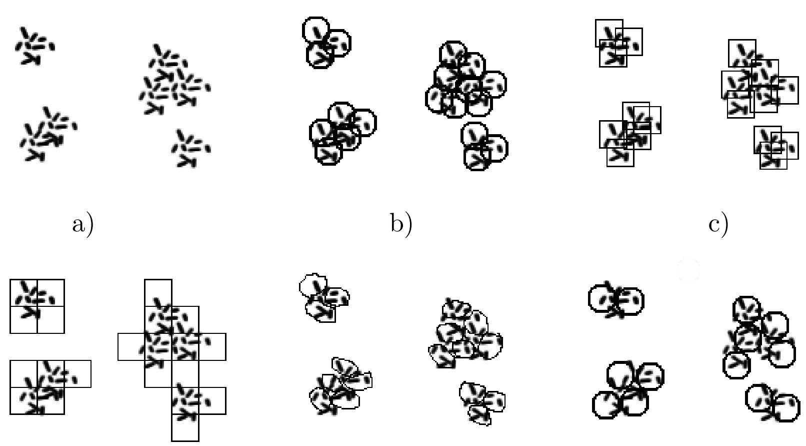


Figure 5.10: *Different ways to find the Box-counting dimension. We have the set F in a), b) the number of closed balls with radius δ that cover F, c) the least number of boxes of side δ that cover F, d) the number of δ-mesh cubes that intersect F, e) the least number of sets of diameter at most δ that cover F and f) the greatest number of disjoint balls of radius δ with centers in F.*

$\delta_k \to 0$ when $k \to \infty$, then each value of s_k will differ from the correct value by just a small amount. We can now estimate the value of s by plotting $\log N_\delta(F)$ against $-\log \delta$ and take the slope of a least-squares fit to the points as the value of s, see Figure 5.9. The dimension is thus the logarithmic rate at which $N_\delta(F)$ increases as $\delta \to 0$, and the best we can do here is estimate the slope of the points we get in the bi-logarithmic plot.

It doesn't matter what shape the boxes have when we estimate the Box-counting dimension. Actually, it doesn't even have to be boxes, any set with diameter δ will do equally well. In [Fal90], an argument for five different shapes can be found, but as stated, the list could be made longer, and in the end it is the particular application that decides which to use. In Figure 5.10 we can see the different methods with which we can cover the set F.

Chapter 6

Generating Fractals

In this chapter we describe two methods, or algorithms, to create fractals. The methods generate fractals from a set of transformations using a method called Iterated Function Systems. The Iterated Function Systems approach is explained and proofs to its validity are given.

There is a dimension connected with the Iterated Function System that is the general case of the similarity dimension explained in Chapter 5. This dimension is easily fooled, however, why a constraint called an *open set condition* is needed. With the constraint active, the similarity dimension equals the Hausdorff and Box-counting dimensions.

6.1 Iterated Function Systems

Many fractals are self-similar, i.e. they are made up of parts that are similar to the whole, but often scaled and translated. These self-similarities are not only properties of the fractals but, as we will see, can in fact be used to generate them as well [Fal90].

We begin with the following definitions [Edg90]:

Definition 6.1.1: *If S and T are two metric spaces, then a function $f : S \to T$ is called an* isometry *if*

$$\rho_T(f(x), f(y)) = \rho_S(x, y)$$

for all $x, y \in S$. The metric spaces S and T are isometric.

Definition 6.1.2: *A function $g : S \to T$ is called a* similarity *if there is a positive number r such that*

$$\rho_T(g(x), g(y)) = r\rho_S(x, y)$$

for all $x, y \in S$. The number r is the ratio *of g. The metric spaces S and T are said to be* similar.

Definition 6.1.3: *A function $h : S \to S$ is called a* contraction *or a* contraction mapping *if there is a constant r with $0 < r < 1$ such that*

$$\rho(h(x), h(y)) \leq r\rho(x, y) \tag{6.1}$$

for all $x, y \in S$.

We have the following lemma [Bar88]:

Lemma 6.1.4: *Let $f : S \to S$ be a contraction mapping on the metric space (S, ρ). Then f is continuous.*

Proof. Let $\varepsilon > 0$ be given. Let $s > 0$ be the contractivity factor for f. Then

$$\rho(f(x), f(y)) \leq s\rho(x, y) < \varepsilon \tag{6.2}$$

whenever $\rho(x, y) < \delta$, where $\delta = \varepsilon/s$. □

Definition 6.1.5: *A point x is called a* fixed points *of a function f if and only if $f(x) = x$.*

We now have a useful proposition on these definitions [Bar88]:

Proposition 6.1.6 (The Contraction Mapping Theorem)**:** *Let $f : S \to S$ be a contraction mapping on a complete metric space (S, ρ). Then f has a unique fixed point $x_f \in S$.*

Proof. Omitted, see [Bar88] or [Edg90]. □

The above theorem implies that it is possible to construct this fixed point [Edg90]:

Corollary 6.1.7: *Let f be a contraction mapping on a complete metric space S. If x_0 is any point of S, and*

$$x_{n+1} = f(x_n) \qquad \text{for } n \geq 0,$$

then the sequence x_n converges to the fixed point of f.

An Iterated Function System is defined as follows [Bar88]:

Definition 6.1.8: *An* iterated function system *(IFS) consists of a complete metric space (S, ρ) together with a finite set of contraction mappings (or similarities) $f_i : S \to S$, with corresponding ratios r_i, for $i = 1, \ldots, n$. The contraction factor, or ratio of the IFS is $r = \max\{r_i, i = 1, \ldots, n\}$.*

The functions of an iterated function system are said to *realize*, or be a *realization* of the corresponding ratio list. We note that a similarity is a special case of a contraction, thus all results that are valid for contractions are equally well valid for similarities.

We need the following definition:

Definition 6.1.9: *Let $f_1, \ldots, f_n$ be contractions, then we say that the set F is* transformation invariant *or an* invariant set *for the iterated function system if and only if*

$$F = \bigcup_{i=1}^{n} f_i(F). \tag{6.3}$$

We have the following theorem [Hut81]:

Proposition 6.1.10: *Let (S, ρ) be a complete metric space. For arbitrary $A \subset S$, let $\{f_1, \ldots, f_n\}$ be a set of contractions and $F(A) = \bigcup_{i=1}^{n} f_i(A)$. Let $F^0(A) = A$, $F^1(A) = F(A)$ and $F^p(A) = F(F^{p-1}(A))$ for $p \geq 2$. Then there is a unique compact set K that is invariant with respect to F. I.e.*

$$K = F(K) = \bigcup_{i=1}^{n} f_i(K). \tag{6.4}$$

If $A \subset S$ is any non-empty bounded set, then

$$\lim_{p\to\infty} |F_{i_1}(\cdots F_{i_p}(A)\cdots)| = 0, \tag{6.5}$$

and the limit is the fixed point of the contraction $F_{i_1}(\cdots F_{i_p}(A)\cdots)$. In particular, we have that

$$K = \bigcap_{p=1}^{\infty} F^p(A) \tag{6.6}$$

in which K is called the attractor of the iterated function system.

Proof. Equation 6.5 follows from Proposition 6.1.6. The rest is omitted, see [Hut81] or [Bar88] for details. □

What the above theorem says is that for a contracting transformation (mapping) on a set, there is always some set that is invariant of the transformation, and also, the repeated application of the transformations on *any* set yields better and better approximations to the set which is invariant to the transformations.

The above proposition is illustrated with the following example [Bar88]:

Example 6.1.11: *Consider an IFS on $\mathbb{R}$, with contraction mappings $f_1(x) = \frac{1}{3}x$, $f_2(x) = \frac{1}{3}x + \frac{2}{3}$. We will show that this is an IFS with contractivity factor $r = \frac{1}{3}$, and that if $B = [0,1]$ and $F(K) = \bigcup_{i=1}^{n} f_i(K)$ then the set $C = \lim_{k\to\infty} F(B)$ is the Cantor set (see Figure 5.1 on page 22); verifying that indeed $C = \frac{1}{3}C \cup \{\frac{1}{3}C + \frac{2}{3}\}$.*

Calculation. We first find the contractivity factor, by finding it for f_1 and f_2:

$$\rho(f_1(x), (f_1 y)) = \left|\frac{1}{3}x - \frac{1}{3}y\right| = \frac{1}{3}|x - y| = \frac{1}{3}\rho(x, y) \tag{6.7}$$

$$\rho(f_2(x), (f_2 y)) = \left|\left(\frac{1}{3}x + \frac{2}{3}\right) - \left(\frac{1}{3}y + \frac{2}{3}\right)\right| = \frac{1}{3}|x - y| = \frac{1}{3}\rho(x, y) \tag{6.8}$$

Thus, $r = \max\{r_1, r_2\} = \max\{\frac{1}{3}, \frac{1}{3}\} = \frac{1}{3}$ is the contractivity factor of the IFS. We denote the application of the IFS on B as $B_n = F^{\circ n}(B)$. Now, let $[a, b]$ be a component of B_k, $k < n$, then out hypothesis is that

$$[a, a + \frac{1}{3}(b - a)] \cup [b - \frac{1}{3}(b - a), b] \tag{6.9}$$

and

$$(a + \frac{1}{3}(b - a), b - \frac{1}{3}(b - a)) \cap B_{k+1} = \emptyset. \tag{6.10}$$

This is true for the first case, since for $B = [0, 1]$:

$$B_1 = F(B) = f_1(B) \cup f_2(B) = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]. \tag{6.11}$$

Now suppose that $[a, b] \in B_n$ is a component of B_n, then there is a component of $[a', b'] \in B_{n-1}$ whose image is $[a, b]$ under f_1 or f_2. The component $[a', b']$ has its middle third removed in B_n. Then $[a, b]$ is replaced in B_{n+1} by the set

$$f_i([a', a' + \frac{1}{3}(b' - a')] \cup [b' - \frac{1}{3}(b' - a'), b']) = [a, a + \frac{1}{3}(b - a)] \cup [b - \frac{1}{3}(b - a), b] \tag{6.12}$$

for $i = 1$ or 2. Hence, the operation of f_i removes the middle third of an interval, and the hypothesis follows by induction.

We can see that if $A = [0, 1]$, defining f_1 and f_2 as above yields $S^k(E) = E_k$, where E_k is described in Example 5.1.1 on page 22, and the Cantor set is obtained when k tends to infinity, see Figure 5.1 d). I.e. the attractor of the IFS is the Cantor set. □

6.2 The Deterministic Algorithm

In this section we will describe a deterministic algorithm for generating fractals with an iterated function system.

Let $f_1, \ldots, f_n$ be an iterated function system on a metric space (S, ρ). Choose a compact set $A_0 \subset S$. Then compute each A_n successively as

$$A_{n+1} = \bigcup_{i=1}^{n} f_i(A_n) \tag{6.13}$$

for $i = 1, 2, \ldots$. I.e. construct a sequence of sets $\{A_i\}$. By Proposition 6.1.10 this sequence converges to the attractor of the IFS for $i \to \infty$.

When choosing the initial set $A_0 \subset S$, it does not matter what set we choose, *any* compact set will work equally well. See Figure 6.1 for an illustration of how the algorithm works.

6.3 The Random Iteration Algorithm

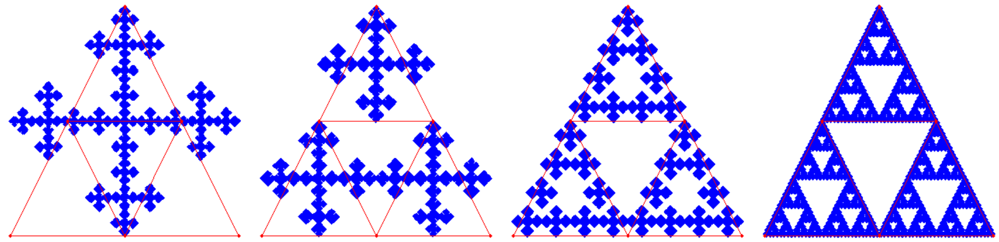
In this section, we will describe two versions of the random iteration algorithm for generating fractals using an iterated function system.

Let $f_1, \ldots, f_n$ be an iterated function system on a metric space (S, ρ). Choose an initial point $x_0 \in S$. Then choose recursively and independently

$$x_n \in \{f_1(x_{n-1}), \ldots, f_n(x_{n-1})\}, \tag{6.14}$$

for $n \geq 1$. I.e. construct a sequence of points $\{x_n\} \subset S$. The sequence $\{x_n\}$ converges to the attractor of the IFS by Proposition 6.1.10 also, when $n \to \infty$.

We can now construct fractals by letting an initial set of only one single point x_0 be transformed in the above way. Select the initial point x_0, let it be *any* point. Randomly select a contraction f_{i_1} from $f_1, \ldots, f_k$ and let $x_1 = f_{i_1}(x_0)$. Iterate this, choosing randomly a contraction in each iteration. For large k, the points will be indistinguishably



a) b) c) d)

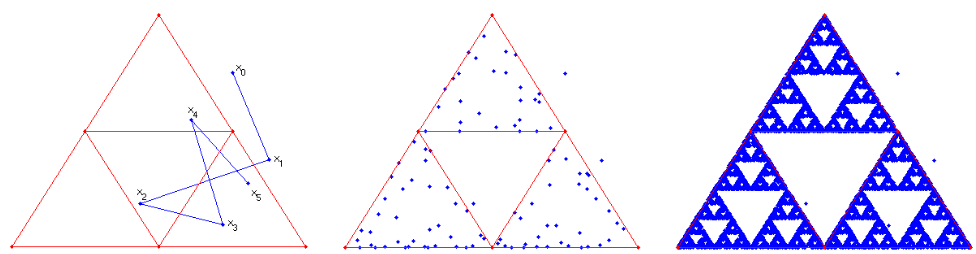
Figure 6.1: *Applying the discrete algorithm to the Box fractal generates the Sierpinski triangle. Successive application of Equation 6.4 yields better and better approximations of the Sierpinski triangle. a) The original Box fractal. b) One* round *of the algorithm, i.e. Equation 6.4 is applied once. c) Three rounds of the algorithm. d) After several iterations, the attracting set is apparent.*

close to the fractal. Thus, plotting the points $x_0, x_1, \ldots, x_k$ will yield an image of the set which is the attractor of the IFS. If the point x_0 was chosen arbitrarily, the hundredth or so first points may be ignored, since they might not have gotten close enough to the attractor, but if x_0 was chosen so that $x_0 \in F$, F the attractor, then it will always stay in F. See Figure 6.2 for illustrations of how the algorithm works.

There is an alternative implementation of the above algorithm. Instead of starting with just one point, and iterate recursively on random contractions, we could start with any set, and for each point in the set apply one of the contractions randomly. More formally: Let $f_1, \ldots, f_n$ be an iterated function system on a metric space (S, ρ). Choose a compact set $A_0 \in S$ and compute

$$A_{n+1} = \{x \in \{f_1(y), \ldots, f_n(y)\} : y \in A_n\} \tag{6.15}$$

for $n \geq 1$. I.e. construct a sequence of sets $\{A_n\}$. The sequence $\{x_n\}$ converges to the attractor of the IFS when $n \to \infty$ just like in the case when we started with one single



a) b) c)

Figure 6.2: *For the Sierpinski triangle we have three contractions:* $S_1(x) = (0.5\ 0.5) \cdot x, S_2(x) = (0.5\ 0.5) \cdot x + (0.25\ 0.5), S_3(x) = (0.5\ 0.5) \cdot x + (0.5\ 0.0)$. *a) The first six points of the IFS are found. Adjacent points are connected by line segments. b) 100 points of the IFS are found, note that only the first few are outside the triangle lines. c) 10,000 points of the IFS are found, the fractal structure is apparent.*

point, x_0, above. See Figure 6.3 for illustrations of how the algorithm works.

This method could be considered as selecting several initial points, and recursively applying a randomly selected contraction on them, but instead of keeping all intermediate points, we keep just the last step for each point of the original set.

6.4 The Dimension of the Attractor of an Iterated Function System

We described the intuitive view of the Similarity dimension in Section 5.2.2 on page 25. What was described there is only a special case of a more general theory which we lay out here.

We have the following definition:

Definition 6.4.1: *Let $f_1, \ldots, f_n$ be an iterated function system on a metric space, (S, ρ), with contraction ratios $r_1, \ldots, r_n$ respectively, with the attracting set F. The Similarity dimension,* $\dim_S F$ *of the attracting set F is the unique number s such that*

$$\sum_{i=1}^{n} r_i^s = 1. \tag{6.16}$$

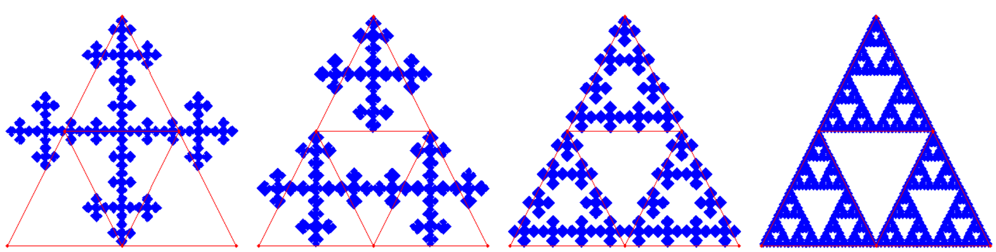
In Section 5.2.2 we only discussed the special case in which $r_1 = \cdots = r_n$, with the common value r, and defined the similarity dimension to be the number s satisfying

$$nr^s = 1, \tag{6.17}$$

or perhaps better recognized when solving for s as

$$s = \frac{\log n}{\log 1/r}. \tag{6.18}$$

However, it is not in general true that $\dim_S F = s$, where F is the attractor of an IFS. Consider the IFS $f_1(x) = (0.5\ 0.5) \cdot x, f_2(x) = (0.5\ 0.5) \cdot x + (0.25\ 0.5), f_3(x) = (0.5\ 0.5) \cdot x + (0.5\ 0.0)$ for the Sierpinski triangle, with the ratio list $(0.5, 0.5, 0.5)$. Now,



a) b) c) d)

Figure 6.3: *Applying the alternative random iteration algorithm on the set that constitutes the Box fractal yields the Sierpinski triangle. a) The original Box fractal. b) About $\frac{1}{3}$ of the points are in each of the three smaller versions of the Box-fractal. c) Three rounds of the algorithm. d) After several iterations, the attracting set is apparent.*

if we create another IFS by adding the function $f_4(x) = (0.5\ 0.5) \cdot x$ to the IFS for the Sierpinski triangle. The attractor is, of course, the same, but since we now have the ratio list $(0.5, 0.5, 0.5, 0.5)$, we get a similarity dimension of the attractor as

$$0.5^s + 0.5^s + 0.5^s + 0.5^s = 1$$

which yields

$$s = \frac{\log 4}{\log(1/0.5)} = 2,$$

while the attracting set obviously is the same as before (and in fact, as we will see, the set has Hausdorff dimension $\log 3/\log 2$).

The problem is, obviously, that the contractions f_1 and f_4 overlap too much. We need the following definition, called the *open set condition* [Mor46, Hut81]:

Definition 6.4.2: *An iterated function system satisfies Moran's* open set condition *if there exists a non-empty open set O such that*

a) $f_i(O) \cap f_j(O) = \emptyset$ *for* $i \neq j$;

b) $O \subseteq \bigcup_{i=1}^{n} f_i(U)$.

Without the open set condition, we have the following proposition for the Hausdorff dimension [Edg90]:

Proposition 6.4.3: *Let K be the invariant set of an iterated function system with similarity dimension* $\dim_{\mathrm{S}} K = s$ *in a complete metric space. Then* $\dim_{\mathrm{H}} K \leq \dim_{\mathrm{S}} K$.

Proof. Omitted, see [Edg90]. □

And we have the following for the Box-counting dimension [Fal90]:

Proposition 6.4.4: *Let K be the invariant set of an iterated function system with similarity dimension* $\dim_{\mathrm{S}} K = s$ *in* $\mathbb{R}^n$. *Then* $\underline{\dim}_{\mathrm{B}} K \leq \overline{\dim}_{\mathrm{B}} K \leq \dim_{\mathrm{S}} K$.

Proof. Omitted, see [Fal90]. □

But if the open set condition is satisfied, we have the following relation [Edg90, Fal90]:

Proposition 6.4.5: *Let K be the invariant set of an iterated function system with similarity dimension* $\dim_{\mathrm{S}} K = s$ *in* $\mathbb{R}^n$. *If Moran's open set condition is satisfied, then* $\dim_{\mathrm{H}} K = \dim_{\mathrm{B}} K = \dim_{\mathrm{S}} K$. *Moreover, for this value of s,* $0 < \mathscr{H}^s(K) < \infty$.

Proof. Omitted, see [Edg90] and [Fal90]. □

If Moran's open set condition is not satisfied, we can in fact say even more than we did above. We have the following proposition [Fal90]:

Proposition 6.4.6: *Let f_i be contractions on $\mathbb{R}^n$ with ratios r_i. If F is the invariant set of the IFS with similarity dimension* $\dim_{\mathrm{S}} F = s$, *then, if the open set condition is not satisfied, we have* $\dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F \leq s$.

Proof. Omitted, but can be found in [Fal90]. □

The above theorem says that for sets that do not overlap *too much*, and that are made up of similar copies of the whole, the Hausdorff and Box-counting dimensions are equal. E.g. this holds for the Cantor set, the Sierpinski triangle, the von Koch curve, and all other such self-similar fractals.

We can now state the following beautiful corollary about the relations between the different dimensions we have discussed in this thesis:

Corollary 6.4.7: *For any set $F \subseteq \mathbb{R}^n$:*

$$\dim_{\mathrm{T}} F \le \dim_{\mathrm{H}} F \le \underline{\dim}_{\mathrm{B}} F \le \overline{\dim}_{\mathrm{B}} F \le \dim_{\mathrm{S}} F, \tag{6.19}$$

when all dimensions are defined for F.

Let K be the invariant set of an iterated function system with similarity dimension $\dim_{\mathrm{S}} K = s$ in a complete metric space. Then the following is true

$$\dim_{\mathrm{T}} K \le \dim_{\mathrm{H}} K = \underline{\dim}_{\mathrm{B}} K = \overline{\dim}_{\mathrm{B}} K \le \dim_{\mathrm{S}} K. \tag{6.20}$$

And if Moran's open set condition is satisfied, then we have the stronger relation

$$\dim_{\mathrm{T}} K \le \dim_{\mathrm{H}} K = \underline{\dim}_{\mathrm{B}} K = \overline{\dim}_{\mathrm{B}} K = \dim_{\mathrm{S}} K. \tag{6.21}$$

Proof. This is immediate from the previous relations stated, Proposition 6.4.4 and Proposition 6.4.5. □

Thus, if the set is self-similar, and the open set condition is satisfied, we have a very simple method to compute the Hausdorff and Box-counting dimension of the set.

Chapter 7

Graph-directed Constructions

The self-similar sets that have been described earlier is actually just a special case of a much broader class of sets – graph-directed constructions. The graph-directed constructions that are described in this chapter are created by a recurrent scheme that creates fractal sets by composition of different and different shaped sets.

Firstly, the Hausdorff dimension of the classical self-similar sets is found by algebra, then the composition of two self-similar sets is described as a way to intuitively describe how the graph-directed sets look and work. Then the general case is described as well as how to find the Hausdorff dimension of such sets.

The reader might need to read Chapter 4 first to fully appreciate this chapter.

7.1 The Hausdorff Dimension of Self-similar Sets

With the following calculations, we can easily deduce that the Hausdorff dimension of the Cantor set is $\frac{\log 2}{\log 3}$. The Cantor set, C, is constructed using two similarity transformations, $S_1(x) = \frac{x}{3}$ and $S_2(x) = \frac{x}{3} + \frac{2}{3}$. We have that $C = S_1(C) \cup S_2(C)$. Since $S_1(C) \cap S_2(C) = \emptyset$ we have that

$$\mathscr{H}^s(C) = \mathscr{H}^s(S_1(C)) + \mathscr{H}^s(S_2(C)). \tag{7.1}$$

Now, by the scaling property of the Hausdorff measure, Proposition 5.2.23 on page 32, we have that

$$\mathscr{H}^s(S_1(C)) = \left(\frac{1}{3}\right)^s \mathscr{H}^s(C) \tag{7.2}$$

and

$$\mathscr{H}^s(S_2(C)) = \left(\frac{1}{3}\right)^s \mathscr{H}^s(C), \tag{7.3}$$

thus

$$\mathscr{H}^s(C) = \left(\frac{1}{3}\right)^s \mathscr{H}^s(C) + \left(\frac{1}{3}\right)^s \mathscr{H}^s(C) = 2\left(\frac{1}{3}\right)^s \mathscr{H}^s(C). \tag{7.4}$$

This means that

$$1 = 2\left(\frac{1}{3}\right)^s \tag{7.5}$$

and thus

$$s = \frac{\log 2}{\log 3}, \tag{7.6}$$

the Hausdorff dimension of C, the Cantor set. Obviously, this argument is invalid if not $0 < \mathscr{H}^s(C) < \infty$, but it can easily be proven that if $s = \frac{\log 2}{\log 3}$ then $0 < \mathscr{H}^s(C) < \infty$, and since it is finite and non-zero the Hausdorff dimension of C is necessarily $\log 2/\log 3$.

Equation 7.5 is perhaps recognized from before. We write the contraction factors $\frac{1}{3}$ as r_1 and r_2 for S_1 and S_2 respectively. Then the formula becomes

$$1 = r_1^s + r_2^s = \sum_{i=1}^{2} r_i^s. \tag{7.7}$$

Thus, in a more general setting, we get

$$1 = \sum_{i=1}^{n} r_i^s, \tag{7.8}$$

which is exactly Equation 6.16 on page 48.

7.2 Hausdorff Dimension of Recurrent Self-similar Sets

Consider the set, F, in Figure 7.1. The set is constructed using two sets, U and V, with two associated similarity transformations each; namely $T_{U,1}$, $T_{U,2}$, $T_{V,1}$ and $T_{V,2}$ respectively. The transformation $T_{U,1}$ has contraction $r_1 = \frac{1}{2}$ and rotates U by 30 degrees counterclockwise. The transformation $T_{U,2}$ has contraction $r_2 = \frac{1}{4}$ and rotates U by 60 degrees clockwise. For V, we have the transformation $T_{V,1}$ which has contraction $r_3 = \frac{1}{2}$ and rotates V by 90 degrees counterclockwise. The transformation $T_{V,2}$ has contraction $r_4 = \frac{3}{4}$ and rotates V by 120 degrees clockwise.

We have that

$$U = T_{U,1}(U) \cup T_{U,2}(V) \tag{7.9}$$

and

$$V = T_{V,1}(U) \cup T_{V,2}(V). \tag{7.10}$$

Now, since $T_{U,1}(U) \cap T_{U,2}(V) = \emptyset$ and $T_{V,1}(U) \cap T_{V,2}(V) = \emptyset$ we have that

$$\mathscr{H}^s(U) = \mathscr{H}^s(T_{U,1}(U)) + \mathscr{H}^s(T_{U,2}(V)) \tag{7.11}$$

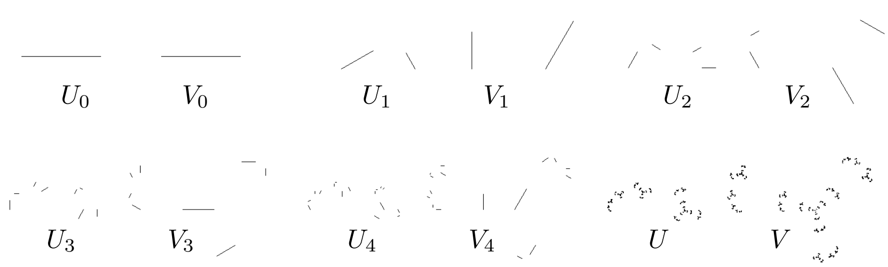


Figure 7.1: *Two-part dust, a recurrent self-similar set.*

and

$$\mathscr{H}^s(V) = \mathscr{H}^s(T_{V,1}(U)) + \mathscr{H}^s(T_{V,2}(V)). \tag{7.12}$$

Thus, by the scaling property of the Hausdorff measure we get

$$\mathscr{H}^s(U) = r_1^s \mathscr{H}^s(U) + r_2^s \mathscr{H}^s(V) \tag{7.13}$$

and

$$\mathscr{H}^s(V) = r_3^s \mathscr{H}^s(U) + r_4^s \mathscr{H}^s(V). \tag{7.14}$$

This is a linear relationship, and thus we can write the above equations as

$$\mathbf{v} = \begin{pmatrix} \mathscr{H}^s(U) \\ \mathscr{H}^s(V) \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} r_1^s & r_2^s \\ r_3^s & r_4^s \end{pmatrix}, \tag{7.15}$$

which gives the neat linear equation

$$\mathbf{v} = \mathbf{M}\mathbf{v}. \tag{7.16}$$

We notice that $\mathbf{v}$ is an eigenvector of $\mathbf{M}$, with eigenvalue 1 and move right along to the next section.

7.3 Hausdorff Dimension of Graph-directed Constructions

There is a generalization to self-similar sets that allow us to study the dimension of a much larger class of sets. The theory of the last two sections is generalized to directed multigraphs, representing a set, from which the Hausdorff and Box-counting dimensions are extracted.

With a finite set of vertices and directed edges, allowing more than one edge between vertices, self-similar sets can be described. Each node corresponds to a subset of the set, and the weight on an edge corresponds to a similarity ratio. Given such a graph, $G = (V, E)$, an iterated function system realizing the graph is set up as follows. Each vertex, v, corresponds to a compact metric space, U_i, and each edge, $e \in E$, corresponds to similarities S_e, with similarity ratio $w(e)$ such that $T_e : \mathbb{R}^n \to \mathbb{R}^n$. The invariant set for such an iterated function system is such that for each i

$$U_i = \bigcup \{T_{ij}(U_j) : (i,j) \in E\}, \tag{7.17}$$

and the construction object is defined by

$$F = \bigcup_{i=1}^{n} U_i. \tag{7.18}$$

We assume that all similarity mappings are disjoint.

The set of similarities, $\{T_e : e \in E\}$, is called a *graph-directed iterated function system*, and the sets $\{U_1, \ldots, U_n\}$ are called *graph-directed sets*.

The dimension of such graph-directed sets is given by an associated $n \times n$ adjacency matrix with elements

$$A_{i,j}^{(s)} = \sum_{e \in E_{i,j}} w(e)^s. \tag{7.19}$$

Such a matrix is easily created from a given graph. Remember the set, F, from the last section, see Figure 7.1 on page 52. The graph of this set would look like the graph in Figure 7.2. The matrix for the graph in Figure 7.2 would thus be

$$A^{(s)} = \begin{pmatrix} \frac{1}{2}^s & \frac{1}{4}^s \\ \frac{1}{2}^s & \frac{3}{4}^s \end{pmatrix}. \tag{7.20}$$

As we saw in the previous section, the eigenvalues and eigenvectors of the adjacency matrix are important. Indeed they are, and we need the following definition of the spectral radius of a matrix before we continue:

Definition 7.3.1: *Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of a matrix M. Then the spectral radius, $\rho(M)$ of M is defined as*

$$\rho(M) := \max_{1 \le i \le n} |\lambda_i|. \tag{7.21}$$

It is easily shown that $\rho(M) = \lim_{k\to\infty} \|M^k\|^{1/k}$, for any matrix norm $\|\cdot\|$, see Proposition 8.4.9.

We need the following lemma to continue [Axe96]:

Lemma 7.3.2 (The Perron-Frobenius Theorem)**:** *Suppose A is a real, non-negative $n \times n$ matrix whose underlying directed graph, G, is strongly connected. Then*

- ***a)*** *$\rho(A)$ is a positive real eigenvalue of A.*
- ***b)*** *There is a positive eigenvector that corresponds to $\rho(A)$.*
- ***c)*** *$\rho(A)$ increases if any element a_{ij} of A increases.*
- ***d)*** *$\rho(A)$ is a simple eigenvalue of A.*

We know the following about the spectral radius of a matrix $A^{(s)}$ [MW88]:

Lemma 7.3.3: *Let $\Phi(s) = \rho(A^{(s)})$. $\Phi(0) \ge 1$, Φ is continuous, strictly decreasing and $\lim_{k\to\infty} \Phi(k) = 0$.*

Thus, by Lemma 7.3.3, there is a unique value for which $\rho(A^{(s)}) = 1$, which is what we need from Equation 7.16 on the preceding page. This value of s is, as we shall see, the Hausdorff dimension of the set.

For the matrix in Equation 7.20 it is easily verified that $\rho(A^{(1)}) = 1$, and thus the Hausdorff dimension of the two-part dust should be $\dim_{\mathrm{H}} F = 1$.

We have the following theorem [Fal97]:

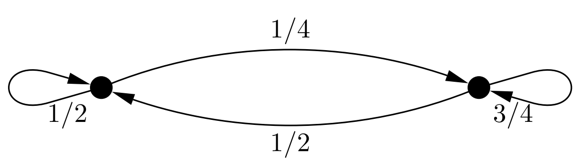


Figure 7.2: *Graph construction for the two-part dust.*

Theorem 7.3.4: *Let* $E_1, \ldots, E_n$ *be a family of graph-directed sets, and* $\{T_{(i,j)}\}$, *be a strongly connected graph-directed iterated function system without overlaps. Then there is a number* s *such that* $\dim_{\mathrm{H}} E_i = \underline{\dim}_{\mathrm{B}} E_i = \overline{\dim}_{\mathrm{B}} E_i = s$ *and* $0 < \mathscr{H}^s(E_i) < \infty$ *for all* $i = 1, \ldots, n$. *Also,* s *is the unique number satisfying* $\rho(A^{(s)}) = 1$.

Proof. Omitted, but can be found in [Fal97].

The proof establishes that, indeed

$$\begin{pmatrix} \mathscr{H}^s(E_1) \\ \vdots \\ \mathscr{H}^s(E_n) \end{pmatrix} = A^{(s)} \begin{pmatrix} \mathscr{H}^s(E_1) \\ \vdots \\ \mathscr{H}^s(E_n) \end{pmatrix}, \tag{7.22}$$

for $s = \dim_{\mathrm{H}} E_i$. It uses Lemma 7.3.2 to say that $(\mathscr{H}^s(E_1), \ldots, \mathscr{H}^s(E_n))^T$ is a positive eigenvector of $A^{(s)}$ with eigenvalue 1, and that this must be the largest eigenvalue of $A^{(s)}$. Thus, $\rho(A^{(s)}) = 1$, and since $\rho(A^{(s)})$ is strictly decreasing with s by Lemma 7.3.3, s is uniquely specified by the condition $\rho(A^{(s)}) = 1$. □

The condition that the graph be strongly connected can be omitted. We have the following theorem [MW88]:

Theorem 7.3.5: *Each graph-directed construction has dimension* $s = \max\{s_H : H \in SC(G)\}$, *where* s_H *is the unique number such that* $\rho(H^{(s_H)}) = 1$. *The construction object,* F, *has positive and* σ*-finite* $\mathscr{H}^s$ *measure. Further,* $\mathscr{H}^s(F) < \infty$ *if and only if* $\{H \in SC(G) : s_H = s\}$ *consists of pairwise incomparable elements. This number* s *is such that* $\dim_{\mathrm{H}} F = \underline{\dim}_{\mathrm{B}} F = \overline{\dim}_{\mathrm{B}} F = s$.

Proof. Omitted, but can be found in [MW88].

It follows from Theorem 7.3.4 that $\dim_{\mathrm{H}} F = \underline{\dim}_{\mathrm{B}} F = \overline{\dim}_{\mathrm{B}} F = s$, since each strongly connected component is a graph-directed iterated function system in its own right. □

These theorems are best illustrated with an example.

Example 7.3.6: *Consider the curve in Figure 7.3. The curve in Figure 7.3 a), is made up of a reflected half size copy of a) and a reflected half size copy of b). The curve in Figure 7.3 b), is made up of a rotated copy of c) and a one-third size rotated copy of*

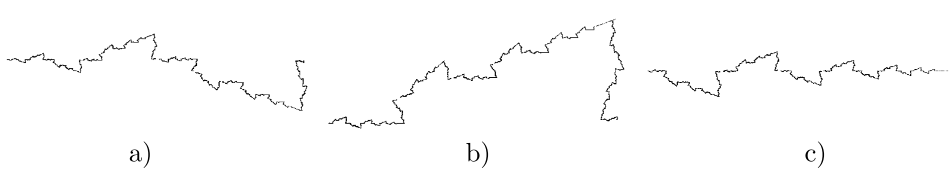


Figure 7.3: *An example of a graph-directed fractal set. The set in a) is made up of a reflected half size copy of itself and a reflected half size copy of b). The set in b) is made up of a rotated copy of c) and a one-third size rotated copy of a). The set in c) is made up of a reflected quarter size copy of b) and a reflected three-quarter size copy of itself.*

a). The curve in Figure 7.3 c), is made up of a reflected quarter size copy of b) and a reflected three-quarter size copy of c). The graph looks like the graph of Figure 7.4.

The matrix for this graph is thus

$$A^{(s)} = \begin{pmatrix} \frac{1}{2}^s & \frac{1}{2}^s & 0 \\ \frac{1}{3}^s & 0 & 1^s \\ 0 & \frac{1}{4}^s & \frac{3}{4}^s \end{pmatrix}. \tag{7.23}$$

Finding $\rho(A^{(s)}) = 1$ *yields that* $s \approx 1.128$*, and thus, this is the dimension of the graph.*

We can also note that Equation 7.1 on page 51 is exactly the one-node case of Equation 7.22 with two edges, i.e. Equation 7.8 on page 52 is the general case for the one-node case, the case with one node with n edges, each corresponding to a similarity or a contraction with contraction ratio r_i. Thus, Figure 7.5 is the graph for the set described by Equation 7.1.

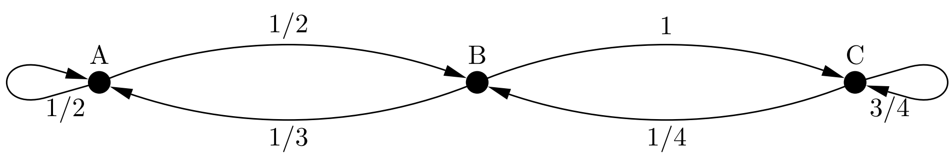


Figure 7.4: *A graph-directed construction of the set in Figure 7.3*

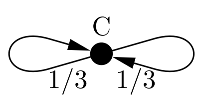


Figure 7.5: *The graph for the one-node case*

Chapter 8

Tree Constructions

The recursive structure of the iterated function system that was described in Chapter 6 suggests a tree structure as a means to describe the recursion. This is all right, but trees can in fact be seen as metric spaces in their own right, and therefore they have for example both Hausdorff measure and dimension.

Trees can also be described with two properties called *branching number* and *growth rate*. These numbers relate, as we will see, to the Hausdorff dimension and to the lower and upper Box-counting dimensions.

Some of the theory in this chapter is not as complete as it is in the other chapters. This thesis does not go in-depth on measure theory, and some of the more advanced topics that are needed for some of the arguments are not covered here. The reader should hopefully not have any problems to follow the text, however, and if more information is wanted, citations for further studies are given where appropriate.

The reader might need to read Chapter 4 first to fully appreciate this chapter.

8.1 The Tree – A Space of Strings

Most of the notation, and theory in this chapter (especially in this and the next section) is from [Edg90]. The theory is added here for completeness and out of courtesy to the reader. It is understood that it is more convenient to have the theory spelled out, than to have the reader refresh the notation and results from the source. However, it should also be noted that many of the proofs in this section are done by the author.

Consider finite rooted trees, i.e. trees in which each vertex has a finite number of edges connected to them, and which has a unique distinguished root. We denote the root as Λ and will consider the tree as a directed graph, in which each edge has the direction away from the root.

Consider a finite set of at least two *symbols*, or *letters*, e.g. $E = \{0, 1\}$, called an *alphabet* and consider *strings* made up of these symbols, e.g. 1011011. The symbols can be anything, but we will in general have N symbols and number them as $0, 1, \ldots, N-1$. The number of symbols in a string is called the *length* of the string and is written as $|\alpha|$ where α is a string. There is a unique string of length 0, the *empty string*, denoted by Λ.

If α and β are two strings, then we may form the string $\alpha\beta$ by *concatenating* the strings α and β, i.e. the symbols of α followed by the symbols of β.

We can let the edges of a tree be labeled with the symbols of our alphabet. For example, a *binary tree* is a tree in which each vertex has exactly two children. Let the two edges from a vertex be labeled with one of the symbols, 0 and 1, respectively.

The strings under consideration are then paths in the tree with Λ as its root. I.e., the set of all finite strings from the alphabet E can be identified with an infinite tree. In the binary tree example, the root is Λ, and if α is a string, $\alpha 0$ is the left child, and $\alpha 1$ is the right child of α.

We will write $E^{(n)}$ for the set of all strings of length n from the alphabet E. We write $E^{(*)}$ for the set of all finite strings. I.e.

$$E^{(*)} = E^{(0)} \cup E^{(1)} \cup E^{(2)} \cup E^{(3)} \cup \cdots$$

is the set of all finite strings.

The string α is called an *ancestor* of the string β if we can write $\beta = \alpha\gamma$, for some γ. I.e. α is the *initial segment*, or *prefix* of β. If so, we will write $\alpha \leq \beta$. If $|\alpha| \geq n$, then $\alpha \restriction n$ is the initial segment of α of length n. We denote the set of all infinite strings of the alphabet E as $E^{(\omega)}$.

The *longest common prefix* for the strings in a set $A \subset E^{(\omega)}$ is the *greatest lower bound* for the set, i.e. the unique longest string β such that $\beta \leq \gamma$ for all $\gamma \in E^{(\omega)}$. The following proposition proves that the greatest lower bound in fact is unique:

Proposition 8.1.1: *Every nonempty subset A of an infinite tree $E^{(*)}$ has a unique greatest lower bound.*

Proof. Let $\gamma \in A$ and $n = |\gamma|$. From the integers $0 \leq k \leq n$, there is some k which is the unique lower bound, $\gamma \restriction k$, for A. Let $\gamma \restriction k_0$ be the greatest lower bound for A and let β be any other lower bound. Then $\beta \leq \gamma$ and thus $\beta = \gamma \restriction k$, for some $0 \leq k \leq n$. We know that $k \leq k_0$, and therefore $\beta \leq \gamma \restriction k_0$. This implies that $\gamma \restriction k_0$ is the greatest lower bound of A. If both α and β are greatest lower bounds for A, then each is less than or equal to the other, and thus they are equal. □

We can also state the following equivalent proposition:

Proposition 8.1.2: *Every nonempty subset A of a tree $E^{(\omega)}$ of infinite strings has a unique greatest lower bound.*

Proof. Let $\gamma \in A$. There is some integer k for which $\gamma \restriction k$ is a greatest lower bound for A. Let $\beta \in A$ and $\beta \restriction l$ be some other lower bound for A. Then $\beta \restriction l \leq \gamma \restriction k$ and thus $\beta \restriction l = \gamma \restriction l$ and therefore $0 \leq l \leq k$ for some l. Then $\gamma \restriction k$ really is the greatest lower bound of A. If two strings $\alpha, \beta \in E^{(*)}$ are both greatest lower bounds for A, then each is less than or equal to the other, and therefore they are equal. □

If $\alpha \in E^{(*)}$, let

$$[\alpha] = \{\sigma \in E^{(\omega)} : \alpha \leq \sigma\}$$

be the set of all strings from the alphabet E that begins with the string α. We have the following proposition for the sets $[\alpha]$:

Proposition 8.1.3: *The set $[\alpha]$ has diameter $r^{|\alpha|}$, for all $\alpha \in E^{(*)}$.*

Proof. Any two strings $\sigma, \tau \in [\alpha]$, where $\alpha \in E^{(*)}$ have at least $|\alpha|$ letters in common. I.e. the diameter of $[\alpha]$ cannot be greater than $r^{|\alpha|}$ since for any integers l and k such that $l \geq k$ we have $r^k \geq r^l$ and $|\sigma|, |\tau| \geq |\alpha|$. This is of course true for all α. □

We can define a function, $h : E^{(\omega)} \to \mathbb{R}$, to act on the set of strings of $E^{(\omega)}$, and map each string to a real number. Consider for example the binary tree: If h adds a decimal point to the left of the string, and is considered a decimal expansion in base 2, the range of h is exactly $[0, 1]$.

The sets of interest, such as $[0, 1]$, are related to $E^{(\omega)}$, the "model", by a function $h : E^{(\omega)} \to \mathbb{R}$, called the "model map". Sometimes, the string σ is called the address of the points $h(\sigma)$.

We define a metric, ρ_r, $0 < r < 1$, on $E^{(\omega)}$ as follows. We want strings that have the same prefix to be considered similar, and, of course, if they are equal, they should have distance zero, i.e. $\rho(\sigma, \tau) = 0$ if $\sigma = \tau$. If two strings are not equal, they have some number of letters in common, possibly none. We can write

$$\begin{aligned} \sigma &= \alpha\sigma' \\ \tau &= \alpha\tau', \end{aligned}$$

and α is the longest common prefix of σ and τ. Let $k = |\alpha|$, then we define the metric to be

$$\rho(\sigma, \tau) = r^k.$$

We have the following proposition:

Proposition 8.1.4: *The set $E^{(\omega)}$ is a metric space with metric ρ_r.*

Proof. Obviously $\rho_r(\sigma, \tau) \geq 0$, since $k \geq 0$. $\rho_r(\sigma, \tau) = \rho_r(\tau, \sigma)$ is also clear. If $\sigma \neq \tau$, then $\rho_r(\sigma, \tau) = r^k > 0$.

Let $\sigma, \tau, \theta \in E^{(\omega)}$. If any of the three strings are equal, the following is trivial, so assume that they are all different. Let α be the longest common prefix of σ and θ, and let β be the longest common prefix of θ and τ. Let $n = \min\{|\alpha|, |\beta|\}$. Then we know that the first n letters of σ are equal to the first n letters of θ, and also that the first n letters of θ are equal to the first n letters of τ. Thus, the longest common prefix of σ and τ is of length at least n, i.e.

$$\begin{aligned} \rho_r(\sigma, \tau) &\leq r^n = r^{\min\{|\alpha|, |\beta|\}} \\ &= \max\{r^{|\alpha|}, r^{|\beta|}\} \\ &= \max\{\rho_r(\sigma, \theta), \rho_r(\theta, \tau)\} \\ &\leq \rho_r(\sigma, \theta) + \rho_r(\theta, \tau). \end{aligned}$$

□

For the next proposition we first need the following two definitions [Edg90]:

Definition 8.1.5: *A* Cauchy sequence *in a metric space S is a sequence (x_n) satisfying: for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ so that $\rho(x_n, x_m) < \varepsilon$ for all $n, m \geq N$.*

Definition 8.1.6: *A metric space S is called* complete *if and only if every Cauchy sequence in S converges in S.*

Now we claim that [Edg90]:

Proposition 8.1.7: *The space $(E^{(\omega)}, \rho_r)$ is complete.*

Proof. Let (σ_n) be a Cauchy sequence in $E^{(\omega)}$. Let $\sigma \in E^{(\omega)}$; we will construct a sequence that converges to τ. For each k, there is an $n_k \in \mathbb{N}$ so that for all $n, m \geq n_k$ we have $\rho_r(\sigma_n, \sigma_m) < r^k$. Thus, $\sigma_{n_k} \upharpoonright k = \sigma_m \upharpoonright k$ for all $m \geq n_k$. We define τ as: The kth letter of τ equals the kth letter of σ_{n_k}, thus $\tau \upharpoonright k = \sigma_{n_k} \upharpoonright k$, for all k. Let $\varepsilon > 0$ and choose k so that $r^k < \varepsilon$. Then for $m > n_k$ we have $\sigma_m \upharpoonright k = \tau \upharpoonright k$, so $\rho_r(\sigma_m, \tau) \leq r^k < \varepsilon$. I.e. $\sigma_n \to \tau$. □

For the next proposition we first need some definitions and a lemma [Edg90]:

Definition 8.1.8: *Let S be a metric space and let $A \subseteq S$. A point $x \in S$ is an* accumulation point *of A if and only if, for every $\varepsilon > 0$, the open ball $B^o_\varepsilon(x)$ contains points of A other than x.*

Definition 8.1.9: *A metric space S is called* countably compact *if and only if every infinite subset of S has at least one accumulation point in S.*

Definition 8.1.10: *A metric space S is called* compact *if it is countably compact.*

Lemma 8.1.11: *A countably compact metric space is separable.*

Proof. Omitted, see [Edg90] for details. □

Proposition 8.1.12: *The space $(E^{(\omega)}, \rho_r)$ of N symbols, $E \in \{0, \ldots, N-1\}$ and $N \geq 2$, is compact and separable.*

Proof. We prove that $E^{(\omega)}$ is countably compact, then it follows from Definition 8.1.10 and Lemma 8.1.11 that it is compact and separable.

Let $A_0 \subseteq E^{(\omega)}$ be infinite. Then at least one of $A_0 \cap [0], A_0 \cap [1], \ldots, A_0 \cap [N-1]$ is infinite. Select one of the infinite subsets, and call it $A_1 = A_0 \cap [i]$, where i is the first symbol of the infinite subset. Now, since A_1 is infinite, at least one of $A_1 \cap [\sigma]$, $\sigma \in E^{(2)}$ is infinite. Select one of the infinite subsets and call it $A_2 = A_1 \cap [\sigma]$. This argument holds forever since every A_i is infinite, and thus, the distance between strings in $[\sigma]$ tends towards zero. Therefore, there is some $\varepsilon > 0$ for which an open ball centered in A_i contains more than one point. □

A more useful way of defining a metric on $E^{(\omega)}$, at least for our purposes, than we have done before is the following. Assign a positive real number w_α to each node α of the tree $E^{(*)}$. With the correct conditions, this is a metric on $E^{(\omega)}$ such that $[\alpha]$ have diameter exactly w_α. We have the following proposition [Edg90]:

Proposition 8.1.13: *Let a family w_α of real number be given for each node α of the tree $E^{(*)}$. Define a metric ρ as follows. If $\sigma = \tau$ then $\rho(\sigma, \tau) = 0$. If $\sigma \neq \tau$, then $\rho(\sigma, \tau) = w_\alpha$, where α is the longest common prefix of σ and τ. If*

$$w_\alpha > w_\beta \text{ when } \alpha < \beta$$

and

$$\lim_{n\to\infty} w_{\alpha\restriction n} = 0 \text{ for } \alpha \in E^{(\omega)},$$

then ρ *is a metric on* $E^{(\omega)}$ *such that* $\operatorname{diam}[\alpha] = w_\alpha$ *for all* α.

Proof. $\rho(\sigma,\tau) \geq 0$ from the definition. If $\sigma \neq \tau$, then $\rho(\sigma,\tau) = w_\alpha > 0$, where α is the longest common prefix of σ and τ. $\rho(\sigma,\tau) = \rho(\tau,\sigma)$ is immediate. Thus, left to prove is the triangle inequality. Assume that σ,τ,θ are all different. Let α be the longest common prefix of σ and θ, β be the longest common prefix of θ and τ, and γ be the longest common prefix of σ and τ. Thus, since both α and β are prefixes of θ, then if $\alpha \leq \beta$ then α is a prefix of both σ and τ, so $\alpha \leq \gamma$, and therefore

$$\rho(\sigma,\tau) = w_\gamma \leq w_\alpha = \rho(\sigma,\theta) \leq \max\{\rho(\sigma,\theta),\rho(\theta,\tau)\} \leq \rho(\sigma,\theta) + \rho(\theta,\tau).$$

If $\sigma,\tau \in [\alpha]$, then the longest common prefix of σ and τ is $\beta \geq \alpha$, so $\rho(\sigma,\tau) = w_\beta \leq w_\alpha$. Thus $\operatorname{diam}[\alpha] \leq w_\alpha$. Let $E = \{\lambda_1,\ldots,\lambda_n\}$. Choose any $\sigma \in E^{(\omega)}$. Then $\alpha\lambda_1\sigma,\ldots,\alpha\lambda_n\sigma \in [\alpha]$ and $\rho(\alpha\lambda_i\sigma,\alpha\lambda_j\sigma) = w_\alpha$, for any $1 \leq i,j \leq n$. Therefore $\operatorname{diam}[\alpha] \geq w_\alpha$. □

The particular metric used for the metric space $E^{(\omega)}$ is of no importance when it comes to topological properties of the space. This is proven for our two different kind of measures in the following propositions (the first proposition is from [Edg90]):

Proposition 8.1.14: *The metric spaces constructed from* $E^{(\omega)}$ *using the metric* ρ_r*, for all* r*, are all homeomorphic to each other.*

Proof. Let $0 < r,s < 1$. If $h : E^{(\omega)} \to E^{(\omega)}$ is the identity function $h(\sigma) = \sigma$, then h is a homeomorphism from $(E^{(\omega)},\rho_r)$ to $(E^{(\omega)},\rho_s)$. It is enough to show that h is continuous, since interchanging r and s shows that h^{-1} is continuous.

Let $\varepsilon > 0$, and choose k such that $s^k < \epsilon$ and $\delta = r^k$. If $\sigma,\tau \in E^{(\omega)}$ with $\rho_r(\sigma,\tau) < \delta$, then $\rho_r(\sigma,\tau) < r^k$, so σ and τ have at least their k first letters in common. But then $\rho_s(\sigma,\tau) < s^k < \varepsilon$. Thus, h is continuous. □

Proposition 8.1.15: $(E^{(\omega)},\rho)$ *is homeomorphic to* $(E^{(\omega)},\rho_r)$.

Proof. Let $h : E^{(\omega)} \to E^{(\omega)}$ be $h(\sigma) = \sigma$, the identity function. Given $\varepsilon > 0$, choose a k such that $r^{-k} < \varepsilon$. Also let $\delta = w_\alpha$ when $|\alpha| = k$. Thus, if $\sigma,\tau \in E^{(\omega)}$ with $\rho(\sigma,\tau) < \delta$ then σ and τ have at least k letters in common, and therefore $\rho_r(\sigma,\tau) \leq r^{-k} < \varepsilon$. I.e. h is continuous.

Now, let $h^{-1} : E^{(\omega)} \to E^{(\omega)}$ is $h^{-1}(\sigma) = \sigma$. Given $\delta > 0$, choose k such that $w_\alpha < \delta$, when $|\alpha| = k$. Let $\varepsilon = r^{-k}$. Then, if $\sigma,\tau \in E^{(\omega)}$ with $\rho_r(\sigma,\tau) < \varepsilon$ then σ and τ have at least k letters in common, and therefore $\rho(\sigma,\tau) \leq w_\alpha < \delta$. Hence, h^{-1} is also continuous, and therefore $(E^{(\omega)},\rho)$ and $(E^{(\omega)},\rho_r)$ are homeomorphic. □

The following proposition tells us a little more about the sets from $E^{(\omega)}$.

Proposition 8.1.16: *The set* $[\alpha]$ *is an open ball in the space* $(E^{(\omega)},\rho_r)$.

Proof. Let $x \in E^{(\omega)}$ and $z \in [\alpha]$ where $\alpha = x \upharpoonright k$ for some k. Also, let $y \in B^o_{r^{|\alpha|}}(x)$. Then $\rho_r(z,x) \leq \max\{\rho_r(z,y), \rho_r(y,x)\} < r^{|\alpha|}$. Thus, for any z the distance to x is $\rho_r(z,x) < r^{|\alpha|}$, hence $[\alpha] \subseteq B^o_{r^{|\alpha|}}(x)$.

But also, we have that $\rho_r(y,z) \leq \max\{\rho_r(y,x), \rho_r(x,z)\} \leq r^{|\alpha|}$. I.e. all points of $B^o_{r^{|\alpha|}}(x)$ is within distance $r^{|\alpha|}$ of any point of $[\alpha]$. Hence, $[\alpha] \supseteq B^o_{r^{|\alpha|}}(x)$. □

Surprisingly enough, we have the following proposition as well:

Proposition 8.1.17: *The set $[\alpha]$ is a closed ball in the space $(E^{(\omega)}, \rho_r)$.*

Proof. We know from Proposition 8.1.16 that $[\alpha]$ is an open ball. Then, if σ is the center of $[\alpha]$, any other point, $\tau \in [\alpha]$, is within distance $\rho(\sigma,\tau) < r^{|\alpha|}$ from σ. Let $B_{r^{|\alpha|}}(\sigma)$ be a closed ball centered in σ, then $[\alpha] \subseteq B_{r^{|\alpha|}}(\sigma)$.

But also, if we let $\gamma \in B_{r^{|\alpha|}}(\sigma)$ then $\rho_r(\gamma,\tau) \leq \max\{\rho_r(\gamma,\sigma), \rho_r(\sigma,\tau)\} \leq r^{|\alpha|}$. I.e. all points of $B_{r^{|\alpha|}}(\sigma)$ is within distance $r^{|\alpha|}$ of any point of $[\alpha]$ and therefore $[\alpha] \supseteq B_{r^{|\alpha|}}(\sigma)$. □

Thus, the sets $[\alpha]$ are *clopen* sets. In fact we can say even more about the open balls of $E^{(\omega)}$. We have the following proposition.

Proposition 8.1.18: *The countable set $\{[\alpha] : \alpha \in E^{(*)}\}$ is equal to the set $\{B^o_\varepsilon(\sigma) : \sigma \in E^{(\omega)}, \varepsilon > 0\}$ of all open balls, and to the set $\{B_\varepsilon(\sigma) : \sigma \in E^{(\omega)}, \varepsilon > 0\}$ of all closed balls.*

Proof. The only possible distances are r^k for all k, and for each k there is an open, and equally closed, ball that is a set $[\alpha]$ for some $\alpha \in E^{(*)}$. Therefore there are no other open, or closed, balls. □

Now, the sets $[\alpha]$ makes up a special class of sets. We have the following proposition:

Definition 8.1.19: *A family $\mathscr{B}$ of open subsets of a metric space S is called a* base for the open sets *of S if and only if for every open set $A \subseteq S$, and every $x \in A$, there is $U \in \mathscr{B}$ such that $x \in U \subseteq A$.*

Proposition 8.1.20: *In the metric space $(E^{(\omega)}, \rho_r)$, the set*

$$\{[\alpha] : \alpha \in E^{(*)}\}$$

is a countable base for the open sets.

Proof. The set $\{[\alpha] : \alpha \in E^{(*)}\}$ is trivially countable. The number of strings of length k is finite, and thus enumerable. Start with $k = 0$ and let $k \to \infty$.

Let $A \subset E^{(\omega)}$ be an open set and let $\sigma \in A$. Now, by the definition of an open set, there is some $\varepsilon > 0$ such that

$$x \in B^o_\varepsilon(\sigma) \subseteq A.$$

Now, since $[\alpha]$ is an open ball by Proposition 8.1.16, select an α for which the radius is $< \varepsilon$, and thus $\{[\alpha] : \alpha \in E^{(*)}\}$ is a base for the open sets by Definition 8.1.19. □

An *ultrametric space* S is a metric space in which the metric ρ satisfies the *ultrametric triangle inequality*:

$$\rho(x, y) \le \max\{\rho(x, z), \rho(z, y)\}.$$

We state the following proposition, which we actually have already proven:

Proposition 8.1.21: *The metric spaces* $(E^{(\omega)}, \rho_r)$ *and* $(E^{(\omega)}, \rho)$ *of two or more symbols are ultrametric spaces.*

Proof. We proved in Proposition 8.1.4 that in fact, for $\sigma, \tau, \theta \in E^{(\omega)}$,

$$\rho_r(\sigma, \tau) \le \max\{\rho_r(\sigma, \theta), \rho_r(\theta, \tau)\}.$$

and in Proposition 8.1.13 that

$$\rho(\sigma, \tau) \le \max\{\rho(\sigma, \theta), \rho(\theta, \tau)\}.$$

□

Ultrametric spaces are somewhat exotic, and fulfills some non-intuitive properties. We state and prove some of these in the following lemma:

Lemma 8.1.22: *Let* S *be an ultrametric space with metric* ρ*. Then*

- ***a)*** *Every triangle is isosceles: If* $x, y, z \in S$*, then at least two of* $\rho(x, y)$*,* $\rho(x, z)$*,* $\rho(y, z)$ *are equal.*
- ***b)*** *A ball* $B^o_r(x)$ *of radius* r *has diameter at most* r*.*
- ***c)*** *Every point of a ball is a center: If* $y \in B^o_r(x)$*, then* $B^o_r(x) = B^o_r(y)$*.*
- ***d)*** *A closed ball is an open set.*
- ***e)*** *An open ball is a closed set.*

Proof. a) If $\rho(x, y)$, $\rho(x, z)$, $\rho(y, z)$ are all equal, the result is trivial, so assume that at least two of them are different. We must then have $\rho(x, y) \le \rho(x, z)$. If not, interchange the variables to fit the argument. Then we have

$$\begin{aligned} \rho(x, y) \le \rho(x, z) &\le \max\{\rho(x, y), \rho(y, z)\} \\ &\le \rho(y, z) \\ &\le \max\{\rho(y, x), \rho(x, z)\} \\ &\le \rho(x, z), \end{aligned}$$

and thus $\rho(x, z) = \rho(y, z)$.

b) Assume that $y, z \in S$ are as far away from each other as possible, i.e. the diameter of $B^o_r(x)$ apart. Then $r = \rho(x, y) = \rho(x, z)$, but $\rho(y, z) \le \max\{\rho(y, x), \rho(x, z)\} \le \rho(y, x) = \rho(x, z) = r$. I.e., the diameter is at most r.

c) We know from the definition of an open ball, $B^o_r(x) = \{z \in S : \rho(x, z) < r\}$, that $\rho(x, y) < r$ and $\rho(x, z) < r$. We also know that $\rho(x, y) \le \max\{\rho(x, z), \rho(z, y)\} \le r$ by b), and thus also $\rho(y, z) \le \max\{\rho(y, x), \rho(x, z)\} \le r$. I.e. all points within distance r from x are within distance r from y.

d) We know from the definition of a closed ball, $B_r(x) = \{z \in S : \rho(x, z) \le r\}$, that $\rho(x, y) \le r$ and $\rho(x, z) \le r$. We also know that $\rho(x, y) \le \max\{\rho(x, z), \rho(z, y)\} \le r$ by b), and thus also $\rho(y, z) \le \max\{\rho(y, x), \rho(x, z)\} \le r$. I.e. all points within distance r

from x is within distance r from y. Thus, all points of $B_r(x)$ is an interior point, since for all $y \in S$, $B_\varepsilon(y) \subseteq B_r(x)$, for $\varepsilon = r$. I.e. $B_r(x)$ is an open set.

e) Follows directly from c). □

Since both $(E^{(\omega)}, \rho_r)$ and $(E^{(\omega)}, \rho)$ are ultrametric spaces, they have the above properties.

We will now define measures on the space $E^{(\omega)}$. Let E be an alphabet with at least two symbols. Suppose a non-negative number w_α is given for each finite string α. The set $[\alpha]$ is the disjoint union of the sets $[\beta]$ when β is anyone of the children of α, i.e. $\beta = \alpha e$, where $e \in E$. We have the following proposition [Edg90]:

Proposition 8.1.23: *Suppose the non-negative number w_α satisfy*

$$w_\alpha = \sum_{e \in E} w_{\alpha e} \tag{8.1}$$

for all $\alpha \in E^{(*)}$. Then there is a measure on $E^{(\omega)}$ with $\mu([\alpha]) = w_\alpha$.

Proof. Omitted. The particular measure theory required is not covered by this thesis. See [Edg90] for details. □

We could deduce many useful results from the above proposition, but for out purposes, we are more interested in *path forests*, and therefore dedicate the next section to such constructions.

8.2 Hausdorff Dimension of Path Forests

Consider the constructions by graphs in Chapter 7 – the same sets can be represented by trees. We will consider the set $E^{(*)}$ of all finite paths in a graph, $G = (V_G, E_G)$, which naturally has the structure of a tree. If α is a path, then the children of α are αe, where $e \in E$. Since we have a finite number of "natural" roots of the tree, i.e. each note of the multigraph, this if not *a* tree, but more accurately a *disjoint union of trees*, one tree $E_v^{(*)}$ corresponding to each node $v \in V_G$ of the graph. A disjoint union of trees is sometimes called a *forest* and we will therefore call them *path forests*.

Naturally, we will also denote the set of all infinite path starting in $v \in V$ by $E_v^{(\omega)}$. Thus, there will be one of these path spaces for each note in the graph or equivalently, one for each tree in the path forest.

Using the terminology we have already introduced, the set of all paths that begins with α are the paths of the set $[\alpha] = \{\sigma \in E^{(\omega)} : \alpha \leq \sigma\}$.

We can introduce metrics on these path spaces $E_v^{(\omega)}$ just as was done before, but we must be aware of such cases as when some node does not have any children, or only one child. If $[\alpha]$ have no children, then naturally $[\alpha] = \emptyset$, and therefore its diameter must be 0. If α has only one child β, then $[\alpha] = [\beta]$ and so $\operatorname{diam}[\alpha] = \operatorname{diam}[\beta]$.

We define the metric similarly to how we did before. Let w_α be a family of positive real numbers, one for each node in the path forest $E^{(*)}$. We define several disjoint metrics, one for each metric space $E_v^{(\omega)}$. The family w_α satisfies

$$w_\alpha > w_\beta \ \text{ if } \alpha < \beta \tag{8.2}$$

and

$$\lim_{n\to\infty} w_\alpha = 0 \quad \text{for } \sigma \in E^{(\omega)}. \tag{8.3}$$

The definition for the metric is as before. If $\sigma, \tau \in E_v^{(\omega)}$, and $\sigma \neq \tau$, then they have at least the longest common prefix Λ_v. So we define $\rho(\sigma, \tau) = w_\alpha$, for α the longest common prefix of σ and τ.

Now, let $(r_1, \ldots, r_n)$ be a contraction ratio list, and $E^{(\omega)}$ be the space of infinite strings from the alphabet E as before. The letters of E are of no importance, but we need to have a one-to-one mapping such that $(r_e)_{e\in E}$ for $(r_1, \ldots, r_n)$.

For each letter $e \in E$ there is a function $f_e : E^{(\omega)} \to E^{(\omega)}$, called a *right shift*, which is defined as

$$f_e(\sigma) = e\sigma. \tag{8.4}$$

Now, there is a metric on $E^{(\omega)}$ such that the right shifts form a realization of the ratio list.

The metric is defined as follows. For each node $\alpha \in E^{(*)}$, if there are at least two letters in E, then there are numbers w_α such that $\operatorname{diam}[\alpha] = w_\alpha$. We define this as

$$w_\Lambda = 1, \tag{8.5}$$

$$w_{\alpha e} = w_\alpha r_e \quad \text{for } \alpha \in E^{(*)} \text{ and } e \in E. \tag{8.6}$$

Thus, w_α is the product of the ratios r_e corresponding to the letters e that make up α.

We now have the following proposition [Edg90]:

Proposition 8.2.1: *The functions $(f_e)_{e\in E}$ is an iterated function system that realizes the ratio list $(r_e)_{e\in E}$.*

Proof. Suppose $\sigma, \tau \in E^{(\omega)}$ have longest common prefix α. If $e \in E$, then the longest common prefix of $e\sigma$ and $e\tau$ is $e\alpha$, so

$$\rho\big(f_e(\sigma), f_e(\tau)\big) = w_{e\alpha} = r_e w_\alpha = r_e \rho(\sigma, \tau). \tag{8.7}$$

Thus, f_e is a similarity on $(E^{(\omega)}, \rho)$ with ratio r_e. □

The metric space $(E^{(\omega)}, \rho)$ is complete, and thus the right-shift realization $(f_e)_{e\in E}$ has a unique nonempty compact invariant set. The invariant set is the whole space $(E^{(\omega)}, \rho)$. The space $(E^{(\omega)}, \rho)$ together with the right-shifts is called the *string model* of the ratio list $(r_e)_{e\in E}$.

We have the following proposition [Edg90]:

Proposition 8.2.2 (The String Model Theorem)**:** *Let S be a nonempty complete metric space and let $(f_e)_{e\in E}$ be any iterated function system realizing the ratio list $(r_e)_{e\in E}$ in S. Assume that $r_e < 1$ for all e. Then there is a unique continuous function $h : E^{(\omega)} \to S$ such that*

$$h(e\sigma) = f_e\big(h(\sigma)\big) \tag{8.8}$$

for all $\sigma \in E^{(\omega)}$. The range $h(E^{(\omega)})$ is the invariant set of the iterated function system $(f_e)_{e\in E}$.

Proof. Omitted, since it rely on some theorems that we have not stated and some theory that we have not explained. It can be found in [Edg90], however. □

With the map h, the infinite string $\sigma \in E^{(\omega)}$ is sometimes referred to as the *address* of the point $x = h(\sigma)$.

Now, just as in Proposition 8.1.23, we can define measures on the path spaces E_v^{ω}. Select a vertex v. Suppose that non-negative numbers w_α are given, one for each $\alpha \in E_v^{(*)}$. The set $[\alpha]$ is the disjoint union of the sets $[\beta]$ when β is anyone of the children of α, i.e. $\beta = \alpha e$, where $e \in E$. We have the following proposition [Edg90]:

Proposition 8.2.3: *Suppose the non-negative number w_α satisfy*

$$w_\alpha = \sum_{\alpha \to \alpha e} w_{\alpha e} \tag{8.9}$$

for all $\alpha \in E_v^{(*)}$. Then there is a measure μ on $E_v^{(\omega)}$ with $\mu([\alpha]) = w_\alpha$.

Proof. Omitted. The particular measure theory required is not covered by this thesis. See [Edg90] for details. □

When we have a measure in a metric space, we can compute dimensions of sets of that space. We have the following example [Edg90]:

Example 8.2.4: *Let $E = \{0, 1\}$ be a two-letter alphabet, let $E^{(\omega)}$ be the space of all infinite strings using E, and let $\rho_{1/2}$ be the metric for $E^{(\omega)}$. The Hausdorff dimension of $(E^{(\omega)}, \rho_{1/2})$ is 1.*

Calculation. We will not give a complete proof of this, since we have not covered all theory, but the idea is as follows. We define a set function $C([\alpha]) = \left(\frac{1}{2}\right)^{|\alpha|}$, and create a measure, $\mu_{1/2}$, from this function. Now, the measure of $E^{(\omega)}$ is $\mu_{1/2}\left(E^{(\omega)}\right) = 1$, using this set function; this can be shown fairly easy. So the proof is showing that $\mathscr{H}^1 = \mu_{1/2}$, which implies that $\dim_{\mathrm{H}} E^{(\omega)} = 1$. By the definition of the kind of measure we created we can show that $\mathscr{H}^1 \leq \mu_{1/2}$ and $\mathscr{H}^1 \geq \mu_{1/2}$. See the details in [Edg90]. □

We have the following proposition [Edg90]:

Proposition 8.2.5: *Suppose non-negative numbers w_α satisfy*

$$w_\alpha = \sum_{\alpha \to \alpha e} w_{\alpha e} \tag{8.10}$$

for $\alpha \in E_v^{(*)}$. Let μ be the measure (mentioned above in Proposition 8.2.3) such that $\mu([\alpha]) = w_\alpha$. If ρ is a metric on $E_v^{(*)}$ and $s > 0$ satisfy $\mu([\alpha]) = (\mathrm{diam}[\alpha])^s$ for all $\alpha \in E_v^{(*)}$, then $\mu(B) = \mathscr{H}^s(B)$ for all Borel sets $B \subseteq E_v^{(*)}$.

We will now give an example where the above proposition is used [Edg90]:

Example 8.2.6: *We will consider the Cantor set. The ratio list is $(r_1, r_2) = (\frac{1}{3}, \frac{1}{3})$. The string model is the set $E^{(\omega)}$, with the alphabet $E = \{0, 1\}$, and the metric $\rho_{1/3}$. The similarities are the right-shifts, f_0 and f_1, defined as*

$$f_0(\sigma) = 0\sigma \tag{8.11}$$

and

$$f_1(\sigma) = 1\sigma. \tag{8.12}$$

Thus, (f_0, f_1) *is a realization of the ratio list* $(\frac{1}{3}, \frac{1}{3})$*, with invariant set* $E^{(\omega)}$*, by Proposition 8.2.2.*

Now, the Hausdorff dimension for $E^{(\omega)}$ *with metric* $\rho_{1/3}$ *is* $\log 2/\log 3$*. This, with the correct mapping, tells us that the Hausdorff dimension of the Cantor set is also* $\log 2/\log 3$*.*

Calculation. Let $s = \log 2/\log 3$. The measure $\mu_{1/2}$ is used as before without further explanation. If the length $|\alpha| = k$, then $\mu_{1/2}([\alpha]) = 2^{-k} = (3^{-k})^s = (\text{diam}[\alpha])^s$. Thus, by Proposition 8.2.5 and Example 8.2.4, $\mathscr{H}^s(E^{(\omega)}) = \mu_{1/2}(E^{(\omega)}) = 1$. So, $\dim_{\text{H}} E^{(\omega)} = s = \log 2/\log 3$.

Now, the model map $h : E^{(\omega)} \to \mathbb{R}$ that satisfy

$$h(0\sigma) = \frac{h(\sigma)}{3} \tag{8.13}$$

$$h(1\sigma) = \frac{h(\sigma)+2}{3}, \tag{8.14}$$

is bi-Lipschitz with

$$\frac{1}{3}\rho_{1/3}(\sigma,\tau) \leq |h(\sigma) - h(\tau)| \leq \rho_{1/3}(\sigma,\tau). \tag{8.15}$$

(See [Edg90] for details.) A bi-Lipschitz function preserves Hausdorff dimension, as was stated in Section 5.2.5, therefore the Hausdorff dimension of the Cantor set is $\log 2/\log 3$. □

Now, to generalize this, we use the metric defined before, where we defined the measure ρ so that $\text{diam}[\alpha] = w_\alpha$ for each node $\alpha \in E^{(*)}$. We define

$$w_\Lambda = 1, \tag{8.16}$$

$$w_{\alpha e} = w_\alpha r_e, \tag{8.17}$$

for $e \in E$.

Also, a ratio list $(r_1, \ldots, r_n)$ is given, with $n > 1$, such that

$$\sum_{i=1}^{n} r_i^s = 1, \tag{8.18}$$

where s is the Similarity dimension (remember Section 6.4). Then ρ is the metric such that the right-shifts realize the ratio list.

Now we create a measure on the string space $E^{(\omega)}$ such that

$$\mu([\alpha]) = w_\alpha^s = \sum_{i=1}^{n} (w_\alpha r_i)^s \tag{8.19}$$

for all α. I.e., s was chosen so that $\mu([\alpha]) = (\text{diam}[\alpha])^s$, and hence, by Proposition 8.2.5, we have $\mu(B) = \mathscr{H}^s(B)$ for all Borel sets $B \subseteq E^{(\omega)}$. Thus, we have proven the following theorem [Edg90]:

Theorem 8.2.7: *The Hausdorff dimension of the string model $E^{(\omega)}$ is equal to the similarity dimension s.*

The translation to other metric spaces, such as $\mathbb{R}^n$, is governed by the the model map $h : E^{(*)} \to S$. The dimension of the realization in other metric spaces is given by the following theorem [Edg90]:

Theorem 8.2.8: *Let K be the invariant set of a realization of similarities or contractions in a complete metric space S of a ratio list with Similarity dimension s. Then* $\dim_{\mathrm{H}} K \leq s$.

Proof. This follows immediately from the fact that the model map is a Lipschitz function. See [Edg90] for details. □

But if we put some restrictions on the iterated function system, such as an open set condition, then we get a much nicer result. Moran's open set condition, Definition 6.4.2 on page 49, implies for strings that $f_\alpha(U) \cap f_\beta(U) = \emptyset$ for two strings $\alpha, \beta \in E^{(*)}$ unless one is an initial segment of the other. We have the following theorem [Edg90]:

Theorem 8.2.9: *Let $(r_e)_{e\in E}$ be a ratio list. Let s be its dimension, and let $(f_e)_{e\in E}$ be a realization in $\mathbb{R}^n$. Let K be the invariant set. If Moran's open set condition is satisfied, then* $\dim_{\mathrm{H}} K = s$.

Now, going back to the path forests again, that we discussed above. Consider Figure 7.1 in Section 7.2. Each of the subsets U and V corresponds to one of the trees of the path forest, and the full set F is described by the entire path forest, see Figure 8.1.

The dimension of the path model can be computed in much the same way as was done for the single tree case above. However, a separate metric for each path space, i.e. for each tree in the path forest, need to be defined. Also, the diameters of $[\alpha]$ will be defined similarly, but with a small difference.

The letters $e \in E$ are thought of as *leading* to a node $\alpha \in E_v^{(*)}$. Thus, if e is a child of alpha, then we can write $\alpha \to \alpha e$. We have done this already above, and it does not

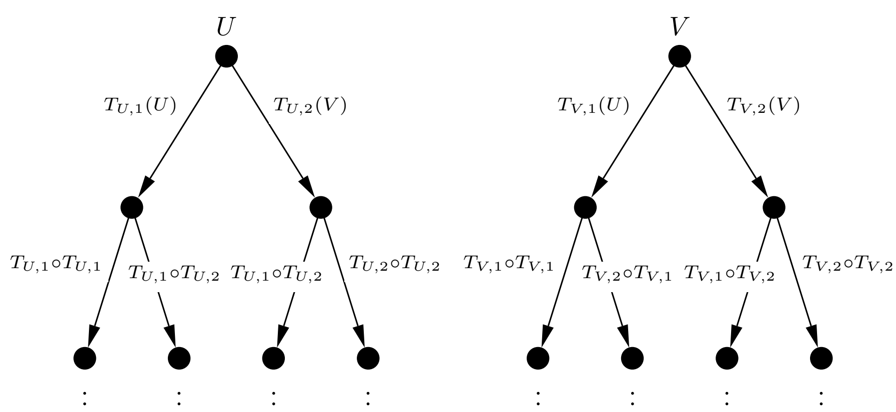


Figure 8.1: *Tree construction for the two-part dust. Each tree represent one of the sets U and V, and the forest represent the set F.*

change any of the theory from before, but will make the arguments here make much more sense.

We define metrics for each path space, so that $\mathrm{diam}[\alpha] = w_\alpha$, as before. But the diameters of the sets $[\alpha]$ is now defined as

$$w_{\Lambda_v} = q_v, \tag{8.20}$$

$$w_{e\alpha} = r_e w_\alpha, \tag{8.21}$$

for $e \in E$ and where q_v is the diameter of the tree rooted at the node v. Thus, $\mathrm{diam}[\Lambda_v] = w_{\Lambda_v} = q_v$ and $\mathrm{diam}[e\alpha] = r_e \cdot \mathrm{diam}[\alpha] = w_{e\alpha} = r_e w_\alpha$.

We need the right-shifts to realize the ratios r_e corresponding to the letters of the string space alphabet, i.e. the $e \in E$, and such that f_e is a similarity with ratio r_e. The right-shifts are defined as before by

$$f_e(\sigma) = e\sigma, \tag{8.22}$$

but are now though of as mapping a path space to another such that if $\sigma \in E_v^{(\omega)}$ then the mapping is to $E_u^{(\omega)}$, when $e = u$.

For each path α, we let w_α be defined as above, so that it is the product of the numbers r_e for each letter of α. For $\alpha \in E_{uv}^{(*)}$ we have $\mathrm{diam}[\alpha] = w_\alpha q_v$, where q_v is the constant mentioned above. Now, the metrics for the spaces $E_v^{(*)}$ satisfy

$$\rho(e\alpha, e\tau) = r_e \rho(\sigma, \tau) \tag{8.23}$$

for $\sigma, \tau \in E_v^{(*)}$.

The measures on the path spaces are defined as follows. Suppose w_α satisfy

$$w_\alpha = \sum_{\alpha \to \alpha e} w_{\alpha e} \tag{8.24}$$

for all $\alpha \in E_v^{(*)}$, as in Proposition 8.2.3, then the diameters of the sets $[\alpha]$ satisfy

$$(\mathrm{diam}[\alpha])^s = \sum_{\alpha \to \alpha e} (\mathrm{diam}[\alpha e])^s, \tag{8.25}$$

for some s. Then there exists a measure on each of the spaces $E_v^{(\omega)}$ satisfying $\mu([\alpha]) = (\mathrm{diam}[\alpha])^s$ for all $\alpha \in E_v^{(*)}$. Thus, by Proposition 8.2.5, $\mathscr{H}^s(E_v^{(\omega)}) = \mu(E_v^{(\omega)}) = q_v^s$ and since $0 < q_v < \infty$ we have $\dim_{\mathrm{H}} E_v^{(\omega)} = s$.

We now want to translate this to sets in S a general metric space, such as $\mathbb{R}^n$, that we are really interested in. This is done with the model maps, just as is done in the single tree case above. We have the following theorem [Edg90]:

Theorem 8.2.10: *Let (K_v), where v represents a tree in the path forest, be non-empty compact sets in $\mathbb{R}^n$. Let s be the number defined above. Then $\dim_{\mathrm{H}} K_v \leq s$ for all v.*

For the lower bound we also need the following definition [Edg90]:

Definition 8.2.11: *If (f_e) is a realization of a ratio list in $\mathbb{R}^n$, then it satisfies the* open set condition *if and only if there exist non-empty open sets U_v, one for each tree v, with*

$$U_u \supseteq f_e(U_v) \tag{8.26}$$

for all trees u *and* v *and* $e \in E_{uv}$ *and*

$$f_e(U_v) \cap f_{e'}(U_{v'}) = \emptyset \tag{8.27}$$

for all trees u *and* v *and* $e \in E_{uv}$, $e' \in E_{uv'}$ *where* $e \neq e'$.

Now we can state the following nice result [Edg90]:

Theorem 8.2.12: *If, in addition to Theorem 8.2.10, the realization, which consist of similarities or contractions, satisfies the open set condition of Definition 8.2.11, then* $\dim_{\mathrm{H}} K_v = s$.

8.3 Equivalence of Graph and Tree Constructions

Rearranging Equation 8.25, by equations Equation 8.20 and Equation 8.21, we see that

$$q_u^s = \sum_{\substack{e \in E_{uv} \\ v \text{ a tree}}} r_e^s \cdot q_v^s, \tag{8.28}$$

for all trees u.

Expanding the above equation for each tree $v_1, \dots, v_n$ we get

$$\begin{aligned} q_{v_1}^s &= r_{e_{v_1 v_1}}^s \cdot q_{v_1}^s + \cdots + r_{e_{v_1 v_n}}^s \cdot q_{v_n}^s \\ &\vdots \\ q_{v_n}^s &= r_{e_{v_n v_1}}^s \cdot q_{v_1}^s + \cdots + r_{e_{v_n v_n}}^s \cdot q_{v_n}^s \end{aligned}$$

which can be rewritten in matrix form as

$$\begin{pmatrix} q_{v_1}^s \\ q_{v_1}^s \\ \vdots \\ q_{v_1}^s \end{pmatrix} = \begin{pmatrix} r_{e_{v_1 v_1}}^s & r_{e_{v_1 v_2}}^s & \cdots & r_{e_{v_1 v_n}}^s \\ r_{e_{v_2 v_1}}^s & r_{e_{v_2 v_2}}^s & \cdots & r_{e_{v_2 v_n}}^s \\ \vdots & \vdots & \ddots & \vdots \\ r_{e_{v_n v_1}}^s & r_{e_{v_n v_2}}^s & \cdots & r_{e_{v_n v_n}}^s \end{pmatrix} \begin{pmatrix} q_{v_1}^s \\ q_{v_1}^s \\ \vdots \\ q_{v_1}^s \end{pmatrix}. \tag{8.29}$$

Hence, by Proposition 8.2.5, we can rewrite the above equation as

$$\begin{pmatrix} \mathscr{H}^s(v_1) \\ \mathscr{H}^s(v_2) \\ \vdots \\ \mathscr{H}^s(v_n) \end{pmatrix} = \begin{pmatrix} r_{e_{v_1 v_1}}^s & r_{e_{v_1 v_2}}^s & \cdots & r_{e_{v_1 v_n}}^s \\ r_{e_{v_2 v_1}}^s & r_{e_{v_2 v_2}}^s & \cdots & r_{e_{v_2 v_n}}^s \\ \vdots & \vdots & \ddots & \vdots \\ r_{e_{v_n v_1}}^s & r_{e_{v_n v_2}}^s & \cdots & r_{e_{v_n v_n}}^s \end{pmatrix} \begin{pmatrix} \mathscr{H}^s(v_1) \\ \mathscr{H}^s(v_2) \\ \vdots \\ \mathscr{H}^s(v_n) \end{pmatrix}, \tag{8.30}$$

and this equation can in turn be rewritten as

$$\begin{pmatrix} \mathscr{H}^s(E_{v_1}^{(\omega)}) \\ \mathscr{H}^s(E_{v_2}^{(\omega)}) \\ \vdots \\ \mathscr{H}^s(E_{v_n}^{(\omega)}) \end{pmatrix} = A^{(s)} \begin{pmatrix} \mathscr{H}^s(E_{v_1}^{(\omega)}) \\ \mathscr{H}^s(E_{v_2}^{(\omega)}) \\ \vdots \\ \mathscr{H}^s(E_{v_n}^{(\omega)}) \end{pmatrix}, \tag{8.31}$$

where

$$A^{(s)} = \begin{pmatrix} r^s_{e_{v_1v_1}} & r^s_{e_{v_1v_2}} & \cdots & r^s_{e_{v_1v_n}} \\ r^s_{e_{v_2v_1}} & r^s_{e_{v_2v_2}} & \cdots & r^s_{e_{v_2v_n}} \\ \vdots & \vdots & \ddots & \vdots \\ r^s_{e_{v_nv_1}} & r^s_{e_{v_nv_2}} & \cdots & r^s_{e_{v_nv_n}} \end{pmatrix}. \tag{8.32}$$

We note that

$$\begin{pmatrix} \mathscr{H}^s(E^{(\omega)}_{v_1}) \\ \mathscr{H}^s(E^{(\omega)}_{v_2}) \\ \vdots \\ \mathscr{H}^s(E^{(\omega)}_{v_n}) \end{pmatrix} \tag{8.33}$$

is an eigenvector of $A^{(s)}$ with eigenvalue 1, and conclude that this is exactly Equation 7.22 on page 55 as described in Chapter 7.

Thus, if the ratios are known, we can find the eigenvector with only positive values corresponding to eigenvalue 1. Following the arguments in Chapter 7: By the Perron-Frobenius theorem, Lemma 7.3.2, there is only one vector with positive values that corresponds to eigenvalue 1. Finding the s for which this is fulfilled gives the Hausdorff dimension of the underlying set.

Rellick, Edgar and Klapper used this fact in [REK91] to find the Hausdorff dimension of certain trees occurring naturally when describing enzymatic reaction pathways. Each edge of the tree represents the probability of a monomer (a constituent of a polymer, a chain of molecules) being added to the chain. Such a setting can be rewritten as a, not necessarily strongly connected, graph. This graph has an associated Hausdorff dimension, which can be used to determine the thermodynamic and kinetic consequences of a certain reaction. The authors show that it is possible to determine the Hausdorff dimension of systems which are modeled as tree structures. This thesis shows that results in the opposite direction are also possible, i.e. systems which are naturally modeled as graphs can be transformed to trees and investigated that way.

8.4 Representation by Trees

We will in general use the notation of [Lyo90] in this section. We label an edge by the label of its vertex farthest away from the root, i.e. a vertex has the same label as its preceding edge.

A *cutset* Π of a tree Γ is a finite set if vertices not including Λ such that for every vertex $\sigma \in \Gamma$, either $\sigma \leq \tau$ for some $\tau \in \Pi$ or $\tau \leq \sigma$ for some $\tau \in \Pi$, or $\{\tau \in \Gamma : \sigma \leq \tau\}$ is finite. I.e., there is no pair $\sigma, \tau \in \Pi$ with $\sigma < \tau$. A special cutset is the *sphere* of radius n, $S_n = \{\sigma \in \Pi : |\sigma| = n\}$. We write $|\Pi| = \min\{|\sigma| : \sigma \in \Pi\}$ and $M_n = \text{card } S_n$. A special type of tree is the n-tree, where Γ is said to be an n-tree if every vertex of Γ has exactly n children.

With the above definitions at hand, we state the following definitions on trees [Lyo90]:

Definition 8.4.1: *The* branching number *of a tree* Γ*, denoted by* $\operatorname{br}\Gamma$ *is defined by*

$$\begin{aligned} \operatorname{br}\Gamma &= \inf\left\{\lambda > 0 : \liminf_{|\Pi|\to\infty} \sum_{\sigma\in\Pi} \lambda^{-|\sigma|} = 0\right\} && (8.34)\\ &= \sup\left\{\lambda > 0 : \liminf_{|\Pi|\to\infty} \sum_{\sigma\in\Pi} \lambda^{-|\sigma|} = \infty\right\} && (8.35)\\ &= \inf\left\{\lambda > 0 : \inf_{\Pi} \sum_{\sigma\in\Pi} \lambda^{-|\sigma|} = 0\right\}. && (8.36)\end{aligned}$$

Definition 8.4.2: *The* upper *and* lower growth rate *of a tree* Γ *is defined by*

$$\underline{\operatorname{gr}}\,\Gamma = \liminf_{n\to\infty} M_n^{1/n} \tag{8.37}$$

and

$$\overline{\operatorname{gr}}\,\Gamma = \limsup_{n\to\infty} M_n^{1/n} \tag{8.38}$$

respectively. When their values are equal, we call the common value the growth rate *of the tree and denote it by*

$$\begin{aligned} \operatorname{gr}\Gamma &= \liminf_{n\to\infty} M_n^{1/n} && (8.39)\\ &= \inf\left\{\lambda > 0 : \liminf_{n\to\infty} \sum_{\sigma\in S_n} \lambda^{-|\sigma|} = 0\right\} && (8.40)\end{aligned}$$

Neither of the above definitions depend on the choice of root. We have the following relation

Proposition 8.4.3: *Let* Γ *be a tree. If* $\operatorname{gr}\Gamma$ *is defined, then*

$$\operatorname{br}\Gamma \le \operatorname{gr}\Gamma. \tag{8.41}$$

Proof. Let $\theta(\sigma)$ be the sum of the weights of all the edges leaving σ. If there is some σ for which $\theta(\sigma) \le \lambda^{-|\sigma|}$, then

$$M_n \ge \sum_{|\sigma|=n} 1 \ge \sum_{|\sigma|=n} \lambda^{|\sigma|}\theta(\sigma) = \lambda^n \theta(\Lambda). \tag{8.42}$$

Take the nth root of each side, and let $n \to \infty$, we get

$$\liminf_{n\to\infty} M_n^{1/n} \ge \liminf_{n\to\infty} \left((\lambda^n)^{1/n}\theta(\Lambda)^{1/n}\right) = \lambda. \tag{8.43}$$

□

Corollary 8.4.4: *Obviously, by the above proposition, we have also that*

$$\operatorname{br}\Gamma \le \underline{\operatorname{gr}}\,\Gamma \le \overline{\operatorname{gr}}\,\Gamma. \tag{8.44}$$

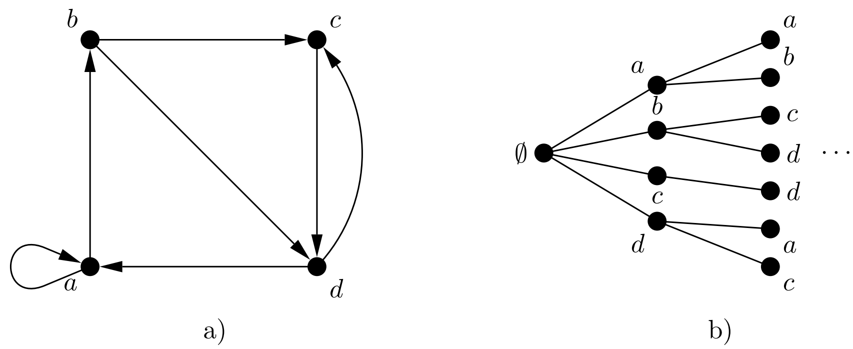


Figure 8.2: *An example of a) a finite graph, and b) its directed cover.*

Now, there are several interesting results for a special type of tree – periodic trees. The periodic tree is defined as follows [Tak97]:

Definition 8.4.5: *A* periodic tree Γ *is a rooted tree with a finite number of edge types. I.e. the root is of type* Λ *and for each vertex type* σ *there are* $n(\sigma, \tau)$ *children of type* τ*.*

Periodic trees can be constructed from finite directed graphs. Let the graph, G, have at least one cycle, then the *directed cover*, Γ, of G is the set of finite directed paths in G. Γ is a tree rooted at the empty set, with each path formed by the possible paths from the current vertex. See Figure 8.2 for an example of a finite graph and its directed cover.

Obviously, every directed cover Γ is a periodic tree, since the only possible children for a particular type of vertex, σ, are the vertices which have directed edges from σ. It then turns out that every periodic tree is isomorphic to a directed cover of a finite directed graph [Lyo90].

The periodic tree is a special case of a type of trees called *spherically symmetric* trees. A spherically symmetric tree is a tree in which the degree of a vertex is only dependent on its distance from the root. We have the following lemma:

Lemma 8.4.6: *If* Γ *is a spherically symmetric tree, then*

$$\operatorname{br}\Gamma = \underline{\operatorname{gr}}\,\Gamma. \tag{8.45}$$

Proof. This is immediate from the definitions. □

Now, if we want to compute the branching number of the directed cover, Γ, of a directed graph, G, we can go about as follows. Let A be the directed adjacency matrix of G, where A_{uv} is the number of edges from u to v. We know from Lemma 7.3.2 that the spectral radius of A is equal to its largest eigenvalue, $\rho(A)$. We will soon prove that this coincides with $\operatorname{br}\Gamma$, where Γ is the directed cover of G. We need the following lemmas:

Lemma 8.4.7: *Let* A *be an* $n \times n$ *matrix, and let* $\rho(A)$ *be its spectral radius. Then*

$$\lim_{k\to\infty} A^k = 0 \tag{8.46}$$

if and only if $\rho(A) < 1$.

Proof. Take any non-zero vector, x, and express it by the eigenvectors, v_i, $1 \leq i \leq n$, of A. I.e.

$$x = c_1 v_1 + \cdots + c_n v_n. \tag{8.47}$$

Multiply both sides in the above equation by A. By the definition of eigenvectors we get

$$Ax = c_1 A v_1 + \cdots + c_n A v_n = c_1 \lambda_1 v_1 + \cdots + c_n \lambda_n v_n, \tag{8.48}$$

where λ_i, $1 \leq i \leq n$, are the eigenvalues of A. If we repeatedly multiply by A, i.e. k times, we get

$$A^k x = c_1 A^k v_1 + \cdots + c_n A^k v_n = c_1 \lambda_1^k v_1 + \cdots + c_n \lambda_n^k v_n. \tag{8.49}$$

Now, assume that eigenvalues of A are sorted such that

$$\rho(A) = |\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|. \tag{8.50}$$

Then, if and only if $\rho(A) = |\lambda_1| < 0$, the terms $c_i \lambda_i^k v_i$ will tend towards zero, and thus so will also the left-hand side do. □

Lemma 8.4.8: *Let A be an (possibly directed) adjacency matrix, with entry*

$$A_{uv} = \{\textit{number of edges from } u \textit{ to } v\}.$$

Then, if the entry of A^k in row i and column j, $A^k_{i,j}$, is non-zero, there are $A^k_{i,j}$ paths of length k from i to j.

Proof. The i, j entry of A^2 is the sum $A_{i1}A_{1j} + \cdots + A_{in}A_{nj}$. The number $A_{ik}A_{kj}$ is 0 if there is no path of either i to k or k to j. If there is a path, the product of the number of different paths between them corresponds to all possible combinations of paths between them. I.e. the number of two-node paths between i and j. If we want to find all three-node paths, we multiply A^2 by A, and so on. □

Now we have the following proposition [Lyo90, Wik07]:

Proposition 8.4.9 (Gelfand's Formula)**:** *Let $G = (V, E)$ be a directed graph, and Γ be its directed cover. Then the branching number of Γ, $\operatorname{br}\Gamma$ equals the spectral radius of the directed adjacency matrix with $A_{uv} = \{$number of edges from u to $v\}$. I.e., for $M_n = \operatorname{card}\{\sigma \in \Pi : |\sigma| = n\}$, where Π is a cutset, we have*

$$\operatorname{br}\Gamma = \lim_{n\to\infty} M_n^{1/n} = \lim_{n\to\infty} \|A^n\|^{1/n} = \rho(A), \tag{8.51}$$

where $\|\cdot\|$ is any matrix norm.

Proof. That $\operatorname{br}\Gamma = \lim_{n\to\infty} M_n^{1/n} = \rho(A)$ is proven in [Lyo90]. We note that the number of paths of length k in G, is A^k by Lemma 8.4.8, which is obviously the same value as M_{n+1}. Thus, we have

$$\lim_{n\to\infty} M_n^{1/n} = \lim_{n\to\infty} \|A_n\|^{1/n}, \tag{8.52}$$

with the appropriate matrix norm.

For the right-hand part of the equality, and a proof for any matrix norm, we go about as follows. For any $\epsilon > 0$, consider the matrix

$$B = (\rho(A) + \epsilon)^{-1} A. \tag{8.53}$$

B has spectral radius

$$\rho(B) = \frac{\rho(A)}{\rho(A) + \epsilon} < 1. \tag{8.54}$$

This means, by Lemma 8.4.7, that

$$\lim_{k \to \infty} B^k = 0. \tag{8.55}$$

Thus, there exists a positive integer n_1, such that $\|B^k\| < 1$ for all $k \geq n_1$. This means that $\|A^k\| < (\rho(A) + \epsilon)^k$, or equivalently that

$$\lim_{k \to \infty} \|A^k\|^{1/k} < \rho(A) + \epsilon \tag{8.56}$$

for all $k \geq n_1$. Now we consider the matrix

$$C = (\rho(A) - \epsilon)^{-1} A \tag{8.57}$$

instead. Then we get

$$\rho(B) = \frac{\rho(A)}{\rho(A) - \epsilon} > 1, \tag{8.58}$$

which means that $\|C^k\|$ grows without bounds for increasing k, by Lemma 8.4.7. By the same argument as above, there exists a positive n_2 for which

$$\lim_{k \to \infty} \|A^k\|^{1/k} > \rho(A) - \epsilon, \tag{8.59}$$

when $k \geq n_2$. Let $n = \max\{n_1, n_2\}$ and combine Equation 8.56 and Equation 8.59 to get

$$\rho(A) - \epsilon < \lim_{k \to \infty} \|A^k\|^{1/k} < \rho(A) + \epsilon \tag{8.60}$$

for all $k \geq n$. Let $\epsilon \to 0$, and we get the desired result

$$\lim_{k \to \infty} \|A^k\|^{1/k} = \rho(A), \tag{8.61}$$

for $k \geq n$. □

Using the above proposition, we can easily find the branching number of a periodic tree, Γ, by transforming it to its corresponding graph, G. We then generate the appropriate adjacency matrix, A, from the graph and compute its spectral radius, $\rho(A)$. This method is deterministic and does not involve finding limits, which the other definitions do.

The last equality in the above proposition is a very useful result, and is also very common in the literature. More information about it, and another proof, can be found in [Rud66].

There is a very useful condition under which we have equality in Equation 8.44 on page 72. Let Γ^x denote the subtree of Γ formed by the descendants of $x \in \Gamma$ rooted at x. We have the following definition [LP05]:

Definition 8.4.10: *A tree is called* subperiodic *if, for all vertices $x \in \Gamma$, there is an isomorphism of Γ^x as a rooted tree to a subtree of Γ rooted at $y \in \Gamma$.*

We have the following lemma [LP05]:

Lemma 8.4.11: *If Γ is a subperiodic tree, then*

$$\operatorname{br}\Gamma = \operatorname{gr}\Gamma. \tag{8.62}$$

We now have the following corollary to Proposition 8.4.9:

Corollary 8.4.12: *Let $G = (V, E)$ be a directed graph defined as in Proposition 8.4.9, and Γ be its directed cover. Then the branching number of Γ, $\operatorname{br}\Gamma$, and the growth of Γ, $\operatorname{gr}\Gamma$, equals the spectral radius of the directed adjacency matrix. I.e., for $M_n = \operatorname{card}\{\sigma \in \Pi : |\sigma| = n\}$, where Π is a cutset, we have*

$$\operatorname{br}\Gamma = \operatorname{gr}\Gamma = \lim_{n\to\infty} M_n^{1/n} = \lim_{n\to\infty} \|A^n\|^{1/n} = \rho(A), \tag{8.63}$$

where $\|\cdot\|$ is any matrix norm.

Proof. This follows from Proposition 8.4.9 and Lemma 8.4.11. □

If we have a tree in which each vertex has a bounded degree, the following interpretation is possible. Define an alphabet $E = \{0, 1, \ldots, r-1\}$; let the number of successors of each node in the tree $E^{(\omega)}$ be at most r and label them with numbers of E. We have the following proposition [Fur70, Lyo90]:

Proposition 8.4.13: *If $E = \{0, \ldots, r-1\}$ and Γ is a closed subset of $E^{(\omega)}$. Let*

$$\overline{\Gamma} = \left\{ \sum_{i=1}^{\infty} \sigma_i r^{-i} : \sigma_i \in \Gamma \right\} \subseteq [0,1], \tag{8.64}$$

then $\operatorname{br}\Gamma = r^{\dim_{\mathrm{H}} \overline{\Gamma}}$ or equivalently $\frac{\log \operatorname{br}\Gamma}{\log r} = \dim_{\mathrm{H}} \overline{\Gamma}$.

Proof. Omitted, see [Fur70], [Lyo90] and [LP05] for details. □

Every infinite path from Λ is a string composed of integers from $[0, r-1]$, which is interpreted as the base r expansion of a real number in $[0,1]$. The set $\overline{\Gamma}$ is the set of all such numbers. See Example 8.4.14 for an illustration of how the proposition can be used. The above proposition is naturally extended to any subset of $\mathbb{R}^n$.

Example 8.4.14: *The Cantor set can be described by a 2-tree, Γ, in a base three expansion. We use the alphabet $E = \{0, 1, 2\}$, but omit the number 1 in the expansion. Every node of the tree has two children, namely 0, the left child, and 2, the right child. Since it is a 2-tree, the branching number is $\operatorname{br}\Gamma = 2$. We have $r = 3$, and therefore we get*

$$\dim_{\mathrm{H}} \overline{\Gamma} = \frac{\operatorname{br}\Gamma}{\log r} = \frac{\log 2}{\log 3} \approx 0.6309. \tag{8.65}$$

This is consistent with Equation 7.6 on page 52 in Chapter 7.

Proposition 8.4.13 can be extended even further. The following proposition also relate the growth rate of a tree to the Box-counting dimension [LP05]:

Proposition 8.4.15: *If $E = \{0, \ldots, r-1\}$ and Γ is a closed subset of $E^{(\omega)}$. Let*

$$\overline{\Gamma} = \left\{ \sum_{i=1}^{\infty} \sigma_i r^{-i} : \sigma_i \in \Gamma \right\} \subseteq [0,1], \tag{8.66}$$

then

$$\operatorname{br} \Gamma = r^{\dim_{\mathrm{H}} \overline{\Gamma}}, \tag{8.67}$$

$$\underline{\operatorname{gr}} \, \Gamma = r^{\underline{\dim}_{\mathrm{B}} \overline{\Gamma}} \tag{8.68}$$

and

$$\overline{\operatorname{gr}} \, \Gamma = r^{\overline{\dim}_{\mathrm{B}} \overline{\Gamma}}. \tag{8.69}$$

Proof. Omitted, see [LP05] for details. □

Chapter 9

Equivalence for Union

In this chapter we will discuss the arithmetic operation *union* that can be applied to sets. We will describe how the operation works on the different types of representations and how the results of graph-directed constructions relate to the classical theory of union of fractal geometry.

9.1 Classical Fractal Geometry

We remember from Section 5.2.5 that the Hausdorff and upper Box-dimensions are *finitely stable*. We formalize this property in the following theorems [Edg90]:

Theorem 9.1.1: *The upper Box-counting dimension is* finitely stable*, i.e.*

$$\overline{\dim}_{\mathrm{B}} \bigcup_{i=1}^{k} F_i = \max_{1 \le i \le k} \overline{\dim}_{\mathrm{B}} F_i, \tag{9.1}$$

for any finite collection of sets $\{F_1, \ldots, F_k\}$.

Proof. We prove first that $\dim_{\mathrm{B}}(F_1 \cup F_2) = \max\{\dim_{\mathrm{B}} F_1, \dim_{\mathrm{B}} F_2\}$. The general proof then follows from iteration. Let $N_\delta(F)$ be the number of cubes of side length δ required to cover the set F. We have that

$$\max\{N_\delta(F_1), N_\delta(F_2)\} \le N_\delta(F_1 \cup F_2) \le N_1(F_1) + N_2(F_2)$$

That is

$$\max\{N_\delta(F_1), N_\delta(F_2)\} \le N_\delta(F_1 \cup F_2) \le 2 \cdot \max\{N_1(F_1), N_2(F_2)\}.$$

Since a logarithm is monotonically increasing, we can rewrite this as

$$\log \max\{N_\delta(F_1), N_\delta(F_2)\} \le \log N_\delta(F_1 \cup F_2) \le \log 2 + \log \max\{N_1(F_1), N_2(F_2)\},$$

which we can rewrite as

$$\max\{\log N_\delta(F_1), \log N_\delta(F_2)\} \le \log N_\delta(F_1 \cup F_2) \le \log 2 + \max\{\log N_1(F_1), \log N_2(F_2)\}.$$

Now, we can divide by $\log\delta$, and we get

$$\begin{aligned}\max\left\{\frac{\log N_\delta(F_1)}{\log\delta},\frac{\log N_\delta(F_2)}{\log\delta}\right\} &\le \frac{\log N_\delta(F_1\cup F_2)}{\log\delta}\\ &\le \frac{\log 2}{\log\delta}+\max\left\{\frac{\log N_\delta(F_1)}{\log\delta},\frac{\log N_\delta(F_2)}{\log\delta}\right\}\end{aligned}$$

Let $\delta\to 0$ and for some $\epsilon>0$ we have

$$\begin{aligned}\max\{\overline{\dim}_{\mathrm{B}}F_1,\overline{\dim}_{\mathrm{B}}F_2\}-\epsilon &\le \overline{\dim}_{\mathrm{B}}(F_1\cup F_2)\\ &\le 0+\max\{\overline{\dim}_{\mathrm{B}}F_1,\overline{\dim}_{\mathrm{B}}F_2\}+\epsilon\end{aligned}$$

If we let $\epsilon\to 0$, we get

$$\overline{\dim}_{\mathrm{B}}(F_1\cup F_2)=\max\{\overline{\dim}_{\mathrm{B}}F_1,\overline{\dim}_{\mathrm{B}}F_2\}, \tag{9.2}$$

and the theorem is proven. □

Theorem 9.1.2: *The Hausdorff and dimension is* finitely stable*, i.e.*

$$\dim_{\mathrm{H}}\bigcup_{i=1}^{k}F_i=\max_{1\le i\le k}\dim_{\mathrm{H}}F_i, \tag{9.3}$$

for any finite collection of sets $\{F_1,\ldots,F_k\}$.

Proof. We prove first that $\dim_{\mathrm{H}}(F_1\cup F_2)=\max\{\dim_{\mathrm{H}}F_1,\dim_{\mathrm{H}}F_2\}$. The general proof for the Hausdorff dimension follows from iteration. Let $s>\max\{\dim_{\mathrm{H}}F_1,\dim_{\mathrm{H}}F_2\}$. Then $s>\dim_{\mathrm{H}}F_1$, so $\mathscr{H}^s(F_1)=0$. We get $\mathscr{H}^s(F_2)=0$ similarly. Then $\mathscr{H}^s(F_1\cup F_2)\le\mathscr{H}^s(F_1)+\mathscr{H}^s(F_2)=0$, and thus $\dim_{\mathrm{H}}(F_1\cup F_2)\le s$. This is true for all $s>\max\{\dim_{\mathrm{H}}F_1,\dim_{\mathrm{H}}F_2\}$, so we have $\dim_{\mathrm{H}}(F_1\cup F_2)\le\max\{\dim_{\mathrm{H}}F_1,\dim_{\mathrm{H}}F_2\}$. But by the monotonicity property, Proposition 5.2.28 a), we have $\dim_{\mathrm{H}}(F_1\cup F_2)\ge\max\{\dim_{\mathrm{H}}F_1,\dim_{\mathrm{H}}F_2\}$, and the theorem follows. □

For the Hausdorff dimension we have the stronger property of *countable stability* by the following theorem [Fal90]:

Theorem 9.1.3: *If $F_1,F_2,\ldots$ is a countable sequence of sets, then*

$$\dim_{\mathrm{H}}\bigcup_{i=1}^{\infty}F_i=\sup_{1\le i<\infty}\dim_{\mathrm{H}}F_i. \tag{9.4}$$

We say that the Hausdorff dimension is countably stable*.*

Proof. It has to be that $\dim_{\mathrm{H}}\cup_{i=1}^{\infty}F_i\ge\dim_{\mathrm{H}}F_j$, for each j, since the Hausdorff dimension is monotonic. But if $s>\dim_{\mathrm{H}}F_i$ for all i, then $\mathscr{H}^s(F_i)=0$, so that $\mathscr{H}^s(\cup_{i=1}^{\infty}F_i)=0$, and thus we have the opposite inequality. □

The above theorem implies that the dimension of any countable set has Hausdorff dimension zero, which the Box-counting dimension does not need to say. See the following example [Fal97]:

Example 9.1.4: $F = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ *is a compact set with* $\dim_{\mathrm{B}} F = \frac{1}{2}$.

Calculation. If $|U| = \delta < \frac{1}{2}$ and k is the integer satisfying $1/(k-1)k > \delta \geq 1/k(k+1)$ then U can cover at most one of the points $\{1, \frac{1}{2}, \ldots, \frac{1}{k}\}$. Thus at least k sets of diameter δ are required to cover F, so

$$\frac{\log N_\delta(F)}{-\log \delta} \geq \frac{\log k}{\log k(k+1)}.$$

Letting $\delta \to 0$ gives $\underline{\dim}_{\mathrm{B}} F \geq \frac{1}{2}$. On the other hand, if $\frac{1}{2} > \delta > 0$, take k such that $1/(k-1)k > \delta \geq 1/k(k+1)$. Then $(k+1)$ intervals of length δ cover $[0, 1/k]$, leaving $k-1$ points of F which can be covered by another $k-1$ intervals. Thus

$$\frac{\log N_\delta(F)}{-\log \delta} \leq \frac{\log 2k}{\log k(k-1)}.$$

giving $\overline{\dim}_{\mathrm{B}} F \leq \frac{1}{2}$, and the result follows. □

9.2 Graph-Directed Constructions

The union of two graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, is $G = G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2) = (V, E)$. See Figure 9.1 for an example. The following theorem states the dimension of the union of graphs.

Theorem 9.2.1: *Let G_i, $i = 1, \ldots, n$, be graph-directed constructions, and let A_i, $i = 1, \ldots, n$, be the corresponding adjacency matrices. For the union graph*

$$G = \bigcup_{i=1}^{n} G_i = (\bigcup_{i=1}^{n} V_i, \bigcup_{i=1}^{n} E_i) \tag{9.5}$$

it holds that

$$s = \max\{s_H : H \in SC(G)\}, \tag{9.6}$$

where s_H is the unique number such that $\rho(H^{(s_H)}) = 1$. The construction object, F, has positive and σ-finite $\mathscr{H}^s$ measure. Further, $\mathscr{H}^s(F) < \infty$ if and only if $\{H \in$

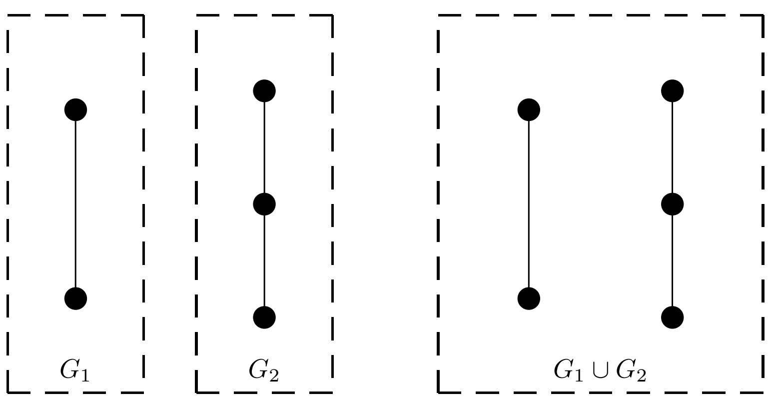


Figure 9.1: *An example of graph union.*

$SC(G) : s_H = s\}$ consists of pairwise incomparable elements. The number s is such that $\dim_{\mathrm{H}} F = \underline{\dim}_{\mathrm{B}} F = \overline{\dim}_{\mathrm{B}} F = s$.

Proof. Each of the G_i can be considered a strongly connected component of G. Apply Theorem 7.3.5 on G. If all G_i are not strongly connected, apply Theorem 7.3.5 to each of the non-strongly connected G_i. □

Finding the dimension of the union of two graphs is a trivial problem, by the following corollary:

Corollary 9.2.2: *Let two graphs, G_1 and G_2 be represented as adjacency matrices, A_1 and A_2, respectively. The union graph, $G_1 \cup G_2$, then has the adjacency matrix*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix},$$

i.e., the block-diagonal matrix with A_1 and A_2 on the diagonal. Let

$$A^{(s)} = \begin{bmatrix} A_1^{(s)} & 0 \\ 0 & A_2^{(s)} \end{bmatrix}.$$

Then the value of s for which $\rho(A^{(s)}) = 1$ is such that $\dim_{\mathrm{H}} F = \underline{\dim}_{\mathrm{B}} F = \overline{\dim}_{\mathrm{B}} F = s$.

Proof. The eigenvalues of $A^{(s)}$ is trivially the eigenvalues of $A_1^{(s)}$ and $A_2^{(s)}$. Thus, the maximum eigenvalue of $A^{(s)}$ is from the strongly connected component with the largest eigenvalue, and the result follows from Theorem 9.2.1. □

Thus, the theory of set union for traditional sets is extended to graph-directed constructions in a natural way, and the dimension of the union of graph-directed sets is easily found from the block-diagonal matrices constructed from the disjoint parts of the union graph.

Chapter 10

Results and Conclusions

In this chapter we discuss the results of the thesis. Results include testing the algorithm for finding the Box-counting dimension and testing graph union.

We also give suggestions for future work, how this thesis and the areas it considers could be used in other contexts.

10.1 Results

As we could see in Chapter 7, the classical self-similar sets are just special cases, with one node, of the graph-directed constructions.

We also noted that the dimension of the union of two disjoint graphs is the maximum of the two graph's dimensions and that this is found from the block-diagonal matrix created directly from the two graphs adjacency matrices. This result is equivalent to the results obtainable from looking the actual sets, as was also proven in Chapter 9. Even though this result was obtained using rather elementary methods, it should be noted that the author has not seen the particular proposition, Proposition 9.2.2, stated anywhere before.

Also, as was concluded in Section 8.3, it turns out that the tree constructions are equivalent to the graph-directed construction approach; at least when it comes to finding the Hausdorff dimension of the underlying set. But since a directed graph can be transformed to an infinite tree, its directed cover, the methods of *Graph-directed constructions* and *Tree constructions* are interchangeable, and we can obtain the dimension of a set from either one of them describing the set. The approach was the same in [Edg90], but the actual conclusion that any of the representation for recurrent sets can be used interchangeable has not been found elsewhere, and is therefore thought to be new by this thesis.

10.2 Conclusions

The area of fractal geometry is indeed very interesting and intriguing. The sets that are created are often mind-blowing and fascinating. This thesis has merely scratched the surface of the area, which grows for each day that goes by.

Many more results are likely to appear in the years to come, if they are not already out there, for other equivalences than union between the classical fractal geometry and the graph-directed constructions. How is it, for example, with set intersections and Cartesian product? These operations can very likely be solvable in a similar way as the solution for union.

The most prominent conclusion that can be drawn from this work is that there is a very close relationship between fractal geometry, graph theory and linear algebra.

A natural extension for the graph-directed constructions is to describe measures on graph-directed sets using multifractal theory, just as has been done for the sets of the classical fractal geometry, see for instance [Löf07] and [HJK+86]. This has been done by Edgar and Mauldin in [EM92], but that is the only article found by the author, but there is likely much more work done, and certainly more work will be done in this area in the future.

10.3 Estimating the Box-Counting Dimension of Self-Similar Sets

To see how well the Box-counting estimation algorithm works (see Section 5.3 on page 40), test runs have been performed where the Box-counting dimension of a number of self-similar sets were estimated. The *true* dimension of the test sets is of course known. When the estimated dimensions are found for all test sets, the values can be compared with the known true dimensions and conclusions can be drawn.

Tests were performed on point sets in $\mathbb{R}^1$, $\mathbb{R}^2$ and $\mathbb{R}^3$. The results can be seen in Table 10.1. The point sets that were created for testing consisted of 50,000 points. The theory of Section 5.3 was used. The grid size was successively reduced seven times by two, i.e. there were seven points in the estimate if the slope of $\log N_\delta$ against $\log \delta$.

Table 10.1: *Experimental results for estimating the Box-counting dimension of sets with known dimensions. The results are grouped in 1D, 2D and 3D sets in the table, starting with the 1D sets.*

True dimension	Estimated dimension	Difference
0.631	0.703	0.072
1.000	1.000	0.000
1.262	1.406	0.144
1.465	1.524	0.059
1.585	1.567	0.018
1.893	1.923	0.030
2.000	2.000	0.000
1.893	2.072	0.179
2.000	1.994	0.006
2.585	2.449	0.136
2.727	2.564	0.163
3.000	2.661	0.339

The results of Table 10.1 are illustrated in Figure 10.1, and, as can be seen, the estimation follows the true values rather well. Between dimensions zero and two there

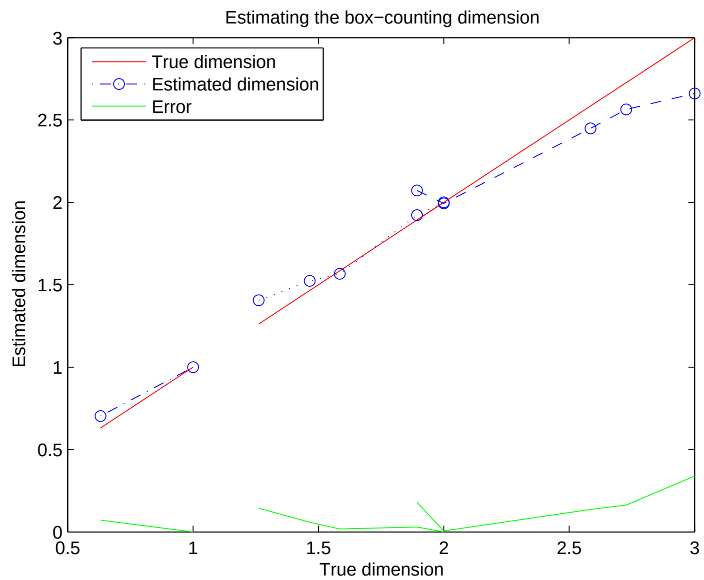


Figure 10.1: *Experimental results for estimating the Box-counting dimension of sets with known dimensions. Lines are connecting 1D, 2D and 3D results. The solid line is the true dimension of the set, the dashed and dotted lines are the estimated dimensions, and the solid line at the bottom is the error of the estimations.*

is a very small error, but when the true dimension increases to between two and three, the error in the estimated dimension increases. This is a known problem with the box-counting algorithm, as noted in i.e. [HLD94], but since, in general, the estimated dimensions increases when the true dimension increases (i.e., there is a one-to-one correspondence between the true values and the estimated values), the results still imply that the algorithm can be used in e.g. a segmentation process.

10.4 Finding the Hausdorff Dimension of the Union of Graph-Directed Self-Similar Sets

The Hausdorff dimension for a graph-directed construction can readily be found by applying Theorem 7.3.5. This can be done numerically using the method of bisection. This method is guaranteed to converge, but does so slowly. We select $s_{\text{low}} < s$ and $s_{\text{high}} > s$, such that $\rho(A^{(s_{\text{low}})}) > 1$ and $\rho(A^{(s_{\text{high}})}) < 1$. Then we select

$$s_{\text{new}} := \frac{s_{\text{low}} + s_{\text{high}}}{2}$$

and if $\rho(A^{(s_{\text{new}})}) < 1$ then

$$s_{\text{high}} := s_{\text{new}}$$

else we set

$$s_{\text{low}} := s_{\text{new}}.$$

Thus, by repeating the above calculations the $\rho(A^{(s_{\text{new}})})$ will become closer and closer to 1, at the same time that s_{new} become closer and closer to the true s. When $|\rho(A^{(s_{\text{new}})}) - 1| < \varepsilon$, for some ε, the algorithm can be stopped, knowing that the error is less than ε.

Now, a number of graph-directed sets have been constructed and the dimensions of their union have been found using the method just described. The result an be seen in Table 10.2, where the maximum of a particular value in the top row, and the left-most column should be equal to the corresponding value in the table. This is so for all combinations that were tested.

Table 10.2: *Finding the fractal dimension of the union of two graphs. The true dimensions are in the top row and the left-most column. When graphs with these dimensions are joined, the resulting dimension is calculated. The value in the ith row and jth column should be equal to the maximum of the corresponding values in the top row and left-most column. It is so for all combinations.*

	0.6309	**0.7369**	**1.0000**	**1.1279**	**1.5236**
0.6309	0.6309	0.7369	1.0000	1.1279	1.5236
0.7369	0.7369	0.7369	1.0000	1.1279	1.5236
1.0000	1.0000	1.0000	1.0000	1.1279	1.5236
1.1279	1.1279	1.1279	1.1279	1.1279	1.5236
1.5236	1.5236	1.5236	1.5236	1.5236	1.5236

Chapter 11

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