Abstract - Eigenvalues of a power system give a picture of the stability in the current operating point. The eigenvalues are calculated from the system matrix of a dynamic system. To create the system matrix, the dynamic system has to be linearized and to do that, linearization methods are utilized. Three linearization methods are compared in the paper. The performance of two software which use two of the methods are also compared and evaluated. It is shown how the linearization methods influence the results. Conclusions drawn from eigenvalue analysis are thus not only dependent on the properties of the investigated system, but also on used linearization method. It is shown how results obtained from three engineers working in parallel, studying the same power system but using three different linearization methods differ.

Keywords - Eigenvalues, Linear Analysis, Linearization Methods, PSS/E, Simpow, Small-signal Stability

1 INTRODUCTION

Linearization of a power system is necessary if its small-signal stability should be examined. Small-signal stability of a power system is the ability of the power system to maintain in synchronism when subjected to small disturbances.

There are several software packages on the market that can extract eigenvalues [1]. However, the various programs are seldom evaluated and compared. Since the software packages use different linearization methods as well as different models of the power system components, it is important to understand that there exist differences in the extraction of the eigenvalues. In [2, 3], two of the linearization methods have been investigated and in this paper a third method is added as well as more analysis of the differences between the methods.

In [4] different models of synchronous machines and how they influence the small-signal stability have been investigated. In [5] four different power system simulation software’s small-signal stability tool have been investigated and differences have been found. In [5] the different synchronous machine models in the software are stated as possible influencing factors.

In the paper three linearization methods are compared manually and also two software [6, 7] which use two of the methods are compared. The investigated methods are,

- Analytical Linearization, abbreviated as the AL method in the following text,
- Forward-Difference Approximation, FDA method,
- Center-Difference Approximation, CDA method.

In the paper, linearizations of a classical synchronous machine model in a power system are done. Since the models of the machine and the power system are identical in all linearizations it is possible to see the pure influence of the linearization methods.

It is shown how linearization methods influence the results, i.e., the location of the obtained eigenvalues in the complex plane. The conclusions drawn from eigenvalue analysis are thus not only dependent on the properties of the investigated system, but also on which linearization method that is used since software packages use different methods. Therefore, conclusions can differ when studying the same power system in different software.

2 ORGANIZATION OF THE PAPER

The paper is organized as follows. Section 3 contains a description of the linearization methods. Section 4 describes how linearization methods are utilized when linearizing a power system. Section 5 makes comparisons of the three methods and section 6 contains conclusions.

3 LINEARIZATION METHODS

In this section the three linearization methods are described. Since we later will study the linearization of a power system containing a classical model of a synchronous machine, we here mention one of the differential equations of that component,

\[ \Delta \omega = \frac{1}{2H} (T_m - T_e - D \cdot \Delta \omega) \] (1)

where \( \Delta \omega \) is the per unit speed deviation, \( T_m \) is the mechanical torque in per unit, \( T_e \) is the electrical torque in per unit, \( H \) is the rotor inertia in seconds, and \( D \) is the damping torque coefficient of the synchronous machine in p.u. torque/p.u. speed deviation.

Equation (1) contains one state variable, \( \Delta \omega \), and one algebraic variable, \( T_e \). The mechanical torque \( T_m \) (provided from a turbine) is constant for the classical machine model and is therefore not defined as an algebraic variable.

All algebraic and state variables can be put in an algebraic variable vector \( \mathbf{v} \) and a state vector \( \mathbf{x} \) respectively,

\[ \mathbf{v}^T = \begin{bmatrix} v_1 & \cdots & v_r \end{bmatrix} \] \hspace{1cm} (2)

\[ \mathbf{x}^T = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \] \hspace{1cm} (3)

The time-derivatives of the state variables are put in a vector \( \dot{\mathbf{x}} \). The equations of the dynamic system consists of both differential equations as well as algebraic equations. Algebraic equations are for instance expressions of node voltages and currents when simulating in phasor mode.

All differential equations can be re-organized and written on the following form,

\[ \dot{x}_i = f_i(x_1, \ldots, x_n; u_1, \ldots, u_p; v_1, \ldots, v_r; t) \] \hspace{1cm} (4)
for all $i = 1, \ldots, n$. Equation (4) is an expression for the time-derivative of state variable $i$. Expression $f_i$ may contain $n$ state variables $x_1, \ldots, x_n$, $r$ input variables $u_1, \ldots, u_p$, and $r$ algebraic variables $v_1, \ldots, v_r$. All $r$ algebraic equations can be combined to the following form,

$$v_i = g_i(x_1, \ldots, x_n; u_1, \ldots, u_p; v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_r, t)$$  \hspace{1cm} (5)

for all $i = 1, \ldots, r$. Equation (5) consists of an expression $g_i$ which may contain the $n$ state variables, the $p$ input variables, and the $r-1$ algebraic variables on the right-hand side. The left-hand side of equation (5) is the algebraic variable $v_i$ itself. Expressions $f_i$ and $g_i$ can be non-linear. The input signals are from now on left out since they are treated as constant.

The aim when performing linearization of a dynamic system is to create a dynamic model on a linear form from an equilibrium point $x_0$ (a steady state of the system). The linear form shows how the system responds linearly on small disturbances $\Delta x$, i.e., a form that shows how the time-derivatives of the state variables $\dot{x}$ are varying depending on small disturbances $\Delta$ of $x$. The relation between $\dot{x}$ and $x$ is then described with a linear system matrix $A$ and since it is by definition only valid for small disturbances, $\Delta$ can be found on both sides of equation (6),

$$A\Delta x = \Delta \dot{x}.$$  \hspace{1cm} (6)

$\Delta \dot{x}$ in equation (6) describes the contribution in the time-derivatives for small disturbances of the state variables in $\Delta x$. To create equation (6) we need to consider the feedback of the algebraic variables $v$. This is done in different ways for the three linearization methods. The AL method contains a linear feedback and the FDA and CDA methods contain non-linear feedback of $v$. The process when generating the linear form in equation (6) is done in different ways for the linearization methods which is briefly described here. Details can be found in [8].

### 3.1 Analytical linearization, AL method

When performing linearization with the AL method, all differential equations and algebraic equations are linearized by their analytical expressions. For instance, the differential equation (1) describing the time-derivative of the speed deviation, $\Delta \omega$, for a synchronous machine (in this case a classical machine model) is done as,

$$\Delta \omega = -\frac{1}{2H} \Delta T_e - \frac{D}{2H} \Delta \omega.$$  \hspace{1cm} (7)

The linearized form of equation (4) for all $i = 1, \ldots, n$ is,

$$\Delta \dot{x}_i = \frac{\partial f_i}{\partial x_1} \Delta x_1 + \cdots + \frac{\partial f_i}{\partial x_n} \Delta x_n + \frac{\partial f_i}{\partial u_1} \Delta u_1 + \cdots + \frac{\partial f_i}{\partial u_p} \Delta u_p$$  \hspace{1cm} (8)

The linearized form of equation (5) is,

$$\Delta v_i = \frac{\partial g_i}{\partial x_1} \Delta x_1 + \cdots + \frac{\partial g_i}{\partial x_n} \Delta x_n + \frac{\partial g_i}{\partial v_1} \Delta v_1 + \cdots + \frac{\partial g_i}{\partial v_r} \Delta v_r$$  \hspace{1cm} (9)

for all $i = 1, \ldots, r$. In equations (8) - (9) we have differentiated equations (4) - (5) depending on the $n$ state variables and the $r$ algebraic variables.

The linearized form of equation (8) - (9) can be identified with the following equation,

$$\begin{bmatrix} 0 \\ \Delta \dot{x} \end{bmatrix} = \begin{bmatrix} J_{aa} & J_{as} \\ J_{sa} & J_{ss} \end{bmatrix} \begin{bmatrix} \Delta v \\ \Delta x \end{bmatrix}$$  \hspace{1cm} (10)

where

$$\Delta v^T = [\Delta v_1 \cdots \Delta v_r]$$  \hspace{1cm} (11)

$$\Delta x^T = [\Delta x_1 \cdots \Delta x_n].$$  \hspace{1cm} (12)

The 0 in equation (10) is a vector containing 1 column and $r$ rows with all elements equal to 0.

The linearized system in equation (10) is described with a Jacobian-matrix consisting of four sub-matrices. The sub-matrix $J_{aa}$ contains a linear feedback and the FDA and CDA methods contain partial derivatives of the $r$ algebraic variables in the $r$ algebraic equations from equation (9). Sub-matrix $J_{as}$ contains partial derivatives of the $n$ state variables in the $r$ algebraic equations from equation (9). Sub-matrix $J_{sa}$ contains partial derivatives of the $r$ algebraic variables in the $n$ differential equations from equation (8). Sub-matrix $J_{ss}$ contains partial derivatives of the $n$ state variables in the $n$ differential equations from equation (8).

In [2] it has been shown how we can arrive to the following equation, combining the relations in equation (10),

$$A \Delta x = -J_{sa}J_{aa}^{-1}J_{as} \Delta x + J_{ss} \Delta x$$  \hspace{1cm} (13)

and in the right-hand side of equation (13) the $A$-matrix can be identified as,

$$A = -J_{sa}J_{aa}^{-1}J_{as} + J_{ss},$$  \hspace{1cm} (14)

and with that we have arrived to equation (6).

Section 4.1 contains a numerical example when linearizing the introduced classical model of a synchronous machine in a small system using the AL method.

### 3.2 Forward-difference approximation, FDA method

The FDA method is a numerical method, see [9], p. 54 used to linearize dynamic systems. Starting from a valid equilibrium condition $x_0$, a second state vector is created $x_{i+}$ in which the $i$th component of the state vector $x$ is perturbed from the equilibrium point $x_0$ by adding a small perturbation $h$ for state variable $i$.

The difference between $x_{i+}$ and $x_0$ is denoted as,

$$\Delta x_i = x_{i+} - x_0.$$  \hspace{1cm} (15)

All elements except for element $i$ are equal to zero in vector $\Delta x_i$. Element $i$ is equal to the perturbation $h$. With the new state vector $x_{i+}$, new values of the $r$ algebraic variables in vector $v$ are solved using the non-linear system of equations (5). All state variables except state variable $i$ remain constant at the values from the equilibrium point $x_0$. In an equilibrium point the $p$ input signals $u_1, \ldots, u_p$ in equations (4) and (5) are constant.

With the new values of the $r$ algebraic variables and state variable $i$, new time-derivatives are calculated for all $n$ state variables with equation (4). These $n$ time-derivatives when perturbing state variable $i$ are denoted $\dot{x}_{i+}$ and the difference between $\dot{x}_{i+}$ and $\dot{x}_0$ is denoted as,

$$\Delta \dot{x}_{i+} = \dot{x}_{i+} - \dot{x}_0 = \dot{x}_{i+}.$$  \hspace{1cm} (16)
Since all elements of \( \dot{x}_0 \) are equal to zero we can omit \( \dot{x}_0 \) in the most right-hand side of equation (16).

Now, when \( \Delta \dot{x}_{i+} \) is known we can calculate each element of column \( i \) of the \( A \)-matrix, denoted as \( A_i \). By using equation (16) we can write

\[
A_i = \frac{1}{h} \dot{x}_{i+}.
\]

Equation (17) is used to calculate values of the \( i \)th column of the system matrix \( A \) and \( h \) is the perturbation that we added to state variable \( i \).

By sequentially perturbing all entries of state vector \( x \) and get all \( \Delta \dot{x}_{i+} \) for every \( i \) we can identify all columns of the \( A \)-matrix and create the full \( A \)-matrix as was formulated in equation (6).

The size of the perturbation \( h \) affects the obtained result, i.e., the elements of the \( A \)-matrix and as a consequence, its eigenvalues. This is shown later in section 5.

### 3.3 Center-difference approximation, CDA method

The CDA method is as the FDA method a numerical method, see [9], p. 55 used to linearize a dynamic system. Starting from a valid equilibrium condition \( x_0 \), two other state vectors are created, \( x_{i+} \) and \( x_{i-} \) respectively. In \( x_{i+} \) the \( i \)th component of the state vector \( x \) is perturbed from the equilibrium point \( x_0 \) by adding a small perturbation \( h \) for state variable \( i \) as was shown in section 3.2. In \( x_{i-} \), the \( i \)th component of state vector \( x \) is perturbed from the equilibrium point \( x_0 \) by subtracting the same small perturbation \( h \) for state variable \( i \). The difference between \( x_{i+} \) and \( x_{i-} \) is here denoted as,

\[
\Delta x_i = \frac{x_{i+} - x_{i-}}{2}.
\]

All elements except for element \( i \) in \( \Delta x_i \) are equal to zero. Element \( i \) is equal to the perturbation \( h \).

With the state vectors \( x_{i+} \) and \( x_{i-} \), new values of the \( r \) algebraic variables in vector \( v \) are solved using the non-linear system of equations (5). All state variables except state variable \( i \) remain constant at the values from the equilibrium point \( x_0 \). Here we can see that the non-linear system of equations (5) has to be solved twice as many times when using the CDA method compared to when using the FDA method.

With the new values of the \( r \) algebraic variables and state variable \( i \), new time-derivatives are calculated for all \( n \) state variables with equation (4). These \( n \) new time-derivatives when perturbing state variable \( i \) are denoted as \( \dot{x}_{i+} \) and \( \dot{x}_{i-} \) respectively.

The difference between \( \dot{x}_{i+} \) and \( \dot{x}_0 \) is denoted as earlier with equation (16) and the difference between \( \dot{x}_0 \) and \( \dot{x}_{i-} \) is denoted as,

\[
\Delta \dot{x}_{i-} = \dot{x}_0 - \dot{x}_{i-} = -\dot{x}_{i+}.
\]

The average of the two vectors \( \Delta \dot{x}_{i+} \) and \( \Delta \dot{x}_{i-} \) is,

\[
\frac{\Delta \dot{x}_i}{2} = \frac{\Delta \dot{x}_{i+} + \Delta \dot{x}_{i-}}{2} = \frac{\dot{x}_{i+} - \dot{x}_{i-}}{2}.
\]

Now when \( \Delta \dot{x}_i \) is known we can calculate each element of column \( i \) of the \( A \)-matrix by using,

\[
A_i = \frac{1}{h} \Delta \dot{x}_i.
\]

or if we use equation (20) in equation (21), then \( A_i \) is,

\[
A_i = \frac{1}{2h} [\dot{x}_{i+} - \dot{x}_{i-}].
\]

Equation (22) is used to calculate values of the \( i \)th column of system matrix \( A \) and \( h \) is the perturbation.

By sequentially perturbing all entries of state vector \( x \) with perturbations \( h \) and \(-h\) and get all \( \Delta \dot{x}_{i+} \) and \( \Delta \dot{x}_{i-} \) for every \( i \) we can identify all columns of the \( A \)-matrix and create the full \( A \)-matrix. The size of the perturbation affects the obtained result, i.e., the elements of the \( A \)-matrix and its eigenvalues. This is shown later in section 5.

### 4 LINEARIZATION OF A CLASSICAL MACHINE

With the three linearization methods a small test system containing an infinite bus, a transmission impedance \( x_{11re} \), and a synchronous machine represented by the classical machine model are modelled and linearized, see figure 1 and [10], p. 732, where all system data can be found. The power-flow solution is given in the appendix.

The reason for applying the three different linearization methods to a power system is to come as close as possible to a real situation when engineers are working with linearization but utilize different linearization methods. Therefore we will in this section show results obtained from three engineers working in parallel, studying the same power system but using three different linearization methods.

![Figure 1: The classical machine model in a small test system.](image)

The models of the power system components are identical in all linearizations and therefore it is possible to observe what impact the different linearization methods have on the obtained eigenvalues. In figure 1 the classical machine model as well as its connection to the rest of the power system is shown. The machine contains two state variables, the speed deviation \( \Delta \omega \) and the machine angle \( \delta \) (DELTAD in figure 1). There are no other state variables in the power system. The initial values of all variables are calculated in the power-flow solution and in the initialisation of the dynamic simulation, see appendix.

For a classical machine model, two differential equations exist. These are equations (1) and (23),

\[
\dot{\delta} = \omega_0 \cdot \Delta \omega.
\]

The following two state variables are included in the state vector \( x \).
\[ x^T = [ \Delta \omega \quad \delta ] . \] (24)

The algebraic equations for the machine and the rest of the power system can be found from [3] and derived from figure 1. We include the following six algebraic variables in the algebraic variable vector \( v \): \( i_d, i_q, i_{re}, v_{re}, i_{im}, i_{re} \). They can be found in figure 1. In section 4.1 – 4.3 the system is linearized with the three methods. The results from the AL method are obtained from manual calculations (by hand) and the software Simpow [6], results from the FDA method are obtained from manual calculations and the software PSS/E [7], and results from the CDA method are obtained from manual calculations.

4.1 Linearization using the AL method

The necessary steps when linearizing the classical machine with the AL method has been shown in [3], so here only the result is shown. In [3] the following \( A \)-matrix was calculated with the four sub-matrices of equation (10), values from the current operating point, machine, and network parameters. The \( A \)-matrix is constructed with equation (14) and becomes,

\[
A = \begin{bmatrix}
0 & -0.108131 \\
376.991118 & 0 
\end{bmatrix}
\] (25)

where \( \omega_0 = 376.991118 \). The eigenvalues of \( A \) are,

\[ \lambda_{1,2} = 0 \pm j6.38471(1/s, rad/s). \] (26)

The results are the same in Simpow where the eigenvalues are calculated using the Quick Response-method (QR-method), see [11]. The inverse iteration method [12] is used to calculate the eigenvectors as well as to improve the real and imaginary parts of the eigenvalues.

In [10], the results have been rounded off several times in the calculation process and since then, the eigenvalues are calculated as \( \lambda_{1,2} = 0 \pm j6.39 \) (1/s,rad/s). If six digits accuracy is used in example 12.2 in [10], p. 732, the bus voltage at the machine bus (indicated as \( E \) in [10]) is 0.999818±36.0185°. By re-calculating example 12.2, the state matrix is then equal to equation (25) and since then, also the eigenvalues.

4.2 Linearization using the FDA method

The necessary steps when linearizing the classical machine with the FDA method has already been shown in [3], so here only the result and comment about matrix element \( A_{12} \) in equation (27) are shown.

The linear system that should be identified is,

\[
\begin{bmatrix}
\Delta \dot{\omega} \\
\Delta \delta
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
\Delta \omega \\
\Delta \delta
\end{bmatrix}
\] (27)

where the \( A \)-matrix is,

\[
A = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}.
\] (28)

In [3] the \( A \)-matrix was manually calculated as,

\[
A = \begin{bmatrix}
0 & -0.107480 \\
376.991118 & 0
\end{bmatrix}.
\] (29)

The eigenvalues of \( A \) in equation (29) are,

\[ \lambda_{1,2} = 0 \pm j6.36545(1/s, rad/s). \] (30)

In PSS/E the \( A \)-matrix is,

\[
A = \begin{bmatrix}
0 & -0.107370 \\
376.991118 & 0
\end{bmatrix}.
\] (31)

The eigenvalues are in PSS/E calculated using the Quick Response method (QR-method) [11]. The eigenvalues of \( A \) in equation (31) are,

\[ \lambda_{1,2} = 0 \pm j6.3622(1/s, rad/s). \] (32)

The eigenvalues in equations (30) and (32) were calculated when the state matrix \( A \) was identified with a perturbation size \( h \) of 0.01.

The differences between the eigenvalues in equations (30) and (32) has not been possible to investigate further. The differences can be a result from propagation of round-off errors in PSS/E, both when calculating the elements of the \( A \)-matrix as well as when applying the QR-method to calculate the eigenvalues. However, the differences are very small and the linearization procedure as described in [7] in section 22.2 is indeed the FDA method.

Of certain interest when deriving the different matrix elements is matrix element \( A_{12} \) since it is the only element which contains non-linear feedback of the equations of the system. The left-hand side of equation (1) when perturbing \( \delta \), noted as \( \Delta \delta \), is used to identify matrix element \( A_{12} \) in equation (27). \( \delta \) is not direct included in equation (1) and therefore \( \delta \)'s feedback to the variable \( T_e \) must be calculated using the algebraic equations that are shown in figure 1. \( A_{12} \) is manually calculated as,

\[ A_{12} = \frac{\Delta \dot{\omega}}{\Delta \delta} = \frac{-1.07480 \cdot 10^{-3}}{0.01} = -0.107480 \] (33)

The numerator in equation (33) has a non-linear relation to the perturbation that we will show in section 5.

4.3 Linearization using the CDA method

Here we apply the CDA method described in section 3.3. For the CDA method the \( A \)-matrix is column by column identified by applying two perturbations in each state variable. As in section 4.2, the perturbation is \( h \) but for the CDA method also perturbation \(-h\) is added to the state variables, see below. In the example the state variables are \( \Delta \omega \) and \( \delta \). The perturbation size \( h \) is as earlier 0.01. By adding it to each state variable from an equilibrium point, it is calculated how the left-hand sides of differential equations (1) and (23) change including feedback of the algebraic equations in figure 1. Equations (1) and (23) together with the algebraic equations are used to derive a state matrix describing linear relations of the left-hand sides \( \Delta \dot{\omega} \) and \( \Delta \delta \) and the right-hand sides \( \Delta \omega \) and \( \Delta \delta \) as in equation (27). This is shown in the following.

Perturbing the speed deviation \( \Delta \omega \)

The left-hand sides of equations (1) and (23) when perturbing \( \Delta \omega \) are used to identify matrix elements \( A_{11} \) and \( A_{21} \). When perturbing \( \Delta \omega \), the other state variable \( \delta \) is assumed to be unperturbed. The feedback of \( \delta \) to the variable \( T_e \) in equation (1) is therefore neglected. Since the damping constant \( D \) is zero, the left-hand side of equation (1) is equal to zero when perturbing \( \Delta \omega \). Therefore, matrix element \( A_{11} \) is by using equation (22) equal to 0.
When perturbing $\Delta \omega$, the left-hand side of equation (23) changes proportional to the perturbation multiplied with $\omega_0$. Therefore, matrix element $A_{21}$ is,

$$A_{21} = \frac{\delta_{i+} - \delta_{i-}}{2 \cdot \Delta \omega} = \frac{2 \cdot 0.01 \cdot \omega_0}{2 \cdot 0.01} = 376.991118$$ (34)

Perturbing the machine angle $\delta$

The left-hand sides of equations (1) and (23) when perturbing $\delta$, noted as $\Delta \delta$, are used to identify matrix elements $A_{12}$ and $A_{22}$. $\delta$ is not direct included in equation (1) and therefore $\delta$’s feedback to the variable $T_e$ must be calculated using the algebraic equations that can be found in figure 1. The matrix element $A_{12}$ is,

$$A_{12} = \frac{\Delta \omega_{i+} - \Delta \omega_{i-}}{2 \cdot \Delta \delta} =$$

$$= \frac{1.07480 \cdot 10^{-3} - 1.08766 \cdot 10^{-3}}{2 \cdot 0.01} = -0.108123$$ (35)

The feedback to the state variable $\Delta \omega$ in equation (23) is neglected when perturbing the state variable $\delta$, see section 3.3, and therefore matrix element $A_{22}$ is equal to 0.

The $A$-matrix gets with the CDA method,

$$A = \begin{bmatrix} 0 & -0.108123 \\ 376.991118 & 0 \end{bmatrix}.$$ (36)

The eigenvalues of $A$ are,

$$\lambda_{1,2} = 0 \pm j6.38446(1/s, rad/s).$$ (37)

5 COMPARISON OF THE THREE METHODS

When comparing the eigenvalues in sections 4.1 – 4.3 differences are seen only in the imaginary parts and therefore we proceed with varying the perturbation size $h$ while observing the imaginary parts of the obtained eigenvalues.

Figure 2 shows the absolute value of the imaginary part of the eigenvalues obtained with the three methods.

The curve indicated as 1) in figure 2 shows the absolute value of the imaginary part of the eigenvalues obtained from the AL method. Linearization with the AL method is done without any perturbation $h$ and therefore the result is independent of the perturbation, i.e., a horizontal line.

The curve indicated as 2) shows the absolute value of the imaginary part of the eigenvalues obtained from the FDA method for perturbation sizes $h$ varied in the range between 0.0001 and 0.1000 in 1000 steps.

The curve indicated as 3) shows results obtained from PSS/E wherein also the FDA method is used. $h$ in curve 3) is varied from 0.0001 to 0.1000 in 32 steps. Since curves 2) and 3) are very close to each other it shows that PSS/E’s linearization method is as shown in section 4.2. Figure 2 shows that the smaller $h$, the closer the result from the FDA method (curves 2 and 3)) is to the AL method (curve 1)). The discrepancies between curve 2) and 3) have not been possible to further investigate in this work. The differences can be a result from propagation of roundoff errors in PSS/E, both when calculating the elements of the $A$-matrix as well as when applying the QR-method to calculate the eigenvalues.

The curve indicated as 4) shows the absolute value of the imaginary part of the eigenvalues obtained from the CDA method for perturbation sizes $h$ varied in the range between 0.0001 and 0.1000 in 1000 steps. In figure 2 we can see that the resulting imaginary part is very close to the AL method also for large values of $h$.

Figure 2: Imaginary parts of the eigenvalues when using the AL, FDA, and CDA methods.

5.1 Truncation errors

In figure 2 it is clearly shown that there exist differences in the results obtained with the three linearization methods and that the size of the perturbation $h$ influences the calculated eigenvalues. If all equations of the system had been linear, curves 2), 3), and 4) in figure 2 had perfectly followed curve 1) also for large values of $h$, but since non-linear block diagrams are included when the four matrix elements in equation (27) are identified, the results are depending on the perturbation size $h$.

As can be seen by comparing the $A$-matrices in equations (25), (29), and (36) it is only element $A_{12}$ that is changing between the methods. Figure 3 shows how matrix element $A_{12}$, see equations (33) and (35), varies with the perturbation size $h$ for the three methods. If the system had been linear, the value of $A_{12}$ had been independent of $h$, i.e., horizontal lines had been obtained for all three linearization methods, perfectly following matrix element $A_{12}$ obtained from the AL method, $A_{12,AL}$ in figure 3.

As can be seen in figure 3, matrix element $A_{12}$ varies more with $h$ for the FDA method compared to the CDA
method. This since equation (33) includes a truncation error where higher orders of the Taylor expansion is neglected in the FDA method as well as in the CDA method [9], p. 54.

Figure 3 also shows the truncation errors $\Delta A_{12FD}^a$ and $\Delta A_{12CDA}$. We can see that $\Delta A_{12FD}^a$ is larger than $\Delta A_{12CDA}$ for all $h$. It can be shown that the truncation error obtained with the FDA method is,

$$\Delta A_{12FD} = \frac{E_f}{4H} \frac{V_{reinfinitebus} - V_{line}}{h} \cdot \sin \delta_0 + O(h^2). \quad (38)$$

In equation (38) we have neglected second and higher orders of $h$. In general when using the FDA method, truncation errors are of first order for small values of $h$ [9],

$$\Delta = O(h). \quad (39)$$

In figure 3 we can see that the truncation error $\Delta A_{12FD}^a$ indeed has almost a linear relation with $h$ as in equation (38).

The other matrix elements of the $A$-matrix are independent of the size of $h$ since the used equations are linear. It can be shown that the truncation error obtained with the CDA method $\Delta A_{12CDA}^a$ is,

$$\Delta A_{12CDA} = \frac{E_f}{2H} \frac{V_{reinfinitebus} - V_{line}}{h} \cdot \cos \delta_0 + O(h^3). \quad (40)$$

In equation (40) we have neglected fourth and higher orders of $h$. In general when using the CDA method, truncation errors are of second order for small values of $h$ [9],

$$\Delta = O(h^2). \quad (41)$$

5.2 Large-disturbance stability analysis

Curves 2) and 3) in figure 2 indicate the situation when perturbations are large. The FDA and CDA method include non-linear feedback of the algebraic equations and it was shown in figure 2 that the imaginary part of the eigenvalue-pair is as low as 6.2 (rad/s) for $h \approx 0.1$ with the FDA method.

This has also been verified with time-domain simulations performed with Simpow. For large perturbations in the system, the remaining oscillation frequency is indeed decreasing as in figure 2 and is as low as 4.6 (rad/s) for very large perturbations, see figures 4 and 5.

In the time-domain simulations a solid three-phase fault has been applied on the machine bus in figure 1. The duration of the fault has been varied between 0.0002 (s) and 0.10 (s). In figure 4 we can see that the longest fault duration $t_c$ possible is 0.10 (s). For longer fault durations the generator loses synchronism and the generator falls out of step. For the largest fault durations we can also observe that the response is not purely sinus-formed. This is a consequence of the non-linear feedback of system of equations.

Often the example single machine infinite bus (SMIB) is compared with the motion for a mass connected to a wall with a spring [13], p. 125. The spring constant $K$ in such mechanical system corresponds to matrix element $A_{12}$ in equation (28) (after a multiplication with $-2H$). The spring constant $K$ in the analogously with the mechanical system varies and for the positive swing (when $\delta > 50^\circ$ in the upper part of figure 4) the spring constant is smaller the farther we are from $\delta = 50^\circ$ and since then, the curve of $\delta$ is smoother for values of $\delta > 50^\circ$. For the negative swing (when $\delta < 50^\circ$ in the lower part of figure 4) the spring constant is larger the farther we are from $\delta_0 = 50^\circ$ and since then, the curve of $\delta$ is sharper.

Figure 4: Rotor angle $\delta$ and speed deviation $\Delta \omega$ for different fault durations $t_c$.

In figure 5 the duration of the fault is shown on the horizontal axis and the oscillation frequency (rad/s) of the recovering system is shown on the vertical axis. The oscillation frequency has been calculated with Fast Fourier Transform with a time window of 500 (s) which gives the frequency solution $1/500 = 0.002$ (Hz). The analyzed signal in figure 5 is $\Delta \omega$.

Figure 5: Frequency of the remaining oscillation after a solid three-phase fault with varying fault duration has been applied to the power system.

From figure 5 we can see that the result is as with the FDA method that the larger perturbations, the smaller imaginary parts of the eigenvalue pair and the slower the remaining oscillation in the power system is when it is recovering after the fault has been removed. When the FDA method is utilized for linearizations, non-linear relations in the algebraic equations are included when deriving the $A$-matrix. This is also the situation when performing time-domain simulations. One conclusion we can draw from this is that the FDA method can give us an idea of possible values of an eigenvalue that we can experience in the power system for both small and large disturbances. Whether the same conformity are obtained also in studies of other power systems could be further investigated.
for the FDA method, i.e., to show that the result from eigenvalue analysis using the FDA method for large perturbations provides the same result as large disturbances in time-domain simulations.

However, when small-signal stability of systems are studied, the perturbation is assumed to be so small so that the system can be linearized in the current working point. Figure 2 shows that the FDA and CDA methods are sensitive to the degree of non-linearity of the equations of the studied system as well as to the size of the perturbation $h$. This sensitivity is largest for the FDA method.

6 CONCLUSIONS

Three linearization methods have been evaluated in this paper. The analytical linearization is achieved by writing the models in terms of the most elementary components for which a linearization formula is known. The forward-difference and center-difference approximation methods are numerical methods.

When analyzing the electromechanical mode of a small power system, the impact of the difference in linearization methods were investigated.

It has been shown that the perturbation size $h$ in the forward-difference approximation method influences the calculated eigenvalues. However, for small values of the perturbation, results from all methods are very similar. The results from the numerical center-difference approximation method are very close to the analytical linearization method also for large values of the perturbation.

The larger perturbation size when using the forward-difference approximation method, the smaller imaginary parts of the eigenvalue pair is obtained. This is in conformity with time-domain simulations of the power system; the larger perturbations the power system is going through, the slower the responding oscillation is when the system is recovering after the fault has been removed. Whether the same conformity are obtained also in studies of other power systems could be further investigated for the forward-difference approximation method.

The forward-difference and center-difference approximation methods are sensitive to the degree of non-linearity of the equations of the studied system as well as to the size of the perturbation $h$. This sensitivity is largest for the forward-difference approximation method.

REFERENCES


APPENDIX

The power-flow solution of the classical machine in section 4 is shown in table 1. The solution is as in [10].

| $i_{d0}$ (p.u.) | 0.507750 |
| $i_{q0}$ (p.u.) | 0.801572 |
| $i_{r0}$ (p.u.) | 0.904523 |
| $i_{m0}$ (p.u.) | 0.286645 |
| $\delta_0$ (rad) | 0.871538 |
| $\delta_d$ (rad) | 0.240471 |
| $\delta_q$ (rad) | 0.970469 |
| $u_{d0}$ (p.u.) | 0.808680 |
| $u_{q0}$ (p.u.) | 0.587940 |
| $E_{ref}$ (p.u.) | 1.122794 |
| $T_{e0}$ (p.u.) | 0.9 |
| $T_{m0}$ (p.u.) | 0.9 |

Table 1: Power-flow solution of the classical machine in section 4.