

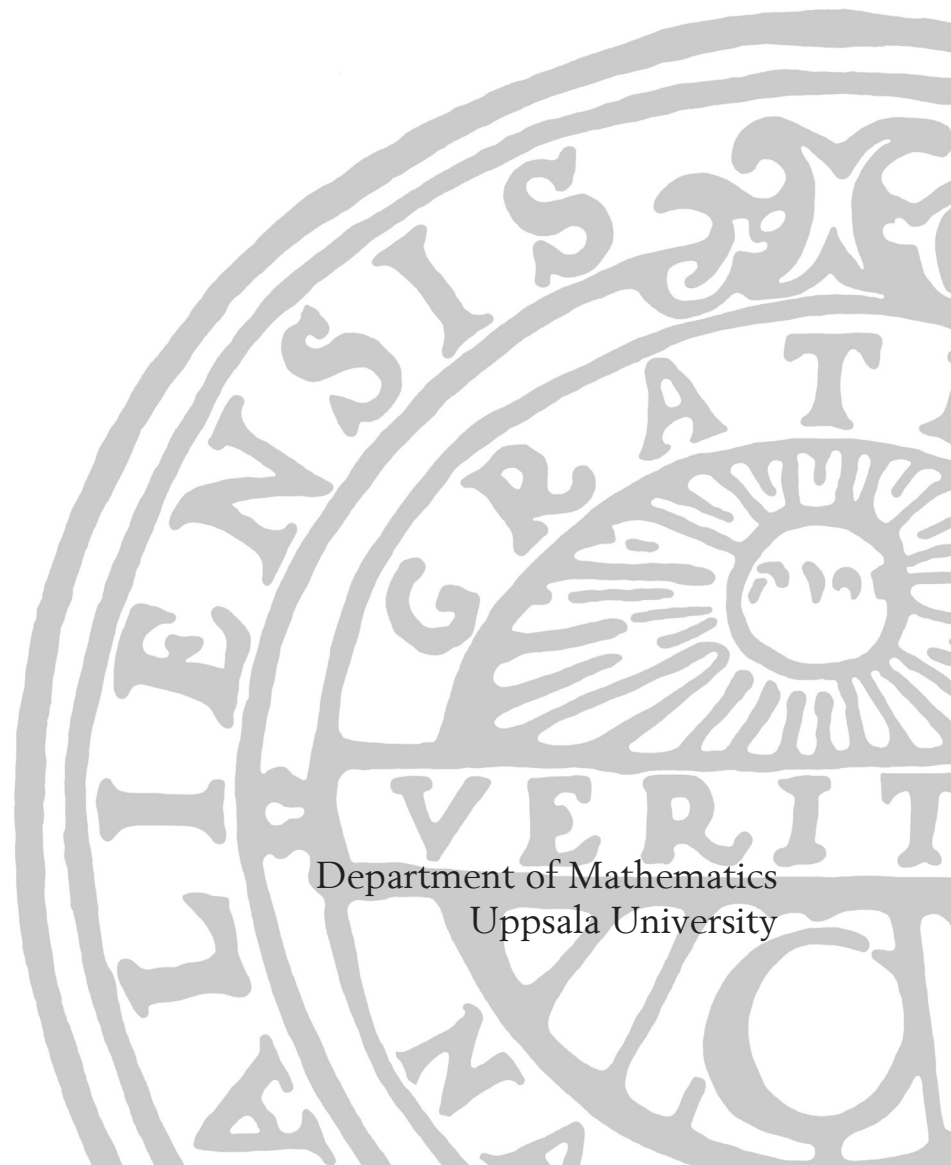


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On the Hsu-Robbins-Erdős-Spitzer-Baum-Katz theorem for random fields

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Abstract

The by now classical results on convergence rates in the law of large numbers involving the sums $\sum_{n=1}^{\infty} n^{\alpha r - 2} P(|S_n| > n^\alpha \varepsilon)$, where $r > 0$, $\alpha > 1/2$, such that $\alpha r \geq 1$ has been extended to the case $\alpha = 1/2$ by adding additional logarithms. All of this has been generalized to random fields by the first named author in *Ann. Probab.* (1978, 1980). The purpose of the present paper is to treat the case when the α 's differ in the different directions of the field, as well as mixed cases with some α 's equal to $1/2$ with added logarithms and/or iterated ones.

1 Introduction and results

The point of departure of this paper is the pioneering paper in 1947 by Hsu and Robbins [11], in which they introduced the concept of complete convergence, and proved that the sequence of arithmetic means of independent, identically distributed (i.i.d.) random variables converges completely to the expected value of the variables, provided their variance is finite. The necessity was proved somewhat later by Erdős [2, 3]. The result was later generalized during a process which led to the now classical paper by Baum and Katz [1], in which the equivalence of (1.1), (1.2), and (1.4) below was established; the equivalence of (1.2) and (1.3) is trivial one way and follows via the Lévy inequalities (more precisely via the standard Lévy inequalities as given in e.g. [6], Theorem 3.7.1 in conjunction with Proposition 3.6.1 there).

Theorem 1.1 *Let $r > 0$, $\alpha > 1/2$, and $\alpha r \geq 1$. Suppose that X, X_1, X_2, \dots are i.i.d. random variables with partial sums $S_n = \sum_{k=1}^n X_k$, $n \geq 1$. If*

$$E|X|^r < \infty \quad \text{and, if } r \geq 1, \quad EX = 0, \quad (1.1)$$

then

$$\sum_{n=1}^{\infty} n^{\alpha r - 2} P(|S_n| > n^\alpha \varepsilon) < \infty \quad \text{for all } \varepsilon > 0; \quad (1.2)$$

$$\sum_{n=1}^{\infty} n^{\alpha r - 2} P\left(\max_{1 \leq k \leq n} |S_k| > n^\alpha \varepsilon\right) < \infty \quad \text{for all } \varepsilon > 0. \quad (1.3)$$

If $\alpha r > 1$ we also have

$$\sum_{n=1}^{\infty} n^{\alpha r - 2} P\left(\sup_{k \geq n} |S_k/k^\alpha| > \varepsilon\right) < \infty \quad \text{for all } \varepsilon > 0. \quad (1.4)$$

Conversely, if one of the sums is finite for all $\varepsilon > 0$, then $E|X|^r < \infty$ and, if $r \geq 1$, $EX = 0$.

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The result in this form was generalized to the multiindex case in [4]. In order to formulate it we need to define our setup. Toward that end, let \mathbf{Z}_+^d , $d \geq 2$, denote the positive integer d -dimensional lattice with coordinate-wise partial ordering \leq , viz., for $\mathbf{m} = (m_1, m_2, \dots, m_d)$ and $\mathbf{n} = (n_1, n_2, \dots, n_d)$, $\mathbf{m} \leq \mathbf{n}$ means that $m_k \leq n_k$, for $k = 1, 2, \dots, d$. The “size” of a point equals $|\mathbf{n}| = \prod_{k=1}^d n_k$, and $\mathbf{n} \rightarrow \infty$ means that $n_k \rightarrow \infty$, for all $k = 1, 2, \dots, d$.

Here is now the promised background result.

Theorem 1.2 *Let $r > 0$ and $\alpha > 1/2$ with $\alpha r \geq 1$, and suppose that $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbf{Z}_+^d\}$ are i.i.d. random variables with partial sums $S_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}}$, $\mathbf{n} \in \mathbf{Z}_+^d$. If*

$$E|X|^r (\log^+ |X|)^{d-1} < \infty \quad \text{and, if } r \geq 1, \quad EX = 0, \quad (1.5)$$

then

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha r - 2} P(|S_{\mathbf{n}}| > |\mathbf{n}|^{\alpha \varepsilon}) < \infty \quad \text{for all } \varepsilon > 0; \quad (1.6)$$

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha r - 2} P(\max_{\mathbf{k} \leq \mathbf{n}} |S_{\mathbf{k}}| > |\mathbf{n}|^{\alpha \varepsilon}) < \infty \quad \text{for all } \varepsilon > 0. \quad (1.7)$$

If $\alpha r > 1$ we also have

$$\sum_{j=1}^{\infty} j^{\alpha r - 2} P(\sup_{j \leq |\mathbf{k}|} |S_{\mathbf{k}}|/|\mathbf{k}|^{\alpha} > \varepsilon) < \infty \quad \text{for all } \varepsilon > 0. \quad (1.8)$$

Conversely, if one of the sums is finite for all $\varepsilon > 0$, then $E|X|^r (\log^+ |X|)^{d-1} < \infty$ and, if $r \geq 1$, $EX = 0$.

REMARK 1.1 Here, and throughout, we define $\log^+ x = \max\{1, \log x\}$, for $x > 0$, although, for convenience, we shall permit ourselves to be sloppy about the additional $+$ -sign within computations. Moreover, we use the convention that products such as $\prod_{i=1}^0 = 1$. We also let C , with or without indices, denote numerical constants of no significance, which, in addition may change between appearances. \square

In the above result every coordinate n_k , $k = 1, 2, \dots, d$, is raised to the same power α . The main point of the present paper is to allow for different powers for different coordinates. As for “the law of the single logarithm”, which is a parallel to the LIL in which one considers delayed sums, we refer to [7].

In order to continue we therefore define $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_d)$, where, w.l.o.g., we assume that the coordinates are arranged in non-decreasing order, such that α_1 is the smallest one and α_d the largest one. We further let p denote the number of α 's which are equal to the smallest one, viz.,

$$p = \max\{k : \alpha_k = \alpha_1\}.$$

As is easily seen the domain of interest concerning the α 's becomes

$$\frac{1}{2} \leq \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_d \leq 1, \quad (1.9)$$

where the boundary $1/2$ takes us into the realm of the central limit theorem and the boundary 1 corresponds to the Kolmogorov strong law.

Finally, for ease of notation, we use the notation $\mathbf{n}^{\boldsymbol{\alpha}} = (n_1^{\alpha_1}, n_2^{\alpha_2}, \dots, n_d^{\alpha_d})$ and $|\mathbf{n}^{\boldsymbol{\alpha}}| = \prod_{k=1}^d n_k^{\alpha_k}$.

Theorem 1.2 now extends as follows.

Theorem 1.3 *Let $r > 0$, suppose that $\alpha_1 > 1/2$, that $\alpha_1 r \geq 1$, and let $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbf{Z}_+^d\}$ be i.i.d. random variables with partial sums $S_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}}$, $\mathbf{n} \in \mathbf{Z}_+^d$. If*

$$E|X|^r (\log^+ |X|)^{p-1} < \infty \quad \text{and, if } r \geq 1, \quad EX = 0, \quad (1.10)$$

then

$$\sum_{\mathbf{n}} |\mathbf{n}^{\boldsymbol{\alpha}}|^{\alpha_1 r - 2} P(|S_{\mathbf{n}}| > |\mathbf{n}^{\boldsymbol{\alpha}}| \varepsilon) < \infty \quad \text{for all } \varepsilon > 0; \quad (1.11)$$

$$\sum_{\mathbf{n}} |\mathbf{n}^{\boldsymbol{\alpha}}|^{\alpha_1 r - 2} P(\max_{\mathbf{k} \leq \mathbf{n}} |S_{\mathbf{k}}| > |\mathbf{n}^{\boldsymbol{\alpha}}| \varepsilon) < \infty \quad \text{for all } \varepsilon > 0. \quad (1.12)$$

If $\alpha_1 r > 1$ we also have

$$\sum_{j=1}^{\infty} j^{\alpha_1 r - 2} P(\sup_{j \leq |\mathbf{k}|} |S_{\mathbf{k}}|/|\mathbf{k}^{\alpha}| > \varepsilon) < \infty \quad \text{for all } \varepsilon > 0. \quad (1.13)$$

Conversely, if one of the sums is finite for all $\varepsilon > 0$, then $E|X|^r (\log^+ |X|)^{p-1} < \infty$ and, if $r \geq 1$, $E X = 0$.

The next natural question would be to consider the boundary $\alpha_1 = 1/2$. However, since in this case,

$$\sum_{\mathbf{n}} |\mathbf{n}|^{(r/2)-2} P(|S_{\mathbf{n}}| > |\mathbf{n}^{\alpha}| \varepsilon) \geq \sum_{i=1}^{\infty} i^{(r/2)-2} P(|S_{i,1,1,\dots,1}| > \sqrt{i} \cdot 1 \cdot 1 \cdots 1 \cdot 1 \cdot \varepsilon) = +\infty$$

for any $\varepsilon > 0$, there cannot exist any positive result in this case.

Nevertheless, in the case $d = 1$ Lai [12], Theorem 3, proves a positive result by adding (that is, by multiplying) additional logarithms, cf. also [5]. Our next result is a multiindex extension of his theorem.

Theorem 1.4 *Let $r \geq 2$, suppose that $\alpha_1 = 1/2$ (and thus, in particular, that $\alpha_1 r \geq 1$), and let $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbf{Z}_+^d\}$ be i.i.d. random variables with partial sums $S_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}}$, $\mathbf{n} \in \mathbf{Z}_+^d$. If*

$$E|X|^r (\log^+ |X|)^{p-1-r/2} < \infty, \quad E X = 0, \quad \text{and} \quad \text{Var } X = \sigma^2 < \infty, \quad (1.14)$$

then

$$\sum_{\mathbf{n}} |\mathbf{n}|^{(r/2)-2} P(|S_{\mathbf{n}}| > \sqrt{\prod_{i=1}^p n_i \cdot \log(\prod_{i=1}^p n_i)} \prod_{i=p+1}^d n_i^{\alpha_i} \cdot \varepsilon) < \infty \quad \text{for } \varepsilon > \sigma\sqrt{r-2}; \quad (1.15)$$

$$\sum_{\mathbf{n}} |\mathbf{n}|^{(r/2)-2} P(\max_{\mathbf{k} \leq \mathbf{n}} |S_{\mathbf{k}}| > \sqrt{\prod_{i=1}^p n_i \cdot \log(\prod_{i=1}^p n_i)} \prod_{i=p+1}^d n_i^{\alpha_i} \cdot \varepsilon) < \infty \quad \text{for } \varepsilon > \sigma\sqrt{r-2}. \quad (1.16)$$

If $\alpha_1 r > 1$, i.e. if $r > 2$, then we also have

$$\sum_{j=1}^{\infty} j^{(r/2)-2} P(\sup_{j \leq |\mathbf{k}|} |S_{\mathbf{k}}| / \sqrt{\prod_{i=1}^p k_i \log k_i} \prod_{i=p+1}^d k_i^{\alpha_i} > \varepsilon) < \infty \quad \text{for all } \varepsilon > \sigma\sqrt{r-2}. \quad (1.17)$$

Conversely, suppose that either $r = 2$ and $p \geq 2$, or that $r > 2$. If one of the sums is finite for some $\varepsilon > 0$, then $E|X|^r (\log^+ |X|)^{p-1-r/2} < \infty$ and $E X = 0$.

REMARK 1.2 When $p = d$ one rediscovers [5], Theorems 3.4 and 3.6.

REMARK 1.3 The first and necessary condition in (1.14) implies, in particular, that the variance is finite *except for the case when $r = 2$ and $p = 1$* . However, one can show (cf. [5], page 301) that an intermediate condition is sufficient when $r = 2$ and ($p =$) $d = 1$ in the symmetric case. For the complicated precise condition and for more on this exceptional case we refer to [14]. \square

REMARK 1.4 Note that when $r = 2$ the sums are, in fact, finite for *every* $\varepsilon > 0$.

Our next result is devoted to the loglog-analog of the previous result.

Theorem 1.5 *Suppose that $\alpha_1 = 1/2$, that $p \geq 2$, and let $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbf{Z}_+^d\}$ be i.i.d. random variables with partial sums $S_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}}$, $\mathbf{n} \in \mathbf{Z}_+^d$. If*

$$E|X|^2 \frac{(\log^+ |X|)^{p-1}}{\log^+ \log^+ |X|} < \infty, \quad E X = 0, \quad \text{and} \quad \text{Var } X = \sigma^2, \quad (1.18)$$

then, for $\varepsilon > \sigma\sqrt{2p}$,

$$\sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} P(|S_{\mathbf{n}}| > \sqrt{\prod_{i=1}^p n_i \cdot \log \log \left(\prod_{i=1}^p n_i \right) \prod_{i=p+1}^d n_i^{\alpha_i} \cdot \varepsilon}) < \infty; \quad (1.19)$$

$$\sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} P\left(\max_{\mathbf{k} \leq \mathbf{n}} |S_{\mathbf{k}}| > \sqrt{\prod_{i=1}^p n_i \cdot \log \log \left(\prod_{i=1}^p n_i \right) \prod_{i=p+1}^d n_i^{\alpha_i} \cdot \varepsilon}\right) < \infty. \quad (1.20)$$

Conversely, if one of the sums is finite for some $\varepsilon > 0$, then $E|X|^2 \frac{(\log^+ |X|)^{p-1}}{\log^+ \log^+ |X|} < \infty$ and $EX = 0$.

REMARK 1.5 When $p = d$ one rediscovers [5], Theorems 6.2. \square

One can also mix logarithmic and iterated logarithmic rates as follows.

Theorem 1.6 Suppose that $\alpha_1 = 1/2$, that $2 \leq p < d$, and let $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbf{Z}_+^d\}$ be i.i.d. random variables with partial sums $S_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}}$, $\mathbf{n} \in \mathbf{Z}_+^d$. If

$$E|X|^2 \frac{(\log^+ |X|)^{d-2}}{\log^+ \log^+ |X|} < \infty, \quad EX = 0, \quad \text{and} \quad \text{Var } X = \sigma^2, \quad (1.21)$$

then, for $\varepsilon > \sigma\sqrt{2p}$,

$$\sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} P(|S_{\mathbf{n}}| > \sqrt{\prod_{i=1}^p n_i \cdot \log \log \left(\prod_{i=1}^p n_i \right) \cdot \log \left(\prod_{i=p+1}^d n_i \right) \cdot \varepsilon}) < \infty; \quad (1.22)$$

$$\sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} P\left(\max_{\mathbf{k} \leq \mathbf{n}} |S_{\mathbf{k}}| > \sqrt{\prod_{i=1}^p n_i \cdot \log \log \left(\prod_{i=1}^p n_i \right) \cdot \log \left(\prod_{i=p+1}^d n_i \right) \cdot \varepsilon}\right) < \infty. \quad (1.23)$$

Conversely, if one of the sums is finite for some $\varepsilon > 0$, then $E|X|^2 \frac{(\log^+ |X|)^{d-2}}{\log^+ \log^+ |X|}$ and $EX = 0$.

Our final mix is devoted to logarithms, iterated ones and powers.

Theorem 1.7 Suppose that $\alpha_1 = 1/2$, that $2 \leq p < q < d$, and let $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbf{Z}_+^d\}$ be i.i.d. random variables with partial sums $S_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}}$, $\mathbf{n} \in \mathbf{Z}_+^d$. If

$$E|X|^2 \frac{(\log^+ |X|)^{q-2}}{\log^+ \log^+ |X|} < \infty, \quad EX = 0, \quad \text{and} \quad \text{Var } X = \sigma^2, \quad (1.24)$$

then, for $\varepsilon > \sigma\sqrt{2p}$, we have

$$\sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} P(|S_{\mathbf{n}}| > \sqrt{\prod_{i=1}^q n_i \log \log \left(\prod_{i=1}^p n_i \right) \cdot \log \left(\prod_{i=p+1}^q n_i \right) \cdot \prod_{i=q+1}^d n_i^{\alpha_i} \cdot \varepsilon}) < \infty; \quad (1.25)$$

$$\sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} P\left(\max_{\mathbf{k} \leq \mathbf{n}} |S_{\mathbf{k}}| > \sqrt{\prod_{i=1}^q n_i \log \log \left(\prod_{i=1}^p n_i \right) \cdot \log \left(\prod_{i=p+1}^q n_i \right) \cdot \prod_{i=q+1}^d n_i^{\alpha_i} \cdot \varepsilon}\right) < \infty. \quad (1.26)$$

Conversely, if one of the sums is finite for some $\varepsilon > 0$, then $E|X|^2 \frac{(\log^+ |X|)^{q-2}}{\log^+ \log^+ |X|}$ and $EX = 0$.

REMARK 1.6 The reason for the strict inequalities concerning p , q , and d is that there is no ‘‘continuity’’ in the moment assumptions between those theorems and earlier ones. Some additional comments on this are given in Remark 6.2.

REMARK 1.7 Note that the case $p = 1$ is special almost throughout, sometimes also for the proof of the sufficiency. However, once again it follows from the necessity that the variance is automatically finite unless $p = 1$. A related trouble concerns the necessity in the LIL which is ‘‘easy’’ when $d \geq 2$ and ‘‘hard’’ when $d = 1$. \square

2 Auxiliary tools

In order to get hold of the “size” of the index set \mathbb{Z}_+^d one needs the quantities

$$M_d(j) = \text{Card} \{\mathbf{k} : |\mathbf{k}| \leq j\} \quad \text{and} \quad d_d(j) = \text{Card} \{\mathbf{k} : |\mathbf{k}| = j\},$$

together with their asymptotics

$$\frac{M_d(j)}{j(\log j)^{d-1}} \rightarrow \frac{1}{(d-1)!} \quad \text{and} \quad d_d(j) = o(j^\delta) \text{ for any } \delta > 0 \quad \text{as } j \rightarrow \infty. \quad (2.1)$$

We begin with some purely computational tools, the proofs of which exploit the method of “summing along hyperbolas” (cf. [4, 5, 7]), partial summation and (2.1).

Lemma 2.1 (a) *Let $\theta_i > 0$, $i = 1, 2, 3$. Then*

$$\sum_{\mathbf{n}} \prod_{i=1}^d n_i^{\theta_i-1} \exp\{-\theta_2 \prod_{i=1}^d n_i^{\theta_3}\} < \infty.$$

(b)

$$\sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|(\log |\mathbf{n}|)^\gamma} \begin{cases} < \infty, & \text{for } \gamma > d, \\ = \infty, & \text{for } \gamma \leq d. \end{cases}$$

(c) *Suppose that $\rho_i > 1$, $i = 1, 2, \dots, d$. Then*

$$\sum_{n_1, n_2, \dots, n_d} \prod_{i=1}^d \frac{1}{n_i^{\rho_i}} \begin{cases} < \infty, & \text{if all } \rho_i > 1, \\ = \infty, & \text{if at least one } \rho_i \leq 1. \end{cases}$$

PROOF. (a) follows upon observing that

$$\sum_{\mathbf{n}} \prod_{i=1}^d n_i^{\theta_i-1} \exp\{-\theta_2 \prod_{i=1}^d n_i^{\theta_3}\} = \sum_{j=1}^{\infty} d(j) j^{\theta_1-1} \exp\{-\theta_2 j^{\theta_3}\} < \infty,$$

and (b) is immediate from the fact that

$$\sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|^\rho (\log |\mathbf{n}|)^\gamma} = \sum_{j=1}^{\infty} \frac{d(j)}{j^\rho (\log j)^\gamma},$$

plus partial summation and (2.1). Finally, (c) follows since

$$\sum_{n_1, n_2, \dots, n_d} \prod_{i=1}^d \frac{1}{n_i^{\rho_i}} = \prod_{i=1}^d \sum_{n_i=1}^{\infty} \frac{1}{n_i^{\rho_i}}. \quad \square$$

The following result connects tail probabilities and moments (for more on this topic, cf. e.g. [6], Section 2.12 for the case $d = 1$ and [4, 5] for the case $d \geq 2$).

Lemma 2.2 (a) *Let X be a random variable. Then, for any $r > 0$ and $\varepsilon > 0$,*

$$E|X|^r (\log^+ |X|)^{d-1} < \infty \iff \sum_{\mathbf{n}} |\mathbf{n}|^{r-1} P(|X| > \varepsilon |\mathbf{n}|) < \infty.$$

Suppose in the following that $r > 0$ and that $\alpha_1 \geq 1/2$ with $\alpha_1 r \geq 1$.

(b) *Let $\alpha_1 > 1/2$ and $1 \leq p \leq d$. Then, for all $\varepsilon > 0$,*

$$E|X|^r (\log^+ |X|)^{p-1} < \infty \iff \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha_1 r-1} P(|X| > \varepsilon |\mathbf{n}^\alpha|) < \infty;$$

(c) Let $r \geq 2$, $\alpha_1 = 1/2$, and $1 \leq p \leq d$. Then, for all $\varepsilon > 0$,

$$E|X|^r (\log^+ |X|)^{p-1-r/2} < \infty \iff \sum_{\mathbf{n}} |\mathbf{n}|^{(r/2)-1} P(|X| > \varepsilon \sqrt{\prod_{i=1}^p n_i \cdot \log(\prod_{i=1}^p n_i) \prod_{i=p+1}^d n_i^{\alpha_i}}) < \infty;$$

(d) Let $\alpha_1 = 1/2$ and $1 \leq p \leq q \leq d$. Then, for all $\varepsilon > 0$,

$$E|X|^2 \frac{(\log^+ |X|)^{p-1}}{\log^+ \log^+ |X|} < \infty \iff \sum_{\mathbf{n}} P(|X| > \varepsilon \sqrt{\prod_{i=1}^p n_i \cdot \log \log(\prod_{i=1}^p n_i) \prod_{i=p+1}^d n_i^{\alpha_i}}) < \infty;$$

(e) Let $\alpha_1 = 1/2$ and $1 \leq p < d$. Then, for all $\varepsilon > 0$,

$$E|X|^2 \frac{(\log^+ |X|)^{d-2}}{\log^+ \log^+ |X|} < \infty \iff \sum_{\mathbf{n}} P(|X| > \varepsilon \sqrt{\prod_{i=1}^d n_i \log \log(\prod_{i=1}^p n_i) \cdot \log(\prod_{i=p+1}^d n_i)}) < \infty;$$

(f) Let $\alpha_1 = 1/2$ and $1 \leq p < q < d$. Then, for all $\varepsilon > 0$,

$$\begin{aligned} E|X|^2 \frac{(\log^+ |X|)^{q-2}}{\log^+ \log^+ |X|} < \infty \\ \iff \sum_{\mathbf{n}} P(|X| > \varepsilon \sqrt{\prod_{i=1}^q n_i \log \log(\prod_{i=1}^p n_i) \cdot \log(\prod_{i=p+1}^q n_i) \cdot \prod_{i=q+1}^d n_i^{\alpha_i}}) < \infty. \end{aligned}$$

The proof is deferred to the Appendix.

The following lemma, which (partly) extends [4], Lemma 2.2, is taken from [8], page 1018.

Lemma 2.3 Suppose that $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbf{Z}_+^d\}$ are i.i.d. random variables with mean 0, that, under our usual setting, $\alpha_1 > 1/2$, and set, for $\mathbf{k} \leq \mathbf{n}$ and all \mathbf{n} ,

$$Y_{\mathbf{k},\mathbf{n}} = X_{\mathbf{k}} I\{|X_{\mathbf{k}}| \leq |\mathbf{n}^{\alpha}| \} \quad \text{and} \quad S'_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} Y_{\mathbf{k},\mathbf{n}}.$$

Then

$$E|X|^{1/\alpha_1} (\log^+ |X|)^{p-1} < \infty \implies \sum_{\mathbf{n}} \frac{\text{Var } Y_{\mathbf{1},\mathbf{n}}}{|\mathbf{n}^{\alpha}|^2} < \infty.$$

REMARK 2.1 When $p = (q =) d$, that is, when all α 's are equal, the results so far in this section all reduce to the appropriate ones in [5].

REMARK 2.2 The sum of the truncated, normalized variances diverges if $\alpha_1 = 1/2$. Namely, if, for simplicity, we consider the symmetric case, then

$$\sum_{\mathbf{n}} \frac{\text{Var } Y_{\mathbf{1},\mathbf{n}}}{|\mathbf{n}^{\alpha}|^2} \geq \sum_{n_1=1}^{\infty} \frac{E X^2 I\{|X| \leq 1\}}{n_1 \cdot 1^{\alpha_2} \cdots 1^{\alpha_d}} = E X^2 I\{|X| \leq 1\} \sum_{n_1=1}^{\infty} \frac{1}{n_1} = \infty. \quad \square$$

For some of the proofs to follow we need the following simpler version of the exponential bounds (cf. e.g., [6], Theorem 3.1.2).

Lemma 2.4 Suppose that $P(|X| \leq b) = 1$ for some $b > 0$, that $E X = 0$, and set $\text{Var } X = \sigma^2$. Then, for $0 < t < b^{-1}$, and $x > 0$.

$$P(|X| > x) \leq 2 \exp\{-tx + t^2 \sigma^2\}.$$

3 Proof of Theorem 1.3

By standard symmetrization/desymentrization, the details of which we omit, we may w.l.o.g. suppose that the random variables are symmetric throughout the proof.

The case $\alpha_1 > 1/2$, $\alpha_1 r > 1$

(1.10) \implies (1.11): The key tool in this part of the proof is the Kahane-Hoffmann-Jørgensen inequality (see e.g. [6], Theorem 3.7.5), according to which, for $j \geq 1$ to be conveniently chosen below,

$$\begin{aligned} \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha_1 r - 2} P(|S_{\mathbf{n}}| > 3^j |\mathbf{n}^\alpha| \varepsilon) \\ \leq C_1 \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha_1 r - 1} P(|X| > |\mathbf{n}^\alpha| \varepsilon) + C_2 \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha_1 r - 2} \left(P(|S_{\mathbf{n}}| > |\mathbf{n}^\alpha| \varepsilon) \right)^{2^j}. \end{aligned}$$

Now, the first sum is finite iff the moment condition of (1.10) is fulfilled, and an application of Chebyshev's inequality majorizes the second one by

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha_1 r - 2} \left(\frac{\mathbf{n} \sigma^2}{|\mathbf{n}^\alpha|^2 \varepsilon^2} \right)^{2^j} = C \sum_{\mathbf{n}} \prod_{i=1}^d n_i^{\alpha_1 r - 2} \left(\frac{n_i}{n_i^{2\alpha_i}} \right)^{2^j} = C \prod_{i=1}^d \sum_{n_i=1}^{\infty} n_i^{-(2\alpha_i - 1)2^j - \alpha_1 r + 2} < \infty,$$

since the last exponent > 1 for j sufficiently large.

The remaining parts of the proof follow the usual pattern and are therefore only indicated.

Implications (1.12) \implies (1.11) and (1.13) \implies (1.11) are trivial. An application of the Lévy inequalities proves (1.11) \implies (1.12). In order to prove that (1.11) \implies (1.13) one mimics the analogous part in the proof of Theorem 1.1 (cf. also [6], Section 6.12).

Finally, if (1.11) holds, then finiteness of the required moment follows via obvious modifications of the analog for the classical Hsu-Robbins-Erdős theorem as provided in [6], page 314. The heart of the matter is to show that $P(|S_{\mathbf{n}}| > \dots)$ is of the same order of magnitude as $nP(|X| > \dots)$, after which one applies Lemma 2.2.

The case $\alpha_1 > 1/2$, $\alpha_1 r = 1$

The only part that differs from the previous proof is the step

(1.10) \implies (1.11): For this boundary case we use truncation as follows.

Set, for $\mathbf{k} \leq \mathbf{n}$, and all \mathbf{n}

$$Y_{\mathbf{k}, \mathbf{n}} = X_{\mathbf{k}} I\{|X_{\mathbf{k}}| \leq |\mathbf{n}^\alpha|\} \quad \text{and} \quad S'_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} Y_{\mathbf{k}, \mathbf{n}}.$$

Then, by the truncated Chebyshev inequality (see e.g. [6], Theorem 3.1.5),

$$\begin{aligned} \sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} P(|S_{\mathbf{n}}| > 2|\mathbf{n}^\alpha| \varepsilon) &\leq \sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} P(|S'_{\mathbf{n}}| > |\mathbf{n}^\alpha| \varepsilon) + \sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} |\mathbf{n}| P(|X| > |\mathbf{n}^\alpha|) \\ &\leq \sum_{\mathbf{n}} \frac{\text{Var } Y_{\mathbf{1}, \mathbf{n}}}{|\mathbf{n}^\alpha|^2 \varepsilon^2} + \sum_{\mathbf{n}} P(|X| > |\mathbf{n}^\alpha|) < \infty. \end{aligned}$$

The finiteness of these sums follows from Lemmas 2.3 and 2.2(b), respectively. \square

4 Proof of Theorem 1.4

The case $\alpha_1 = 1/2$, $r = 2$, $p \geq 1$

We first recall that now,

$$\frac{1}{2} = \alpha_1 \cdots \alpha_p < \alpha_{p+1} \leq \cdots \leq \alpha_d < 1.$$

The general idea of the proof is the same as that of the first part in the previous proof, and, once again, it "suffices" to treat the symmetric case.

(1.14) \implies (1.15): Toward that end, the Kahane-Hoffmann-Jørgensen inequality, for $j \geq 1$ conveniently chosen, followed by an application of Chebyshev's inequality (recall that we assumed

finite second moment in all cases) tells us that,

$$\begin{aligned} \sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} P(|S_{\mathbf{n}}| > 3^j \sqrt{\prod_{i=1}^p n_i \cdot \log(\prod_{i=1}^p n_i)} \prod_{i=p+1}^d n_i^{\alpha_i} \cdot \varepsilon) \\ \leq C_1 \sum_{\mathbf{n}} P(|X| > \sqrt{\prod_{i=1}^p n_i \cdot \log(\prod_{i=1}^p n_i)} \prod_{i=p+1}^d n_i^{\alpha_i} \cdot \varepsilon) \\ + C_2(\sigma) \sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} \left(\frac{\prod_{i=1}^d n_i \sigma^2}{\prod_{i=1}^p n_i \cdot \log(\prod_{i=1}^p n_i) \cdot \prod_{i=p+1}^d n_i^{2\alpha_i}} \right)^{2^j}. \end{aligned}$$

The first sum in the RHS is finite for all $\varepsilon > 0$ iff the moment condition of (1.14) is fulfilled (note that $r = 2$ and that $p - 1 - r/2 = p - 2$ in this case), and the second sum can be rewritten as

$$\begin{aligned} & \sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} \cdot \left(\frac{1}{\log(\prod_{i=1}^p n_i)} \right)^{2^j} \cdot \left(\frac{1}{\prod_{i=p+1}^d n_i^{2\alpha_i-1}} \right)^{2^j} \\ &= \sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} \cdot \left(\frac{1}{(\log(\prod_{i=1}^p n_i))^p} \right)^{2^j/p} \cdot \frac{1}{\prod_{i=p+1}^d n_i^{(2\alpha_i-1)2^j}} \\ &\leq \sum_{\mathbf{n}} \prod_{i=1}^p \frac{1}{n_i (\log n_i)^{2^j/p}} \cdot \prod_{i=p+1}^d \frac{1}{n_i^{(2\alpha_i-1)2^j+1}} \\ &= \prod_{i=1}^p \left(\sum_{n_i=1}^{\infty} \frac{1}{n_i (\log n_i)^{2^j/p}} \right) \cdot \prod_{i=p+1}^d \left(\sum_{n_i=1}^{\infty} \frac{1}{n_i^{(2\alpha_i-1)2^j+1}} \right). \end{aligned}$$

Now, the second portion of sums converges since the exponents in the denominator $(2\alpha_i - 1)2^j + 1 > 1$, and the first sums converge by choosing j such that $2^j > p$.

An application of Lemma 2.1, finally, establishes the desired implication.

The remainder of the proof follows as in the proof of Theorem 1.3 with Lemma 2.2(c) replacing Lemma 2.2(b) in the very final step. \square

The case $\alpha_1 = 1/2$, $r > 2$

Once again, the only part that significantly differs from the usual procedure is the implication (1.14) \implies (1.15).

The main feature in comparison to the proofs so far is that this one is more closely related to LIL-type of proofs. This is due to the fact that positive results are not valid *for all* $\varepsilon > 0$, but only for all ε which are bounded below by some positive number. One consequence of this is that symmetrization is not possible, since one always loses a factor 2 during the desymmetrization process. Another complication is that this necessitates two truncations.

Let $0 < \eta < \alpha_{p+1} - 1/2$, set $\beta_i = \alpha_i - \eta$, for $i = p+1, p+2, \dots, d$, so that $\beta_i > 1/2$, and introduce the truncations

$$b_{\mathbf{n}} = b_{|\mathbf{n}|} = \frac{2\delta\sigma^2}{\varepsilon} \sqrt{\frac{\prod_{i=1}^p n_i}{\log(\prod_{i=1}^p n_i)}} \cdot \prod_{i=p+1}^d n_i^{\beta_i} \quad \text{and} \quad c_{\mathbf{n}} = c_{|\mathbf{n}|} = \delta \sqrt{\prod_{i=1}^p n_i \cdot \log(\prod_{i=1}^p n_i)} \cdot \prod_{i=p+1}^d n_i^{\alpha_i},$$

for $\mathbf{k} \leq \mathbf{n}$ and $\delta > 0$ small. Furthermore, set

$$X'_{\mathbf{k}} = X_{\mathbf{k}} I\{|X_{\mathbf{k}}| \leq b_{\mathbf{n}}\}, \quad X''_{\mathbf{k}} = X_{\mathbf{k}} I\{b_{\mathbf{n}} < |X_{\mathbf{k}}| < c_{\mathbf{n}}\}, \quad X'''_{\mathbf{k}} = X_{\mathbf{k}} I\{|X_{\mathbf{k}}| \geq c_{\mathbf{n}}\},$$

and let all objects with primes or multiple primes refer to the respective truncated summands (and recall from above that $\log(\cdot) = \log^+(\cdot)$ throughout our computations).

Next, letting

$$A_{\mathbf{n}} = \{|S_{\mathbf{n}}| > (\varepsilon + 2\delta) \sqrt{\prod_{i=1}^p n_i \cdot \log(\prod_{i=1}^p n_i)} \cdot \prod_{i=p+1}^d n_i^{\alpha_i}\}, \quad (4.1)$$

and setting

$$A'_n = \left\{ |S_n| > \varepsilon \sqrt{\prod_{i=1}^p n_i \cdot \log\left(\prod_{i=1}^p n_i\right) \cdot \prod_{i=p+1}^d n_i^{\alpha_i}} \right\}, \quad A''_n = \{\text{at least two } \mathbf{k} \leq \mathbf{n} : X''_{\mathbf{k}} \neq 0\},$$

$$A'''_n = \{\text{at least one } \mathbf{k} \leq \mathbf{n} : X'''_{\mathbf{k}} \neq 0\},$$

we note that

$$A_n \subset A'_n \cup A''_n \cup A'''_n, \quad (4.2)$$

from which we conclude that

$$P(A_n) \leq P(A'_n) + P(A''_n) + P(A'''_n). \quad (4.3)$$

• $P(A'_n)$:

Since, by standard procedures,

$$|E S'_n| \leq |\mathbf{n}| \int_{|x|>\mathbf{n}} |x| dF(x) \leq \frac{|\mathbf{n}|}{b_n} \int_{|x|>\mathbf{n}} x^2 dF(x) = o\left(\sqrt{\prod_{i=1}^p n_i \cdot \log\left(\prod_{i=1}^p n_i\right) \cdot \prod_{i=p+1}^d n_i^{1-\beta_i}}\right),$$

an application of Lemma 2.4 with $x = \varepsilon \sqrt{\prod_{i=1}^p n_i \cdot \log\left(\prod_{i=1}^p n_i\right) \cdot \prod_{i=p+1}^d n_i^{\alpha_i}}$ and $t = 2\delta/b_n$ (noticing that $1 - \beta_i < \beta_i$) yields

$$\begin{aligned} P(|S'_n| > \varepsilon(1 + \delta)) &\leq \sqrt{\prod_{i=1}^p n_i \cdot \log\left(\prod_{i=1}^p n_i\right) \cdot \prod_{i=p+1}^d n_i^{\alpha_i}} \\ &\leq P(|S'_n - E S'_n| > \varepsilon \sqrt{\prod_{i=1}^p n_i \cdot \log\left(\prod_{i=1}^p n_i\right) \cdot \prod_{i=p+1}^d n_i^{\alpha_i}}) \\ &\leq \exp\left\{-\frac{\varepsilon^2}{\sigma^2} \log\left(\prod_{i=1}^p n_i\right) \prod_{i=p+1}^d n_i^{\alpha_i - \beta_i} + \frac{\varepsilon^2}{2\sigma^2} \cdot \frac{\log\left(\prod_{i=1}^p n_i\right)}{\prod_{i=p+1}^d n_i^{2\alpha_i - 1}} \cdot \prod_{i=p+1}^d n_i^{2(\alpha_i - \beta_i)} (1 + \delta)\right\} \\ &\leq \exp\left\{-\frac{\varepsilon^2}{2\sigma^2} \log\left(\prod_{i=1}^p n_i\right) \prod_{i=p+1}^d n_i^\eta (1 - \delta)\right\}, \end{aligned}$$

which we summarize as

$$P(A'_n) \leq \exp\left\{-\frac{\varepsilon^2(1 - \delta)}{2(1 + \delta)^2\sigma^2} \cdot \log\left(\prod_{i=1}^p n_i\right) \prod_{i=p+1}^d n_i^\eta\right\} \quad \text{for } \mathbf{n} \text{ large.} \quad (4.4)$$

As a preliminary for the final step we rewrite the estimate for this contribution to (1.15) as

$$\begin{aligned} \sum_{\mathbf{n} > \mathbf{n}_0} |\mathbf{n}|^{(r/2)-2} P(A'_n) &\leq \sum_{n_1, n_2, \dots, n_p} \prod_{i=1}^p n_i^{(r/2)-2} \sum_{n_{p+1}, n_{p+2}, \dots, n_d} \prod_{i=p+1}^d n_i^{(r/2)-2} \\ &\quad \times \exp\left\{-\frac{\varepsilon^2(1 - \delta)}{2(1 + \delta)^2\sigma^2} \cdot \log\left(\prod_{i=1}^p n_i\right) \prod_{i=p+1}^d n_i^\eta\right\}. \end{aligned} \quad (4.5)$$

In order to estimate the RHS it is necessary to distinguish between four cases, namely whether $\log(\prod_{i=1}^p n_i)$ and/or $\prod_{i=p+1}^d n_i^\eta$ are < 2 or ≥ 2 .

• $\log(\prod_{i=1}^p n_i) \geq 2$ and $\prod_{i=p+1}^d n_i^\eta \geq 2$

In this case,

$$\log\left(\prod_{i=1}^p n_i\right) \cdot \prod_{i=p+1}^d n_i^\eta \geq \log\left(\prod_{i=1}^p n_i\right) + \prod_{i=p+1}^d n_i^\eta,$$

so that the RHS in (4.5) is dominated by

$$\sum_{n_1, n_2, \dots, n_p} \prod_{i=1}^p n_i^{(r/2)-2-\frac{\varepsilon^2(1-\delta)}{2(1+\delta)^2\sigma^2}} \cdot \sum_{n_{p+1}, n_{p+2}, \dots, n_d} \prod_{i=p+1}^d n_i^{(r/2)-2} \exp\left\{-\frac{\varepsilon^2(1-\delta)}{2(1+\delta)^2\sigma^2} \cdot \prod_{i=p+1}^d n_i^\eta\right\},$$

where the first factor converges whenever $\varepsilon > \sigma \frac{1+\delta}{\sqrt{1-\delta}} \sqrt{r-2}$, and the second factor converges in view of Lemma 2.1(a) (with d replaced by $d-p$).

- $\log(\prod_{i=1}^p n_i) < 2$ and $\prod_{i=p+1}^d n_i^\eta \geq 2$

Since $n_i \rightarrow \infty$ for all i it follows that summation over the first p coordinates involves only a finite number of terms, and summation over the latter ones is convergent due to Lemma 2.1(a).

- $\log(\prod_{i=1}^p n_i) \geq 2$ and $\prod_{i=p+1}^d n_i^\eta < 2$

In this case summation over the latter coordinates involves a finite number of terms, and summation over the former ones turns into

$$\sum_{n_1, n_2, \dots, n_p} \prod_{i=1}^p n_i^{(r/2)-2} \exp\left\{-\frac{\varepsilon^2(1-\delta)}{2(1+\delta)^2\sigma^2} \cdot \log(\prod_{i=1}^p n_i)\right\} = \prod_{i=1}^p \sum_{n_i=1}^{\infty} n_i^{(r/2)-2-\frac{\varepsilon^2(1-\delta)}{2(1+\delta)^2\sigma^2}},$$

and convergence holds whenever $\varepsilon > \sigma \frac{1+\delta}{\sqrt{1-\delta}} \sqrt{r-2}$.

- $\log(\prod_{i=1}^p n_i) < 2$ and $\prod_{i=p+1}^d n_i^\eta < 2$

The sum converges trivially, since it contains only a finite number of terms.

As for the first contribution to (4.3) we have thus shown that

$$\sum_{\mathbf{n}} |\mathbf{n}|^{(r/2)-2} P(A'_{\mathbf{n}}) < \infty \quad \text{for } \varepsilon > \sigma \frac{1+\delta}{\sqrt{1-\delta}} \sqrt{r-2}. \quad (4.6)$$

- $P(A''_{\mathbf{n}})$:

Recalling that $\beta_i > 1/2$ we find that

$$\begin{aligned} P(A''_{\mathbf{n}}) &\leq |\mathbf{n}|^2 (P(|X| > b_{\mathbf{n}}))^2 \leq |\mathbf{n}|^2 \left(\frac{E|X|^r (\log^+ |X|)^{p-1-r/2}}{b_{\mathbf{n}}^r (\log b_{\mathbf{n}})^{p-1-r/2}} \right)^2 \\ &\leq C(r, \delta) |\mathbf{n}|^2 \left(\frac{\{\log(\prod_{i=1}^p n_i)\}^{r/2}}{\prod_{i=1}^p n_i^{r/2} \prod_{i=p+1}^d n_i^{r\beta_i} \left\{ \frac{1}{2} \log(\prod_{i=1}^p n_i) + \log(\prod_{i=p+1}^d n_i^{\beta_i})^{p-1-r/2} \right\}} \right)^2 \\ &\leq C(r, \delta) |\mathbf{n}|^{2-r} \frac{\{\log(\prod_{i=1}^p n_i)\}^r}{\log(|\mathbf{n}|^{1/2})^{p-1-r/2}}, \end{aligned}$$

which tells us that

$$\sum_{\mathbf{n}} |\mathbf{n}|^{(r/2)-2} P(A''_{\mathbf{n}}) \leq C(r, \delta) \sum_{\mathbf{n}} |\mathbf{n}|^{-r/2} \frac{\{\log(\prod_{i=1}^p n_i)\}^r}{\log(|\mathbf{n}|^{1/2})^{p-1-r/2}} < \infty \quad \text{for all } \delta > 0. \quad (4.7)$$

- $P(A'''_{\mathbf{n}})$:

Since

$$P(A'''_{\mathbf{n}}) \leq |\mathbf{n}| P(|X| > c_{\mathbf{n}}),$$

it follows, via Lemma 2.2(c), that

$$\sum_{\mathbf{n}} |\mathbf{n}|^{(r/2)-2} P(A'''_{\mathbf{n}}) \leq \sum_{\mathbf{n}} |\mathbf{n}|^{(r/2)-1} P(|X| > \delta \sqrt{\prod_{i=1}^p n_i \cdot \log\left(\prod_{i=1}^p n_i\right) \cdot \prod_{i=p+1}^d n_i^{\alpha_i}}) < \infty, \quad (4.8)$$

likewise for all $\delta > 0$.

By combining (4.1) and (4.3) with (4.6) – (4.8) we finally conclude that

$$\sum_{\mathbf{n}} |\mathbf{n}|^{(r/2)-2} P(|S_{\mathbf{n}}| > (\varepsilon + 2\delta) \sqrt{\prod_{i=1}^p n_i \cdot \log\left(\prod_{i=1}^p n_i\right) \cdot \prod_{i=p+1}^d n_i^{\alpha_i}}) < \infty \quad \text{for } \varepsilon > \sigma \frac{1+\delta}{\sqrt{1-\delta}} \sqrt{r-2},$$

which, in view of the arbitrariness of δ , finishes the proof of the implication (1.14) \implies (1.15).

The remaining parts of the proof follow along the usual lines, and are therefore omitted. We are thus done with the sufficiency part as well as with the necessity part with respect to the moment condition. Concerning the mean, if one would assume that $E(X) = \mu \neq 0$, then $S_{n_1, \dots, 1} \approx n_1 \cdot \mu$, which means that $P(|S_{n_1, 1, \dots, 1}| > \sqrt{n_1 \log n_1} \varepsilon)$ does not tend to zero for any $\varepsilon > 0$, and, hence, that the sums cannot converge, which, in turn forces the mean to be equal to zero.

5 Proof of Theorems 1.5, 1.6, and 1.7

The proof of Theorem 1.5 follows by modifying the previous one, crudely speaking by replacing logarithms by iterated ones. We therefore confine ourselves to providing the skeleton of the main steps.

Once again we let $\delta > 0$ be small, $0 < \eta < \alpha_{p+1} - 1/2$, and set $\beta_i = \alpha_i - \eta$, for $i = p+1, p+2, \dots, d$, so that $\beta_i > 1/2$. We truncate via

$$X'_{\mathbf{k}} = X_{\mathbf{k}} I\{|X_{\mathbf{k}}| \leq b_{\mathbf{n}}\}, \quad X''_{\mathbf{k}} = X_{\mathbf{k}} I\{b_{\mathbf{n}} < |X_{\mathbf{k}}| < c_{\mathbf{n}}\}, \quad X'''_{\mathbf{k}} = X_{\mathbf{k}} I\{|X_{\mathbf{k}}| \geq c_{\mathbf{n}}\},$$

for $\mathbf{k} \leq \mathbf{n}$, where

$$b_{\mathbf{n}} = b_{|\mathbf{n}|} = \frac{2\delta\sigma^2}{\varepsilon} \sqrt{\frac{\prod_{i=1}^p n_i}{\log \log(\prod_{i=1}^p n_i)}} \cdot \prod_{i=p+1}^d n_i^{\beta_i}, \quad c_{\mathbf{n}} = c_{|\mathbf{n}|} = \delta \sqrt{\prod_{i=1}^p n_i \cdot \log \log(\prod_{i=1}^p n_i)} \cdot \prod_{i=p+1}^d n_i^{\alpha_i},$$

Next, by introducing

$$\begin{aligned} A_{\mathbf{n}} &= \{|S_{\mathbf{n}}| > (\varepsilon + 3\delta) \sqrt{\prod_{i=1}^p n_i \cdot \log \log(\prod_{i=1}^p n_i)} \cdot \prod_{i=p+1}^d n_i^{\alpha_i}\}, \\ A'_{\mathbf{n}} &= \{|S'_{\mathbf{n}}| > \varepsilon \sqrt{\prod_{i=1}^p n_i \cdot \log \log(\prod_{i=1}^p n_i)} \cdot \prod_{i=p+1}^d n_i^{\alpha_i}\}, \quad A''_{\mathbf{n}} = \{\text{at least three } \mathbf{k} \leq \mathbf{n} : X''_{\mathbf{k}} \neq 0\}, \\ A'''_{\mathbf{n}} &= \{\text{at least one } \mathbf{k} \leq \mathbf{n} : X'''_{\mathbf{k}} \neq 0\}. \end{aligned}$$

it follows (of course) that

$$P(A_{\mathbf{n}}) \leq P(A'_{\mathbf{n}}) + P(A''_{\mathbf{n}}) + P(A'''_{\mathbf{n}}). \quad (5.1)$$

- $P(A'_{\mathbf{n}})$:

By estimating $|E S'_{\mathbf{n}}|$, and thereafter applying Lemma 2.4 as above, we obtain the following analog of (4.4):

$$P(A'_{\mathbf{n}}) \leq \exp \left\{ -\frac{\varepsilon^2(1-\delta)}{2(1+\delta)^2\sigma^2} \cdot \log \log \left(\prod_{i=1}^p n_i \prod_{i=p+1}^d n_i^{\eta} \right) \right\} \quad \text{for } \mathbf{n} \text{ large}, \quad (5.2)$$

and consequently that

$$\begin{aligned} \sum_{\mathbf{n} > \mathbf{n}_0} \frac{1}{|\mathbf{n}|} P(A'_{\mathbf{n}}) &\leq \sum_{n_1, n_2, \dots, n_p} \prod_{i=1}^p \frac{1}{n_i} \sum_{n_{p+1}, n_{p+2}, \dots, n_d} \prod_{i=p+1}^d \frac{1}{n_i} \\ &\quad \times \exp \left\{ -\frac{\varepsilon^2(1-\delta)}{2(1+\delta)^2\sigma^2} \cdot \log \log \left(\prod_{i=1}^p n_i \prod_{i=p+1}^d n_i^{\eta} \right) \right\}. \end{aligned} \quad (5.3)$$

Next we have to deal with the four cases.

- $\log \log(\prod_{i=1}^p n_i) \geq 2$ and $\prod_{i=p+1}^d n_i^{\eta} \geq 2$

The RHS of (5.3) is dominated by

$$\sum_{n_1, n_2, \dots, n_p} \prod_{i=1}^p \frac{1}{n_i} \left(\log \left(\prod_{i=1}^p n_i \right) \right)^{\frac{-\varepsilon^2(1-\delta)}{2(1+\delta)^2\sigma^2}} \cdot \sum_{n_{p+1}, n_{p+2}, \dots, n_d} \prod_{i=p+1}^d \frac{1}{n_i} \exp \left\{ -\frac{\varepsilon^2(1-\delta)}{2(1+\delta)^2\sigma^2} \cdot \prod_{i=p+1}^d n_i^{\eta} \right\}.$$

The first factor converges whenever $\varepsilon > \sigma(1 + \delta)\sqrt{2p/(1 - \delta)}$, and the second one converges in view of Lemma 2.1(a).

- $\log \log(\prod_{i=1}^p n_i) < 2$ and $\prod_{i=p+1}^d n_i^\eta \geq 2$

Summation over the first p coordinates involves only a finite number of terms, and summation over the latter ones is convergent due to Lemma 2.1(a).

- $\log \log(\prod_{i=1}^p n_i) \geq 2$ and $\prod_{i=p+1}^d n_i^\eta < 2$

Now summation over the latter coordinates involves only a finite number of terms, and summation over the former ones turns into

$$\sum_{n_1, n_2, \dots, n_p} \prod_{i=1}^p \frac{1}{n_i} \cdot \left(\log \left(\prod_{i=1}^p n_i \right) \right)^{-\frac{\varepsilon^2(1-\delta)}{2(1+\delta)^2\sigma^2}} < \infty \quad \text{whenever } \varepsilon > \sigma(1 + \delta)\sqrt{2p/(1 - \delta)}.$$

- $\log \log(\prod_{i=1}^p n_i) < 2$ and $\prod_{i=p+1}^d n_i^\eta < 2$

The sum converges trivially, since it contains only a finite number of terms.

By combining the four pieces we have thus shown that

$$\sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} P(A'_{\mathbf{n}}) < \infty \quad \text{for } \varepsilon > \sigma(1 + \delta)\sqrt{2p/(1 - \delta)}. \quad (5.4)$$

- $P(A''_{\mathbf{n}})$:

We argue as before to obtain ($1/2 < \beta < 1$ again)

$$\begin{aligned} P(A''_{\mathbf{n}}) &\leq |\mathbf{n}|^3 (P(|X| > b_{\mathbf{n}}))^3 \leq |\mathbf{n}|^3 \left(\frac{E|X|^2 (\log^+ |X|)^{p-1}}{b_{\mathbf{n}}^2 (\log b_{\mathbf{n}})^{p-1} / \log \log b_{\mathbf{n}}} \right)^3 \\ &\leq C(\delta) |\mathbf{n}|^3 \left(\frac{\log \log(\prod_{i=1}^p n_i^{1/2} \prod_{i=p+1}^d n_i^{\beta_i})}{\prod_{i=1}^p n_i \prod_{i=p+1}^d n_i^{2\beta_i} (\log(\prod_{i=1}^p n_i^{1/2} \prod_{i=p+1}^d n_i^{\beta_i})^{p-1})} \right)^3 \\ &\leq C(\delta) \frac{(\log \log |\mathbf{n}|)^3}{\prod_{i=p+1}^d n_i^{2\beta_i-1} (\log |\mathbf{n}|)^{3(p-1)}} \leq C(\delta) \frac{1}{\prod_{i=p+1}^d n_i^{2\beta_i-1} (\log |\mathbf{n}|)^{3(p-1)-\delta}} \\ &\leq C(\delta) \frac{1}{\prod_{i=p+1}^d n_i^{2\beta_i-1} (\log(\prod_{i=1}^p n_i))^{3(p-1)-\delta}}, \end{aligned}$$

and, hence, for $\delta > 0$ small enough by Lemma 2.1(b) and (c) (recall that $p \geq 2$),

$$\sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} P(A''_{\mathbf{n}}) \leq C(\delta) \sum_{n_1, \dots, n_p} \frac{1}{\prod_{i=1}^p n_i (\log(\prod_{i=1}^p n_i))^{3(p-1)-\delta}} \cdot \sum_{n_{p+1}, \dots, n_d} \frac{1}{\prod_{i=p+1}^d n_i^{2\beta_i}} < \infty. \quad (5.5)$$

- $P(A'''_{\mathbf{n}})$:

Since, once again, $P(A'''_{\mathbf{n}}) \leq |\mathbf{n}| P(|X| > c_{\mathbf{n}})$, it follows, however this time via Lemma 2.2(d), that

$$\sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} P(A'''_{\mathbf{n}}) \leq \sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} P(|X| > \delta \sqrt{\prod_{i=1}^p n_i \cdot \log \log \left(\prod_{i=1}^p n_i \right) \cdot \prod_{i=p+1}^d n_i^{\alpha_i}}) < \infty \quad \text{for all } \delta > 0. \quad (5.6)$$

By combining the different contributions as in the proof of Theorem 1.4, we finally conclude that

$$\sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} P(|S_{\mathbf{n}}| > (\varepsilon + 2\delta) \sqrt{\prod_{i=1}^p n_i \cdot \log \log \left(\prod_{i=1}^p n_i \right) \cdot \prod_{i=p+1}^d n_i^{\alpha_i}}) < \infty \quad \text{for } \varepsilon > \sigma \sqrt{2p},$$

where in the last step we also exploited the arbitrariness of δ .

The proof of Theorem 1.5 thus is complete. \square

The proofs of Theorem 1.6 and 1.7 amount to modifying the proof of Theorem 1.5 “in the obvious way”. In fact, Theorem 1.6 is a particular case of Theorem 1.7 ($q = d$).

6 Concluding remarks

First, some observations.

1. A preliminary inspection of (1.11), together with a glimpse on Theorem 1.2, tells us that, since

$$\begin{aligned} \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha_d(\alpha_1 r/\alpha_d)-2} P(|S_{\mathbf{n}}| > |\mathbf{n}|^{\alpha_d} \varepsilon) &\leq \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha_1 r-2} P(|S_{\mathbf{n}}| > |\mathbf{n}^{\alpha}| \varepsilon) \\ &\leq \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha_1 r-2} P(|S_{\mathbf{n}}| > |\mathbf{n}|^{\alpha_1} \varepsilon), \end{aligned}$$

it follows immediately that $E|X|^{\alpha_1 r/\alpha_d} (\log^+ |X|)^{d-1} < \infty$ and $EX = 0$ is necessary for Theorem 1.3 to hold and that $E|X|^r (\log^+ |X|)^{d-1} < \infty$ and $EX = 0$ is sufficient.

As for Theorem 1.4 an analogous argument, together with an appeal to [5], Section 3, shows that $E|X|^{r/2\alpha_d} (\log^+ |X|)^{d-1} < \infty$ and $EX = 0$ is necessary for Theorem 1.3 to hold, however, under the additional assumption that $\alpha_d > 1/2$ (consider, instead, the boundary $|\mathbf{n}|^{\alpha_d} \varepsilon$), and that $E|X|^r (\log^+ |X|)^{d-1-r/2} < \infty$ and $EX = 0$ is sufficient, (compare with the boundary $\sqrt{|\mathbf{n}| \log |\mathbf{n}|} \varepsilon$).

For Theorem 1.5 we observe, via [5], Theorem 6.2, that $E|X|^{1/\alpha_d} (\log^+ |X|)^{d-1} < \infty$ and $EX = 0$ is necessary (with $\alpha_d > 1/2$), and that $E|X|^2 \frac{(\log^+ |X|)^{d-1}}{\log^+ \log^+ |X|} < \infty$ and $EX = 0$ is sufficient for the theorem to hold.

2. The moment conditions in Theorems 1.4 and 1.5 depend only on the shortest edge(s). This is because more smoothening occurs during longer stretches, or, equivalently, more oscillations and “wild” behavior is visible in short ones. The moment assumptions are therefore adapted to “taming” the latter ones. On the other hand, this is not the case in Theorems 1.6 and 1.7. An explanation for this would be that in those cases all α ’s are equal to $1/2$, and the expansion rates only differ by different slowly varying functions, which is not sufficiently dramatic.

This might also elucidate the fact that there is no continuity in the conditions between the last two theorems and the earlier ones cf. e.g. Remarks 1.2, 1.5, and 1.6.

3. The “reason” that there is no third equivalent sum in the boundary cases $r = 2$ is because the weights of the probabilities are equal to $|\mathbf{n}|^{-1}$ which introduces the appearance of the harmonic series which grows logarithmically and not as “the next power”, which is the case as soon as $r > 2$. In order for a third sum to enter one has to introduce additional logarithms in the weights; cf. [5], Theorem 3.5.
4. A frequently convenient method of proof is to use symmetrization and desymmetrization. One reason for this is that truncated means remain zero, another one is that the Lévy inequalities are much easier to handle, since the medians are equal to zero. However, by doing so one always “looses” at least one factor 2 in the lower bound for ε as mentioned above. Now, when a sum is finite for all $\varepsilon > 0$ this does not cause any problem, but otherwise, as in some of our results, it does. In order to circumvent this one typically has to resort to two truncations and exponential inequalities, that is, the proofs become LIL-type proofs.
5. As just mentioned one normally needs two truncations in LIL-type proofs. Namely, one truncation adapted to the application of the exponential inequalities and one in order to match the moment requirement. Unfortunately these truncations do not coincide, and, typically, it is a thin central slice that is the hard one to deal with. It is, however, always possible to obtain a weaker result requiring only the lower truncation $b_{\mathbf{n}}$, a consequence of this being that the moment assumption equals “the inverse of the lower truncation” $b_{\mathbf{n}}$ (instead of the usual $c_{\mathbf{n}}$). For example, a sufficient condition for Theorem 1.5 would then correspond to evaluating the relevant moment condition for

$$\sum_{\mathbf{n}} P(|X| > \sqrt{\frac{\prod_{i=1}^p n_i}{\log \log (\prod_{i=1}^p n_i)}} \prod_{i=p+1}^d n_i^{\beta_i}) < \infty,$$

where $0 < \eta < \alpha_{p+1} - 1/2$ is chosen so that $\beta_i = \alpha_i - \eta > 1/2$. For the proof one would then prove the exponential part as before. The remaining part would correspond to $P(A_{\mathbf{n}}''')$ with $b_{\mathbf{n}}$ playing the rôle of $c_{\mathbf{n}}$ before. Comparing with Lemma 2.2(d) one finds that the corresponding moment assumption turns out as $E X^2 (\log^+ |X|)^{p-1} \log^+ \log^+ |X| < \infty$, which is (slightly) stronger than that of the theorem.

A similar reasoning applies in various results in the realm of the central limit theorem, where an additional assumption of a third moment permits one to apply the Berry–Esseen theorem.

6. The analog of “last exit times” related to the LLN and LIL have been investigated in various papers.

The analog with respect to Theorem 1.3 is

$$L(\varepsilon) = \sup\{|\mathbf{n}| : |S_{\mathbf{n}}| > \varepsilon |\mathbf{n}^{\alpha}|\},$$

for which we have the relation

$$\{L(\varepsilon) \geq j\} = \left\{ \sup_{j \leq |\mathbf{k}|} \left| S_{\mathbf{k}} / |\mathbf{k}^{\alpha}| \right| > \varepsilon \right\},$$

which tells us that

$$E(L(\varepsilon))^r \asymp \sum_{j=1}^{\infty} j^{r-1} P\left(\sup_{j \leq |\mathbf{k}|} \left| S_{\mathbf{k}} / |\mathbf{k}^{\alpha}| \right| > \varepsilon\right), \quad r > 0. \quad (6.1)$$

Combining this with Theorem 1.3 yields precise conditions for the moments of $L(\varepsilon)$ to be finite, namely:

Theorem 6.1 *Let $r > 0$, $\alpha_1 > 1/2$, and $\alpha_1 r > 1$. If $E X = 0$, $\text{Var } X = \sigma^2$, and $E|X|^r (\log^+ |X|)^{p-1} < \infty$, then*

$$E(L(\varepsilon))^{\alpha_1 r - 1} < \infty \quad \text{for all } \varepsilon > 0.$$

Conversely, if $E(L(\varepsilon))^{\alpha_1 r - 1} < \infty$ for all $\varepsilon > 0$, then $E|X|^r (\log^+ |X|)^{p-1} < \infty$ and, if $r \geq 1$, $E X = 0$.

REMARK 6.1 For $p = d$ this reduces to [5], Theorem 8.1. □

By arguing similarly, but with respect to Theorem 1.4, we set

$$L(\varepsilon) = \sup \left\{ |\mathbf{n}| : |S_{\mathbf{n}}| > \varepsilon \sqrt{\prod_{i=1}^p n_i \cdot \log \left(\prod_{i=1}^p n_i \right)} \cdot \prod_{i=p+1}^d n_i^{\alpha_i} \right\},$$

for which we have the relation

$$\{L(\varepsilon) \geq j\} = \left\{ \sup_{j \leq |\mathbf{k}|} \left| S_{\mathbf{k}} / \sqrt{\prod_{i=1}^p k_i \log k_i} \prod_{i=p+1}^d k_i^{\alpha_i} \right| > \varepsilon \right\},$$

which implies that

$$E(L(\varepsilon))^r \asymp \sum_{j=1}^{\infty} j^{r-1} P\left(\sup_{j \leq |\mathbf{k}|} \left| S_{\mathbf{k}} / \sqrt{\prod_{i=1}^p k_i \log k_i} \prod_{i=p+1}^d k_i^{\alpha_i} \right| > \varepsilon\right), \quad r > 0.$$

Combining this with Theorem 1.4 yields the following analog of Theorem 6.1.

Theorem 6.2 *If $E X = 0$, $\text{Var } X = \sigma^2$, and $E|X|^r (\log^+ |X|)^{p-1-r/2} < \infty$, then, for $r > 2$,*

$$E(L(\varepsilon))^{(r/2)-1} < \infty \quad \text{for } \varepsilon > \sigma \sqrt{r-2},$$

Conversely, if $E(L(\varepsilon))^{(r/2)-1} < \infty$ for all $\varepsilon > 0$, then $E|X|^r (\log^+ |X|)^{p-1-r/2} < \infty$ and $E X = 0$.

REMARK 6.2 For $p = d$ this reduces to [5], Theorem 8.2. □

In spite of the fact that all moments of $L(\varepsilon)$ are infinite one can show that if $EX = 0$, $\text{Var } X = \sigma^2$, and

$$EX^2 \frac{(\log^+ |X|)^p}{\log^+ \log^+ |X|} < \infty,$$

then

$$E \log L(\varepsilon) < \infty \quad \text{for all } \varepsilon > \sigma \sqrt{2(p+1)},$$

by extending [5] Theorem 8.1 in the obvious way to the analog of Theorem 1.5 above.

Similar results can, of course, be obtained with respect to the last exit times relative to Theorems 1.7 and 1.6.

There also exist results for the counting variable

$$N(\varepsilon) = \text{Card} \left\{ \mathbf{n} : |S_{\mathbf{n}}| > \varepsilon \sqrt{\prod_{i=1}^p n_i \cdot \log \left(\prod_{i=1}^p n_i \right) \cdot \prod_{i=p+1}^d n_i^{\alpha_i}} \right\},$$

albeit less precise ones.

As for analogous results related to the multiindex LIL, we observe that, for any p ,

$$\begin{aligned} N(\varepsilon) &= \text{Card} \left\{ \mathbf{n} : |S_{\mathbf{n}}| > \varepsilon \sqrt{\prod_{i=1}^p n_i \cdot \log \log \left(\prod_{i=1}^p n_i \right) \cdot \prod_{i=p+1}^d n_i^{\alpha_i}} \right\} \\ &\geq N_1(\varepsilon) = \text{Card} \left\{ \mathbf{n} : |S_{n_1, 1, 1, \dots, 1}| > \varepsilon \sqrt{n_1 \log \log n_1} \right\}, \end{aligned}$$

and since in view of Slivka's result [13] $E(N_1(\varepsilon))^r = \infty$ for all $r > 0$ and $\varepsilon > 0$, it follows immediately that

$$E(N_1(\varepsilon))^r = \infty \quad \text{for all } r > 0 \text{ and } \varepsilon > 0,$$

and all the more so for the last exit time.

For more on this we refer to [5], Section 8, and some additional references given there.

A Appendix

Here we present the proof of Lemma 2.2. Since ε acts as a scaling parameter it suffices to consider the case $\varepsilon = 1$. The proof amounts to, what is frequently called, "routine calculations", which, nevertheless, can be somewhat delicate and tricky.

(a) Cf. e.g. [4], Lemma 2.1 and further references given there.

(b) We generalize the method of summing along hyperbolas in that we treat the coordinates 1 through p in one portion and those from $p+1$ to d in another one. Summing over all values of $j = \prod_{i=1}^p n_i$ and n_k , $k = p+1, \dots, d$ separately, an upper bound of the sum becomes

$$\begin{aligned} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} d_p(j) d_{d-p}(k) \cdot (jk)^{\alpha_1 r - 1} P(|X| > j^{\alpha_1} k^{\alpha_d}) &\leq \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha_1 r - 1} P(|X| > |\mathbf{n}^{\alpha}|) \\ &\leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} d_p(j) d_{d-p}(k) \cdot (jk)^{\alpha_1 r - 1} P(|X| > j^{\alpha_1} k^{\alpha_{p+1}}). \end{aligned} \quad (\text{A.2})$$

Hence, with $\alpha = \alpha_{p+1}$ or $= \alpha_d$, respectively, we have to control

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} d_p(j) d_{d-p}(k) \cdot (jk)^{\alpha_1 r - 1} P(|X| > j^{\alpha_1} k^{\alpha}),$$

or, equivalently, using partial summation twice together with (2.1),

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (\log j)^{p-1} (\log k)^{d-p-1} \cdot (jk)^{\alpha_1 r - 1} P(|X| > j^{\alpha_1} k^{\alpha}).$$

This sum converges iff the following integral converges:

$$\iiint_{x^{\alpha_1} y^{\alpha} \leq u} (xy)^{\alpha_1 r - 1} (\log x)^{p-1} (\log y)^{d-p-1} dx dy dF_{|X|}(u).$$

In order to prove the desired conclusion it therefore suffices to show that

$$I(u) = \iiint_{x^{\alpha_1} y^{\alpha} \leq u} (xy)^{\alpha_1 r - 1} (\log x)^{p-1} (\log y)^{d-p-1} dx dy \asymp u^r (\log^+ u)^{p-1} \quad \text{as } u \rightarrow \infty.$$

Omitting second- and higher order terms—for a more careful treatment we refer to the proofs of the harder parts (e) and (f)—we obtain

$$\begin{aligned} I(u) &\sim \iint_{x^{\alpha_1} y^{\alpha} \leq u} (xy)^{\alpha_1 r - 1} (\log x)^{p-1} (\log y)^{d-p-1} dx dy \\ &\sim \int^{u^{1/\alpha}} y^{\alpha_1 r - 1} (\log y)^{d-p-1} \left(\int^{(u/y^\alpha)^{1/\alpha_1}} x^{\alpha_1 r - 1} (\log x)^{d-1} dx \right) dy \\ &\sim C \int^{u^{1/\alpha}} y^{\alpha_1 r - 1} (\log y)^{d-p-1} \left[x^{\alpha_1 r} (\log x)^{d-1} \right]^{(u/y^\alpha)^{1/\alpha_1}} dy \\ &\asymp \int^{u^{1/\alpha}} y^{\alpha_1 r - 1} (\log y)^{d-p-1} \frac{u^r}{y^{\alpha r}} (\log(u/y^\alpha))^{d-1} dy \\ &\asymp u^r (\log u)^{d-1} \int^{u^{1/\alpha}} y^{(\alpha_1 - \alpha)r} (\log y)^{d-p-1} \sim C u^r (\log u)^{d-1}, \end{aligned}$$

since the last integral is convergent (recall that, by assumption, $(\alpha - \alpha_1)r > 1$).

(c) Using the arguments from part (b), in particular inequalities (A.2), we have to deal with

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (jk)^{(\tau/2)-1} d_p(j) d_{d-p}(k) P(|X| > \sqrt{j \log j} k^\alpha),$$

with $\alpha = \alpha_{p+1}$ or $\alpha = \alpha_d$, respectively, which, in particular, implies that $\alpha > 1/2$.

This means that we have to check the convergence of the integral

$$\iiint_{\sqrt{x \log x} \cdot y^\alpha \leq u} (xy)^{(\tau/2)-1} (\log x)^{p-1} (\log y)^{d-p-1} dx dy dF_{|X|}(u),$$

which, is done by the same procedure as in the previous part.

(d) In this case the evaluation of

$$\sum_{\mathbf{n}} P(|X| > \sqrt{\prod_{i=1}^p n_i \cdot \log \log \left(\prod_{i=1}^p n_i \right) \prod_{i=p+1}^d n_i^{\alpha_i}})$$

with α as before, reduces to

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} d_p(j) d_{d-p}(k) P(|X| > \sqrt{j \log \log j} k^\alpha),$$

and the corresponding evaluation of the integral

$$\iiint_{\sqrt{x \log \log x} \cdot y^\alpha \leq u} (\log x)^{p-1} (\log y)^{d-p-1} dx dy dF_{|X|}(u)$$

leads to the desired moment condition.

(e) The last two parts have to be treated differently, because we do not end up with a convergent sum in the end as before.

By estimating the sum with the corresponding integral we obtain

$$\begin{aligned} \sum_{\mathbf{n}} P(|X| > \varepsilon) &\sqrt{\prod_{i=1}^d n_i \log \log \left(\prod_{i=1}^p n_i \right) \cdot \log \left(\prod_{i=p+1}^d n_i \right)} \\ &\asymp \int_{e^\varepsilon}^\infty \iint_{\sqrt{x \log \log x \cdot y \log y} \leq u} d_p(x) \cdot d_{d-p}(y) dx dy dF_{|X|}(u), \end{aligned}$$

where we define $d_\cdot(x) = d_\cdot(\lfloor x \rfloor)$ for $x > 0$.

Although it is *not* true that $d_\cdot(j)$ behaves as a power of $\log j$ asymptotically, it is all the same enough (equivalent) to show that

$$I(u) = \iint_{\sqrt{x \log \log x \cdot y \log y} \leq u} (\log x)^{p-1} (\log y)^{d-p-1} dx dy \asymp u^2 \frac{(\log u)^{d-2}}{\log \log u} \quad \text{as } u \rightarrow \infty,$$

in order to prove the desired conclusion.

Toward that end, omitting non-influential (iterated) logarithms, we have

$$\begin{aligned} I(u) &\sim \iint_{\sqrt{x \log \log x \cdot y \log y} \leq u} (\log x)^{p-1} (\log y)^{d-p-1} dx dy \\ &\sim \int^{u^2 / \log \log u} (\log x)^{p-1} \left(\int^{u^2 / (x \log \log x \cdot \log(u^2/x))} (\log y)^{d-p-1} dy \right) dx \\ &\sim \int^{u^2 / \log \log u} (\log x)^{p-1} \frac{u^2}{x \log \log x \cdot \log(u^2/x)} \cdot (\log(u^2/x))^{d-p-1} dx \\ &= u^2 \int^{u^2 / \log \log u} (\log(u^2/x))^{d-p-2} \cdot \frac{(\log x)^{p-1}}{x \log \log x} dx. \end{aligned}$$

Suppose first that $p \leq d-2$. For an upper bound we then have

$$\begin{aligned} I(u) &\leq u^2 (\log(u^2))^{d-p-2} \int^{u^2 / \log \log u} \frac{(\log x)^{p-1}}{x \log \log x} dx \\ &\sim C u^2 (\log u)^{d-p-2} \left[\frac{(\log x)^p}{\log \log x} \right]^{u^2 / \log \log u} \\ &\sim C u^2 (\log u)^{d-p-2} \frac{(\log u)^p}{\log \log u} = u^2 \frac{(\log u)^{d-2}}{\log \log u}, \end{aligned}$$

and for the lower bound,

$$\begin{aligned} I(u) &\geq u^2 \int^u (\log(u^2/x))^{d-p-2} \cdot \frac{(\log x)^{p-1}}{x \log \log x} dx \\ &\geq u^2 (\log u)^{d-p-2} \int^u \frac{(\log x)^{p-1}}{x \log \log x} dx \sim u^2 \frac{(\log u)^{d-2}}{\log \log u}. \end{aligned}$$

This concludes the proof for that case.

If $p = d-1$, then

$$I(u) \sim u^2 \int^{u^2 / \log \log u} \frac{1}{\log(u^2/x)} \cdot \frac{(\log x)^{p-1}}{x \log \log x} dx,$$

and the argument is “reversed, so that, for the lower bound we have

$$\begin{aligned} I(u) &\geq u^2 \frac{1}{2 \log u} \int^{u^2 / \log \log u} \frac{(\log x)^{p-1}}{x \log \log x} dx \\ &\sim C u^2 \frac{1}{\log u \log \log u} \frac{(\log u)^p}{\log \log u} = u^2 \frac{(\log u)^{p-1}}{\log \log u} = u^2 \frac{(\log u)^{d-2}}{\log \log u}. \end{aligned}$$

For the upper bound we split the original integral in two pieces.

$$I(u) = I(\sqrt{u}) + \int_u^{u^2/\log \log u} = I_1(u) + I_2(u).$$

Now,

$$I_1(u) \leq u \int_{u/\log \log(\sqrt{u})}^{u/\log \log(\sqrt{u})} \frac{(\log x)^{p-1}}{x \log \log x} dx \sim Cu \frac{(\log u)^p}{\log \log u} = o(u^2) \quad \text{as } u \rightarrow \infty,$$

and

$$\begin{aligned} I_2(u) &\leq u^2 \frac{1}{\log(u/\log \log(\sqrt{u}))} \cdot \int_{u/\log \log(\sqrt{u})}^{u^2/\log \log u} \frac{(\log x)^{p-1}}{x \log \log x} dx \\ &\sim Cu^2 \frac{1}{\log u} \cdot \frac{(\log u)^p}{\log \log u} = Cu^2 \frac{(\log u)^{d-2}}{\log \log u}. \end{aligned}$$

Combining the two integrals establishes the desired estimate.

(f) The proof of this part amounts to the estimation of

$$I(u) \sim \iiint_{\sqrt{x \log \log x \cdot y \log y \cdot z^\alpha \leq u}} (\log x)^{p-1} (\log y)^{q-p-1} (\log z)^{d-p-1} dx dy dz$$

following the ideas of the proof of part (e).

This concludes the proofs of the sufficiencies. The converses are immediate, since the proofs of the sufficiencies throughout are based on \sim , \asymp , and upper and lower bounds. \square

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