Numerical Algorithms for Free Boundary Problems of Obstacle types

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Abstract

This thesis consists of four scientific papers concerning numerical methods of certain free boundary problems. This consists of mathematical analysis of different approximations for problems and the description of numerical implementation along with the numerical results.

Paper I deals with a free boundary problem that appears in biology modeling. Two novel iterative methods for a class of population models of competitive type are introduced. These numerical solutions are related to the positive solution as the competitive rate tends to infinity. Furthermore, the first method is applied to an optimal partition problem.

In Paper II we study perturbation of the free boundary problem

$$\begin{cases} \triangle u_i = \lambda^+ \chi_{\{u_i>0\}} - \lambda^- \chi_{\{u_i<0\}} & \text{in } \Omega, \\ u_i = g_i & \text{on } \partial\Omega. \end{cases} \tag{1}$$

We perturb the data in the right-hand side of the two-phase equation. The main result of the paper is that continuity and differentiability of the solution with respect to the coefficients are proved. Also the stability of the solution and the free boundary with respect to perturbation in coefficients and boundary values, is shown.

In paper III different numerical approximations for a two-phase membrane problem are discussed. In the first method a new iterative method with different examples is presented. We study the regularization method and give an a-posteriori error estimate which is needed for the implementation of the regularization method. Moreover, an efficient algorithm based on the finite element method is given. It is shown that the sequence constructed by the algorithm is monotone and converges to the solution of a given free boundary problem.

The last paper deals with numerical solutions for the m-membrane problem. We consider minimization of the functional

$$I = \int_\Omega \sum_i^m (\frac{1}{2}|\nabla u_i|^2 + f_i \cdot u_i)\, dx, \tag{2}$$

over the set $\{(u_1, ..., u_m) \mid u_i - g_i \in H_0^1(\Omega),\, u_1 \geq u_2 \geq ... \geq u_m\}$.

Sammanfattning

Denna avhandling innehåller fyra vetenskapliga artiklar som studerar numeriska metoder för olika frirandsproblem.

Den första artikeln behandlar ett frirandsproblem som uppstår vid modellering inom biologi. Vi introducerar två nya iterativa metoder för klassen av populationsmodeller av konkurrenstyp.

De numeriska metoderna är speciellt avsedda för att studera den positiva lösningen då konkurrenssituationen omöjliggör samexistens. Därutöver är metoderna tillämpade på ett så kallat "optimal partition problem".

I den andra artikel studerar vi störningar av data i högerledet till följande frirandsproblem:

$$\begin{cases} \triangle u_i = \lambda^+ \chi_{\{u_i>0\}} - \lambda^- \chi_{\{u_i<0\}} & \text{i } \Omega, \\ u_i = g_i & \text{på } \partial\Omega. \end{cases} \tag{3}$$

Vi diskuterar kontinuiteten och deriverbarheten med avseende på koefficienterna λ^- och λ^+. Dessutom studerar vi stabiliteten av lösningen och fria randen med avseende på störningar i koefficienter och randvillkor.

I tredje artikeln presenterars olika numeriska approximeringar av tvåfashinderproblemet.

Den fjärde, och sista artikeln behandlar numeriska lösningar till m-membranproblemet. Vi betraktar följande minimisering av funktionalen

$$I = \int_\Omega \sum_i^m (\frac{1}{2}|\nabla u_i|^2 + f_i \cdot u_i) dx, \tag{4}$$

över mängden $\{(u_1, ..., u_m) \mid u_i - g_i \in H_0^1(\Omega), \quad u_1 \geq u_2 \geq ... \geq u_m\}$.

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Scientific papers

Paper I
Numerical algorithms for spatial segregation of competitive systems

To appear in SIAM Journal on Scientific Computing

Paper II
Perturbation formula of the two-phase membrane problem

Submitted

Paper III
Numerical solutions of the two-phase membrane problem
Submitted

Paper IV
Numerical solution of the m-membranes problem
(joint with Marzena Franek)
Manuscript

Chapter 1

Introduction to free boundary problems

1.1 Some examples of free boundary problems

The topics of free boundary problems is an important branch of non linear partial differential equations (PDEs). More precisely, free boundary problems are boundary value problems in which some parts of the boundaries of the considered domains have to be determined as a part of the solution. Such problems deal with solving PDEs in a domain, such that a part of the boundary is not known in advance; that part of the boundary is called the free boundary. In order to have uniqueness, in addition to the standard boundary conditions an extra condition must be imposed on the free boundary.

The theory of free boundary problems has been greatly developed in the last forty years. Besides the progress in theory of free boundary, many problems arising in mechanics, physics, biology, financial mathematics etc, can be formulated as free boundary problems. In this section some examples of free boundary problems and applications are given.

A classical example is the one-phase obstacle problem. Given an elastic membrane u attached to a fixed boundary $\partial\Omega$ and an obstacle ψ, we seek the equilibrium position of the membrane when it is moved towards the obstacle (for more details see Section 1.3).

The second example is the Stefan problem [12], describing the process of melting and solidification. In the Stefan problem the free boundary is the boundary of a solid material and it can therefore be characterized as a liquid/solid interface.

Others examples are tumor growth, coating flows, chemical vapor deposition, image development in electro-photography and financial mathematics (see [9, 15]).

For most of the free boundary problems it is not possible to derive explicit solutions. In many cases the geometry of free boundary is complicated. Therefore, numerical methods are needed to compute approximation of the solutions. Accurate

approximations of such problems demand a strong combination of analysis and approximation methods. Note that since the free boundary has to be solved as a part of the solution, standard methods for boundary value problems can not be applied directly.

1.2 Basic facts and notations

The following notations are fixed throughout.

Ω : an open subset of $\mathbb{R}^n$,

V : a real Hilbert space with a scalar product $(\cdot,\cdot)$ and associated norm $\|\cdot\|$,

V^* : the dual space of V,

$a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is a bilinear, continuous and V-elliptic form on $V \times V$,

$L : V \to \mathbb{R}$ is a continuous, linear functional,

K : is a closed convex nonempty subset of V,

$j(\cdot) : V \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is a convex lower semi conscious functional.

The notion of weak derivative will be used in the development of the variational formulation of partial differential equations.

Definition 1. *The function $u \in L^1_{\text{loc}}(\Omega)$ possesses a weak derivative, if there exists a function $v \in L^1_{\text{loc}}(\Omega)$ such that*

$$\int_\Omega v(x)\phi(x)\,dx = (-1)^{|\alpha|}\int_\Omega u(x)D^\alpha\phi(x)\,dx \text{ for all } \phi(x) \in C_0^\infty(\Omega).$$

Here, the differential operator D^α is given by

$$D^\alpha u = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} u(x_1, ..., x_n) \text{ with } |\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n.$$

Fix $1 \le p \le \infty$ and let m be a nonnegative integer. We define the space $W^{m,p}(\Omega)$ to consist of all locally summable functions $u : \Omega \to \mathbb{R}$ such that for each multi-index α with $|\alpha| \le m$, $D^\alpha u$ exists in weak sense and belongs to $L^p(\Omega)$. The space $W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the space $W^{m,p}(\Omega)$.

1.3 Finite element approximation

The finite element method has become popular and is an efficient method for solving partial differential equations over complex domains, also when the domain changes, or the desired precision varies over the entire domain, or when the solution lacks smoothness. In this section the main idea of the finite element method and the Galerkin discretization are explained.

Consider the following Poisson problem

$$-\Delta u = f \text{ in } \Omega, \tag{1.1}$$

$$u = 0 \text{ on } \partial\Omega. \tag{1.2}$$

If we multiply equation (1.1) by a test function $v \in H_0^1$ and then integrate by parts we obtain

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx. \tag{1.3}$$

The solution of (1.3) is called the weak or variational solution. Set

$$V = H_0^1(\Omega),\ a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx \text{ and } \langle f, v\rangle = \int_\Omega f v \, dx.$$

Now we can reformulate (1.1) in the weak sense as follows.

Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = \langle f, v\rangle \text{ for all test functions } v \in H_0^1(\Omega). \tag{1.4}$$

The existence of the weak solution follows from Lax-Milgram's Theorem [16]. Then discretization is obtained by replacing V with a finite dimensional subspace V_h. Let h be a given parameter converging to 0 and $\{V_h\}$ be a family of closed subspaces of V satisfying for any $v \in V$

$$\inf_{w \in V_h} \|v - w\| \to 0 \text{ as } h \to 0.$$

The idea of the Galerkin method is, for a given f find $u_h \in V_h$ such that

$$a(u_h, v_h) = \langle f, v_h\rangle \text{ for all } v_h \in V_h. \tag{1.5}$$

The existence of a discrete solution $u_h \in V_h$ follows from minimizing the functional in the finite dimensional (complete subspace). Let $\phi_1, ..., \phi_n$ be a basis of V_h. Since problem (1.5) is satisfied for any $v_h \in V_h$, we can replace v_h by basis functions ϕ_i for $i = 1, 2, .., n$. We have

$$a(u_h, \phi_i) = \langle f, \phi_i\rangle,\ i = 1, \cdots, n. \tag{1.6}$$

Note that an approximate solution u_h can be written in the form

$$u_h = \sum_{i=1}^{n} u_i \phi_i. \tag{1.7}$$

Substituting the expression (1.7) into (1.6), the following system of equations is obtained

$$\sum_{j=1}^{n} a(\phi_j, \phi_i) u_j = \langle f, \phi_i\rangle \quad \text{for } i = 1, ..., n. \tag{1.8}$$

The expression (1.8) can be written in the form

$$SU = F, \tag{1.9}$$

where the elements of the matrix S are given by $s_{ij} = a(\phi_j, \phi_i)$, the elements of the vector F are $F_i = \langle f, \phi_i \rangle$ and the vector U consists of unknown values u_j. The matrix S is called the stiffness matrix and F is called the load vector. Thus, to find the numerical solution of a second order elliptic problem one should compute the stiffness matrix S, the vector F and subsequently solve the system (1.9). To summarize, in the finite element method the problem is written first in weak formulation. Then the domain is subdivided into smaller pieces (triangles or rectangles). Note that in each element the test functions are given in a simple form (normally polynomials).

1.4 Elliptic variational inequalities

A class of nonlinear problems arising in mechanics and physics can be formulated as variational inequalities. Some examples are the problem of lubrication, the steady filtration of a liquid through a porous membrane in two and tree dimensions, the motion of a fluid past a given obstacle profile and the small deflections of an elastic beam (for more details see [10, 16, 17]). We consider mainly two kinds of elliptic variational inequalities (EVI).

1. EVI of first kind:

   Find $u \in V$ such that u is the solution of the following problem

   $$a(u, v-u) \geq L(v-u), \quad \forall v \in K, \quad u \in K, \tag{1.10}$$

   where K is defined in Section 1.2.

   Remark 1.1. The obstacle problem can be written as EVI. Indeed,

   $$V = H_0^1(\Omega), \quad V^* = H^{-1}(\Omega), \quad a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx,$$

   and

   $$L(v) = \langle f, v \rangle, \quad K = \{v \in H_0^1(\Omega) \mid v \geq \psi \text{ a.e. in } \Omega\}.$$

   Other examples are the Elasto-Plastic Torsion problem (see [14]) and Signorini problem, which can be obtained from EVI of the first kind and have physical motivation.

2. EVI of the second kind:

   Find $u \in V$ such that u is the solution of the following problem

   $$a(u, v-u) + j(v) - j(u) \geq L(v-u), \quad \forall v \in K, \quad u \in K. \tag{1.11}$$

Remark 1.2. If $K = V$ and $j = 0$, then problems (1.10) and (1.11) reduce to the following variational equation

$$a(u, v) = L(v), \quad \forall v, u \in V.$$

The existence and uniqueness results for EVI of the first and the second kind have been shown by Lions and Stampacchia [17].

Assume that $a(\cdot, \cdot)$ is symmetric. Let $I : V \to \mathbb{R}$ be defined by

$$I(v) = \frac{1}{2}a(v, v) - L(v).$$

Then I is strictly convex, continuous and $\lim_{\|v\| \to \infty} I(v) = \infty$. Consider the following minimization problem : Find u such that

$$I(u) \leq I(v), \quad \forall v \in K, \quad u \in K. \tag{1.12}$$

Since I is strictly convex, continuous and proper, then problem (1.12) has a unique solution (see Cea [5]). Therefore the above minimization problem (1.12) is equivalent to finding u such that

$$(I'(u), v - u) \geq 0, \quad \forall\, v \in K, \quad u \in K,$$

where $I'(u)$ is the Gateaux derivative of I at u. Note that

$$(I'(u), v - u) = a(u, v) - L(v),$$

which shows if $a(\cdot, \cdot)$ is symmetric then (1.10) and (1.12) are equivalent.

To conclude this section the approximation of EVI problems is briefly explained. Let $\{V_h\}$ be a family of closed subspaces of V. Also let $\{K_h\}$ be a family of closed, convex and nonempty subsets of V with $K_h \subset V_h$, such that if $v_h \in K_h$ and $\{v_h\}$ is bounded in V, then the weak cluster points of $\{v_h\}$ belong to K. Problem (1.10) is approximated by

$$a(u_h, v_h - u_h) \geq L(v_h - u_h), \quad \forall v_h \in K_h, \quad u_h \in K_h. \tag{1.13}$$

Theorem 1.3. *[10] Problem (1.13) has a unique solution. Furthermore,*

$$\lim_{h \to 0} \| u_h - u \|_V = 0,$$

where u_h and u are solutions of (1.10) and (1.13), respectively.

Another alternative for solving elliptic variational inequality is the penalty method [10].

1.5 An introduction to obstacle-type problems

Assume that a function $\psi \in C^2(\Omega)$, called the obstacle is given such that $g \geq \psi$ on $\partial\Omega$. Consider the minimizer of functional

$$I(u) = \int_\Omega |\nabla u|^2 dx, \tag{1.14}$$

over the constrained set

$$K = \{u \in W^{1,2}(\Omega) : u - g \in W_0^{1,2}(\Omega), u \geq \psi \text{ a.e. in } \Omega\}.$$

Since I is continuous and strictly convex on a convex subset K of the Hilbert space $W^{1,2}(\Omega)$, it has a unique minimizer on K. Therefore, the following theorem holds (see[2]).

Theorem 1.4. *There exists a unique minimizer for the obstacle problem (1.14). Moreover, the minimizer u is super-harmonic in Ω and harmonic in $\{u > \psi\}$. Furthermore, u is the least super-harmonic function above the obstacle and u is the only super-harmonic function that is harmonic whenever it is above the obstacle.*

The set where the membrane touches the obstacle, denoted by $\Lambda = \{u = \psi\}$, is known as the coincidence set. Setting $\Omega^+ = \Omega \setminus \Lambda$, the boundary

$$\Gamma = \partial\Lambda \cap \Omega = \partial\Omega^+ \cap \Omega$$

is called the free boundary, as it is a priori unknown. Assume that ψ is smooth then what is the best regularity one can get for u? The answer is that regardless of how smooth the obstacle is, $u \notin C^2$. The optimal regularity for the solution is $C^{1,1}$. To avoid technicalities, we shall assume $\Delta\psi = -1$. Let $w = u - \psi$. Assume that w is a $C^{1,1}$ function defined on B_1 that satisfies

$$\begin{aligned} &w \geq 0 \text{ in } B_1, \\ &\Delta w \leq 1, \text{ in } B_1, \\ &\Delta w = 1, \text{ in } \{w > 0\}, \\ &0 \text{ belongs to free boundary.} \end{aligned}$$

With the above assumptions, the following theorem says about the regularity of the free boundary [2].

Theorem 1.5. *($C^{1,\alpha}$ smoothness of the free boundary) Let w be a normalized solution. There exists a universal modulus of continuity $\sigma(r)$, such that if for some r, $\Lambda(w) \cap B_r$ cannot be enclosed in a strip of width $r\sigma(r)$, then in an r^2-neighborhood of the origin, the free boundary is a $C^{1,\alpha}$ hyper-surface for $0 < \alpha < 1$. The free boundary, $\partial\Omega$, has a locally finite $\mathcal{H}^{n-1}$ Hausdorff measure and*

$$\mathcal{H}^{n-1}(\partial\Omega^+ \cap B_r(x_0)) \leq Cr^{n-1}.$$

In general we consider solutions of the equations of the type

$$\Delta u = f(x, u, \nabla u) \text{ in } \Omega, \tag{1.15}$$

where the right hand side term is supposed to be piecewise continuous, having jumps at some values of the arguments u and ∇u. We will assume that $u \in L^{\infty}_{\text{loc}}(\Omega)$ and $f \in L^{\infty}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ and that the equation (1.15) is satisfied in the sense of distributions, that is,

$$\int_{\Omega} u\Delta\eta \, dx = \int_{\Omega} f(x, u, \nabla u)\eta \, dx \text{ for all } \eta \in C_0^{\infty}(\Omega).$$

Two phase membrane problem

Given a bounded open set Ω in $\mathbb{R}^n$, with smooth boundary and let $\lambda^{\pm} : \Omega \to \mathbb{R}^+$ be non-negative bounded functions. Consider the minimization of the functional

$$I(v) = \int_{\Omega} (\frac{1}{2}|\nabla v|^2 + \lambda^+ \max(v, 0) - \lambda^- \min(v, 0))dx, \tag{1.16}$$

over $K = \{v \in W^{1,2}(\Omega) : v - g \in W_0^{1,2}(\Omega)\}$. The functional $I(v)$ is convex, weakly lower semi-continuous and attains its infimum at some $u \in K$. Then the Euler-Lagrange equation corresponding to the minimizer is (see [20])

$$\begin{cases} \triangle u = \lambda^+ \chi_{\{u>0\}} - \lambda^- \chi_{\{u<0\}} & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{1.17}$$

We use the following notations:

χ_A : the characteristic function of the set A.
$\Gamma(u) = \partial\{x \in \Omega : u(x) > 0\} \cup \partial\{x \in \Omega : u(x) < 0\} \cap \Omega$: free boundary.
$\Gamma'(u) = \Gamma(u) \cap \{x \in \Omega : \nabla u(x) = 0\}$, and $\Gamma''(u) = \Gamma(u) \cap \{\nabla u \neq 0\}$.
$\Gamma^+ = \partial\{x \in \Omega : u(x) > 0\}$, and $\Gamma^- = \partial\{x \in \Omega : u(x) < 0\}$.
$\Omega^+(u) = \{x \in \Omega : u(x) > 0\}$, $\Omega^-(u) = \{x \in \Omega : u(x) < 0\}$ and $\Lambda(u) := \{x \in \Omega : u(x) = 0\}$.
$\Gamma^+ \cap \Gamma^- \cap \{x \in \Omega : \nabla u(x) = 0\}$: branch points.

The physical interpretation of this problem is the consideration of a thin membrane (film) which is fixed on the boundary of a given domain, and some part of the boundary data of this film is below the surface of a thick liquid (heavier than the film itself). Now the weight of the film produces a force downwards λ^+ on that part of the film which is above the liquid surface. On the other side the part in the liquid is pushed upwards by a force λ^-, since the liquid is heavier than the film. Obviously the equilibrium state of the film is given by a minimization of the above mentioned functional. One of the difficulties one confronts in this problem is that the interface $\partial\{x : u(x) = 0\}$ consists in general of two parts : one where the

gradient of u is nonzero and one where the gradient of u vanishes. Close to points of the latter part we expect the gradient of u to have linear growth.

The regularity of the solution and the free boundary are discussed in [18, 20]. In [19] it has been shown that if λ^+ and λ^- are Lipschitz, then in higher dimensions the free boundary in a neighborhood of each branch point is the union of two C^1-graphs.

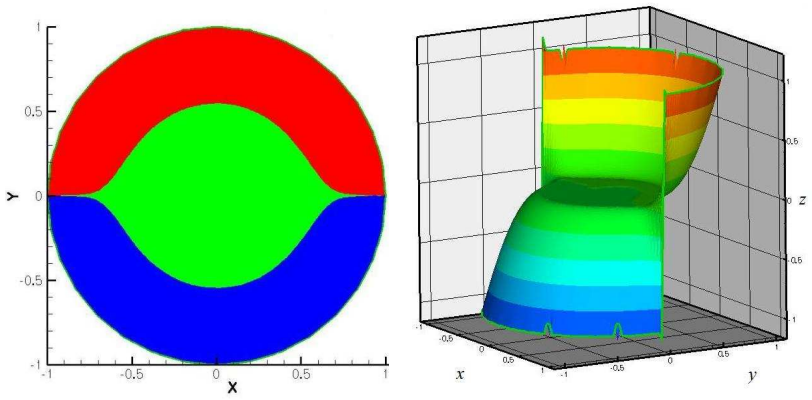


Figure 1.1: The left figure shows the zero set, positivity and negative part of u. The right figure shows the surface u. The boundary conditions are $u = 1$ for $y > 0$ and $u = -1$ when $y < 0$.

1.6 Spatial segregation of competitive systems

In this section the asymptotic behavior of a family of singularly perturbed systems of elliptic type, is discussed. All of the proofs and theorems can be found in [3, 4, 6, 7, 8]. In this part some proofs are simplified. The methods in the first paper are based on these theorems, which has motivated us to include them in this part.

1. A central problem in population ecology is studying of the interactions between biological components. To achieve this aim, different models based on reaction diffusion equations are studied, see [6, 7].

   Let $\Omega \subset \mathbb{R}^n$ be a connected, open bounded domain with smooth boundary. We consider the system of m differential equations

$$\begin{cases} -\triangle u_i = -\frac{1}{\varepsilon} u_i(x) \sum_{j \neq i}^{m} a_{ij} u_j(x) + f_i(x, u_i(x)) & \text{in } \Omega, \\ u_i \geq 0 & \text{in } \Omega, \\ u_i(x) = \phi_i(x) & \text{on } \partial\Omega, \\ i = 1, 2, ..., m. \end{cases} \tag{1.18}$$

This system describes the stationary states of the evolution of m components diffusing and competing for resources. Here u_i shows the populations density of the i-th components, whose internal dynamic is given by f_i. The positive constants $\frac{1}{\varepsilon} \cdot a_{ij}$ determine the interaction between the population u_i and u_j. The boundary values ϕ_i are positive functions with disjoint supports, $\phi_i \cdot \phi_j = 0$ for $i \neq j$, almost everywhere on $\partial\Omega$.

2. The second related problem occurs when the grouping is nontrivial. Such a problem arises in a more general segregation of competing species, see [3, 4]. Let $u^\varepsilon = (u_1^\varepsilon, \cdots, u_m^\varepsilon)$. Consider the functional

$$\int_\Omega [\, |\nabla u^\varepsilon|^2 + 2F^\varepsilon(u^\varepsilon)\,]\, dx \tag{1.19}$$

where $F^\varepsilon(u) = \sum_{i<j} \frac{\beta_{ij}}{\varepsilon} u_i^2 u_j^2$ with $\beta_{ij} \geq 0$ and $u^\varepsilon = \phi$ on the boundary of Ω, such that each of the components of ϕ is nonnegative.

We state some basic facts about (1.18). The problem (1.19) has the same properties. First we prove some a priori estimates for the solutions of (1.18). For simplicity assume that $f_i = 0$. Let $U = (u_1, ..., u_m)$ be a solution of (1.18) for fixed ε. Since the right hand side of the equation for u_i is non negative,

$$\Delta u_i \geq 0 \text{ in } \quad \Omega. \tag{1.20}$$

For all i, define

$$\widehat{u_i} = u_i - \sum_{i \neq j}^{m} \frac{a_{ij}}{a_{ji}} u_j.$$

We have

$$\Delta \widehat{u_i} = -\frac{1}{\varepsilon} \sum_{j \neq i} \frac{a_{ij}}{a_{ji}} \sum_{k \neq j} a_{jk} u_j u_k,$$

which implies

$$\Delta \widehat{u_i} \leq 0. \tag{1.21}$$

Consider the class

$$F := \{U \in (H^1(\Omega))^m : u_i \geq 0,\, \Delta u_i \geq 0,\, \Delta \widehat{u_i} \leq 0,\, u_i = \phi_i \text{ on } \partial\Omega, i = 1, ..., m\}.$$

The functions in F have the following properties (see [6]).

Lemma 1.6. *Let v_i be the harmonic extension in Ω with boundary data ϕ_i and let h_i be the harmonic extension on Ω with boundary data $\hat{\phi}_i = \phi_i - \sum_{j \neq i}^m \phi_i$. Then*

1. $h_i \leq \hat{u_i} \leq u_i \leq v_i$ *in* Ω.

2. $\frac{\partial v_i}{\partial n} \leq \frac{\partial u_i}{\partial n} \leq \frac{\partial h_i}{\partial n}$ *on* $\partial\Omega$.

3. U is bounded in $(H^1(\Omega))^m$.

Proof. By (1.21) we have $\Delta(\widehat{u}_i - h_i) \leq 0$ in Ω and $\widehat{u}_i - h_i = 0$ on $\partial\Omega$. By the maximum principle,

$$\widehat{u}_i \geq h_i \quad \text{in} \quad \overline{\Omega}, \text{ which implies } \frac{\partial \widehat{u}_i}{\partial n} \leq \frac{\partial h_i}{\partial n} \text{ on } \partial\Omega. \tag{1.22}$$

A similar argument implies that $u_i \leq v_i$. Note that if $x \in \partial\Omega$ and $u_i(x) = \phi_i(x) > 0$, then $u_j(x) = 0$ for all $j \neq i$ and since $u_j \geq 0$ this gives $\frac{\partial u_j(x)}{\partial n} \leq 0$. From (1.22), it follows that

$$\frac{\partial u_i(x)}{\partial n} \leq \sum_{j \neq i} \frac{a_{ij}}{a_{ji}} \frac{\partial u_j(x)}{\partial n} + \frac{\partial h_i(x)}{\partial n} \leq \frac{\partial h_i(x)}{\partial n}.$$

Now we show that F is uniformly bounded. Let U be a fixed. Multiplying the inequality $\Delta u_i \geq 0$ by u_i and using Green's formula

$$\int_\Omega |\nabla u_i|^2 dx - \int_{\partial\Omega} \phi_i \frac{\partial u_i}{\partial n} ds \leq 0.$$

By part (2) in this lemma, $\frac{\partial u_i}{\partial n} \leq \frac{\partial h_i}{\partial n}$ on the boundary. Therefore $\|u_i\|_{H^1(\Omega)}$ is bounded by a constant depending only on the boundary value. □

Theorem 1.7. *[6] For every fixed $\varepsilon > 0$ there exists at least one solution U^ε of (1.18). Moreover, U^ε is of class $W^{1,\infty}(\overline{\Omega})$ and it belongs to the class F.*

By Theorem 1.7 we know that for every fixed $\varepsilon > 0$, a solution exist. We are interested in the behavior of the system as $\varepsilon \to 0$. By the following theorem we find that a limit always exists (up to subsequences), and it belongs itself to the class F.

Theorem 1.8. *[6] Let $U = (u_1, ..., u_m)$ be a solution of (1.18) at fixed ε. Let $\varepsilon \to 0$. Then, there exists $U \in F$ such that, for all $i = 1, ..., m$*

(i) up to subsequences, $u_i^\varepsilon \to u_i$ strongly in $H^1(\Omega)$,

(ii) if $i \neq j$ then $u_i \cdot u_j = 0$ a.e. in Ω,

(iii) $\Delta u_i = 0$ in the set $\{x \in \Omega : u_i(x) > 0\}$.

Proof. The last part in Lemma 1.6 shows the existence of a weak limit U such that, up to subsequences, $u_i^\varepsilon \to u_i$ in $H^1(\Omega)$. The weak limit U belongs to F, since the differential inequalities (1.20) and (1.21) for u_i^ε, pass to the weak limit, and by the trace theorem $u_i = \phi_i$ on $\partial\Omega$. Multiplying the differential equation for u_i^ε by u_i^ε we get

$$\int_\Omega |\nabla u_i^\varepsilon|^2 dx - \int_{\partial\Omega} \phi_i \frac{\partial u_i^\varepsilon}{\partial n} ds = -\frac{1}{\varepsilon} (u_i^\varepsilon)^2 \sum_{j \neq i} a_{ij} u_j^\varepsilon.$$

By Lemma (1.6), the left hand side is bounded independently of ε. Hence,

$$(u_i^\varepsilon)^2 \sum_{j \neq i} a_{ij} u_j^\varepsilon \to 0 \text{ as } \varepsilon \to 0.$$

Therefore $u_i^\varepsilon \cdot u_j^\varepsilon \to 0$ a.e. in Ω when $i \neq j$ and consequently, $u_i \cdot u_j = 0$ a.e. in Ω. This implies that in the set $\{x : u_i(x) > 0\}$, we have $\widehat{u_i} = u_i$ and consequently it holds $\Delta u_i = \Delta \widehat{u_i}$ on $\{x \in \Omega : u_i(x) > 0\}$. But by (1.20) and (1.21) we obtain that

$$\Delta u_i = 0 \text{ in the set } \{x \in \Omega : u_i(x) > 0\}.$$

To prove the strong convergence in H^1, let us test the inequality $\triangle \widehat{u_i} \leq 0$ with u_i. Then

$$\int_\Omega |\nabla u_i|^2 dx - \int_{\partial\Omega} \phi_i \frac{\partial u_i}{\partial n} ds \geq 0 \tag{1.23}$$

If we multiply $\Delta u_i^\varepsilon \geq 0$ by u_i^ε and integrating by parts it holds

$$\int_\Omega |\nabla u_i^\varepsilon|^2 dx - \int_{\partial\Omega} \phi_i \frac{\partial u_i^\varepsilon}{\partial n} ds \leq 0. \tag{1.24}$$

By Trace Theorem, $\frac{\partial u_i^\varepsilon}{\partial n} \to \frac{\partial u_i}{\partial n}$ in $L^2(\partial\Omega)$. passing to limit in (1.24) and using (1.23), gives

$$\int_\Omega |\nabla u_i|^2 dx \geq \limsup_{\varepsilon \to 0} \int_\Omega |\nabla u_i^\varepsilon|^2 \, dx.$$

The above inequality above and Fato's Lemma give the strong convergence. □

Chapter 2

Overview of Papers

2.1 Overview of Paper I

The major goal of the first paper is to obtain numerical approximations of the following system of m differential equations, as ε tends to zero:

$$\begin{cases} -\Delta u_i^\varepsilon = -\frac{1}{\varepsilon} u_i^\varepsilon(x) \sum_{j\neq i}^m a_{ij} u_j^\varepsilon(x) + f_i(x, u_i^\varepsilon(x)) & \text{in } \Omega, \\ u_i^\varepsilon \geq 0 & \text{in } \Omega, \\ u_i^\varepsilon(x) = \phi_i(x) & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

for $i = 1, 2, ..., m$. For simplicity let $f_i = 0$.

First Iterative Method

If the number of components is $m = 2$, then we can solve the problem explicitly. The first method is based on the following theorem that can be found in [6].

Theorem 2.1. *Let $m = 2$ and let W be the harmonic extension on Ω with the boundary data $\phi_1 - \phi_2$. Then, for every $\varepsilon > 0$, problem (2.1) has a unique positive solution $u_1^\varepsilon > W^+$, $u_2^\varepsilon > W^-$. Moreover, if we set $u_1 = W^+$, $u_2 = W^-$, then there exists $C > 0$ such that*

$$\left(\frac{1}{\varepsilon}\right)^{1/6} \| u_i^\varepsilon - u_i \|_{H^1(\Omega)} \leq C \quad \text{as } \varepsilon \to 0, \text{ for } i = 1, 2,$$

where $W^+ = \max(W, 0)$ and $W^- = \max(-W, 0)$.

Note that in this case the free boundary is $\{x \in \Omega : W(x) = 0\}$. The iterative method for $m = 3$ is as follows. Assume that we knew in advance the support of one of the components then we could solve for the two remaining components explicitly by using Theorem 2.1. So in each iteration the support of one component is fixed and the problem is solved for the two others components. The proof of convergence

of method in the case $m = 3$ is given. The proof is based on the observation that when $m = 3$, the limiting configuration minimizes the energy associated to the system (see [7]):

$$E(U) = \int_\Omega \sum_i \mid \nabla u_i(x) \mid^2 dx.$$

The minimization is over all $U = (u_1, u_2, u_3) \in (H^1(\Omega))^3$ such that $u_i \geq 0$, $u_i = \phi_i$ on $\partial\Omega$ and $u_i \cdot u_j = 0$ for $i \neq j$. By the construction of the method,

$$E(U^0) \geq E(U^1) \geq E(U^2) \geq ...$$

Therefore, $\| U^j \|_{(H^1(\Omega))^3} \leq C$. Hence there exists a subsequence denoted again by the same notation such that

$$\lim_{j\to\infty} U^j \rightharpoonup U^* \text{ weakly in } (H^1(\Omega))^3.$$

Also we show that this convergence is in fact strong.

An optimal partition problem

Consider the Laplace operator Δ on a bounded domain $\Omega \subset \mathbb{R}^2$ with the Dirichlet boundary condition. The eigenvalues of Δ in Ω are denoted by

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ... \leq \lambda_n ...$$

For any $u \in C^0(\overline{\Omega})$ we define

$$N(u) = \overline{\{x \in \Omega | u(x) = 0\}}.$$

The nodal domains of u are the components of $\Omega \setminus N(u)$. The number of nodal domains of function u is denoted by $\mu(u)$. We are interested in the nodal domains of the eigenfunctions of the two-dimensional Dirichlet Laplacians.

Definition 2. *Let $1 \leq k \in \mathbb{N}$. A family $\mathcal{D} = \{D_i\}_{i=1}^k$ of subsets of Ω is called k-partition if*

$$D_i \cap D_j = \emptyset, \forall\, i \neq j \quad \text{and} \quad \bigcup_{i=1}^k D_i \subseteq \Omega.$$

We denote by $\mathfrak{D}_k$ the set of open connected k-partitions of Ω.

Definition 3. *For $\mathcal{D} \in \mathfrak{D}_k$, we introduce*

$$\Lambda(\mathcal{D}) = \max_i \lambda_1(D_i),$$

where $\lambda_1(D_i)$ denotes the first eigenvalue of the Laplace operator in D_i.

Definition 4. *For any $k \geq 1$, we define $\mathfrak{L}_k = \inf_{D \in \mathfrak{D}_K} \Lambda(D)$. We call the sequence $\{\mathfrak{L}_k\}_{k\geq 1}$ the spectral minimal partition of Δ. For given k, we say that $\mathcal{D}$ is k-minimal partition if $\mathfrak{L}_k = \Lambda(\mathcal{D})$.*

Theorem 2.2. *[13] $\mathfrak{L}_2$ is the second eigenvalue λ_2 and the minimal 2-partition is the nodal partition associated to the second eigenvector. Thus, $\mu(u_2) = 2$.*

Algorithm for the minimal 3-partition

1. **Initialization:**
   Let $\mathcal{D}^0 = (\Omega_1^0, \Omega_2^0, \Omega_3^0)$ be a 3-partition of Ω.
2. **Step (n)** For $n \geq 1$, we define the partition $D^n = (\Omega_1^n, \Omega_2^n, \Omega_3^n)$ by
   $\Omega_1^n = \Omega_3^{n-1}$,
   (Ω_2^n, Ω_3^n) is the nodal partition associated to the second eigenvector of the Δ on $\mathrm{Int}(\Omega \setminus \Omega_1^n)$.

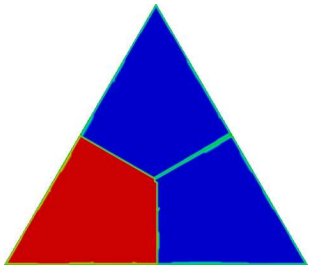


Figure 2.1: 3-minimal partitions.

Second method

The second method exploits the following properties.

$$\text{If } i \neq j \text{ then } u_i \cdot u_j = 0 \text{ } a.e. \text{ in } \Omega,$$
$$\Delta(u_i - u_j) = 0 \text{ in the set supp } \{u_i > 0\} \cup \text{supp } \{u_j > 0\}.$$

Using finite difference and imposing above conditions the second method is as follows. For $l = 1, 2, ..., m$, let $\overline{u_l}(x_i, y_j)$ denote the average of u_l for all neighbors of the point (x_i, y_j).

1. **Initialization:** For $l = 1, ..., m$,

$$u_l^0(x_i, y_j) = \begin{cases} 0 & (x_i, y_j) \text{ is an interior point,} \\ \\ \phi_l(x_i, y_j) & (x_i, y_j) \text{ is a boundary point.} \end{cases}$$

2. **Step $k+1$, $k \geq 0$.** iterate for all interior points by

$$u_l^{(k+1)}(x_i, y_j) = \max\left(\overline{u_l}^{(k)}(x_i, y_j) - \sum_{p \neq l} \overline{u_p}^{(k)}(x_i, y_j), 0\right) \quad l = 1, ...m.$$

2.2 Overview of Paper II

In this paper we study the perturbation of coefficients λ^+, λ^- and the boundary value g for the two-phase obstacle problem.

$$\begin{cases} \triangle u = \lambda^+ \chi_{\{u>0\}} - \lambda^- \chi_{\{u<0\}} & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

For given $(\lambda^+, \lambda^-) \in L^p(\Omega) \times L^p(\Omega)$, with $1 < p < \infty$ equation (2.2) has a unique solution $u \in W^{2,p}_{\text{loc}}$. Define the map $T : (\lambda^+, \lambda^-) \mapsto u$ where u is the corresponding solution of (2.2) related to coefficients λ^+ and λ^-.

Main results

The stability of solution in L^∞-norm is proved.

Proposition 2.3. *Let u_i for $i = 1, 2$ be the solutions of following equation*

$$\begin{cases} \triangle u_i = \lambda^+ \chi_{\{u_i>0\}} - \lambda^- \chi_{\{u_i<0\}} & \text{in } \Omega, \\ u_i = g_i & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

If $g_1 \leq g_2 \leq g_1 + \varepsilon$, then $u_1 \leq u_2 \leq u_1 + \varepsilon$. In particular,

$$\|u_2 - u_1\|_{L_\infty} \leq \|g_1 - g_2\|_{L_\infty}.$$

Lemma 2.4. *Let u solve*

$$\Delta u = \lambda^+ \chi_{\{u>0\}} - \lambda^- \chi_{\{u<0\}} \text{ in } B_1,$$

and let u^ε satisfy the following equation

$$\Delta u^\varepsilon = (\lambda^+ + \varepsilon)\chi_{\{u^\varepsilon>0\}} - (\lambda^- - \varepsilon)\chi_{\{u^\varepsilon<0\}} \text{ in } B_1,$$

with $u = u^\varepsilon = g$ on ∂B_1. Then

$$|u^\varepsilon - u| \leq C\varepsilon.$$

Both of the proofs are applications of the maximum principle. We prove the continuity and differentiability of the map T.

Theorem 2.5. *The map $(\lambda^+, \lambda^-) \mapsto u$ is Lipschitz continuous in the following sense. If u_i for $i = 1, 2$ solve*

$$\begin{cases} \triangle u_i = \lambda_i^+ \chi_{\{u_i>0\}} - \lambda_i^- \chi_{\{u_i<0\}} & \text{in } \Omega, \\ u_i = g & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

then

$$\|u_2 - u_1\|_{H^1(\Omega)} \leq C(\| \lambda_1^+ - \lambda_2^+ \|_{H^{-1}(\Omega)} + \| \lambda_1^- - \lambda_2^- \|_{H^{-1}(\Omega)}),$$

and for $p > \frac{n}{2}$

$$\|u_2 - u_1\|_{L_\infty} \leq C(\| \lambda_1^+ - \lambda_2^+ \|_{L_p} + \| \lambda_1^- - \lambda_2^- \|_{L_p}).$$

Let $\overline{\lambda} = (\lambda^+, \lambda^-)$, $\overline{h} = (h_1, h_2)$. By $u^{\overline{\lambda}+\varepsilon\overline{h}}$, we mean the solution of the problem (2.2) with coefficients $(\lambda_1 + \varepsilon h_1)$ and $(\lambda_2 + \varepsilon h_2)$.

Theorem 2.6. *The mapping $T : C^{0,1}(\Omega) \times C^{0,1}(\Omega) \longrightarrow W^{2,p}(\Omega)$, defined by $u = T(\lambda^+, \lambda^-)$ is differentiable. Furthermore, if $\overline{\lambda}, \overline{h} \in C^{0,1}(\Omega) \times C^{0,1}(\Omega)$, then*

$$\frac{u^{\overline{\lambda}+\epsilon\overline{h}} - u^{\overline{\lambda}}}{\epsilon} \rightharpoonup w^{\overline{\lambda},\overline{h}} \text{ in } H_0^1(\Omega) \text{ as } \epsilon \to 0,$$

where

$$\Delta w^{\overline{\lambda},\overline{h}} = h_1 \chi_{\{u^{\overline{\lambda}}>0\}} - h_2 \chi_{\{u^{\overline{\lambda}}<0\}} + \frac{(\lambda^+ + \lambda^-)}{| \nabla u^{\overline{\lambda}} |} w^{\overline{\lambda},\overline{h}} \mathcal{H}^{n-1} \lfloor_{\Gamma''(u)}. \tag{2.5}$$

In (2.5) $\mathcal{H}^{n-1}$ denotes the $(n-1)$ dimensional Hausdorff measure.

The idea of the proof of Theorem 2.5 is first to show that $\frac{u^{\overline{\lambda}+\epsilon\overline{h}} - u^{\overline{\lambda}}}{\epsilon}$ is uniformly bounded in $H_0^1(\Omega)$. Hence the weak convergence to the limit denoted by $w^{\overline{\lambda},\overline{h}}$, follows. Next we show that

$$\Delta(w^{\overline{\lambda},\overline{h}}) = h_1 \text{ in } \{u^{\overline{\lambda}} > 0\}$$

and

$$\Delta(w^{\overline{\lambda},\overline{h}}) = -h_2 \text{ in } \{u^{\overline{\lambda}} < 0\}.$$

In the case of one phase obstacle problem we have the following measure stability [1].

$$\mathcal{L}^n(\Lambda(u_2) \triangle \Lambda(u_2)) = \mathcal{L}^n(\Omega^+(u_2) \triangle \Omega^+(u_1)) \leq C(n, \mu) \| \lambda_2 - \lambda_1 \|_{L^n},$$

where $0 < h \leq \lambda_i \leq \mu$ and u_i for $i = 1, 2$ satisfy

$$\begin{cases} \triangle u_i = \lambda_i \chi_{\{u_i>0\}} & \text{in } \Omega, \\ u_i = g & \text{on } \partial\Omega. \end{cases}$$

Question: Can we obtain a similar estimate for two-phase problem?

2.3 Overview of Paper III

In this papers we study different numerical approximations for two-phase membrane problem. Consider the following problem

$$\begin{cases} \triangle u = \lambda^+ \chi_{\{u>0\}} - \lambda^- \chi_{\{u<0\}} & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

First method

This method is introduced by author in Paper I. In equation (2.6) let

$$u_1 = u^+,\ u_2 = u^-,$$
$$g_1 = g^+,\ g_2 = g^-,$$

where $u^+ = \max(u, 0)$ and $u^- = \max(-u, 0)$. Then

$$u_1 \cdot u_2 = 0, \quad u_1, u_2 \geq 0.$$

We have

$$\Delta(u_1 - u_2) = \lambda^+ \chi_{\{u_1>0\}} - \lambda^- \chi_{\{u_2>0\}}. \tag{2.7}$$

Now for a given mesh on Ω, let $\triangle x = \triangle y = h$. One can use standard finite difference for equation (2.7),

$$\begin{aligned} &\frac{1}{h^2}[u_1(x_{i-1}, y_j) + u_1(x_{i+1}, y_j) - 4u_1(x_i, y_j) + u_1(x_i, y_{j-1}) + u_1(x_i, y_{j+1})] \\ &- \frac{1}{h^2}[u_2(x_{i-1}, y_j) + u_2(x_{i+1}, y_j) - 4u_2(x_i, y_j) + u_2(x_i, y_{j-1}) + u_2(x_i, y_{j+1})] \\ &= \lambda^+ \chi_{\{u_1(x_i,y_j)>0\}} - \lambda^- \chi_{\{u_2(x_i,y_j)>0\}}. \end{aligned} \tag{2.8}$$

If we obtain $u_1(x_i, y_j)$ and $u_2(x_i, y_j)$ from (2.9) and imposing the conditions

$$u_1(x_i, y_j) \cdot u_2(x_i, y_j) = 0 \quad \text{and} \quad u_1(x_i, y_j) \geq 0, \quad u_2(x_i, y_j) \geq 0,$$

then the iterative method for u_1 and u_2 will be as follows.

•**Initialization:**

$$u_1^{(0)}(x_i, y_j) = \begin{cases} 0 & \text{if } (x_i, y_j) \text{ is an interior point,} \\ g_1(x_i, y_j) & \text{if } (x_i, y_j) \text{ is a boundary point.} \end{cases}$$

$$u_2^{(0)}(x_i, y_j) = \begin{cases} 0 & \text{if } (x_i, y_j) \text{ is an interior point,} \\ g_2(x_i, y_j) & \text{if } (x_i, y_j) \text{ is a boundary point.} \end{cases}$$

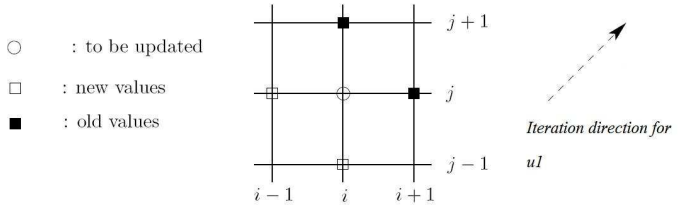
• **Step** $k + 1$**,** $k \geq 0$**:**

Let $\overline{u_1}(x_i, y_j)$ denote the average of u_1 for all neighbors of the point(x_i, y_j). Then

$$u_1(x_i, y_j) = \max\left(\frac{-\lambda^+ h^2}{4} + \overline{u_1}(x_i, y_j) - \overline{u_2}(x_i, y_j), 0\right)$$

$$u_2(x_i, y_j) = \max\left(\frac{-\lambda^- h^2}{4} + \overline{u_2}(x_i, y_j) - \overline{u_1}(x_i, y_j), 0\right)$$

For implementation aspects, in each point we use update value for neighboring points like in the Gauss-Seidel method.



A general way to accelerate the convergence of the method is possible by the Multi-grid method [11]. Another alternative can be to solve the problem for coarse mesh then the obtained solution is used as initial values for fine mesh and so on.

Regularization method

One of the well known methods to solve variational inequalities and obstacle problems is the regularization method. The idea of the regularization method is to approximate the non-differentiable terms by a sequence of differentiable terms. Convergence is obtained when the parameter ε tends to 0. Consider the family of the regularized equation

$$\Delta u^\varepsilon = (\lambda^+)\chi^\varepsilon(u^\varepsilon) - (\lambda^-)\chi^\varepsilon(-u^\varepsilon), \tag{2.9}$$

where $\chi^\varepsilon(t)$ is a non decreasing, smooth approximation of Heaviside function $\chi_{\{u>0\}}$ such that

$$\chi^\varepsilon(t) = \begin{cases} 1 & t \geq \varepsilon, \\ 0 & t \leq -\varepsilon. \end{cases}$$

In paper (II) it has shown that that for the solution u^ε of equation (2.9), $u^\varepsilon \to u$. In addition, we showed the following estimates for the regularized solution for equation

(2.9):

$$\sup_{\Omega}|u^{\varepsilon_1} - u^{\varepsilon_2}| \leq 2\max\{\varepsilon_1, \varepsilon_2\}, \tag{2.10}$$

$$\|\nabla(u^{\varepsilon_1} - u^{\varepsilon_2})\|_{L^2(\Omega)} \leq C \cdot \max\{\varepsilon_1, \varepsilon_2\} \max\{\lambda^+, \lambda^-\}.$$

In numerical experiments, the term $|u|$ can be regularized by the following sequences

1. $\phi_1^\varepsilon(u) = \sqrt{u^2 + \varepsilon^2}$.

2. $\phi_2^\varepsilon(u) = \frac{|u|^{\varepsilon+1}}{\varepsilon+1}$.

3. $\phi_3^\varepsilon(u) = \begin{cases} u & u \geq \varepsilon, \\ \frac{1}{2}(\frac{u^2}{\varepsilon} + \varepsilon) & |u| \leq \varepsilon, \\ -u & u \leq -\varepsilon. \end{cases}$

4. $\phi_4^\varepsilon(u) = \begin{cases} u - \varepsilon/2 & u \geq \varepsilon, \\ \frac{u^2}{2\varepsilon} & |u| \leq \varepsilon, \\ -u - \frac{\varepsilon}{2} & u \leq -\varepsilon. \end{cases}$

The regularized equation can be written in the form

$$\begin{cases} -\Delta u^\varepsilon + (\frac{\lambda^+ + \lambda^-}{2})(\phi^\varepsilon)'(u^\varepsilon) + (\frac{\lambda^+ - \lambda^-}{2}) = 0 & \text{in } \Omega, \\ u^\varepsilon = g & \text{on } \partial\Omega. \end{cases} \tag{2.11}$$

In order to obtain a posteriori error estimates of the approximate solution, we use duality method by conjugate functions. We proved the following the a-posteriori error estimate:

Theorem 2.7. *Assume u and u^ε be the solution of two-phase equation (2.6) and regularized equation (2.11) respectively, then*

$$\frac{1}{2}\|\nabla(u^\varepsilon - u)\|^2_{L^2(\Omega)} \leq \int_\Omega \frac{1}{2}(\lambda^+ + \lambda^-)[\mid u^\varepsilon \mid -u^\varepsilon(\phi^\varepsilon)'(u^\varepsilon)]\, dx. \tag{2.12}$$

Steady state problem

The two-phase membrane problem can be regard as a steady state of the parabolic equation

$$\begin{cases} u_t - \Delta u = -\lambda^+\chi_{\{u>0\}} + \lambda^-\chi_{\{u<0\}} & \text{in} \quad \Omega \times (0,T), \\ u = g & \text{on} \quad \partial\Omega \times [0,T], \\ u = u_0 & \text{on} \quad \Omega \times \{t = 0\}. \end{cases} \tag{2.13}$$

For given mesh (x_i, y_j) with $\Delta x = \Delta y$ and using standard finite difference the explicit method as follows.

$$\begin{aligned}&\frac{u^{n+1}(x_i,y_j)-u^n(x_i,y_j)}{\Delta t} - \frac{u^n(x_{i+1},y_j)+u^n(x_{i-1},y_j)-2u^n(x_i,y_j)}{(\Delta x)^2}\\&+\frac{u^n(x_i,y_{j+1})-2u^n(x_i,y_j)+u^n(x_i,y_{j-1})}{(\Delta y)^2}\\&=-\lambda^+\chi_{\{u^n(x_i,y_j)>0\}}+\lambda^-\chi_{\{u^n(x_i,y_j)<0\}}.\end{aligned} \tag{2.14}$$

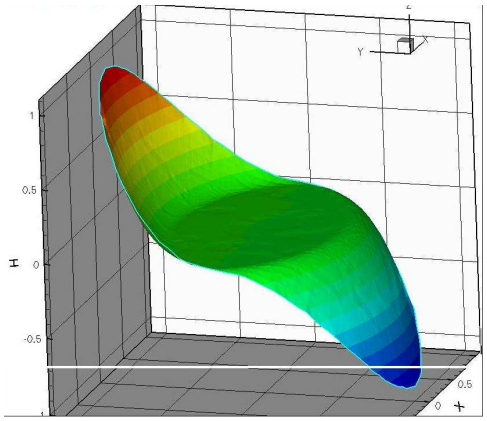


Figure 2.2: Steady state solution.

2.4 Overview of Paper IV

Consider the problem of minimizing the functional

$$I = \int_\Omega \sum_i^m (\frac{1}{2}|\nabla u_i|^2 + f_i \cdot u_i)\, dx, \tag{2.15}$$

over the set $\{(u_1, ..., u_m) \mid u_i - g_i \in H_0^1(\Omega),\ u_1 \geq u_2 \geq ... \geq u_m\}$. In Paper IV, we study problem for general m. In order to simplify, assume $m = 2$, the boundary values g_1 and g_2 are given with $g_1 \geq g_2$. The Euler-Lagrange equation corresponding to the minimizer (u_1, u_2) is:

$$\begin{aligned}\Delta u_1 &= f_1 + \frac{f_2 - f_1}{2}\chi_{\{u_1=u_2\}} \text{ in } \quad \Omega,\\ \Delta u_2 &= f_2 + \frac{f_1 - f_2}{2}\chi_{\{u_1=u_2\}} \text{ in } \quad \Omega,\\ u_i &= g_i \text{ on } \partial\Omega, \text{ for } i = 1, 2.\end{aligned}$$

From the above equations one obtains

$$\begin{cases} \Delta(u_1 - u_2) = (f_1 - f_2)\chi_{\{u_1 \geq u_2\}} & \text{in } \Omega, \\ u_1 - u_2 = g_1 - g_2 & \text{on } \partial\Omega, \end{cases} \tag{2.16}$$

and

$$\begin{cases} \Delta(u_1 + u_2) = (f_1 + f_2) & \text{in } \Omega, \\ u_1 + u_2 = g_1 + g_2 & \text{on } \partial\Omega. \end{cases} \tag{2.17}$$

Let $v_1 = u_1 - u_2$ and $v_2 = u_1 + u_2$. The equation (2.16) is one phase obstacle problem, while (2.17) is a PDE. By solving (2.16) and (2.17) for v_1 and v_2 one can obtain u_1 and u_2. Also this is motivation to obtain optimal regularity for two-membrane problem. For general m set

$$\begin{aligned} &v_i = u_i - u_{i+1}, \text{ for } i = 1, ..., m-1 \\ &v_m = u_1 + \cdots + u_m. \end{aligned} \tag{2.18}$$

The conditions $u_1 \geq u_2 \geq u_3 \geq ... \geq u_m$ will give

$$v_1 \geq 0, \quad v_2 \geq 0, \quad ..., \quad v_{n-1} \geq 0.$$

From (2.18), one gets the following experisons

$$\begin{aligned} u_1 &= \frac{1}{m}(\sum_{k=1}^{m-1}(m-k)v_k + v_m), \\ u_k &= \frac{1}{m}[-\sum_{i=1}^{k-1} iv_i + \sum_{i=k}^{m-1}(m-i)v_i + v_m] \text{ for } k = 2, .., m. \end{aligned} \tag{2.19}$$

Substituting these values of u_i in (2.15) gives

$$\min I = \min(I_1 + I_2),$$

where the minimization of functional I_1 is over the set

$$\{(v_1, v_2, ..., v_{m-1}) \mid v_i \geq 0,\ i = 1, 2, ..., m-1\}.$$

The second minimization I_2 is over v_m without constraint. Finite element discritization implies that the minimization problem I_1 can be written as a quadratic form

$$\min_{X \geq 0}[\frac{1}{2}(AX, X) - (F, X)], \tag{2.20}$$

The bound constraint minimization (2.20) is solved by gradient projection method.

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