Improved Approximation of Two Watchmen's Routes in Simple Polygons

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Abstract

We study the watchman route problem for a set of two watchmen for the objective of minimizing the length of the longest route (min-max) inside a simple polygon P having n vertices, which is known to be weakly NP-hard. First, we consider seeing a discrete set of m points in the interior of P. We show that even this problem is weakly NP-hard and present an approximation algorithm with approximation ratio $2+\varepsilon$ that runs in $O(m^5 n)$ time, assuming that a starting point for each of the two routes is given. We generalize the algorithm to see all of the interior of P in $O(n^6)$ time with approximation ratio $2+\pi/2 \approx 3.571$, improving on the previously known best algorithm that has an approximation ratio of ≈ 6.922 and runtime $O(n^2)$ [8]. Finally, we describe how to extend this algorithm to the case where no starting points are given, this taking $O(n^8)$ time, yielding an approximation ratio of $3+\pi/2 \approx 4.571$, improving on the previously known best approximation algorithm with ratio ≈ 5.969 also having runtime $O(n^8)$ [8].

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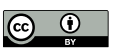
1 Introduction

The problem of finding a shortest watchman route goes back to Chin and Ntafos, who also provided algorithms for rectilinear polygons [2]. Algorithms for simple polygons were first given in [3] and corrected in [10]. The shortest watchman route problem has two variants, the *anchored* or *fixed* variant, where the route must pass through a given point sometimes specified to lie on the boundary, and the *non-anchored* or *floating* variant with no requirement to pass a specific point. The currently best algorithm for the anchored version requires $O(n^3)$ time in the worst case [11], if the anchor point is assumed to lie on the boundary of the polygon. The floating variant can be computed in $O(n^4)$ time by combining the result in [11] with the method in [9].

For two watchmen, the so-called *two-watchman route* problem, Mitchell and Wynters [7] showed that computing routes with the objective of minimizing the length of the longer route (min-max) is weakly NP-hard. Nilsson and Packer [8] recently provided approximation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algorithms for computing min-max two-watchman routes with an approximation ratio of ≈ 5.969 for the floating case taking $O(n^8)$ time, and ≈ 6.922 for the case of two given starting points, not necessarily on the boundary, taking $O(n^2)$ time.

We first consider the following variant of the two-watchman route problem. We are given m point sized treasures (the set $\mathcal{S}$) in the simple polygon that need to be guarded [1, 4]. For the single watchman route case, an appropriate modification of the algorithm of Tan and Jiang [11] obtains a shortest route that visits the visibility polygons of the treasures in $O(m^2n)$ time for the boundary anchored version and can be extended to $O(m^3n)$ time for the floating version using Tan's technique [9]. We want to compute the shortest two routes V_s and $V_{s'}$, starting at two points s and s', respectively, not necessarily on the boundary of the polygon, such that every point $p_i \in \mathcal{S}$ is seen from at least one point on one of the routes. Two natural optimization criteria have previously been considered for these problems: the *min-max* measure and the *min-sum* measure. For the *min-max* measure we wish to obtain two routes such that the longer of the two is as short as possible. As we prove, this problem is weakly NP-hard not only for the objective of seeing all of the simple polygons [7], but also for the objective of seeing all points in $\mathcal{S}$. The *min-sum* measure on the other hand asks for two routes such that the sum of their lengths is minimized. The complexity status for this version is unknown.

We develop two algorithms for this anchored discrete problem, optimizing on the min-max criterion, one having approximation ratio $2 + \pi/2$ and running in $O(m^5n)$ time and an improved version that, for any fixed $\varepsilon > 0$, gives an approximation ratio of $2+\varepsilon$, also running in $O(m^5n)$ time.

Secondly, we also consider the problem of guarding the complete polygon P and modify the first algorithm developed for a discrete point set using results from Nilsson and Packer [8]. We obtain algorithms having approximation ratio $2 + \pi/2$ for the anchored and $3 + \pi/2$ for the floating versions of the two-watchman route problem. The algorithm for the anchored version runs in $O(n^6)$ time and the algorithm for the floating version runs in $O(n^8)$ time.

2 Preliminaries

Let P be a simple polygon with n vertices $\mathcal{V} = \{v_1, \ldots, v_n\}$ labeled in clockwise order, and let $\mathcal{S} = \{p_1, \ldots, p_m\}$ be a set of m points in P, the *treasures*. We assume that no three points among the polygon vertices and $\mathcal{S}$ are collinear. Furthermore, let s and s' be the two starting points for the watchman routes in P, not necessarily lying on the boundary.

For a point $p \in P$, let $VP(p)$ be the *visibility polygon* of p, the set of points in P that are visible to p. An edge of $VP(p)$ that is not contained in an edge of P is called a *window* of p.

Without loss of generality, we may assume that $s, s' \notin VP(p_i)$, for any $p_i \in \mathcal{S}$, as those points are already seen from the starting points. Let C_s^i be the window that separates the subpolygon of $P \setminus VP(p_i)$ containing s from $VP(p_i)$. Analogously, let $C_{s'}^i$ be the window that separates the subpolygon of $P \setminus VP(p_i)$ containing s' from $VP(p_i)$. Now, observe that a watchman who starts walking at s (s') sees point p_i as soon as it visits C_s^i ($C_{s'}^i$). We assign a direction to each window C_s^i ($C_{s'}^i$) so that $VP(p_i)$ lies locally to the left of C_s^i ($C_{s'}^i$) if the polygon is rotated so that the window is directed upwards, see also Figure 1. We denote the lower endpoint of the window the *bottom point* of the window and the upper endpoint the *top point*.

We study the problem of computing two routes such that each of the treasures can be seen from at least one of the routes. We denote the length of a route W by $\|W\|$.

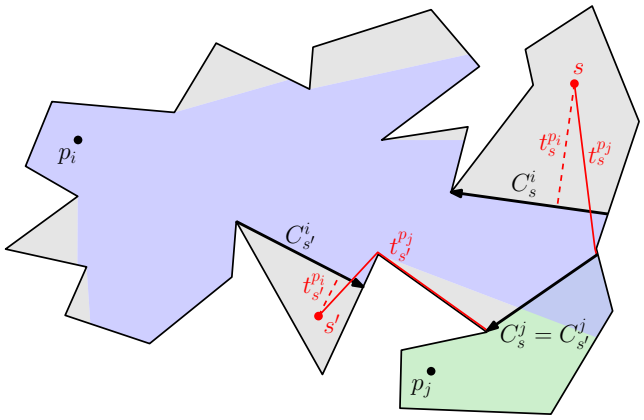

Figure 1 Illustration of the visibility polygon of a point and the windows (black arrows). The windows point towards their corresponding top points. For point p_j, $C_s^j = C_{s'}^j$, hence $p_j \in \mathcal{S}_1$, while for p_i, $C_s^i \neq C_{s'}^i$, thus $p_i \in \mathcal{S}_2$.

▶ **Problem 1** (Min-Max Two-Watchman Route Problem for Treasures). *Given a simple polygon P, a set of points $\mathcal{S}$ in P, (and two starting points s and $s' \in P$,) compute two routes W and W' (where W passes through s and W' passes through s'), such that every point of $\mathcal{S}$ is visible from at least one point on W or W' and such that $\max\{\|W\|, \|W'\|\}$ is minimized.*

▶ **Theorem 1.** *Problem 1 is weakly NP-hard.*

Proof. The reduction is from 2-partition [6]. Let $A = \{a_1, a_2, \ldots, a_n\}$ be a set of integers. We construct a polygon P with points $p_1, \ldots, p_n$ where an optimal watchman route that sees all points has length $\sum_{i=1}^{n} a_i + \delta$ for some small constant δ, and an optimal solution to Problem 1 consists of two tours with length $1/2 \cdot \sum_{i=1}^{n} a_i + \delta$ each, corresponding to an optimal partition of A.

Let $\delta \ll \min_{a_i \in A} a_i$. We start with a regular convex polygon with n vertices and perimeter δ, which we call the *center* of P. The starting points s and s' are placed on the boundary of the center. On the i^{th} side of the center, we attach a rectilinear tube of width δ/n and length $a_i/2$. At the end of each corridor, there is a small notch that contains point p_i such that the window of p_i has distance $a_i/2$ to the center. See Figure 2 for an illustration of the polygon.

The shortest watchman route that sees all points $p_1, \ldots, p_n$ runs along the boundary of the center, and thus passes through both s and s', and walks through each tunnel up to the window C_s^i of point p_i, for $1 \leq i \leq n$. For two watchman routes, only one of them needs to visit the window of a point p_i. As each route is connected and starts at its corresponding starting point, the watchmen also walk a portion within the center of the polygon. However, note that even walking along all of the boundary of the center extends the total route length only by an additive constant of δ, which does not affect the choice of points assigned to each route. Thus, for an optimal pair of routes T_s, $T_{s'}$ for Problem 1, the points $p_1, \ldots, p_n$ are partitioned into two sets $\mathcal{X}_s$ and $\mathcal{X}_{s'}$, such that T_s visits the windows of the points in $\mathcal{X}_s$ and $T_{s'}$ visits the windows of the points in $\mathcal{X}_{s'}$. Hence, $(\mathcal{X}_s, \mathcal{X}_{s'})$ corresponds to a 2-partition of A. ◀

We can extend this problem and require the whole polygon to be seen.

▶ **Problem 2** (Min-Max Two-Watchman Route Problem for a Polygon). *Given a simple polygon P (and two starting points s and $s' \in P$), compute two routes W and W' (where W passes through s and W' passes through s'), such that every point in P is visible from at least one point on W or W' and such that $\max\{\|W\|, \|W'\|\}$ is minimized.*

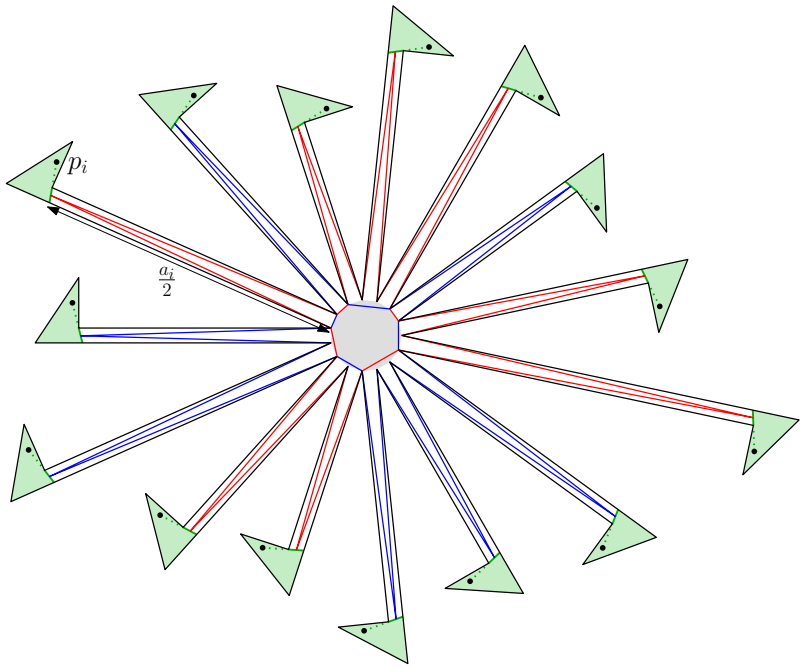

Figure 2 Schematic representation of the polygon constructed from an instance of 2-partition. Each of the two watchmen (red and blue) guards a portion of the points p_i, such that the route lengths of both watchmen are (almost) equal.

Let W_s ($W_{s'}$) be the shortest route that sees all the points in $\mathcal{S}$ and passes through s (s'). If s (s') lies on the boundary of P, we can use the modified algorithm by Tan and Jiang to compute it in $O(m^2n)$ time. If s (s') does not lie on the boundary, we need to modify the algorithm to obtain which window is visited first along a clockwise traversal of the tour starting at s (s'). By trying all possible windows, we can compute the optimal route in $O(m^3n)$ time.

We denote the min-max shortest pair of routes that together see all the points in $\mathcal{S}$, one passing through s and the other passing through s', by T_s and $T_{s'}$. We can without loss of generality assume that each of T_s and $T_{s'}$ is as short as possible. Furthermore, we define $OPT \stackrel{\text{def}}{=} \max\{\|T_s\|, \|T_{s'}\|\}$.

For each $p_i \in \mathcal{S}$, let $t_s^{p_i}$ be the shortest path from starting point s to $VP(p_i)$, i.e., to the window C_s^i, and analogously let $t_{s'}^{p_i}$ be the shortest path from starting point s' to $VP(p_i)$, i.e., to the window $C_{s'}^i$; these paths are marked in red in Figure 1. Let $t_{\min}^{p_i} = \arg\min\{\|t_s^{p_i}\|, \|t_{s'}^{p_i}\|\}$ be the shorter of the two paths to see p_i and let $t_{\max}^{p_i} = \arg\max\{\|t_s^{p_i}\|, \|t_{s'}^{p_i}\|\}$ be the longer of these two paths.

We remark that finding an (approximate) min-max optimal pair of routes is a question of correctly partitioning the points to be seen between the route passing through s and the route passing through s'. Once the partitioning is obtained, the routes can be computed in $O(m^3n)$ time as previously described.

3 The Algorithm

We describe an algorithm to compute two watchman routes to see a set of treasures $\mathcal{S}$ in a polygon P (Problem 1). This is also the basis for the algorithm for seeing the whole polygon (Problem 2) that we present in Section 4, and then generalize to the floating version in Section 5.

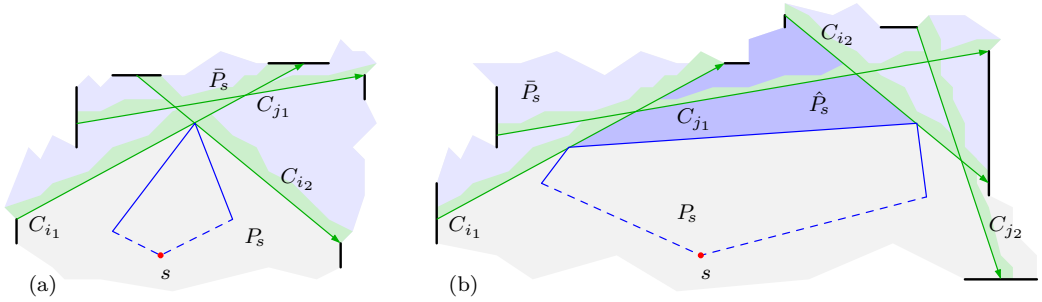

Figure 3 Illustrating the proof of Lemma 2.

First, compute a shortest watchman route W_s that sees all points in $\mathcal{S}$ and starts at s, and a shortest watchman route $W_{s'}$ that sees all points in $\mathcal{S}$ and starts at s'. If the two routes T_s and $T_{s'}$ of an optimal two-watchman route intersect, then either of W_s and $W_{s'}$, together with a route of length 0 at the respective other starting point, gives a 2-approximation. This follows from the fact that we can trace the subpaths of the routes between their intersection points to obtain a route that sees all the points in $\mathcal{S}$. Thus, $\max\{\|W_s\|, \|W_{s'}\|\} \leq \|T_s\| + \|T_{s'}\| \leq 2 \cdot \max\{\|T_s\|, \|T_{s'}\|\}$. All the remaining steps use the assumption that the routes T_s and $T_{s'}$ do not intersect.

We partition the points in $\mathcal{S}$ into two sets, depending on where s and s' lie with respect to $VP(p_i)$. Let $\mathcal{S}_1$ be the set of points in $\mathcal{S}$ for which $C_s^i = C_{s'}^i$, and let $\mathcal{S}_2$ be the remaining set of points in $\mathcal{S}$; see Figure 1 for an illustration. To compute the two watchman routes, we consider $\mathcal{S}_1$ and $\mathcal{S}_2$ separately in Sections 3.1 and 3.2, respectively.

3.1 Handling the Points in $\mathcal{S}_1$

Let $t_s^{p_i}$ and $t_{s'}^{p_i}$ be the shortest paths from s and s', respectively, to $VP(p_i)$. The endpoints of $t_s^{p_i}$ and $t_{s'}^{p_i}$ different from s and s' lie on windows C_s^i and $C_{s'}^i$, respectively. If $C_s^i = C_{s'}^i$, then p_i belongs to the set $\mathcal{S}_1$ and we denote the window C_i. Our objective in this section is to show how to compute two routes V_s^1 and $V_{s'}^1$ of optimal min-max length such that they together see all the points in $\mathcal{S}_1$.

Consider the set $P \setminus VP(p_i)$ of disjoint subpolygons of P. Let P_i be the subpolygon that contains s and s', for each point $p_i \in \mathcal{S}_1$. If $P_i \subset P_j$, we say that p_j *dominates* p_i, $i \neq j$, since any route through s or s' that visits $VP(p_j)$ must also visit $VP(p_i)$. We call any point in $\mathcal{S}_1$ *dominating* if it is not dominated by any other point in $\mathcal{S}_1$ and assume henceforth that $\mathcal{S}_1$ contains only dominating points.

For the sake of convenience in this section, we relabel each window C_i associated to the point $p_i \in \mathcal{S}_1$ based on the first appearance of its bottom point on the boundary of P along a clockwise traversal of the boundary. To do this, start at vertex v_1 and traverse the boundary in clockwise direction. Then, relabel the windows to $C_0, C_1, \ldots, C_{|\mathcal{S}_1|-1}$ in the order as the bottom point of each window is encountered. We order the associated points in $\mathcal{S}_1$ correspondingly.

Let T_s^1 and $T_{s'}^1$ be two min-max length routes that together see all points in $\mathcal{S}_1$.

▶ **Lemma 2.** *Let $C_0, C_1, \ldots, C_{|\mathcal{S}_1|-1}$ be the windows of the dominating points in $\mathcal{S}_1$, ordered as their bottom points appear along the boundary in clockwise order. If T_s^1 and $T_{s'}^1$ do not intersect, then there exist at most two indices i and j such that T_s^1 visits $C_i, C_{i+1}, \ldots, C_{j-1}$ and $T_{s'}^1$ visits $C_j, C_{j+1}, \ldots, C_{i-1}$. (Implicitly assuming that indices are taken modulo $|\mathcal{S}_1|$.)*

Proof. For three distinct indices i, i', and i'', we have that $i \prec i' \prec i''$ if one of the following sequences of inequalities holds: $i < i' < i''$, or $i'' < i < i'$, or $i' < i'' < i$, i.e., there exists a boundary point on P such that a clockwise traversal of the boundary starting at this point reaches the bottom points of the windows in the order C_i, $C_{i'}$, $C_{i''}$.

Assume that there are indices $i_1 \prec j_1 \prec i_2 \prec j_2 \prec i_1 \prec j_1$ such that T_s^1 intersects C_{i_1} and C_{i_2}, but not C_{j_1} and C_{j_2}, and $T_{s'}^1$ intersects C_{j_1} and C_{j_2}, but not C_{i_1} and C_{i_2}. Consider the polygonal chain defined by the part of C_{i_1} between its bottom point and its intersection with T_s^1, the part of T_s^1 until its intersection with C_{i_2}, and the part of C_{i_2} until its top point; see Figure 3. This chain separates P into two regions. By definition, s lies in one of these regions P_s. As T_s^1 does not visit C_{j_1}, the window C_{j_1} lies in the other region, $\bar{P}_s = P \setminus P_s$. Let $\hat{P}_s \subseteq \bar{P}_s$ be the region enclosed by C_{i_1}, C_{i_2}, T_s^1, and, if C_{i_1} and C_{i_2} do not intersect, possibly part of the polygon boundary; see Figure 3(b). The region $\hat{P}_s$ may be empty; see Figure 3(a) for such an example. Since s' lies outside the visibility polygons associated to the four windows we consider, s' either lies in P_s or in $\hat{P}_s$.

If s' lies in P_s, then $T_{s'}^1$ has to cross C_{i_1} or C_{i_2} in order to visit C_{j_1}, contradicting the stated assumptions; see Figures 3(a) and (b). If s' lies in $\hat{P}_s$, then in order for $T_{s'}^1$ to visit C_{j_2}, it must cross the boundary of $\hat{P}_s$, again contradicting our stated assumptions. Hence, T_s^1 (and $T_{s'}^1$) visits an ordered subsequence of the windows as stated. ◀

The previous lemma gives us an immediate algorithm to compute T_s^1 and $T_{s'}^1$. By trying every index pair $0 \le i, j \le |\mathcal{S}_1| - 1$, computing the shortest route starting at s that visits the windows $C_i, C_{i+1}, \ldots, C_{j-1}$ and the shortest route starting at s' that visits the windows $C_j, C_{j+1}, \ldots, C_{i-1}$, indices always taken modulo $|\mathcal{S}_1|$, and selecting the min-max shortest pair of these, we obtain the two desired routes $V_s^1 = T_s^1$ and $V_{s'}^1 = T_{s'}^1$. There are $O(m^2)$ index pairs, and computing the shortest route takes $O(m^3 n)$ for each of these pair, as described in Section 2. Thus, the time complexity is bounded by $O(m^5 n)$. We state this result as a theorem.

▶ **Theorem 3.** *The shortest min-max pair of routes in P that see the points in $\mathcal{S}_1$ can be computed in $O(m^5 n)$ time under the assumption that they do not intersect.*

3.2 Handling the Points in $\mathcal{S}_2$

Let C_s^i and $C_{s'}^i$ be the windows of $VP(p_i)$ that $t_s^{p_i}$ and $t_{s'}^{p_i}$ reach, respectively. If $C_s^i \neq C_{s'}^i$, then p_i belongs to the set $\mathcal{S}_2$. The visibility polygon $VP(p_i)$ must intersect the shortest path $SP(s, s')$ since the starting points lie in different components of $P \setminus VP(p_i)$. Our objective in this section is to show how to compute two routes V_s^ε and $V_{s'}^\varepsilon$ of almost optimal min-max length such that they together see all the points in $\mathcal{S}_2$.

First, we introduce a concept that we will use extensively in the sequel. Let L be some length $L \le \|SP(s, s')\|$. If p_L is the point at distance L from s on $SP(s, s')$, we can trace the geodesic quarter circle Q_U of radius L upwards from p_L, the connected locus of points having shortest paths in P of length L to s, ending the trace if it encounters the polygon boundary. Similarly, we can trace the geodesic quarter circle Q_D of radius L downwards from p_L; see Figure 4(a). Define $H_s^L = Q_U \cup Q_D$ as the *geodesic half circle* of radius L from s. We call the region $F_s^L = \{SP(s, p) \mid p \in H_s^L\}$ that is bounded by $Q_U \cup Q_D$ and the two shortest paths from s to the endpoints of H_s^L the *funnel* from s of radius L. We symmetrically define the geodesic half circle $H_{s'}^L$ and the funnel $F_{s'}^L$ from s' of radius L.

Recall that $t_{\min}^{p_i} = \arg\min\{\|t_s^{p_i}\|, \|t_{s'}^{p_i}\|\}$ is the shorter of the two paths from s and s' to the visibility polygon $VP(p_i)$, for every point $p_i \in \mathcal{S}$. Let $B = \max_{p_i \in \mathcal{S}_2}\{\|t_{\min}^{p_i}\|\}$ be the longest among all such paths for the points in $\mathcal{S}_2$, and let F_s^B and $F_{s'}^B$ be the funnels of radius B from s and s', respectively. We can show that for every point $p_i \in \mathcal{S}_2$, one of its windows C_s^i, $C_{s'}^i$ intersects the funnel (of radius B) from the corresponding starting point.

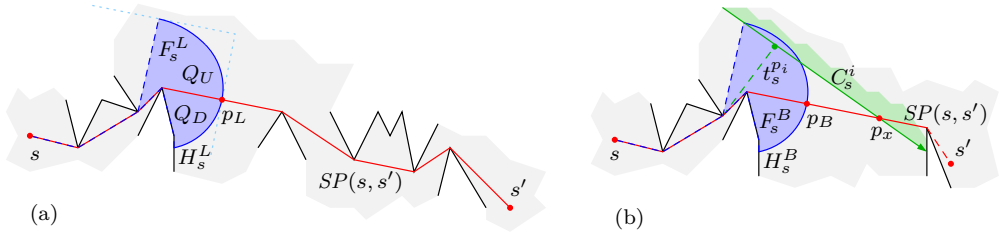

Figure 4 (a) Illustrating the definition of a funnel. (b) Illustrating the proof of Lemma 4.

▶ **Lemma 4.** *For any point $p_i \in \mathcal{S}_2$, either $VP(p_i) \cap F_s^B \neq \emptyset$ or $VP(p_i) \cap F_{s'}^B \neq \emptyset$.*

Proof. Consider the point $p_i \in \mathcal{S}_2$. Both windows C_s^i and $C_{s'}^i$ of $VP(p_i)$ intersect $SP(s,s')$. Without loss of generality assume that $t_s^{p_i} = t_{\min}^{p_i}$, the other case is proved in the same way.

If C_s^i intersects the subpath $SP(s,p_B)$, where p_B is the point at distance B from s along $SP(s,s')$, then C_s^i intersects the funnel F_s^B and the lemma holds. If C_s^i does not intersect the subpath $SP(s,p_B)$, then it intersects the subpath $SP(s',p_B)$ at p_x; see Figure 4(b). The pseudotriangle bounded by $t_s^{p_i}$, C_s^i, and $SP(s,p_x)$ must have an acute angle at p_x, since the angle between $t_s^{p_i}$ and C_s^i is right or obtuse. Therefore, C_s^i must intersect the geodesic half circle H_s^B, since $\|t_s^{p_i}\| \leq B$ and $t_s^{p_i}$ is a subpath of one of the paths $F_s^B = \{SP(s,p) \mid p \in H_s^B\}$, hence $VP(p_i)$ intersects F_s^B also in this case. ◀

Next, we compute an initial solution for which we can bound the min-max length. Let $\mathcal{S}_s^{\text{std}}$ and $\mathcal{S}_{s'}^{\text{std}}$ be a partitioning of the points in $\mathcal{S}_2$ that we obtain as follows. If for a point $p_i \in \mathcal{S}_2$ $\|t_s^{p_i}\| \leq \|t_{s'}^{p_i}\|$, then p_i belongs to $\mathcal{S}_s^{\text{std}}$, otherwise p_i belongs to $\mathcal{S}_{s'}^{\text{std}}$. We call this the *standard partitioning* of the points in $\mathcal{S}_2$. It follows from Lemma 4 that each visibility polygon $VP(p_i)$, for every $p_i \in \mathcal{S}_2$, is at distance at most B from one of s or s'.

Let V_s^{std} and $V_{s'}^{\text{std}}$ be the two shortest routes that see the points in $\mathcal{S}_s^{\text{std}}$ and $\mathcal{S}_{s'}^{\text{std}}$, respectively, that we can compute with the modified method of Tan and Jiang [11].

▶ **Lemma 5.** *The length of the two routes V_s^{std} and $V_{s'}^{\text{std}}$ is bounded by*

$$\max\left\{\|V_s^{\text{std}}\|, \|V_{s'}^{\text{std}}\|\right\} \leq \left(1 + \frac{\pi}{2}\right) OPT.$$

Proof. Consider the boundary of the funnel F_s^B. It consists of two paths of length B and a sequence of connecting arcs of circles having radius at most B; the sum of their lengths is bounded by πB. The perimeter is a route that intersects all the visibility polygons of the points in $\mathcal{S}_s^{\text{std}}$. Since V_s^{std} is the shortest route that intersects these visibility polygons, its length is bounded by $(2+\pi)B$. The length of the perimeter of $F_{s'}^B$ is bounded in the same way and since $OPT \geq 2B$, the lemma follows. ◀

Next, we improve on the previous result by updating the standard partitioning and obtain a $(1+\varepsilon)$-approximation, for any $\varepsilon > 0$. We consider ε to be a constant, sufficiently small, and define the integer $k = \lceil 69/\varepsilon \rceil$.

We partition the points into two sets in such a way that for each point the shortest path from the corresponding starting point is within a $(1+\varepsilon)$ factor from the optimum. In order to do this, we make a decomposition of the points in $\mathcal{S}_2$ into few subsets such that those points for which their corresponding windows are "similar" are placed in the same subset. By then choosing one representative point from each subset, we can compute the optimal partition of them using brute force, since there are only a few points to partition. The partition of the representative points indicates in which partition each similar point should be in. Once

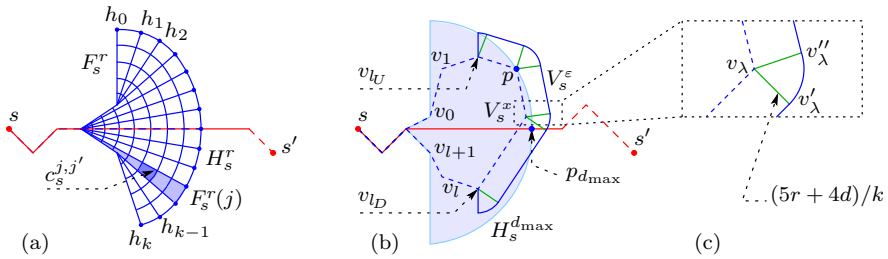

Figure 5 (a) Description of the cells in the similarity decomposition. (b)-(c) Construction of V_s^ε (blue) from V_s^x (dashed blue).

the partition for the full set of points has been computed, we can compute the two shortest watchman routes using the algorithm described in Section 2. We provide the details of the construction below.

For any point $p_i \in \mathcal{S}_2$ where $\|t_{\max}^{p_i}\| > 3B$, we can immediately assume that p_i must be seen by the route from the starting point for which the geodesic is shorter. Otherwise, $OPT > 6B$, as the optimal route must at least be as long as the geodesic $t_{\max}^{p_i}$, followed to $VP(p_i)$ and back. In this case, the standard partitioning together with the optimal routes for $\mathcal{S}_1$, i.e., the solution $(V_s^1 \cup V_s^{\text{std}}, V_{s'}^1 \cup V_{s'}^{\text{std}})$ gives at worst a 2-approximation to our problem, since $\max\left\{\|V_s^{\text{std}}\|, \|V_{s'}^{\text{std}}\|\right\} \leq (2+\pi)B < 6B \leq OPT$. Let $\mathcal{S}_2^{\text{L}}$ be the subset of the points in $\mathcal{S}_2$ for which $t_{\max}^{p_i}$ is "too long", in particular, $\|t_{\max}^{p_i}\| > 3B$, and assume they are partitioned according to the starting point of their shorter geodesic $t_{\min}^{p_i}$. Furthermore, let $\mathcal{S}_2^{\text{S}} = \mathcal{S}_2 \setminus \mathcal{S}_2^{\text{L}}$ be the remaining points in $\mathcal{S}_2$ for which $t_{\max}^{p_i}$ is "short". It remains to find an appropriate partition of the points in $\mathcal{S}_2^{\text{S}}$. To do this, we define a refinement of the funnel, and partition it into k^2 cells.

Define $r \stackrel{\text{def}}{=} \min\{3B, \|SP(s,s')\|/2\}$. For the points $p_i \in \mathcal{S}_2^{\text{S}}$, we have $\|t_{\max}^{p_i}\| \leq r$ by definition, giving us that $\|t_s^{p_i}\|, \|t_{s'}^{p_i}\| \leq r$. Let F_s^r and $F_{s'}^r$ be the two funnels of radius r from s and s', respectively.

We make a similarity decomposition of the set of points in $\mathcal{S}_2^{\text{S}}$ into at most k^4 distinct subsets $\mathcal{S}_2^{(1)}, \mathcal{S}_2^{(2)}, \ldots, \mathcal{S}_2^{(k^4)}$ as follows. H_s^r is the geodesic half circle of radius r centered at s; see Figure 5(a). Consider the $k+1$ consecutive equidistant points h_j, for $0 \leq j \leq k$, along H_s^r, h_0 and h_k being the two endpoints of H_s^r. The points of F_s^r between the two geodesics $SP(s, h_{j-1})$ and $SP(s, h_j)$, for each $1 \leq j \leq k$, define a *subfunnel* of F_s^r, denoted $F_s^r(j)$, lightly shaded in Figure 5(a). We can further decompose each subfunnel $F_s^r(j)$ into k cells $c_s^{j,j'} = \left\{p \in F_s^r(j) \,\middle|\, r \cdot (j'-1)/k < \|SP(s,p)\| \leq r \cdot j'/k\right\}$, for $1 \leq j' \leq k$, depending on the distance to s. A cell $c_s^{j,j'}$ is more heavily shaded in Figure 5(a). The funnel F_s^r thus decomposes into k^2 cells. Similarly, we decompose the funnel $F_{s'}^r$ into k^2 cells $c_{s'}^{j,j'}$, for $1 \leq j, j' \leq k$.

We say that a window C_s^i *belongs to* cell $c_s^{j,j'}$ if the intersection point of $t_s^{p_i}$ and C_s^i is a point in the cell $c_s^{j,j'}$. The point $p_i \in \mathcal{S}_2^{\text{S}}$ lies in the subset $\mathcal{S}_2^{(\kappa)}$ if C_s^i belongs to $c_s^{j_s,j'_s}$, $C_{s'}^i$ belongs to $c_{s'}^{j_{s'},j'_{s'}}$, and $\kappa = j_s + (j'_s - 1)k + (j_{s'} - 1)k^2 + (j'_{s'} - 1)k^3$. In this way we have partitioned all the points in $\mathcal{S}_2^{\text{S}}$ into at most k^4 distinct subsets.

For each non-empty subset, we select one point as the representative for the subset and let $\mathcal{R} \subseteq \mathcal{S}_2^{\text{S}}$ be the set of all the chosen representative points. By trying all $2^{|\mathcal{R}|} \leq 2^{k^4}$ partitions of the representative points of $\mathcal{R}$ into two subsets, we compute the two shortest tours V_s^x and $V_{s'}^x$ that visit the visibility polygons of the points in $\mathcal{R} \cup \mathcal{S}_2^{\text{L}}$, over all possible partitions of $\mathcal{R}$.

Let V_s^{ref} and $V_{s'}^{\text{ref}}$ be the two shortest routes obtained from the partition given by placing each point in a subset $\mathcal{S}_2^{(\kappa)}$ in the same partition as the representative point of this subset. The two routes can be computed in $O(m^3 n)$ time. To bound the lengths of V_s^{ref} and$V_{s'}^{\text{ref}}$, we introduce two auxiliary routes V_s^{ε} and $V_{s'}^{\varepsilon}$ that are easier to bound and prove that V_s^{ref} and $V_{s'}^{\text{ref}}$ are not longer than V_s^{ε} and $V_{s'}^{\varepsilon}$, respectively.

Let $d = \max\left\{\|V_s^x\|, \|V_{s'}^x\|\right\}$. We construct the route V_s^{ε} from V_s^x as follows ($V_{s'}^{\varepsilon}$ is constructed from $V_{s'}^x$ in the same way). Let $d_{\max}$ be the supremum value for which the geodesic half circle $H_s^{d_{\max}}$ intersects V_s^x, i.e., $d_{\max} = \sup_{L>0}\left\{H_s^L \cap V_s^x \neq \emptyset\right\}$. Let p be a point of contact between $H_s^{d_{\max}}$ and a convex vertex of V_s^x; see Figure 5(b). From p follow V_s^x in counterclockwise order until a reflex vertex v_0 of V_s^x is encountered. From p now follow V_s^x in clockwise order until a reflex vertex v_{l+1} of V_s^x is encountered, where l is the number of convex vertices on the clockwise path of V_s^x from v_0 to v_{l+1}. Let $p_{d_{\max}}$ be the point of intersection of $H_s^{d_{\max}}$ and $SP(s, s')$ and let θ be the angle between the tangent of $H_s^{d_{\max}}$ at $p_{d_{\max}}$ and the x-axis. For simplicity of construction, assume P, $F_s^{d_{\max}}$, and V_s^x to be rotated so that $\theta = \pi/2$, i.e., the tangent of $H_s^{d_{\max}}$ at $p_{d_{\max}}$ is vertical. Let v_{l^*}, for some $1 \leq l^* \leq l$, be the vertex of V_s^x such that the edges $[v_{l^*-1}, v_{l^*}]$ and $[v_{l^*}, v_{l^*+1}]$ lie on the same side of a vertical line through v_{l^*}, i.e., the rightmost vertex of V_s^x, if s lies locally to the left and s' lies locally to the right of $p_{d_{\max}}$. For some $1 \leq l_U \leq l^*$, assume the edges $[v_{\lambda-1}, v_\lambda]$ have non-negative slope, for all $1 \leq \lambda \leq l_U$. Similarly, for some $l^* \leq l_D \leq l$, assume the edges $[v_\lambda, v_{\lambda+1}]$ have non-positive slope, for all $l_D \leq \lambda \leq l$. Let the clockwise subpath from s to v_{l_U} of V_s^x also be the first part of V_s^{ε}. Add a vertical edge of length $(5r + 4d)/k$ at v_{l_U} upward to a new vertex v'_{l_U} and a vertical edge of length $(5r + 4d)/k$ at v_{l_D} downward to a new vertex v''_{l_D}. While $l_U \leq \lambda < l_D$, the segment $[v_\lambda, v_{\lambda+1}]$ has slope angle θ_λ, and we offset the segment by a distance $(5r + 4d)/k$ in the direction $\theta_\lambda + \pi/2$ (increasing angles going clockwise around a circle), which gives the segment $[v''_\lambda, v'_{\lambda+1}]$; see Figures 5(b) and (c). We connect v'_λ with v''_λ by a circle arc with origin v_λ and radius $(5r + 4d)/k$, for each $l_U \leq \lambda \leq l_D$ and finally connect v_{l_D} back to s with the corresponding subpath of V_s^x; again see Figures 5(b) and (c). If at some point the construction intersects the polygon boundary we shortcut the path so that it is the shortest to connect to in the free space of the polygon. We show that $(V_s^{\varepsilon}, V_{s'}^{\varepsilon})$ sees all points in $\mathcal{S}_2$ and bound the min-max length.

▶ **Lemma 6.** *The two-watchman route $(V_s^{\varepsilon}, V_{s'}^{\varepsilon})$ sees all points in $\mathcal{S}_2$ and*

$$\max\left\{\|V_s^{\varepsilon}\|, \|V_{s'}^{\varepsilon}\|\right\} \leq d + \frac{30r + 24d}{k}.$$

Proof. Recall that $(T_s, T_{s'})$ is an optimal min-max solution to the shortest two-watchman route problem for the points in $\mathcal{S}$, and let $(T_s^2, T_{s'}^2)$ be an optimal min-max solution for the points in $\mathcal{S}_2$. Since $(V_s^x, V_{s'}^x)$ is an optimal min-max solution for the problem on the points in $\mathcal{R} \cup \mathcal{S}_2^{\text{L}} \subseteq \mathcal{S}_2$, it follows that $\|V_s^x\| \leq \|T_s^2\| \leq \|T_s\| \leq OPT$ and $\|V_{s'}^x\| \leq \|T_{s'}^2\| \leq \|T_{s'}\| \leq OPT$, whereby $d \leq OPT$.

Let p_i be a point in $\mathcal{R} \cap \mathcal{S}_2^{(\kappa)}$, for some κ, $1 \leq \kappa \leq k^4$, and assume that V_s^x sees p_i. This implies that V_s^x intersects the window C_s^i at some point p. Consider any other point p_j in the same subset $\mathcal{S}_2^{(\kappa)}$ as p_i. Since the endpoints of $t_s^{p_i}$ and $t_s^{p_j}$ lie in the same cell, the angle between the windows C_s^i and C_s^j is at most π/k and the distance between the two windows within the cell is at most $(1 + \pi)r/k$ as this is an upper bound on the farthest distance between any two points in a cell (the height of the cell plus the width of the cell); see Figure 6. Let $\hat{p}_i$ be the endpoint of $t_s^{p_i}$ intersecting the window C_s^i. Thus, the distance from p to C_s^j is bounded by

$$\left\|p, C_s^j\right\| \leq \frac{(1 + \pi) \cdot r}{k} + \left\|\hat{p}_i, p\right\| \cdot \sin\frac{\pi}{k} \leq \frac{(1 + \pi) \cdot r}{k} + d\frac{\pi}{k} < \frac{5r + 4d}{k}.$$

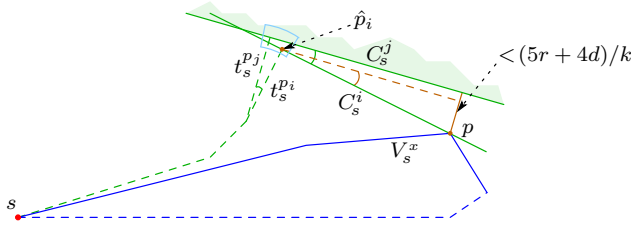

Figure 6 Bounding the distance between C_s^i and C_s^j at p.

Hence, if V_s^x sees p_i, then V_s^ε sees all points in $\mathcal{S}_2^{(\kappa)}$ as p_j was chosen arbitrarily in $\mathcal{S}_2^{(\kappa)}$. We can argue the other possibility when $V_{s'}^x$ sees p_i in the same way and therefore, since V_s^x and $V_{s'}^x$ see all representative points in $\mathcal{R}$, V_s^ε and $V_{s'}^\varepsilon$ together see all points in $\mathcal{S}_2$.

It remains to bound the lengths of V_s^ε and $V_{s'}^\varepsilon$. We do this only for V_s^ε as the argument for $V_{s'}^\varepsilon$ is the same. V_s^ε consists of translated segments from V_s^x (recall that $\|V_s^x\| \leq d$), two additional segments of length $(5r+4d)/k$, and circular arcs, also of radius $(5r+4d)/k$, where the angle sum is at most π; see Figures 5(b) and (c). The total length of V_s^ε is therefore bounded by

$$\|V_s^\varepsilon\| \leq \|V_s^x\| + \frac{2(5r+4d)}{k} + \frac{\pi(5r+4d)}{k} \leq d + \frac{30r+24d}{k},$$

as claimed. ◀

We summarize the results on handling the points in $\mathcal{S}_2$.

▶ **Theorem 7.** *The pair of watchman routes $(V_s^{\text{ref}}, V_{s'}^{\text{ref}})$, that see the points in $\mathcal{S}_2$, can be computed in $O(m^3 n)$ time, and has length bounded by*

$$\max\left\{\|V_s^{\text{ref}}\|, \|V_{s'}^{\text{ref}}\|\right\} \leq (1+\varepsilon)\,OPT.$$

Proof. By construction, the routes V_s^{ref} and $V_{s'}^{\text{ref}}$ visit the same windows as the routes V_s^ε and $V_{s'}^\varepsilon$ respectively. As V_s^{ref} and $V_{s'}^{\text{ref}}$ are the shortest routes that visit the windows in this way, $\|V_s^{\text{ref}}\| \leq \|V_s^\varepsilon\|$ and $\|V_{s'}^{\text{ref}}\| \leq \|V_{s'}^\varepsilon\|$. Since $r \leq 3B \leq 3OPT/2$ and $d \leq OPT$, we obtain

$$\begin{aligned}\max\{\|V_s^{\text{ref}}\|, \|V_{s'}^{\text{ref}}\|\} &\leq \max\{\|V_s^\varepsilon\|, \|V_{s'}^\varepsilon\|\} \leq d + \frac{30r+24d}{k} \leq OPT + \frac{45\,OPT + 24\,OPT}{k} \\ &\leq \left(1 + \frac{69}{k}\right) OPT \leq (1+\varepsilon)\,OPT\end{aligned}$$

by our choice of k. Moreover, we can compute V_s^{ref} and $V_{s'}^{\text{ref}}$ in $O(m^3 n)$ time, as mentioned in Section 2. ◀

3.3 Putting It All Together

Finally, we chose the shortest two routes among the possible pairs: a watchman route W_s starting at s and a stationary watchman at s'; a watchman route $W_{s'}$ starting at s' and a stationary watchman at s; and the combined watchman routes $V_s^1 \cup V_s^\varepsilon$ and $V_{s'}^1 \cup V_{s'}^\varepsilon$, where V_s^1 and $V_{s'}^1$ are the two routes that see the points in $\mathcal{S}_1$ as defined in Section 3.1, and V_s^ε and $V_{s'}^\varepsilon$ are the improved routes that see the points in $\mathcal{S}_2$ as computed in Section 3.2. Let

$$(V_s, V_{s'}) = \arg\min\Big\{\max\left\{\|W_s\|, \|s'\|\right\}, \max\left\{\|s\|, \|W_{s'}\|\right\}, \max\left\{\|V_s^1 \cup V_s^\varepsilon\|, \|V_{s'}^1 \cup V_{s'}^\varepsilon\|\right\}\Big\}.$$

By combining the algorithms developed in Sections 3.1 and 3.2 together with the bounds in Theorems 3 and 7, we immediately have the following theorem. Since ε is considered to be a constant, it does not affect the total running time.

▶ **Theorem 8.** *The two routes V_s and $V_{s'}$ see all points in $\mathcal{S}$, can be computed in $O(m^5 n)$ time, and*

$$\max\left\{\|V_s\|, \|V_{s'}\|\right\} \leq (2+\varepsilon)\, OPT,$$

for any fixed constant $\varepsilon > 0$.

We remark that the two routes V'_s and $V'_{s'}$ given by

$$(V'_s, V'_{s'}) = \arg\min\Big\{\max\left\{\|W_s\|, \|s'\|\right\}, \max\left\{\|s\|, \|W_{s'}\|\right\}, \max\left\{\|V_s^1 \cup V_s^{\text{std}}\|, \|V_{s'}^1 \cup V_{s'}^{\text{std}}\|\right\}\Big\}$$

using the standard partitioning instead of the improved partitioning can be computed in $O(m^5 n)$ time, have a min-max length bound of $(2+\pi/2)\,OPT$, but do not require exhaustive search to find the correct partition of the representative points in set $\mathcal{R}$.

4 Computing the Shortest Watchman Routes That See All of P

We can generalize the method obtained in Section 3 to find two watchman routes $(V_s, V_{s'})$ that together see the whole polygon P. We assume that the two optimal routes T_s and $T_{s'}$ for Problem 2 do not intersect, as otherwise a shortest watchman route W_s starting at s, or a shortest watchman route $W_{s'}$ starting at s', together with a route of length 0 at the respective other starting point, gives a 2-approximation (as explained in Step 1 in Section 3).

We carefully choose a set $\mathcal{S}$ of $O(n)$ points along the boundary of P with the property that whenever V_s and $V_{s'}$ see these points, they see all of the boundary of P, and for two watchmen, this is enough to see the polygon.

▶ **Theorem 9** (Lemma 3.1 in [8]). *If two watchmen together see all of the boundary of a polygon P, then they see all of P.*

Consider a point p on the boundary of P that is not seen from s or s', and let $VP(p)$ be the visibility polygon of p. We use the notation P_s^p to denote the subpolygon in $P \setminus VP(p)$ that contains s, and $P_{s'}^p$ to denote the subpolygon in $P \setminus VP(p)$ that contains s'. Analogous to before, we call the edge $C_s^p = VP(p) \cap P_s^p$ the *window* of p with respect to s, and the edge $C_{s'}^p = VP(p) \cap P_{s'}^p$ the *window* of p with respect to s'.

We want to see every point on the boundary. To be able to apply the strategy from Section 3, we start by defining two infinite sets $\mathcal{S}_1$ and $\mathcal{S}_2$, and allocate every boundary point into one of them based on the windows of it. Let $\mathcal{S}_1$ be the set of boundary points for which $C_s^p = C_{s'}^p$, and let $\mathcal{S}_2$ be the remaining set of boundary points. This partitions every polygon edge into intervals of points that lie in the same set, $\mathcal{S}_1$ or $\mathcal{S}_2$. We call such an interval a *subedge* and denote the set of subedges with points in $\mathcal{S}_1$ by $\mathcal{E}_1$ and the set of subedges with points in $\mathcal{S}_2$ by $\mathcal{E}_2$. The endpoints of the subedges are either vertices of P, i.e., points in $\mathcal{V}$, or interior points of the edges. We call these interior points *transition points*, and denote the set of all transition points by $\mathcal{T}$. Observe that each transition point is defined by one of the following:

- The intersection of the polygon boundary with a line through two polygon vertices
- A vertex of the visibility polygons $VP(s)$ and $VP(s')$

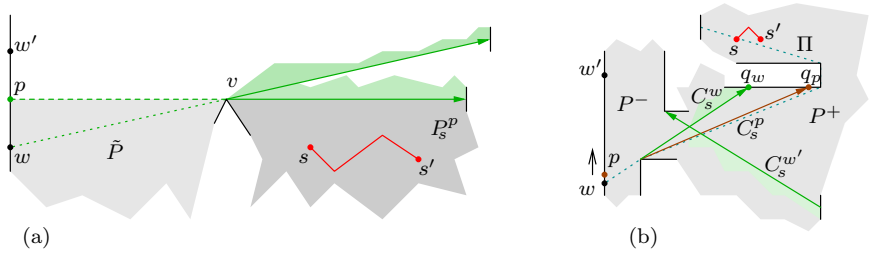

Figure 7 (a) Illustrating the proof of Lemma 10. (b) Determining the split point of a subedge in $\mathcal{E}_1$.

We can trivially construct the arrangement of lines through polygon vertices in $O(n^4)$ time, using a brute-force algorithm. This arrangement has complexity $O(n^2)$, and the polygon consists of n edges. Thus, the intersection points of the arrangement and the polygon edges can be computed in $O(n^3)$ time. The visibility polygon of a point in a polygon can be computed in $O(n)$ time, see [5]. With this, $\mathcal{T}$ can be computed in $O(n^4)$ time (note that this is not optimal, but does not affect the final complexity of our algorithm).

The visibility polygon of every point in $\mathcal{S}_2$ intersects the shortest path between the two starting points $SP(s, s')$. Conversely, the visibility polygon of any point in $\mathcal{S}_1$ cannot intersect $SP(s, s')$. As the area seen by a point that moves along a polygon boundary edge changes continuously, we observe the following.

▶ **Lemma 10.** *If w and w' are two points in $\mathcal{S}_2$ that lie on the same edge of P then, every point between w and w' is also in $\mathcal{S}_2$.*

Proof. Assume for a contradiction that there exists a point p in the interior of ww' that lies in $\mathcal{S}_1$. Then, the shortest path $SP(s, s')$ between the starting points lies fully inside $P_s^p = P_{s'}^p$. Let v be the vertex endpoint on C_s^p that is closest to p, and consider the segment pv. This segment splits P into two subpolygons, one containing w and one containing w'. Assume w.l.o.g. that the subpolygon $\tilde{P}$ that contains w does not contain P_s^p, see Figure 7(a). Then, $VP(w) \cap SP(s, s') = \emptyset$, since no ray emanating from w can intersect both pv and C_s^p except at v and w then belongs to $\mathcal{S}_1$, a contradiction. ◀

Thus, every edge contains at most one subedge in $\mathcal{E}_2$ and at most two subedges in $\mathcal{E}_1$, and there is at most a linear number of transition points.

To see the polygon boundary, we need to ensure that every subedge is seen. In a first step, we apply the strategy from Section 3 to $\mathcal{T} \cup \mathcal{V}$. This ensures that all points in $\mathcal{T} \cup \mathcal{V}$ are seen. To do so, we partition the points in $\mathcal{V}$ into two sets $\mathcal{V}_1 = \mathcal{V} \cap \mathcal{S}_1$ and $\mathcal{V}_2 = \mathcal{V} \cap \mathcal{S}_2$, and consider the two discrete, linear-sized sets, $\tilde{\mathcal{S}}_1 = \mathcal{V}_1 \cup \mathcal{T}$ and $\tilde{\mathcal{S}}_2 = \mathcal{V}_2 \cup \mathcal{T}$. We take care of the subedges in $\mathcal{E}_1$ and $\mathcal{E}_2$ separately. By adding the transition points to both sets $\tilde{\mathcal{S}}_1$ and $\tilde{\mathcal{S}}_2$, we ensure that the endpoints of each subedge in $\mathcal{E}_1$ lie in $\tilde{\mathcal{S}}_1$, and the endpoints of each subedge in $\mathcal{E}_2$ lie in $\tilde{\mathcal{S}}_2$.

Again, our goal is to take care of each subedge separately. We assign a direction to C_s^p so that $VP(p)$ lies locally to the left of C_s^p, if the polygon is rotated so that the window is directed upwards. The shortest path t_s^p that visits C_s^p is fully contained in the subpolygon P_s^p, and visits C_s^p locally from the right. We analogously assign a direction to $C_{s'}^p$.

We refine our set of subedges by splitting every subedge in $\mathcal{E}_1$ into segments that we call *fragments* such that for every point on a fragment, the windows of the points have the same orientation, either towards the subedge or away from the subedge.

▶ **Lemma 11.** *Every subedge ww' in $\mathcal{E}_1$ is split into at most two fragments, that is, a transition between points with windows of different orientations happens at most once when sliding along ww'.*

Proof. Consider the path $SP(s, w)$. Extending the first segment of $SP(s, w)$ (adjacent to s) backwards until it hits the boundary of P gives us a shortest path Π connecting two boundary points of P. The path Π separates P into two simply connected polygons P^+ and P^-, P^+ lying locally to the left of (or on) Π as we move from s towards w along Π and P^- lying locally to the right of (or on) Π as we move from s towards w. Assume, without loss of generality, that the farthest endpoint q_w of C_s^w lies in P^-, otherwise we reverse the roles of P^+ and P^-. An illustration is given in Figure 7(b). The point q_w is a bottom or top point of C_s^w. Let p be a point that moves from w to w' along the subedge and consider the window C_s^p. As p moves from w to w', the corresponding top (bottom) point q_p of the window C_s^p traverses the polygon boundary in P^-. If C_s^p aligns with a segment of Π at $p = p^*$, the point q_p moves from P^- to P^+ and C_s^p flips its orientation, q_p goes from being a bottom (top) point in the open ended interval wp^* to being a top (bottom) point in the open ended interval p^*w'. We call such a point p^* a *split point*. Since Π is a shortest path, the alignment of C_s^p and a segment of Π can occur only once. Hence, every subedge is split into at most two fragments at a split point. ◀

For an edge ww', the split point is given by the intersection of the polygon boundary and a line through two polygon vertices, and can therefore be computed in the same way as the transition points. We replace the subedge in $\mathcal{E}_1$ by these fragments, add the split points to $\mathcal{T}$, and update $\tilde{\mathcal{S}}_1$ and $\tilde{\mathcal{S}}_2$ accordingly.

With this modification, we handle the points in $\tilde{\mathcal{S}}_1$ in the same way as described in Section 3.1, and obtain the two routes V_s^1 and $V_{s'}^1$. The following Lemma guarantees that every subedge in $\mathcal{E}_1$ is seen.

▶ **Lemma 12.** *Let ww' be a fragment in $\mathcal{E}_1$. Either V_s^1 or $V_{s'}^1$ sees all of ww'.*

Proof. Since w and w' lie on the same fragment of $\mathcal{E}_1$, one of the two subpolygons $P_s^w, P_s^{w'}$ is fully contained in the other, say w.l.o.g. $P_s^w \supset P_s^{w'}$. The situation is as depicted in Figure 7(b), with w' at p. Thus, if V_s^1 visits C_s^w to see w, it also visits $C_s^{w'}$, and since ww' is part of a polygon edge, V_s^1 sees every point on ww'. The case for $V_{s'}^1$ is analogous. ◀

Next, we handle the points in $\tilde{\mathcal{S}}_2$ as described in Section 3.2 by computing the standard partitioning of the points. We make use of the following lemma.

▶ **Lemma 13** (Lemma 4.1 in [8])**.** *For a subedge $ww' \in \mathcal{E}_2$, if a watchman sees w and w' then it sees the whole subedge ww'.*

Thus, only those subedges in $\mathcal{E}_2$ where each endpoint is seen by a different watchman, meaning that its two endpoints are separated in the standard partitioning, are not yet seen. Let $\mathcal{E}_2^{\text{diff}}$ be the set of these subedges. For our approximate solution, we aim to find a good way for both watchmen to cooperatively guard the edges in $\mathcal{E}_2^{\text{diff}}$. To do so, we refine our discretization by splitting every subedge in $\mathcal{E}_2^{\text{diff}}$ into two, and let each watchman guard one of the resulting subedges.

For each subedge ww' in $\mathcal{E}_2^{\text{diff}}$, we compute the point p on ww' for which both watchmen have an equal distance from their starting point to the window of p's visibility polygon, i.e., $\|t_s^p\| = \|t_{s'}^p\|$, see Figure 8. Then, we replace ww' with wp and pw'. This can be done in $O(n)$ time per subedge, using the strategy described in [8]. We also add p to a third set of

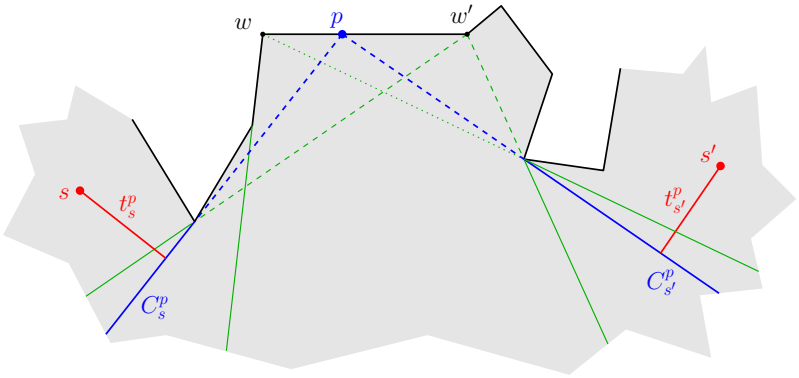

Figure 8 Splitting ww' at p where $\|t_s^p\| = \|t_{s'}^p\|$.

points $\tilde{\mathcal{S}}_3$, and modify the partitioning of $\tilde{\mathcal{S}}_2$ by adding all points in $\tilde{\mathcal{S}}_3$ to both partitions. With this, we simultaneously partition the subedges in $\mathcal{E}_2^{\text{diff}}$ based on their endpoints. Finally, we compute the shortest watchman routes $(V_s^{\text{std}}, V_{s'}^{\text{std}})$ for this new set of points. The two endpoints of a subedge $ww' \in \mathcal{E}_2^{\text{diff}}$ are seen by the same watchman. Hence, by Lemma 13, the whole subedge is seen.

Furthermore, recall that we construct the standard partitioning by assigning every point to a starting point based on its distance to the starting point. Putting the points in $\tilde{\mathcal{S}}_3$ into both partitions does not contradict the bound $B = \max_{p \in \tilde{\mathcal{S}}_2 \cup \tilde{\mathcal{S}}_3} \min\left\{\|t_s^p\|, \|t_{s'}^p\|\right\}$ on the maximum length within each partition, since for every point $p \in \tilde{\mathcal{S}}_3$, $\min\left\{\|t_s^p\|, \|t_{s'}^p\|\right\} = \|t_s^p\| = \|t_{s'}^p\| \leq B$. Therefore, the bound of Lemma 5 also holds for the modified set $\tilde{\mathcal{S}}_2$, hence we get

$$\max\left\{\|V_s^{\text{std}}\|, \|V_{s'}^{\text{std}}\|\right\} \leq \left(1 + \frac{\pi}{2}\right) OPT.$$

Finally, as in Section 3.3, we choose the shortest set of watchman routes as our solution

$$(V_s, V_{s'}) = \arg\min\Big\{\max\left\{\|W_s\|, \|s'\|\right\}, \max\left\{\|s\|, \|W_{s'}\|\right\}, \max\left\{\|V_s^1 \cup V_s^{\text{std}}\|, \|V_{s'}^1 \cup V_{s'}^{\text{std}}\|\right\}\Big\}.$$

Since the set $\mathcal{T}$ contains at most $O(n)$ points, we obtain a set of two watchman routes for P of length at most $(2 + \pi/2)OPT$ in $O(n^6)$ time.

▶ **Theorem 14.** *The two routes V_s and $V_{s'}$ see all of P, can be computed in $O(n^6)$ time, and*

$$\max\left\{\|V_s\|, \|V_{s'}\|\right\} \leq \left(2 + \frac{\pi}{2}\right) OPT.$$

5 Adaptations for the Floating Variant

Nilsson and Packer [8] show how to compute two points s and s' having the property that: for any point p on the boundary of P, the shortest of the paths t_s^p and $t_{s'}^p$ has length bounded by $OPT^F/2$, where OPT^F is the min-max length of the optimum floating two-watchman route. Recall that in Section 4, we defined a set of points $\mathcal{T}$ on the boundary. Therefore, for the two points s and s', the stated property holds in particular for the points in $\mathcal{T}$. The computation of s and s' takes $O(n^8)$ time.

Using the method derived in Section 4, we split the points in $\mathcal{T}$ into $\mathcal{S}_1$ and $\mathcal{S}_2$ and can obtain a pair of watchman routes V_F and V_F' through the points s and s', respectively, in additional $O(n^6)$ time. These two routes consist of two subroutes $V_F = V_s^1 \cup V_s^{\text{std}}$ and

$V'_F = V^1_{s'} \cup V^{\text{std}}_{s'}$, where V^1_s and $V^1_{s'}$ see the points in $\mathcal{S}_1$, and the subroutes V^{std}_s and $V^{\text{std}}_{s'}$ see the points in $\mathcal{S}_2$. The length of each of V^1_s and $V^1_{s'}$ is bounded by $2 \cdot OPT^F$, since the optimal routes are not guaranteed to pass through s and s' but there is some point on each of the two optimal routes at a distance at most $OPT^F/2$ from s and s', respectively. The length of V^{std}_s and $V^{\text{std}}_{s'}$ is bounded by the length of the perimeter of its funnel, i.e., $(1 + \pi/2)OPT^F$; see Lemma 5. We therefore conclude

▶ **Corollary 15.** *For a polygon P, the pair (V_F, V'_F) of watchman routes that see all of P can be computed in $O(n^8)$ time, and*

$$\max\left\{\|V_F\|, \|V'_F\|\right\} \leq \left(3 + \frac{\pi}{2}\right) OPT^F,$$

where OPT^F is the maximum length of an optimal pair of watchman routes for the floating two-watchman route problem in P.

6 Conclusion

We have addressed the two-watchman route problem in simple polygons by providing an approximation algorithm that is based on a discretization of the polygon boundary. The hardness in finding a good solution lies in determining a partition of the polygon boundary pieces into two subsets, such that each piece is seen by one of the two watchmen. For more than two watchmen, to see a discrete point set, a generalization of our partitioning scheme could be a promising strategy. However, to see the whole polygon with three (or more) watchmen, seeing the polygon boundary is not sufficient. Obtaining such approximation algorithms therefore requires different methods.

Since optimal watchman routes for the min-sum objective behave differently than for the min-max objective, our strategy does not apply, except for the crude observation that the longest of two min-max optimal routes is at most as long as the longest of two min-sum optimal routes. This adds a multiplicative factor of 2 on our results for the min-max measure, giving an approximation factor of ≈ 7.142 for the anchored version, and a factor of ≈ 9.142 for the floating version – better than the currently best known factors of ≈ 13.844 for the anchored version and ≈ 11.939 for the floating version [8]. Nilsson and Packer's approximation factor for the anchored version is larger than for the floating version as their anchored routes result from enforcing the floating routes to pass through the starting points, while in our algorithms the anchored routes are constructed from the starting points. Finally, the computational complexity of the min-sum two-watchman route problem in simple polygons remains open.

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