Approaching Proof in a Community of Mathematical Practice

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Abstract

This thesis aims to describe how students encounter proof in a community of mathematical practice at a mathematics department and how they are drawn to share mathematicians' views and knowledge of proof. Considering the department as a *community of practice* where the joint enterprise is learning mathematics in a broad sense made it possible to perceive the newcomers as active participants in the practice. The combination of a socio-cultural perspective, Lave and Wenger's and Wenger's social practice theories and theories about proof offers a fresh framework in understanding and describing the diversity of the culture involving such a complex notion as proof. Proof is examined from historical, philosophical and didactical points of view and considered as *reification* and as an *artefact* from a socio-cultural perspective. The metaphor of *transparency* of artefacts that refers to the intricate dilemma about how much to focus on different aspects of proof at a meta-level and how much to work with proof without focusing on it, both from teacher and student perspectives, is a fundamental aspect in the data analysis. The data consists of transcripts of interviews with mathematicians and students as well as survey responses of university entrants, protocols of observations of lectures, textbooks and other instructional material. Both qualitative and quantitative methods were applied in the data analysis. A theoretical model with three different teaching styles with respect to proof could be constructed from the data. The study shows that the students related positively to proof when they entered the practice. Though the mathematicians had no explicit intention of dealing so much with proof in the basic course, students felt that they were confronted with proof from the very beginning of their studies. Proof was there as a mysterious artefact and a lot of aspects of proof remained invisible as experienced by students when they struggled to find out what proof is and to understand its role and meaning in the practice. The students who proceeded further experienced a mix of *participation* and *non-participation* regarding proof depending on their capacity to follow lectures and on how much they invested themselves in the *negotiation of meaning* of proof. The first oral examination in proof seems to be significant in drawing students to the practice of proof.

Keywords: proof, university mathematics, mathematical practice, community of practice, participation, reification, transparency, artefact

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Introduction

The purpose of this thesis is to describe and characterise the culture of proof in a mathematical practice at a mathematics department and how students are engaged in proof and proving activities. The main issue is how students encounter proof and how they are drawn to share mathematicians' views and knowledge of proof.

"Proof is the soul of mathematics" as a mathematician in this study expressed it. Proof is a method of getting acceptance for and generating new mathematical knowledge. Proof is a multi-faceted notion, difficult to define and on which different persons have different views. According to mathematicians in my study, proof actually permeates all mathematics. Nevertheless, proof is also a part of mathematics that has been considered as difficult to teach and learn (e.g. Bell, 1976b; Moore, 1994; Selden & Selden, 1995; Weber, 2001). For all these reasons it has been a great challenge for me to approach and examine proof and the teaching and learning of proof.

I approach the issue of proof from different directions. Firstly, I study mathematicians' views and pedagogical intentions concerning proof. Secondly, I examine students' backgrounds, views and how they experience proof in their mathematical practice at the mathematics department. Finally, I contrast the results of these two parts. So, approaching proof in the title of the thesis refers not only to my own approaching to proof, but mathematicians' and students' approaching to it.

To understand these two sides and their interaction better, I have developed a theoretical perspective on proof by combining a socio-cultural perspective and the social practice theory of Lave and Wenger (1991) and Wenger (1998), with theories about proof obtained from didactical research. According to Wenger (1998), structuring resources for learning come from a variety of sources, not only the pedagogical activity. *Pedagogical intentions create a context in which learning can take place.* Teachers, lectures, lessons and instructional materials, like textbooks, become resources for learning in complex ways. Much of what students learn is not intended and much of what teachers want to convey is not captured by students. Also, there are not very clear aims set up for the learning of proof in mathematics either in the school curriculum or the curriculum for the university courses. In my study, I contrast the mathematicians' views and intentions with the students learning experiences in order to shed light on how the structuring resources become resources for learning.

Further, I use Wenger's (1998) theory to give structure to the practice I am studying. As a unit/level of analysis I use *a community of practice of mathematics at a mathematics department*. In this community, I include all people exercising and learning mathematics at the department which is the focus of my study. There are mathematicians, doctoral students, teaching assistants and students. It is a dynamic practice and the joint enterprise for all participants is the learning of mathematics in a broad sense. Learning is conceived as increasing participation in the community of practice of mathematics which leads to changing identities (Wenger, 1998). There is a richness of competence, and learning in this community occurs on different levels. Many students learn mathematics on a basic level but there are also doctoral students who are learning to carry on research in mathematics. Mathematicians are researching and obtaining new mathematical knowledge, teaching, examining and supervising students, improving teaching etc. But not only mathematicians are teaching mathematics to students, also doctoral students and teaching assistants take part of this enterprise. There are also pedagogical and didactical seminars, discussions and activities that aim to develop the teaching of mathematics. There are lectures, lessons, seminars and other kinds of meetings for the participants where teaching and learning of mathematics takes place. All these activities are included in the exercising of mathematics and are important for maintaining and developing the community of mathematical practice at the academic department.

There is a diversity of experience about mathematics among those who participate in this practice; there are *old-timers* and *newcomers*. However, it is not possible to exactly define when a newcomer becomes an old-timer as the character of these notions is relative. My thesis gives a contribution to knowledge in this area by describing how the enculturation of newcomers to the practice takes place with a special focus on their access to proof. An important theoretical aspect that I put forward in my work is that I consider proof as an *artefact* in mathematical practice, not only in the *community of practice of mathematics at the mathematics department* that is the object of my study but in mathematics as whole, and examine how different aspects of proof can be focused on in the teaching of proof. According to the theory of Lave and Wenger (1991) there is an intrinsic balance in the teaching of mathematics between the use of artefacts on the one hand, and how to focus on artefacts at a meta-level, on the other hand. My thesis sheds light on this dilemma in the teaching of proof.

Besides putting forward these theoretical arguments in Chapter 2, I test them empirically against the data obtained from surveys with university entrants and interviews with mathematicians and students. There are also themes that emerge from the data, for example different styles in mathematicians' utterances concerning the teaching of proof and problems for newcomers' engagement with proof. These themes are included in the examination about how the structuring resources become resources for learning.

My research and the theory I have put forward in the thesis draws together the earlier results in the field and offers a fresh perspective on proof in mathematics education. It also contributes to the illumination of how different aspects of proof can be focused on in the teaching of proof. The theoretical perspective and the empirical findings of the thesis open up new possibilities for research in the field of proof in mathematics education.

The **general research questions** are formulated in the following way:

- How do students meet proof in the community of mathematical practice at a mathematics department?
- How are students drawn to share mathematicians' views and knowledge of proof?

The structure of the thesis

In **Chapter 1**, I first give an account of how the research questions have developed during the study. I then very briefly describe the changes in the community of practice of mathematics at the mathematics department that is the object of my study regarding the role of proof during the last three decades. In the third section, I illustrate the complexity of the notion of proof in mathematics by approaching proof from historical and philosophical points of view. I conclude the chapter with a short review on research about proof in mathematics education.

In **Chapter 2,** I clarify the theoretical positions of my study. I justify and describe the choice of the theoretical frame for the thesis and elaborate the central theoretical notions applied in my work. I put forward the basic theoretical arguments concerning proof in mathematical practice and define and describe the unit of analysis for the object of my study. Besides the learning theories relating to ontological and epistemological assumptions and commitments, I also looked at theories and research about proof in mathematicians' practice and in mathematics education, in order to link my study to previous studies. Thus, the last section of the chapter is a deepening of the issue of proof in mathematical practice.

Chapter 3 is divided into three main parts. In the first section, I define the specific research questions through which I want to examine the issue as well as the methods used for shedding light on these specific research questions. The second section provides an epistemological account of the methodology. In the last section, I give a detailed account of the research methods and the procedures used for data gathering and the data analysis.

Chapter 4 is about mathematicians' practice. I report the results of the analysis of the interviews with the mathematicians. The first section is about the mathematicians' views on proof. In the second section, I describe how the mathematicians talked about proof as a tool in their practice. In the third section, I give an account of how the mathematicians in my study spoke about the changes in the practice concerning proof. I conclude the chapter by describing a theoretical model with three pedagogical approaches to the teaching and learning of proof.

Chapter 5 is about students' practice. In the first section, I give an account of students' stated upper secondary school experiences concerning proof. In the second section, I describe how the students related to proof at the beginning of their studies. In the following sections, I describe what kind of participation regarding proof there was available to students and how students talked about their experiences in their mathematical practice. I give examples of utterances expressing participation as well as non-participation concerning the meaning of proof.

In **Chapter 6**, I draw together the different parts of the results. I contrast the results of mathematicians' practice with the results regarding students' practice and discuss both consistencies and inconsistencies in the data. I also describe how the three different teaching styles in the theoretical model can be experienced by students.

In **Chapter 7**, I present the main conclusions of the study. I discuss how the thesis illuminated the main research questions and how the theoretical frame developed in the thesis helped to shed light on these questions. Finally, I suggest some implications to the educational practice and broach problems to focus on in further studies.

Appendix:

1. Course descriptions
2. The questionnaire
3. Tables of some survey results
4. Questions for the oral examination in *Mathematical Analysis 3*
5. The table of the three teaching styles
6. An example of how I have worked with NVivo

1 Background

Developing research questions perhaps requires the most complicated thinking of research (Stake, 1995). Most often, they have to be dug out and worked over and, according to Stake (1995), the best research questions evolve during the study (*ibid.,* p. 33). I start the chapter by very briefly describing the development of the research questions and how my study has gradually been limited. In the next section, I describe the changes in practice concerning the treatment of proof. In the third section of the chapter, I approach proof from a historical and philosophical point of view in order to shed light on the complexity of the notion of proof. I conclude the chapter with a brief review of research about proof in mathematics education.

1.1 The development of the research questions

I started to study proof in mathematics education in 2002 and the aim of the thesis at the time was to examine how proof was treated both in upper secondary school and in undergraduate university courses in Sweden as well as what kind of prior knowledge students had regarding proof, when they entered the mathematical practice at different universities in Sweden. I also wanted to explore if there was continuity between the school mathematics and the university mathematics concerning the issue of proof.

The first data collection was a pilot survey among a hundred university entrants who started to study mathematics at the mathematics department that is the focus of my study in the thesis. The aim of the pilot study was to give an overall view on students' stated upper secondary school experiences, how students related to proof at the beginning of their university studies including their views and feelings, as well as their proving abilities, an overview which would be supplemented by in-depth studies carried out within the global project (Nordström, 2003). I also studied upper secondary school textbooks in order to see to what extent in different mathematical domains proofs occurred in the textbooks and what special kinds of proofs were treated in them (Nordström & Löfwall, 2005). I created and distributed a questionnaire for upper secondary school teachers about how they related to proof and got about 40 responses. As there are no studies about the role of proof in mathematics curricula and classrooms in Sweden, I studied debate articles, old curricula and other official documents in order to obtain a pic-

ture about the main changes in the treatment of proof during the last decades. I also interviewed persons who had followed the development for a long time. In autumn 2003, I conducted a survey among university entrants in different parts of Sweden and also started to interview mathematicians in five different departments about their views on proof and the teaching and learning of proof (Nordström, 2004).

In parallel to the data gathering I studied the relevant pedagogical theories in order to find an appropriate frame for my study and to define the level/unit of analysis. I also created a conceptual frame from the literature to be able to link the data to previous studies about proof. All these activities influenced the development of my research questions. I found a socio-cultural approach appropriate for my study and in Lave and Wenger's (1991) and Wenger's (1998) social practice theory I found a frame that at least partially described the teaching and learning conditions at the mathematics department that I was studying, in a way coherent to my views. One of the main theoretical challenges in my study turned out to be the examination of how to apply the social practice theory of Lave and Wenger (1991) and Wenger (1998) to the practice of mathematics at a mathematics department with a special focus on students' access to proof and how to combine the social practice theory and the theories about proof.

I started to examine proof as an artefact in the mathematical practice and explore the strengths of the metaphor of *transparency* (see p. 40) in the case of proof. I tested the theoretical ideas and the conceptual frame in a pilot study about five mathematicians' views on proof and the teaching and learning of proof and also in a textbook study (Nordström, 2004; Nordström & Löfwall, 2005). I went on investigating the ideas against the data obtained from some focus group interviews with students as well. During the first data analysis of the focus group interviews with students, the issue of students' access to proof turned out to be central for my study.

Hence, in order to deepen the issues, I decided to limit my study to concern one university only. For the same reason, I also left aside the research on students' proving abilities. I decided to concentrate on students' backgrounds, how they related to proof when they entered the practice and how they talked about their experiences in the practice rather than their proving abilities. This was also partly because there was so much evidence about students' difficulties with proving tasks that was documented in the examinations and didactical research, and partly because of the time limitation for my study. Hence, a lot of data (survey responses from university entrants in different parts of Sweden, interviews with mathematicians in four other departments and survey responses from upper secondary school teachers) that I have gathered have been postponed for possible later analysis.

The purpose of the thesis is now to describe and characterise the culture of proof in a mathematical practice at a mathematics department and how newcomers become engaged in proof and proving activities. The main issue

is how students meet proof in a community of mathematical practice and how they are drawn to share mathematicians' knowledge and views of proof. I approach the issue from different directions (*Figure 1*, p. 13).

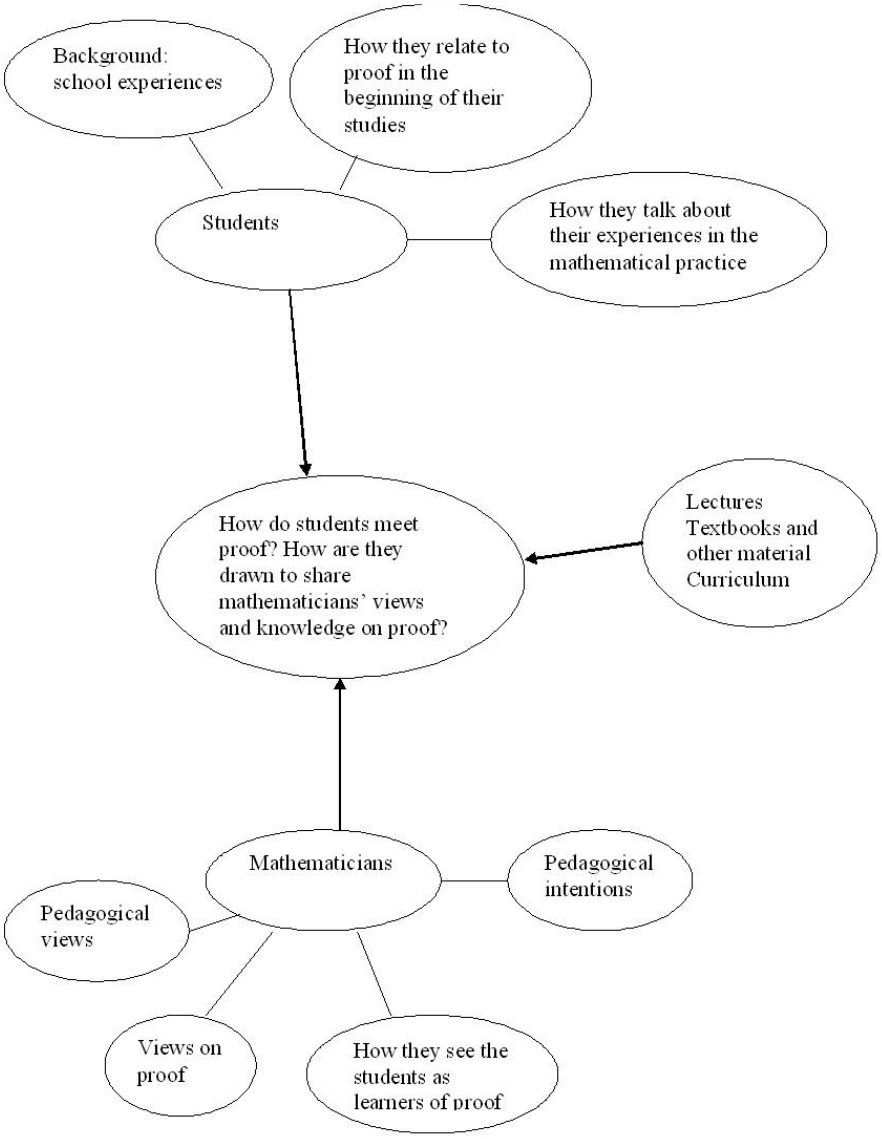

Figure 1 Approaching the issue from different directions

On the one hand, I am interested in mathematicians' views on proof and their pedagogical perspectives concerning the teaching and learning of proof, for example how they view the changes in the practice and how they talk about students and their own intentions concerning the teaching of proof. On

the other hand, I explore newcomers' backgrounds when they enter the practice: their declared upper secondary school experiences regarding proof and how they related to proof. I also want to examine what kind of occasions there are available in the community of mathematical practice for students to engage in proof and proving and how students in different phases of their studies talk about their experiences in the practice. Hence, students' access to proof is one of the central issues in my thesis. The main question here is how students meet proof in undergraduate courses and what possibilities students are offered to enhance their learning and understanding of proof.

1.2 Changes in practice

Communities of practice develop in larger contexts – historical, social, cultural and institutional with special resources and constraints (Wenger, 1998). In order to situate the study on proof into its socio-cultural and historical backgrounds I first provide the reader with a brief account, from an international perspective, of the changes in the practice regarding the role of proof during the last three decades.

Undergraduate mathematical curricula are always in some state of change. Some of the changes follow from the research of new mathematics but most of them are put forward because of factors outside the community of mathematical practice, like changes in the school curriculum[1], changes in the economical support system for the academic departments, changes in the demands of mathematical knowledge from other practices. Also the fact that university education has become more accessible to a larger part of the population has changed the practice and made it more heterogeneous. Today, there are a lot of students in mathematical practice who are registered in teacher education, social sciences or natural sciences. Hence, departments of mathematics have been "faced with the challenge of having to teach students whose background preparation, learning styles, study habits, and career ambitions are more and more at odds with the traditional lecture-style mathematical training with its Bourbaki-like curriculum, particularly in pure mathematics." (Hillel, 2001, p. 63)

At the mathematics department that I have studied, there have been changes in the courses, in the organisation of the teaching, in the choice of the course literature as well as in the curriculum as a response to the changes

[1] The status of proof in school mathematics has changed during the last decades and proof has had a diminished place in the secondary school mathematics curriculum in many countries (Hanna, 1995; Niss, 2001). There is no research about such changes in Sweden. However, in the national curriculum 1994 for upper secondary school, the word proof was not mentioned (Grevholm, 2003). There are signs that proof is coming back to the school curriculum in many countries, also in Sweden (Knuth, 2002; Skolverket, 2006; Waring, 2001).

outside the mathematical practice described above. In the 70s a course in Euclidean geometry was introduced as a part of the basic course of 20 study points as a consequence of the fact that geometry had got a diminished place in school curriculum (Strömbeck, 2006). At the time, there was a concern among university teachers about students' lacking the capability to answer questions in examinations concerning proof and the theories of mathematics (Boman, 1979) and, gradually, such questions were moved to intermediate and more advanced courses (Appendix 1).

At the beginning of the 90s the introductory courses in calculus were reformed. This reform was a response to the demands of other practices, like those of natural sciences. A part of the theory, for example, epsilon – delta proofs were moved to intermediate courses. Instead, more applications and multivariable calculus were included in the basic course in analysis (Strömbeck, 2006). In the middle of the 90s the number of applicants who wanted to study mathematics at the department was much bigger than the number of students who could be accepted. Hence, it was possible to choose the students with the highest marks in the subject. Now the number of applicants has diminished and all of them are accepted (Johansson, 2006).

Hillel (2001a) reports in the ICMI-study about the teaching and learning of mathematics at university level that the transition problem from secondary to tertiary level has led to the appearance of so called bridging courses aiming to facilitate students' entry into university mathematics. The lacks in students' prior knowledge in mathematics at the beginning of the tertiary level are well documented in Sweden (e.g. Bylund & Boo, 2003; Högskoleverket, 1999; Thunberg & Filipsson, 2005). As a consequence introductory courses were introduced to curricula in many universities in Sweden, also at the department that is the focus of my study (Appendix 1). There are differences in the character of bridging courses concerning the role of proof. For example, at KTH (The Royal Institute of Technology) proof is a central issue in a 4-point bridging course (Thunberg, 2005), whereas the introductory course at the mathematics department which I am studying, is largely a repetition of upper secondary school mathematics and an introduction of some new calculation techniques. There is also an online course available for university entrants the aim of which is to facilitate students' transition from school mathematics to university mathematics.

At the same time as the introductory course was offered for the first time at the department that is the object of my study, the course in Euclidean geometry was not included in the curriculum any more (Strömbeck, 2006). Some changes in the course literature also took place at the time. Vretblad's (1999) textbook was no longer used in the basic course. In Vretblad's book students were introduced to proof and elementary proof techniques in Swedish. The book was also mentioned by some mathematicians and students in my interviews as a significant help for students' understanding of proof. Instead of Vretblad's book, a book with repetition of upper secondary school

mathematics (Wallin, Lithner, Jacobsson, & Wiklund, 1998) is now used in the introductory course together with literature for the following courses (Appendix 1).

The number of teachers in relation to the number of students at the department has steadily diminished during the last decade. Because of this, in 1997, lectures with about a hundred beginner students were introduced. The time for lectures diminished at the same time from three to two hours. Instead, a group of students were offered one hour with a teaching assistant/lecturer to go through the exercises. Prior to 1997 a mathematician had a group of about 30 students for three hours and it was possible for the teacher to shift between theory and applications (Johansson, 2006; Strömbeck, 2006). From 2002, lessons with about 10 students and a teaching assistant/lecturer were introduced with the aim of giving students the opportunity to present mathematics both orally and in written form.

Mathematicians in my study related to the changes in the practice concerning the treatment of proof in the curriculum in various ways (see p. 96). I will come back to the issue in Section 4.2.

1.3 What is mathematical proof?

Proof constitutes the means for justifying knowledge in mathematics. The purpose of this section is to shed light on the complexity of the notion of proof by first giving a brief account of how the view on proof has changed during its history. I then discuss philosophical, ontological and epistemological aspects of mathematics and proof and how a working mathematician relates to these philosophical issues. My aim is not to make an exhaustive examination of the subject, but just to focus on some main changes and controversies concerning the notion of proof.

When Greek philosophers started to apply philosophical methods to mathematics, they analysed the results in mathematics and systematised the contemporary mathematical knowledge in a deductive manner (e.g. Eves 1997; Katz, 1998). They developed the idea of dividing a theory into *axioms* and *definitions* followed by *statements* derived from these, using the chains of logical reasoning which is still characteristic of mathematics. For the Greeks, the elementary concepts of geometry, like points and lines, were regarded as idealisations of certain actual physical entities. Then the postulates were accepted statements about these idealisations. These statements would be so carefully chosen that their truths were “evident”. This view is called *material axiomatic* (e.g. Eves, 1997). This has also been seen as a natural view for pupils when they work with parts of the Euclidian geometry in school (e.g. Jahnke, 2005).

Before the introduction of algebraic symbols proofs were mostly *generic examples* or based on geometry. Even Euclid proved that the number of

primes is infinite with a generic example using specific geometrical entities (Heath, 1956). In the sixteenth century, Viète (1540-1603) started to use letters as well as numbers and came part way towards modern symbolism. This enabled him to leave specific examples and verbal algorithms and, instead, treat general examples. Descartes (1596-1650) similarly noted that it was not necessary to imagine line segments, but instead it was sufficient to assign each by a single letter. Descartes also started to use the terms a^2 and a^3 as line segments, rather than as geometric squares and cubes as demanded by Euclidean geometric algebra. This enabled him to mix higher powers without worrying about their lack of geometric meaning (e.g. Katz, 1998). This development, together with the systematisation of arithmetical laws in the 19th century enabled the progress of algebraic proofs. The first encounter with proofs for many students in Sweden is, besides geometric proofs, algebraic derivations of formulae.

Calculus as it was developed in 17th and 18th centuries, was a powerful tool for applications and led to an expansive development of mathematics. During the period applications were more important than proofs and at the time, the idea of function itself was not understood/defined in the manner it is now and notions such as limit, continuity, differentiability, integrability, and convergence were unclear and lacked exact definitions (e.g. Eves, 1983). Gradually contradictions and paradoxes arose and in the early nineteenth century the first steps were taken towards replacing a method of infinitesimals by a more precise method of limits within the so-called *arithmetisation* of analysis (e.g. Katz, 1998). In Sweden, students first meet calculus in a way more similar to the calculus used in the 18th century than to later formalisations. Most representations lean on pictures and intuition rather than on exact definitions. The first time students at the department which I am studying meet the modern definitions, for example the one for the notion of limit, is in the intermediate course *Mathematical Analysis 3* (Appendix 1), where students have to prove the theories they have applied earlier.

The development of non-Euclidean geometry during the first half of the nineteenth century and the liberation of algebra (development of a non-commutative algebra) led to a deeper study and refinement of the axiomatic procedure. Hence, from the *material axiomatic* of the ancient Greeks evolved the *formal axiomatic* of the twentieth century (e.g. Eves, 1998). In modern mathematical theories, axioms are not seen as basic universal truths any more but as contingent assumptions that are used as the starting point of a theory.

Recently, there have been some new trends in mathematical proofs due to the growing use of computers in mathematical practice. A computer has been used to validate enormously long proofs, for example the four-colour theorem but also to "prove" statements with experimental methods. There have been controversies among mathematicians concerning the computer "proofs" because they are at odds with the traditional view of mathematical

proof, where every single statement should be open to verification (e.g. Jaffe & Quinn, 1993; Thurston, 1994).

I have, so far, very briefly described some aspects in the history of mathematics that have relevance for the modern view on proof. I will next present the three schools that studied the foundations of mathematics during the so called classical period (1879-1931), *logicism*, *intuitionism*, and *formalism* (e.g. Benacerraf & Putnam, 1998; Eves, 1997). These three schools have different views on the nature of mathematics and proof but all of them have influenced mathematical practice.

The *logicist* thesis is that mathematics is a branch of logic. All mathematical concepts are to be formulated in terms of logical concepts, and all theorems of mathematics are to be developed as theorems of logic. Logicism reduced all of classical mathematics to a single formal system and thus, certainly influenced the way mathematical statements and proofs are formalised (e.g. Eves, 1997).

The *intuitionist* thesis is that mathematics is to be built solely by finite constructive methods on the intuitively given sequence of natural numbers. One of the main principles of the intuitionist school is that an entity whose existence is to be proved must be shown to be constructible in a finite number of steps; it is not sufficient to show that the assumption of the entity's non-existence leads to contradiction. More generally, the intuitionists deny the universal acceptance of the law of the "excluded middle". Intuitionism produces its own type of logic, and mathematical logic, as a consequence is a branch of mathematics (Benacerraf & Putnam, 1998).

Intuitionism is also called *constructivism* but in the philosophy of mathematics means something different from constructivism in pedagogical research (see p. 25). However, there are those who advocate constructivism in mathematics as an inspiring source for mathematics educators because it considers mathematics as a mental activity that produces explicit constructions (e.g. Fosgerau, 1992). The intuitionist view of mathematics has also been significant within the field of computer science. There are also ideas coming from persons working with programming (Back, Peltomäki, Salakoski, & von Wright, 2004) how to help students' understanding of mathematical reasoning and proof by so called structural derivations.

The logicist achievement of reducing all of classical mathematics to a single formal system was much admired by *formalists* (Eves, 1997). The formalists pushed the axiomatic method to its extreme. Mathematics is viewed as a formal system consisting of axioms, definitions, statements and proofs. Mathematics is a collection of such abstract developments, in which the terms are mere symbols and the statements are formulas involving these symbols. The ultimate base of mathematics does not lie in logic but only in a collection of prelogical marks and symbols and in a set of operations with these marks. The consistencies of various branches of mathematics are an important and necessary part of the formalist program. Freedom from con-

tradictions is only guaranteed by consistency proofs. However, Gödel showed in 1931 by *methods acceptable to the followers of any of the three principal schools of the philosophy of mathematics*, that the consistency of the formal systems known to be adequate for the derivation of mathematics cannot be demonstrated by finitary methods formalised within the system, whereas any system known to be safe in this sense is totally inadequate to describe a significant part of mathematics (e.g. Benacerraf and Putman, 1998).

Thus, the three schools presented above hold different epistemological views on proof but, at the same time, proof is very central in all of them. They also hold different *ontological* perspectives on mathematics. An ontological question is whether we for example, consider mathematics to be the discovery of truths about structures that exist independently of the activity or thought of mathematicians (a platonistic view). Then the truth of mathematical propositions is not determined by the rules we adopt, but rather by the correspondence between the propositions and the mathematical structures to which the terms in those propositions refer (e.g. Benacerraf and Putman, 1998). This is a common working perspective for mathematicians (Davis & Hersh, 1981) and often connected to the formalist school. Opposite to this view, mathematics can be seen as an activity in which the mathematicians play a more creative role. Then propositions are true if they follow from the assumptions and definitions we have made. The assumptions, definitions and methods of proof constitute the rules determining the truth or falsity of the propositions formulated in their terms. This is called a conventionalist view on mathematics (Fosgerau, 1992).

The three philosophical schools described above, deal with the question of what an acceptable mathematics should be like: what methods, practices, proofs, and so on, are legitimate and therefore justifiably used. Characteristic of the creators of the three schools is that they are *mathematicians* rather than philosophers, and they criticise the foundations of their subject (Benacerraf & Putnam, 1998). In contrast, there are those who do not want to propagate certain mathematical methods as the only ones acceptable, but who want to describe the accepted and used ones (Benacerraf & Putnam, 1998). Hersh (1998) and Ernest (1991; 1998b) have defended, by building on Lakatos' ideas, a *fallibilist* approach to the philosophy of mathematics. Both Hersh and Ernest have influenced the discussions and research on proof in the field of mathematics education[2]. Hersh (1998) considers the criteria for a philosophy of mathematics and claims that a socio-historical approach gives better answers to the main philosophical questions concerning mathematics than the philosophy of the three schools presented above. He criticises the creators of "foundationist" philosophy of mathematics for turning

[2] Paul Ernest has even developed a social constructivist philosophy of mathematics education where he draws on the ideas about his philosophy of mathematics (Ernest, 1991).

philosophical problems into mathematical problems. Hersh wants to think of philosophy of mathematics, not as a branch of mathematics, but as a philosophical enterprise based on mathematical experience.

Whatever one thinks about this, the classical period was a dynamic period and the three schools influenced and criticised each other's work. What is important to my work is that all these schools have also influenced mathematical practice and proof as they are today. Using a socio-cultural perspective to my object of study does not entail an agreement with Ernest (1998b) who questions the grounds for, not only mathematical but even logical assumptions made in proofs. Moreover, the question about the fallibility of mathematical knowledge is irrelevant for my study. In my study it is important to describe how ordinary working mathematicians relate to their practice and proof and what the character of the mathematics and proofs is that students are expected to learn and participate in, in the practice. Further, everyday mathematicians seem to not bother about the philosophical discussions about the foundations of mathematics but agree on the certainty of a great part of mathematical knowledge. They think that the criteria for accepting new theorems are internationally similar and thus more objective than criteria for other sciences.

> *"Mathematics as we practice it is much more formally complete and precise than other sciences, but it is much less formally complete and precise for its content than computer programs… Mathematicians can and do fill gaps, correct errors, and supply more detail and more careful scholarship when they are called on or motivated to do so. Our system is quite good at producing reliable theorems that can be solidly backed up. It's just that the reliability does not primarily come from mathematicians formally checking formal argument; it comes from mathematicians thinking carefully and critically about mathematical ideas."* (Thurston[3], 1994, p. 170)

This is also important for the newcomers in the community of mathematical practice. They have to learn the commonly accepted rules of reasoning and the body of mathematical knowledge that is exercised in the community of practice of mathematics at the department they enter. In my work, I look at proof very broadly and include derivations of formulas in the notion of proof. This is in line with the view held by many mathematicians and students in my study.

To sum up this section, I first described some aspects in the history of mathematics relevant for the contemporary view of proof. I then presented the three philosophical schools, logicism, constructivism and formalism and their epistemological views on mathematics and proof. Finally, I declared

[3] William Thurston is one of the most famous contemporary mathematicians and winner of the Fields Medal.

my stances concerning how I view proof in mathematical practice in my study.

In the next section, I will introduce the reader, very briefly, to the field of mathematics education research on proof.

1.4 Proof in mathematics education research

Proof is a vital issue in mathematics education research today. There has been an explosion of articles and research papers published on this topic during the last two decades. The rich variety of meanings and uses of mathematical proof in mathematical practice corresponds to a complexity in the educational field. But, as Mariotti (2004) points out, while for mathematicians the mathematical complexity is the foremost problem regarding proof, for the students proof is above all a problem of meaning, and educators have to devise teaching contexts which make proof meaningful to them.

Many mathematics education researchers have discussed different functions of proof and considered their significance for mathematics education. Bell (1976) was one of the first in mathematics education research to deal with the nature and the role of proof in mathematics in relation to mathematics education. De Villiers (1990) presented the following model for the functions of proof which is an expansion of Bell's (1976) original distinction between the functions of *verification*, *illumination* and *systematisation*.

- Verification (concerned with the **truth** of a statement)
- Explanation (providing insight into **why** it is true)
- Discovery (the discovery or invention of **new** results)
- Systematisation (the **organisation** of various results into a deductive system of axioms, major concepts and theorems)
- Communication (the **negotiation** of **meaning** and **transmission** of mathematical knowledge) (de Villiers, 1990, p. 18)

De Villiers' categories above have found a shared consensus among researchers in the mathematics education community and have been applied in many research projects and articles (e.g. Almeida, 2000; de Villiers, 1991; Hanna, 2000; Knuth, 2002; Weber, 2002). Hanna (2000) added to the model of de Villiers the following three functions:

- Construction of an empirical theory
- Exploration of the meaning of a definition or the consequences of an assumption
- Incorporation of a well-known fact into a new framework and thus viewing it from a fresh perspective (Hanna, 2000, p. 8)

Further, Weber (2002) considers the functions of proof in teaching of mathematics and states that besides proofs that convince or/and explain there

are proofs that justify the use of definition or an axiomatic structure and proofs that illustrate technique. The functions of conviction/explanation in connection to proof, in particular, have been discussed in mathematics education research (e.g. de Villiers, 1990; Hanna, 2000; Hersh, 1993) and these considerations have led to a lot of empirical studies. I will come back to these considerations later in Chapter 2.

The epistemological distinctions between different functions of proof described above have also been important for my study, since I apply a sociocultural approach in my work and, consider proof as an *artefact* (see p. 38) in mathematical practice. Thus, proof is considered as a tool, not only for generation of new mathematical knowledge but for all the functions presented above.

So far, studies on a variety of topics relating to proof have been conducted in the mathematics education community. These topics include the following aspects: students' difficulties when constructing proofs (e.g. Bell, 1976; Moore, 1994; Selden & Selden, 1995; Weber, 2001), different levels of proving identified in students' efforts (Balacheff, 1988; Bell, 1976; Godino & Recio, 2001; Miyazaki, 2000), how to renew the treatment of proof using new approaches with students' investigations (Alibert, 1988; Haddas & Hershkowitz, 1998, 1999; Schalkwijk, Bergen, & Rooij, 2001), the use of technology in teaching of proof (Jones, 2000; Laborde, 2000; Mariotti, 2000) and how to help students in transition to more formal proof (Chin & Tall, 2000; Moore, 1994). There are also studies about students' and teachers' beliefs and conceptions about proof (Almeida, 2000; Chazan, 1993; Dreyfus, 2000; Knuth, 2002) and the role of logic and/or structure in understanding and constructing proofs (Leron, 1983; Selden & Selden, 1995). There have also been micro level studies about students' argumentation (e.g. Garuti, Boero, & Lemut, 1998; Pedemonte, 2001; Reid, 2003; Simon, 1996), for example the relation between inductive, abductive and deductive thinking and students' understanding of conditionality. Proof and applications have been focused on by Hanna and Jahnke (1993). Besides all these topics there are research studies where the focus is on special kinds of proofs, for instance proof by contradiction, proof by mathematical induction, proof in calculus, geometrical proofs, informal proofs, visual proofs and so forth.

Most of the studies mentioned above were conducted within a cognitive paradigm. Many researchers in the field have been influenced by Piaget's stage theory and constructivism[4] when they have set up different stages in the pupils' reasoning abilities (e.g. Balacheff, 1988; Harel & Sowder, 1998; Miyazaki, 2000). Furthermore, van Hiele's levels about the developmental stages in a child's geometrical learning are based on a view of an individual

[4] Constructivism here refers to a learning theory and means something else than constructivism in mathematics (see p. 18).

who goes through different phases in a certain order and thus, are related with Piaget's theory and constructivism. Van Hiele levels have led to a lot of cognitive, empirical studies (e.g. Silfverberg, 1999) and also influenced the mathematics teacher education in Sweden.

Studies on proof have recently been carried out employing socio-cultural approaches (e.g. Herbst, 2002a; Herbst, 2002b). Hoyles (1997) questions the existence of a universal hierarchy of students' ability of proving and points out that there are big differences between different countries concerning the treatment of proof. She also shows how curriculum changes influence students' views on proof. Knipping (2001a) compares French and German classrooms and analyses the differences in form and function of proof in these environments. She examines the impact of culturally-embedded classroom practices on the teaching and learning of proof. Proof in textbooks has also been in focus of some studies. Hanna and de Bruyn (1999) investigate the opportunity to learn proof in Ontario grade twelve mathematics texts. Cabassut (2005) compares argumentation and proof in French and German curricula and upper secondary school textbooks. I have studied how proof is dealt with in Swedish upper secondary school textbooks (Nordström & Löfwall, 2005).

There are not many studies on proof in mathematics education conducted in Sweden. However, there has obviously been concern about development in the upper secondary school curriculum where geometry got a diminished place after the "new math" period (Råde, 1986). Råde conducted a survey among university students at Chalmers University of Technology about how students described their upper secondary school experiences regarding proof. I found some similarities between his survey and my pilot survey (Nordström, 2003) 20 years after his survey. For example, there was a retrospective question about how often students had met proof in upper secondary school. The answers were quite similar to those in my study. There was a group of students in Råde's study, who stated that they never dealt with proof in upper secondary school. This was also the case in my pilot study. Another question concerned how students related to proof and if they wanted to have more proof in upper secondary school than they themselves had had. A majority of them answered yes to this question. In my questionnaire, an even bigger percentage of students related positively to the statement "I would like to have learned more about proof in upper secondary school". However, it is difficult to compare the results more deeply because of the differences in the design of the questionnaires.

Two decades after Råde's survey, some qualitative studies concerning students' ability to examine, make conjectures and justify their conjectures were conducted by Bergqvist (2001) who applied Balacheff's (1988) classification of different levels of students' "proofs". His study is about how upper secondary school students explore mathematics and verify their solutions. How students explore and verify using mathematical induction in a co-

operative setting has been studied by Wistedt and Brattström (2005) and Pettersson (2004). Pettersson focused on the interplay between the formal and the intuitive when students struggled to find a solution to a proving task.

My thesis is not about how individual students learn or work with specific proofs but the focus is more at meta-mathematical level and on the socio-cultural context of knowledge growth. However, in order to shed light on different aspects of proof that students might meet in the community of mathematical practice at a mathematics department, I use the results and the theories obtained from didactical studies conducted in the field and combine them with a socio-cultural perspective and Lave and Wenger's (1991) and Wenger's (1998) social practice theory. I will come back to some of the studies mentioned above in Section 2.3, when describing different aspects of proof in mathematics and in the teaching and learning of mathematics.

1.5 A summary

In this chapter, I provided some background for the thesis. In the first section, I gave an account of the development of my research. I went on describing very briefly the changes regarding the role of proof in the curriculum, during the last three decades, at the department that is the focus of my study. I then provided a short description about proof from a historical and philosophical point of view in order to shed light on the complexity of the notion of proof. Finally, the fourth section was a brief introduction to the field of proof in mathematics education research. In the following chapter, I will clarify the theoretical positions of my study.

2 Theoretical framework

My thesis is about proof in mathematical practice at a mathematics department. I examine the role of proof in this practice, mathematicians' pedagogical views and intentions, students' experiences and how students are drawn to share the views and the knowledge of proof of mathematicians. Hence, the key elements in my study are proof, mathematical practice, mathematicians' pedagogical views and students' learning experiences. In order to create a theoretical frame that addresses these elements I first looked at learning theories. In the first section of the chapter, I justify and describe the choice of the theoretical frame for the thesis. In the second section I elaborate the central theoretical notions applied in my work and define and describe the unit of analysis for the object of my study that is a community of practice of mathematics at a mathematics department.

Besides the learning theories relating to ontological and epistemological assumptions and commitments, I also looked at theories and research about proof in mathematical practice and in mathematics education, so that I could place my concerns in historical and cultural contexts and locate sources of similar ideas in the past. Thus, in the last section of the chapter, I create a conceptual frame about aspects of proof and relate these aspects to the central theoretical notions presented in Section 2.2.

2.1 The overall theoretical frame in which the research questions are embedded

A lot of research on proof in mathematics education is conducted within a constructivist paradigm (see p. 22). Constructivism deals with the *cognitive* aspects of the *individual learner*. Piaget whose model of human mental operations was essential for the constructivist theories, focused on the adaptive and constructive activity of the individual across development stages and not so much on influences of the environment (Bruner, 1996; Renshaw, 2002). Wood (1988) describes the shift from *behaviourism* in the learning theories to adoption of Piaget's theory in the following way: Piaget's theory places *action* and *self-directed problem-solving* at the heart of learning and development.

> "As psychologists studying learning began to entertain ideas about intrinsic motivation and the importance of activity and mastery for its 'own sake', Piaget's theory provided a compatible and already well-developed approach to the study of learning development." (Wood, 1988, p. 5)

During the 1980s Ernst von Glasersfeld, strongly influenced by Piaget presented a view of "coming to know" in mathematics, which he referred as Radical Constructivism (Jaworski, 1999). He laid out its two basic principles:

- Knowledge is not passively received but built up by a cognising subject;
- The function of cognition is adaptive and serves the organisation of the experiential world, not the discovery of ontological reality (Glasersfeld, 1995).

Radical constructivism of von Glasersfeld has influenced mathematics education research in particular, during the last three decades (e.g. Hanna & Jahnke, 1996; Nickson, 2000).

In my research, I am not exploring the cognitive aspects of the individual learner concerning her ability to construct proof in isolation from her learning histories and the socio-cultural context, but the analytical focus is more on the roles of socio-cultural contexts in knowledge growth. However, the knowledge about Piaget's theory and its influence on the radical constructivism of von Glasersfeld and the social constructivist theories is important for me when analysing the research articles on the teaching and learning of proof. Many of the studies framed by constructivist theories on the teaching and learning of proof offer valuable aspects of proof for the conceptual frame for analysing the mathematicians' and the students' utterances and linking them to the previous results in the field of proof. I will come back to these studies later in the last section of the chapter.

Vygotsky places far more emphasis than Piaget does on the role played by culture and its systems of symbols, for example language (Wood, 1988). Most of the approaches that are called socio-cultural are associated with the Vygotskian school of thought and they all promote a vision of human thinking as essentially social in its origins and dependent on historical, cultural and situational factors (Kieran, Forman, & Sfard, 2002). Vygotsky stresses that the individual learns by being socialised into a culture.

> "It is necessary that everything internal in higher forms was external, that is, for others it was what it now is for oneself. Any higher mental function necessarily goes through an external stage in its development because it is initially a social function…Any higher mental function was external because it was social at some point before becoming an internal, truly mental function." (Vygotsky, 1981, p. 162).

According to the socio-cultural perspective, learning is an aspect of interrelated historical, cultural, institutional and communicative process (Renshaw, 2002). In my study, I want to situate the issue of proof in mathematics education in its historical and cultural context, so I consider a socio-cultural approach as more appropriate for my area of study and the units of analysis, than the purely cognitive approaches.

However, different theories have been developed from Vygotsky's ideas and there is not only one socio-cultural theory. For me it is important to view a person as an active part of the world but at the same time, to a certain extent, formed and influenced by the environment. The perspective of Lave and Wenger (1991) provides a bridge between cognitivist perspectives and sociological perspectives because their theory of social practice "emphasizes the relational interdependency of agent and world, activity, meaning, cognition, learning, and knowing." (*ibid.*, p. 50) Lave and Wenger's theory of *legitimate peripheral participation* (LPP) emphasises "connecting issues of socio-cultural transformation with the changing relations between newcomers and old-timers in the context of a changing shared practice." (*ibid.,* p. 49) Their theory aims to shift the discussion about learning beyond the issues of cognition to those of *participation* and *identity*. Lave and Wenger consider learning as increasing participation in *communities of practice*, which concerns the whole person acting in the world. This is something I found relevant for both the mathematicians and the students in my study, when learning, teaching, practicing/exercising and experiencing mathematics. Further, the primary unit of analysis in Lave and Wenger's (1991) and Wenger's (1998) theory is neither the individual nor social institutions but communities of practice.

> "… a community of practice is a living context that can give newcomers access to competence and also can invite a personal experience of engagement by which to incorporate that competence into an identity of participation." (Wenger, 1998, p. 214)

Wenger (1998) points out that participation in any cultural practice in which any knowledge exists is an epistemological principle of learning. For example, mathematicians do not stop learning mathematics when they have taken all the obligatory courses, since they stay in the community of mathematical practice. In my research, I also include researching and obtaining new mathematical knowledge in learning (see p. 34).

In my study, I explore mathematicians' views and intentions, textbooks and lectures and the organisation of teaching of proof and students' participation in and their experiences with proof in their mathematical practice. According to Wenger, structuring resources for learning come from a range of different sources, not only the intentional teaching. Hence, much learning takes place without teaching, and much teaching takes place without learn-

ing. Pedagogical intentions create a context in which (the intended) learning *can* take place. Teachers, lectures, lessons and instructional materials, like textbooks, become resources for learning in complex ways and, as Wenger points out, an important question is how the *planned and the emergent interact.*

> "Pedagogical debates traditionally focus on such choices as authority versus freedom, instruction versus discovery, individual versus collaborative learning, or lecturing versus hands-on experience. But the real issue underlying all these debates is the interaction of the planned and the emergent." (Wenger, 1998, p. 267)

I do not mean that the organisation of learning environments mentioned in the quotation above would not be important to discuss and research on in the mathematics education community. However, in my work none of them are especially in focus. All of them combined in various ways, together with instructional materials and together with mathematicians' intentions, create the structuring resources (the planned) for learning. I contrast the mathematicians' views and intentions with students learning experiences to examine the interaction between the planned and the emergent (Chapter 6). This interaction, which is an ongoing process, may be exemplified as follows. A teacher, who is just one but a very important actor, is planning a lecture and makes a lot of choices of what to focus on in the presentation. Students who are following the lecture may focus on those aspects but also on aspects that were not at all intended by the teacher. I will come back to this when describing the condition of transparency of proof (see p. 60).

Hence, I use a socio-cultural perspective and the social practice theory of Lave and Wenger (1991) and Wenger (1998), to examine and give structure to the object of my study. The practice I am studying is *the practice of people exercising/sharing university mathematics at a mathematics department.* I want to define mathematicians' participation as well as students' participation in this practice with a special focus on students' access to proof. The notion of *community of practice* provides me with an appropriate level of analysis.

There are studies in mathematics education that apply Lave and Wenger's and Wenger's theories, for example to describe and explain student and teacher learning in the field of mathematics (e.g. Adler, 2000; Boaler, 1999; Graven, 2004; Santos & Matos, 1998). Wenger's theory about communities of practice has also been employed by Burton (2004) in her study on how mathematicians talk about their practice. Santos & Matos (1998) apply the theory on how students use the Pythagorean theorem in problem solving. However, I have not found examples of research on proof embracing a social practice approach.

It happens that learning theories are connected with different teaching methods. This is natural because learning theories offer views on how a person comes to know and what knowledge is. For me a socio-cultural approach means that a social construction of knowledge has always taken place everywhere where people have had something to do with each other and whatever approaches researchers have used in their research. Even reading a book or quietly struggling with a mathematics problem or trying to follow what a lecturer is talking about can be seen as a social action or *negotiation of meaning* (see p. 35). Therefore, I want to distinguish a learning theory from teaching methods and I do not want to advocate certain methods solely because I embrace a certain approach to examine the issue of proof. For me, the socio-cultural perspective and the theories of Lave and Wenger and Wenger is an analytical viewpoint on learning and I hope it helps me to shed a new light on the key aspects of students' learning experiences and the problems they talk about regarding the learning and the understanding of proof. The theory for me is not a recipe but "it can act as a guide about what to pay attention to, what difficulties to expect and how to approach problems". (Wenger, 1998, p. 9)

In this section, I have introduced the theoretical stances in my study concerning the learning theories and the ontological and epistemological assumptions and commitments. In the next section I elaborate the central notions of the socio-cultural perspective and Lave and Wenger's and Wenger's theories as applied in my study.

2.2 Key assumptions and central notions

I start the examination of the central notions of the socio-cultural perspective and Lave and Wenger's (1991) and Wenger's (1998) theories very broadly by focusing on the notion of culture; follow it by zooming in on a community of practice and the issue of learning and how it is defined in the social practice theory of Lave and Wenger and Wenger. I go on to examine fundamental notions like *negotiation of meaning*. I exemplify the central notions with my area of study. I conclude the section by defining and describing proof as an artefact in mathematical practice.

2.2.1 Culture and communities of practice

According to Vygotsky, all human development is learning from others in some sense, from the *culture* that precedes us. As we grow up we become socialised in a culture. How does the notion of culture relate to *communities of practice* that are the units of analysis in Lave and Wenger's and Wenger's theories?

The notion of *practice* is defined as "doing in a historical and social context that gives structure and meaning to what we do." (Wenger, 1998, p. 47) Hence, mathematical practice is doing in a historical and social context and includes, for example, its special language, symbols, tools, documents, specified criteria and well-defined roles, that give structure and meaning to what people in that practice do. Lave and Wenger (1991) talk about culture as something that influences the lives of the communities of practice. For example, in my study, the community of mathematical practice at the mathematics department has its own culture, which is influenced by the culture in which it is embedded and by the cultures that the individuals participating in the practice come from. In my study, some of the mathematicians and the students, for instance, come from countries other than Sweden, and they may have different traditions concerning proof in mathematics education than the Swedish mathematicians and students have. All the members influence the culture, each in an individual way. At the same time they are all influenced by the culture of the mathematical practice.

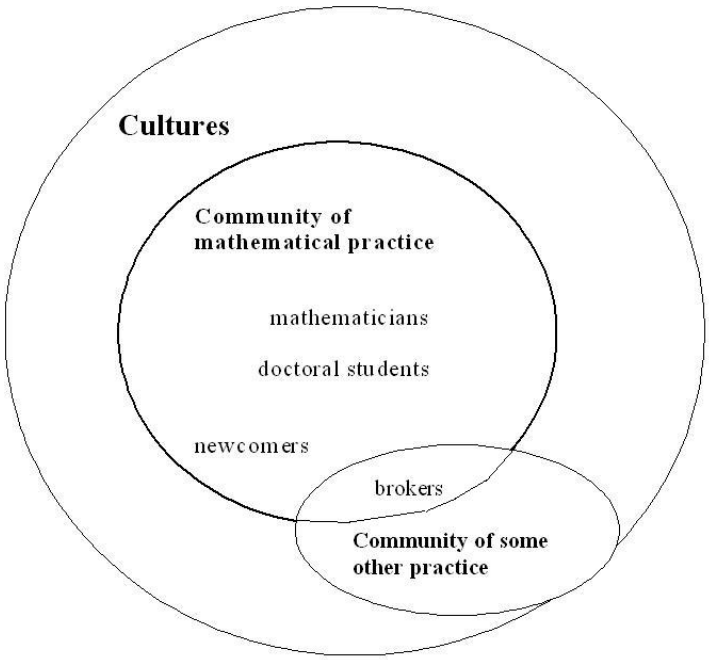

Figure 2 Cultures and communities of practice

2.2.2 The community of practice of mathematics at a mathematics department

According to Wenger (1998), a practice defines a community through three dimensions: *mutual engagement*, *a joint enterprise* and *a shared repertoire*. Next, I relate the practice of mathematics at the mathematics department to these three dimensions. I want to study mathematicians' participation as well as students' participation in this practice with a special focus on students' access to proof. Hence, I include the newcomers in the practice.

The mathematical practice resides in a community of people and the relations of *mutual engagement* by which they are engaged in studying, teaching/explaining, learning and communicating mathematics. Mutual relations of engagement give rise for both differentiation and for homogenisation. The members in the community of mathematics at the mathematics department distinguish themselves as well as they develop shared ways of doing things. A mathematician and a student as members of the community have very different status with respect to daily work and authority. But also each mathematician as well as each student has a unique place and gains a unique identity in the community. There can, for example be a variety of views on proof and its role in mathematics and the teaching and learning of mathematics in the mathematical practice.

Communities of practice develop in larger contexts – historical, social, cultural, institutional – with specific resources and constraints. The mathematical practice at a mathematics department is institutionally a part of the academic world with all its traditions. It is also located in a special historical stage of the development of mathematics (see the introduction) and its relations to other disciplines. Further, it is a part of the Swedish culture conforming to the demands of democracy and justice and all that this entails. Some of the conditions and requirements of these larger contexts are explicitly articulated, some are implicit relations and tacit conventions. In organising the learning environments for the newcomers, there are a lot of traditions, both articulated and not articulated which both guide the enterprise but also constrain it. Mathematicians and the students when sharing with mathematics share it in the frame of the historical conditions of the practice including the lectures, textbooks, lessons, individual home-works, preparations of the lectures, research, seminars, examinations and other forms of participation. Practice is defined by the participants in the very process of pursuing it.

What is shared by both mathematicians and students as a *joint enterprise* in the practice? I argue that the learning in the sense Wenger (1998) defines it (see Section 2.2.3) and the enhancing of learning and hence, developing and maintaining the practice can be seen as a joint enterprise. All the members are engaged in the learning of mathematics and all of them use partly the same tools even if they are on different levels of learning. Newcomers are learning on a basic level and being enculturated (as active agents) in the

practice. Many students are also teaching mathematics for other students. Mathematicians are teaching, supervising, researching and, at the same time learning and enhancing the learning of mathematics. I argue that, in accordance with Wenger's theory of learning, researching mathematics can be also seen as learning (see p. 34).

The *shared repertoire* in a mathematical practice reflects its history of mutual engagement (see Section 1.3). It includes routines like organising certain courses, seminars and examinations, but also words and symbols specific for the mathematical language, particular computer software, specified criteria for justifying knowledge in mathematics (proof), regulations and contracts, for example about how to proceed in the practice, and all the concepts the community has produced and adopted in the course of its existence and which have become part of its practice. The repertoire combines both *reificative* and *participative* (see Section 2.2.4) aspects. It is by its very practice – not by other criteria – that a community establishes what it is to be competent participant, an outsider, or somewhere in between.

Participation in the community of practice influences the identities of all the participants in relation to other practices and communities. Their position in the community also influences their identities in relation to the own practice. Learning events and participation depend on the engagement they afford and their location on the *trajectories*.

There can be various types of trajectories when proceeding in the communities of practice and the concepts of centrality and peripherality have a relative character. For old-timers there are insider trajectories because the formation of identities does not end with full membership. The evolution of practices continues; new demands, new technology, new generations all create occasions for negotiating one's identity. There are newcomers who aim to become full participants in the practice even though their present participation may be peripheral. By choice or by necessity, most trajectories in the community of mathematical practice at the mathematics department never lead to full participation, but they may well provide a kind of access to a community and its practice that becomes significant enough to contribute to one's identity. Lerman (2002) points out that when a person enters a practice, there is a sense in which he or she has already changed. A person who starts to study mathematics has an orientation towards the practice from the beginning, or has goals that have led the person to the mathematical practice, even if he or she leaves the practice after a short time. This is something I see very clearly in the university entrants' responses to the questionnaire about how they relate to proof and proving (see Section 5.2.1).

Finally, there are also trajectories for so called *brokers*. They do not aim for full participation but a multimembership in two or more communities of practice. Brokering requires the ability to link practices by introducing into one practice elements of the other. Most of the members in the community of mathematical practice at the mathematics department stay on the periph-

ery for a while and some of them might, after that, become brokers between the mathematical practice and some other practices, for example other academic institutions, practices of physics, biology, economy or schools.

Of course, we all participate in many different communities and constellations. Hence, our membership in a community of practice is only a part of our identity and identity is more than a single trajectory. So we can be viewed as a "nexus of a multimembership." (Wenger, 1998, p. 158)

Burton (2004) discusses the community of practice of mathematics from the perspective of researching mathematicians. Also Wenger (1998) uses academic communities as an example of how the doctoral students get access to these communities (*ibid.*, p. 101). The practice in my study is somewhat different from those of Burton's study and Wenger's example even if it is overlapping with them. I include in the practice also the newcomers who will never become mathematicians but will stay in the community of mathematics for a short time.

My interest is the role of proof in this practice and how mathematicians and newcomers approach it. I examine what intentions and pedagogical perspective mathematicians have regarding the teaching of proof, on the one hand, and how students in different locations in the practice talk about their experiences and their engagement on the other hand. As proof is a central part of mathematical practice at a mathematics department and the university entrants consider proof as an important part of mathematics (see p. 150), students' relation to proof can be significant for how they relate themselves to the practice.

2.2.3 Knowing and learning

Because participation in social practice suggests a very specific focus on the person, not as an isolated unit of analysis but as a person-in-the-world and as a member of a socio-cultural community, *knowing* is seen to be an activity by specific people in specific circumstances, in my study persons doing, teaching, learning and communicating mathematics. The primary focus in this theory is on *learning as social participation in practices* of social communities and constructing *identities* in relation to these *communities* and *experiencing* the world and our engagement in it *meaningful* (Wenger, 1998).

Wenger defines four components necessary to characterise social participation as a process of learning and knowing. They are learning as *doing* (practice), learning as *belonging* (community), learning as *experience* (meaning) and learning as *becoming* (identity). These four components are seen to be mutually defining and interconnected. For example participation in the mathematical practice is doing and learning mathematics and in that way belonging to the community of people who learn and practice mathematics. The practicing of mathematics and belonging to the community are

experienced in various ways by newcomers and old-timers, and these experiences influence their identities in different manners.

A defining characteristic of participation is the possibility of developing an *identity of participation*. Building an identity consists of negotiating meanings of our experience in social communities (Wenger, 1998, p. 145). One's identity is always changing; it is a constant becoming. When we come into contact with new practices we do not know how to interact, we cannot make use of the repertoire of the practice and so on. Our non-membership shapes our identities through our confrontation with the unfamiliar. *Peripheral participation* involves a mix of participation and non-participation where the participation-aspect is dominating whereas *marginality* involves a restricted form of participation where non-participation is dominating and disabling participation (Wenger, 1998). For example, for students the possibility to participate in different kinds of activities around proof can develop their identity of participation in the mathematical practice if they can follow and experience meaning in them. Conversely, not being able to follow and find a meaning in the activities can lead to the development of an identity of non-participation in mathematical practice, because the students already at the beginning of their studies view proof as a central part of mathematics (see p. 150).

Lave and Wenger (1991) state that newcomers' *legitimate peripherality* involves participation as a way of learning, which is both absorbing and being absorbed in the culture of practice. For example, the students in the mathematical practice have a possibility to make the culture of practice theirs when they gradually assemble a general idea of what constitutes the practice of the community. They increase their understanding of how and what mathematicians (old-timers) do, what they respect and admire. Participation offers examples about how the masters or teaching assistants (more advanced apprentices) work, how the finished products, like proofs, look etc. All these examples are, according to Lave and Wenger, grounds and motivation for learning activity.

I argue that Wenger's (1998) definition of learning entails that also mathematicians, by participating in the practice are learning. Learning in the community of practice occurs on different levels. Newcomers are learning on a basic level but, when they, for example struggle with a proving task, they are engaged in an enterprise, closely related to that of researching mathematicians. The distinction is that students are struggling with mathematics that is known by the community, whereas mathematicians are working with creating new mathematics. In my study, in accordance with Wenger's definition of learning, the creating of new mathematics is also seen as learning, since for example finding new theorems and proofs leads to more intense participation in practice and leads to changing identities in relation to other participants as well as to people outside the practice. Re-

searching mathematicians are also learning from other mathematicians and new fields when they participate in conferences and read articles.

This is an epistemological question about the character of teaching and learning and their mutual relation. Within the transmission paradigm (behaviourism) researching and finding of new knowledge is not seen as learning, because teaching and learning is viewed as transmission of the known (quite stable) body of knowledge from experts to those who do not "own this knowledge". Within a constructivist paradigm researching and constructing of new knowledge is seen as learning by its very definition of learning. According to social practice theory, learning is enhancing participation in practice that leads to changing identities. As described above, researching new mathematics can be seen as learning within this "paradigm". Further, all the four components of learning defined by Wenger (1998), presented at the beginning of this subsection (*doing, belonging, experience and becoming*), are involved in researching mathematics.

Does the view on researching as learning entail a platonistic view (see p. 19) on mathematics? Not necessarily. Given the mathematical body of knowledge and the rules of reasoning, there are possibilities to create certain relations and combinations. This does not necessarily imply a platonistic view of an ideal world, where mathematical truths can be discovered.

2.2.4 Negotiation of meaning

A central notion for social practices is the process of *negotiation of meaning*. It is seen to be a fundamental process on different levels and in different manners in all social practices. The negotiation of meaning involves the interaction of two constituent processes, *participation* and *reification.* These processes are fundamental to the human experience of *meaning* (Wenger, 1998, p. 52). Meaning here is not to be interpreted as meaning of life in a philosophical sense but as an experience of something in everyday life. Negotiation does not necessarily refer to something going on between people but can as well be conceived as processes going on silently in one's head. Negotiation constantly changes the situations to which it gives meaning and affects the participants. It entails both interpretation and action. This process always creates new circumstances for further negotiation and further meanings (*ibid.,* p. 54).

In my study the core issue is the negotiation of meaning concerning proof and the role and the meaning of proof in mathematics. Next, I attempt to clarify how I conceive the notions of participation and reification, the two parts that are involved in the negotiation of meaning.

Participation refers to a process of taking part of the practice in different ways. It also refers to the relations with others that reflect this project of participation. For example, when students grapple with their lecture notes and try to make sense of proofs and mathematical arguments, or when they

discuss a new concept with some other students or listen to the lecturer they participate in the practice. Participation is both personal and social. It is a complex process that combines doing, talking, thinking, feeling, and belonging. It involves our whole person, including our bodies, minds, emotions, and social relations. For example, students when talking about their experiences concerning proof, besides their views and thoughts, often express also different kinds of feelings. But as well, feelings are present in mathematicians' utterances (see p. 90).

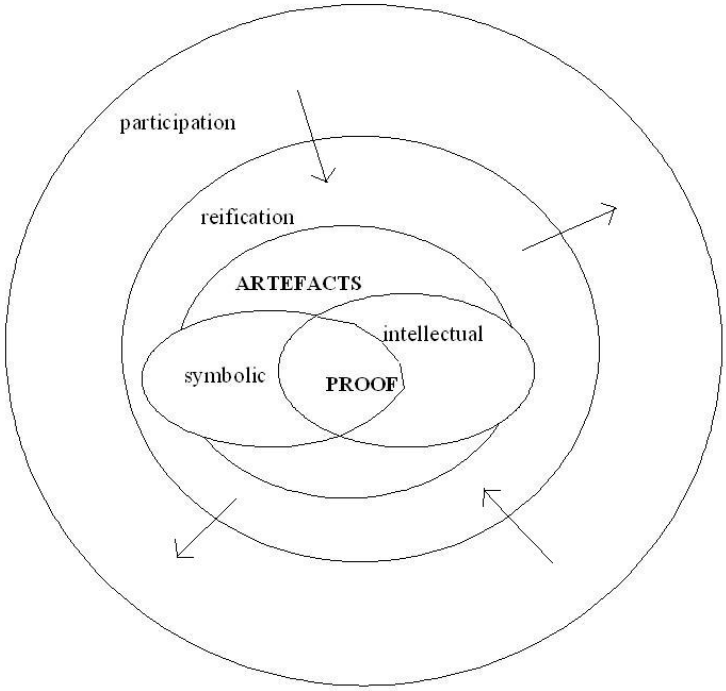

Figure 3 Negotiation of meaning

With reification Wenger refers to projection of our meanings into the world and then perceiving them as existing in the world and having a reality of their own. Wenger uses the concept of reification very generally to refer to the process of giving form to our experience by producing objects that "congeal" this experience into "thingness" that does not only refer to matters or material objects but also thoughts and ideas. In doing so we create *points of focus around which the negotiation of meaning becomes organised*. So, for example, students can reify proof in certain ways (see p. 149) and mathematicians can reify the changes in the practice in various ways (see p. 96).

Any community of practice produces abstractions, tools, symbols, stories, terms, and concepts that reify something of that practice in a "congealed" form (Wenger, 1998). In the practice of mathematics, for example mathe-

matical symbols, definitions, theories and proofs can be seen as different kinds of reifications. Proof as reification can refer both to a process of proving and its product, proof. This is important for my study because I want to view proof also as a dynamic notion and I approach it from different directions (*Figure 4*, p. 42).

> "Articulating an emotion or building a tool is not merely giving expression to the existing meanings, but in fact creating the conditions for new meanings. As a consequence, such processes as making something explicit, formalizing, or sharing are not merely translations; they are indeed transformations – a production of a new context of both participation and reification, in which the relations between the tacit and the explicit, the formal and the informal, the individual and the collective, are to be renegotiated." (*ibid.,* p. 68)

In a similar way, a proof is not only formalising mathematics and organising it in a deductive manner but also creating conditions for new theorems and proofs and also a means of communication and thus production of a new context of both participation and reification. The newcomers in the mathematical practice have not designed the mathematical theories and proofs, yet they must absorb a part of them into their practice. According to Wenger (1998) the reifications coming from outside, have to be reappropriated into a local process in order to become meaningful (*ibid.*, p. 60).

The concept of reification has been used in a variety of ways in social theory. There is an affinity between Wenger's use of the concept and Sfard's (1991) use of it. With reification Sfard refers to the structural description of mathematics. Wenger's use of the word is more general; with reification he refers to both the process and the product whereas Sfard defines reification as "an ontological shift – a sudden ability to see something familiar in a totally new light." (Sfard, 1991, p. 19) Actually, the entire duality of operational/ structural conceptions (of the same mathematical notion) that Sfard discusses can be classified as reification in Wenger's sense. Being complementary they are a process and an object at the same time and then serve as an example of reification in mathematics. Further, Sfard (1991) argues that "the ability of seeing a function or a number both as a process and as an object is indispensable for a deep understanding of mathematics, whatever the definition of understanding is." (*ibid.*, p. 5) I will come back to Sfard's dualism and her thesis about a "vicious circle" in the next chapter about the aspects of proof when discussing the notion of transparency of artefacts.

According to Wenger's theory, participation and reification cannot be considered in isolation: they come as an interacting pair. Reification always rests on participation: for example proof always assumes a history of participation as a context for its interpretation. In turn, participation always organises itself around reification because it always involves artefacts, words, and concepts that allow the negotiation of meaning to proceed.

Wenger stresses that in general, viewed as reification, a more abstract formulation will require more intense and specific participation to remain meaningful, not less. It could mean that, for example, to experience proof as meaningful students would have to participate in different kinds of activities, involving the negotiation of meaning of proof. However, I do not fully agree with Wenger that a higher level abstraction in reifications in general needs to require more participation than struggling with a concrete problem with a lot of details. Abstraction in mathematics can often help us to see connections and structures and in that way we can use them in many occasions without participating in the negotiation of meaning at all levels. These kinds of abstractions also offer us a means to understand problems in new contexts. On the other hand, one can say that behind every abstract mathematical formula and proof, there is a lot of participation during the history of mathematics. So, new generations do not need to start from a scratch. Some students in the focus groups that expressed participation identity regarding proof stated that studying of proof (a very abstract reification) made everything in mathematics simpler (see p. 187). An important question for mathematics education is how to obtain a level of being able to take advantage of the general results and understand the power of them. This issue is connected to the question whether and when it is better to start from concrete/abstract, and what advantages there might be in starting by examples/theories (see also the about inductive/deductive approaches (see, p. 47) in the last section of this chapter and Lerman (2000)).

2.2.5 Proof as an artefact

Artefact is a central concept within all socio-cultural theories although there are slightly different interpretations of the notion in different research projects. Artefacts are the concrete and abstract *tools* that *mediate* between the social and the individual (Säljö, 2005a). Vygotsky developed the concept of mediation in human-environment interaction to the use of signs as well as tools. Tool systems and sign systems like language, writing and number systems are created by societies during human history (Cole & Scribner, 1978). We *come to know* the world and the culture by *mediation* through artefacts: for example meanings are known through language, which is also seen to be an artefact, and in mathematical practice meanings are mediated through symbols and language, for example by a teacher, another student, or a textbook that can be the mediator. Hence, mediation between the individual and the social occurs through artefacts. I argue in my thesis that proof can be seen as an important artefact in the mathematical practice and subsequently relate it to Säljö's classification of artefacts.

Säljö divides artefacts into two groups, *intellectual tools* like discourses and systems of ideas and *physical tools* like texts, maps and computers. He also talks about *primary tools* (for example a hammer) and *symbolic tools*

(used for communicating ideas). He states that artefacts are carriers of information. As proof is a system of ideas and used for communicating ideas it can be seen as an intellectual and symbolic tool (*Figure 3*, p. 36).

Further, Säljö states that artefacts serve as tools for mediating in social practice, stabilise human practice, facilitate continuities across generations, co-ordinate and discipline human reasoning by suggesting how to do things.

Proofs have mediating character about mathematical knowledge, regarding how the knowledge is connected. Proof also stabilises the practice of mathematics, because it offers mathematicians common criteria for accepting and generating new mathematical knowledge (see p. 20). The systematisation of mathematical results into a deductive system of axioms, definitions and theorems, unifies and simplifies mathematical theories by integrating unrelated statements, theorems and concepts with one another, thus leading to an economical presentation of results (de Villiers, 1990, p. 20). Hence, proof facilitates continuities across generations because the axiomatic deductive way of organising mathematics makes it easier for new generations to *reappropriate* (Wenger, 1998) the mathematical knowledge obtained by the previous generations (see p. 37). For the same reason, proof also allows new generations to further new problems in mathematical practice. Hence, the idea of mathematical proof has made it possible to create a body of knowledge, a core of mathematics that is relatively stable from generation to generation. Finally, proof also co-ordinates and disciplines mathematical reasoning because of the severe demands it has on precision in reasoning and justifying results. Proof helps with identification of inconsistencies, circular arguments and hidden and not explicitly stated assumptions (de Villiers, 1990).

Proof as an artefact in mathematical practice has specific functions in that practice. Several researchers in mathematics education have examined these functions and their significance for the teaching of mathematics (see p. 21). Also mathematicians in my study talked about proof as a tool in their mathematical practice in various ways (see Section 4.2).

According to the theory of Lave and Wenger (1991) the key to legitimate peripheral participation in a practice is *access* by newcomers to its ongoing activity, to old-timers, and other members of the community as well as to information, resources, and opportunities for different kinds of participation (*ibid.,* p. 101). Access to artefacts both through their use and through understanding their significance is crucial in order to facilitate students' access to the practice of mathematics.

Lave and Wenger introduce the concept of *transparency* of the artefacts. They use it in connection to technology but I will examine its strengths for describing conditions of intellectual and symbolic artefacts as well, in my case, in particular, proof.

> "The significance of artifacts in the full complexity of their relations with practice can be more or less transparent to the learners. Transparency in its simplest form may just imply that the inner workings of an artefact are available for the learners." (*ibid.,* p. 102).

The term *transparency* refers to the way in which using artefacts and understanding their significance interact to become one learning process. It describes the intricate relation between using and understanding artefacts (*ibid.,* p. 103). There is a duality inherent in the concept of transparency; it combines the two characteristics *invisibility* and *visibility*. Invisibility is the form of "unproblematic" interpretation and integration to the activity. Visibility is the form of extended access to information. This is not a simple dichotomous distinction but these two characteristics are in a complex interplay and their relation is one of both conflict and synergy (*ibid.,* p. 103). Lave and Wenger (1991) illustrate this interplay by analogy to a window.

> "A window's invisibility is what makes it a window, that is, an object through which the world outside becomes visible. The very fact, however, that so many things can be seen through it makes the window itself highly visible, that is, very salient in a room, when compared to, say, a solid wall." (*ibid.,* p. 103)

Invisibility of mediating artefacts is necessary for allowing focus on, and thus supporting visibility of, the subject matter. Conversely, visibility of the significance of the artefacts is necessary for allowing its unproblematic use. This interplay of conflict and synergy is central to all aspects of learning in practice and "makes the design of supportive artifacts a matter of providing a good balance between these two interacting requirements." (*ibid.,* p.103)

The condition of transparency is a metaphor that I find relevant in describing the dilemma of how to introduce students to proof. It is impossible to focus on proof without some experience of "unproblematic" use of it. Conversely, it can be difficult for students to understand the meaning of proof or learn to produce own proofs in the mathematical practice without any explicit focus on it. I will come back to the notion of transparency in the next section of this chapter, where I describe the conceptual frame about different aspects of proof in mathematical practice (see p. 54).

In this section, I have described how I conceive the central notions in sociocultural perspective and Lave and Wenger's (1991) and Wenger's (1998) theories which I apply in my study. At the beginning of the section I described the unit of analysis for my study: the community of practice of mathematics at a mathematics department. I examine newcomers' participation in this practice with a special focus on proof. The issue of proof is embedded in the theoretical frame described in Section 2.1 and in this section. In order to look more deeply in the special *properties/approaches* and *func-*

tions of this artefact, I need to examine the research on and the theories about proof, especially in the teaching of mathematics. This is the subject of the next section.

2.3 Proof in mathematical practice – the conceptual frame

In the first chapter, I provided the reader with a section about the complexity of the notion of proof from a historical and philosophical point of view (see Section 1.3). In the previous section of this chapter, I first considered proof as reification (see p. 37). The view on proof as reification allows a focus on both the process of proving and the final products, proofs.

In Section 1.4, p. 21, I presented the different roles and functions of proof that have gained a wide consensus in the field of mathematics education research. The epistemological distinctions on functions of proof are also important for my study because I consider proof as an artefact in mathematical practice (see p. 38). Thus, proof is considered to be a tool, not only for acceptance/generation of new mathematical knowledge but for all the other functions as well, such as systematisation and communication. The meaning of proof in mathematical practice is involved in all the functions of proof, and therefore, according to the theory of Wenger (1998), they are important for how newcomers experience the practice. The functions of conviction and explanation have particularly been in focus in mathematics education research because of their relevance to mathematics teaching. But there are other aspects of proof that have been in focus as well. They are not functions of proof but rather properties of proof and how to approach proof. They illuminate the dynamic character of proof as reification, as both a process and a product (*Figure 6*, p. 62).

Because the aim of the thesis is to describe what opportunities there are for the newcomers to learn proof, I want to, in this section, present and examine different ways of approaching proof. The aim of this examination is to create a conceptual frame which I can use to link my study and the data to the previous studies and to the main themes and controversies within the research on proof and the teaching and learning of proof. I have summed up the main themes and issues in mathematics education research on proof along the following aspects. All of them had an important role in the data analysis:

- Conviction/Explanation
- Induction/Deduction
- Intuition/Formality
- Invisibility/Visibility

These aspects involve two different interacting components. In Figure 4 (p. 42), I illustrate these pairs with hints and examples of what I mean with them. For the illumination of these aspects, I provide examples both from literature concerning mathematical practice, from mathematics education research and from some empirical studies illuminating the concerns in the pedagogical debates. In doing so, I go on describing features in the communities of mathematical practice with special focus on proof as well as possible problems that newcomers may encounter when entering this practice and approaching proof.

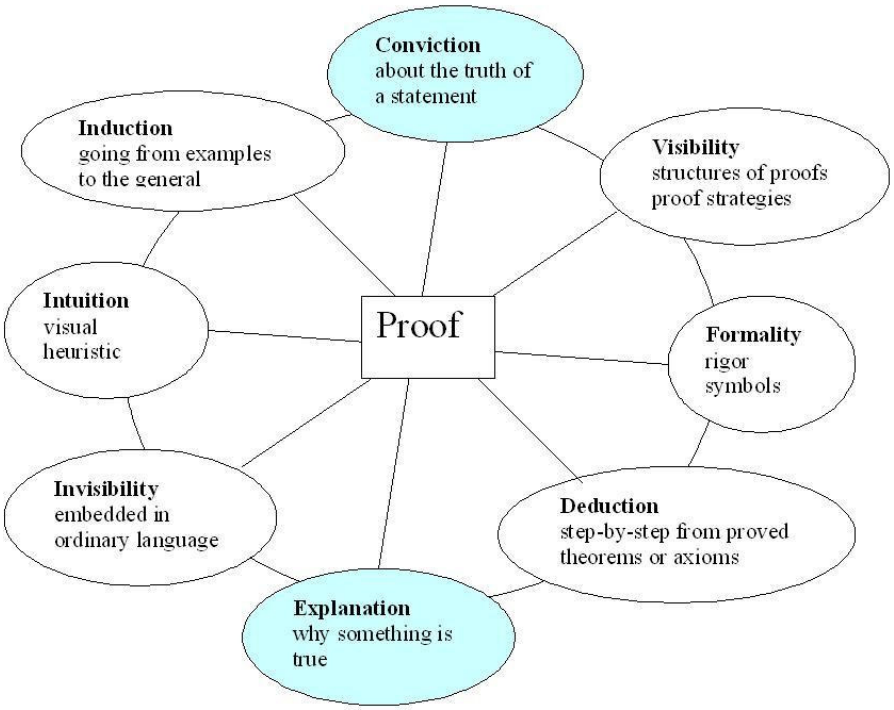

Figure 4 The interacting aspects of proof

Conviction/Explanation has a different color from the other aspects in the figure, since, as mentioned above, this pair is different from the other aspects in this model in a sense that the others deal with properties of proof and how to approach proof whereas Conviction/Explanation refers to the functions of

proof. There are other functions of proof, which I will include in the conceptual frame. I present them in the end of this section. All the aspects in the frame are partly overlapping and intertwined.

2.3.1 Conviction/Explanation

I start the presentation of the frame by considering the notions of conviction and explanation. Conviction is to believe that something is true in mathematics. Explanation is about why something is true in mathematics. Conviction/Explanation can be obtained by different means (communication) where all the other aspects are involved to various extents. For example, one can be convinced by examples or deductive proofs. One can get an explanation by a heuristic argument or a formal presentation. All this can be more or less visible.

The interplay between explanation and conviction has significance to the component of experiencing meaning in learning. Explanation should enhance the personal understanding of mathematics. However, as Wenger (1998) points out, words like "understanding" require some caution because there is not a universal standard of the knowable but there is an intricate relation between the abstract notion of knowledge and what is understood in practice. If proof is to be an explanation for a person, it also depends on the person's earlier experiences. Conviction offers confidence to the people working in the practice of mathematics because they can trust the earlier results and go on building new theories. In the field of mathematics education research, discussions and research have often been concerned about whether inductive/deductive ways of reasoning can offer explanation and/or conviction. Next, I present and discuss these concerns.

Proof as explanation

Many mathematicians have emphasised and discussed the explanatory aspects of proof (e.g. Hersh, 1993; Rota, 1997). Mathematics educators (e.g. de Villiers, 1990; Hanna, 1995) have been concerned about whether the role of conviction or the role of explanation is prior in mathematics teaching and learning. They agree that rather than conviction, explanation is the main function of proof in education. De Villiers (1991), for example, states that students are easily convinced by textbooks, teachers or a couple of examples. It is certainly true that the function of proof as explanation is important in mathematics teaching. However, I argue that the function of conviction and explanation are both intertwined in a critical process of accepting mathematical knowledge in mathematical practice and that is something that could also be focused on in the teaching of mathematics, especially at the higher levels.

Next, I discuss the view on proof as entirely an explanation and the view that examples cannot serve as explanations. Alibert and Thomas[5] (1991) point out that proving and explaining seem to be two different kinds of mathematical activities and give an example of a remark made by Deligne[6] who wrote after having produced a very formal proof about derived functors and categories:

> "I would be grateful if anyone who understood this demonstration would explain it to me." (Deligne, 1977, cited in Alibert and Thomas (1991))

Hence, there are proofs in mathematical practice that are correct and accepted but do not serve as explanations even for mathematicians. Further, even if a proof can often serve as an explanation for a mathematician there is no guarantee that the same proof is an explanation for a student. It depends on the level of proof and the experiences of the students. As Mancosu[7] (2001) points out, the concept of explanation in the classroom is not always the same as explanation in mathematical research. Some proofs might be perfect explanations for the professional mathematician but not for the student.

Rowland[8] (1998) argues the case for wider acceptance of the appropriateness and validity of generic arguments for the purpose of enlightening and explanation, and for more attention to the deliberate deployment of generic examples as didactical tools. I agree with him that generic examples often give an explanation, and maybe it is sometimes easier for the students to understand that kind of explanations than a complete proof (see p. 157).

Hence, proof can serve as an explanation depending on the proof and on the prior knowledge of those who study the proof. Generic examples can serve as explanations too; in fact, the newcomers often prefer them as explanations. Even if a proof does not always serve as an explanation, it can involve other aspects that can be experienced as meaningful for the reader or the listener or those constructing a proof, for example aesthetics and useful methods for other contexts in mathematics.

Proof as conviction

De Villiers (1990) argues that traditionally proof has been seen almost exclusively in terms of verification[9] of correctness of mathematical statements, also among mathematics education researchers. He argues that proof is not necessarily a prerequisite for conviction for mathematicians but conviction is

[5] Daniel Alibert and MichaelTomas are researchers in mathematics education.
[6] Pierre Deligne is one of the most famous contemporary mathematicians and winner of the Fields Medal.
[7] Paolo Mancosu is a researcher in the philosophy of mathematics.
[8] Tim Rowland is a researcher in mathematics education.
[9] De Villiers (1990) uses the word verification as synonym for conviction and justification.

probably far more frequently a prerequisite of proof. Both de Villiers and Hersh[10] (1993) describe the confidence mathematicians feel when they have verified the theorem in particular cases and gathered strong inductive evidence for it. When they have satisfied themselves that the theorem is true they start to prove it. Their point is that in such situations the function of such proof cannot only be verification/conviction. The mathematicians want to have an explanation as, according to de Villiers, quasi-empirical verification does not provide an explanation as to why results are true. However, de Villiers (1990) recognises that proof can be an extremely useful means of verification, "especially in the case of surprising non-intuitive or doubtful results" (*ibid.*, p. 19).

Some mathematicians I talked with in my study claimed that proof was to convince them about the truth of the statements. They could not be convinced and go on if they did not have a proof (see p. 91). I argue that conviction by proof can also be important for the students depending on what we mean by proof and what we mean by conviction. If we think that proof exists in all mathematical activity where we justify every step, then the conviction by proof is essential. In Sweden, for example, it is usual that mathematics educators complain that students pursue the right answer without convincing themselves about the correctness of their reasoning. We are able to distinguish between three types of achieving conviction: conviction that comes from authority, conviction achieved by getting an explanation and conviction that comes from seeing how the facts are derived from other mathematical results. So, conviction as a result of *critical thinking* and questioning can be seen as desirable qualities for persons working in mathematical practice. It can also be something we can focus on in the teaching of mathematics, learning to question the "evident".

Proof creates critical debate

An important value of proof is that it creates a forum for critical debate (e.g. Davis & Hersh, 1981). Proof is a unique way of communicating mathematical results between professional mathematicians. Selden and Selden[11] (2002) describe how this communication takes place through proofs by examining the way in which mathematicians read others' proofs. They call this reading for *validation* of proof. They claim that when mathematicians read proofs they act *as if the theorem were in question*. Further, they emphasise that validation appears to be instrumental in mathematicians' learning of new mathematics. The validation is, according to Selden and Selden, a form of reflection that can be as short as a few minutes or stretch into days or more, but in general, it is much more complex and detailed than the corresponding

[10] Ruben Hersh is a mathematician who is interested in education and philosophy of mathematics.
[11] Annie Selden and John Selden are mathematicians interested in mathematics education.

written proof. Scrutinising proofs creates critical debate that is significant for the mathematical practice. There is a clear connection from the act of validation to the aspect of conviction. Critical thinking, and questioning the evident seems to be a part of the mathematical practice. Critical thinking can be also connected to the *function of proof of systematisation* (de Villiers, 1991) with the identification of inconsistencies, circular arguments as well as hidden and not explicitly stated assumptions.

But also in exercising mathematics in mathematics classrooms proofs can foster critical thinking. I agree with Hanna (1995) who argues against mathematics educators who have accused proof being authoritarian. She emphasises the character of proof as a transparent argument, in which all the information used and all the rules of reasoning are clearly displayed and open to criticism. Proof conveys to students the message that they can reason for themselves and that they do not need to "bow down to authority." (*ibid.*, p. 46)

Examples of empirical studies concerning the aspect of Conviction/Explanation

The reason for the concerns among mathematics educators about the aspect of conviction/explanation seems to be that they have noticed that students do not feel the need for proof because they are easily convinced of the truth of the statement by the authority of a teacher or a textbook or by a couple of examples. Considerations of the relation between the aspects of conviction and explanation have led to various empirical studies, for example, activities designed at putting students into situations (often by working with a geometry software) where they would feel the *need for proof* as an explanation for their findings, and cognitive research on students' actions in such activities (e.g. de Villiers, 1991; Haddas & Hershkowitz, 1998, 1999). Within these studies students explore mathematical connections inductively, so these studies also involve the next aspect, the aspect of Inductive/Deductive approaches.

2.3.2 Induction/Deduction

Traditionally proof and deductive reasoning were taught in the domain of Euclidean geometry. The rationale for including formal geometry in the school curriculum was twofold: it was seen as a vehicle for teaching and learning of deductive thinking and as a first encounter with a formal axiomatic system (de Villiers, 1986). As a result of the school reforms in the 60s Euclidean geometry as it was taught before almost vanished from the school curriculum in Sweden (Håstad, 2003). The axiomatic deductive approach that is still usual in mathematics teaching and textbooks, especially at a higher level, has been criticised since the 70s by Freudenthal, Hersh, Human,

Kline, Fischbein, Lakatos, Van Hiele and others (de Villiers, 1986). Lakatos (1976) called this style deductivist. "…, deductivist style tears the proof-generated definitions of their "proof-ancestors", presents them out of the blue, in an artificial and authoritarian way. It hides the global counterexamples which led to their discovery." (*ibid.*, p. 144) Lakatos in turn advocates a *heuristic style* that, on the contrary, highlights and emphasises the problem-situation, the logic which gave birth to the new concept (*ibid.*, p. 144). De Villiers (1986) suggests a variability of approaches. "The axiomatic deductive approach may, in terms of time-saving, perhaps become more and more essential as students progress into higher mathematics, provided they had already acquired a sound understanding of axiomatic structures by their own participation in its construction (or as re-enacted by the teacher)." (*ibid.*, p. 23) This can be compared to Wenger's (1998) description about reifications coming from outside, for example proofs, that students have to "reappropriate" into a local process in order make them meaningful (see p. 37).

However, it is important to point out that discussions about the relations between these different teaching styles are in no way a recent phenomenon but have been more or less common during several hundred years. The emphasis in mathematics education during the centuries has moved back and forth, lying sometimes more on the practical and heuristic approaches, sometimes on the theoretical and deductive approaches (e.g. Nykänen, 1945). For example, recently Lerman (2000) described an approach, which runs contrary to the tendency of working inductively, like from everyday examples to general principles. Vygotsky called this approach the *ascent from the abstract to the concrete* (Lerman, 2000, p. 65). Lerman also gives an account of the results of studies about the teaching of general principles before the applications that support the argument for a "theoretical learning approach". This questions Wenger's assumption that the more abstract reification in general would always require more participation (see p. 38). Starting from the abstract and general may sometimes require less participation than starting from the concrete.

Attempts to find out new methods for the teaching of proof led to various studies on students' own investigations, conjectures and proofs (e.g. Alibert, 1988; Almeida, 2003; Bell, 1976; de Villiers, 1991; Haddas & Hershkowitz, 1998, 1999; Schalkwijk et al., 2001). Bell (1976) suggests that the best way to achieve pupils' appreciation of proof is likely to be *cooperative, research-type activity by the class, where investigations would lead to different conjectures by different pupils, and the resolution of conflicts would be made by arguments and evidence*. This Lakatos-inspired view started a trend in mathematics education and in mathematics education research in the 80s and a lot of studies have been conducted in this spirit. These ideas have also been in harmony with the constructivist paradigm according to which teaching and learning is not the same as transmission of knowledge but knowledge has to be actively constructed by the learner (see p. 25). Very often the re-

searchers also refer to the authentic ways in which mathematicians work and advocate such working manners for students in an early stage of their mathematical studies. The idea is that students should not just meet ready-made proofs and formulae but should be able to participate in constructing them.

It has to be emphasised that theorems in mathematics are not always first discovered by means of intuition and/or quasi-empirical methods before they are verified by the production of proofs. Even within the context of formal deductive processes as *a priori* axiomatisation and defining, proof can frequently lead to new results. So, for the mathematicians proving can also be a means of *exploration*, *analysis*, *discovery* and *invention* (e.g. de Villiers, 1990). Further, the role of empirical examples for a research mathematician is (besides to find out conjectures) to find the possible counterexamples before starting the attempts to find a proof for the conjecture.

Examples of some empirical studies

Alibert (1988) conducted a research project for undergraduate students in Lakatos' spirit but not all the students were satisfied with the investigative working manner; they felt that it was inaccessible, not sufficiently ordered. They stated that they were not interested in conjectures if they did not know where the investigation would lead. The students who stated that they were satisfied with investigations also wanted to have traditional lectures. Many of them stated that debate was interesting when new concepts or new properties were first encountered. However, they wanted a teacher to give a clear summary of the lecture in order to "institutionalise the knowledge worked on in the debates". This can be compared to results in two Swedish case studies where students first investigated some problems and then tried to prove them. After working together they wanted to see the "correct proof" formulated by the teacher (Pettersson, 2004; Wistedt & Brattsström, 2005). The interplay between inductive and deductive approaches in mathematics can be related to the interplay defined by Wenger (1998) between participation and reification (see p. 37). As a pair, participation and reification refer to a duality that is fundamental to the negotiation of meaning. With inductive ways of working, and by conjecturing and trying to justify the conjectures, students participate in constructions of proofs and do not just meet the complete reifications. Of course they cannot create all the mathematics themselves but they do have to participate in appropriating the reifications presented to them as complete deductive proofs. To study a deductive proof and make sense of it is also a form of negotiation of meaning that involves both participation and reification (compare with Selden and Selden's description of validation in the previous section).

How students experience the difference between empirical evidence and mathematical truth has been studied by many mathematics education researchers (e.g. Balacheff, 1988; Chazan, 1993; Schoenfeld, 1985). Chazan

(1993) identifies from the literature two sets of problematic student beliefs about argumentation in mathematics. The first view is that students contend that measuring is enough to conclude that a statement is true for sets that have infinite number of members (Evidence is Proof). The second view is that students view deductive proofs in geometry valid only for a single case (that is pictured in the associated diagram). This means that deductive proof is simply evidence for them. In Chazan's study these aspects were explicitly focused in teaching on proof in order to enhance students' understanding of the meaning of proof.

The studies inspired by the new ways of approaching proof have also led to examinations of students' levels of "proofs" (e.g. Balacheff, 1988; Godino & Recio, 2001; Harel & Sowder, 1998; Miyazaki, 2000; Nordström, 2003). These studies attempt to characterise students' reasoning and set up levels or hierarchies about the qualities in the reasoning. The results of these studies also involve the aspect of Intuition/Formality, because the criteria for the different levels in some of the studies address the students' ability to produce deductive proofs with general symbols as well. Further, some researchers have conducted micro level analyses about students' reasoning concerning the relations (for example, if there is continuity) between inductive, abductive and deductive reasoning (e.g. Pedemonte, 2001).

At the beginning of my thesis work I also conducted a pilot study with 100 university entrants inspired by the study of Hoyles (1997), Almeida (2000) and Recio & Godino (2001) about students experiences, views and proving abilities. It showed that the students did not consider examples as proof but had great difficulties with producing deductive proofs (Nordström, 2003).

2.3.3 Intuition/Formality

The aspect of Intuition/Formality is overlapping with the aspect of Induction/Deduction in a sense that working in an investigative, inductive level is often associated with intuitive and informal ways of reasoning. Intuitive/formal representations also have connections to Conviction/Explanation, communication and aesthetic. The interplay between the intuitive and the formal in mathematics also has relevance for the condition of transparency, which is discussed in the next subsection.

Formality and rigour in the practice of mathematics

Formality and rigour in mathematics are relative and context dependent concepts. Hersh (1993) describes some variations in proof standards in applied mathematics and pure mathematics and finds great differences in the rigour between them, but even in pure mathematics itself. He shows with some examples such as computer proofs and probabilistic algorithms (see p. 20) how standards of rigour in the mathematical practice have changed. Further,

he argues that the passage from an informal, intuitive theory to a formalised theory (in the sense of predicate calculus) entails some loss of meaning. He takes an opposite standpoint of those who claim that logic can verify mathematical discoveries. "What mathematicians in large sanction and accept is correct. Their work is the touchstone of logic, not *vice versa*" (*ibid.*, p. 392). He concludes that what is done in day-to-day mathematics has little to do with formal logic.

Thurston (1994) discusses proof and progress in mathematics and states that when people are doing mathematics, the flow of ideas and the social standard of validity is much more reliable than formal documents. He claims that mathematicians are not usually very good in checking formal correctness of proof, but that they are quite good at detecting potential weaknesses or errors in proofs. However, he stresses that attempts to make mathematical arguments more explicit and formal are important for mathematics (*ibid.*, p. 169).

Language and symbols

The aspect of Intuitive/Formal is closely connected to the use of language and symbols. The mathematical language and symbols are an important part of communicating mathematics and understanding the deductive ways of presenting mathematics. Engagement in practice requires access to reifications like symbols and language. Thurston (1994) criticises the habits of communication in the mathematical practice. He points out that much of the difficulty has to do with language and culture of mathematics, which is divided into subfields.

> "Organizers of the colloquium talks everywhere exhort speakers to explain things in elementary terms. Nonetheless, most of the audience at an average colloquium talk gets little value from it. Perhaps they are lost within the first 5 minutes, yet sit silently through the remaining 55 minutes. Or perhaps they quickly lose interest because the speaker plunges into technical details without presenting any reason to investigate them. At the end of the talk, the few mathematicians who are close to the field of the speaker ask a question or two to avoid embarrassment." (*ibid.*, pp.165-166)

He also states that the pattern is often similar to situations in classrooms, where mathematicians go through what they think the students ought to learn, while the students are trying to grapple with the more fundamental issues of learning, of language and guessing at mathematicians' mental models. This is something I could see very clearly in many students' experiences concerning the lectures (*ibid.*, p. 177).

Alibert and Thomas (1991) stress the importance of, not only letting the students actively engage in discovering and constructing their own mathematical knowledge but finding better ways of communicating the products of such mathematical activities to others and improving the formalism itself.

This is also an aspect students in my study talked about, difficulties in finding out the formal demands of the community. As mentioned in Section 1.2, in the practice I am studying, there are lessons for students at the basic level, where they have a possibility to exercise writing and communicating mathematics in a small group whilst receiving guidance from a more experienced person, teaching assistant or a mathematician.

Intuition in the practice of mathematics

Many mathematicians have written about intuition (e.g. Hersh, 1998; Thurston 1994). Intuition is difficult to define and there are different interpretations of it. Instead of trying to define intuition, I describe how mathematicians talk about it.

Hersh (1998) calls intuition an essential part of mathematics and relates it to visual, plausible, convincing in absence of proof, incomplete, based on physical model or on some special examples (close to heuristic). Further, intuitive is the opposite of rigorous, intuition is holistic or integrative as opposed to detailed or analytic (*ibid.*, pp. 61-62). He also points out that in all these usages intuition is vague and changes from one usage to another. Thurston (1994) describes intuition in the following way:

> "Intuition, association, metaphor. People have amazing facilities for sensing something without knowing where it comes from (intuition); for sensing that some phenomenon or situation or object is like something else (association); and for building, and testing connections and comparisons, holding two things in mind at the same time (metaphor). These facilities are quite important for mathematics. Personally, I put a lot effort into "listening" to my intuitions and associations, and building them into metaphors and connections. This involves a kind of simultaneous quieting and focusing in my mind. Words, logic, and detailed pictures rattling around can inhibit intuitions and associations." (*ibid.*, p. 165)

In terms of negotiation of meaning, intuition seems to play an important role in the participation around reifications. An important epistemological question is where intuition comes from. In Burton's (2004) study mathematicians often related intuition to aesthetic. But instead of intuition many mathematicians preferred to talk about *insights*. For almost all of the seventy mathematicians in Burton's study intuition was something important when working with mathematics. For most, the combination of knowledge and experience was exactly what did explain their intuitions (*ibid.*, p. 80).

Transition from intuitive to formal

In mathematics, it is important to strive to come from intuitive to explicit presentation. The relation between the formal and the intuitive has appealed to many researchers in the field of mathematics education in different ways.

Fischbein (1987) argues that the educational problem is not the elimination of intuition, which, according to Fischbein, is impossible, but to "develop new, adequate intuitive interpretations as far as possible, together with developing the formal structures of logical reasoning." (*ibid.*, p. 211) He also stresses that the students have to clearly understand that not everything in mathematics lends itself to an intuitive interpretation. Mathematics is, by its very nature, a formal, deductive system of knowledge. He criticises the two opposite didactical strategies, the one which emphasises the intuitive, pictorial components on one hand, and the one in which the body of knowledge is presented axiomatically on the other hand. He argues that both strategies were mistaken "because each of them considered only a half of the complex structure of mathematical concepts which, psychologically, are both intuitively and formally based." (*ibid.*, p. 214)

We can compare Fischbein's concerns to those of Wenger (1998). Wenger suggests that his perspective regarding participation/reification has pedagogical implications for teaching of complex knowledge: an excessive emphasis on formalism without corresponding levels of participation, or conversely a neglect of explanations and formal structure, can easily result in an experience of meaninglessness. Further, Wenger connects the ability to bring the two together with creativity: on the one hand, the ability to intensely involve with the *reificative formalisms* of a discipline; and on the other, to obtain a deep *participative intuition* of what those formalisms are about (*ibid.*, p. 67). Further, he states that explicit knowledge is not freed from the tacit, as formal processes are not freed from informal. Hence, Wenger connects the intuitive more to participation and the formal to reification.

Applications of Lakatos' ideas described in the previous subsection has, according to Hanna (1995), led many mathematics educators to downplay the role of formal mathematics and in particular formal proof. She agrees with those who stress the importance of informal methods in curriculum. However, she points out that the total exclusion of formal methods leads to a curriculum unreflective of the richness of current mathematical practice and a denial to both teachers and students of "accepted methods of justification which in certain situations may also be the most appropriate and effective teaching tool." (*ibid.*, p. 46) Hanna argues that rigour is a question of degree and the level of rigour is often quite a pragmatic choice. She points out that the teacher must judge the proper level and a more rigorous argument may sometimes be more enlightening. "It might be a calculation, a visual demonstration, a guided discussion observing proper rules of argumentation, a preformal proof, an informal proof, or even a proof that conforms to strict norms of rigour, all depending on the grade and level and the context of instruction." (*ibid.*, p. 47) I agree with Hanna when she stresses that a proof would not succeed with students who never learned to follow an argument. "It fits the cultural context because it is aimed at an audience that has the

required level of experience, understands the language and has been taught to follow a mathematical argument" (*ibid.*, p. 48.) These considerations also are relevant to the condition of transparency of proof discussed in Section 2.3.4, in this chapter.

Some examples of empirical studies addressing the aspect of Intuition/Formalism

The considerations above have inspired mathematics education researchers to conduct various empirical studies (e.g. Chin & Tall, 2000; Moore, 1994; Pettersson, 2004). Chin and Tall (2000) studied the mathematical concept development of novice university students introduced to formal definitions and formal proof. They argue that the introduction to formal proof in mathematics involves a significant shift from the computation and symbol manipulation of elementary arithmetic and algebra to the use of formal definitions and deduction. They talk about the change in the language, from everyday informal register to formal mathematical register and from informal loosely speaking to formal strictly speaking mathematics. Further they describe the successive development from "definition-based" proofs to "theorem-based" proofs. Even on this level informal mental images may be used side by side with formal concepts. Pettersson (2004) studied the interplay between the intuitive ideas and formal requirements with a group of undergraduate students when working on a task in calculus. The students created a proof by induction putting heavy demands upon the formalisation of their ideas. These demands sometimes hampered the problem solving process but also encouraged the students to expand their search for a solution for the problem.

Nardi (1996) looked at the same tension from another point of view when studying students' encounter with mathematical abstraction. Students' interaction with the new (formal) concept definitions was obstructed by their unstable previous knowledge. Students' "concept image" construction was characterised by a tension between Informal/Intuitive/Verbal and the Formal/Abstract/Symbolic. Nardi showed that students had difficulties with the mechanics of formal mathematical reasoning and with applying these mechanics. The difficulties were linked to the fragility of students' knowledge with regard to the nature of rigour in formal mathematics.

Moore (1994) followed students during a so-called transition course where the students met more formal mathematical language and learned to construct definition based proofs. He noticed that the main difficulty for the students was getting started. He points out that at many colleges and universities students are expected to write proofs in real analysis, abstract algebra, and other abstract courses with no explicit instruction in how to write proofs. These considerations also have significance for the visibility of proof in the following section.

2.3.4 Invisibility/Visibility

I originally obtained the idea of considering proof as an artefact from Adler (1999). She studied a bilingual mathematics classroom where she considered *talk* as a resource for mathematical learning. She argued that Lave and Wenger's concept *transparency* (see p. 40) captured the dual function (visibility and invisibility) of talk as a learning resource in the practice of school mathematics. Further, Adler argued that the dual functions, visibility and invisibility, of talk in mathematics classrooms created dilemmas for teachers. The first data analysis in my study, showed that the condition of transparency could be a useful tool in analysing the data in order to shed light on the dilemma of how to treat proof in mathematics teaching and how to enhance students' access to proof and thus to mathematics. In this subsection I describe the condition of transparency regarding the treatment of proof and also relate it to notions closely related to it.

The condition of transparency in relation to the teaching of proof

Proof considered as an artefact can be seen as a resource for mathematical learning. According to this theory then, it needs to be both seen (be visible) and to be used and seen through (be invisible) in order to provide access to mathematical learning (Lave and Wenger, 1991). Next, I will examine how the concept of transparency can be related to different aspects of proof and proving activities and discuss the possibilities/hindrances of seeing both conditions (visibility and invisibility) and the interplay between them. Both characteristics are needed and, according to the theory, they support each other. To be able to focus on proof you have to have some experience of an "unproblematic" use of proof. Conversely, when we have gained insights into, say logical structures of proofs, it is easier for us to use them without thinking of the structure. Parallels can be drawn to language and Adler's study because proof in mathematics is used as a means of communication and explanation.

By the first condition, visibility, I refer to the different ways of focusing on the significance of proof. What is the logical structure of proof? What is the historical role and function of proof in mathematics? How were proofs created for the first time? How is it possible to differentiate and define various proofs? What is the meaning of proof in mathematics? How does one construct a proof, what are the main components in specific proofs?

With the second condition, invisibility is more difficult to capture. I refer to the opposite, not focusing on different aspects of proof, not discussing the logical structure of proofs and so on, but sharing with proof as derivations of formulas or explanations, not focusing on the process of proving but the products like formulae and theorems and the justifying of the solutions of problems without thinking it as proving. A lot of proving activities can be made and learned implicitly without focusing on the process of proving.

There are many examples in the mathematics textbooks in Sweden where the argumentation is hidden in the text, the proof has not an explicit beginning or end and its logical structure is not emphasised (Nordström & Löfwall, 2005). In older textbooks it was usual to give proofs with clear structure. Now the tendency seems to be the opposite, proofs are hidden in the text. They can be seen as kinds of explanations but they are not been focused on as proofs.

The structure of proofs

Authors of textbooks for upper secondary school and basic university courses seem to attempt to use informal language when presenting mathematics and particularly proofs to the readers. Selden and Selden (2002) point out that students often have difficulties interpreting the logical structure of informally written statements. However, visibility is not necessarily a matter of rigour and formality even if a rigorous treatment of proof activities can sometimes help one to see the logical structure of the reasoning and proof. But as well a rigorous, very detailed presentation of proof can obscure the structure of proof. Some mathematicians (eg. Leron, 1983), have discussed how to make the structures and key ideas of proof visible and, in that way, facilitate the communication between mathematicians as well as between mathematicians and students.

Alibert and Thomas (1991) point out that students lack appreciation of proof as a functional tool. They advocate Leron's structural method of proof exposition which helps the prefacing of a long, complex proof with a short, intuitive overview. It also makes visible the ideas behind the proof and its connections with other mathematical results.

> "The linear formalism of traditional proof may be described as the minimal code necessary for the transmitting of the mathematical knowledge. It appears, however, that in several important respects, it is a sub-minimal code, resulting in an irretrievable loss of information vital for understanding." (*ibid.*, p. 220)

Further, Alibert and Thomas discuss the benefits of helping students to understand the structure of proof instead of letting them by themselves discover it, which according to them is beyond the capacity of most undergraduates.

> "They are simply unable to decode the proof and are reduced to meaningless manipulation of the formal code itself, with no awareness of the ideas and concepts it represents…The major difference between the approach outlined above and the traditional linear proof style is that the students are given a means of understanding the choices that, generally, the teacher presents without any indication that there had actually been a choice involved." (*ibid.*, p.224)

Even if it is difficult to give a definition of what a proof is there are several ways of focusing on different aspects of proof in the teaching of proof. The following example from a Finnish upper secondary school textbook shows how, for example, the logical structure in geometrical proofs can be made more visible (*Figure 5*, p. 56). There is first a discussion about how to find out from the formulation of the theorem ("The base angles in an isosceles triangle are equal.") what the assumption is and what the statement is, which is not necessarily easy for the students to decide. The proof then begins with the *assumption* (*Antagande*): "The triangle ABC is isosceles." It follows by the *statement* (*Påstående*): "The base angles *DAC* and *DBC* are equal." After the proof the logical structure of the proof is illustrated with a figure. The figures illuminate the process of proving by showing its logical structure and how the necessary arguments needed for the conclusion are obtained from the assumptions, definitions, constructions, axioms or theorems.

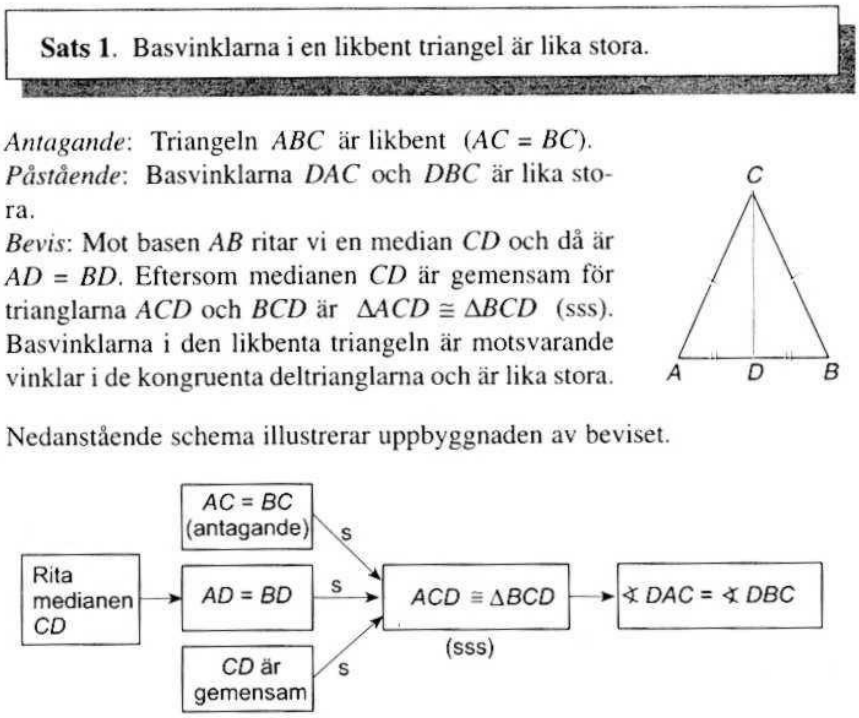

Figure 5 The structure in geometrical proofs

This kind of treatment makes the role of definitions, axioms and construction in the logical reasoning visible and, thus, may help the student to see the fundamental logical structures of geometrical proofs.

Also a focus on proving techniques, like working backwards (Polya, 1981) makes the procedures behind the finished proofs visible. Polya pre-

sented a method to solve a geometrical problem from Euclid's Elements (Proposition 4 in the Eleventh Book) and illustrated how to work backwards to diminish the gap between the hypothesis and the conclusion.

Proof is learned implicitly?

Among some mathematics education researchers, knowledge about proof and proving is sometimes seen to be learned implicitly by a kind of tacit enculturation into the mathematical practice. Ernest's (1998a) model (Table 1, p. 58) is about what in mathematical knowledge is mainly explicit, mainly tacit. Accepted propositions and statements, accepted reasoning and proofs are categorised as mainly explicit whereas meta- mathematical views, including *views of proof and definition* and the structure of mathematics as whole are categorised as mainly tacit knowledge. What Ernest means by tacit is that mathematicians get a sense of them and build them up incidentally through experience and he states that they are not and *can probably not be fully taught explicitly* (*ibid.*, p. 15). Further, Ernest states that these elements are usually acquired from experience and are tacit. Language and symbolism are important aspects in construction of proof but they are also seen by Ernest largely as tacit knowledge. Further, Ernest claims that mathematical knowledge shown in Table 1 is a broadening and an extension of the traditional view of knowledge as primarily explicit.

Also other mathematics education researchers have assumed that the learning of proof occurs mainly by enculturation, not by deliberately focusing on it. Nardi, Jaworski and Hegedus (2005) in their study on undergraduate mathematics tutors' conceptualisations of students' difficulties, classify proof into the abilities in mathematics that are mostly learned by enculturation. This kind of view of proof as something "you just get used to" was also expressed by some mathematicians I interviewed (see p. 113).

Table 1

Mathematics Knowledge Component	**Explicit or Tacit**
Accepted propositions and statements	Mainly explicit
Accepted reasoning and proofs	Mainly explicit
Problems and questions	Mainly explicit
Language and symbolism	Mainly tacit
Meta-mathematical views: proof & definition standards, scope and structure of mathematics	Mainly tacit
Methods, procedures, techniques, strategies	Mainly tacit
Aesthetics and values	Mainly tacit

Selden and Selden (2002) state that skill at validation, a kind of critical examination of proofs (see p. 45), is an implicit part of mathematics curriculum, and is rarely explicitly taught. They claim that when beginning undergraduate mathematics, students may well be unaware of its existence and importance. Several kinds of logic-related topics are not emphasised, perhaps because these are seen as unimportant or apparent. Selden and Selden discuss substitution, interpreting the logical structure of informally written statements, applying theorems and definitions to situations in proofs, understanding the language of proofs, and recognising logical structures in the context of mathematics. They advocate explicit introduction of all these, because they are difficult for students just beginning their work with proofs and "unfortunately these have been considered part of 'mathematical maturity' in the past". (*ibid.*, p. 7) By "explicit instruction" they mean a variety of instructional techniques including explorational activities and group work, as well as the more traditional lecturing and homework exercises.

But visibility is not only a matter of logic and structure. Chazan (1993) studied the possibilities of focusing on and making visible for students the difference between empirical evidence and deductive proofs in an upper secondary school geometry course. Hanna (2000) stressed the importance of discussing with students the functions of proof in mathematics. Furinghetti & Paola (2002) focus on the problem of defining and stress the importance of *awareness* in students' approach to theoretical thinking. They use it in a sense that students should be active participants in the process of constructing a theory and understand the meaning of what they are doing.

Induction/Deduction

There is an instructive example in Vretblad's (1999) textbook, that was earlier used in the basic course, about formulating a conjecture, the process of finding a proof for the conjecture and, finally, formulating a theorem and the proof (*ibid.*, p. 25). Vretblad uses as an example the relation between the arithmetic and geometric means. He starts by defining these mean values and encourages the reader to test some examples to find a pattern. Vretblad carries on a conversation with the reader about how to proceed and uses word like "Aha!" in order to show the most important points of the solution. He first explains why this experiment is not enough and shows how to pose a hypothesis. He calls the hypothesis a conjecture.

The conjecture:

If $a > 0, b > 0, A = \frac{a+b}{2}$ and $G = \sqrt{ab}$, then $A \geq G$.

Then Vretblad asks how we can convince ourselves and humankind about the truth of this statement. He puts forward the solution by reformulating the problem to show that $A - G \geq 0$ and by asking the reader what this would be good for. After this informal and instructive account of the whole process, Vretblad states that now we have achieved the goal and can give the result the force of a theorem. Then he formulates the theorem and gives a proof for the theorem. He concludes the presentation by the following sentence: "The way in which we have reasoned here is an example of an *inductive method* or *induction* (in a broad sense): one starts by an observation, one formulates a *conjecture,* and one proves a *theorem*." (*ibid.*, p. 26) In the end of the chapter, Vretblad offers students exercises with similar procedures.

Focusing on different aspects of proof

The condition of transparency is the intricate dilemma about how and how much to focus on the different aspects of proof in relation to how and how much to let students participate in different proving activities without focusing on the process in order to enhance students' access to proof (see pp. 40 and 54). All the aspects of proof can be focused on explicitly in discussions and activities, just as well as they can be an implicit part of the practice. For example, it is possible to discuss the notion of aesthetics, but the feeling of beauty could just as well grow from participation without explicit focus on it. Further, it is possible to work both inductively and deductively conjecturing and validating the conjectures with or without explicitly focusing on the aspect of inductive/deductive approaches or discussing the nature of deductive reasoning and proof. On the other hand, it is difficult to focus on them if you do not have any experience about the practice. Parallels to Sfard's (1991) "vicious circle" thesis can be drawn. It implies that "a person must be quite skilful at performing algorithms in order to attain a good idea of the "objects" involved in these algorithms; on the other hand, to gain full techni-

cal mastery, one must already have these objects, since without them the process would seem meaningless and thus difficult to perform and to remember." (*ibid.*, p. 32)

This is a pedagogical paradox that makes the teaching of proof difficult. It is not easy to talk about proof without some experience about it. But it can be difficult for students to understand the meaning of proof or learn to produce their own proofs without an explicit focus on them.

The condition of transparency has also to be considered both from the teaching and the learning perspective (see p. 27). According to Wenger (1998) pedagogical intentions and other structuring resources become resources for learning in a complex way and learning is but *"a response of the pedagogical intentions of the setting."* (*ibid.*, p. 266). As I have shown in my thesis in Chapter 6, what is intended to be in focus in teaching is not necessarily the focus of students.

So far, I have dealt with some main themes in research on proof in mathematics and in mathematics education along with four aspects that involve two interacting components: Conviction/Explanation, Induction/Deduction, Intuition/Formality and Invisibility/Visibility. I conclude the section by describing some **functions** of proof that I include in the conceptual frame.

2.3.5 Some further functions of proof included in the frame

Next, I very briefly, present some functions of proof that emerged from the data mainly during the pilot study (Nordström, 2004). They are *Communication, Aesthetic, Systematisation, Intellectual Challenge* and *Transfer.* All but Transfer have been dealt with in the earlier models about the functions of proof (de Villiers, 1990, 1996). There are other functions that have been discussed in the mathematics education community (see p. 21), which I do not include in the conceptual frame. This is because they were not touched in the interviews. It can depend on the fact that the focus on the interviews with mathematicians were in the teaching of proof, not on their own research.

Proof is a means for communicating mathematical ideas. The function of *Communication* is related to the other aspects in the frame presented earlier, in the following way: Explanation and conviction can be obtained by communication; communication can also occur via a deductive proof or inductive ideas. Important for communication is also the language and how explicit everything is expressed so it is also interlinked to the aspect of formality, level of rigour and the language/intuition.

Aesthetic is a very personal experience. De Villiers (1990) mentioned also an aesthetic function of proof in his examination of the functions of proof although it was left outside the model presented in Chapter 1. Burton (2004) interviewed seventy mathematicians about their views on mathematics.

Forty-three of them introduced aesthetics, beauty or elegance into the discussions. (*ibid.*, p. 65) and for the majority, the aesthetics was connected to proof.

Systematisation is a way of organising mathematics in a deductive manner. The function of systematisation is also interlinked to the other aspects of proof in the conceptual frame. Systematisation demands a certain level of rigour and critical thinking. The function of systematisation can also enhance understanding and conviction.

Intellectual challenge refers to self-realisation and fulfilment derived from constructing a proof (de Villiers, 1996).

Transfer is a function of proof that especially the mathematicians in my study touched with when talking about the meaning of the learning of proof. Transfer is not discussed in the earlier models of functions of proof. It is close to and partly overlapping the aspect Weber (2002) describes but not exactly the same. I will come back to transfer in Section 7.3. The function of transfer refers to two basically different things.

- Transfer can refer to a possibility of drawing advantages of working and understanding with mathematical proofs to problems in different situations outside mathematics. The question if it is possible to learn logical reasoning that is useful in other contexts than mathematics, when working with mathematical proofs, is worth a discussion and research in the mathematics education community. There have been periods in the history when an educated person was to learn geometry, since according to a general view it was a way to learn to reason logically also outside mathematics.
- Transfer can refer to the benefits of learning proof for other mathematical contexts, since proofs can sometimes offer new techniques to attack other problems or offer understanding for something different from the original context. For example, Galois' result that the fifth degree equation cannot be solved by radicals has had much less importance to mathematics than his *proof* for the theorem, which opened a possibility to develop a new theory. But also in teaching contexts at an elementary level, for example, the method of completing the square in deriving the formula for the solution of the second degree equation may be also used in problem solving in other mathematical contexts.

In this section, I have described the conceptual frame about the aspects of proof. This frame was continuously used in the data analysis and I will refer to these aspects when reporting the results. The aspects in the conceptual frame consist of both functions and properties/approaches of proof.

Functions:
- Conviction/Explanation
- Communication
- Aesthetic
- Systematisation
- Intellectual challenge
- Transfer

Properties/Approaches:
- Induction/Deduction
- Intuition/Formality
- Invisibility/Visibility

In the figure below, I illustrate how these aspects of proof relate to the notions of artefact and reification described in Section 2.2.4 and 2.2.5 The figure also illuminates how I have combined a socio-cultural perspective, Lave and Wenger's social practice theories and didactical research on proof.

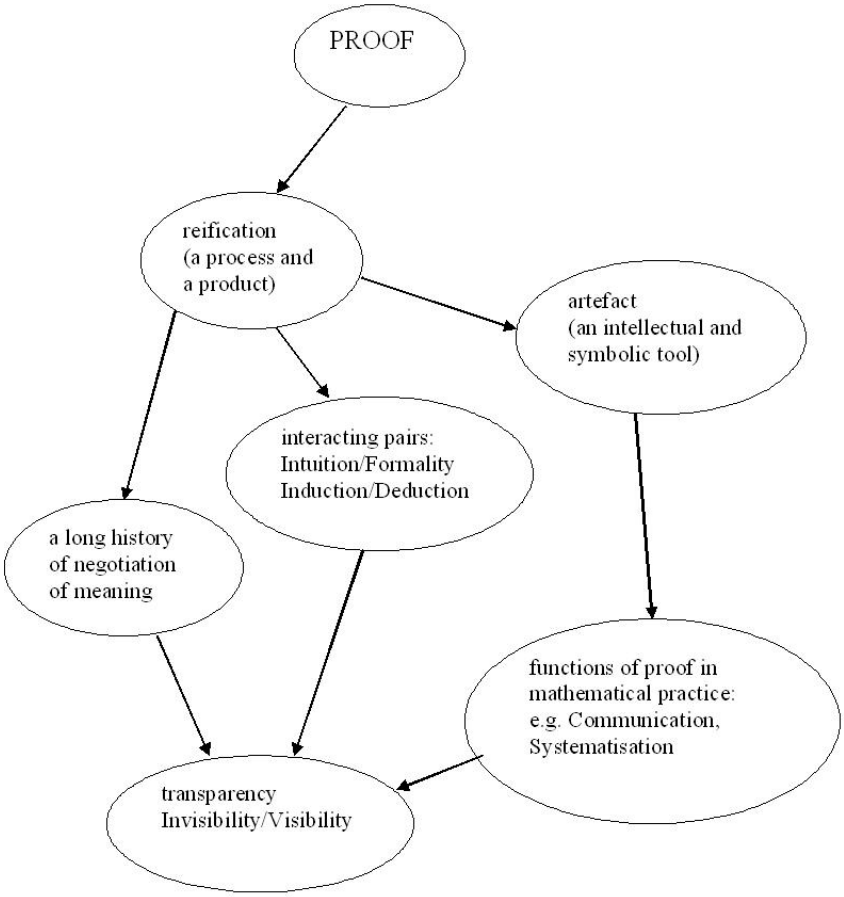

Figure 6 Proof as dynamic notion

2.4 A summary

In this chapter, I first described my choice of the theoretical frame. After that I elaborated with central notions of the socio-cultural perspective and the theories of Lave and Wenger (1991) and Wenger (1998) describing how I apply these notions in my work. I started by describing the community of mathematical practice at the mathematics department that is the focus of my study. I went on examining notions like learning, knowing and negotiation of meaning. I examined proof as reification and as an artefact in mathematical practice and concluded the chapter by describing a conceptual frame created from literature about the aspects of proof. This frame was helpful in the data analyses when structuring the results and in linking them to previous research.

In the next chapter, I go on describing the methodology of my study.

3 Methodology

Wellington (2000) defines methodology as the activity of choosing, reflecting upon, evaluating and justifying the methods one uses. I have divided my methodology chapter into three main parts. In the first part, I present the specific research questions and the design of the study. In the second section, I discuss different research paradigms in relation to my study and how various theories and paradigms relate to my choice of research methods and ways of analysing and interpreting the data. In the third part of the chapter I give a detailed account of the different methods as well as the different procedures for the data analyses applied in the study. I include ethical considerations in the description of the methods and explain how they have influenced the way in which I report the results.

3.1 The design of the study

Bassey (1999) compares a research question to the engine which drives the train of inquiry.

> "It should be formulated in such a way that it sets the immediate agenda for research, enables data to be collected and permits analysis to get started. – If this 'engine' is found to be under-powered, or breaks down or is pulling the train to wrong direction, it should be replaced." (*ibid.*, p. 67)

The purpose of my study is to describe and characterise the culture of proof in a community of mathematical practice and how students are engaged in proof. I approach the issue from different directions (*Figure 1*, p. 13). I examine mathematicians' views and pedagogical perspectives on proof. I also study students' backgrounds and experiences in their mathematical practice.

I presented the **general research questions** in the introduction. They are:

- How do students meet proof in the community of mathematical practice at a mathematics department?
- How are students drawn to share mathematicians' views and knowledge of proof?

The specific research questions, through which I aimed to gain insights to the general research questions, evolved to be the following:

1) How do mathematicians talk about proof and its role in mathematical practice?
2) What pedagogical approaches towards the teaching of proof can be identified in mathematicians' utterances?
3) What kind of experiences regarding proof do students have from their upper secondary school mathematics?
4) How do students relate to proof and the learning of proof?
5) What kind of participation in proof and proving is there available for students in the practice?
6) How do students talk about their experiences regarding proof in their mathematical practice?

Educational practice is a complex phenomenon and different sorts of questions require different sorts of research. Pring (2000) claims that researchers must be eclectic in their search for truth. Some research questions might demand several methods, others only one. As I approached the issue of proof from different directions, I used various approaches and associated methods. I tried to be creative in choosing the methods and combined both quantitative and qualitative methods. In Table 2, I present the methods and describe the specific issues I hope they will help me to shed light on. The table gives an overall picture about the design and the methods of my study.

The data for shedding light on how mathematicians talked about proof and their pedagogical intentions in mathematical practice were transcripts of interviews with mathematicians. The main data for illuminating students' background, their experiences, their views, and how they related to proof and the learning of proof were survey responses and transcripts of focus group interviews. Surveys offered me some rough background data whereas focus group interviews with students in different phases of their studies complemented it and provided me with more personal and richer information than mere figures.

Finally, I contrasted the results concerning the mathematicians' practice and the results concerning the students' practice in order to shed light on how the structuring resources and mathematicians' intentions became resources for learning.

As complementary data, I used interviews with experts and field notes from observations of lectures as well as documents like annual department reports, examinations, textbooks and curricula.

In this section, I have presented the specific research questions and provided the reader with an overall picture about the design of my study. In the next section, I give an epistemological account of the research methods.

Table 2 Design of the methods

	MAIN DATA			COMPLEMENTARY DATA	
Methods / Research questions	**Interviews with mathematicians**	**Surveys with university entrants**	**Focus group interviews with students**	**Observations of lectures**	**Document analysis and interviews with experts**
Mathematicians' views and pedagogical perspectives	X			X	textbooks curriculum examinations other material
Students' upper secondary school background		X	X		upper secondary school textbooks
How students relate to proof		X	X	X	
What kind of participation in proof is there available in the practice?	X	X	X	X	curriculum textbooks other material examinations interviews with experts
How students talk about their experiences			X		
How do students meet proof? How are they drawn to share mathematicians' views and knowledge of proof?	Results of the analysis of interviews with mathematicians	Results of the survey analysis	Results of the analysis of the focus group interviews with students	X	X

3.2 An epistemological account of my area of study

I start the epistemological considerations by relating different research paradigms in relation to my study. I go on by discussing how various theories and paradigms relate to my choice of research methods and ways of analysing and interpreting the data. I examine the benefits and disadvantages of quantitative/qualitative methods in terms of what kind of knowledge they provide and how such notions as validity, reliability, trustworthiness and generality relate to these methods which have quite different character. Finally, I discuss how quantitative and qualitative methods have been connected to different epistemologies and how to combine them.

3.2.1 The thesis in relation to different research paradigms

My thesis can be defined as a picture-drawing case study (Bassey, 1999). It is primarily a descriptive account where I draw together the results of explorations and analyses of the phenomenon that is proof and the teaching and learning of proof in all its diversity in the context of university mathematics at a university in Sweden. Case studies can be placed both in the *interpretive* paradigm and in the *positivist* paradigm.

According to positivism there is a reality in the world that exists irrespective of the observer. This reality can be discovered by people observing with their senses (Bassey, 1999). To the positivist the entire world is rational and the researcher can explain the reality s/he has discovered to others with factual statements. Language is seen as an agreed symbolic system for describing reality in an unproblematic manner. Positivist researchers do not consider themselves as significant variables in their research and they expect other researchers to come to the same conclusion that they find. Positivist knowledge is deemed to be objective, value-free, generalisable and replicable (Wellington, 2000). The methodology of the positivists is often described as quantitative.

The interpretive researcher, on the contrary, does not accept the idea of a reality which exists irrespective of people, but that concepts of reality vary from one person to another (Bassey, 1999). The observers are part of the world which they are observing. They can also, by observing, influence what they are trying to observe. They see themselves as potential variables in the enquiry and so, in writing reports, may use personal pronouns. The language is seen as a more or less agreed symbolic system, but different people may have some differences in their meanings and the rationality of one observer may not be the same as the rationality of another observer. In consequence the sharing of accounts of what has been observed is always to some extent problematic. The data collected by interpretive researchers are usually verbal. Even if interpretive data can be analysed numerically the quantitative statistical analysis used by positivists is not usual (*ibid.*, p. 43).

In my study, I cannot follow the traditional rules of scientific inquiry, because the case is too complicated. It is also impossible for me to put myself totally outside the mathematical practice. I participate, both as a teacher and as a student, in the community of practice of mathematics at the department that I am studying. Furthermore, I have to interpret the qualitative data. So, I recognise myself as an instrument in the inquiry, an instrument that is influenced by the very practice that I am studying. My purpose is to advance knowledge of the teaching and learning of proof in undergraduate university courses at a university in Sweden by analysing and interpreting different sorts of data. Some of the results in the quantitative part in my study were obtained by statistical analyses and can, to some extent, be generalised, and the results of the data analyses can be obtained by other researchers. However, the study as a whole may offer possibilities and insights, not certainties. I agree with Pring (2000) who argues that it is possible to reject "naïve realism" (for example that there is an unproblematic correspondence between the language and the reality) without abandoning the realism of the physical and social sciences and without therefore concluding that reality is but a social construction or that correspondence between language and reality is to be thrown overboard completely.

According to the social practice theory of Lave and Wenger (1991) and Wenger (1998) that I apply in my study, the world is seen to consist of objective forms and systems of activity, on the one hand, and agents' subjective and intersubjective understanding of them, on the other hand. These mutually constitute both the world and its experienced forms. Further, cognition and communication in, and with, the social world are situated in the historical development of ongoing activity (Lave and Wenger, 1991, p. 51). There is, for example, a certain kind of social and historical structure that constrains the old-timers (see Section 1.2 and 1.3) and the students in a way that limits the range of actions open to them. The mathematical practice at the department that I study is the participants' response to the conditions in their enterprise. Mathematicians and students act in the frame of historical conditions of the practice including the lectures, textbooks, individual homework, seminars, examinations and other forms of participation and the long history of practice of mathematics (see Section 1.3).

Further, knowledge about proof and the teaching and learning of proof is not simply in individual teachers' minds: it is tied to their identities and evolves in and through co-participation in the practices of the community. Hence, I consider the mathematicians and the students as participants in the community of mathematical practice and interpret their utterances, not entirely as their own opinions but to some extent as reproduction of views belonging to the community, utterances that are influenced by the social, cultural and historical context of the same mathematics environment but also from other possible environments they are members of.

3.2.2 Theories and the data analysis

An important question for data analysis is how and when the theory comes into the process. Glaser and Strauss (1967) developed a method of systematically discovering theories from data called grounded theory. Instead of starting with a body of theoretical propositions about social relations, the idea was first to observe those relations, collect data on them, and then proceed to generate our theoretical propositions. There are problems with this procedure. It is difficult to enter the data without any a priori assumptions. May (2001) points out that the method of grounded theory ignores the idea of theory altogether and entails that our presuppositions about social life remain hidden, but still influence decisions and interpretations (*ibid.*, p.31).

In my study, it was impossible for me to enter the data without any *a priori* thoughts and expectations. There is a lot of research on proof in mathematics education and my aim was to relate the data to these previous studies and to historical and philosophical issues as well. One of the criticisms of educational research is that it is non-cumulative (Bassey, 1999; Wellington, 2000; Bryman, 2001). Wellington (2000) questions whether the researchers have to recreate theory every time they collect and analyse data. For me the role of theory was to help to understand events in my area of study and see them in a new or a different way. It helped me to focus on different aspects of proof in the community of practice of mathematics. From the literature I created a conceptual frame for understanding and making sense of aspects regarding proof and the teaching and learning of proof emerging from the data (see Section 2.3, p. 61). Yet, besides relating the data to the research questions and the theoretical frame, I used an open approach and explored new themes emerging from the data. The first data analyses also influenced the improvement of the theoretical frame.

3.2.3 Quantitative/qualitative methods

I employed both quantitative and qualitative methods in my study. I started from a quantitative basis and then selected a smaller group for a more detailed study when zooming from surveys with the newcomers to focus group interviews. For the focus group interviews, I chose students with different kinds of relation and experiences (according to their responses to the questionnaire). With the help of quantitative inquiries, I could, for example by calculating percentages and correlations, get rough information about the aspects I was exploring. From focus group interviews I obtained data that were richer and shed more light on the uniqueness of individuals beyond the figures in the surveys.

The employment of quantitative and qualitative methods has been connected to different epistemological approaches. Quantitative methods are often associated with the positivist paradigm whereas qualitative methods

are associated with the interpretive paradigm (e.g. Stake, 1995). The sharp contrast between quantitative and qualitative methods has been recently criticised by some researchers (e.g. Bryman, 2001; Gorard, 2001; Pring, 2000). They point out that qualitative research has quantitative features, just as quantitative research has qualitative features, and that the research methods are much more free-floating in terms of epistemology and ontology than is often supposed.

In my study, choosing to pose certain questions in the questionnaire for the quantitative survey was already a personal act and a lot of decisions had to be made before the questionnaire was drawn up. Also decisions concerning what kinds of data analysis were conducted were personal and depended on my theoretical perspectives and how I had posed the research questions. In the quantitative part of the study, I used descriptive statistics with percentages and correlations. The proceeding of the statistical analyses with SPSS software could be regarded quite impersonally and the numerical results of the analyses did not depend on the researcher. The way of interpreting the numerical results, however, are again personal. For example, I do not consider the relation between the statements and the questions that the students responded to on the one hand, and the reality on the other hand as unproblematic.

Next, I examine notions of validity, reliability, trustworthiness and generality in relation to my study and to the quantitative and qualitative methods that I have employed.

3.2.4 Reliability, validity, objectivity, and generality

Stability is the extent to which a research fact can be repeated, given the same circumstances. The surveys among the university entrants could be repeated and were also repeated three times in my study and gave the same kind of results each time. Surveys were conducted among a similar population with roughly defined similar backgrounds, because there have not been any changes lately, in the school curriculum regarding proof, changes that would have influenced the experiences of the samples. Further, the statements were focused on certain issues and led the students to certain reflections. If the surveys were conducted in another country or after ten years, they could give different results. In the context of the issues I was exploring with the surveys I also calculated the so called *internal reliability* between the items within the issues. Stability and internal reliability are two factors connected to *reliability* (Bryman, 2001). I will come back to it when describing the methods in more detail.

Concerning the interviews, it is impossible to create exactly the same circumstances several times even if you gather the same persons to talk about the same issues again. Premises of qualitative studies include the uniqueness and idiosyncrasy of the situations, such that the study cannot be replicated

and as Cohen et al (2000) point out – that is their strength rather than their weakness.

Validity is the extent to which a research fact or finding is what it is claimed to be. In the questionnaire, there were some different questions addressing the same aspects so I could check the correlation between the responses to these pairs. In the focus group interviews I also had a possibility to check that the students had understood the survey questions in the way I had intended them to, when I posed the question.

The problem of how qualitative research findings can be validated is much discussed in the literature (Ernest, 1998). Instead of reliability and validity it is usual within the context of qualitative research to talk about *trustworthiness* (Bassey, 1999). Burton (2002) argues that utilising so-called objective methods does not make a research study *objective* as little as the subjective information makes the study *subjective*. According to her, objectivity is gained through the internal consistency and coherence with which the story is told. The researcher must be able to convince the reader of their trustworthiness and of the authencity of what they have done as well as of the conclusion that they have reached and the resultant implications they have drawn (*ibid.*, p. 9). Next, I discuss aspects of my study and relate them to the issues that Bassey (1999) defines as criteria for trustworthiness. He draws his criteria on Lincoln and Guba's (1985) account.

During the inquiry, I had prolonged engagement in the field and continuously observed emerging issues influencing my study and the results of it. I also kept a diary of how my research developed. Bassey (1999) states, that it is important to check the interview reports with the data sources to give the interviewees a possibility to put the record straight if they think something they have said has not been understood correctly (*ibid.*, p. 76). I did it when I interviewed experts for the background facts but not when I interviewed the mathematicians and the students about their views and experiences. It would have changed the character of my study. As I described earlier in this chapter (see p. 68), I considered the mathematicians and the students as participants in the community of mathematical practice and interpreted their utterances, not entirely as their own opinions but to some extent as reproductions of views belonging to the community. According to the theory, the views and stances are not static but I was not studying the changes in them. What was interesting for my study was the way in which the mathematicians, when talking about proof defined the role of proof in their practice at the moment of the interviews and what pedagogical considerations could be discerned in the utterances. Focus group interviews with students offered my study, for instance, examples of identities of participation and non-participation. According to the social practice theory (Wenger, 1998) identities are not static but temporal and always becoming. So, if I would have gone back and asked the same questions again, I would have received different results and my study would have been different.

Parts of the data analyses were done in cooperation with one of my supervisors so different possible interpretations could be considered. From the different data analyses I formulated analytical statements and checked them once again against the data. Then the story and the results reported were systematically tested against the analytical statements and the data. I have repeatedly asked other researchers to critically read my reports and I attempt to give an account of my research that is sufficiently detailed to give the reader confidence in the findings. I made the qualitative data analyses in Swedish because it was the language of the interviews. At the beginning, I considered providing the Swedish original examples in the reports of the results but after my decision to neutralise the language of mathematicians (see p. 83) it was not important any more. Anyway, utterances had lost the original form. However, I decided to offer the reader some expressions also in Swedish when it was difficult to translate the utterance or a part of the utterance to English. All the translations of the quotations given as evidence in the results have been checked by a bilingual person, Tristan Tempest, who also could consider them in their original contexts.

The data has been related to the theoretical framework, which I described thoroughly in Chapter 2. This helped me to make the data analyses and the conclusions transparent, and hence, more objective. My approach to the research questions also involved triangulating the data. There were the students and the mathematicians, the textbooks/syllabuses and other documents, as well as the observations of the lectures. I limited the study to one university only, in order to be able to triangulate a great part of the data and deepen some issues. However, it was not always possible to observe the lectures of every mathematician whom I interviewed. Besides, the students in the focus groups had experiences about teachers whom I had not interviewed and so on. Some parts of the data were triangulated in the following way. I observed the lectures, interviewed the students taking part of the course and the mathematician who held the lecture. I contrasted the results obtained from different data sources (interviews with mathematicians, students and surveys) with each other. Further, I supported some of the conclusions with the complementary data, for example, the analysis of the field notes from the observations of lectures.

Is it possible to generalise the results that I have obtained from my study? The samples in the surveys were convenience samples (Cohen, Manion, & Morrison, 2000), but about half of the university entrants responded to questionnaires, so the data can be seen as representative for the whole population that was students who started to study ordinary courses in mathematics. From the qualitative data, I have formulated some analytical statements and obtained results which can later be challenged or refined and developed by me or some other researcher and in that way made more general. Bassey (1999) calls these kinds of generalisations *fuzzy generalizations*. He defines fuzzy generalisation as a kind of prediction, arising from empirical enquiry,

that says something may happen, but without any measure of its probability. It is a qualified generalisation, carrying the idea of possibility, not certainty. He advocates a wider use of these kinds of analytical statements in pedagogical research. It is then easier for another researcher to start where the first has ended and try to refine and develop the results and maybe make them even more general.

In this section, I discussed different research paradigms in relation to my study and to my choice of research methods and ways of analysing the data. I discussed quantitative and qualitative methods and what kind of knowledge they provide and ended up with the notions of validity, reliability, trustworthiness and generality. I then discussed how these notions relate to these methods, which have quite different characters. In the next section, I describe and evaluate each method employed in the study and relate them to the theoretical issues described in this section.

3.3 A description about the specific methods and the associated data analyses

I conducted surveys with university entrants at the mathematics department that I am studying at the beginning of the term in August 2003 and in January 2004. In the first subsection, I describe these surveys and the procedures of the quantitative data analysis connected to the surveys. In the following subsections, I give an account on the qualitative methods: interviews with mathematicians and focus group interviews with students. Finally, in the last subsection, I describe the methods for collection of the complementary data: observations of lectures, document analyses and interviews with experts about changes in the curriculum, in the organisation of teaching, in the contents of the courses and in the course literature.

3.3.1 The surveys

I start the subsection by explaining the background for the surveys in 2003 and 2004. Hence, I first very briefly describe the pilot survey in 2002 and the development of the final questionnaire. I go on with a detailed description about the questionnaire for the surveys in 2003 and 2004, the data-collection and the procedures of the data-analysis. I conclude the subsection with critical considerations of the method and with some ethical remarks.

Pilot survey and the development of the final questionnaire for the surveys in 2003 and in 2004

I started my data collection with a pilot survey among university entrants at a Swedish university in autumn 2002 (Nordström, 2003). I considered this survey as an appropriate method of gathering some initial information about students' backgrounds, attitudes and proving abilities because the population I wanted to study was big. The population consisted of 170 university entrants who started to study ordinary courses in mathematics and the sample I gathered and analysed was 100 students. I handed out the questionnaire to all the students at the registration in the very beginning of the term, and gathered the questionnaires at the same occasion. The aim of the pilot study was to get some overall information about how students related to proof when they entered the practice and what they stated about their school experiences concerning proof and how they managed to prove some elementary statements. At the time of the pilot study, I had just started to study proof and had a broad approach to the issue, so I wanted to get a lot of information about different issues concerning proof. I created a questionnaire guided by Cohen, Manion and Morrison's (2000) book *Research Methods in Education* and Oppenheim's (1998) book *Questionnaire Design, Interviewing and Attitude Measurement*. The majority of questions and statements in the pilot study came from previous studies (Almeida, 2000; Godino & Recio, 2001; Hoyles, 1997). The dichotomous statements, with which students could agree or disagree, addressed students' views on proof, how they related to proof and the learning of proof, and what they stated about their experiences about proof.

There are some elementary problems in using questionnaires. The heart of the problem is that different respondents interpret the same words differently. The wording of questionnaires is of paramount importance and pretesting is crucial to its success (Bryman, 2001). A pilot has several functions, principally to increase the reliability, validity and practicability of the questionnaire. That is why the questionnaire for the pilot study was tested with several groups.

- I started with a small group of young people in my neighborhood, who had just finished their natural science program in upper secondary school. I got several important comments from them.
- I let some experienced researchers check the questionnaire and got good advice.
- At last I tested the questionnaire with a group of university students in connection with a summer examination before the final pilot survey with all the university entrants in 2002.

I personally distributed the questionnaire to the students at the beginning of their first term at the university. I do not give a detailed description about the pilot questionnaire here because I do not include the pilot survey in the the-

sis. This is because the research questions evolved during the time I worked with the thesis and some parts of the results of the pilot study were not relevant for the thesis. The results presented in this thesis are based on the analyses of the surveys in 2003 and 2004.

After the analysis of the pilot study (Nordström, 2003), which I made manually, I improved the questionnaire by adding some multiple choice questions about students' experiences (11-15) in order to get a more varied picture about them. I also added some more statements in order to double check the students' statements and in that way be able to check the validity of the questionnaire. I changed the dichotomous part of the questionnaire to one with five possible responses: fully disagree, partially disagree, no opinion, partially agree, and fully agree in order to obtain more precise responses to the statements.

As I described in Section 1.1, I limited the study at the beginning in order to be able to study some issues more deeply and hence, decided to focus on students' stated upper secondary school experiences and their relation to proof rather than their proving abilities. That is why I omitted the proving tasks in the main surveys. I also changed the order of the questions and moved the personal questions to the end of the questionnaire because, although important, they could appear intrusive (Gorard, 2001). Having them at the end would encourage people to start the questionnaire, and once started be more likely to complete the task (*ibid.*, p. 99). Thus, the final questionnaire contained open questions, multiple choice questions and statements on a five-point scale running from totally agree to totally disagree.

It is possible to get access to a wider range of aspects of the issue by asking a number of questions and in that way get a lot of indirect indicators of the issue (Bryman, 2001). I was interested in the students' background. Hence, I stated various questions about their upper secondary school experiences regarding proof. I also wanted to know how they related to proof including their feelings and views on proof. So I posed many different kinds of questions trying to cover a wide range of indicators. Next, I will describe the contents of the final questionnaire in more detail (Appendix 2).

The final questionnaire

1. The first question was a background question about students' motives for studying mathematics. I categorised the answers into three categories, *pragmatic* (for example if the student wrote he needed mathematics for some other purposes than mathematics itself), *subject oriented* (for example if the student stated that mathematics was fun) and *mixed/ does not know* if the student did/could not answer or mentioned both the pragmatic and subject oriented reasons for the mathematical studies.
2. The second question addressed students' feelings. Students' responses were classified into three categories, *negative*, *positive* and *mixed*. The negative responses consisted of alternatives b) nervous, d) dull, e) inse-

cure and some of their own descriptions, like "anxious". The positive responses consisted of the alternatives a) curious, c) eager and some of their own descriptions, like "This will be easy". The mixed group consisted of those who had chosen both kinds of alternatives.

3. The third question was about the students' views on proof. It was an open question and gave numerous different aspects of proof mentioned by students. I listed the various aspects that the students mentioned as different variables. It gave me information about how students perceived proof.
4. The fourth question was a modification of a question from Celia Hoyles' (1997) study about students' views on proof. The choices of the fourth question were categorised into five categories: 1) Lina or no answer, 2) Tove, 3) Mattias, 4) A mixture of Lisa/Peter and some other, 5) Lisa/Peter. The aim of this question was to help students to enter the context of proof. Many of them might have had a break in their mathematical studies. The responses also gave me information about what students considered as a valid proof.

5. – 9. The multiple-choice-questions (5-9) were about students' upper secondary school experiences. They gave me various kinds of information about students' stated upper secondary school experiences concerning proof. The statements were assigned five different values when coded in SPSS software.

10. This part of the questionnaire consisted of 30 statements with a five-point scale from fully disagree to fully agree. Some of the statements (1, 2, 5, 10, 11, 12 and 28) were adapted from Almeida's (2000) study on students' perception of proof. The statements 3, 8, 16, 21 and 29 addressed students' stated upper secondary school experiences. The statements 1, 2, 4, 5, 6, 7, 9, 10, 11, 12, 13, 18, 19, 20, 22, 23, 24, 25, 26, 27, 28, 30 addressed how students related to proof, including their views and feelings. The rest of the statements, 14, 15 and 17 were background questions. I coded the statements with a five-point scale as ordinal variables with five categories. In the correlation calculation I reversed the values for the statements 3, 5, 11, 12, 20, 22, 24 and 26.

The last part of the questionnaire consisted of personal questions about gender, age, the year when the student finished his/her upper secondary school, the programme in upper secondary school, the marks in mathematics courses, foreign upper secondary school background and studies after upper secondary school. Finally, I asked the students if they wanted to volunteer and be interviewed. If they agreed, they left their contact information.

One disadvantage of using closed questions is a loss of spontaneity in respondents' answers. In the last version of the questionnaire, which I used in

January 2004, I added some lines after the multiple choice questions and after the statements on a five-point scale and suggested to the respondents that they write any further comments on those lines. These lines were intended for those respondents who were not able to find a category that they felt applied to them or who wanted to add something they found important concerning an issue. However, there were not many comments in the students' responses. Someone pointed out that it was difficult to remember the upper secondary school experiences because they had had a long break in their mathematical studies.

Reliability and validity

Stability and *internal reliability* are factors involved when considering whether a measurement is reliable.

Stability is the extent to which a research fact can be repeated, given the same circumstances. I had a possibility to check the results of the surveys against the results of the pilot study that had been conducted one year earlier than the surveys. They were quite similar.

In my questionnaire, there were two topics that were measured by several indicators: Students' stated upper secondary school experiences about proof and how positively students' related to proof. When designing questionnaires with multiple-indicator measures, there is a risk that the indicators do not relate to the same thing and thus, lack coherence (Bryman, 2001). So we need to be sure that all our indicators are related to each other (*internal reliability*). For this purpose, I conducted *Cronbach's alpha* test for the two issues I was studying. The results for both issues were over 0.80, which implies an acceptable level of internal reliability.

Validity is the extent to which a research fact of finding is what it is claimed to be. For the validity of the questionnaire, I sometimes had pairs of indicators measuring almost the same aspects in order to be able to calculate the correlation between them and in that way investigate if students had understood the questions. These pairs were the following: 5-10:16; 6-10:21; 9-10:8; 10:10-10:27; 19-10:22; 10:20-10:6 (Appendix 2). The correlations (with Spearman's rho, on 0.01 significance level) between these pairs were between 0.47 and 0.73 depending on how similar the questions were. Also the focus group interviews offered me a possibility to check that the students had interpreted the questions in a way coherent to my aims, because three of the focus groups had responded to my questionnaire.

Distribution and the sample

I personally distributed the questionnaires to the students during their first lecture of their first term in August 2003 and in January 2004. I handed them out at the end of the lectures and collected them on the same occasion. Thus, the sample was a convenience sample (Cohen et al., 2000). At the beginning of the term most of the students, especially those who seriously want to in-

vest themselves in mathematical studies, usually participate in the lectures. However, it is difficult to speculate how the choice of the sample influenced the results. The population in the surveys in 2003 and 2004 was about 340 university entrants who would study the ordinary university courses in mathematics and the sample who responded to the questionnaires was 168 students. Twelve of them had a foreign upper secondary school background.

Data analysis

I analysed the results of the last two surveys together with SPSS- software using descriptive statistics. I calculated percentages and tested the correlations using a two-tailed Spearman's rho test.

Besides between the similar pairs of questions for the validity, I also tested the correlation between other statements and questions to discover relations between different items concerning students' backgrounds and how they related to proof, including their feelings and views. There were also some background questions in the questionnaire and I wanted to check if they correlated to students' statements about the two issues I was studying.

Students' declared upper secondary school experiences about proof

The multiple choice questions (5-9) and the statements 3, 8, 16, 21 and 29 addressed students' stated upper secondary school experiences. When analysing this part of the study I separated the students with foreign upper secondary school backgrounds from students with a Swedish upper secondary school background. Of course it is impossible to draw certain conclusions concerning upper secondary school teaching from students' responses to the statements. But together with textbook studies and the focus group interviews they gave a more varied picture about how students had experienced proof in their mathematical studies. Concerning students' stated upper secondary school experiences there is a natural scale between very little experience and a lot of experience of different kinds of activities regarding proof. The internal reliability between the indicators of students' stated upper secondary school experiences was 0.89 (*Cronbach's alpha*).

How students related to proof

The questions 2, 3 and 4 and the statements 10: 1, 2, 4, 5, 6, 7, 9, 10, 11, 12, 13, 15, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 28, 30 (Appendix 2) addressed how students related to proof, including their views on proof. It is difficult to draw a strict distinction between views and how one relates to something. The students' views on proof involved a wide range of different aspects, for example whether they viewed proof as an explanation or conviction. These kinds of views are not measurable in a way the other issues in the questionnaire were (how much experience the students had about proof and how positively they related to proof). The parts in the questionnaire addressing these kinds of views were the open question (3) "What do you think is char-

acteristic of a correct proof?", the multiple choice question (4) about what kind of proof students would choose for a certain statement and, finally, the statements 1, 4, 24, 28 and 30. Hence, these items were left outside the calculation for internal reliability. Internal reliability for the rest of the statements measuring how *positively* students' related to proof was 0.88 (*Cronbach's alpha*).

Most of the aspects concerning students' views on proof included only one indicator. I compensated these aspects with information about students' views obtained from qualitative data from focus group interviews with students.

Critical considerations of the method

The goal of the surveys was to get some background information about students' declared experiences and how they related to proof at the beginning of their university studies. Using a questionnaire makes it possible to gather quite crude data which often need to be complemented with other kinds of research. Most of the questions that I used in the questionnaire were closed and easy to analyse but there was a risk that they left out some important aspects. There are other problems, too. We have no way of making sure, whether the respondents were telling the truth. I hope that a face-to-face delivery and a brief personal presentation of my study have encouraged the students to respond honestly to the questions. However, with retrospective questions, a wide range of life variables and events may have been difficult for the respondents to recall (Gorard, 2001) and that is something I had to take into account when interpreting the data. I combined the results of the surveys with focus group interviews with university students.[12] I had the possibility of meeting three groups that had responded to the questionnaire. I chose these groups according to how they had responded to the questionnaire in order to get a varied picture of the students' experiences, views and feelings.

Ethical aspects

In surveys, anonymity helps to protect a person's privacy. I personally distributed the questionnaires to the students and tried to clearly present myself and the aims of my study both orally and at the beginning of the questionnaire. No one was forced to respond in any way. In such a face-to-face delivery, the students had the possibility of asking me questions about the questionnaire and about my study.

[12] I also conducted a textbook study about how proof was dealt with in upper secondary school textbooks (Nordström & Löfwall, 2005) as well as a pilot survey among upper secondary school teachers about their views and intentions regarding proof, in order to get a varied picture about students backgrounds. I have not included these studies in the thesis.

3.3.2 Interviews with mathematicians

The individual interviews with mathematicians, which took place in 2003-2004, were semi-structured and lasted from 30 minutes to two hours. The sample was 13 mathematicians at a department with about 40 mathematicians. I tried to choose both mathematicians who were engaged in the basic course as well as mathematicians engaged in intermediate and advanced courses at the time of the interviews. This was because I aimed to gather focus groups of students from all levels and observe lectures on different levels. The first two interviews were not tape-recorded but I carefully took notes. The rest of the interviews were tape-recorded. I invited the mathematicians to reflect on the items presented below but they could talk quite freely about other issues as well during our discussions. The items I focused on were the following:

- The teaching experience, the current course
- Changes in the contents of undergraduate courses concerning proof
- Changes in students' prior knowledge concerning proof
- How do students meet proof in their lectures and lessons
- Why should students learn proof
- How do we/students/pupils learn proof
- How students relate to proof
- Discussions about proof or proof techniques
- Students' own investigations
- Further issues

The focus in the interviews was not on the role of proof in mathematicians' own research but in the teaching and exercising of mathematics. It is also important to point out that my aim was not to categorise mathematicians but their utterances. I considered mathematicians' utterances as representative of various views in the community, views that were influenced by the social, cultural and historical context of the practice. I considered semi-structured interviews as the best method to gain insights in mathematicians' views and intentions. Kvale (1996) points out that there is always definite asymmetry of power in a research interview. The interviewer defines the situation, introduces the topics and steers the course of the interview. However, mathematicians were more familiar with proof and the teaching of proof than I was as a doctoral student, so I thought they would feel quite free to express their thoughts and ideas about proof and the teaching and learning of proof.

Data analysis

At the beginning of the data analysis, I considered the following three aspects of proof obtained from literature about mathematics education research: *conviction/explanation*, *inductive/deductive approaches* and *formality, level of rigour and language*. During the time of the interviews and the

pilot study with mathematicians, I started to analyse the first interviews with students. I found the metaphor of *transparency* in relation to the teaching of proof and to students' access to proof appealing to examine and I added it to the conceptual frame about the aspects of proof (see Section 2.3). I tested the frame in a pilot study about five mathematicians' views on proof and the teaching and learning of proof (Nordström, 2004). I included these five interviews in the global study.

I used NVivo software for the qualitative analysis in the following way. The interviews were transcribed and imported to NVivo. The last interview was not completely transcribed but I listened to it and identified the topics that added something new to my research. In some of the interview transcripts I first identified the topics that were significant for my subject of study and left aside the items where mathematicians talked about subjects which were irrelevant for my study, before importing the transcripts to NVivo.

I first created free nodes (i.e., labels like "Conviction/Explanation") representing the different aspects of proof in the conceptual frame and some free nodes for topics regarding mathematicians' pedagogical perspectives (like "mathematicians' intentions"). After the first coding, I studied the transcripts once again to find other themes emerging from the data and created new free nodes. For example the functions of *Aesthetic, Systematisation, Intellectual challenge* and *Transfer* were dealt with by mathematicians and I included them in the conceptual frame (see p. 62). During the data analysis, I continuously examined the relations between the different nodes and organised them into trees (Appendix 6). The pilot study with five mathematicians helped me to see these relations and hierarchies and thus, influenced the theoretical frame of my study. The a priori categories which I had created for the conceptual frame at the time of the pilot study proved to be relevant for the analysis of the interviews and I complemented the frame by the functions of proof mentioned above.

In parallel to the pilot study I continued the work with the theoretical frame. I analysed the data obtained from all the interviews with the mathematicians and related it to the conceptual frame (see Section 2.3) and to new aspects that had emerged in the pilot study. In parallel to the data analysis of the interviews with the mathematicians, I also studied curricula and statistics about the changes in the courses and the organisation of the teaching in the mathematical practice and interviewed Peter Strömbeck (director of studies) and Jan Johansson (head administrator) about these changes (see Section 3.3.4). I then related the ways in which mathematicians talked about the changes to the data I had obtained from these complementary sources and found different styles in how mathematicians related to these changes. These styles were also connected to the three main pedagogical styles (described below) that were constructed from the data. I also formulated analytical statements concerning the mathematicians' views on proof and how they

related to the changes in the practice and how they talked about their pedagogical intentions and checked them against the data.

The development of a theoretical model

Three main pedagogical approaches could be constructed from the data. As early as in the pilot study (Nordström, 2004), I had noticed a difference between the mathematicians who stated that they had an intention of teaching proof and the mathematicians who stated that they avoided proof for a variety of reasons. I divided the reasons for not having the intention of dealing with proof into *internal* and *external* reasons. Internal reasons refer to mathematicians' own pedagogical choices to avoid proof, for example if they state that students do not need to learn proof yet. External reasons refer to circumstances like the lack of time or the lack of students' prior knowledge, as reasons for not intending to deal with proof in the teaching. I started to look at the data from that point of view and noticed that those who stated that they wanted to avoid proof often had the same kinds of views about students and often related to the aspects in the conceptual frame (see Section 2.3) in a similar way. This was also the case for those who stated that they had intention of dealing with proof. This was the starting point for the constructing of three different positions.

After analysing all the interviews with the mathematicians, I set up a table about the three different styles of how to approach proof in the teaching (Appendix 5). I decided to call the first style *progressive* or *I don't want to foist the proofs on them*, the second style *deductive* or *It is high time for students to see real mathematics* and the third style *classical* or *I can't help giving some nice proofs*. It is difficult to choose proper labels. Next, I very briefly explain, why these labels were chosen. The label "progressive" was chosen because, in the utterances categorised into the first style, there could be discerned a tendency towards/willingness to reform the educational practice in various ways. It is an approach normally contrasted to the "traditional" one (Edwards & Mercer, 1987). There are also features of constructivism (learning theory) visible in the utterances categorised into the progressive style (see p. 209). The label "deductive" was chosen because in the utterances categorised into the second style, the deductive character of mathematics was often visible. The label "classical" refers to a view on mathematics as a "fine art", to a style of a professional mathematician who enjoys the beauty of mathematics and proofs.

Hence, I had got three positions. I checked the positions which I had defined so far against the data. There were mathematicians who often expressed views belonging to one of these positions and there were mathematicians whose utterances and views could be characterised as a mixture of them. Hence, it was impossible to map each mathematician into one of these positions and as I mentioned above, that was never my aim. Teaching styles and intentions of one person can also vary from day to day and from one

moment to another. However, the categories were helpful in organising and giving structure to the results. So, I did not try to categorise every mathematician's way of talking exactly into one position but instead considered and developed the styles as ideal types, as a theoretical model. I then analysed the utterances using this model. Sometimes also utterances contain features from various styles. There is not a spectrum between these categories and in certain aspects they are overlapping. However, depending on different criteria, for example Intuition/Formality, expressions of greater or lesser intensity could be discerned. In Section 4.4, I present this idealised theoretical model and exemplify the three styles with utterances from the data.

Hence, the main criteria for the different categories were the pedagogical intentions, the views on students and the relation to the aspects in the conceptual frame (see Section 2.3, p. 61).

Ethical aspects

There were some ethical aspects I had to consider when proceeding with the interviews and when reporting the results. Firstly, I tried to clearly present the aims of my study to the mathematicians who I interviewed and tell them how their contribution was going to be dealt with.

Secondly, it was important when reporting the results, to do it in a way that would protect the anonymity of the persons in the study. On occasion I had to omit facts that might have been enlightening for the case. After the decision to limit the study to just one university, the issue of anonymity became even more important. To protect the mathematicians' anonymity, I took the following decisions:

- I neutralised the mathematicians' language. I do not give the authentic examples of mathematicians' utterances. Some of the mathematicians came from other countries than Sweden and it could be possible to recognise the persons from their ways of expressing themselves. The same could be said for some of the Swedish mathematicians, some of whom had certain characteristic ways of expressing themselves. Therefore, I omitted such traits from their remarks.
- I do not reveal the mathematicians' gender. This is because there are few female mathematicians, and in case I had interviewed some of them it would have been easy to identify them.
- I do not reveal from which country each mathematician comes from.
- When describing the ideal types, I combined quotations from different persons. I only labelled the quotations with M (mathematician) when needed, to distinguish them from students'. This was because I did not want the individuals to become visible in the presentation. The most important thing for the results was what and how the mathematicians as representatives of the mathematical practice, not as individuals talked about

their practice and what kind of approaches there could be discerned among the utterances.

Critical considerations of the method

There were differences on how much and how deeply mathematicians talked about pedagogical issues. It would have been preferable to organise focus group interviews also with mathematicians after the individual interviews in order to stimulate reflections between different mathematicians. Some of the interviews offered very little data about mathematicians' pedagogical concerns.

3.3.3 The focus group interviews

I organised focus group interviews during 2004 among students who studied the ordinary courses in mathematics in different phases of their studies. The interviews were semi-structured according to the items I had piloted with one student. I invited the students to have lunch together before their lectures and tried to create an informal atmosphere and be in the background during the discussions as much as possible. Of course, I had to intervene now and then in order to focus on the items described below. The interviews lasted from one to two hours and were tape recorded.

The aim of the focus group interviews with students was, partly, to complement the results from the surveys about the students' stated experiences, views and feelings with qualitative data that would help me to give a richer description about the students' background, and how they related to proof. But the focus group interviews would also give information about students' experiences about proof during the university courses in different phases of their studies. I started with a pilot study with one student studying the continuous courses in order to test the relevance of the items I planned to introduce to the groups and the theoretical frame. The items I focused on in the interviews were:

- Students' upper secondary school experiences concerning proof
- Students' responses to the questionnaire (if they had responded to it)
- Students' university experiences concerning proof
- Items from observed lectures
- Further thoughts

Because of a possible power asymmetry in the interview situation there is a danger that the students answer the questions in a way they feel they are expected to do, especially when the interviewer is a doctoral student (Kvale, 1996). That is why I chose to use focus group interviews instead of individual interviews with the students. Furthermore, focus group discussions offer data about how students talk about their experiences with each other and

there is another kind of spontaneity in the utterances than in the ordinary interviews.

The sample

Six focus group interviews were conducted during 2004 among students who studied the ordinary courses in mathematics. In each group, there were three to five participants. Students with both Swedish and foreign upper secondary school backgrounds were represented in the focus groups.

- Two of the groups had recently started to study mathematics and were taking their first courses. They had also responded to the survey questionnaire and I had chosen them according to their responses. I tried to get into touch with students with different kinds of experiences and relationship to proof. I contacted the students whom I wanted to interview and who had left their contact information in the questionnaire, by e-mail or via telephone.
- Two of the focus groups were studying intermediate courses. One of these groups had responded to the survey one year earlier. I contacted that group via e-mail and succeeded in gathering three students to discuss the items above and reflect on their responses to the questionnaire they had responded to one year earlier, at the very beginning of their studies. The other focus group studying intermediate courses was brought together after observations of lectures where I presented my study to students and asked them to volunteer by participating in a focus group discussion.
- Two focus groups studying advanced courses were gathered together after observations of lectures in a similar way as the previous one.
- After the data analysis I gathered together a group of doctoral students in June 2005, to reflect on their experiences in relation to the results of the data analysis of the focus group interviews with the other students.

Data analysis

The interviews of the six focus groups in 2004 were transcribed and imported to NVivo- software. I analysed the interviews in Swedish and first used the free nodes that were created during the analysis of the pilot interview. I included the pilot study in the whole sample. I then tried to read the data afresh and created new free nodes. Afterwards, I organised the free nodes into trees according to how the items were connected to each other and the earlier items. I formulated analytical statements and, finally, in June 2005, gathered a group of doctoral students and confronted them with the results of the qualitative data analysis. Their experiences in their mathematical practice confirmed the main parts of the results.

In parallel to the data analysis, I went on developing the theoretical frame for the thesis. The first data analysis of the interviews with students influenced the focus on the research questions as well as the development of the theoretical frame. For example, the metaphor of transparency and students'

access to proof turned out to be one of the central issues for my thesis. However, the deepening of the theory also provided new insights for the data analysis. Hence, at last, I analysed the data against Wenger's (1998) theory of changing identities and the location in the practice. I identified and analysed the utterances expressing participation and non-participation concerning proof in the mathematical practice. These utterances often concerned how the meaning of proof was experienced in the practice.

Ethical aspects

The ethical aspects I had to consider when proceeding with the student interviews and the ways in which the results were recorded were quite similar to those used when dealing with the mathematicians (see p. 83). I tried to clearly present the aims of my study to the students I interviewed and tell them how their contribution was going to be dealt with. Also, to protect the anonymity of the students who volunteered in the focus groups, when reporting the results, I took the following decisions:

- I only gave information about the level on which the student studied at the time of the interview.
- In the dialogues, I used letters to distinguish different persons.
- I did not reveal either the gender or what country the student came from.

Critical considerations of the method

I wanted to get a rich picture about students' views and experiences. There is a risk that students who did not participate in the focus groups had views that were not expressed by those who participated in the interviews. Also, the fact that the students could freely discuss the subject made the data analysis time consuming.

3.3.4 Gathering of the complementary data

As described before, I gathered some complementary data in order to triangulate the data and give a varied picture of the issue. I conducted **observations of lectures** (about two lectures of almost all the mathematicians I interviewed) during the same period I organised the focus group interviews with the students, in 2004. In that way I could relate the students' utterances to the lectures I had observed.

Bryman (2001) defines *structured observation* or *systematic observations* as a technique in which the researcher employs explicitly formulated rules for the observation and recording of behaviour. It is a method that works best when accompanied with other methods because it can rarely provide reasons for observed patterns of behaviour. In my observations of lectures, I paid attention to the different ways mathematics was presented. I also checked how the proofs were presented and how the students were stimulated. However, I did not use a standardised observation instrument so I can-

not call my observations purely *structured* or *systematic* observations. Yet, there were features of systematic observations because I focused on a special issue, proof. They were *simple observations* because I had no influence over the situation being observed. An interesting issue is if the observations would be called *participant* or *non-participant observations*? I was not a student but in a way I was in the same situation as they were because I was listening to the lectures and at the same time making the notes. So I could draw certain conclusions of how the students might experience the mathematicians' behaviour and the lectures. At the same time, I also observed the students' behaviour, which gave me insights in how mathematicians experienced the students in the lectures. Hence, I got some complementary data about the issues the students in the focus groups and the mathematicians in the interviews talked about. Here, this kind of triangulation helped me to give a richer picture of the treatment of proof and the students' access to it. The field notes from the observations of the lectures serve as complementary data and helped me triangulate a part of the data. They also offered items to the focus group discussions.

The second source of complementary data was the interviews with experts, Matts Håstad (secretary of a Nordic Committee for the Modernising of School Mathematics in the 60's) and Barbro Grevholm (professor in didactics of mathematics) about changes in the school curriculum and Peter Strömbeck (director of studies) and Jan Johansson (head administrator) about the changes in the curriculum, organisation of teaching and statistics at the department that I am studying. I wanted to gain insights into the historical background of the current situation and used the interviewees' utterances as oral references in my study. So the interviewees were not anonymous. According to Bryman (2001) the term *focused interview* refers to an interview using predominantly open questions to ask interviewees questions about a specific situation or event that is relevant to them and of interest to the researcher.

The third source of complementary data was documents like textbooks, curricula and statistics. I explored the issues in the university textbooks, extra material and examinations that mathematicians or students talked about. I also studied the official documents and statistics about the changes in the organisation of the teaching and changes in the course literature.

Some ethical remarks: I asked the mathematicians in advance if I could observe their lectures. I made clear that I was observing proof and how proof was dealt with in the lectures and I emphasised that I was not there to judge their teaching competence. I also tried to clearly present the aims of my study to those whose lectures I observed as well as to those who I interviewed and told them how their contribution was going to be dealt with.

Critical considerations

In the observations of the lectures, it is possible that my presence in the classroom somehow influenced the lecturer. One problem with focused interviews is the memory of the interviewees because my questions concerned events during the last four decades. Thus, I combined the interviews with document analyses. Here my aim was to gain insights into the historical events which were significant for the current situation in the mathematical practice concerning the treatment of proof.

3.4 A summary

To sum up this chapter, I first presented the general and the specific research questions and provided the reader with a design of my study. In the second section, I discussed different research paradigms in relation to my study and how they related to my choice of research methods and ways of analysing and interpreting the data. In the third section, I gave a detailed account of the different methods, as well as the different procedures for the data analyses applied in the study. I included ethical considerations in the description of the methods and explained how they had influenced the way in which I reported the results.

4 Mathematicians' practice

I begin the report of the results with the mathematicians – the old-timers in the practice. The structure of the chapter is as follows:

In the first two sections, I describe how the mathematicians in my study talked about proof and its significance to their practice and how they dealt with various functions of proof. In the third section, I deal with the changes that the mathematical practice has undergone during the last decades regarding the treatment of proof in the basic course and how the mathematicians related to these changes. In the last section, I describe the three different styles I created from the interview data, concerning mathematicians' pedagogical perspectives and intentions.

The results reported in this chapter are based mostly on the data analysis of the interviews with mathematicians. The focus in the interviews was not on the role of proof in mathematicians' own research but in the teaching and exercising of mathematics. Not all of the mathematicians talked about all the aspects that I have dealt with in the theoretical frame; different aspects dominated their talk in various ways.

4.1 The soul of mathematics

Very clearly, the mathematicians in my study considered proof as an essential part of mathematics. All of them showed an appreciation of proof in different ways. However, they talked about proof in slightly different manners and stressed different aspects of it. The mathematicians in my study had various backgrounds which might have influenced their ways of viewing proof. Some of them, for example, came from other countries than Sweden and had their school backgrounds in these countries.

The centrality of proof in mathematical practice was obvious in all of the interviews. The following quotations can be given as characteristic and representative of this view.

> *"I suppose that proof is a fundamental idea of mathematics..."; "But that is the soul of mathematics."*

Proof is something that most of the mathematicians consider as *real* **mathematics** in contrast to upper secondary school mathematics that is often asso-

ciated with the learning of rules without understanding. This is something that infiltrates the mathematicians' talk in various ways and can be recognised in a lot of the utterances:

> *"…There is always someone who is truly interested in mathematics and then the proofs become important."*

There is also often a feeling of appreciation and admiration in the mathematicians' way of speaking about proof.

> *"…you have to learn to understand and appreciate what you might call the triumph of logical thinking of human beings and the ability to draw conclusions."*

One of the mathematicians very clearly declared his view of mathematics as different from many other sciences because it is built up from axioms in a unique way.

> *"… you cannot go through a mathematical education without experiencing this feeling at least once, otherwise you might as well study theology, philosophy, nothing against them, or politics if you want, there are so many subjects where everyone can have an opinion and argue, and in a way, everyone is right as long as they put their feet down, and present arguments that others accept. But in mathematics there is no law saying that the strongest wins but the one who never makes any mistakes is right, after having been confronted with all possible counter arguments following the axiomatic method, and still…. To live so you don't rely upon a vague statement but have a solid ground."*

Some of the mathematicians also expressed the idea that proof actually exists in all mathematics.

> *"…as I myself learned already at an early stage of my education in* […]*, proof is a natural part of mathematical studies, it was impossible to distinguish the solving of problems and proving, but they quite simply come together."*

What the mathematician states here, is that the idea of solving problems and proof comes together. This is an important aspect for mathematics education and has to do with the aspect of Conviction/Explanation. In the presentation of the conceptual frame, I pointed out that conviction could also be viewed as an important aim of all mathematical activities even in school mathematics, if we hold the view that proof permeates the whole of mathematics, as a tool for justifying every step in our solutions (see p. 45).

4.2 Proof as an artefact

In Section 2.2.5, I argued that proof can be seen as an artefact in mathematical practice. Mathematicians talked about communication and systematisation (see p. 93) and about other important functions of proof in their practice. They also talked about proof as a tool for deriving formulas and checking the correctness of statements (see p. 92). Hence, the data also supports the idea of considering proof as an artefact in mathematical practice.

Next, I will describe the way in which the mathematicians talked about various functions (see p. 62) of this artefact in their practice. I provide the reader with some quotations to illustrate their ways of talking about each function.

Conviction

In connection with the deductive character of mathematics, some mathematicians spoke about *conviction*. No one questioned the value of proof as a means for validating mathematical knowledge (function of verification p. 21) and some of the mathematicians stated it explicitly:

> *" I suppose proof is a fundamental idea in mathematics… that we can arrive at various results, to build on definitions, which sometimes come from applications and then they are not completely meaningless. Then we start from them and derive new results and there we need proof. In mathematics you can't be convinced and continue without a proof. It is the very proof that leads to conviction."*

One of the mathematicians expressed the view that proof was not needed in *teaching* in order to convince students since students were already convinced.

> *"Not for conviction, they are already convinced."*

This view is similar to that of de Villiers (1990) and Bell (1976) who argue that students' conviction in mathematics is often obtained by quite other means than that of following a logical proof.

Proof for mathematicians seemed to be connected to a kind of critical thinking, questioning and checking the "evident".

> *"When one absorbs the critical way of thinking and reasoning in an early phase one will never be fooled to accept things or statements without checking them."*

Hence, in real mathematics we have to be critical and proceed in a deductive manner so we can become convinced about the results. Proof gives us confidence because we can be convinced that our reasoning is correct and that we

have arrived at true results. This can be compared to Selden and Selden's (1995) description about validation, a kind of critical checking of proofs by mathematicians (see p. 45).

One of the mathematicians talked explicitly about proof as a tool for deriving new results in mathematics.

> *"… this ability to make conclusions is as a tool, so if we meet something that we do not really know if it is true, if we have worked and thought in this way, then we have this toolbox, this fundamental, these rules, or the theorems we have got by thinking in this way, then we can take them and look if we can derive also this statement, this is that we have got a tool."*

The quotation above is also an example of considering proof as a tool for checking the correctness of statements.

Explanation

Conviction was also connected to the *explanation* proofs could give. This supports the idea described in the conceptual frame (see p. 43) that conviction and explanation in mathematical practice are intertwined.

> *"It's the same thing here; it would be strange to believe all your life in something if you don't get an explanation why it's true."*

The aspect of explanation that proof would provide was present in mathematicians' utterances in different ways. Proof would help to clarify:

- mathematical constructions
- mathematical structures
- relations between different concepts in terms of connections or hierarchies

> *"Learning of proof enhances conceptual understanding…to see how mathematics is constructed, how things are connected with each other."; "Gives insight how mathematics works…gives understanding for the hierarchies between different concepts like continuity and differentiability."*

Some of the mathematicians, however, pointed out that not all the proofs enhanced understanding. This was also exemplified in the conceptual frame (see p. 44).

> *"…there are actually proofs that give understanding why it is like this, and can give structure for the minor parts, that this really is something universal that is valid for all cases, I think that can offer something but far from all proofs do that. There are even proofs that leave mathematicians in a kind of dissatisfaction: yes, yes, yes, I understand this but why is it like this?"*

Understanding is a personal experience and depends on a person's earlier experiences. Hence, proof is not always an explanation that enhances understanding of mathematics, not even for mathematicians.

Communication, aesthetic and intellectual challenge

One mathematician talked about proof as a means of *communication*. But communication was restricted to that between mathematicians.

> *"It's important to learn proof because it's the language all mathematicians can communicate with."*

When mathematicians presented mathematics to students, it was sometimes possible to view the presentation as communication through proof. Mathematics was often presented in a manner, where all the steps were made visible and justified.

Some mathematicians talked also about *aesthetics* in connection to proof.

> *"Proofs can be beautiful."; "Calculus contents very classical material and the proofs are beautiful."*

One of the mathematicians talked about proof as an *intellectual challenge* in contrast to calculating with specific numbers and doing sums/arithmetic.

> *"And if they* (students) *only calculate with numbers it's not a big intellectual feat..."*

Systematising mathematical knowledge

According to the mathematicians, proof also rationalises mathematics because we do not need to prove every single case any more if something general has been proved.

> *"…if one proves once for all that every polynomial and every trigonometric function is differentiable, one understands the point of proof, because if one constructs such a proof there is no need to prove the concrete examples any more and the life becomes easier…"*

Proof was also seen to be a tool for systematising mathematical knowledge so we do not need to memorise everything.

> *"It's hardly possible to learn everything by memorising it, it's easy to get it all muddled up whereas if one tries to get a system of it all, things come together…"*

Hence, proof is seen to be something opposite to the learning of rules or memorising formulas.

"To come away from cookery book thinking."

The notions like "cookery book thinking" (kokbokstänkande) and "recipes" were usual when the mathematicians talked about the opposite to proof.

Transfer

Transfer is a function of proof that mathematicians in my study touched on when talking about the meaning of the learning of proof. Transfer refers to two basically different things (see p. 61).

Firstly, proof teaches us the logical thinking that is needed in other contexts outside mathematics.

> *"It's simply an exercise in formal reasoning that is more or less useful regardless of what we do when it comes to more theoretical issues. I mean even if you study other things I believe it's useful with formal training to construct things logically, to express yourself logically."*

Hence, mathematicians talked about the benefits of learning proof because it was exercise in formal reasoning and, therefore, also useful for all of us also in other than mathematical contexts where logical reasoning was needed. It was seen to be especially important for programming but desirable also in other branches.

> *"Mathematical logic and algorithms and programming, mathematical proof is connected to algorithmising."*

The second meaning to which I refer with transfer is the usefulness of proof techniques themselves in other mathematical contexts. This was also stressed as a reason for learning proof by one mathematician. Proofs could also offer useful techniques and structures that could be applied in other mathematical situations and could help to obtain new mathematical knowledge. In connection to the basic course, the derivation of the formula for second degree equations was mentioned as an example about such proofs.

> *"To take something that still occurs at the lower level, solving of second degree equations and say that this is the reason for why pq-formula or things like that work. And you have to learn the technique because there are situations where you need to do it in this way, where it does not work to apply the formulas you have learned in upper secondary school…"*

The mathematician quoted above points out that learning of the derivation of the formula for the solving of a second degree equation gives techniques that can be used in other mathematical situations. Indeed, there are many problems where it is useful to be able to complete the square.

I will come back to this function in Section 7.3.

When talking about the reasons for why the students should learn proof the mathematicians also expressed their own views on proof as an essential part of mathematics.

A means to come to grips with the essence of mathematics

In a more abstract manner, mathematicians talked about proof as a means to come to grips with the essence (väsen) of mathematics.

> *"…to understand the essence of mathematics."; "Gives insight into the essence of mathematics."*

This is interesting because mathematicians, by these utterances, convey the view that the understanding of proof enhances the understanding of mathematics itself. They view proof as a tool by which to get insights about what mathematics is about. The learning of proof enhances access to mathematics. The meaning of proof in education would be then, to help give the students an insight into the essence of mathematics. At the same time, proof is regarded as a fundamental idea of mathematics itself, the soul of mathematics (see p. 89).

Hence, to sum up the mathematicians' views, proof is seen to be essential in mathematical practice. The view of proof as an artefact found support in the data in a sense that mathematicians considered proof as a tool for various functions. The following functions of proof were identified in mathematicians' utterances:

- Proof also gives conviction about the truth of mathematical statements and allows the mathematicians to proceed and investigate new theories.
- Proof explains and clarifies mathematical connections, hierarchies and relations between different notions.
- Proof is a means of communication and gives intellectual challenge and aesthetic experience.
- Proof is a tool for deriving results in mathematics so one does not need to memorise everything.
- Proofs give the general results that can be applied in other contexts in mathematics.
- Proofs can give techniques that can be used in other mathematical contexts (transfer).
- Proof also teaches us logical thinking that is needed in other contexts than mathematical practice (transfer).

In Section 1.2, I described the changes in the curriculum regarding the role of proof in the community of mathematical practice that is the focus of my

study. In the next section, I describe how the mathematicians in my study related to these changes.

4.3 Changes in the curriculum/changes in the newcomers?

Changes take place in communities of practice all the time and people relate themselves differently to them. In this section, I describe how mathematicians talked about some of the main changes in the basic course during the last decades regarding the treatment of proof. I base the section on analysis of the interview transcripts, a textbook review and documents and interviews with experts about curriculum changes.

Some of the changes in the curriculum are reified in forms of official documents (see Section 1.2). New courses have been introduced and others have vanished. The character of the examinations has changed, for example earlier in the 80's there was a problem solving part and a theory part in the examination for the basic calculus courses. Textbooks also reveal differences in teaching and learning styles in the practice. Computers have impacted on the methods and the possibilities to obtain new results and so on. But, obviously, there are also changes the character of which can be hard to reveal because "constant change is so much a day-to-day engagement in practice that it largely goes unnoticed." (Wenger, 1998, p. 94) Even if the mathematicians' views on and response to the changes vary, from one person to the next and, to some extent, from one day to the next their responses to the changing conditions are interconnected because they are engaged together in the joint enterprise of enhancing mathematical learning. I start the section by describing how the mathematicians who I interviewed talked about the changes and the reasons for the changes in the curriculum concerning the treatment of proof.

4.3.1 How did the mathematicians talk about the changes in the curriculum?

All the mathematicians who I interviewed had been in practice several years but had, of course, different kinds of experiences depending on what courses they had taught during those years. Some of them were more familiar with the basic course than the others. However, all of them agreed that some changes had been made in the contents regarding the status of proof in the courses for the first 20 study points during the time they had been working at the department, even if those who had worked there for a shorter time, were not so sure about these changes. To my question about possible changes in the treatment of proof one of them answered in the following way:

"Devalued constantly even if we still try to maintain a certain level but it has been postponed. Earlier we had a theory part in the examinations of Analysis 1 and 2 that were the same course then… We try to motivate some simpler theoretical things such as theorems about continuous functions and some other theorems where we only tell the students that a proof exists but we do not go through them."

However, some of the mathematicians pointed out that the changes regarding the treatment of proof concerned only the lower level courses.

"If you go up to the courses we call D-level or doctoral courses, I do not think there is a crucial difference."

That is something that can also be seen when examining the examinations. The basic course seems to have changed more than intermediate and advanced courses.

Reasons for the changes

The mathematicians had different views as to why the changes at the lower level regarding the treatment of proof had taken place. They mentioned students' lacking of prior knowledge regarding proof, students' bad calculation skills, students' lacking of interest, new course literature and economical aspects as well as changes in the examinations as reasons for the changes in the status of proof in teaching of mathematics to the undergraduates.

1. Students' prior knowledge about proof

The most usual explanation was that the students who started to study at the university had little experience about proof from upper secondary school and thus, it was impossible to deal so much with proof in the basic course.

"Elements of proof in upper secondary school and in basic courses at the university have diminished, it's perfectly obvious. We have to adjust to the fact that the students usually have almost no experience when they come here."

In many of the utterances the dissatisfaction with school mathematics was obvious. It was seen as rule learning (superficial) contrary to real mathematics (proof). I will come back to this standpoint in the next section when describing the mathematicians' pedagogical perspectives.

"It's natural because school mathematics has become more and more superficial. The students who come to us have usually no experience about proof."

Some of the mathematicians also referred to the fact that more students now come to university than earlier and that was one of the reasons for lower

standards (see p. 15). Some of the mathematicians also talked about students' lack of maturity as a reason for the changes.

> *"This successive postponing of proof to higher courses has been a consequence of the fact that we noticed that it did not function, the students couldn't learn the proofs. They were not mature enough for it so early."*

There were some mathematicians who were more careful in their judgement about changes in students' prior knowledge about proof and were aware of the fact that they might have been influenced by the common opinion in the community. To my question about possible changes in students' prior knowledge about proof, one of the mathematicians answered in the following way:

> *"Regarding that question I think all agree and that is why it is difficult to say if it is true because you from the beginning are filled with preconceptions…But naturally my impression is also that the standard is going down. The standard in upper secondary school has obviously declined."*

Finally, some mathematicians stated that proof had always been difficult for students, so that could not be the reason for the changes in the contents.

> *"Same kind of variation as in climate, there are better and worse years."*

However, there was a clear dissatisfaction visible in the mathematicians' utterances concerning students' school experiences. For example the lack of geometry studies was mentioned by many.

2. Lack of time

The introductory course was introduced in 2000 because students had difficulties for example with elementary algebra and the manipulations of fractions. At the same time contents in the basic course changed, for example a course in Euclidean geometry in *Algebra and geometry 2* disappeared (see p. 15). That is something some of the mathematicians reflected on and pointed out as a reason for why there was no space for discussion about proof or proof techniques any more.

> *"We used to have more discussions about proof and proof techniques earlier. I think it was among other things something we had in the course Algebra and geometry 2 where we had Euclidean geometry as well. There was a discussion about proof as a method. We can say that the latest reform we made aimed to improve students' calculation skills and elementary problem solving skills, and all the other things, like why mathematics is needed and how it really works was pushed sort of into the background…"*

Hence, according to some mathematicians, the changes in students' prior knowledge in mathematics in general had led to a reduction in the time allowed for the teaching of proof. Priority was given instead to calculation skills and the time that was previously spent on proving activities, now had to be used for helping students to gain a solid base in their basic calculation skills. Also the organisation of teaching, with less time for the lecturer was mentioned as a reason for not dealing with proof (see p. 16).

3. Students' lack of interest

Several mathematicians stated that students were not interested in proof and in the question "Why?".

> *"Students have worked too much with collections of formulas; they are not interested in the question "Why?". They do not understand what mathematics actually is, that proof somehow exists in all mathematics."*

That was one of the reasons why they did not deal with proof.

> *"I present a lot of theorems without proof because of the lack of time and the lack of students' interest."*

According to these mathematicians students wanted to get their study points and were also used to get a set of formulas in upper secondary school instead of deriving them themselves.

4. New course literature

The courses are often designed in line with the contents in the textbooks even if some other materials are offered besides the textbooks. According to three of the mathematicians the changes of textbooks have also influenced the role of proof in the basic course. However, one could argue that maybe the textbooks were changed because mathematicians did not want to deal with proof in the way it was dealt with earlier.

> *"We used to have another textbook in analysis and I think there were more proofs in it, so my feeling is that the courses are simplified and proofs occur more and more seldom, I believe."*

Vretblad's (1999) textbook was no longer included in the course literature for the basic course after the latest reform in 2000 (see p. 15). Two mathematicians mentioned the disappearance of Vretblad's book as a partial reason for changes, for example they pointed out that there were some meta-level discussions about proof and proof techniques in the book.

> *"There is another question you did not ask, if teaching now contains less proof than earlier and I think it does compared to how it was five six years*

ago. And it's partly because we have changed the course literature and partly because we have adapted to what is seen to be a lower level from upper secondary school."

In Vretblad's textbook, there are some discussions and exercises on proof and proof techniques in Swedish (see p. 59). It was earlier used, not only in the ordinary courses but also in courses for prospective teachers. The literature used in the introductory course is partly the same as in the ordinary calculus courses. Besides, a book with repetition of upper secondary school mathematics (Wallin et al., 1998) is used as course literature (Appendix 1).

5. Economical reasons

Even economical reasons were identified in these discussions as one of the reason for why less time was spent on proof and proving activities:

"…the lower the demands on students the more economical support to the institution…"

The department gets support according to the number of students who have passed the examinations.

6. Examinations

The lack of proving tasks in the examinations that mathematicians set to the students can be traced to the lack of treatment of proof in the lectures and the lessons.

"We now may have some tasks connected to theories but proving tasks are lacking. We cannot give such tasks because we do not deal with them in the lessons."

But some mathematicians also put this the other way around: because there are not many proving tasks in the examinations, students and mathematicians are not interested in dealing with them in the lectures and lessons.

"The exams also rule the contents."

There are not many proving tasks in the examinations for the basic course as a whole (see Section 6.2.3). Earlier, there was a theory part in the examinations for the basic course in calculus (see p. 15). Now the theories and proofs for calculus are demanded for the first time in an oral examination during the intermediate course *Mathematical Analysis 3* (Appendix 1 and 4).

These are the reasons identified in mathematicians' utterances for why less proof was dealt with in the basic course. Mathematicians related to the changes in the role of proof in the basic courses in different ways. Next, I

describe some of the differences between the ways in which they related to the changes.

4.3.2 How did the mathematicians relate to the changes?

There were differences between mathematicians as to whether or not they were satisfied with the development they talked about. Some of the mathematicians hardly saw any problems with the changes in curriculum and stated that not all students needed to learn proof, it was for those who were going to become mathematicians, whereas others would have liked to have more "real mathematics" from the very beginning and regretted the "lower standard" and were concerned about students' possibilities of becoming familiar with proof. These mathematicians often talked about the usefulness of proof and logical reasoning in all contexts, also outside of the mathematical practice. There were different standpoints concerning the benefits of Euclidean geometry in the mathematicians' utterances. The course in Euclidian geometry was introduced in the curriculum in the 70s and excluded from the curriculum when the introductory course was implemented in 2000 (see p. 15).

Euclidean geometry

Many of the mathematicians told stories about nice school memories of working with geometry tasks, and learning deductive thinking through them. They advocated geometry rather than algebra as the first contact with proof.

> *"Geometry is good, it's so easy to get acceptance, geometrical proofs give something, aha, that's why I get this, whereas at the similar level in elementary algebra or in number theory proof is either unconceivable or evident."*

Proofs in elementary algebra were, according to the quotation above, either unconceivable or evident and for that reason not good as the first contact with proof. Yet, many of those who related positively to Euclidean geometry stated that geometry came too late when they dealt with it at the university and complained that it was not dealt with in school mathematics where it would be better suited.

> *"I think it* (Euclidean geometry) *came too late. It would be nice if they could do something nice with Pythagorean Theorem in school. Then they would have some positive experiences when they start to study at university. But the reality is not like that now."*

Some mathematicians stated that students had difficulties with proofs of "evident statements". I find it interesting that students' proving of evident statements is regarded as a problem by both those who criticise geometry as well as those criticising algebra as the first contact with proof.

The following is an example of the reason given by a mathematician for why geometry was not appropriate as the first contact with proof.

> *"Euclidean geometry is not appropriate as the first contact with proof and it's because you prove many statements that seem to be evident and that's a difficulty for many students"*

Further, another reason given by the critics was that in geometry, it is difficult to formalise everything profoundly. Students' capability of judging what must be proved and what could be taken for granted in the domain of geometry was also pointed out as a difficulty.

> *"… to be honest because, anyway, it is completely unthinkable to formalise everything profoundly. You have to a certain point, to a certain level accept intuition but where we put the boundaries is arbitrary, so it's a difficulty for the students to understand why they should prove some evident things while other evident things can be accepted without a proof."*

One has to accept intuition to a degree, and what to take for granted was a convention that mathematicians knew but not the students.

> *"We who have taken part of these courses have some kind of tradition that it's natural to draw the boundary precisely somewhere there, it's actually not evident and it's not strange that the students become confused and wonder: "why shall I show this?", "why can't I take it for granted?"."*

Therefore algebra was to be preferred as a first contact with proof for the students. There was also a view of geometry as something old-fashioned in mathematics.

> *"One turns a little into an old Greek when one works with it* (Euclidian geometry)*."*

Hence, those mathematicians who advocated algebra as the first contact with proof criticised the arbitrariness of what to take for granted in geometry as well as the proving of evident statements, whereas others saw a lot of benefits in the learning of proof in geometry and instead pointed out that in elementary algebra statements that are proved are often evident. There were also those who advocated the use of both geometry and algebra in the teaching of proof. Yet, as it is now, many students never seriously meet geometrical proofs in Sweden because there is only a short course in geometry in upper secondary school mathematics today and the basic university course in geometry was excluded from the curriculum in 2000.

Pedagogical trends

Some of the mathematicians were critical of the way proof was taught earlier even if they stated that the situation today was not satisfactory either. Here we can recognise the ideas of students' lack of feeling meaning of proof and the need for proof (see p. 46). For one of these mathematicians, the recipe was to wait with proofs until the students felt that they needed proof instead of foisting on them arguments that they did not value.

> *"Well, it's been a clear tendency that one should wait with rigorous mathematics. In the 70s all these epsilon-delta tasks were obligatory in all the examinations. To know analysis, one had to be able to use definitions of the limits, that's nothing we demand now. The basic course, in general, demands very little proof. And I think, on the whole, that it's good because I think there was earlier a tendency to prove things before one had understood the point of proof, and before one had this experience and maturity."*

There was also criticism of the way in which proof was dealt with earlier that, according to a mathematician, led to the learning of proof by heart without understanding. The proving of evident statements was also criticised.

> *"… I had a feeling that very many students in the 70s used to learn proof by heart without understanding. I think it is the worst possible method of studying. The proofs were not of the type, that was suited to enhance understanding…one proves many statements that seem evident for students…"*

In Chapter 2, I described the new trends in the teaching of proof (see p. 47). Inspired by for example Lakatos (1976), mathematics educators have advocated explorative activities for students. These activities would be closer to the way in which mathematicians work. Students' investigations would lead to different conjectures by different students and the resolution of conflicts would be made by arguments and evidence. The idea is that students should not just meet "readymade proofs" and formulae but would be able to participate in constructing them from the very beginning, by exploring, finding patterns, finding counter examples or constructing proofs.

Some mathematicians stated that such tasks and working manners were used in *Project programme*. This programme is not given any more at the department. All the mathematicians in my study, related positively to this kind of working manner, stating for example, that it was the way in which mathematicians worked. At the same time, they saw a lot of hindrances and disadvantages in applying the way of working with mathematics, for example the lack of time and lack of students' competence. Some of the mathematicians were afraid that only some students would succeed whereas others never would. Calculus in particular, was pointed out as problematic for this kind of working manner.

"…in the introductory course in analysis, for example, one has no time to say to students: You can play here for a while until you encounter the supremun axiom."

Mathematicians pointed out that such problems could be a good complement for ordinary contents but emphasised that the problems had to be carefully selected in order to engage all students, and offer many students a chance to succeed.

"It's certainly lots of fun for those who manage to find out something, but I think that if it (the working manner) *is to be successful one has to think through profoundly and take things where it's not too difficult to state that something is true. I mean that if something is too hidden, it can take too long a time and then it's not meaningful. It's fun to succeed but not fun to fail."*

One mathematician suggested that these kinds of tasks could be well suited for lessons, if they were not too time-consuming. However, according to the quotation below, there was too little time for the planning of the teaching together with other teachers.

"In the lessons, we could have problems where the solutions are not visible at once but problems that would demand a little more studying of the theory and investigations to arrive at the right formulations. Our problems are more of the type: Prove a formula or solve an equation. They might not inspire students in the same way as problems where students feel that they themselves have arrived at something essential, […] *Small problems, conjectures and proof do not need to take so much time and could probably be used in the basic course. But we have too little time for the planning of the teaching and not so much time to talk with each other."*

Lessons (introduced in 2002) with about 10 students and a teaching assistant/lecturer aim to give students the opportunity to present mathematics both orally and in written form. Some mathematicians were sceptical of applying investigative working manner in mathematics if students lacked elementary tools with which to explore and find patterns, or if they lacked the knowledge needed prove their conjectures.

Dissatisfaction with the basic course

There was also criticism of the basic course for containing too much material. There was no time for deeper discussions about "real mathematics". The following extract is an example of a view according to which students wanted to learn and understand but there was too little time and the courses had become some kind of brief orientation courses.

"…it means that one actually has more stuff to learn and less time to digest the stuff. And then it becomes very difficult to, at the same time, give them the

explanation. I mean that they really wonder and listen and think, yes, are we going to know now why. We take them in a little."

There was also an example about a vision to offer the students a kind of orientation course, but not for the repetition of what the school has done but for meeting the content in a qualitatively different way, to question and to get the idea of deductive thinking and relations between different mathematical contents. The aim of the course would also be to reveal the difference between the two attitudes towards mathematics, one as calculating or applying the formulas, the other as understanding and being able to prove and derive the results.

"So what I actually would like to see, even if I can see counter arguments against that, is a completely different planning for a course where we would deal with quite a little stuff but demand a full understanding of that stuff. I mean really slowly take up these things and really differently and make the leap, make the difference in attitude visible and obvious […] *And it is not a good milieu when we both try to get them to think differently and deal with a huge amount of new stuff. And I think, this is my view, I do not know if this has been practiced somewhere. I think that such a ground could enhance the tempo later after they* (students) *would have got some time to absorb this shock and the way of seeing things in comparison to this mish-mash method."*

The feeling of giving up

Many mathematicians, including some of those, who were critical of the way proof was taught earlier, stated that it was a pity not to be able to deal with proof to the extent they would like to do. They blamed the lack of time, students' low level and the lack of experiences in upper secondary school, the lack of students' interest and economical reasons, new course literature and the lack of proving tasks in the examinations, for not dealing so much with proof in the basic course. As described in the previous section about mathematicians' views on proof, most of the mathematicians exhibited an appreciation of proof and often regarded their own positive experiences as something they wished the students could be able to experience. Here, a kind of a feeling of giving up can be interpreted in many utterances:

"The role of proof in our teaching has clearly diminished, now you hardly prove anything. Teaching has become more like giving cooking advice and formulas. We adapt our teaching to the students' low level and there's no room for proof, we've also got fewer lessons than earlier."

Because students had great difficulties in their basic algebraic and computational skills there had not been any choice, according to one of the mathematicians, but to offer an introductory course, not for learning of mathematical

reasoning, language and proof but for repetition of the upper secondary school mathematics and consequently, another course had to be sacrificed.

> *"And there are both advantages and disadvantages with that. In some sense we did not have any choice, we had to give this introductory course because they had too poor a prior knowledge…"*

To sum up this section, the mathematicians who I interviewed talked about the changes in the curriculum concerning the treatment of proof and the reasons for these changes in various manners. They agreed about some changes in the curriculum that had led to a diminished place for proof in the basic course but had slightly different interpretations about why these changes had taken place. They also related to the changes in various ways. Some of them thought it was a pity whereas the others did not see any problems. How they related to the changes had to do with their pedagogical perspectives, this I will deal with in the next section.

4.4 Mathematicians' pedagogical perspectives

I have set up a table about three different styles of how to approach proof in the teaching based on the data (Appendix 5). I call them:

- *Progressive style or "I don't want to foist the proofs on them"*
- *Deductive style or "It's high time for them to see real mathematics"*
- *Classical style or "I can't help giving some nice proofs"*

The styles are idealised, no individual could perfectly fit into one of them. They constitute a theoretical model to give structure to the results. As main criteria for different categories, I used pedagogical intentions, the views on students and the aspects in the conceptual frame (see Section 2.3, p. 61). I did not use the model for categorising mathematicians, just their utterances (see Section 3.3.2, p. 82). That is why the quotations are not labelled. In the same way as individuals cannot fit into one style, utterances sometimes have features of several styles.

In this section, I first describe the main characteristics of each style and then exemplify the characteristics of the style with quotations from the data.

4.4.1 The progressive style (*"I don't want to foist the proofs on them"*)

The progressive style is characterised by pedagogical reflections with a kind of sensitivity towards students. Proof and especially the word proof should be avoided in order not to frighten students. Typical for this style is an attempt to be flexible and adapt the teaching to students' level and try to give students what they think students need.

An inductive approach is preferred in the presentation of mathematics to newcomers. Natural language is preferred before formal symbols and it is unnecessary to confront students with formal mathematics. That is why long, technical and formal proofs should be avoided. Proof is used invisibly in calculations and in the derivations of formulas. This style emphasises the explanatory aspects of proof. The enhancing of *understanding* is the most important in teaching. Conviction is also seen as an important function in the following sense: proofs should offer conviction to students so they can deem them worthwhile. That is why proofs for evident statements should be avoided.

According to the view held within the progressive style, only few students can value and understand proof. Most of the students do not even need to learn it. They need to learn to calculate. Proof is difficult for students, they are afraid of proving tasks, they are not interested in proof and they do not understand the meaning of proof. Further, discussions about proof, the formal demands of the practice or proof techniques are not the aim of the teaching, because it is impossible to "transmit" knowledge to students who are not interested in it. The small minority, who are capable and interested in proof, are able to find out for themselves what is accepted as proof in the practice.

The label "progressive" was chosen because, in this style, there is a tendency towards/willingness to reform the educational practice (see Section 3.3.2, p. 82). There are also influences of constructivism as a learning theory within this style. I will discuss it in Section 6.3.1 (p. 209).

Next, I will exemplify some of the features characterising the progressive style with utterances from the data.

The meaning of proof

Utterances expressing this style do not contain the same kind of expression of emotions and enthusiasm about proof as utterances characterising the two other styles but there is more concern about the pedagogical problems for the need for proof. The utterances categorised into the progressive style often express a criticism of the earlier ways of dealing with proof and theories (see the previous subsection, p. 103). There is also reflection on the meaning of teaching students proof. Mathematicians should not give "unnecessary" or long and technical proofs but proofs that enlighten something essential and

give some aha-experience to students. It is important that students feel that the proofs are worthwhile. Proof for its own sake is not the focus of the teaching. The following is an example of an answer to my question: Why should one learn proof?

> *"And my answer to that, if there is to be any reason to teach it at all, it must be deemed worthwhile, which is easier said than done. Anyway, you do what you can in order to reach it. I do not torment them with proofs that I conceive as really unnecessary, anyway not on that level. If they are interested they will return to this eventually."*

The quotation above exemplifies the view within which one questions if it is any meaning at all to teach proof *"…if there is to be any reason to teach it …"*. One should not *"torment"* students with unnecessary proofs. What constitutes intellectual need among students has been discussed, for example, by Harel (1998).

When is proof needed? The idea of proof within this style is for example, to enhance understanding (aha-experience) (see p. 43).

> *"And you can also concentrate on proofs that have some kind of core that they can understand as a kind of aha-experience."*

The utterances characteristic to this style, often deal with giving proof as enhancing understanding**.** One should not give proofs that no one understands, not long proofs either.

> *"I definitely don't give any long proofs if I don't believe that there is understanding."*

Proofs for important theorems that students can apply as a method of problem solving can be given to students because students accept such proofs better. The next quote is an example of this. The factor theorem is seen to be worthwhile for students and as the theorem itself is important and useful for students, the proof for it can be given.

> *"…something I thought was good as the first proof was the factor theorem. Partly because it is a natural question that they accept, that one should solve equations and then they see that one can solve equations of the second degree and then we ask what to do with equations of higher degrees. And I think that is something that goes down* (studenterna sväljer) *quite well, that one needs to solve equations and that the factor theorem can be of assistance."*

As exemplified in the quotes above, within this style, there is sensitivity to what is thought students need or are interested in or "accept" and thus experience as worthwhile.

A small minority of students need to learn proof

The following quotation is an example of the view that only a small minority of students needs to learn proof. They are the students who will become mathematicians and maybe even the students studying computer science.

> *"The small minority, that really is going to become mathematicians has to get used to proof, but this they do in general for their own interest* […] *People who are going to become computer scientists, they really need to learn to think in this way because so does a computer…"*

For the others, for example, chemists and physicists, proof can be given if it serves as an explanation or can easily be used in problem solving.

> *"There is an essential difference between those who are going to use mathematics for modelling, chemists or physicists and so on, for those I think proof is not so important, one can give proofs if that enhances understanding but no more."*

According to the progressive style, learning proof is not important for most of the students, although there is a small minority who need to learn proof.

Conviction

According to the progressive style, conviction is an important function because it helps students feel that proof is needed. The perspective is that of students, so there is a desire to awaken a need for proof for students by confronting them with something that is hard for them to believe. The following quotation exemplifies this view.

> *"As a matter of fact, the only justification for proof is in situations when it's not as one has believed it would be. You discover or possibly get help to discover that in some cases the things you believed in are true, in other cases not at all. So you really have a need to sort out, when the things that seem to be reasonable a priori, hold and when they do not hold."*

According to the progressive style, proofs are needed when they offer conviction. In order to provide students with the experiences of conviction and the feeling of a need for proof, evident statements should be avoided.

> *"But if one can meet mathematics in this way instead, and actually understand that there are such relations that are not evident. If one can be convinced that there is a relation I think one gets another attitude towards proof than if one starts with epsilon and delta."*

The need for proof can be enhanced by confronting students with relations in mathematics that are not evident. The considerations in the examples above are similar to those presented in the conceptual frame (see p. 46) when I

described the concerns among researchers and mathematics educators about the aspect of conviction/explanation regarding students' feeling of the need for proof because, according to these researchers, students were easily convinced of the truth of the statement by the authority of a teacher or a textbook or by a couple of examples.

Inductive approaches

Typical for the progressive style is to prefer an inductive approach in the teaching. Teachers should start with calculations and teach logical reasoning via them. The following quotation is an example of this view. The mathematician in the quote states that calculations are a natural way of arriving at proof instead of "fobbing it off on students".

> *"I think it's good to first learn to calculate and in that way arrive at natural questions and in that way, if you are lucky, discover that proofs are actually needed. Instead of foisting the proofs on them* (students) *when they do not see any point with them."*

Working with examples rather than general results is also a means to hide proof in the calculations and not to frighten students. I will come back to this when dealing with visibility/invisibility.

Intuition/Formality

According to the progressive style, besides avoiding evident and abstract theorems, teachers should try to avoid formal mathematics and formal proofs because students can not see any meaning in them. The following quotation exemplifies this view. According to this utterance, proving the triangle inequality is totally meaningless and "pure, abstract nonsense" for students. It is also an example of a criticism against the way in which proof was earlier taught.

> *"There are many proofs that in some sense are easier, but tend to become very formal. Some years ago we had this idea that they* (students) *would have questions about theories in the examinations in the first term. Often, they were to show, for example, the triangle inequality, which I conceive as totally meaningless. Because for them it is pure abstract nonsense and, in the end, they do not understand what they have done."*

It is also typical for the progressive style to be more careful with the use of symbols than in the two other styles. Everyday language is preferred and, especially at the beginning, it is important to avoid dealing with formal theorems and formal symbols.

> *"Concerning mathematical language and signs and logical symbols, some mathematicians want to, from the very beginning, write everything with mathematical symbols. I have kind of the opposite attitude; I start by writing*

as much as I can with words. And then eventually, I say that it is time-consuming to write everything so maybe we could use these symbols… My aim is that they won't sense them like a burden when they meet them, but as an easier way to write, how nice, now we don't have to write so much."

This way of proceeding, by eventually introducing symbols also shows certain sensitivity towards students as they were expected to be. In the same manner as naturally arriving at the feeling that proof is needed, the use of symbols should be justified so the students would feel that the symbols are needed for making their life easier.

Invisibility/Visibility

How does the progressive style relate to the condition of transparency? Discussions about proof and formal symbols are avoided and natural language preferred according to the views belonging to this style. Even the word "proof" is avoided. It seems that proof exists quite invisibly in the lectures and lessons. However, there are some aspects that are made visible. For example, the way of focusing on the significance of symbols in mathematical language that was described in the extract above is an attempt to make their role in mathematics visible. Also trying to arrive at proof through "natural questions" can enlighten the meaning of proof in mathematics in a different way than for example, just telling the students why we need proof.

The next example illustrates how proof can be dealt with quite invisibly without focusing on it as proof. In this utterance, it is stressed that calculating in a way in which one understands where the different components come from is crucial for students' learning of mathematics. It is stated that this kind of understanding is easier to reach than understanding by working with "proofs".

"…instead of hanging the question of understanding on proof, I want to connect it to the difference between on the one hand, being able to calculate and, on the other hand doing that and understanding where the different components come from, understanding that this is an effort with a goal, and this kind of understanding is more often easier to gain, and is often even more crucial…"

This is also an example of the way of thinking that proof actually exists in all mathematics if one justifies the steps one takes.

"This is a step towards proof; it's proof, exactly, even if we don't have the headline "proof". So this is not an alternative to proof but an alternative way of working towards the same goal or we produce a proof because this is an argumentation that is correct."

Proof as quite invisible in the derivations of formulas or in calculations with numbers, is obviously seen to be a tool for bringing understanding and conviction about the correctness of the calculations.

> *"We shall become convinced about the truth, that we can see it ourselves, but also for the reason that the mass of knowledge we learn becomes clearer and we learn what is more fundamental. And then we learn to derive the one from the other,…"*

Within all the three styles, proof is seen to be an essential part of mathematics and a tool for deriving formulas and critically checking the correctness of calculations.

However, proof is not the focus of the teaching, and there are no intentions of discussing the formal demands of the correctness of proof with students within the progressive style.

> *"I have not felt a need for some more profound discussion about the formal demands of proof, but rather that one often gets questions as all of us do from the students: "Does this do as a proof?" and then they are waiting for a formal answer, but I want instead that they will have an answer from inside of themselves where the proof fits if they understand. So I do not want to go too far regarding these formal discussions."*

This style does not reveal important aspects of proof that could make the idea of proof and proof techniques more available for students.

Views of newcomers

According to the progressive style, students are afraid of proof, so they have to be led to proof invisibly via calculations (see the previous paragraphs). In general, students cannot appreciate proof. There is, however, a small minority who are interested in proof and able to learn it, which the beginning of the following quote exemplifies.

> *"There are, however, every year students that think that proving is something evident. And they, naturally, need a totally different kind of challenge. But for the overwhelming majority that is, of course, not the case. One must first learn to at least understand that proof can have some value in itself. That it does not work any more only using recipes."*

But, as the end of the quotation above exemplifies, the majority of students do not value proof; they just want to have recipes. According to the progressive style, students can "swallow/accept" certain proofs, if the theorems that are presented to them are not evident and if the proofs are not long or technical but explain useful aspects of mathematics or can be used in problem solving. More formal proofs are to the majority of students only "pure abstract nonsense" and, in the end, they do not understand what they have

done. Those, who are capable and interested in proof, will return to proofs for their own interest *"If they are interested they will return to this eventually."* (see p. 108)

4.4.2 The deductive style (*"It's high time for students to see real mathematics"*)

A common view within the deductive style is that it is natural to use a deductive approach in the teaching of proof. There is no intention of avoiding the word proof or of avoiding mathematical symbols, often the opposite. Right from the outset, students should get used to symbols and formal language. There is no fear of confronting the students with something unknown; students first maybe memorise and just follow the arguments but at the same time they learn the rules of the game. Proof is something that students should know and it is high time for them to become familiar with "real mathematics" and to get the answer for why something is true in mathematics, how everything is connected in mathematics. Nothing is evident in mathematics; one should not accept or trust anything without first proving it. Hence, proof is connected to critical thinking. We also have to trust in students' ability to follow and learn. Students are interested but have no experience of proof. Discussions about proof and proof techniques are considered as worthwhile (even if they are often impossible because of external circumstances).

There is a desire to convey positive experiences and feelings regarding proving activities and proof and advantages connected to the axiomatic-deductive method without hiding proof e.g. in calculations. The general should be shown to the students immediately and there is a conviction that abstract thinking is not more difficult than concrete thinking, sometimes easier. There is no desire to avoid proving evident statements either. The learning of proof is compared to the learning of language, and learning by heart is not rejected but it is seen to be one part of the learning process, to imitate. Within this style, it is important to make the formal demands of the practice visible to students. For example, it is important to present the logical structure of a course to students at the beginning. Rigour and careful presentation is emphasised because students at the beginning of their studies need to clearly see every step.

Next, I will exemplify this style by giving some quotes illustrating various characteristics of the style.

Nothing concealed?

The following example is enlightening because the approach in this utterance is very different from those expressing the progressive style, nothing is concealed: axioms, definitions, logical steps, the abstract and the general. The pedagogical idea here seems to be that students will get used to nota-

tions and mathematical language, when students confront them from the very beginning even if they may not understand everything first. There are no intentions of avoiding proof or the word "proof", rather the opposite. Mathematics should be presented in a deductive manner from the very beginning.

> *"We base everything on certain axioms, certain definitions, assumptions, and from them we derive new things, in derivations one takes logical steps, a profound justification and how one refers in every place, to what one uses and checks if it is correct or not. And this is something they see by and large from the first lecture and this is sound, I think, to do so, because even if they do not really understand from the beginning they get home already from the first day, this is how it is going to be, they get used to, they have to do that, because they have not created the rules of the game but they have to accept them and we have to show them to them."*

The quotation above is also an example about the view according to which students have to accept the rules of mathematics in the practice, because they have not created the rules.

Conviction/Explanation

In the deductive style proof is connected to critical thinking but is not only viewed as a tool for conviction, but also explanation as an answer to the question why. In contrast to the progressive style, within which the students' requirements to learn proof is questioned, within the deductive style there are clear answers to the question about why students should learn proof. The main reason for the learning of proof is to learn to question the truth of the statements and to become convinced when one gets the answer to the question why something is true, to get a system and to see how everything is connected.

> *"It would be the same as, if in my natural life someone would claim something, and then they would just say, believe in that for all eternity. One would not be really satisfied with that and… it is the same here, that they would believe but never understand why something is true."*

Aspects of conviction and explanation are intertwined in the utterances belonging to this style. One should learn to question until one gets the answer to the question why.

Critical thinking and intellectual challenge

It is important for students to learn to think critically and question the truth of the statements (see p. 45). Proofs transform assumptions into theorems and theorems are also seen as a tool for simplifying life in mathematical practice.

> *"If you don't prove that a theorem is true - no muddled things are allowed in mathematics - you cannot go on, you cannot take anything for granted if a statement is not proved and this step of assuring yourself that a certain statement is true, transforming an assumption into a theorem just simplifies the life."*

Further, if this kind of critical examination, questioning, and convincing oneself with rigorous steps is lacking in the activities, they are not real mathematics.

> *"If they don't learn to think critically and question things and convince themselves with rigorous logical steps that the statements they encounter are true so it is incomplete, it does not fit in mathematics, you have to think critically, rigorously and not with muddled statements..."*

The quote above exemplifies the view that accepting statements without checking them with rigorous logical steps does not fit in mathematics.

Derivations of formulas

Similar to the progressive style, proof is also seen to be a tool for deriving results from earlier results and seeing how everything is related. Students should also learn to use this tool themselves so they do not need to memorise everything.

> *"For the students it can be a question of conviction concerning the correctness of formulas. It is impossible to learn by heart all the formulas and all the theorems, that's where we need proof. We have to remember some of them and then derive the others from these basic…"*

According to the deductive style, it is possible to show students the benefits of thinking in this manner and to show them that it is not difficult to decide what is true and why.

> *"So I try to talk a little about it, there is so much to memorise so it is impossible to learn everything by heart…I tried to tell by giving power laws as an example that if one understands from the very beginning what the laws mean, it is actually quite easy to decide what is true and why."*

There is a view within the deductive style that there is a qualitative difference between school mathematics and university mathematics. A teacher should challenge the view students have when they come to the practice ('calculating with recipes') and there is a view that students are interested and capable of learning "real mathematics" and of finally deducing an answer to the question "why?".

"What I have said to the students in the introductory course is that we are interested in why things are true, we are not interested in seeing it as a collection of recipes even if they are going to learn a lot of them too, but how things are connected to each other. When they leave, we hope that they themselves have enough knowledge to be able to, to some extent, not just take a readymade recipe but in a situation where you cannot apply the formulas they might have in the collection or to see what could be true here and to have enough tools to be able to derive from one form to another.

In the quotation above, there is also a view of proof as an artefact and a desire that students would learn to use proof themselves as a tool for deriving formulas (see p. 95). It is an example of the view that students should learn to derive formulas and understand how everything is connected. This view is also held within the progressive style (see p. 112).

Language and rigour

Concerning the language, abstractions expressed by symbols are not avoided like in the progressive style, rather the opposite. Students should get used to them as early as possible. The following quotation is an example of the view that students have seen too many simplifications and special cases. It is time for them to see the power of the abstract and general in mathematics.

"…Here we shall prove, in what sense, yes, concrete examples, for example in linear algebra, assume that some linearly independent vectors can be completed with vectors until we get a basis for a vector space, that is a statement, it is not an axiom, we have to prove it. And what is meant by that, yes, we prove the theorem in a way – it is a general statement – without simplifying, they have seen all too many simplifications, for example that everything works only in R^3 or a function defined on a certain interval, two to five, without accustoming them to see the things in an abstract way, interval a, b or as in the fundamental of analysis, metric space, in linear algebra, not only in R^2 or in R^3 but in infinite dimensional vector space, that are general and a bit more difficult to handle and create a clear picture about them and the earlier they see them, from the first day, the better and simpler later."

The quote above also exemplifies the view, according to which students should meet the general from the very beginning as it makes life easier later. Rigour is preferred in the presentation of mathematics because students at the beginning of their studies need to clearly see every step.

"When you are a beginner you have to get very clear presentations because in that phase it is difficult to fill the details by yourself."

The difference between intuitively true statements and rigorously proved statements should be made visible for students. The following quotation is an example of the view according to which a teacher should tell students when a proof is needed.

> *"…there is a little danger to confuse say in the simple theorem about a function that has the same value in two points, then there is a point in between those points where the derivative equals zero. Then they* (the students) *say, OK, we draw the line and see that it is clear. This is something one has to stress that here we have this intuitive idea of where the ideas for the theorem come from but then one must formulate a logical chain of statements."*

Instead of waiting until students experience the need for proof mathematicians should tell them when it is needed. The difference between intuitive presentations and formal proofs has interested researchers in mathematics education and is much discussed in the literature (see Section 2.3.3). It is also connected to the question of proving "evident" statements that some mathematicians pointed out as a difficulty for students.

The problem with giving the "big picture" at the beginning

Important for this style, is to try present the logical structure of the course to students at the beginning of the course.

> *"When you describe first what the course is about and what the goals of the course are and what is expected of the students to do to reach the goals, so there proof comes as a natural part of certain things."*

At the same time, when the goal is to introduce students to the course contents, according to the deductive style, it is impossible to give the students the "big picture" from the very beginning, they just have to believe in certain things and accept the rules of the game. The intellectual satisfaction will come afterwards when everything will fall into place. This is something a teacher should tell students at the beginning of the course.

> *"…one has to first believe in certain things, give a certain credit to the subject and it is first after a while it is possible to get a revelation about how everything are connected in a certain way. It is impossible to give the big picture at the beginning, but as soon as they realise it…it is important to point out already at the beginning that they have to have patience and accept the rules of the game in order to later get the intellectual satisfaction. To see the big picture about how everything is connected in a structure and not fall because one step is missing. They have been very satisfied with such a planning in my courses."*

This quotation also expresses something about the dilemma of transparency in the teaching of proof; how to talk about the contents, proofs and theorems which can help students' access to the practice before students have any experience about them.

How students learn proof

How do students learn proof? According to the deductive style, the learning of proof can first occur by memorising the proofs, but there are many steps in the learning of proof. The importance of the knowledge of terminology is stressed as is the ability to follow and understand when others present proofs. The following quotation is an example of the steps that have to be taken.

> *"There are of course a lot of steps to learn proof; first, you have to learn terminology in order to understand what it is all about. The second step is to be able to follow when someone else proves something and think that it seems to be right. The third phase might be that you have understood the proof so well you can give it yourself without learning by heart but more understanding how the things hang on each other. How you learn these different steps can vary between the students."*

The learning of proof demands a lot of time and exercise, so it is important to start early to learn deductive reasoning, to use definitions and axioms and to justify every step.

> *"The earlier the better. For example, with help of geometry, to learn deductive reasoning, to use definitions, other theorems and axioms and justify every step. In that manner you drill this way of reasoning into your head. For example, if the three angles in two triangles are equal, they are similar. Without justifying one gets no points. Then it is easier for the students to solve problems with circles etc."*

According to the last sentence of this quotation, the learning of proof also helps students in problem solving.

Further, there is also a desire to enhance the students' learning of constructing own proofs.

> *"We have to exercise students' ability to construct proofs step by step. In the course Foundations of analysis, there is a theorem about compact sets and… in the first glance one does not know enough, so one imitates the others. Most often one can learn, I think about my own learning, how one starts and then go on. After a while when you have trained it you feel it's simple and convincing. It's a little like learning a language, maybe partly by memorising first. Demands much exercise. Memorising is the first step in the learning."*

Even if the explanatory aspects of proof are stressed, drill, imitation and learning by heart, are also seen to be worthwhile methods for the learning of proof. The learning of proof is also compared to the learning of language. It demands a lot of exercise.

Concern for school mathematics

The importance of the language and the logical symbols are seen as crucial for the learning of proof within this style. There is concern about students' difficulties understanding mathematical texts and formulating their own statements. On the whole, there is a worry about students' ability to be exact with their presentations of mathematics.

> *"Much depends on the language: I think that students in school get too little training with the mathematical language. They cannot formulate and they have difficulties understanding the texts. For example, they want to omit the sign of equivalence and write an arrow instead. It also often happens that students write the value of the limit without the sign of "limes" and put the sign of equivalence freely. I have seen with my own children that they do not learn what a sign of equivalence means; they say it means "follows"."*

Similar to all the styles, there is dissatisfaction with school teaching within the deductive style as well. However, according to the progressive style, the school should deal with proof in a way that would enhance students' understanding of the need for proof in mathematics, whereas according to the deductive style the school should teach mathematical language and go properly through some elementary notations, like the use of equivalence symbol and the symbol of equality.

> *"…So I wonder how teachers in upper secondary school teach this part of mathematics. There are many problems we have inherited from the upper secondary school. We have to somehow influence the teaching in the school so everything are dealt properly from the beginning so we do not need to go through, I am continually asking students for the sign of equality "Is there an equality here?" Sometimes they succeed to put it right but for example in connection of equations one can never write that they are equal, there you should use the sign of equivalence. Such notions are important for proof."*

The quotation above also expresses a willingness to interfere and teach students the right notations and the correct use of symbols and hence, a belief in the students' ability to learn as long as they get enough feedback.

Discussion about proof

Concerning the discussion about proof there is a desire to discuss proof and proof-techniques although it seldom happens in the practice for various reasons.

> *"We seldom study structures of proofs or discuss them on meta-level; we should raise these things more in the basic course."*

The lack of time and the disappearance of Vretblad's (1999) textbook are the reasons for not discussing proof or proof techniques.

"It is partly because we do not have time and the textbooks are different. Vretblad's textbook that we used earlier, discusses proving, logic terminology and such things. Now we do not have that... there are not as many occasions at the beginning of the courses as we used to have."

According to the deductive style, it is preferable to discuss proof at the very beginning of the course, when the contents are outlined.

"It is often rather good to give a kind of introductory lecture at the beginning of a course, and outline what is coming…and on that occasion it is not so bad to talk a little about proof and what role it has."

Within this style there is a desire to make many aspects of proof visible for students (language, logical steps, critical thinking). However, in practice, there is no time for discussions even if there is an intention. Proof techniques are not in focus either because of the changes in the course literature or the lack of time.

Invisibility/Visibility

How does the deductive approach relate to the condition of transparency? There is no intention of avoiding proof or anything to do with proof. There is a desire to make various aspects of proof visible and to both discuss proof and to teach all the students proof. The expectations of the practice should also be made visible for the students according to this style. Abstract thinking does not need to be more difficult than concrete thinking.

"I very often experience a gap between what expectations students have of mathematics, on the one hand and what we want them to learn on the other hand. Students do not have an understanding of proof we would like them to have and they do not always profit by it either. I mean that…one of the problems is that we do not tell the students what we want. I mean my personal belief is that abstract thinking does not need to be more complicated but in many ways simpler than more concrete thinking or if we call it the related thinking versus free thinking.

There is also a desire to focus on different aspects of proof, like language, deductive reasoning and definitions. The following extract is about the importance of clarifying for the students the difference between a mathematical definition and an everyday description.

"And I think a little that we have the problem that people sit and search something else, we give a definition and a lot of students sit there and see the definition and think that we have given a description and a lot of features about what we are talking about and they are waiting for the continuation. And we think impatiently that "Here is the definition, I have told what this is and everything is clear". If one does not even make the difference between a

definition and a description clear for the student, which I think is often the case; we have "let them down"."

The difference between intuitively true statements and rigorously proved statements should also be made clear to students according to this style. However, at the same time as this style does not attempt to hide anything, one can question if everything is going to be revealed for the students. This is something I am going to discuss in Section 6.3.2.

The view of newcomers

According to the deductive style students are capable of understanding proof and learning abstract thinking if they get the right presentation. There is a belief in students' desire to understand and learn proof.

> *"I do not think that it is about a genetically hereditary ability to think abstractly. I think it is partly because they have not got to know what we want them to do."*

Within this style, abstract thinking is not seen as something that only few can learn but it is more about making clear for students what is expected of them.

4.4.3 The classical style (*"I can't help giving some nice proofs"*)

One characteristic of the classical style is that there is a great admiration of proof. Proof is considered to be an essential part of mathematics in the mathematical practice, the "soul of mathematics". Proofs can be beautiful and offer intellectual challenge. According to the classical style, there are a lot of benefits in the learning of proof for everyone, not only for mathematicians and computer scientists, since proofs teach us logical reasoning[13]. Similar to the two other styles, within the classical style, proof is seen as an opposite to recipes, as real mathematics and as an explanation for why something is true and how everything is connected.

However, there is not so much intention of teaching students proof, particularly in the basic course, because of external reasons, for example students' lack of prior knowledge and the lack of time. Most of the students are not seen to be capable or interested and it is a pity for those few who are capable. Nevertheless, sometimes some "nice proofs" are given to students when there is time for it, proofs which mathematicians themselves appreciate even if only a small minority of students would understand the proofs. Thus,

[13] Logical reasoning here does not refer to formal logic.

there is not the same kind of sensitivity towards students as within the progressive style. Symbols are used when they are needed. The learning of proof is considered to happen quite implicitly without a focus on proof. The presentation is not rigorous[14], sometimes intuitive, but mostly of a deductive character; it is like presenting a proof to another mathematician. There is no intention of proving elementary details and evident statements. On the contrary, there is a desire to convey the great ideas in order to inspire students. Within the classical style, there is not much reflection on the problems of teaching or pedagogy. Pedagogical considerations concern mostly the question whether or not to give a proof.

Next, I give some examples of utterances characterising this style.

The meaning of learning proof

Similar to the deductive style, within the classical style the benefits of learning proof are not questioned. It is important for students to learn to appreciate proof for a variety of reasons.

> *"I do not consider it important to learn proofs for some theorems by heart but one must learn to understand and value something you can call a Triumph for human logical thinking and the ability to draw conclusions."*

The quotation above is an expression of the appreciation of proof as an important tool for logical reasoning and for drawing conclusions held within this style. According to the classical style, proof also offers an **intellectual challenge**.

> *"And if they only calculate with numbers it is not a big intellectual feat."*

Within this view, calculating only with numbers without the general results, does not offer students the same kind of intellectual challenge as proof.

Transfer

According to the classical style, proof is also useful in contexts other than mathematics. Contrary to the progressive style, according to which proof is needed only by those who will become mathematicians or computer scientists, within the classical style the learning of proof is seen to be important for many different kinds of professions and everyone can benefit and learn logical reasoning by working with proofs.

> *"Through proofs students learn systematic thinking, to argue for their opinion."*

[14] Rigorous refers here to a careful step-by-step presentation.

Within the progressive style, there is a view that proof is not so important for chemists or physicists and so on (see p. 109), whereas the view in the following quotation is the opposite.

> *"A simple question, the meaning of mathematics, there are not so many who use integral calculus and such things later in their lives. On the contrary, logic and the habit of thinking logically is training for the brain and the logical thinking is absolutely very important, and not just calculation skills, and there proof is an important part of it. For those who are not going to become pure mathematicians or pure physicists there is yet a great value to practice proof because one gets used to thinking in a special way. It's very important I think…even concerning finances or all the natural science subjects. There is a lot of logical structure to learn even for those becoming chemists or those writing articles. That there is a kind of theory, not just experiments."*

Again, the quotation above is an example of the respect and admiration towards proof and what it can teach us, which is typical for this style.

Aesthetic

As described and exemplified above, there is an admiration towards proof in many ways; also the aesthetic aspect of proof is appreciated.

> *"Analysis contains very classical material and handsome proofs, elegant ones and one can at least give some of the theorems completely with proofs even if students are not going to continue their mathematical studies on higher level."*

> *"Proofs can be beautiful."*

Even if the mathematicians expressing the classical style do not really intend on dealing with proof because of the lack of students' interest, prior knowledge and time, they sometimes give some proofs they themselves appreciate.

> *"The aim of the course Analysis 1 is to enhance intuitive understanding and calculation and problem solving skills, so I have given very few proofs in the lectures. But I can't help giving some handsome and short proofs, often in a simpler manner than in the textbook."*

There is not the same kind of sensitivity against students within the classical style as in the progressive style. Proofs are given if they are considered as elegant and nice and if there is time for that, no matter if students understand the elegance of the proofs.

Intuition/Formality

In contrast to the deductive style, rigour in the presentation of mathematics is not seen as important. It is enough to present the important points and steps

of the proofs to students and let them fill in details by themselves. Otherwise, maybe the magic and beauty of mathematics fades away.

> *"No rigorous proofs, too formalised proofs are unbearable. A piece of poetry,* (proof) *can be as attractive as the entire theorem."*

> *"Complete proofs are unbearable, to learn important points and to be able to fill in the details."*

Rather than the rigorous treatment of theories one can enhance students' intuitive understanding.

> *"I prefer pictures and this geometrical intuition, in contrast to the formal calculations, at least for the beginners it is important…"*

The teaching and learning of proof

A characteristic of the classical style is not to reflect very deeply on pedagogical issues involved in the teaching and learning of proof.

> *K: How do students learn proof?*

> *M: As anything else elementary.*

The learning of proof is seen to happen quite implicitly and there are no aims to teach students systematically precise formulations or mathematical language. Proofs are beautiful and it is a pity that students can not experience the intellectual challenge that proof can offer. But why is there no intention of teaching proof for newcomers?

The reasons for why proof is not dealt with, especially in the basic course, are external: students' lack of interest, the lack of time and the lack of students' earlier experiences. Also because there is a lack of proofs in examinations, there is no use dealing with them so much in the lectures.

> *"I introduce several theorems without proofs because of the lack of time and the lack of students' interest. Also examinations rule contents."*

Also within the classical style there is scepticism against the school mathematics. Because of the low level of school mathematics it is not possible to deal with proof at a higher level.

> *"What is basically wrong, is the idea that teaching should be adapted to the level of the weakest students so we deal more and more with upper secondary school mathematics and in upper secondary school they deal with lower secondary school mathematics. Everything changes; the stuff you could deal with in the sixth grade is dealt with later and later."*

The criticism against school mathematics is a common feature for all the styles although the concerns vary between the styles.

Invisibility/Visibility

How does the classical style relate to the condition of transparency? In this style, there are no aims to discuss proof with students. To my question concerning the discussions about proof, the answer could be as follows:

> *"Not as far as I know, the natural place for such discussions for understanding of proof would be in courses in logic but we do not have such courses on that level."*

Thus, the learning of proof would occur, if it occurs at all, implicitly without discussions about proof. It is possible that aspects of proof that were admired by the old-timers, for example aesthetic, are visible in the teaching.

The view of newcomers

How are students seen as learners of proof within the classical style? Most of the students are not interested, as they are not capable to follow but this is mostly because of their poor school background.

However, according to this style there are some "good students" in every group, for the others proof is actually a waste of time.

> *"…and I feel anyway that a tenth part of a class are capable and they can absorb this but I think that for the most of them this is lost time."*

The view exemplified above is somewhat similar to that within the progressive style. Finally, there is also dissatisfaction because there is no program for pure mathematics at the department and it is a shame for the "good students" because the level of teaching at the department is so basic.

> *"…there is not a programme for pure mathematics, so in every class there are one or two who are capable and I feel pity for them because they have the requirements…I try to stimulate them. Then there are quite many who are capable to calculating but are totally uninterested in the question why."*

In this section, I described the theoretical model with three different teaching styles that was constructed from the interview data, the *progressive,* the *deductive* and the *classical* style. The styles are exemplified with quotations from interviews with all mathematicians. The theoretical model was created in order to structure the results of mathematicians' pedagogical views. The styles are ideal in a sense that no mathematician can perfectly fit into one style. Teaching styles and intentions can vary from day to day and from one

moment to another. The intentions are not stable either. The main difference between the progressive and the classical style, on the one hand, and the deductive style on the other hand, is that in the two first-mentioned styles there is no intention of teaching proof to newcomers, although some proofs are offered nevertheless.

4.5 A summary

In this chapter, I first described how the mathematicians in my study talked about proof and its significance to their practice and how they dealt with various functions of proof. The data supported the view of proof as an artefact in mathematical practice. In the second section, I described how mathematicians talked about the changes that the mathematical practice has undergone during the last decades regarding the treatment of proof and how they related to these changes. In the last section, I presented the theoretical model of three different teaching styles and exemplified them with quotations.

In the next chapter, I will describe students' practice of proof.

5 Students' practice

In this chapter, I report the results concerning the students' entering into and participating in the community of mathematical practice. I start by describing students' backgrounds and report what they stated about their upper secondary school experiences regarding proof when they entered the practice. The second section is about how newcomers related to proof at the beginning of their studies including their views and feelings. The results in sections 5.1 and 5.2 are based on the quantitative analysis of the surveys among 168 university entrants and the qualitative analysis of the focus group interviews with 6 groups in different phases of their studies. When reporting the results in these sections, I first provide the reader with some quantitative results from the survey analyses and then exemplify them, whenever possible, with qualitative results from the qualitative analysis of the focus group interviews.

The following four sections are about students' participation in the practice. In section 5.3, I describe the lectures and newcomers' participation in them. Section 5.4 is about how students were to learn to construct proofs. In section 5.5, I go on giving examples of how students' feeling of meaning was connected to their experiences of participation or non-participation. I conclude the chapter by describing the examinations and how students' relation to proof changed after the first examination on proof. The results in sections 5.3 – 5.6 are based on the qualitative analysis of all the focus group interviews with students and on the field notes.

All the quotes from the focus group discussions in this chapter are marked with S – B (a student studying basic courses), S – I (a student studying intermediate courses), or S – A (a student studying advanced courses) depending on what courses the students were studying at the time of the interviews. Students are labelled only in dialogues (see p. 86). When reporting the results about students' practice, I continuously contrast them with the results about mathematicians' practice that were reported in the previous chapter but I will return to these issues in Chapter 6 when bringing together the results from different parts of the study.

I commence the chapter by describing the newcomers' backgrounds and their school experiences concerning proof.

5.1 Students' background

The contemporary Swedish curriculum for upper secondary school does not clearly state the aims of introducing the students to proofs and proving activities. Only the main goals are stated.

> "The school in its teaching of mathematics should aim to ensure that pupils develop their ability to follow and reason mathematically, as well as present their thought orally and in writing." (Skolverket, 2002)

Local schools and teachers have the possibility to apply these goals in their own way. However, one of the criteria for *Pass* (lowest mark of a three-level grading scale: Pass, Pass with distinction, Pass with special distinction) for any of the five courses A-E, into which upper secondary school mathematics is divided, is that "pupils differentiate between guesses and assumptions from given facts, as well as deductions and proof". Furthermore, one of the criteria for *Pass with special distinction* is that "pupils participate in mathematical discussions and provide mathematical proof, both orally and in writing." (Skolverket, 2002)

The mathematicians, who I interviewed, assumed that students had very little experience about proof when they started to study mathematics at the university. According to the analysis of the surveys and focus group interviews this is true for a lot of students. However, the pilot survey (Nordström, 2003) had already indicated that there was a variety of experiences among students concerning proof[15] when they started to study mathematics. The results of the survey analyses in 2003 and 2004 confirmed the result of the pilot study. Next, I will give an account of the variety of experiences that the newcomers reported. It seems that students are in very different situations regarding their experiences and knowledge about proof when they enter the practice.

According to the students with Swedish upper secondary school backgrounds there are still a lot of upper secondary school teachers who prove statements to the pupils. About one half of the students who responded to the surveys stated that their upper secondary school teachers proved statements once a week or every lesson (*Figure 7*).

[15] There were questions about students' views on proof and I also met three focus groups with students who had responded to the questionnaire so I could check that we talked about the same thing when we talked about proof (see also section 5.2.2. about students' views on proof).

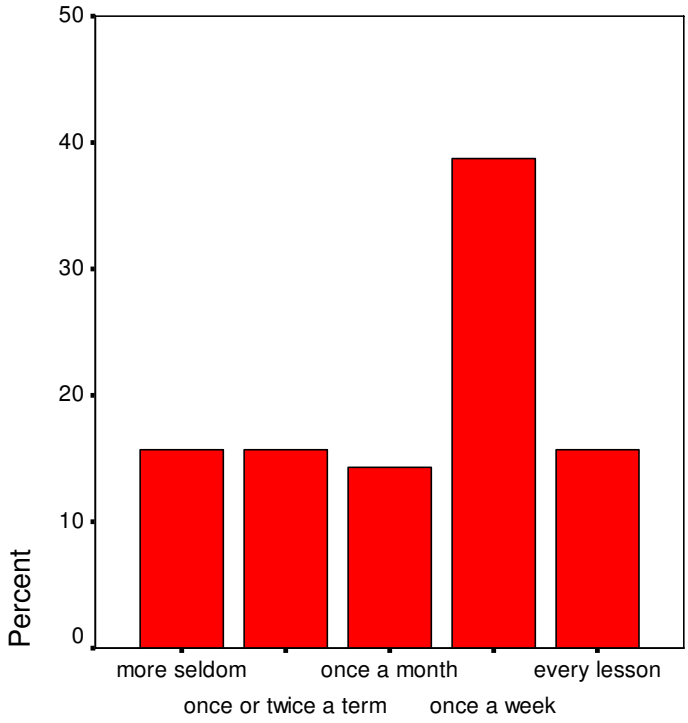

Figure 7 How often did your upper secondary school teacher prove statements to your class?

Further, about 36 percent of the students agreed or fully agreed with the statement: *My upper secondary school teacher often used to prove statements to us.* (*Figure 8*)

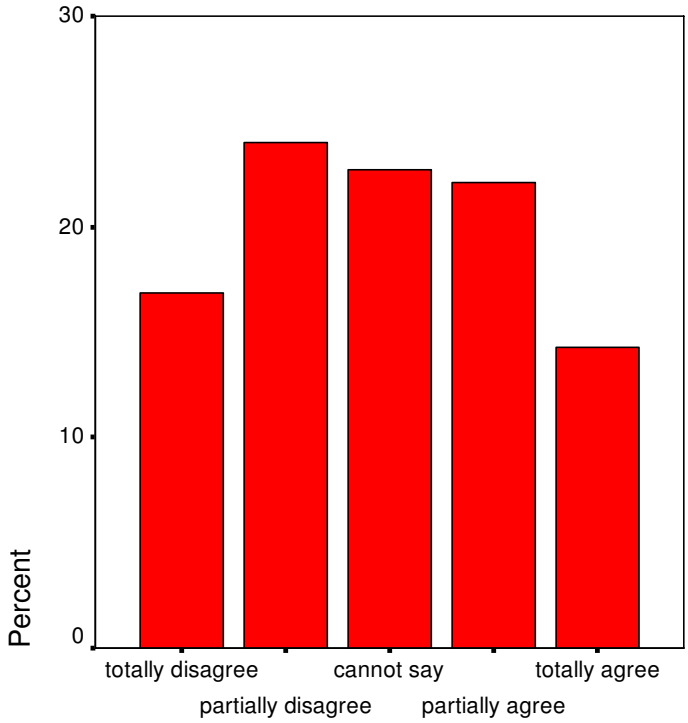

Figure 8 My upper secondary school teacher often used to prove statements to us.

About 40 percent agreed, either partially or fully, with the statement *I have had the possibility to familiarise myself with different kinds of proofs in school*. (*Figure 9*)

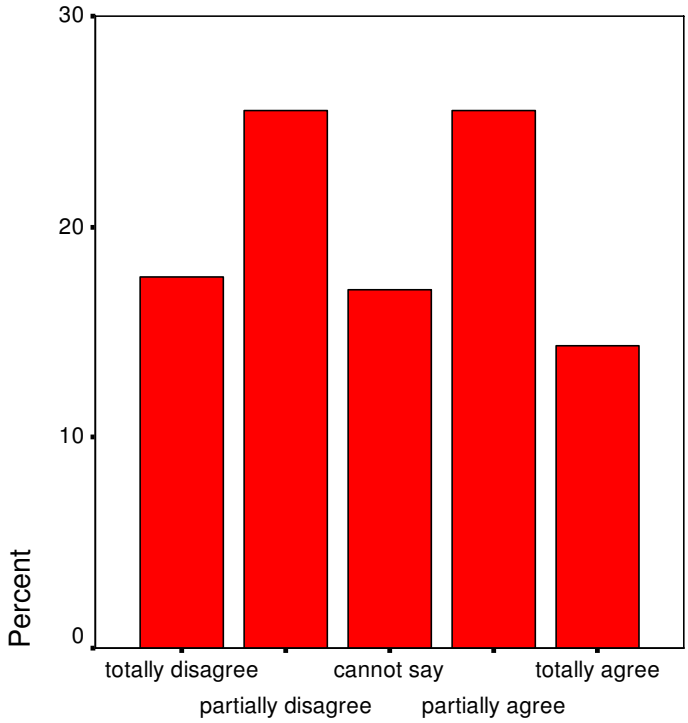

Figure 9 I have had the possibility to familiarise myself with different kinds of proofs in school.

About one third of the newcomers who responded to the questionnaires stated that their upper secondary school teachers rarely proved statements: once or twice a term or more seldom. In the focus groups, students with a Swedish upper secondary school backgrounds had various experiences about proof. Some of them did not have any recollection of proving anything.

> *S: When I think about upper secondary school I do not remember any proofs. Most of it was just doing sums.*
>
> *K: But the teacher did not prove?*
>
> *S: No, they definitely did not, not in my school.*
> (S – A, 2004)

Some of them remembered the formula for the solving of second degree equations. It was done by particular numbers.

"But otherwise, I don't remember that they proved anything. Admittedly, when they presented how to solve second degree equations, then, I suppose, they did a little proof with numbers on the board, I think."
(S – I, 2004)

But there were also those whose teachers regularly proved statements to the class.

"I think my maths teacher gave a presentation once a week and went through the proofs, showed us on the board and derived formulas."
(S – I, 2004)

Similar to some of the mathematicians who I interviewed, there were students both in the surveys and in the focus groups who had their school backgrounds in countries other than Sweden. They came from countries with different cultures and, thus, different traditions regarding the treatment of proof in the lessons (Examples: China, Arabic countries, Finland, …). There was a noticeable difference between the students with Swedish upper secondary school backgrounds and the students with a foreign upper secondary school backgrounds concerning their declared experiences about proof. Those with foreign backgrounds seemed to have more experience about proof. For instance, all the students with foreign backgrounds stated that their teachers proved statements once a week or every lesson. However, there were only twelve students with foreign school backgrounds in the sample of 168 students so it is not possible to generalise the results.

Even if many of the students with Swedish upper secondary school backgrounds stated that they had seen teachers' proofs including derivations of formulas, very few of them had participated in the practice of constructing proofs, according to the responses to the questionnaire and the focus group discussions. Responses to the question *"How often did you practice proving statements by yourself in upper secondary school?"* show that over half of the Swedish students have had very little own practice, once or twice a term (19 percent) or even more seldom (40 percent) (*Figure 10*).

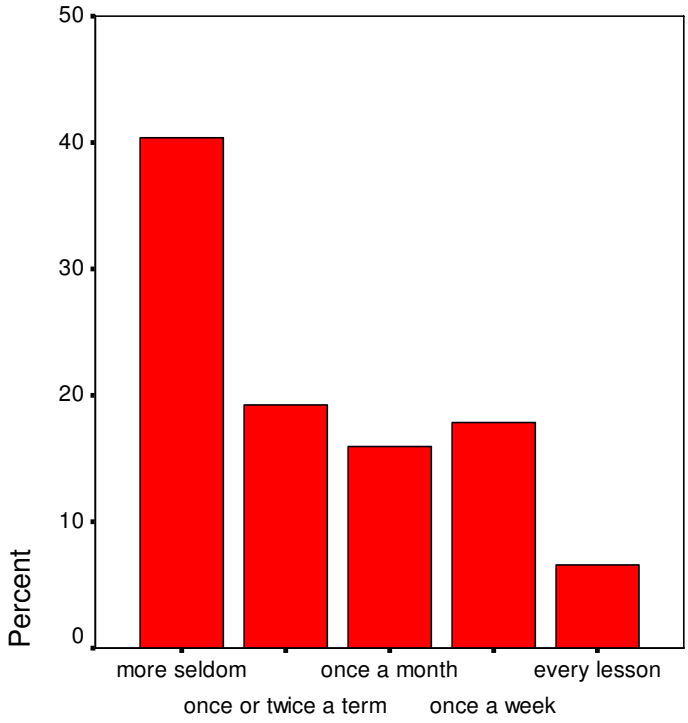

Figure 10 How often did you practice proving statements by yourself in upper secondary school?

Further, about 60 percent of the students disagreed or strongly disagreed (28 percent) with the statement *I have had the possibility to practice proving by writing in school.* (*Figure 11*)

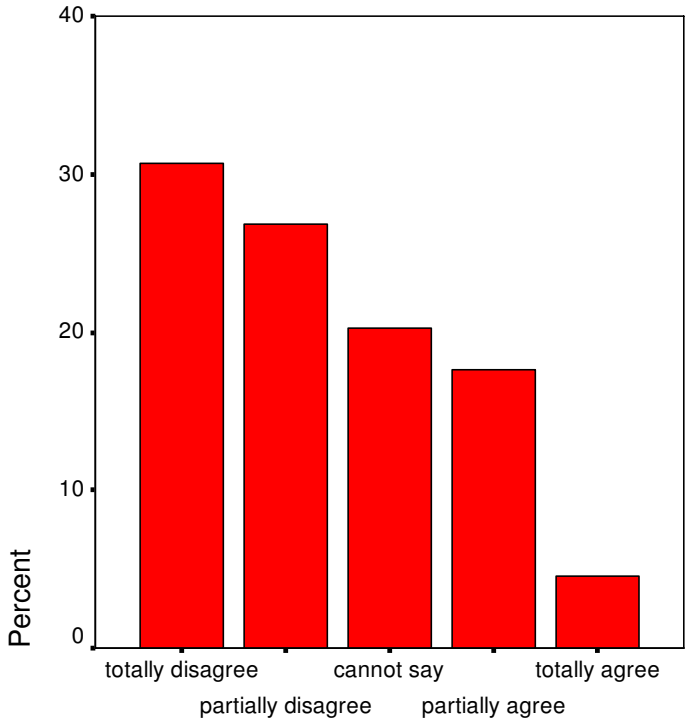

Figure 11 I have had the possibility to practice proving by writing in school.

In contrast, ten of the twelve students with foreign backgrounds agreed or strongly agreed with the statement: *I have had the possibility to practice proving by writing in school* and none of them strongly disagreed with it. Ten of the twelve also stated that they had exercised proving statements at least once a month, and five of them every lesson.

Yet, there was a small minority with Swedish school backgrounds who stated that they had practiced proof very often. Seven percent of the Swedish students stated that they had practiced proving every lesson and also strongly agreed with the statement *I have had the possibility to practice proving by writing in school*. Two students with Swedish backgrounds in the focus groups told that they had got a lot of exercise in proving in school.

> *"We had to be able to know them (derivations of formulas) anyway. But then it's also that there are always such pupils who are not interested and, but I liked proof very much and learned a lot so I have been doing proofs from the end of lower secondary school more or less so that I can't say that I really agree with you (that there was very little proof in upper secondary school)."*
> (S – B, 2004)

Some of the students with Swedish upper secondary school backgrounds in the focus groups reported that those who wanted to get the best marks in mathematics sometimes worked with proving tasks. This is in accordance with the curriculum criteria for different marks in upper secondary school mathematics courses (see p. 128). According to the students the proofs were often of the type where one would show a formula by algebraic manipulation showing that the left side was equal to the right side. Students called them "simple proofs".

> *A: But then we didn't have these theoretical results. It was more those kinds of rules we worked with.*
>
> *B: Yes, they were simple proofs.*
>
> *A: Yes, like show that this formula…or something. And that is something you can do by calculating a little and then you get it.*
> (S – A, 2004)

Some of the students pointed out that these tasks were actually easier than ordinary problems because one already knew the answer.

> *L: I had top marks, so I did them as exercises.*
>
> *K: How did you find working with the tasks?*
>
> *L: They were so easy, just to calculate some things and go on.*
>
> *J: I don't remember it as hard either. I think I even thought it was fun. They were easier than other tasks because you already knew the answer. I am good at careless mistakes and then I know that I have got the right answer and if not I just have to check where my careless mistakes are.*
> (S – I, 2004)

These findings were supported by the upper secondary school textbook analysis (Nordström & Löfwall, 2005). It is not surprising that most of the students with Swedish upper secondary school backgrounds did not have so many memories about proving activities from their school period. We found that proof was often presented invisibly in the textbooks. Further, the space given to proving tasks was minimal compared to practical applications and routine exercises (about 2 percentages). However, there were some special mathematical domains where proving tasks were more common: in geometry, in the context of verifications of solutions of differential equations and verification of formulas of trigonometric functions.

Example (Björk & Brolin, 2000): Show that $2\sqrt{x}$ is a solution for the differential equation $2xy'-y=0.$

Students who had some memories about proving tasks talked about tasks where they would show that the left side equaled the right side. There were quite a few tasks of the following type in the textbooks.

Example (Björk & Brolin, 2000)**:**

$$\text{Show that} \quad (\tan x + \frac{1}{\cos x})^2 = \frac{1+\sin x}{1-\sin x}$$

$$\text{LS (left side)} = (\tan x + \frac{1}{\cos x})^2 = (\frac{\sin x}{\cos x} + \frac{1}{\cos x})^2 =$$

$$= (\frac{\sin x + 1}{\cos x})^2 = \frac{(1+\sin x)^2}{\cos^2 x} = \frac{(1+\sin x)^2}{1-\sin^2 x} =$$

$$= \frac{(1+\sin x)^2}{(1+\sin x)(1-\sin x)} = \frac{1+\sin x}{1-\sin x} = \text{RS (right side)}$$

Two students in the focus groups with foreign upper secondary school backgrounds talked about their teachers who continuously gave proving tasks to the students and after that some of the students were expected to account for their solutions and prove statements on the board. They had very positive experiences about proof. The following quotation illuminates a master-apprentice relationship between a newcomer and an old-timer in a mathematical practice in an upper secondary school mathematics classroom.

> *"My teacher was so enthusiastic, came to the lessons and showed interesting theorems and proofs she had found in some book, she really engaged me when she proved theorems on the board, she asked all the time, how do you think, how would you start and very soon I had to do it myself on the board. On one side of the board I would prove and justify what I did and on the other side of the board she then showed how she would have done it and I could see what mistakes I had made. Sometimes I succeeded, sometimes not at all. But you sort of see how you think yourself and how the teacher wants you to think in order to arrive at an answer. I felt it was very good."*
> (S – B, 2004)

Students' own investigations (alone or in groups) that would lead to hypotheses or sometimes to proofs seem to be unusual in the Swedish upper secondary school according to the students. Over 80 percent stated that they had had such activities only once or twice a term or more seldom (70 percent). However, there was a small minority (3 percent) who stated that they had worked in an investigative manner during every mathematics lesson (*Figure 12*).

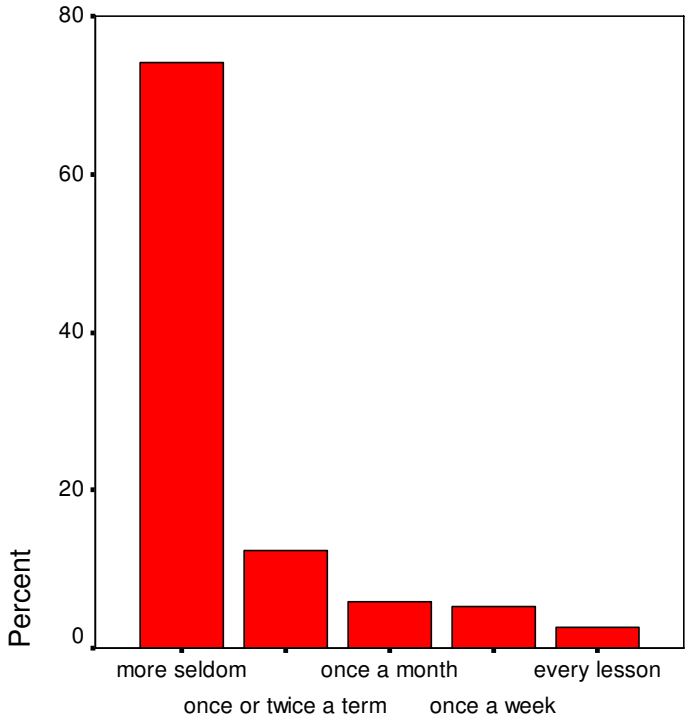

Figure 12 How often did you work on your own investigations (alone or in a group) that led to conjectures and sometimes to proofs?

There was one student with memories about working in an investigative manner in school who participated in a focus group discussion. He had worked in this manner during an optional course (F-course).

> *"We had a lot of group discussions during the F-course. We discussed the proving tasks and a lot of other things. We were maybe 15 students. The F-course is not so long, we didn't have so much time but a part of the course was used to investigate some problem."*
> (S – B, 2004)

Others did not remember working in this way. One of the students answered my question in the following way:

> *"In the pedagogy that I have been exposed to, you don't make any discoveries."*
> (S – B, 2004)

These findings were also supported by the textbook analysis that showed that tasks encouraging students to engage with investigations and conjectures

were largely lacking in the upper secondary school textbooks (Nordström & Löfwall, 2005).

The upper secondary curriculum states that one of the criteria for the best mark "pass with special distinction" is to be able to prove statements both orally and in writing. However, according to the students it was not usual to practice proving statements orally. Almost 80 percent disagreed or strongly disagreed with the statement *I have had the possibility to practice proving orally in school.* (*Figure 13*) Again, there was a small minority who stated that they had worked in that manner. The results are very similar to the responses to the question *How often could you orally prove mathematical statements in upper secondary school?* Over 70 percent of the students stated that they had done it more seldom (*Figure 14*). This is an example of two questions meant to measure the same thing and can be used to check the validity of the questionnaire. There was a (modest) positive correlation (with Spearman's rho 0.57 on the 0.01 significance level) between the responses to the statement (*Figure 13*) and the question (*Figure 14*).

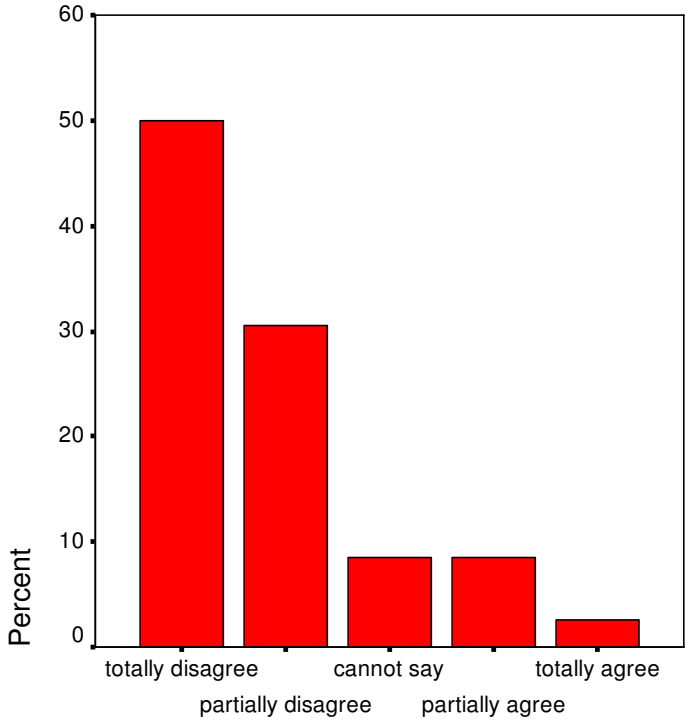

Figure 13 I have had the possibility to practice proving orally in school.

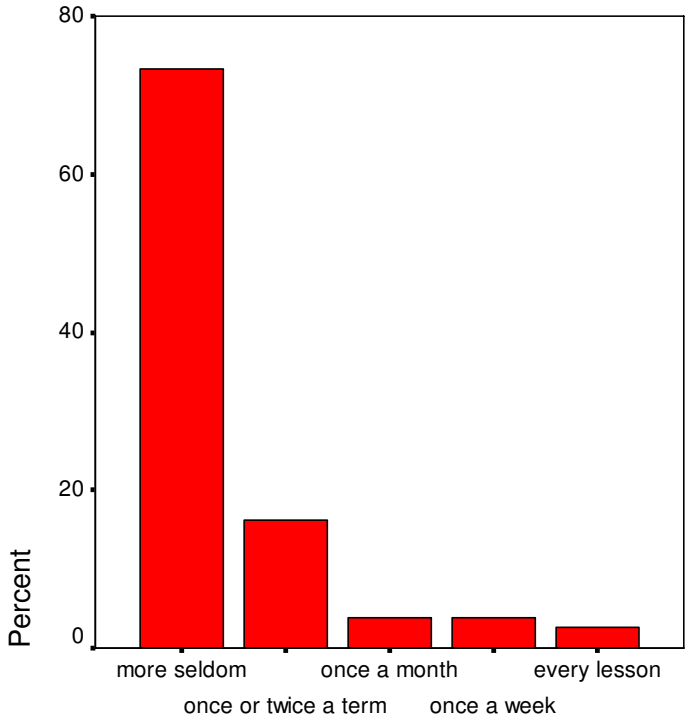

Figure 14 How often could you orally prove mathematical statements in upper secondary school?

Obviously, there were significant differences between the students' school backgrounds, as they remember it, concerning the possibilities of developing their proving abilities and their understanding of proof. There is a positive correlation (0.3-0.6 with Spearman's rho on the 0.01 significance level) pairwise among the questions (5, 6, 7, 9) and the statements (10:8, 21, 29), regarding the various kinds of proving activities posed in the questionnaire (Appendix 2). This shows that the students who stated that they had got exercise in proof also often stated that they had got it in different ways, whereas the students who stated that they were only a little familiar with proof also often stated that they had got experience only by teachers' proofs and derivations of formulas or not at all. Hence, there seems to be a small minority of newcomers who had got a lot of exercise in proof in different ways and a minority who had very little experience about proof. However, the results of the data analysis show that students are in various positions when they start to study mathematics at the tertiary level. An unexpected result for me was that there are still many upper secondary school teachers that, according to the students, prove statements to them.

5.2 How did the newcomers relate to proof when they entered the practice?

In this section, I report how the newcomers related to proof when they entered the practice including their views of proof. The results of the data analyses indicate that most of the students related very positively to proof when they started to study mathematics.

I base the results reported in this section on the analyses of both the surveys in 2003 and 2004 and the focus group interviews in 2004. I exemplify the results of the analyses of the surveys with newcomers' utterances from the analysis of the focus group interviews. I also report some correlations between the students' stated experiences about proof and how they related to proof when they entered the practice.

5.2.1 Newcomers enter the practice

Many of the mathematicians' utterances conveyed a view of students as not being interested in proof. Nevertheless, already the analysis of the pilot study in 2002 showed that most of the newcomers related positively to proof (Nordström, 2003). The students wanted to learn more about proof and stated that they would have liked to have learned more about proof in upper secondary school. The surveys in 2003 and in 2004 confirm the results of the pilot study.

Want to learn more about proof

Over 80 percent partially agreed or totally agreed with the statement *I would like to learn more about mathematical proof.* (*Figure 15*)

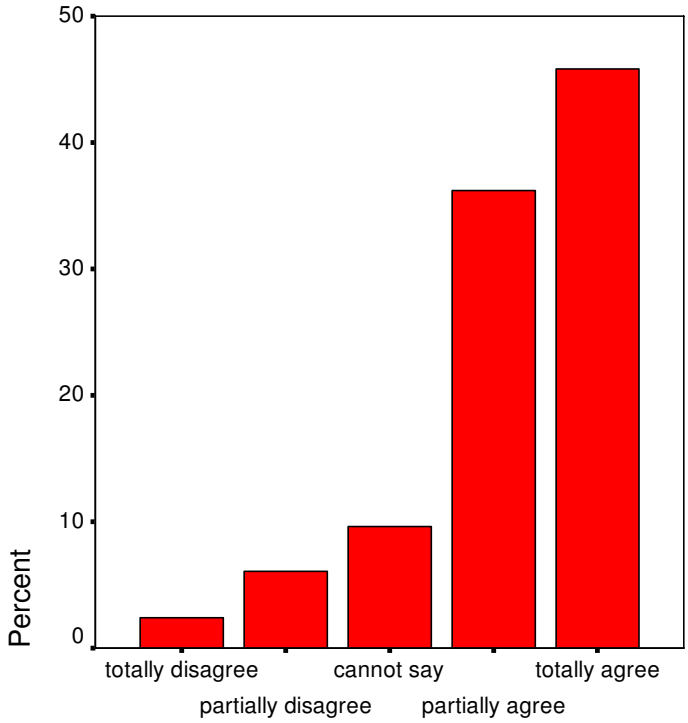

Figure 15 I would like to learn more about mathematical proof

Further, over 80 percent of the Swedish students partially agreed or totally agreed with the statement *I would like to have learned more about proof in school* whereas only 5 percent partially disagreed with it. No one totally disagreed with the statement (*Figure 16*).

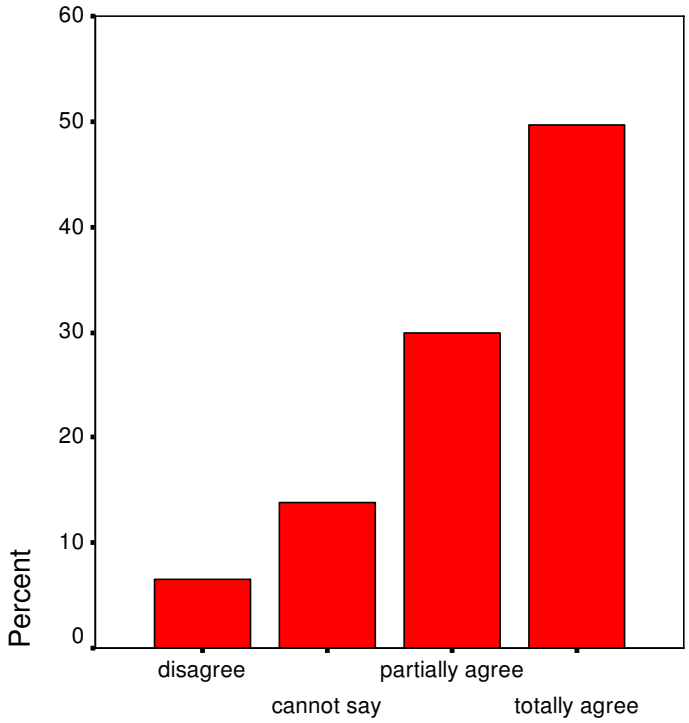

Figure 16 I would like to have learned more about proof in school

It is natural that many students related positively to the learning of proof when they started to study mathematics. A person who starts to study mathematics, has an orientation towards the mathematical practice right from the beginning, or has goals that have led the person to the practice. Students considered proof as an essential part of mathematics (see p. 150) and, hence, they were oriented towards learning more about it.

Want to understand

According to some mathematicians' utterances, most of the students just wanted to get their study points and get recipes but were not interested in the question "why?" (see p. 99) Yet, the survey analysis showed that most of the students when they entered the practice wanted to understand what they did in mathematics. Over 90 percent partially agreed or totally agreed with the statement: *I always want to understand what I do in mathematics.* (*Figure 17*)

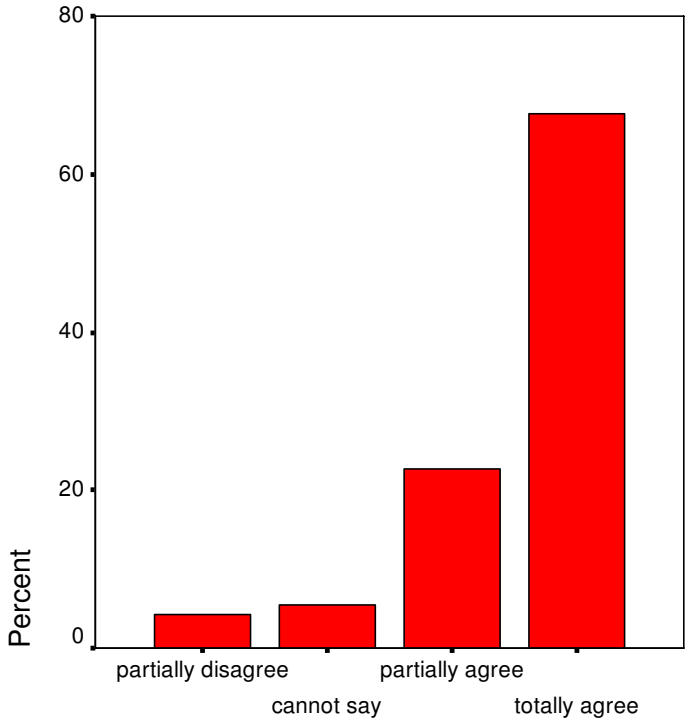

Figure 17 I always want to understand what I do in mathematics.

There were some mathematicians in my study who also stated that students were not willing to understand that it was better to derive formulas instead of memorising them (see p. 200). The survey analysis shows that the newcomers preferred the knowledge about how to derive formulas rather than recipes or memorising the formulas. Over 90 percent partially agreed or totally agreed with the statement: *It is good to be able to derive formulas* (*Figure 18*) whereas less than seven percent partially or totally agreed with the statement: *It is enough to be able to use formulas. It is not so important to understand everything.* (*Figure 19*)

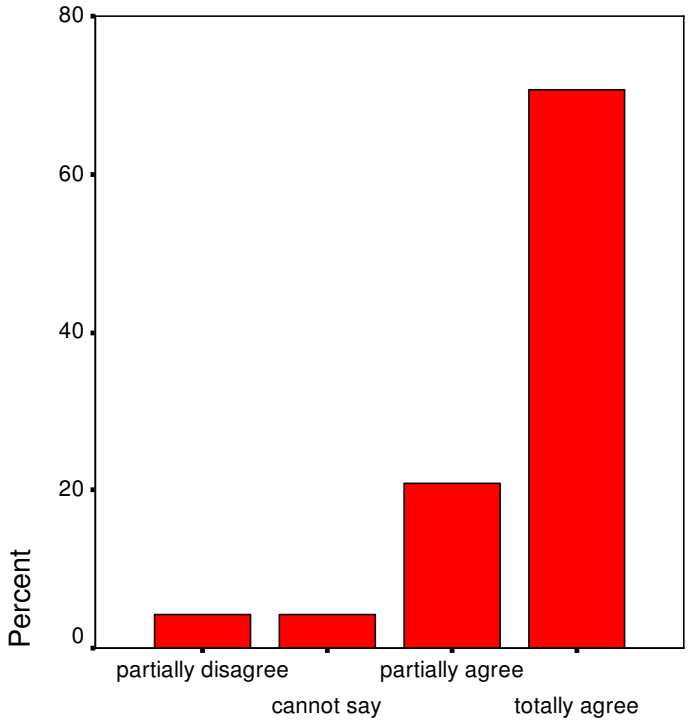

Figure 18 It is good to be able to derive formulas

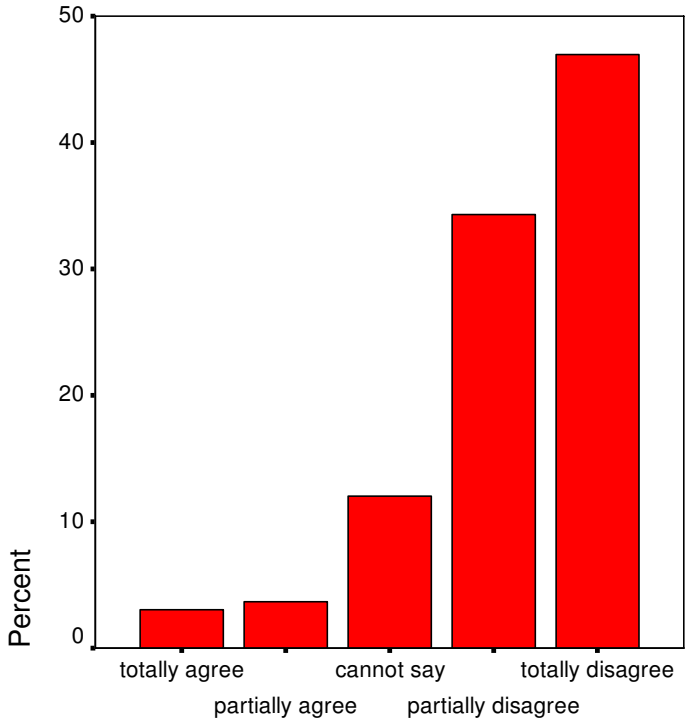

Figure 19[16] It is enough to be able to use formulas. It is not so important to understand everything.

Hence, students seemed to have a positive orientation towards understanding mathematics and the learning of proof, when they entered the practice.

Intellectual challenge

Many of the mathematicians who I interviewed stated that students were afraid of proving tasks. Students' responses to the multiple-choice question (2) (Appendix 2) about how they felt when they got a proving task show that there were slightly more university entrants who showed positive feelings than those who showed negative feelings when confronting a proving task (*Figure 20*). Many of the newcomers expressed a feeling of getting an intellectual challenge when trying to solve proving tasks. This is something that, according to the deductive style, was one of the aims of the proving activi-

[16] In the statistical analysis I reversed the values for some statements because of the correlation calculations. That is the reason for why the scale in the horizontal axis is reversed in Figures 19, 22, 23, 24, 29.

ties (see p. 115). Intellectual challenge is also one of the functions de Villiers (1996) sets up in his later model about the functions of proof.

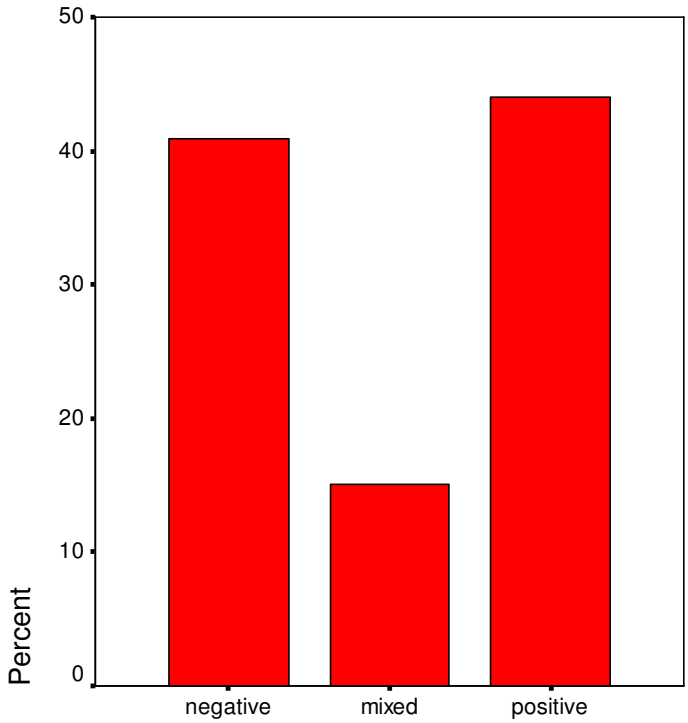

Figure 20 When I get a task that starts with "Show that…", I most often feel…(see Appendix 2)

Furthermore, one half of the newcomers who responded to the surveys stated that they liked to try to show/demonstrate mathematical statements (*Figure 21*).

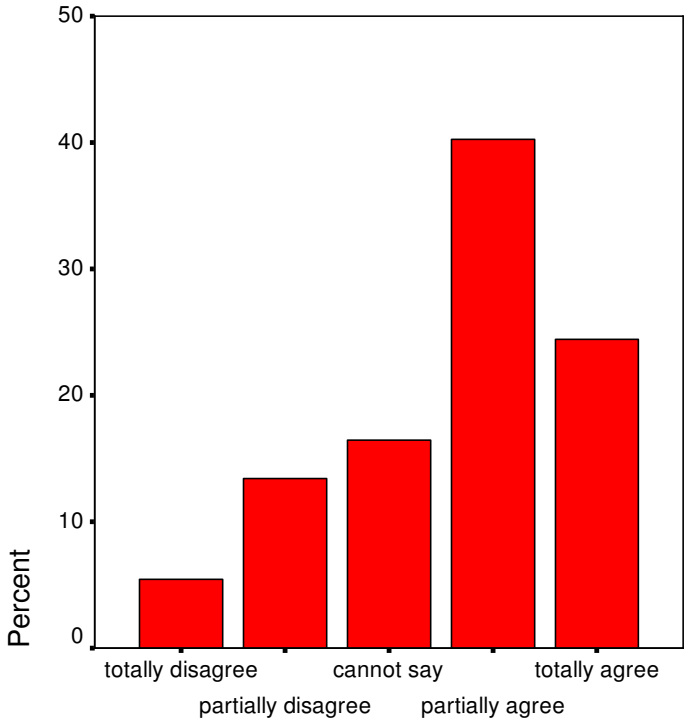

Figure 21 I like to try to show/demonstrate mathematical statements.

In focus groups students reflected on this question and many of them stated that it was interesting. They talked about an intellectual challenge in trying to find an elegant solution to a proving task.

> *"…to arrive at an elegant answer so that one gets it as pure as possible, that is something I find interesting."*
> (S – B, 2004)

About one half of the university entrants also partially agreed or totally agreed with the statement: *It is fun to construct mathematical proofs*. Yet, 36 percent of the beginner students stated that it was more tedious to prove statements than solve computational problems.

The mathematicians whom I interviewed stated that it was more difficult for the students to prove statements than solve problems (see p. 203). This is in accordance with the responses of the newcomers. A majority (85 percent) of the university entrants stated that it was more difficult to prove mathematical statements than solve computational problems (*Figure 22*).

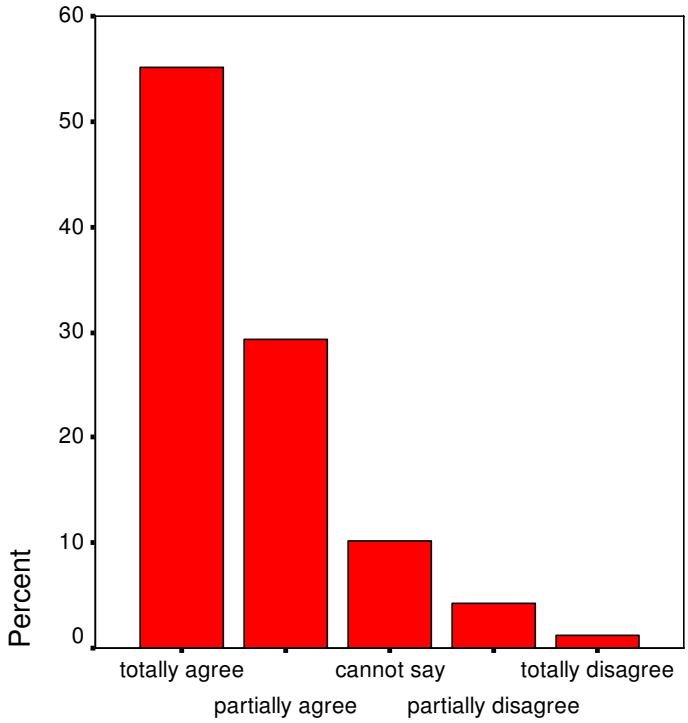

Figure 22 It is more difficult to prove statements than solve computational problems

Students in focus groups in different phases of their studies talked a lot about their difficulties with proving tasks. Here is an example of one who has just started to study mathematics and had experiences about proving tasks in upper secondary school mathematics. She had stated that she felt nervous when confronted with a proving task (Appendix 2, question 2). She also strived to reach an elegant solution.

> *"For me it's difficult to organise all my thoughts in my head, what is going to be first, what step do I take first in order to make it elegant. Often, when I get a task like that, I have to solve it twice, and make a fair copy of it so one can present it maybe in writing for someone else who is to understand how I have proved it. If the task is more difficult… therefore the anxiety."*
> (S – B, 2004)

But as described earlier, some students talked about the proving tasks in upper secondary school as simple and easy, especially after they had been in the practice for some time and met the proofs that they experienced as more difficult.

5.2.2 Newcomers' views of proof

In the previous chapter, I described how mathematicians reified proof (see p. 96). Students' views of proof were in many aspects similar to those of mathematicians already at the beginning of their studies. For example, a major part of the students considered proof to be an essential part of mathematics. They also stated that they appreciated the knowledge about how to derive formulas instead of just memorising them. There were not many newcomers who considered examples as correct proofs. However, they often convinced themselves of the correctness of formulas or statements by using particular numbers.

Learning proof is meaningful

The university entrants showed a participation identity concerning the proving of statements in many ways. Only three percent partially or totally agreed with the following statement: *I see no meaning with proof; Famous mathematicians have already proved all the results.* Almost 90 percent partially disagreed or totally disagreed with the statement (*Figure 23*).

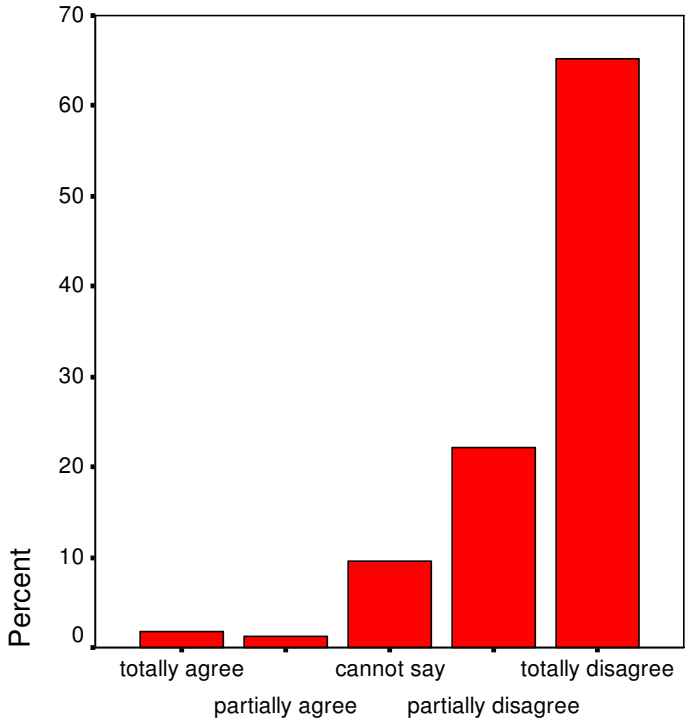

Figure 23 I see no meaning with proof. All the statements have already been proved by famous mathematicians

Further, about 85 percent of the newcomers partially or totally disagreed with the following statement: *If a statement seems to be intuitively true there is no need to prove it.* (*Figure 24*)

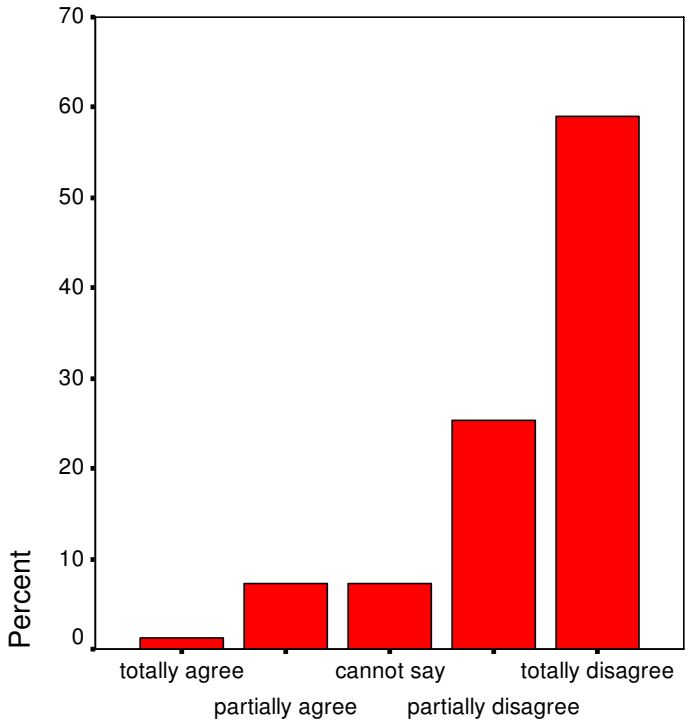

Figure 24 If a statement seems to be intuitively true there is no need to prove it

Most of the students, like the mathematicians in my study, already considered proof to be an essential part of mathematics when they began their studies. About 90 percent of the students in the surveys partially or totally agreed with the statement that proof was an *essential part of mathematics*. (*Figure 25*)

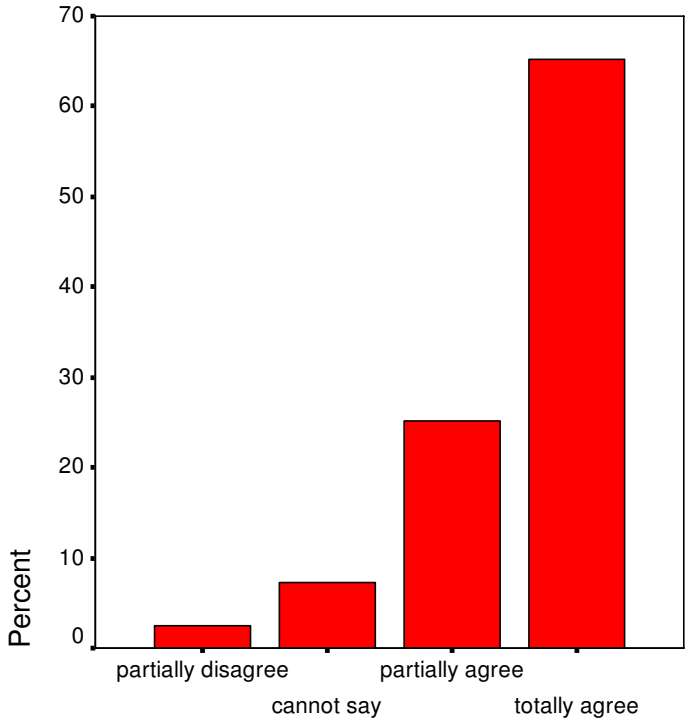

Figure 25 Proof is an essential part of mathematics.

The aspect of transfer (see p. 93) in terms of logical thinking got support from the university entrants. Over 80 percent of the students partially agreed or totally agreed with the statement *Proving statements teaches me logical thinking*. (*Figure 26*)

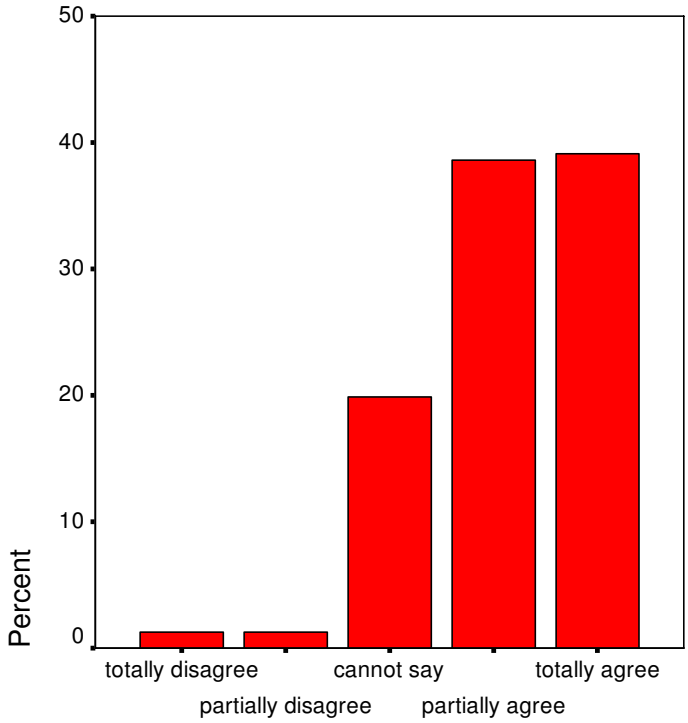

Figure 26 Proving statements teaches me logical thinking.

The aspect that I call transfer was an aspect many mathematicians also talked about as a reason for why one should learn proof (see p. 93). Another aspect that mathematicians talked about was that proofs helped one to understand how everything in mathematics was related. Most of the students shared this view. Over 80 percent of the university entrants partially or totally agreed that proofs helped them to understand mathematical connections (*Figure 27*).

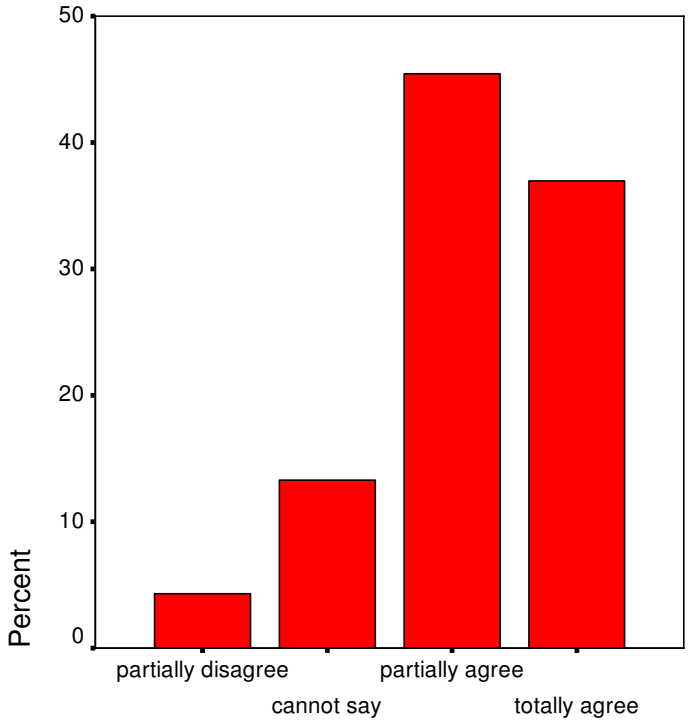

Figure 27 Proofs help me to understand mathematical connections.

Many mathematicians stressed the explanation aspect of proof, and the students in the focus groups also viewed proof as a kind of explanation.

Real mathematics and critical thinking

Further, in the students' focus group discussions, proof was strongly associated with "real mathematics" and understanding in contrast to school mathematics, which was claimed to be rule learning and applications of formulas without understanding. Already some of the newcomers who had just started to study mathematics conveyed this view.

> *"I think it's another thing here. In upper secondary school we had a lot of rules, you learn a lot of rules and then you just go ahead. There is nothing to understand. But here it is more like…he stresses it all the time, to count is not mathematics but mathematics is the understanding of it and that is exactly the point."*
>
> (S – B, 2004)

The previous quote exemplifies how students soon adapted the same way of talking about proof as mathematicians (compare in the previous quote: "…he (the teacher) stresses it all the time…"). Students achieved an increased un-

derstanding of how and what old-timers did and talked about in the mathematical practice, what they respected and admired and in that way got a possibility to make the culture of practice theirs. In particular, after the first examination on proof in *Mathematical Analysis 3* students talked about school mathematics as doing sums and applying formulas, and university mathematics as proof, derivation of formulas and the understanding of mathematics .

Some beginner students also spoke about proof as questioning the evident.

> *E: I agree with you that one should begin early.*
>
> *L: The very idea to question the evident.*
> (S – B, 2004)

That is something that mathematicians also connected to proof (see p. 114). In the conceptual frame (see p. 45), I also discussed the value of proof for students as a way of learning critical thinking and questioning the evident.

Are mathematical proofs different from proofs in other sciences?

About half of the university entrants regarded mathematical proofs as different from other kinds of proofs (*Figure 28*).

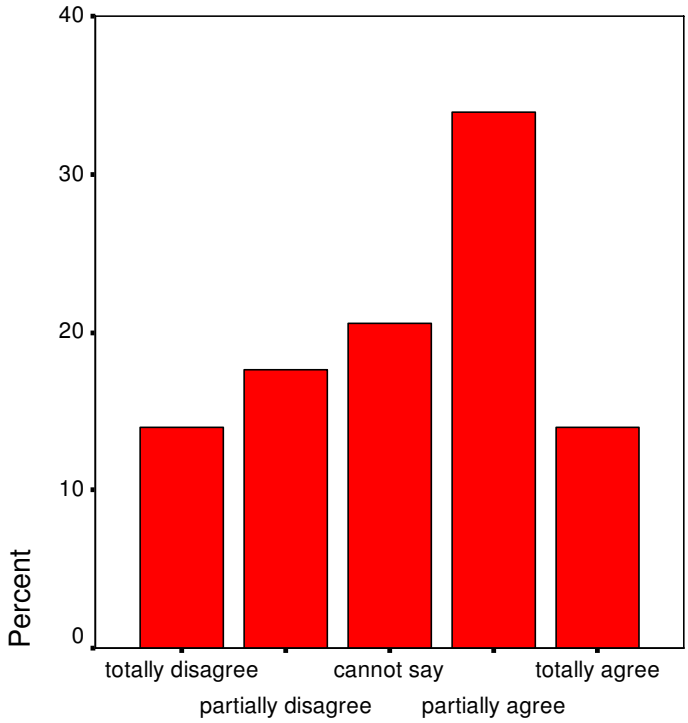

Figure 28 Mathematical proofs are different from other kinds of proofs

Proof was seen by many of the students participating in the focus group interviews at the beginning of their studies as exact and infallible and, therefore different from proofs in other sciences. This is similar to the view conveyed by some mathematicians as well (see p. 90).

> *" …mathematics is by and large an exact science, it is by and large freed from these worldly variables…as many other proofs are based on observations, logical, I mean somehow logical conclusions, and so, and sure they are similar but they are different in a quite fundamental way."*
> (S – B, 2004)

> *"If something seems to be reasonable in mathematics, then it is valid without any doubt."*
> (S – B, 2004)

> *"It's nice* (skönt) *because everything is clear and all the definitions are definition that gives meaning, so to say. You can sit in an empty room with totally white walls and just calculate and calculate and arrive at the most wonderful things* (världens grejer)*. You don't have to enter the outside world at all, and that is what is so cool with maths."*
> (S – B, 2004)

"In physics I became irritated when we came to Einstein's theories and all Newton's laws collapsed in certain circumstances so there are no exact laws either."

(S – B, 2004)

But there was also a student, who had been in the practice for a while, who considered the social dimension of the acceptance of a valid proof. He said he had understood that proofs had to be accepted by somebody.

"When I started to study mathematics I was really convinced that mathematics was the only place where you really could prove something that was true forever. I thought like that quite a long time but when I started to study Foundations of analysis I realised how difficult it was to prove and, above all, understand a proof… But as I think now, proof is not correct forever, but only as long as somebody has accepted it as a proof. And there is always a human being who says "This is valid as a proof".

(S – A, 2004)

The previous quotation illustrates the view that there is an authority, a textbook or a mathematician, who judges when a proof is valid and when it is not.

Induction/Deduction

However, as I discussed in the theory chapter (see p. 44) proof is not always an explanation or verification for the students. The analysis of the responses to the multiple-choice question (4) (Appendix 2) and the focus groups' reflections on the question shows that even if a majority of the newcomers did not consider examples as correct proofs they sometimes convinced themselves with "proofs" with specific numbers. Sometimes they also wanted to put numbers in general formulas or proofs in order to understand them better.

According to the responses to the multiple-choice question (4) university entrants did not consider specific examples as correct proofs. Most of them preferred the algebraic proof rather than the other alternatives. Almost 70 percent of the newcomers chose Lisa's or Peter's proofs (or both of them) as a correct proof. Mattias' proof was chosen by 12 percent of the students. Only 4 percent chose the specific example as a correct proof.

I met three focus groups who had responded to the questionnaire and asked them to comment on their choices of the correct proof. In one of these focus groups the students reflected on Tove's example in the following way.

A: It only shows the special cases.

C: It doesn't say anything about numbers greater than hundred or even greater than six.

(S – I, 2004)

However, this is the way some of the students in these groups checked, for example correctness of rules or formulas. Students in the focus groups talked about testing formulas with particular numbers in the following way:

> *K: No one has chosen Tove's answer...*
>
> *J: But I quite often do it like her. If there is a statement that seems to be a little fuzzy I usually put in some numbers and check if it is right, that I do almost all the time…*
>
> *M: Yes, I also do that, on a scrap paper.*
>
> *J: Yes, exactly.*
>
> *M: If this is correct.*
>
> *J: Yes, exactly.*
>
> *M: But you keep it to yourself.*
>
> *J: Yes, exactly, you have proofs very often as a…*
>
> *N: The teacher has often quite complicated steps and if you want to see clearly how it goes, you can put numbers there as an example and that is sometimes a good way to teach too.*
>
> (S – B, 2004)

The discussion above shows, that newcomers often preferred particular specific numbers when testing some properties or, for example, convincing themselves of the correctness of a formula. However, they felt that that was something private and not accepted in real mathematics. About 60 percent of the beginner students who responded to the surveys stated that examples more or less convinced them about the truth of a mathematical result. Whereas mathematicians in my study stated that they could not be convinced and go on if they lacked a general proof (see p. 91).

N's utterance above shows that putting special numbers in the proofs helped her to understand the steps taken in the arguments (see p. 44). Whilst some of the students preferred algebraic symbols because of the general structure that they illustrated in proofs they felt that the use of them made mathematics more abstract and thus difficult, which the following dialogue shows. This could partially explain why students preferred special numbers rather than algebraic symbols when they wanted to understand a formula or a proof.

> *E: You see the system when you use* a *and* b.

M: But at the same time it becomes more abstract and therefore possibly more difficult to understand.
(S – B, 2004)

The previous example also shows that there are different sorts of understanding. With the help of algebraic symbols you can understand structures and systems. However, when testing formulas with examples students could become familiar with these structures. Thus, it seemed to enhance other kind of understanding of them. About half of the students responding to the surveys partially or strongly agreed with the statement: *It is easier to understand that a statement is true after seeing an example than after seeing a proof.* (*Figure 29*)

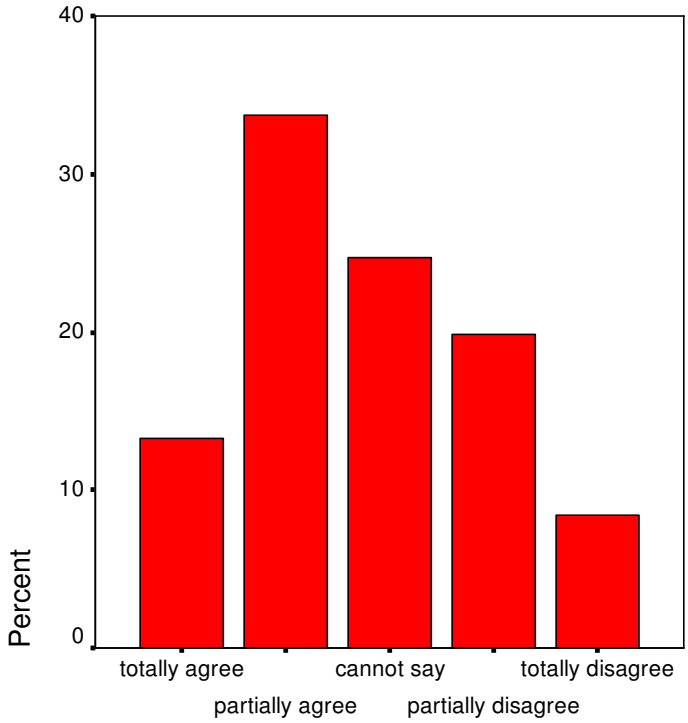

Figure 29 It is easier for me to understand that a statement is true after seeing an example than after seeing a proof

Similar results as reported above are well documented in mathematics education research (e.g. Balacheff, 1988; Chazan, 1993).

Language and rigour

Many of the students in the focus groups in different phases of their studies seemed to think that proofs had to be written in a special language with

mathematical symbols. This is interesting because not many mathematicians stressed the importance of the language or rigour. However, the use of correct mathematical language is one of the grading criteria in upper secondary school mathematics. What distinguishes the best mark from the others is that the students "demonstrate clear thinking in correct mathematical language". The following examples illustrate how the demands/ examples of formal language in the practice had influenced students' views of proof.

> *"…I think the first one* (Lisa's answer, question 4, Appendix 2) *looks most like a proof, mathematically correct. I think I chose it because proof is also a lot of different variables and most often not written in ordinary Swedish. It is shortened by the use of all the strange symbols."*
> (S – I, 2004)

I will come back to the formal demands regarding proof as students talked about them in Section 5.4 where I describe students' participation in the mathematical practice and their difficulties in constructing proofs.

5.2.3 Some correlations

Besides the percentages, I investigated correlations between different statements and questions concerning students' experiences on the one hand and how they related to proof on the other hand. I also examined correlations between statements concerning how students related to proof in order to validate the questions and investigate if the negative and positive responses correlated to each other.

There was a positive correlation (Spearman's rho on the 0.01 significance level) between the statement *I would like to learn more about mathematical proof* and the following statements. The correlations are given below after the statements.

- *Mathematical proof both verifies and explains* (0.29)
- *Proof is an essential part of mathematics* (0.29)
- *Proofs help me to understand mathematical connections* (0.30)
- *It is fun to construct mathematical proofs* (0.49)
- *Studying proofs teaches me logical thinking* (0.50)
- *I like to try to show/demonstrate mathematical statements* (0.49)
- *It is nice to know some mathematical proofs* (0.51)

We can see that the correlations here are from low to modest (Bryman & Cramer, 1990). Hence, those who related positively to proof sometimes showed it in different statements. In accordance with this result, there was a modest negative correlation (Spearman's rho on the 0.01 significance level)

between the following statements and the statement *I would like to learn more about mathematical proof*:

- *If a result seems to be intuitively correct there is no need for proving it* (0.44)
- *I see no meaning with proof; famous mathematicians have already proved all the results* (0.36)
- *It is more boring to prove statements than to solve computational problems* (0.43)

These correlations indicate that there is a small minority of the students who, already at the beginning of their studies, show negative feelings about proof in many aspects and hence, show a non-participation identity regarding proof.

There was a background question in the questionnaire (Appendix 2) about students' motives for studying mathematics (1). The majority of the students stated that they studied mathematics because they liked it and were interested in it (*Figure 30*). I wanted to investigate if there was a relation between the motives for mathematical studies and students' relation to proof. I was interested in finding out if those who studied mathematics for some external purpose were less interested in proof than those who stated that they studied mathematics because they liked it.

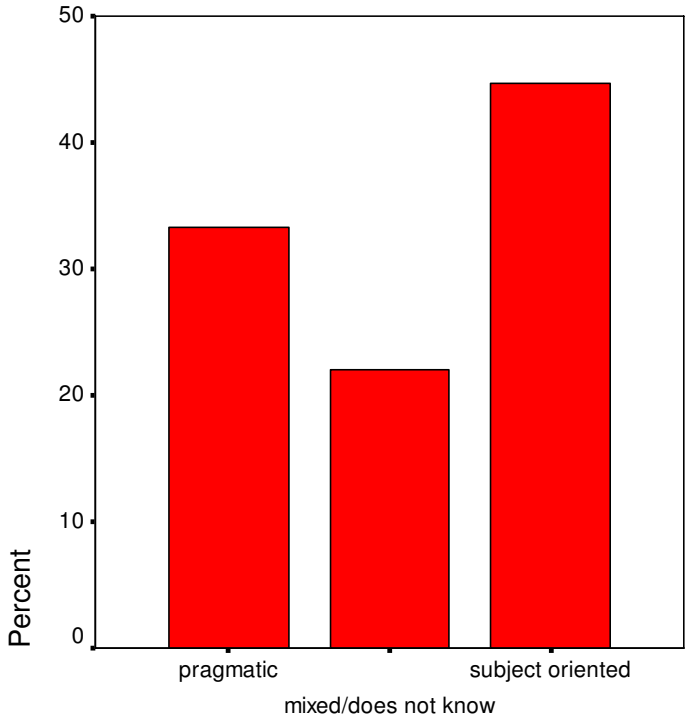

Figure 30 Why do you want to study mathematics?

I found no relation between the reasons for why the students wanted to study mathematics and how they related to proof. On the other hand, there were some interesting relations between students' stated upper secondary school experiences and how the students related to proof. There was a modest negative correlation (0.41 on 0.01 significance level) between the statement *It is difficult for me to prove statements* and on the other hand, the question *How often did you practice to prove statements yourself in upper secondary school*. There was also a negative correlation (0.38 on the 0.01 significance level.) between the statements *It is difficult for me prove statements* and *I have had exercise enough in constructing proofs in school.* Hence, those who stated that they had not had exercise often stated that it was difficult for them to prove statements. This seems quite natural.

When choosing the students for the focus groups, I also tried to get in touch with students who related negatively to proof. It was difficult, because those who wanted to volunteer most often showed an identity of participation concerning proof. However, I managed to engage some students at the beginning of their studies who related negatively to proof. One of them (L) had responded to the question 2 (Appendix 2) above in the following way:

L: *"The tasks dealing with proof very often lead to strenuous lines of reasoning."*
(S – B, 2003)

His experiences about proof in upper secondary school were quite poor according to his responses to the survey questions and what he said during the focus group discussion. He stated that he did not want to learn more about proof and he would not have liked to learn more about proof in upper secondary school either. In the focus group discussion he explained his responses in the following way.

L: *"No, because I didn't have very good or especially fun memories of proving, because in the proving situation the teacher when teaching was even more isolated from the class than usual."*
(S – B, 2004)

The rest of the focus group had more experience of proof and also related positively to proof. In the end of the discussion, student L seemed to have changed his way of talking about proof and showed some curiosity when he at last stated:

L: *"I notice that I've really been starved of proofs* (svältfödd på bevis) *as a matter of fact, I noticed this when we went through the sine-, cosine- and tangent laws here* (at the university)*. I had never seen them, well, I had seen them but not the proofs for them so I had never given it so much thought …"*
(S – B, 2004)

To sum up the results in this section, contrary to what many of the mathematicians thought, most of the students related very positively to proof when they entered the practice. They wanted to learn more about proof. They also viewed proving as a meaningful activity. Most of the newcomers considered proof as an essential part of mathematics. Students seemed to adapt quite quickly to the view conveyed by mathematicians on proof as real mathematics in contrast to upper secondary school mathematics that was seen as rule learning. According to the theory of Wenger (1998), building an identity consists of negotiating meanings of our experiences in social communities. Right from the beginning, when students enter the practice, being in the practice influences their identities. They are forging their identities, and in order to participate, they must gain some access to the history they want to take part in, so they start to make it part of their identities.

According to Wenger, one's identity is never constant, it is forever changing. Although the majority of the students related positively to proof and the learning of proof when they entered the practice, many of them very soon

reported a lot of difficulties in understanding the proofs and the meaning of proofs they met at the university. That is something I will describe and exemplify in the following four sections where I report what different kinds of occasions for negotiation of meaning concerning proof there were available to the students in the mathematical practice and how students in different phases of their studies talked about their experiences in the practice.

5.3 Newcomers' participation in the lectures

Wenger (1998) defines four components to characterise social participation as a process of learning: doing (practice), belonging (community), experience (meaning) and becoming (identity). These components are mutually defining and interconnected. As described in the theory chapter, students participate in the practice when they, for example, grapple with their lecture notes and try to follow proofs and mathematical arguments, or when they discuss a new concept with some other students or when they take part in the examinations. In all these activities students can experience meaning and belonging. The participation also influences their identity. In this section, I describe students' participation in lectures.

There are various ways for mathematicians to present mathematics in the lectures. They can focus on examples and applications, or they can also start with a definition and go on presenting the contents deductively. Mathematicians can also present proofs in various ways in the lectures. The most usual way, according to the observations of the lectures and to the mathematicians' and the students' utterances, is to present proofs by writing on the board without further discussions or structuring. However, there are differences in how carefully mathematicians write all the steps in proofs. Most of the students participate by writing lecture notes and/or by trying to follow the reasoning. Outside the lectures students can study the lecture notes, textbooks, old examinations and other material and try to understand the lines of reasoning in them. Students can also get help from a teaching assistant if there is something they wonder about.

Almost all the mathematicians who I interviewed claimed that they avoided dealing with proof in the introductory course for different reasons (see p. 198). This is interesting because students in the focus groups talked about an experience from the beginning of their university studies where mathematicians were proving a lot of statements during the lectures. I will come back to this issue in Section 6.1.2 when contrasting the results of the mathematicians' practice with the results of the students' practice.

5.3.1 Possible hindrances for students' engagement in the lectures

Engagement in practice gives us certain experiences of participation. Wenger (1998) emphasises that in order to support learning, engagement requires access both to the participative (interaction with other participants) and the reificative (symbols, tools, language, documents, and the like) aspects of practice (*ibid.*, p. 184). This could mean students' possibilities to interact with other participants but also students' confidence with mathematical language, symbols and proofs and other reifications in the practice. There were a lot of possibilities for the students to interact with each other but they had not always absorbed the important reifications or even part of them into their practice. So that was often a hindrance for their participation and also for interaction. Students have not created the rules of reasoning or the mathematical language, theories or proofs but these reifications have to be reappropriated into a local process by students in order to become meaningful for them (see p. 37). One hindrance for students' engagement in the negotiation of meaning during the lectures would also seem to be the high tempo of the lectures.

Tempo

Moreover, the focus groups somehow touched of the high tempo in the presentation of mathematics in the lectures.

> *"…They go through the things very fast in the lectures and if they are to prove something, for example logarithms, you have to be absolutely clear what a logarithm is and how it can be rewritten, all these rules, and if you don't do that you cannot follow* […] *I noticed that the tempo is much faster also with proofs, not so many comments on what the teacher does as in upper secondary school."*
>
> (S – B, 2004)

Also some mathematicians complained that they had to go through a lot of material in a short time and they experienced that they had less time now than earlier for the students. According to the statistics about the organising of the lectures, before 1997 there was one mathematician with about 30 students and three hours for lectures where they could combine the theory with applications (Strömbeck, 2006). Now there is a lecture of two hours and one mathematician with about 100 students, so there is less time for the contents if mathematicians try to do the same within two hours as they did earlier in three hours (see p. 16).

Language

The notion of reification in Wenger's theory refers to abstract and concrete objects that are the projection of our meanings into the world. In the mathe-

matical practice, for example mathematical symbols, definitions, theories and proofs can be seen as typical reifications for that practice. Engagement in practice requires access to all these reifications. According to the students, the mathematical language at the university level was different from that of upper secondary school. Also the lack of experience with mathematical symbols was mentioned as a hindrance for students' participation.

> *"Now it's much more mathematical language, in upper secondary school it was possible to say everything with a little easier language and when the language becomes more difficult it's difficult to fully follow. You have to know these different symbols they write on the board, what it means."*
> (S – I, 2004)

Yet, some of the students stated they had got a short introduction to logical symbols during the introductory course. They also pointed out that it was possible to study them in the textbooks. Vretblad's (1999) book was mentioned as informative in this aspect by some of the students.

> *"In a chapter in Vretblad's book, Algebra och kombinatorik, you can read what those things mean and I think that's the best description I have found or read but that is nothing one follows up in the courses…it's difficult to work with the notions by yourself."*
> (S – I, 2004)

The students studying advanced courses had also experienced the hard tempo and the learning of terminology as a problem. Hence, they must have tolerated a certain amount of experiences of non-participation in order to proceed in the practice.

> *"I've experienced mathematics as a language where you have to learn a lot of words and grammar. And if you forget some words the teacher talks and talks and it can take some minutes before you recall the meaning of the words but then the teacher already talks about other things and you've lost the thread a long time ago. And the rest of the lecture you just try to catch up. So it's very demanding with so many new words and expressions that we are expected to know from the first time we hear them, which we don't. Instead, you have to drill them and that takes time. Often you haven't learned it until the end of the course."*
> (S – A, 2004)

Concerning the mathematical language and symbols, there were two discernable styles in mathematicians' ways of talking about them. According to the deductive style they should be introduced and used from the very beginning in order to accustom the students to them at once whereas within the progressive style a teacher should try to avoid them at first, and then gradually introduce them (see p. 110).

Rigour

Sometimes gaps in the proofs and small careless mistakes in the lecture notes can cause a lot of problems for students' understanding when they come back to the lecture notes at home.

> *"I feel that when I read the proofs, sometimes they don't write all the arguments and I think they assume that we understand that some assumptions have some implications and then they do not write these implications…I'm not so used to, for me it's a gap in my thinking. And then I can stand for hours without understanding the proof until someone explains the implication for me and then I feel: Oh, now I understand! And that's something disturbing that they don't always write all the arguments."*
>
> (S – A, 2004)

That is something, I also observed in the lectures. Mathematicians seem to usually be in a hurry when they write proofs on the board. It is often difficult to follow and control that all the steps in proofs are correct. Also in the textbooks for the basic course, often some steps are left for the reader to justify.

There are differences in the presentation of mathematics between different mathematicians with respect to rigour according to the students and the observations of lectures. With rigour, I refer to how carefully mathematicians write and justify all the steps they take in their presentation of mathematics and proofs.

5.3.2 Different approaches among teachers and students

What makes engagement in practice possible and productive is as much a matter of diversity as a matter of homogeneity (Wenger, 1998). In the community of practice of mathematics at the mathematics department, there are diverse views on how to teach and present mathematics in the best way, both among the old-timers and the newcomers. In this subsection I present two aspects, on which there were different views, rigour and inductive/deductive approaches.

Rigour

Different teaching styles relate differently to rigour. Some of the mathematicians stated that they did not like rigorous proofs (see p. 124), whereas others wanted to show every step especially for the newcomers (see p. 116). It is not only the mathematicians who have various views on how rigorously mathematics and proofs should be presented to the students. As the next dialogue shows, there were differing views among students as well. Some of them wanted the proofs to be properly and rigorously presented whereas some of them wanted to get inspiration from the lecturer in forms of extended information for example about the history of mathematics. They stated that they could study the proofs in the textbooks.

I*: But the most important thing for me is the structure of the lectures, because there are such huge differences between the teachers. A teacher who can really hold the planned structure and has certain techniques for the black board writing and writes the important things and goes through everything step by step, it gives much more than someone who just says "This can be proved with combinatorial methods" and then writes down something incomplete and then goes on to the next subject.*

…

P*: You notice very clearly that certain teachers have very different views on their role as teacher, I mean what is expected of them as teachers. Because some of them, a few of them anyway, think that they are to be there and babble in general terms and like inspire students to study the textbook themselves. And some think that they really should teach everything on the board. And these are quite different things and it's clear that if a student expects one thing and the teacher is of the other sort, then it becomes quite strange…*

I*: …I think that I'm kind of a structure person and I totally lose the appetite for learning maths if they just stand there and prattle and don't even finish the examples.*

P*: It's as if I fall to sleep if they like go through a proof extremely carefully like* […] *But, anyway, it still reads very clearly in the textbook, there are definitions, theorems and proofs on and on and on.*
(S – A, 2004)

I argued in Section 3.4.4 that, according to the theory of Wenger (1998), the condition of transparency can be considered from the point of view of teaching as well as learning. Teachers' intentions of focusing on certain things in their presentation do not necessarily imply that these aspects become visible to students (see pp. 27 and 60). The following dialogue also illustrates how differently two students saw the same presentation. The student *A* did not see any structure in mathematics by studying proofs, theorems and definitions whereas the student *B* stated that it was there she saw the structure.

A: *"I think, I want to get new ideas, want to see patterns and such things, that is what is interesting, not proof, theorem and definition and so on."*

B: *It is there* (definition, theorem, proof) *I see the patterns.*
(S – A, 2004)

The first student (A) gets a structure by patterns and new mathematical ideas and finds them interesting whereas the student B sees the very pattern in definitions, theorems and proofs. There were similar examples of the differences between students in other focus groups as well.

Induction/Deduction

Some students had recognised different styles in mathematicians' approaches also concerning the aspect of Induction/Deduction and stated that some mathematicians use more time presenting examples of different problems to students whereas some others are more concerned with presenting the mathematical theories to them.

> *"Some of them are a little more interested in giving examples and calculation tasks and others think it's more important with the theoretical part of maths."*
> (S – I, 2004)

Some students complained about the abstract level of mathematics and wanted to go straight examples and applications. The following quotation is an example of a student, who, regardless of a lot of different kinds of experiences of proof in upper secondary school, did not appreciate the learning of proof very much. It was enough for her to see the proofs or know that they existed but the applying of formulas was the most important thing for her.

> *"But I think proofs are good but I have never bothered to learn any of them, not here either. When we sit in the lectures and they write a proof I stop writing. It's because, anyway, I don't go back to them but I don't think mathematics is the most exciting subject in the whole universe so I learn the formula and I've an extremely good memory…, and then apply, apply, apply and I know that this formula exists, it is proved and then you can just go on."*
> (S – I, 2004)

For students who had made progress and proceeded in their trajectories further in the mathematical practice, the meaning of mathematics seems less likely to be regarded as the applications of formulas for problem solving as it is for students studying at the basic level. Instead, they spoke of a pleasure to know proofs the results of which could be used in other mathematical contexts.

> *"I think such proofs are still quite fun, where you really can prove something profoundly or a big theorem where the results can be used in other contexts, maybe only to prove other results."*
> (S – A, 2004)

This view is very similar to that of mathematicians in my study. They stated that with the help of proof one could obtain general results that could be used in other mathematical contexts. The following two examples show the complexity regarding the best way to draw the newcomers into the mathematical practice and how to meet the different expectations of the students. Some of the mathematicians proved Taylor's theorem when they presented it to the students during the basic course in calculus, others did not do that.

Some of the students had difficulties seeing the point of studying the theorems.

> *"For example, this is the way the teacher introduced us into Taylor McLaurin…and, "Yes, I will take Taylor's and McLaurin's formulas, yes I start with the proof." So, I start with a proof and then I thought, what's this? I didn't know anything about what Taylor and McLaurin was at all, what are they good for, like Why? We started with the proof and then he went on for half an hour, I mean I didn't grasp anything. Then in the end of the lecture, anyway, he came to the applications. And if you don't grasp, you learn to shut it out because you don't understand anything, anyway."*
>
> (S – I, 2004)

Some students, however, taking advanced courses complained that the teacher did not make it clear from the very beginning that they did not need to memorise all the formulas but they could be derived using the Taylor's formula.

> *"The teacher should have said during the lectures that with this formula you can get all the Taylor developments but he didn't say it and I don't understand it…"*
>
> (S – A, 2004)

Finally, I offer an example about a student who liked to work with mathematical proofs in physics. The way in which the student in the next quote works with the proofs in calculus can be compared to studies that show that many students want to test the proofs empirically.

> *"…for example, in Analysis 4, I first studied all the proofs and all of them were applicable in physics, for example in electricity. So I worked with all these proofs once again in physics and proved them in a…and that gave me another kind of understanding to do the proofs without presuming this most general language at the beginning but I could assume that I really had something important and the proofs could be accomplished, I mean in principle, I wrote the same thing arguing in a convincing way because it was like reality."*
>
> (S – A, 2004)

There were also different views among students on whose responsibility it was to enhance discussion in the lectures or whether it was possible to hold a discussion during the lectures. There are some examples of it in the following two subsections.

5.3.4 Participating in the negotiation of meaning by posing questions

It is possible for students to pose questions during the lectures. This is not usual according to the mathematicians, the students and the observations of lectures. The questions could enhance students' identity of participation/non-participation depending on if they understand the questions that the other students pose and could follow the answers. The following extract of a group discussion exemplifies students with experiences of non-participation and a student with experiences of participation concerning the posing of questions. One of them (B), saw the possibilities to participate whereas the others saw the hindrances. There were also different views among the students in this group about whose responsibility it is to create discussions in lectures.

> *B: Then it's often that nobody dares to ask. We won't know if no one says anything. I think that you still can get help in the math library* (students can get help from a teaching assistant). *You have to go and ask if there's something you don't understand. That is my opinion.*
>
> *T: But there's not a very open atmosphere for questions.*
>
> *G: No.*
>
> *T: It's a little like if you ask something you are stupid.*
>
> *G: Absolutely.*
>
> *T: The atmosphere is very inhibiting.*
>
> *G: Yes.*
>
> *T: Why doesn't anybody ask when no one understands anything?*
>
> *B: But I don't feel like that at all…*
>
> *G: I would put it in this way. You must really be a certain type of a person to pose questions in this place it feels like, curious, not afraid, think it's fun. Many students are not like that.*
>
> *B: It's a pity that so many are not interested, maybe.*
>
> *T: No, but I've tried to ask, in many cases it's difficult to formulate the question, don't know if anybody even understands the question. Then one feels that one's a big problem. Take for example this one hour proof. If one would say at the end of the proof that one didn't understand, we would never get rid of that boring proof.*

G: And you don't want to ask before you have sat at home and tried to understand yourself.

(All agree.)

B: If you have prepared yourself before you come to the lectures and really have wondered, I understand this part of the proof but not the other part and when they give the proof you can ask that question… But I have to say that the mathematicians seem happy when somebody asks something […] *It shouldn't only come from the lecturer.*

G and T: *Mmm, yes*.
(S – I, 2004)

One mathematician also stated that not many students pose questions during the lectures. The groups are big and it is difficult to create discussion.

"One of the goals of the course Mathematical Analysis 3 is to teach students proof but there are too many in a group, about 50, so it's impossible for students to show something on the board. There are few who ask questions."
(M – 2003)

Students had different views on whose responsibility it was to create discussions.

" But it has to come, it can't just come from the lecturer."
(S – I, 2004)

Some students stated that the last lecture was aimed at dealing with students' questions. According to the following student, the questions were about problem solving, not proofs.

S: We have got the last lesson for questions.

K: Will there be any questions on proof?

S: During the last lesson, which is devoted to questions, there is more discussion, but not about proof because they are already forgotten, but calculation tasks and such.
(S – I, 2004)

But there were other experiences as well. Some students said that they had opportunity of posing questions and going through proving tasks during the last lecture.

"We have lessons where we go through calculation tasks and may pose questions and then the teacher takes it again, a little slower. Then one has time to understand a proof or a definition."
(S – I, 2004)

So, it depended on the teacher whether the last hour was used for problem-solving tasks or proving tasks or questions about proofs.

5.3.5 Engaging students in the presentations of proofs

During the lectures a teacher has a possibility to invite students to fill some gaps or in other ways engage them in the proving act. This could lead to enhanced demands on students' engagement in the negotiation of meaning during the lecture if students could follow the discussions. In my observing of the lectures, there was a mathematician who wanted to activate students by giving a proving task to them in the middle of the lecture. It was about mathematical induction. However, the high tempo seemed to be a hindrance again. The time for solving the task was short; the students did not have a chance to do it. After that the mathematician presented the proof himself. I had a focus group interview with four students who had participated in the lecture and they said that the initiative was good but the time too short. Besides, some of them advocated group discussions in the connection of this kind of tasks.

D: I like these kinds of lecturers who try to get into contact with the students and push them a little.

E: It's definitely better than "You understand this anyway".

K: Did you get enough time to solve it?

E: I think it would be better to discuss in small groups, for example, or that you do more than just wait for two minutes, it was far too short a time. You start to try and think, like, and then he gives away the answer himself.

F: The question is if there is time for such things at all. These courses are so intensive.

E: Time for?

F: I mean sit in small groups and so on.

E: You could also say that this is something you can solve in groups for tomorrow. It doesn't need to take time from the lectures. It doesn't need to be run by the teacher, so there are pedagogical refinements.
(S – I, 2004)

Student *F* had apparently been enculturated to the practice and "understood" that there was not so much time, for example for group work, because there were a lot of mathematical contents that needed to be dealt during the courses. However, just as in the example about the posing of questions, there were different views among the students as to whose responsibility it was to create discussions among students. One of the students pointed out that for example, discussing proof in small groups does not need to be run by the teacher, nor take time from lectures. However, he seems to expect that this kind of work would be organised by mathematicians. Participating in the lectures is the students' response to the conditions in the practice and, according to the theory of Wenger (1998) students are active agents in that practice and can make use of the possibilities there. However, students have various identities and backgrounds and they all respond to the conditions in the practice in unique ways.

In this section, I described issues concerning students' possibilities to participate in the negotiation of meaning regarding proof in the lectures. A hard tempo, students lacking knowledge about mathematical symbols and language, and gaps in the presentation of proofs could hamper students' participation. I concluded the section with two examples where students talked about how to create discussion and participation in the lectures and whose responsibility it was to do that.

In the next section, I describe students' participation in constructing their own proofs.

5.4 Constructing their own proofs

Sometimes students get proving tasks that they themselves should solve by constructing their own proofs. This happens by working alone or together with someone outside the lectures. Most of the mathematicians I talked with claimed that proving tasks are difficult for students (see p. 203). The results of the survey analysis also showed that students consider proving tasks to be more difficult than problem solving right from the beginning of their studies (see p. 147).

5.4.1 Students' difficulties

The students in the focus groups talked a lot about difficulties they experienced when constructing their own proofs. They did not know how to start, they did not know what they could take for granted and when they had proved the statement. These difficulties were also identified and reflected on in mathematicians' utterances belonging both to the progressive and the de-

ductive approaches. They are also documented in earlier research (e.g. Moore, 1994).

Proving tasks are more difficult than those in upper secondary school

The students who had had some experiences with proofs in upper secondary school talked about the difference between the proving tasks there and at the university. They claimed that proving tasks were much more difficult at the university.

> *K: Now you have studied here for one year. How do you experience the difference between upper secondary school mathematics and university mathematics regarding proof?*
>
> *L and J: The proofs are much more difficult.*
>
> *J: Now the proving tasks are the difficult ones, it's usually difficult to understand the proofs because they are so abstract and they have to work in so many different cases and then it's impossible to understand how one has derived them, the proofs and how we ourselves…*
>
> *K: Do you agree?*
>
> *L: Yes, I do, it's become much more difficult.*
> (S – I, 2004)

According to the students, the proving tasks in upper secondary school were most often of the type "Show that the left hand side equals the right hand side." (see p. 135). The students appreciated the comments on their proofs that they sometimes got from mathematicians but it was not usual according to the students; it sometimes occurred in the lessons and in the scope of more advanced courses.

Have I really proved it?

One of the difficulties they talked about was deciding when one was ready with the proof, when one had really proved the theorem.

> *"Well, we often get tasks like "Show that" and then I often think afterwards: "Have I really proved it now?" I don't feel sure even though I've really managed to show what I am supposed to show. And it can depend on lack of experience, that you cannot decide yourself that it is enough now, now I've actually shown what I was supposed to show. And I can also be insecure, now I've learned certain things, for example what "if and only if" means, that you have to show that if something is true then something else is true and then also in the opposite direction. So you learn these kinds of tricks, how to construct a proof."*
> (S – A, 2004)

Students seemed to lack knowledge about elementary logic, for example what is meant by "if and only if" which the student in the quote above calls a trick. Some of the difficulties students talked about are well documented in research, for example the understanding of when one has proved something, when the proof is complete (e.g. Pettersson, 2004; Wistedt & Brattström, 2005).

What can I take for granted?

It was also difficult for the students to know what they could take for granted and what they had to justify. The difficulty to recognise what to take for granted and what to prove can sometimes be arbitrary and is not always clearly stated by the old-timers, as a mathematician pointed out in the interview. They are conventions that are seldom discussed (see p. 102).

> *"Yes, I also think it is really hard to decide if you are finished or not. And then some things that you prove you don't have to prove because they are supposed to be evident. There are other things you think are evident that you have to show and to find out what is what, is difficult. And that has to do with experience, I suppose…"*
> (S – A, 2004)

The difficulty to decide what to prove and what to take for granted was especially connected to Euclidean geometry by some mathematicians (see p. 102).

Logic and structure

Focusing on the logical structure of proofs is an aspect of visibility (see p. 54). The role of understanding the definitions for learning and understanding of proof was stressed in mathematicians' utterances belonging to the deductive style (see p. 120). The difficulties concerning the understanding of the role of definitions in proofs was also discussed by the students:

> *"We never met theorems or definitions in upper secondary school. Sometimes I still have difficulties understanding the difference. I think that a theorem can look like a definition."*
> (S – I, 2004)

There is not much discussion about proof or the logical structure of different kinds of proofs according to the mathematicians, the students and the observations of the lectures. One of the mathematicians stated that the opportunity to discuss proof did not present itself earlier than in the course *Foundations of Analysis* (Appendix 1).

> *"Not earlier than during the course Foundations of Analysis one learns to prove oneself and has a possibility to discuss proof."*
> (M – 2004)

Some students also talked about discussions during that course and during the advanced course *Algebra* (see Appendix 1).

> A: *In Foundations of Analysis and Algebra, which are such courses where one is to get understanding for how to prove, I think that one quite deeply goes through and talks about what a proof is and so, more than in other courses anyway.*
>
> B: *Much more than in the other courses.*
> (S – A, 2004)

However, there were differences in the way in which mathematicians planned these courses regarding discussions about proof, according to some students. Not all mathematicians seemed to give precedence to discussions.

> *"But is it really the meaning that one should drill theorem proof, theorem proof, theorem proof, theorem proof all the days? Shouldn't one discuss what one is doing on the blackboard?"*
> (S – A, 2004)

According to this student, the lecturer copied the textbook onto the board without any discussions.

According to the mathematicians, there is no discussion about proof techniques now when Vretbald's textbook is not in use any more (see p. 99). Thus, the newcomers try to find out many things about proof without the systematic guidance of the old-timers. The lack of the knowledge of elementary logic, for example, causes difficulties for the students. The learning of proving statements seems to happen quite randomly, if indeed if it happens at all. Some students on the advanced level said that they had felt that it was expected that they knew what a proof was and how to construct proofs from the very beginning of their studies.

> *"I have felt that it was expected that you know what it is all about. Yes, when we started it was taken for granted that you knew what a proof was. It's nothing you have learnt. I think it was combinatory…when someone happened to say something about "If and only if" that you have to first show it in one direction and then in the other. So it wasn't until then I heard it. So there is nobody who has told you what a proof is."*
> (S – A, 2004)

The quotation above is an example of the feeling that the learning of proof takes place occasionally. Some students also stated that they felt that they were expected to learn to prove statements by trial and error.

> *"Now when we study algebra we get home exercises and sometimes they are "Show that" and then you try and then you make a mistake and then you get back your tasks and the key and then you see: "Oh, it was like this I should*

have done." But no one has told us that "Now when you are going to prove statements, think that…" Rather, you have to try to catch up as much as possible by yourself."
(S – A, 2004)

According to the students, it was more usual to get feedback for proving tasks during the more advanced courses. The contents in these courses have not changed as much as the contents in the basic course so it is possible that students do not have so much practice in constructing their own proofs before it is required of them at a higher mathematical level.

Language and rigour

The students in the focus groups also discussed the formal requirements of proof. They seemed to struggle a lot to find out what was demanded by the community of mathematical practice. They felt they were not capable of using the correct mathematical language that was demanded/used by the community.

"…that is something I am always afraid of when I write proofs. I feel I use too many words. If I only had some more symbols it would look more professional. But what I do in the text, is that I tell that because I have this theorem I can draw these conclusions and this expression and then I get this. Then I'm always afraid of reasoning too much, that that is not OK in mathematics, that only symbols are allowed."
(S – A, 2004)

This is also an example of newcomers' endeavour to be like old-timers in the practice. The next extract is an example about questions the newcomers are struggling with for themselves. In upper secondary school one of the criteria for the different marks is the use of natural language and the use of symbols. To get the lowest mark "pass" you have to be able to reason in natural language. To get the mark "pass with distinction" you are demanded to use both natural language and symbols. To get the best mark "pass with special distinction" you have to only use correct mathematical language. This is something that may have influenced the students' view of symbols as necessary for proofs.

"When I write a proof myself I don't want to use any words, but when I read in the textbook there are expressions like "Because we do like this" I mean it is really written with words. But then I think it is only because of the educational purpose, actually if you are really professional you should omit all these words, but I am surely wrong."
(S – A, 2004)

The previous quotations about the formal requirements of proof are important regarding the condition of transparency. The students struggle to figure

out these by themselves, they build their own theories about the pedagogy in the textbooks and about what the mathematicians demand and what is really professional. This can be contrasted to the different pedagogical styles. The progressive style is to let students themselves decide if a proof is valid and not to discuss the formal requirements of the community (see p. 112). According to the deductive style the expectations of the community are not made clear for students and it is important for teachers to explain, for example the difference between a definition and a description (see p. 120).

Guidance by old-timers

The students talked a lot about the lack of getting some explicit guidance in constructing proofs.

> *"It would not be wrong if we had one lesson at least, where we would go through how one constructs a proof and maybe would get some clear examples and discuss why we do like we do and how we should think. Because there are many ideas and a lot of thinking behind the proofs that are actually not visible."*
>
> (S – A, 2004)

In the textbook for linear algebra (Tengstrand, 1994), there are solutions for many problems but not for proving tasks.

There were also examples of situations when students had got some guidance. One student studying at advanced level talked about the help that she had got during the lessons (see p. 16) belonging to the basic course, *Mathematical Analysis 1* and *Mathematical Analysis 2* (Appendix 1).

> *"In Analysis 1 and Analysis 2, it was quite good; we got two tasks, I think during each course, and the tasks were some kind of proving tasks. Then we would present them on the black board for other students. And that was actually quite good.* […] *I mean we could both ask the lecturer and the teaching assistants who worked on the basic course. Yet, we still didn't get a clear picture of what was actually demanded but more like when one asked the assistant he answered: "Yes, but here you have a little gap, but you can fill it." But I never understood that there were any patterns of how to build up what one would show…"*
>
> (S – A, 2004)

There was also some material about the constructing of proof the teacher in abstract algebra had given to students. The material as well as the individual comments that students had got about their own proofs were really appreciated by them.

> *K: What is that about?*
>
> *A: What you should have in mind when you construct a proof.*

B: He has chosen a proof as an example. It is about factorising into primes. First you have to show that it is possible to factorise and then show that the factorising is unique.

C: I think it was good because it is the only thing I got about proof, actually.
(S – A, 2004)

Hence, in advanced courses the students had sometimes received individually feedback to their own proofs. However, it seemed to be occasional and depended on the teacher.

"In Foundations of Analysis I got… we got tasks to hand in instead of an ordinary exam. And the teacher I had, I think that mathematics always depends so much on the teacher, but from that teacher we got quite difficult tasks, all of them of the type "show that", quite extensive. Then we handed in the tasks individually and got comments personally on everything we had done and that was quite good. And that was the only time you had really constructed a proof and then you got to know that: "Here you made typical logical mistakes, here you should have done like this instead". But the teacher felt it was too demanding to give personal comments on everybody's work so the following tasks were easier group tasks."
(S – A, 2004)

The following extract is an example of a student who stated that he had got a little more self-confidence and guidance in constructing proofs. He had proceeded in the mathematical practice and was already positioned near the doctoral students.

"Now when I study more advanced courses and one notices that they are at another level, we get a lot of tasks every week and then we must prove or show a lot of things. And then one has to believe in oneself a little. I have to check what I have used, and then I have to maybe assume that something is true and so on. And that's actually very good, especially that I have to decide that now I'm ready… you learn a lot, you can write your own proofs and get comments."
(S – A, 2004)

The feedback the student described in the quote above was obtained in a specialised course in *Algebra* (Appendix 1).

According to a socio-cultural perspective students need direction and guidance (Säljö, 2005b) to proceed in their mathematical practice. Figure 31 (p. 180) illustrates how students' experiences of participation swing back and forth when struggling with a proving task.

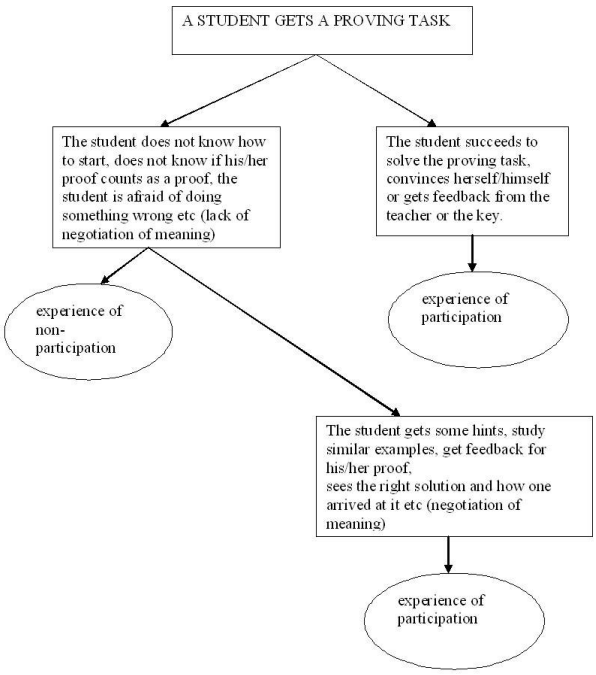

Figure 31 A student gets a proving task.

Invisibility/Visibility

The metaphor of transparency (Lave and Wenger, 1991) refers to the way in which using artefacts and understanding their significance interact in the learning process. Visibility of artefacts is a form of extended access to information about the specific artefact. There is an intricate balance between how much we focus on different aspects of proof at a meta-level and how much we use proof invisibly in the teaching of mathematics (see p. 54). Many quotations in this section illustrate the experience of invisibility regarding some aspects of proof. Utterances like *"And then some things that you prove you don't have to prove because they are supposed to be evident. There are other things you think are evident that you have to show and to find out what is what, is difficult."* is an example of the lack of visibility of the conventions in the mathematical practice regarding where to put the boundaries for what has to be justified and what one can take for granted. The quotation *"I think that a theorem can look like a definition."* illustrates the invisibility of the role of definitions. Further, several of the utterances in this section express the lack of discussion about what proof is. *"So there is nobody who has told you what a proof is.",* The learning of the construction

of one's own proofs seems to happen without any focus on proving techniques or the role of definitions: *"But no one has told us that "Now when you are going to prove statements, think that…". Rather, you have to try to catch up as much as possible by yourself.",* One student stated that it would be good to have at least one lesson with discussion of what to think about when constructing proofs *"because there are many ideas and a lot of thinking behind the proofs that are actually not visible."* Finally, students struggled a lot to understand the demands of the practice concerning the mathematical language and symbols. They wanted to act as professional mathematicians but were uncertain how to do so. *"If I only had some more symbols it would look more professional."*; *"…actually if you are really professional you should omit all these words, but I am surely wrong."*

5.4.2 Working in an investigative manner

In Chapter 2, I described the new trends in the teaching of proof (see p. 47) that advocate explorative activities for students. None of the students in the focus groups had experiences about working inductively with proofs, producing conjectures and then trying to prove their conjectures.

> *"Proof is something you usually learn by heart, anyway, it is not often that you sit and prove something by yourself, something you have not seen before or that you notice a pattern and try to find a proof. That is something I have never been demanded to do, I think."*
> (S – I, 2004)

Most of the students did not have such experiences from upper secondary school mathematics either (see p. 136). According to the textbook study (Nordström & Löfwall, 2005), there were some tasks where students would find patterns in the upper secondary school textbooks, which were in the focus of our study. However, there were not many tasks encouraging students to find a proof to their conjectures and they were often outside the ordinary course. In Vretblad's (1999) textbook that was earlier used in the basic course, there is an introductory section about finding patterns and proving conjectures that is very instructive (*ibid.*, p. 25). It was presented in Section 2.3.4, p. 59, as an example of how to make the aspect of Induction/Deduction visible in the teaching of proof.

Mathematicians related positively to this working manner but, at the same time, saw a lot of possible hindrances to their being able to apply this manner in their teaching, for example the lack of time, the lack of students' competence and the difficulty finding suitable problems that suited the majority of the students (see p. 105).

To sum up this section, I described students' participation in the constructing of proofs. The learning of proving statements seems to happen randomly and many students struggle with the language and the demands of the practice that are not always made visible for students. Students stated that they lacked the support and the guidance of old-timers, especially at the beginning of their studies. According to Wenger (1998), one component of social participation as learning is the doing. It looks like there are not many occasions for students to practice proving. Not until during some of the advanced courses, is there a possibility for students to seriously participate in, and get some feedback from old-timers about constructing their own proofs.

In the next section, I describe how students talked about the meaning of proof and provide the reader with examples from the data about expressions of participation as well as non-participation regarding proof.

5.5. The meaning of proof

A central notion for social practices is the process of negotiation of meaning (see p. 35). The negotiation of meaning involves the interaction of two constituent processes, participation and reification. A defining character of participation is the possibility of developing an identity of participation (see p. 34). In this section, I give examples of how students' feeling of meaning is connected to their experiences of participation or non-participation.

The utterances expressing non-participation in relation to proof often concerned problems about following and understanding the proofs presented to the students in the lectures. The students had various backgrounds and they related to proof in individual ways, even if, at that moment they were all influenced by the culture of the mathematical practice. There could be, of course, a lot of possible factors influencing the students' capacity to develop an identity of participation, for example their earlier experiences and the presentation of the material and how they related to mathematics. The students with participation identity could also have tolerated a certain amount of non-participation in order to proceed.

5.5.1 Expressions of non-participation

Many students stated that they could not understand why the mathematicians wrote the lengthy proofs on the board when they themselves did not need to know them and the knowledge about the proofs were not demanded in the examination. There were also many students in the focus groups who wondered what a proof actually was and why it was needed. They stated that it was never discussed.

What is proof?

Students had some interpretations about proof and the importance of proof in mathematical practice and stated that they wanted to learn more about it when they entered the practice (see p. 140). The students in the focus groups stated that they, in the very beginning of their studies, met a lot of proofs in the lectures but at the same time struggled with the question of what proof actually is and why it is needed. Here is a typical example about how many students felt when they started to study mathematics:

> *"I've an example here. In the basic course, the first time I met a proof, as I remember it anyway, so here's the proof and what's the proof? You never learned what a proof was or that you yourself would struggle with something and show things and then, the teacher used an entire hour for filling three boards with one proof."*
> (S – I, 2004)

Some of these students also seem to feel that mathematicians give proofs as an obligatory ritual, without any real purpose.

> *"I often feel that they have to give the proof whether or not someone understands it, that's how it feels."*
> (S – I, 2004)

The students who showed a lot of expression of non-participation stated that they had difficulties seeing a purpose in studying proof because they could not use them in problem solving or applications.

> *"Most often you don't have to be able to know anything of the proofs in order to solve problems."*
> (S – I, 2004)

They also advocated working manners and tasks where they could use the proofs in some ways in order to enhance their own engagement with proofs.

> *"I mean tasks in which you are supposed to calculate something using proofs. At least for me, it is easier to understand if I really use them for something."*
> (S – I, 2004)

Some students also discussed the lack of studying some proofs in detail, in order to enhance their understanding of the proofs and their meanings.

> *"I think it's wrong to give proofs like that, then it's better to omit them and take the proofs that are inspiring and interesting and then really go through the proof profoundly and make something of it instead of almost always spreading the feeling that they give them because they have to. Better to cover proofs because they are fun and interesting, I think."* (S – I, 2004)

"The lecturer should not just take the proof as it is in the book but make the proof simpler and explain it in a way that makes it easier to understand as well."
(S – A, 2004)

Wenger (1998) discusses what makes information knowledge and what makes it empowering. He states that it is the way in which it can be integrated within an identity of participation. The way, in which the students in the previous examples talked shows that the information about proof they got in the lectures did not build up to an identity of participation but remained alien, fragmented and unnegotiable to them.

The lack of history

Proofs as reifications always assume a long history of participation (see Section 1.3). Students in the focus groups said that they sometimes understood and could follow a proof but at the same time showed a feeling of giving up and they stated that they would never be able to construct such proofs.

"Sometimes I feel, well yes, sure this was evident, but how could I ever find it out myself."
(S – I, 2004)

Thus, the lack of the historical knowledge about how proofs were constructed for the first time could cause a feeling of inadequacy among students. There are concerns among some mathematicians in the practice about the students' lack of historical knowledge. Tambour (2005) points out that proofs have developed over a long time and have in many cases, been made "simpler". So, if there is never a focus on the difficulties with the formulations of proofs that the mathematicians have encountered during the long historical development of mathematics, newcomers can obtain a tainted picture and think that there is something wrong in their own capability to construct such proofs and this leads to experience of non-participation.

Also a student in a focus group advocated information about the situations where the proofs were constructed for the first time.

"It occurred to me that it would be good if the teacher gave some background information about the proofs when teaching them, in what kind of situation the proof was constructed first. For example the proof he took last Tuesday, the half circle and the triangles in it, who found it out and how? Because I would never be able to find such proofs in that way."
(S – B, 2004)

"The proof is left as an exercise for the reader…"

The style in some textbooks also seems to cause a feeling of inadequacy among students and, thus, can lead to an experience of non-participation. Very often, some of the arguments in the textbooks are left for the reader to

prove. They can be suitable exercises for those who know how to prove statements while for others who are not familiar with proof they can enhance the experience of non-participation.

> *"In this book* (Analys i en variabel) *I read the introduction where the authors say in this way: This is evident and clear but you must not feel stupid if you do not understand it. And then I thought, of course not, I mean I also use words like consequently, dada, dada, dada, evident…, but when I started to read this book I noticed that I sometimes don't understand how you come from this step to the next step and* […] *I think you create wrong expectations. At the beginning we have here the hyperbolic functions and you can read: The following theorem is easy to prove and then we had tasks where we would prove these theorems and there I sat and searched and searched in the book, what? No, it's not so easy as they…, but of course if you have studied eleven years and know all these things backward and forward …"*
>
> (S – B, 2004)

I checked the textbook in calculus (Persson & Böiers, 1990) that the student talked about. In the introduction the authors write that in the text one many times finds comments like "follows immediately", "a simple control", "one realises easily" but they warn the reader not to take these comments too seriously, at least not during the first reading. Then the authors give three "good reasons" for these comments. The first reason they give is because they think that a too rigorous and pedantic presentation makes the text difficult to grasp.[17] The second reason is that according to the authors these comments work as a spur for the reader to work actively, which is, according to them, very important in mathematics. The third reason the authors give is that the arguments that have been omitted should be seen as a control about the learning, since the omitted arguments should be experienced as easy and simple when one masters the contents.

The reason for why the student in the quote above talked about wrong expectations can be that although working hard she cannot find enough information about how to tackle the problem. To be able to solve the proving task in question, "Show that the derivative of sinh (x) is equal to cosh (x)", one has to start revealing the definitions, which might be one of the problems the student experienced.

All the others know

Some of the students expressed a feeling that all the others in the classroom knew what a proof was because the teacher did not explain or discuss the issue. They felt that it was implicitly expected that all the students knew what it was all about.

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[17] This dilemma is connected to the condition of transparency in teaching of proof (see p. 54). At the same time, when focusing on the rigorous treatment of proof and when trying to make every step visible, maybe some other aspects, like the structure of the proof can be obscured.

> *"I became so… because when showed something it was not accepted as a proof, but I understood exactly what I was supposed to prove, and why it is wrong when I do it but when the teacher does it, it's right? What is it that makes the difference? How do you define a proof? Because we have never been informed about that, so you think: "OK, the rest of the class knows what a proof is."*
>
> (S – A, 2004)

An assumption that someone else understands what is going on refers to an identity of non-participation in relation to ownership of meaning (Wenger, 1998).

Invisibility/Visibility

I have earlier in the previous section dealt with visibility of language and logic and the formal demands of the practice (see p. 180). In this section, I have exemplified students' struggling with questions like what proof is *"So here's the proof and what's the proof?"*. The lack of discussions about the issues led to a feeling that all the others know what is going on *"How do you define a proof? Because we have never been informed about that, so you think: "OK, the rest of the class knows what a proof is."*. I also described the invisibility of historical knowledge about how the proofs were once constructed and the feeling of inadequacy among students *"sure this was evident, but how could I ever find it out myself"* These utterances illuminate the experience of invisibility of some aspects of proof in the presentation of mathematics.

5.5.2 Expressions of participation

According to the theory of Wenger (1992), participation and reification have to be considered as an interacting pair where reification always rests on participation. In the theory chapter (see p. 38), I discussed Wenger's view that in general, a more abstract formulation viewed as reification, would require more intense and specific participation to remain meaningful, not less. I questioned if a higher level of abstraction in reifications, like in mathematical formulas in general, requires more participation than a concrete problem with a lot of details. Abstraction in mathematics can often help us to see connections and structures and in that way we can use them on many occasions without participating in the negotiation of meaning on all levels. I also pointed out that these kinds of abstractions offer us a means of understanding problems in new contexts. On the other hand, there is a lot of participation before a student learns the mathematical language and learns to follow arguments.

Benefits of the learning of proof

The students who showed expressions of participation regarding proof did not share the negative feelings shown by the others regarding the meaning of proof but declared the advantages of studying them. The following quote is an example about a student who states that mathematics becomes easier when one learns proof. However, she also states that it can be hard to work through proofs but that it is worthwhile. Hence, according to her, one has to struggle to come to a certain level where it is possible to take advantages of the abstract and the general and in that way make life easier.

> *"I think that if you go through the proofs and understand them you get a lot for free, since you can always go back, I mean a proof is often a rather concentrated piece and if you have understood it you hardly have to cram at all* (laughter). *No, I mean that then you don't have to sit with everything else that takes so much time if you want to spare some time. It is clear it can be hard to work through them and really acquaint yourself with them but it can actually be worthwhile."*
>
> (S – I, 2004)

Abstract formulas and concentrated results have naturally demanded a lot of effort and participation during their history and, for an individual, to obtain a level where he/she is able to handle with these abstractions and utilise them requires a lot of participation.

Another student expressing participation states that proofs are sometimes intuitively trivial and not so important whereas the value of more complicated proofs is to see how everything is related.

> *"I think that benefits of studying proofs are that – sometimes the proofs are naturally intuitively trivial like…, yes I cannot recall any but certain proofs we can feel that OK it is clear that they work and then it is not so important with proof. But the more complicated definitions you have to deal with the more difficult it can be to see that it is possible that all these things are related, and then one gets in black and white, that yes, they are related."*
>
> (S – A, 2004)

Hence, the students when showing expressions of participation talked about proofs explaining mathematical relations and giving the general results that made life easier, in a very similar way as the mathematicians did.

I am unique

It happened that the students who strongly expressed participation in the mathematical practice talked about other students as not being interested or capable to understand the benefits of studying proof or abstract mathematics. This has become a part of their identity.

"Last autumn, I had a colleague and his aim was to pass these studies because he would study physics afterwards. And before the examinations he used to say: "I just want to learn a method that works." And I, who will become a teacher, I want to understand so that I can explain to my pupils, so I was extremely oriented to understanding. So we had completely different ways of approaching mathematics then. I went crazy if someone just presented: "this is an algorithm that works" because I want to know why it works."

(S – A, 2004)

Wenger (1998) states that experiences of non-participation do not necessarily build up to an identity of non-participation. The next quote offers an example of a student studying the intermediate courses who expressed participation even if she sometimes had had difficulties following arguments and, thus, there must have been strains of non-participation. She also holds the same point of view about other students (as anxious and afraid of proof) as many of the mathematicians held.

K: How did you feel having a teacher who started with theory when in upper secondary school they often started with examples?

F: I don't think I'm representative of all students but for me it was only fun. I like theory the most. But I know that there were protests at the lectures sometimes and there were very many who said: "How can we understand delta and epsilon; help, this is tough!" Most of the students thought it was enormously difficult and tough to understand where all this would lead. I didn't perhaps understand very much myself all the time but I thought it was so very fascinating, very fun, for me it was more like a spur; I want to learn more about this. For the others it was frightening, "Help, I will never learn this!" In that group I was actually quite alone and it is not easy to be together with others who think a bit differently.

(S – I, 2004)

The previous quote is also an example about the work of imagination in identity building in the mathematical practice *"…I didn't perhaps understand very much myself all the time but I thought it was so very fascinating,…"*. Wenger (1998) talks about this as an ability to "accept non-participation as an adventure." (*ibid.*, p. 185)

In this section, I have dealt with the experience of meaning of proofs and exemplified expressions of participation and non-participation from the data. I discussed the metaphor of transparency in relation to the results and showed with some examples how students' lack of various kinds of extended information about proof, for example the lack of discussions about what proof is and the lack of knowledge about the history of participation in constructions of proof, could lead to experiences of non-participation. I also discussed the relation between reification and participation in connection to

abstractions and generalisations and how students expressing an identity of participation talked about the benefits of learning proof.

5.6 Examinations

Some of the students expressed that there was no meaning in studying and struggling to understand the proofs because they were seldom demanded in the examinations. Examinations can be seen as important reification at the mathematics community. They are crucial for students' future trajectories. In my experience, examinations have a huge influence on students' identities and on how they see other students, what they focus on in their studies and their future. Students have to participate in and pass the examinations in order to get their study points. Thus, students have to prepare themselves to be able to solve the types of tasks demanded in these examinations. Examinations are almost always individual. Textbooks are not allowed to be used in them. A part of the examinations can sometimes be done by handing in assignments or by participating in the lessons (see p. 16) in the basic course.

In Section 1.2, I described the changes in the role of proof in the examinations. The first time students have an examination on proofs and theories is in the course *Mathematical Analysis 3* (Appendix 1 and Appendix 4).

5.6.1 Proof does not concern me

Some of the students felt that proof was nothing that concerned them at the beginning of their studies because of the lack of proofs in the examinations for the basic course (see p. 204). This was even mentioned as a problem by some mathematicians. Hence, examinations seem to have significance for the experience of meaning in learning for some of the students, since mathematicians when posing questions in the examinations, at the same time, convey their view on what is important/reasonable or possible for students to learn. In the next quote a student compares the demands of the examinations in the mathematical practice to the demands in the examinations in upper secondary school where he has worked as a teacher and he states that if a teacher does not demand some parts of course contents in examinations students are not going to study them:

> *"I've learnt from upper secondary school and my pupils there that if I don't say to them that these can pop up in the exams, no one studies the proofs and it's similar here.*
>
> *…when he* (a mathematician) *stands and draws the proofs that you know are not demanded in the exam, I stopped, all the others stopped writing down the proofs*[…] *Then I think I have to try to understand these proofs and then I try to concentrate but at last you give up and concentrate instead when he comes to the applications and the theorems and…But when the teacher says:*

now you have to know these proofs, it's a little like one learns them by heart and then tries to understand them and then these courses have some meaning for understanding. But the proofs that are not demanded in the exams or which you don't understand at all, when he goes through them, you don't understand the meaning of going through so many proofs in Analysis 1 of which no one grasps anything. I don't usually get so high marks on the exams but those who do, they don't understand anything of the proofs either, not a thing."
(S – I, 2004)

This view was criticised by one of the students who seemed to have got insights in the benefits of the understanding of mathematics by learning proof and, thus, showed identity of participation regarding proof.

"I think that sounds terrible: if it doesn't pop up in the exams…, so it's for your own sake, mathematics becomes much easier."
(S – I, 2004)

Although mathematicians said that they had to avoid tasks beginning "Prove that…" in the basic course, because students were afraid of proving tasks, sometimes proving tasks popped up in examinations. Some students claimed that it was difficult to exercise the proving tasks before the examination (see Section 6.2.3).

5.6.2 A change in students' relation to proof

In intermediate courses and in more advanced courses proofs are sometimes demanded in oral examinations, for example in the courses *Mathematical Analysis 3*, *Algebra* and *Foundations of Analysis* (Appendix 1), and to pass these examinations students have to study the proofs and learn to reproduce them. Three of the focus groups had studied *Mathematical Analysis 3* and talked about the change concerning their relation to proof caused by the examination on proofs in analysis. The students were mostly positive towards the examination but some of them complained about the abruptness of the demand that they should know so many proofs (Appendix 4).

I really had my eyes opened

The following dialogue shows how one of the groups reflected on this change.

A: I had the same kind of experiences concerning the first 20 study points; I mean the basic course, when I saw a proof in the textbook I thought: "I won't read that." I learned the theorems and skipped the proofs. But then in Analysis 3 we had an oral examination on a lot of proofs and there I really had my eyes opened to proof and since then I've studied the proofs extremely carefully and really tried to understand the proofs in the textbook, also so that I

can follow and understand why different theorems can be applied and under what circumstances they can be applied, what premises that have to be met. Now I'm meticulous about it.

…

B: I think it started to change during Analysis 3, as A *said. We had to study the proofs and then I discovered that they were interesting and that they actually gave something. And naturally you got more understanding when you arrived at "I see, it is from these facts that the consequences come and if we combine them we get this third one" and then suddenly it all became interesting.*

(S – A, 2004)

The students really appreciated the insights they had got during the intermediate course *Mathematical Analysis 3*. The following example is from another group, a student's comment to another when she heard that the other student had not yet started to study the course *Mathematical Analysis 3*.

"Then I understand why you don't like proofs. I started to like them when I studied Analysis 3. I got an aha-experience because there were so many theorems that we had gone through in Analysis 1 and Analysis 2 and then we saw how everything was connected and it was nice to see that I don't have to memorise a formula, I can derive it and I feel great."

(S – A, 2004)

These results are interesting because they show that the requirements of the students posed by the mathematicians in terms of demanding students to learn a lot of proofs helped them get insights into the benefits of studying proofs. Wenger (1998) discusses demanding alignments by a community of practice, and points out that they do not need to mean the lack of negotiability but demanding alignment itself can be a means of sharing ownership of meaning. *"The reluctant compliance of students with the directives of a demanding teacher can take these students beyond their own limitations, likes, and dislikes, and may result in their reaching new understandings of their own."* (*ibid.*, p. 206)

Why not get this aha-experience earlier

At the same time several of the students also expressed that they would have liked to get the insights earlier.

B: I also liked it but I thought it was tragic that we could not get this aha-experience earlier when we sat and listened to the teacher.

A: I think we should have got this experience as early as in Analysis 1.

C: As early as in the C-course in upper secondary school. (S – A, 2004)

Students talked about school mathematics as rule learning and applying formulas and connected understanding, questioning and logical reasoning to proof. Some of them expressed that it was a pity that they were not given an opportunity to become familiar with this side of mathematics earlier in school.

> *"I often feel that people do not like mathematics because they do not understand anything. If they have never learned to question and understand it is clear that they do not question and then they have nothing to understand. Either it's easy for you to apply the formulas or it's difficult for you to apply formulas. That is what ability in mathematics is all about in upper secondary school and it's really a pity, because if you want to make young people a little more curious about mathematics and teach them to understand mathematics they would, with help of proof and some kind of derivation, I believe, they would stimulate another kind of interest in mathematics."*
>
> (S – A, 2004)

The previous quotation is also an example of the dissatisfaction with school mathematics that several students expressed. The student quoted above believes that another focus in school mathematics, would make young people curious and they *"would get another kind of understanding in mathematics."* There was also a feeling of having been taken in, in some of the students' utterances:

> *"In upper secondary school we didn't do anything in the math lessons. The teacher said: You can do arithmetic* (räkna)*, you will get a good mark you can go now. In Komvux* (adult secondary education) *I was prepared to meet difficulties but nothing really difficult popped up there. When you say about Analysis 3 that it is so fun* (jävla roligt) *with proof, I agree, but what a "kick in the teeth"* (jävla spark i ansiktet) *it was to me after that I had had it so easy, waltzed through and then suddenly, what is this all about, no sums to do, it was really a big changeover* (rejäl omställning)*."*
>
> (S – A, 2004)

Some students were a little sceptical as to whether students in upper secondary school were mature enough for proof whereas most of the students stated that it is possible to deal with proof in school, if it happens on a proper level.

> C*: I think the proofs of the theorems have appeared quite properly, it began quite calm in Analysis 3.*
>
> M*: In upper secondary school I don't think one would absorb it.*
>
> L*: No.*
>
> …

J*: I think, that if we wouldn't have had this oral part in Analysis 3, where they demanded that one would give an account of about 24 proofs, I would not have started to study the proofs then either. One does it, only when one has to do it. And it's the same in upper secondary school…I think that trying to prove statements would suit very well, to sit four and four and argue about, why is it like this, how are things related. And then they can try to give an account of as well as they can. I think it would be a good exercise, since one learns to think about mathematics, that it is not just doing sums, but also reasoning.*

(S – A, 2004)

Students also talked about the difficulty to get the understanding ("the big picture") at the beginning of the studies and the difficulties of conveying an aha-experience to someone else.

S: But I think that it's extremely difficult, to explicit put across the thing one gets aha for since I very often feel in this way when I arrive at something: "Why don't they reveal this, why don't they tell it like it is?"

G: It's so extremely personal too, it's nothing one can share. It's awfully personal like.

(S – A, 2004)

Also some mathematicians talked about the difficulty of giving the "big picture" at the beginning of a course and they claimed that students had to tolerate some amount of ignorance and first after a while everything would fall into place (see p. 117). These considerations can be connected to the condition of transparency as well as Sfard's (1991) vicious circle metaphor (see p. 59).

Aesthetics

Students who had studied the intermediate course *Mathematical Analysis 3* with the oral examination of proofs talked also about an aesthetic experience they gained when reading proofs. This is similar to some mathematicians' views (see p. 93).

"I think that it's common to say that mathematics is one of the fine arts and I would like to have a bit more of that feeling. And that is something I can get when I read a proof and feel "Yes, it fits" and one feels so happy. And then one really feels that mathematics is one of the fine arts, but I would like the teachers to convey a little more of that, the clearness that exists in mathematics."

(S – A, 2004)

But the examination was not the only reason for the insights into proof students had gained on the higher level of their studies. As one of the students

pointed out, they had got some experience of proving themselves, during the advanced course, *Algebra* and in other more advanced courses.

> *"When I started the elementary course, it was the introductory course, and there they stood and proved and proved and I didn't really understand what it would be good for. I thought that for me those proofs were totally unnecessary… Though now when I study the intermediate courses and more advanced courses I have understood how important the proofs actually are. And now I become irritated if a teacher says: "I won't go through this proof now; you can study it in the textbook." Because now I want the proofs presented in a structured way, so I can pose questions and so. Because they are important. But I think it's because we have had to write a little ourselves also; there have been a little more questions of the type "Show that…"."*
>
> (S – A, 2004)

Both activities (doing in practice), studying proofs and learning to reproduce a set of proofs, on the one hand, and exercising the proving of statements, on the other hand, offer students different kinds of occasions for negotiation of meaning. Students on more advanced level seem to have got also more individual guidance and feedback from old-timers. According to the students in the focus groups, this sometimes happened during the courses *Mathematical Analysis 3*, *Algebra* and *Foundations of Analysis* (Appendix 1).

To sum up this section, some of the students expressed that there was no meaning in studying proofs because they were not demanded in the examinations. After the first oral examination in proof, students expressed a changed relation to proof. Many of the students stated that they had got an aha-experience during the time they studied the proofs but complained that they got this experience so late.

5.7 A summary

I started the chapter about students' practice by describing the students' school backgrounds and showed that the students had various backgrounds regarding their experiences about proof when they entered the practice. Further, over 80 percent of the newcomers related very positively to proof. They considered proof as a central part of mathematics and wanted to learn more about it. I then described what possibilities there were for the newcomers to engage in negotiation of meaning with respect to proof in the mathematical practice and how students talked about their experiences in the practice. Several students spoke of the learning of proof as occasional and that they lacked discussions on the subject and guidance from mathematicians. I demonstrated with some examples how students' lack of various kinds of extended information about proof could lead to experience of non-

participation. Students' relation to proof seems to change after the first oral examination on proof in calculus.

6 Contrasting the results regarding mathematicians' and students' practices

Pedagogical activity, pedagogical intentions and instructional material create a context in which learning can take place. Learning is viewed as increasing participation in the practice (Wenger, 1998). In this chapter, I draw together the different parts of the results and discuss them.

My aim in the first two sections is to give a general description of the results concerning students' encounters with proof. I first report what the mathematicians and the students stated about some issues regarding the teaching of proof and how these statements could be supported using the complementary data. I go on contrasting the mathematicians' views on students as learners of proof with what the students themselves stated about their relation to proof.

In the third section, I discuss the three teaching approaches to proof that were constructed, how students may experience these styles and how the styles can be related to the condition of transparency. What opportunities do they offer various students to engage in the negotiation of meaning concerning proof in the practice? I also make some epistemological remarks concerning the styles.

The results reported in this chapter are based on the data analyses of the interviews with mathematicians and students, the surveys with university entrants, as well as the complementary data, like textbooks, examinations and protocols from observations of lectures.

6.1 Proof in the teaching of mathematics

The mathematicians' and the students' views (as I have interpreted them) on several issues concerning the treatment of proof in the lectures and lessons were coherent but sometimes there were inconsistencies in the data. First, I present the topics where their views were quite similar.

6.1.1 Discussion about proof

Both the mathematicians and the students agreed that there was not much discussion about proof or proving techniques in the basic or intermediate

courses (see pp. 183, 176). They pointed out the advanced course *Foundations of Analysis* as the course where it was possible to discuss proof and learn to construct proof. Earlier, Vretblad's (1999) textbook offered university entrants discussion about proof and how to construct proofs on an elementary level (see p. 15). The book was mentioned both by some mathematicians and students studying at more advanced level as helpful in drawing students into the practice of proving and understanding the role of symbols and logic in it (see pp. 99, 165). Some students also talked about the material they had got during the advanced course *Algebra*, where certain aspects of reading proof and constructing own proofs were focused on (see p. 178). In the new edition of the textbook for mathematical analysis, *Analys i en variabel* (Persson & Böiers, 2001), that is used at the basic and intermediate level, there is a new section dealing with proof (*ibid.*, pp. 26-33). However, neither students nor mathematicians mentioned the section. The ignoring of the section suggests that it has not been focused on in the lectures or lessons.

The mathematicians and the students in my study also agreed that few students posed questions during the lectures of basic and intermediate courses (see p. 170). Some of the students stated that it was not easy to pose questions in the lectures:

> *"I would put it this way. You must really be a certain type of a person to pose questions in this place. You have to be like, curious, unafraid, think it's fun. Many students are not like that."*
> (S – I, 2004)

The mathematicians also stated that there were not many students asking questions in the lectures.

> *"There are few students who ask questions."*
> (M, 2003)

This was also true according to the field notes. Questions during the lectures in the basic and intermediate courses were unusual.

Working in an investigative manner was not usual in the teaching of mathematics at the university according to both the mathematicians and (see p. 105) and the students (see p. 181).

Hence, so far the students' views and mathematicians' views were similar. However, there was an inconsistency in data regarding how much proof was dealt with in the lectures for newcomers, which I describe and discuss in the following subsection.

6.1.2 How much proof is dealt with in the lectures?

In general, all the mathematicians in my study conveyed a very careful position concerning the teaching of proof in the basic course. Most of them claimed that they avoided proof in the basic course for different reasons (see Section 4.3.2).

> *"We do not deal with proof much, some simple things, but I do not demand of them that they* (the students) *would be able to present them. Possibly, they can get a task where they have to demonstrate some simple formula."*

Contrary to this, students in the focus groups talked about an experience from the beginning of their university studies about mathematicians proving statements during the lectures even if they mentioned that some mathematicians omitted long and technical proofs (see p. 163).

> *"When I started to study the elementary course, it was the introductory course, and there they stood and proved and proved and I didn't really understand what it would be good for. I thought that for me those proofs were totally unnecessary…"*
> (S – A, 2004)

There can be various reasons for this discrepancy between the mathematicians' declared intentions and how many students experienced the lectures. I suggest two possible ones, which I base on the classroom observations and the analyses of the interviews:

1) Mathematicians cannot help giving some "nice proofs" now and then even if they state that they do not prove so many statements (see p. 123).
2) The mathematicians' and the students' views on proof were similar in many aspects (see pp. 150, 152). Yet, proving statements can mean different things for different persons. It is natural for mathematicians to present mathematics in a deductive way starting with definitions and proceeding in a deductive manner, justifying the most steps they take. They might not always think of this as proving even if many of them who I interviewed talked a lot about derivations of formulas as proving and also stated that proof somehow existed in all mathematics. Students may conceive this as different from the way mathematics was presented to them in upper secondary school and as proving.

It is also possible that students experienced the lectures at the beginning of their studies as containing a lot of proof due to both of the reasons presented above. Anyway, this inconsistency shows that what is intended to be of focus by mathematicians is not necessarily of focus by all students (see p. 60). However, this issue would need further examinations (see 7.3.).

In the following section, I describe and discuss how mathematicians viewed students as learners of proof and what students stated about themselves and their relation to proof.

6.2 Mathematicians' views on students as learners of proof

There were some inconsistencies in the mathematicians' views on students' interest in proof, and what students themselves stated about how they related to proof at the beginning of their studies. In this section, I first describe and discuss these inconsistencies. Then, I go on describing and discussing how mathematicians talked about students' difficulties regarding proof and what students themselves stated about them. I conclude the section by describing the examinations for the basic course and discuss why students did not succeed in the proving tasks in the examinations.

6.2.1 How did students relate to proof?

Mathematicians had a variety of views on students as learners of proof. Several mathematicians assumed that students in general were not interested in proof and assumed that students wanted instead to get "recipes" about how to carry on with mathematical problems. This was not consistent with what students stated about themselves. Next, I discuss these inconsistencies in the results.

Students are not interested in proof

According to several mathematicians, there was only a small minority of students who were interested in proof (see p. 99).

However, right from the beginning of their studies, the newcomers conveyed a view on proof as an essential part of mathematics (see p. 150). A majority of the newcomers (over 80 percents) also stated that they wanted to learn more about proof and they would like to have learned more about proof in school (see p. 140). Contrary to what mathematicians assumed, there was only a small minority of students who related negatively to proof when they entered the practice (see p. 160).

It seems natural that students related positively to proof when they entered the mathematical practice at the department, since they considered proof as an essential part of mathematics. Students' positive responses to the statements in the questionnaire might also reflect a socio-cultural effect: thoughts and feelings are influenced by the culture and the situation at the time for the responses. However, even if students related positively to proof at the beginning of their studies, there seems to be a difficulty in drawing

students into the practice of proving because many students in different phases of their studies experienced difficulties in understanding the meaning of proof in the basic course (see Section 5.5.1). They also indicated that it was often hard for them to follow the presentation in the lectures and many of the students gave up and stopped listening (see p. 190).

Thus, it is not surprising, that many mathematicians felt that students were not interested in proof. According to the analysis of the interviews with students, hindrances for students' ability to follow the proofs and the reasoning in mathematicians' presentations was the lack of knowledge about and confidence with mathematical symbols and the deductive way of presenting mathematics (see pp. 164 and 166). According to the mathematicians and to some earlier studies (e.g. Bylund & Boo, 2003; Högskoleverket, 1999; Thunberg & Filipsson, 2005), students also had difficulties with elementary algebra and they lacked confidence with manipulation of fractions. This also hinders their capability to participate in the lectures. All these problems can also partly explain why students experienced the tempo in mathematicians' presentations as a problem (see p. 164).

Students are not interested in the question "why?"

Several mathematicians assumed that most of the students just wanted to get their study points but were not interested in the question "why?" (see p. 99) whereas most of the students who entered the practice stated that they wanted to understand what they did in mathematics (see p. 142). Students were certainly interested in getting their study points but at the same time they stated that they wanted to understand what they did in mathematics.

There were also mathematicians who stated that students were not willing to understand that it was better to derive formulas (because it has to do with proof) instead of memorising them.

> *"…students in the basic course think that they have to memorise enormously many formulas. And this is because they are unwilling, it has to do with proof, unwilling to understand that twenty formulas are consequences from one. Because they are afraid of proof, if we scare them, they get trapped in relation to proof and this can contribute to a reluctance to see these simple relations, we have to teach them them."*
>
> (M, 2004)

However, most of the newcomers stated that they preferred the knowledge about how to derive formulas rather than recipes or memorising the formulas (see p. 143). It is possible that students lacked the knowledge about how to derive the various formulas because in upper secondary school, students use a table of formulae in exercises and in mathematics examinations instead of memorising or deriving the formulae. One of the mathematicians also pointed this out as a difficulty for students.

"For example, trigonometric formulas are difficult for the students. In school they have used the set of formulas and not been expected to derive them so much. They rarely thought why the formulas are as they are."
(M, 2003)

As the student who had very little experience of proof in upper secondary school, expressed it (see p. 162):

"I notice that I've really been starved of proofs as a matter of fact, I noticed this when we went through the sine-, cosine- and tangent laws here (at the university). *I had never seen them, well, I had seen them but not the proofs for them so I had never given it much thought …"* (S – B, 2004)

Also students' utterances after the first oral examination in proof show that, when studying and learning to reproduce the proofs, they got an aha-experience concerning how everything in mathematics was related (see p. 190).

"I got an aha-experience because there were so many theorems that we had gone through in Analysis 1 and Analysis 2 and then we saw how everything was connected and it was nice to see that I don't have to memorise a formula, I can derive it and I feel great."
(S – A, 2004)

Both students and mathematicians regarded proving tasks as difficult for students. They pointed out several similar reasons for the difficulties.

6.2.2 Students' difficulties

Many mathematicians stated that proving tasks were more difficult for students than problem-solving tasks.

"The most common attitude (among students)*, which I think you know as well as I, is that, you can give a caricature like this: If it* (the task) *says: "Solve the equation* $x^2=4$*" all of them succeed but if it says: "Prove that the solution is* ± 2*" they don't succeed."*
(M, 2004)

When they entered the practice, a majority (85 percent) of the students also stated that it was more difficult for them to prove mathematical statements than solve computational problems (see pp. 147 and 174). There was also a consensus among mathematicians and students regarding some of the reasons for these difficulties. Several mathematicians stated that students had difficulties with exact mathematical language (see p. 119). Students themselves talked a lot about difficulties with the language, which was different from the language that they were used to in their upper secondary school

mathematics classrooms (see p. 164). Some mathematicians as well as some students mentioned the problem of getting started with a proof and knowing when the statement was proved.

> *"I very often get the question from the students: "When is a proof finished?" or How does one start a proof?" These two issues, I think characterise students' difficulties. Either they do not understand how to start or, after making a lot of calculations, do not know if they have proved the statements."*
> (M, 2003)

> *"Well, we often get tasks like "Show that" and then I often think afterwards: "Have I really proved it now?" I don't feel sure even though I have really managed to show what I am supposed to show."*
> (S – A, 2004)

> *"For me it's difficult to organise all my thoughts in my head, what is going to be first, what step do I take first in order to make it elegant."*
> (S – B, 2004)

These difficulties coincide with those Moore (1994) describes in his study of undergraduate students participating in a transition course.

Both students and mathematicians also talked about the difficulty students experienced concerning what had to be justified and what one could take for granted. The difficulty to recognise what to take for granted and what to prove can sometimes be arbitrary and is not always clearly stated by mathematicians, as one of them pointed out. They are conventions that are seldom discussed.

> *"And then some things that you prove you don't have to prove because they are supposed to be evident. There are other things you think are evident that you have to show and to find out what is what, is difficult."*
> (S – A, 2004)

> *"… to be honest because, anyway, it is completely unthinkable to formalise everything profoundly. You have to a certain point, to a certain level accept intuition but where we put the boundaries is arbitrary, so it's a difficulty for the students, to understand, why prove some evident things while other evident things can be accepted without a proof."*
> (M, 2004)

One mathematician also pointed out particularly the proving of evident statements as a difficulty for students.

> *"In the last examination there was a task that if* a<b<c<d *you were to show that you had* (a+b)/(b+d)>a/d *or something like that. And basically, these are totally evident inequalities but they* (students) *experience that as an extra difficulty. The fact that it is evident makes them not really know what to focus on* (sätta stöten på)*."* (M, 2004)

However, no student mentioned the proving of evident statements as particularly difficult. I find the assumption that proving evident statements is more difficult than proving non-evident statement interesting and worth further investigations.

Many students stated that they had difficulties understanding what proof is and lacked discussion about the subject (see p. 183).

> *"In the basic course, the first time I met a proof, as I remember it anyway, so here's the proof and what's the proof? You never learned what a proof was or that you yourself would struggle with something and show things…"*
> (S – I, 2004)

Also one mathematician had recognised this as a problem.

> *"I have noticed that students have difficulties understanding what a proof is."*
> (M, 2003)

As pointed out earlier in this chapter (see Section 6.1.1, p. 196), there was not much discussion about proof or the role of proof in mathematics. Many students did not have much experience of proof in their upper secondary school mathematics (see p. 139) and they stated that learning of proof occurred occasionally (see p. 176).

Many of the mathematicians who I interviewed stated that students were afraid of proving tasks. More than half of the students showed positive feelings than negative feelings when confronting a task that began "Show that…" and several students expressed a feeling they were getting an intellectual challenge when trying to solve proving tasks, when they entered the practice. One half of the university entrants stated that they liked to try to prove mathematical statements (see p. 145). However, some of the students in the focus groups who had experiences with proof in upper secondary school said that proving tasks in school mathematics were much easier than the proving tasks at the university so they might have soon developed a feeling of insecurity instead of feeling of getting an intellectual challenge (see p. 174). In that case, it is natural that many mathematicians had a picture about students of being afraid of proving tasks.

Important didactic questions are what in these tasks could scare students and how mathematicians can help them to overcome these feelings. One mathematician pointed out students' lacking knowledge about how to derive formulas as a reason for students' difficulties with and for their fear for proving tasks.

> *"They* (students) *are afraid of tasks that begin with "Show that…" and leap over them directly. Even tasks where there is to show that formulas are true are difficult even if it is demanded only to calculate from the beginning to the*

end. They are afraid of proving tasks, actually in vain, more a psychological problem than a problem concerning their knowledge. One reason for the fear might be that students are not able to derive formulas."
(M, 2003)

However, there were several students who talked about proving tasks exemplified in the previous quotation as easier than ordinary problems (see p. 135).

"I don't remember it as hard either. I think I even thought it was fun. They were easier than other tasks because you already knew the answer. I am good at careless mistakes and then I know that I have got the right answer and if not I just have to check where my careless mistakes are."
(S – I, 2004)

For many mathematicians the solution to the problem that students were afraid of proving tasks was to avoid them instead of trying to get students used to them.

"Students have difficulties with proving tasks. We have to avoid formulations like "Prove that…" that may frighten and block the students."
(M, 2003)

Yet, there were sometimes proving tasks even in the examinations for the basic course. The proving tasks were not only about mathematical induction which was dealt with at the beginning of the course *Linear Algebra 1* (Appendix 1).

6.2.3 Examinations

Some mathematicians assumed that students' lack of interest towards proof was partly because proofs and theories were not demanded in the examinations. Examinations rule students' life and where they put their efforts.

"For students proofs seem to be abstract and distant, "Do we need to know the proofs in the examination?" they ask."
(M, 2003)

"As far they know that they are not demanded in the examinations they do not care so much…"
(M, 2004)

This was also mentioned by some students as one of the reasons for why they felt the studying of proof lacked meaning (see p. 189).

"But the proofs that are not demanded in the exams or which you don't understand at all, when he goes through them, you don't understand the meaning of going through so many proofs…"
(S – I, 2004)

However, another student who showed a participation identity regarding proof and talked about the benefits of learning proof criticised the view, according to which one should only learn proof for the sake of examinations (see p. 190).

In the *Introductory course*, between 2002 and 2006, three tasks of about 180 tasks began with "Show that…". They were the following:

(1) Show that $^{a^c}\log\left(b^c\right) = {}^a\log b$ for all $a, b, c \in \{x \in \mathrm{R} : x > 1\}$.
(*Introductory course*, 050531)

(2) Show (preferably by using a Venn diagram) that if *A, B* and *C* are subsets of the complex numbers, then $(A \cap B) \cup (C \cap (A \cup B)) = (A \cup B) \cap (C \cup (A \cap B))$.
(*Introductory course*, 040312)

(3) Suppose that $0 < a < b < c < d$. Show that $\frac{a}{d} < \frac{a+b}{c+d} < \frac{b}{c}$.
(*Introductory course*, 040108)

Besides the tasks presented above, there were two tasks in the examinations for the introductory course where students were to "motivate" something, although the solutions of the tasks were proofs. This is similar to upper secondary school textbooks, where the words like show, prove and proof were avoided and replaced by words like "justify" and "explain" (Nordström & Löfwall, 2005). This can be because one does not want to *"frighten and block the students"*.

Some mathematicians stated that the proving tasks that were given to students were easy and just demanded a little self confidence.

"I use to, even if it is not usual in the introductory course, I have given a proving task of a kind that does not demand a sort of deeper mathematical knowledge at all. But if you only have a little self confidence and try then you will also succeed."
(M, 2004)

Yet, for example the few proving tasks in the examinations for the introductory course (1), (2) and (3), represent different kinds of problems, so it is difficult for students to exercise them, for example in old examinations. Though mathematicians may consider them as simple tasks, for students they

are not trivial. Students are used to proving tasks where they have to show that the left hand side equals the right hand side and which just involve direct calculations (see p. 135). The tasks above are not of this type. They demand, for example understanding of the role of definitions when proving statements.

In *Linear algebra 1,* between 2002 and 2006, there were, besides the proving tasks demanding mathematical induction, six proving tasks of about 130. Three of them were ordinary problem solving tasks of the type "Show that the line is parallel with the plane." where the line and the plane are explicitly given. In *Mathematical Analysis 1* the number of proving tasks was also low.

In the examinations for *Mathematical Analysis 2*, proving tasks were more frequent than in the other examinations for the basic course. Only six of 26 examinations lacked proving tasks. There were also several proving tasks of similar kinds (for example about convergence and limits), so students could have met them more often in old examinations.

Most of the students were not familiar with proof at all. They had not exercised proving tasks very much in upper secondary school. Students lacked guidance and feedback from old-timers and they struggled a lot with the formal demands of the practice (see p. 177). The following quotation is an example from a student who complains the lack of exercise and guidance concerning these tasks.

> *"In Linear algebra 2* (Appendix 1)*, there are these kinds of tasks, show that this is a linear map, show that this is a scalar or…One thinks they are damn hard and I don't understand how one can… in exams there are always such tasks of different level of difficulty but in the whole book there are only three or four tasks of a similar level* […] *but my opinion is that they should teach us more, when we get two of these exam tasks, that is what I feel."*
>
> (S – I, 2004)

As mentioned before (see p. 178), in the textbook for linear algebra, there are no solutions to proving tasks, only answers to problem-solving tasks. However, in the old examinations, there are solutions for the proving tasks. There is also a student who privately sells solutions to many textbook problems and this was mentioned by one student in the focus groups.

Since students had not had much guidance or practice in constructing proofs (see pp. 178 and 190), proving tasks in examinations just seemed to confirm what mathematicians and students said; that students had great difficulties with such tasks. In this way, these tasks could enhance students' experience of non-participation regarding proof.

> *"One always knows that it is the proving tasks that the students fail in the examinations."*
>
> (M, 2004)

However, as shown in Section 5.6.2, the students' relation to proof seems to change after the first oral examination on theories and proofs during the intermediate course *Mathematical Analysis 3*. After that, students in the focus groups talked about an aha-experience about how everything is related.

In the two previous sections, I contrasted and discussed the general results about the mathematicians' views with the results about the students' views regarding the teaching and learning of proof and students' relation to proof, as I have perceived the utterances in the data analysis. I also discussed some inconsistencies in the data as well as some pedagogical issues, like students' difficulties. In the following section, I discuss how students' experiences of participation and non-participation can be related to the different teaching styles identified in the data.

6.3 How did students experience the three approaches to the teaching of proof?

I set up a table to illustrate three different styles of how proof van be approached in the teaching, based on the issues emerging from the data (see Section 4.4 and Appendix 5). The main criteria for different categories were the mathematicians' intentions, their views on students and the aspects of proof in the conceptual frame (see Section 2.3). The styles are idealised and no individual could perfectly fit in one of them. Further, a mathematician when demonstrating some of theses styles or a mixture of styles in the teaching does it in his/her very personal manner. Next, I discuss how various students may experience the different teaching styles. What aspects of proof are intended to make visible/invisible on the one hand, and what aspects are experienced as visible/invisible by students, on the other hand (see p. 60)? When may the various styles lead to experiences of participation or non-participation? I also make some epistemologies remarks concerning these styles.

6.3.1 The progressive style

The progressive style (see Section 4.4.1) implies that proof is invisible in the calculations and the derivations of formulas without a focus on them as proofs. However, some aspects of proof are more visible in this style than in the other styles, for example the meaning of proof. According to this style it is preferable to come to proof "via natural ways in calculations" and in that way appeal to students' feeling of the need for proof. Natural language is preferred before formal symbols, which enhances students' possibilities to

participate in the lectures since many students are not familiar with mathematical symbols (see p. 110).

Also, the way of eventually introducing the symbols makes the benefits of the use of symbols visible for students. However, some aspects of proof remain hidden, where evident, long and technical proofs are always avoided and there are no discussions about the formal demands of the community.

Some students talked about teachers, who only chose the "important proofs" or omitted some of the proofs saying that they were boring or technical.

> *"…there were some proofs in Algebra 1 but not so many but that may depend on the teacher, he could say like this: "This proof is quite easy but involves long calculations and I think you consider it so boring, so it's in the book." This he said several times."*
> (S – I, 2004)

According to the theory of Wenger (1998) students by participating in the mathematical practice absorb the culture in different ways. They observe the masters in the practice, what they do, how they talk and how they work, what they enjoy and what they dislike. Hence, utterances, like *"This proof is long and technical and I think you consider it boring…"* suggest to students that proof is not so important for the newcomers.

According to the progressive style, only a small minority of students would need proof and they would learn it themselves. The mathematicians when expressing this style indicate that they have also no intention of trying to awaken students' interest by discussions about proof but they expect the students who are interested in proof, to find out the demands themselves. The following example about confronting an utterance of a student with a mathematician's utterance expressing the progressive style, illustrates very clearly the gap between students' needs and expectations, on the one hand, and mathematicians' intentions within the progressive style, on the other hand.

> *"I became so… because when I showed something it was not accepted as a proof, but I understood exactly what I was supposed to prove, and why is it wrong when I do it but when the teacher does it, it's right? What is it that makes the difference? How do you define a proof? Because we have never been informed about that, so you think: "OK, the rest of the class knows what a proof is."*
> (S – A, 2004)

Whereas a mathematician talks in the following manner:

> *"I have not felt a need for some more profound discussion about the formal demands of proof, but rather that one often gets questions as all of us do from the students: "Does this do as a proof?" and then they are waiting for a for-*

mal answer, but I want instead that they will have an answer from inside of themselves where the proof fits if they understand. So I do not want to go too far regarding these formal discussions."
(M, 2004)

The progressive style can lead to teaching that does not reveal important aspects of proof that could make the idea of proof and proof techniques more available for students. Students are left for themselves to find out and judge if their solutions are correct and why. The example above is interesting, because earlier research (e.g. Pettersson, 2004) and analysis of the focus group interviews show that students want to see the "correct" solutions to the proving tasks which they have been struggling with whereas mathematicians within the progressive style do not want to choose one solution as the only correct solution.

I conclude the subsection with some epistemological considerations on this teaching style. It is possible to discern some features of constructivism (learning theory, see p. 25) – as it has been interpreted in mathematics classrooms – in the utterances categorised as the progressive style. According to constructivism, a learner actively constructs the knowledge and thus finds the meaning of learning proof themselves. It is impossible for teachers to transmit the knowledge about proof, or the meaning of proof, to students. In Sweden, the constructivist ideas have largely influenced the pedagogy and the school work during the last decades (Säljö, 2005a). Hanna and Jahnke (1996) claim that the influence of constructivism has also had a deleterious effect on the teaching of proof, *"if only because it has been interpreted in a way that undermines the importance of the teacher in the classroom."* (*ibid.*, p. 885) As Hanna (1995) remarks, a lot of studies have shown that it is crucial for the teacher to take an active part in helping students understand why a proof is needed and when it is valid. "A passive role for the teacher also means that students are denied access to available methods of proving: It would seem unrealistic to expect students to rediscover sophisticated mathematical methods or even the accepted modes of argumentation." (*ibid.*, p. 45).

6.3.2 The deductive style

Typical for the deductive style (see Section 4.3.3) is to use a deductive approach in the teaching of mathematics and proof. There is no intention of avoiding the word proof, abstractions or mathematical symbols, but rather the opposite. Students should get used to them in the very beginning of their studies. Contrary to the progressive style, there is a desire to discuss proof and students are considered to be capable of learning abstract thinking. It is important to focus on techniques for proving and on logic, whilst at the same time make the formal demands of mathematical practice clear to the students

from the very beginning. The learning of proof occurs partly by first memorising a proof.

There are differences between students' capabilities of following and understanding the deductive presentation of mathematics at the beginning of their studies (see p. 210). A lot of students had very little experience about proof when they entered the practice (see p. 139). Further, newcomers have great difficulties with the general results in mathematics and in understanding and using the algebraic symbols (e.g. Thunberg & Filipsson, 2005). Then the deductive style at the beginning of the studies causes experiences of non-participation caused by too big a gap between students' earlier experiences and the new ones they meet in the mathematical practice.

The following focus group discussion with newcomers, serves as an example of how a group of newcomers experienced their first lecture in one part of the basic course, where the presentation of mathematics could be characterised as deductive according to the field notes. These students had had experience of proof in upper secondary school. Yet, they talked about dry definitions and getting something very hard into their brains. They obviously had difficulties following the deductive and compact way of presenting mathematics. The presentation was very careful, the mathematician justified every step he took so there were no logical gaps in it but the tempo was quite fast.

> *L: We had the first lecture last Monday and the teacher… started with a definition and then gave one or two examples of it, the definition was in a dry mathematical language, very formal* (korrekt) *and…*
>
> *A:* […] *very general terminology and then there were eight pages of lecture notes.*
>
> *V: I threw away those lecture notes.*
>
> *A: It's like getting something very hard in your head* (få nånting väldigt hårt I huvudet)*, it's like, what, help.*
>
> *N: If you in some way would understand that the domain is* x *and the range is* y *but such a dry definition and then you sit there and try to struggle, I struggled maybe some minutes, wait, what does this mean, I mean what does this mean for the things coming after…*
>
> (S – B, 2004)

The last utterance in the previous dialogue also offers an example of how students struggled to follow and experience meaning in what they saw and heard during the lectures.

> *"…I struggled maybe some minutes, wait, what does this mean, I mean what does this mean for the things coming after…"* (S – B, 2004)

Hence, the first confrontation with a deductive teaching style of these students seemed to lead to an experience of non-participation. Some of the students even threw away their lecture notes or stopped writing the notes, so they could not study them outside the lectures either.

However, some of the students, because of their backgrounds and/or their capability of accepting a certain amount of non-participation at the beginning of their studies, succeed in proceeding further in their mathematical practice and gain a participation identity (see p. 194). It is typical in the deductive style for mathematicians to view the students in general as capable of learning proof and deductive reasoning and the results about students' experiences concerning the first oral examination of proof show that the requirements of the students could help them get insights to the benefits of studying proofs (see p. 190). As Wenger (1998) points out, *"a demanding teacher can take these students beyond their own limitations, likes, and dislikes, and may result in their reaching new understandings of their own."* (*ibid.*, p. 206)

It is also stressed within the deductive style that students should get a very detailed presentation because they are not used to following proofs and have difficulties filling the gaps themselves. This enhances the newcomers' ability to follow and learn to follow the presentations if they are familiar with mathematical language, symbols and the deductive presentation of mathematics (see p. 166). So there is a kind of sensitivity towards the newcomers visible in the utterances belonging to the deductive style but in a different way than in the utterances characteristic to the progressive style. However, if the gap is too big between students' competence and the level of presentation of mathematics, it can be difficult for a student to follow and learn.

The deductive style confronts students at once with the mathematical language that is seen as important in the practice. Hence, it can be a more enculturative style than the progressive style and can draw students to the culture of mathematics and proof if students have the proper prerequisites. The learning of proof is compared to the learning of language, and learning by heart is not rejected but it is seen to be one part of the learning process, to imitate.

Finally, some epistemological remarks. In this style, some features of the view on the teaching and learning as a quite unproblematic "transmission of knowledge" can be discerned. At the same time as this style does not attempt to hide anything, one can question if everything is going to be revealed for the students. For example the approach exemplified in Section 4.4.2, p. 114, under the heading "Nothing concealed?" can lead to a situation for newcomers where everything is concealed if students have no knowledge and experience about mathematical language and symbols or the deductive way of presenting mathematics when they enter the practice.

6.3.3 The classical style

Characteristic of the classical style is a great admiration of proof. Proofs can be beautiful and offer intellectual challenge. There are not so much pedagogical considerations regarding the teaching proof. One either gives a proof or does not give a proof. Although the great appreciation of proof, mathematicians express no intention of dealing with proof in lectures and lessons for newcomers because of external circumstances (see p. 123). However, sometimes some "nice proofs" are given if there is time for that. The presentation is often intuitive, not rigorous. Mathematicians want to convey ideas that they themselves consider as fantastic, so they often leap over elementary steps.

This style can lead to participation if students are able to follow the presentation and find a meaning in proofs that are sometimes given to them. The students studying mathematics get increasing understanding of what old-timers enjoy, dislike, respect and admire (see p. 34). For example, a proof that is sometimes given by a mathematician can be experienced as logical, simple, beautiful etc depending on students' prior experiences and how mathematicians present the proof. The following quotation is an example of a student who seems to be satisfied with this kind of presentation.

> *"In Analysis 3 our teacher omitted many of these pedantic and fidgety examples or proofs and only went through the ones which were a little more proper…"*
> (S – A, 2004)

However, there were examples in the data about gaps between what mathematicians' intentions were and how students experienced their presentation. For example a "nice proof" that was considered by a mathematician as simpler than the one in the textbook was not at all appreciated as such by the student who could not understand what was better in that proof.

> *"…so I have given very few proofs in the lectures. But I can't help giving some handsome and short proofs, often in a simpler manner than in the textbook."*
> (M, 2003)

> *"The way in which anyway M used to do: "The book has done it in this way but this is much more exact." And then he used to compare and say: "This is much better but then it's much more complicated than the proof in the book.""*
> (S – I, 2004)

The classical style is epistemologically close to a master-apprentice style of learning, where the learning occurs without any deeper reflections on the teaching and learning by the master. The teacher is a professional mathema-

tician in the first place and by practicing mathematics himself/herself draws students in to the practice. It is also possible that this style can inspire and enthuse students, especially on the higher levels. However, too much of the classical style may lead to disinterest in students and their practice and to a minimal engagement in teaching of newcomers that in turn can make students feel that they are incapable of learning to appreciate proof.

6.3.4 Students have various styles

As shown in the previous section, there are different styles that mathematicians mix and apply in their personal ways in the teaching of newcomers, but also students are individuals with various backgrounds (see Section 5.1). They have different goals with their studies as well as different tastes regarding the presentation of mathematics. Some students may prefer a careful presentation of mathematical contents with definitions, theorems and proofs whereas some others get bored when listening to that kind of presentation (see p. 166).

> Q*: …I think that I'm kind of a structure person and I totally lose the appetite for learning maths if they just stand there and prattle and don't even finish the examples.*
>
> P*: It's as if I fall to sleep if they like go through a proof extremely carefully* (liksom)
> (S – A, 2004)

The student Q might prefer a deductive teaching style, whereas the student P could be more satisfied with the classical style:

> *"No rigorous proofs, too formalised proofs are unbearable. A piece of poetry,* (proof) *can be as attractive as the entire theorem."*
> (M, 2003)

Mathematicians struggle with a lot of difficulties in teaching the newcomers mathematics and the teaching of proof was conceived as particularly problematic by many mathematicians.

> *"Because they conceive proof as a sort of extra burden that don't know how to handle. And I would very much like to help them to get out of this but it is not easy."*
> (M, 2004)

The groups are often big and heterogeneous (see p. 16). Students have a variety of backgrounds regarding their experiences about proof, their confidence with mathematical language, symbols and deductive presentation of

mathematics (see p. 139). There is so much content in the courses that some mathematicians feel that they do not have time enough for discussions about proof. Also other practices, such as the department of physics, have their own demands of the course contents (see Section 1.2).

In this chapter, I have contrasted the results about the mathematicians' practice with the results about the students' practice in order to shed light on how the structuring resources and mathematicians' views and intentions became resources for learning (participation). Utterances categorised as belonging to both the progressive style and to the deductive style expressed deep reflections on the didactic problems in the practice. Yet, my study shows that pedagogical reflections can lead to totally different teaching practice depending on the style of approaching mathematics and proof.

7 Conclusions and discussion

The aim of the thesis was to describe and characterise the culture of proof in a community of mathematical practice at a mathematics department and how newcomers are engaged in proof and proving in this practice. In the first section of this chapter, I describe how the thesis illuminates the general research questions and what conclusions it allows me to draw. In the second section, I summarise the new theoretical ideas and describe the theoretical tools that I have developed in the thesis and which I have used in analysing the material. I conclude the chapter by discussing in what way the thesis can contribute to the educational practice and suggest both some theoretical and empirical issues for further research.

7.1 Conclusions

At the beginning of the thesis, I formulated the following general research questions:

- How do students meet proof in the community of mathematical practice at the mathematics department?
- How are students drawn to share mathematicians' views and knowledge of proof?

Next, I describe and discuss very briefly the conclusions regarding the general research questions that I have drawn from the results reported in the previous chapters.

7.1.1 How do students meet proof in the community of mathematical practice at the mathematics department?

The first encounter

Students meet a mixture of the styles described in Sections 4.4 and 6.3 when they enter the practice. An important result in my thesis is that although most of the mathematicians in my study had no intention of teaching newcomers

proof and they stated that they did not deal so much with proof in the basic course, students in the focus groups talked about an experience where they were confronted with proof from the beginning of their studies (see p. 163). Many of the mathematicians are concerned to present the mathematical contents in a more informal manner and focus on enhancing students' *understanding*. They do not want to scare students as they think students are afraid of proof, so there is no intention of dealing with proof (see p. 203) and they do not think about their presentation as proving. As described in the theory chapter (see p. 60), the condition of transparency does not only concern the intended presentation of mathematics but how it is experienced and what students focus on. According to Wenger's (1998) theory, *"the learning that actually takes place is but a response of the pedagogical intentions of the setting."* (p. 266) The students who can follow and understand the presentation are drawn into the "understanding" that mathematicians want to take them. Yet, those students who can not follow the lines of reasoning may distance themselves and start to look at the presentation as an object and try to figure out what is going on. They observe the structures in the presentations, notice typical repeated utterances and concepts, symbols and so on. All these observations can lead to the experience that mathematicians are *proving*. It seems also that sometimes mathematicians consciously give proofs although they have no intention of doing it (see p. 123). What actually goes on in the lectures when mathematicians state that they are not proving whereas students experience the presentation as proving, needs further research. I will come back to the issue in the last section of this chapter.

Invisibility/Visibility

Following this first encounter, students continuously meet proof in different manners although proof is not so much in the focus of teaching (see Section 6.1.1 and 6.3). One mathematician gives students a complicated proof with formal presentation and does not expect them to fully understand everything, another avoids giving a proof, but instead offers some informal explanation not labelled as proof, and yet another omits the proofs or sometimes gives a "nice proof"[18]. This may be a good balance for students in propitious circumstances and enhance their learning (participation). But many students struggle with the very notion of proof (see Section 5.5.1). They may misinterpret an intuitive presentation as a proof, may not understand that an explanation sometimes is a proof; students may struggle with the mathematical presentation of a complex proof without seeing any structure in it; they might not see that a proof is "nice" and so on.

A lot of aspects of proof remain invisible as experienced by the students. Discussions about proofs or logical structures in proofs seem to be unusual. Students in the focus groups often wondered what a proof was *"So here's*

[18] This can also refer to the same mathematician in different occasions.

the proof and what's the proof?". Constructing own proofs does not seem to be focused on in either the basic or the intermediate courses according to the students, the mathematicians and the observation of lectures *"But no one has told us that "Now when you are going to prove statements, think that…"."* (see pp. 180 and 186). Depending on the teaching style applied in the lectures various aspects and functions of proof could become visible/invisible for students in the presentation of mathematics (see Section 6.3). However, as exemplified under the previous heading, it is not evident that what mathematicians *intend* to focus on in their teaching, becomes the focus of students but rather that students respond to the settings in their own personal ways.

The first oral examination on proof

The first time students have an examination on theories and proofs is during the intermediate course *Mathematical Analysis 3* (see Appendix 1 and 4)[19]. This is the first real encounter between mathematicians and students where they literally talk with each other about proofs and use the same language. Proof is visible for both of them. At a more advanced level there are sometimes also discussions about proof during the lectures (see p. 196).

7.1.2 How are students drawn to share mathematicians' views and knowledge of proof?

Students interested at the beginning

Most students in my study showed interest in proof and the learning of proof at the beginning of their studies (see p. 153) and they were conscious about the centrality of the role of proof in this practice (see p. 150). Mathematicians had no real intention of teaching students proof in the beginning, but for mathematicians proof is a natural part of exercising and presenting mathematics. By merely being mathematicians they offer exemplars to the newcomers. As students observe what and how the old-timers do in the practice, they learn a lot about proof implicitly without a meta-level focus on the activity. In my thesis, I have described how students are forging their identities right from the start of their studies (see Section 5.2, p. 153). Some of them start very soon to talk in a way, similar to the mathematicians about school mathematics as rule learning and university mathematics as real mathematics, understanding and proof (see p. 153). There was also a desire among the students to become professional and particularly the use of

[19] However, we have to bear in mind that a majority of the students who start to study mathematics just take the basic course or a part of it and do not study the intermediate course *Mathematical Analysis 3*.

mathematical symbols in proofs was connected to it. *"If I only had some more symbols it would look more professional."*

However, according to the theory, students are all active agents in the practice and they respond to challenges in the practice in individual ways, depending on their earlier experiences, and on how much and in what ways they invest themselves in the practice. They have also different goals with respect to their mathematical studies. Hence, the expression "drawn into the practice" in the second research question, does not mean that students are passive in the process of enculturation into the practice.

Non-participation/Participation

There are various possible trajectories for students after the first encounter with proof. According to Wenger, one's identity is always changing (see p. 34). Peripheral participation involves a mix of participation and non-participation. The students who cannot follow the presentation may eventually develop a non-participation identity regarding proof. They leap over the proofs in the textbook and stop listening to the lecturer when he/she gives a proof or just presents mathematics in a deductive manner, which students may experience as proving (see p. 189).

It is also possible that students accept non-participation as an adventure (see p. 188) and the encounter with proof for them leads to enhanced participation when they struggle to find out what proof is and to understand its role and meaning in the practice. The students, who had proceeded further in their trajectories, talked both about experiences of participation and experiences of non-participation (see p. 194). The learning of proof seems to happen quite randomly. Students talked about the invisibility of many aspects of proof when trying to grasp and understand the rules and the formal demands of the practice. *"Rather, you have to try to catch up as much as possible by yourself."* Students had various tastes (see Section 5.3.2 and 6.3.4) and mathematicians applied various teaching styles in their own personal ways. This encounter also influenced the way in which students got access to proof.

However, students' capability of participating in different activities regarding proof depends on their earlier experiences and their ability to follow the deductive lines of reasoning, as well as their familiarity with mathematical language and symbols. Students had various school backgrounds regarding their experiences with proof, and, hence they were in very different positions as regards to how they could participate in negotiation of meaning concerning proof (see p. 139). Some of the students stated that they could not even ask questions because, anyway, they could not understand the answers and sometimes they found it hard even to formulate the questions, since there was too much that they did not understand (see p. 170). Since proving tasks are occasional in the examinations (see Section 6.2.3), it is possible to study the basic course in mathematics without much participation in proof.

Proceeding further in the practice

Students in the focus groups on the advanced level talked about an aha-experience they had obtained on how everything was related in mathematics when they studied the theory questions and learned to reproduce the proofs during the preparation for the examination in *Mathematical Analysis 3* (Appendix 4). Hence, mathematicians' demands on students to study and reproduce a lot of proofs helped them get insights into the benefits of studying proofs (see p. 190). Students also talked about getting more insights into proof by constructing own proofs and getting feedback for them (see p. 193).

At the same time, some students in the focus groups on the advanced level talked about the difficulty to achieve the understanding ("the big picture") at the beginning of the studies and the difficulties conveying an aha-experience to someone else (see p. 193). Also some mathematicians talked about the difficulty of offering the "big picture" at the beginning of a course and they claimed that students had to tolerate some amount of ignorance and that first after a while everything would fall into place (see p. 117). These considerations are also illuminated by the metaphor of transparency (see p. 40) in the sense that it is difficult to focus on something you have no experience about, for example at the beginning of a course.

It seems that when students themselves started to use proof as a tool in their mathematical practice and got hold of extended information from more experienced persons, textbooks or other material, they were able to make progress in their mathematical practice and gained an identity of participation. Moreover, the oral examination, where many students, for the first time, not only studied the proofs and how everything was related, but also talked about proofs with a mathematician, seems to be a crucial step that contributes to students' access to proof (see Section 5.6.2).

In the next section, I discuss how the theoretical framework that I have developed in my thesis contributed in shedding light on the research questions described in this section.

7.2 Theoretical contributions of the thesis

I have analysed data from mathematical practice at the mathematics department from a perspective of "community of practice" (Wenger, 1998), where the joint enterprise, the *practice*, is the learning of mathematics in a broad sense. I have found this perspective well suited to considering the newcomers, the *students*, as *active participants*, a role for them which is well attested to in the interview material. I also argue that, in accordance with Wenger's (1998) theory of learning, researching new mathematics can be seen as learning (see p. 34). Hence research into mathematics can be seen as a part of a broad spectrum of mathematics learning within the department. Thus,

both mathematicians and students are participants in the same community of practice. I find this stance a better starting point for didactical research on proof than focusing on different (and possibly conflicting) practices because the boundaries between various positions in this practice are not clear cut. However, this standpoint does not mean that problems or conflicts (e.g. *mathematicians' various pedagogical intentions*) in the broad practice are ignored, rather they are seen as a (necessary) part of the joint enterprise. Sometimes problems can also function as the engine for the learning. In my study, I have investigated the problems of drawing newcomers into the practice of proof.

Wenger's theory of *learning as increasing participation in communities of practice leading to changing identities* offered a straight forward tool for data analysis for discerning students' utterances and expressing participation/non-participation with respect to proof and, thus, helped me to describe how students were drawn into the practice of proof. Both the fundamental assumption about the character of learning and the view on a person in the practice as an active agent who invests more or less of herself in the practice but at the same time, is influenced and absorbed in the culture of practice matched well to the way in which I wanted to look at the teaching and learning conditions of mathematics and, hence, offered an appropriate starting point to the thesis (see Section 2.1). In the data analysis, in coherence with the theory, I considered the mathematicians and the students as participants in the community of mathematical practice and interpreted their utterances, not entirely as their own opinions but also as reproductions of views belonging to the community.

One of the main arguments I have put forward in my thesis is that from a socio-cultural perspective proof can be seen as an artefact in mathematical practice, as a tool which has a lot of functions in this practice. I examined proof as an artefact by using Säljö's (2005) classification (see p. 38). In this way, proof is viewed as a symbolic and intellectual tool. I found support for this view from both the literature and from the data (see p. 95). As an artefact, I see proof also as *reification*[20] in Wenger's terms. This implies a view of proof as both a process and a product, a view that allows me to describe the complex process of working with and creating proofs. Proof is not only formalising mathematics and organising it in a deductive manner but also creating conditions for new theorems and proofs and also a means of communication and thus production of a new context of both participation and reification which are two constituent processes in *negotiation of meaning* (see p. 35). There is an ongoing negotiation of meaning along with the interacting aspects of proof, for example Intuition/Formality and Induc-

[20] Wenger uses the concept of reification very generally to refer to the process of giving form to our experience by producing objects that congeal this experience into "thingness". In doing so we create *points of focus around which the negotiation of meaning becomes organised.*

tion/Deduction in which both mathematicians and students participate (*Figure 4*, p. 42). The didactical research on various functions of proof can be seen as an examination of proof as an artefact whereas didactical research on various approaches/properties of proof, for example Intuition/Formality and Induction/Deduction can be connected to proof as a process of reification (*Figure 6*, p. 62). This is the way I have brought together a sociocultural perspective, the social practice theories of Lave and Wenger (1991) and Wenger (1998) and didactical research on proof.

Lave and Wenger (1991) introduce the metaphor of transparency of artefacts (Invisibility/Visibility) which I examined with respect to proof, since it seemed to be a powerful notion in illustrating the intricate dilemma about how much to focus on various aspects of proof at a meta-level and how much to work with proof without a focus on it as a proof in the teaching of mathematics. That is why I included the aspect of Invisibility/Visibility in the *conceptual frame about the aspects of proof* (see p. 42). This dilemma ought to be a fundamental concern of the teaching of mathematics, as can be seen in my material (see pp. 40, 54, 167 and 213). I created the conceptual frame from didactical literature and used it in the data analysis. In this way, I linked the issues that mathematicians and students talked about, to the main themes in didactical research on proof. The conceptual frame was also the main tool in constructing and characterising the three different styles in the teaching of proof.

The theoretical model that I have presented; of the different idealised styles of mathematicians as teachers of proof; has given structure to my material (see p. 82). The model could be developed further, for example seeing whether it is applicable, at different levels of study, for example in upper secondary school, teacher education and graduate courses, and even for studies in different countries. It would also be interesting to take it as a basis of an analysis of focus group interviews with mathematicians in order to study more closely mathematicians' practice. Finally, it could also serve as a benchmark in an internal discussion at a department of mathematics aimed at developing more effective way to attack the problem of drawing newcomers into the mathematical practice.

In the next section of this chapter, I discuss what insights the empirical findings, as well as the theoretical arguments that I have put forward in the thesis, offer to educational practice and suggest some theoretical and practical items for further studies.

7.3 Challenges to educational practice and issues for further research

The theoretical framework developed in the thesis, as well as the empirical findings suggest the following questions for further research in the field of proof in mathematics education.

Issues from the practice

There was an inconsistency in the data concerning how much proof is dealt with in the lectures (see Section 6.1.2 and Section 7.1.1). What does actually happen when students experience in the lectures that mathematicians are proving whereas the mathematician does not think he/she is proving, just presenting mathematics to students? This has to do with the condition of transparency in a sense that if the activity can be characterised as proving, then it is invisible for the mathematician but visible for students who experience the presentation as proving. Such an inquiry could also make it easier for mathematicians to be conscious about their way of presenting mathematics and how students experience it (see p. 213).

My thesis brings about the importance of the role of examinations. Mathematics examinations are important in that they inform students about the view of mathematicians regarding what is important/possible/desirable for students to learn. There were students who experienced studying proof as meaningless because proofs were not demanded in examinations (see p. 204). Further, the first oral examination in proof seemed to be important for students' relation to proof (see Section 5.6.2). It would be interesting to study closer the impact of this examination and other forms of examinations concerning proof.

What proofs are useful for various purposes?

The conceptual frame about the aspects of proof and the view of proof as an artefact prompts further studies on various proofs with respect to what different functions and aspects of proof these proofs can illuminate in the didactic processes. Questions like: "In what phase of mathematical studies and in what ways certain proofs benefit students' learning of mathematics?" are important to investigate. It could help to improve the teaching of mathematics by asking: What proofs are useful for the illumination of different aspects of proof and mathematics? Why is this very proof important to give on this level? Why is Pythagorean Theorem seen to be the best theorem to start with in school? Here, the conceptual frame I have developed in the thesis makes a contribution by helping to discern and analyse proof from different points of view.

Transfer

The function of *transfer* (see pp. 61, 93 and 151) included in the conceptual frame has not been included in the earlier models of functions of proof (de Villiers, 1990; Hanna & Jahnke, 1996; Weber, 2002) but it partly overlaps the aspect Weber (2002) describes. The function of transfer refers to two basically different things. Firstly, working with proofs can be useful in other contexts than in mathematics. Secondly, some proofs can provide methods or techniques useful in other mathematical contexts. How does the function of transfer relate to other functions of proof in the earlier models? What significance has the function of transfer to mathematics on the one hand, and to mathematics education, on the other hand?

The condition of transparency

Students in my study often wondered what proof was and lacked discussions about the subject (see pp. 180 and 186). Proof was there as a mysterious artefact even if the word proof was avoided and the intention of teaching proof was not always there. Students showed interest towards proof when they started to study mathematics. According to Lave and Wenger (1991), there is an intricate dilemma in the teaching of newcomers regarding the balance between an unconscious use of artefacts on the one hand, and focusing in different ways on these artefacts, on the other hand, by offering some extended information about the artefact. The condition of transparency of proof suggests that proof should not only be used and given in the teaching practice but focused on from different points of view (see the conceptual frame about the aspects of proof Section 2.3). There are various ways of focusing on proof. For example, mathematicians can make it clear to students where they are proving or not proving. They can discuss why an inductive argument based on examples cannot be seen as a mathematical proof. They can offer a meta-level analysis of the complex proofs they demonstrate. They can reflect openly on how such proofs are constructed by mathematicians in the first place. Mathematicians can make it clear to students why they prefer some proofs and omit some others. They can point out when a proof is useful in other mathematical contexts. Finally, they can focus on various aspects of proof, like historical dimensions about how proofs, axiomatic, tricks or special proving techniques. According to students, mathematicians and the observations of lectures, these kinds of focus sometimes take place but are occasional in the teaching of newcomers.

The condition of transparency of proof is an intricate balance and it is not an easy task for mathematicians to decide what to focus on and when. Sometimes, focusing on one aspect leads to the obscuring of another. For example, a very detailed, rigorous, linear presentation of a proof makes all the logical steps visible for students but, at the same time, can obscure the overall logical structure of the proof. The condition of transparency regarding proof

would need to be focused on in further studies. How and how much should the teacher focus on various aspects of proof in the teaching? Is it possible, by making various properties/approaches and functions of this artefact visible, to help students experience proof as worthwhile and enhance their understanding and access to proof?

Evident statements

A question that has also to do with the condition of transparency of proof is the assumption held for example by some mathematicians in my study that proofs for evident statements are unnecessary and that teachers should avoid them because students do not see any meaning in them but consider them as pedantic and do not feel the need for proof. There are others, for example Weber (2002) who claims, that it is not wrong to give a proof for 1+1=2 by using Peano axioms, but the teacher has to make visible why this proof is interesting and what aspects of mathematics it enlightens. In scrutinising proofs for evident statements together with students, the results are not in focus or questioned but the focus is on the actual proof. This kind of treatment makes the role of formal mathematics and formal definitions in mathematics visible. A question for further studies is if these kinds of activities help students to better understand for example, the difference between a description and a definition and the difference between intuition and a deductive proof. Is it possible, by working with some proofs in detail to enhance students' understanding of how proofs are constructed as well as their understanding of the role of critical thinking in exercising mathematics?

How is proof dealt with in school and in teacher education?

Even if most of the students related positively to proof when they entered the practice, many of them lacked the necessary tools to follow and understand the lines of reasoning in them. Many students, especially those with a Swedish upper secondary school background, had minimal or no experience at all about proof, especially the constructing of their own proofs in their earlier mathematical studies and, yet, there was a small minority who had practiced proof in many different ways in their upper secondary school mathematics. Hence, students were in different positions when they started their studies at the university.

My study only gives a hint of Swedish upper secondary school teaching concerning proof. Hence, there is obviously a need for research about how proof is dealt with in upper secondary school classrooms. In the suggestions for a new national curriculum for upper secondary school mathematics, proof is paid more attention than in the previous curriculum (Skolverket, 2006). Teacher education has a key role in communicating new ideas to prospective teachers. The teacher education in Sweden has been reformed many times during the last decades and the extent of mathematical studies at the mathematics department involved in the teacher education has diminished

(e.g. Bergsten et al., 2003). When at the same time the position of proof has become diminished in the basic course at university, there is a risk that the teachers with the new education are not very familiar with the issue of proof. There is no research on how proof is dealt with in teacher education in Sweden. The theoretical framework developed in my thesis could be beneficial, if applied in research both in school mathematics and in teacher education.

7.4 Final words

My study illuminates the teaching and learning conditions of proof in a community of mathematical practice at a mathematics department where the joint enterprise is the enhancing of learning of mathematics in a broad sense (see p. 31). The thesis is a descriptive account and gives a contribution to knowledge in this area by shedding light on the diversity of pedagogical views on proof among mathematicians. It illuminates the complexity of the didactic issue of proof from both students' and from mathematicians' perspective. The fusing of a socio-cultural perspective with the social practice theory of Lave and Wenger (1991) and Wenger (1998) and theories about proof offers a fresh perspective, which I have argued is well suited to understanding and describing the diversity of the culture involving such a complex concept as proof. I hope it will prove useful for further studies.

The results bring about the following reflections. Students related positively to proof and they wanted to learn more about proof when they entered the practice. How then, could mathematicians, in the best way, take care of students' positive relation and expectations regarding proof and help them to proceed in their mathematical practice? Both the students and the mathematicians agreed on the fundamental role of proof in mathematics. Hence, a focus on proof as a dynamic notion could serve as a source of inspiration for both teachers and students. Examining various aspects of proof creates an excellent possibility to look at mathematics as a human enterprise with rules and conventions and definitions that do not have a truth value. But at the same time these activities could allow newcomers to learn to appreciate mathematics as a fantastic body of knowledge that is always growing and changing, a practice where people exercising mathematics investigate, question, criticise, define, test conjectures, prove statements, calculate, solve problems, reason, argue and so on.

I hope this thesis will rouse a debate about the role of proof in mathematics curricula, both in school and at university, because in the end, it is a question of value whether proof is included in the curriculum, a question that has to do with how mathematics is seen and what aspects of mathematics are in the focus of teaching. I also hope that the thesis with both the empirical findings and the theoretical insights about the teaching of proof will enhance consciousness among mathematicians, upper secondary school teachers, the

authors of mathematics textbooks and teacher educators about the role and functions of proof in the teaching of mathematics as well as the problems in drawing students to the practice of proof.

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Appendix 1 (http://www.info.su.se/english/edu/ECTS/Departments/Courses.List.403.shtml)

Mathematics, Basic course (30 ECTS Credits)

Course description:

Introductory course (7.5 ECTS credits)
Polynomial division, The factor theorem, factorizations, inequalities, absolute value, geometric sum, functions, the straight line, power functions, exponential- and logarithmic functions, geometry (congruence and similarity), trigonometry, trigonometric functions. Somewhat about sets. Complex numbers.

Linear algebra (7.5 ECTS credits).
The binomial theorem, proof by induction.
Systems of linear equations, matrix algebra, determinants, vectors in 2 and 3 dimensions, linear independence, dot product, vector product, straight lines and planes, linear mappings.

Mathematical Analysis 1, (7.5 ECTS credits).
Inverse functions, cyclometric functions.
Limits, continuity, derivatives, derivation rules, derivation of elementary functions, extreme value problems, asymptotes, inequlities, integrals, relation between primitive functions and integrals, partial integration, method of substitution, integrals of certain classes of functions.

Mathematical Analysis 2, (7.5 ECTS credits).
Functions of one variable:
Applications of integrals. Differential equations (separable, linear first- and second-order equations), Taylor´s formula.
Functions of several variables: Limits, partial derivation, level curves and level surfaces, tangent plane, linear approximation, extreme value problems over compact domains, double integrals.

Teaching and learning methods:

Lectures: 8 hours per week.
Lessons in small groups: 7 hours per week

Methods of assessment:

Written examination in each of the four sub-courses of 7.5 ECTS credits

Bibliography:

Persson-Böiers, Analys i en variabel, Studentlitteratur

Persson-Böiers, Analys i flera variabler, Studentlitteratur

Tengstrand, Lineär algebra med vektorgeometri,
Studentlitteratur

Linear Algebra 2, Intermediate course (7.5 ECTS Credits)

Course description:
Linear spaces, linear independence, base, dimension, coordinates in different bases. Inner product. Cauchy-Schwarz inequality, orthogonal bases. Matrices, row spaces and column spaces, rank of matrix, invertibility, orthogonal matrices, determinants. Linear mappings, matrix representation in different bases, null space, range, Eigenvectors, diagonalization. Quadratic forms with applictions to curves and surfaces of the second degree.

Teaching and learning methods:
Lectures 6 hours a week

Methods of assessment:
Written examination

Bibliography:

Tengstrand, Lineär algebra med vektorgeometri,
Studentlitteratur

Mathematical Analysis 3, Intermediate course (7.5 ECTS Credits)

Course description:
Functions of one variable: Theory of limits, continuity, differentiation, integration and Taylor´s formula.
Functions of several variables: Limits, continuity, differentialbility, the chain rule, gradient and directed derivative. Higher derivatives, Taylor´s formula, optimization problems, local extrema. Dubble integrals, change of variables.

Teaching and learning methods:
Lectures 6 hours a week

Methods of assessment:
Written and oral examination

Bibliography:

Persson & Böiers, Analys i en variabel, Studentlitteratur

Persson & Böiers, Analys i flera variabler, Studentlitteratur

Mathematical Analysis 4, Intermediate course (7.5 ECTS Credits)

Course description:
Analysis in one variable: Series, generalized integrals and power series.
Analysis in several variables: Triple integrals, curves, line integrals, Greens formula, surfaces, surfce integrals, theorems of Gauss and Stokes.

Teaching and learning methods:
Lectures 6 hours a week

Methods of assessment:
Written examination

Bibliography:

Persson & Böiers, Analys i en variabel, Studentlitteratur

Persson & Böiers, Analys i flera variabler, Studentlitteratur

Algebra and Combinatorics, Intermediate course (7.5 ECTS Credits)

Course description:
Recursion and induction, set theory (functions and relations), combinatorics (combinations and permutations), divisibility and factorization of integers, modular arithmetic, group theory, somewhat about rings and fields

Teaching and learning methods:
Lectures 6 hours a week

Methods of assessment:
Written examination

Bibliography:

Biggs, Discrete Mathematics

Foundations of Analysis, Advanced course (7.5 ECTS Credits)

Course description:
Real numbers. Bolzano-Weierstrass theorem. Theorems for continuous functions on compact intervals.
Derivation and integration i R^n. Series of functions, uniform convergence. Implicit functions.

Teaching and learning methods:
Lectures 6 hours per week

Methods of assessment:
Written and oral examination

Bibliography:

Rudin, Principles of Mathematical Analysis, McGraw-Hill

Analytic Functions, Advanced course (7.5 ECTS Credits)

Course description:
Analytic functions. Integration and series expansion of analytic functions. Residue calculus.
Conformal mappings. Harmonic functions. Applications to physics.

Teaching and learning methods:
Lectures 6 hours per week.

Methods of assessment:
Written examination.

Bibliography: Saff & Snider, Fundamentals of Complex Analysis, Prentice-Hall

Algebra, Advanced course (7,5 ECTS Credits)

Course description:
Group theory: Subgroups, cosets, Lagrange´s theorem, homomorphisms, normal subgroups and factor groups, permutation groups, simple groups.
Rings and fields: Matrix rings, Quaternions, ideals and homomorphisms, quotient fields, polynomial rings, principal ideal domains and Euclidean domains.
Fields and vector spaces: Vector spaces of finite dimension, algebraic extensions, finite fields.

Teaching and learning methods:

Lectures 6 hours per week

Methods of assessment:

Written and oral examination

Bibliography:

Beachy & Blair, Abstract Algebra, Waveland Press

Logic, Advanced course (7,5 ECTS Credits)

Course description:

Predicate logic in axiomatic form and in natural deduction, completeness theorem of predicate logic and its apparent applications, among them the theoretical background of the resolution method.

Teaching and learning methods:

Lectures 6 hours per week

Methods of assessment:

Written examination

Bibliography:

van Dalen, Logic and Structure, Springer

Algebra, Specialized Course (7.5 ECTS Credits)

Course description:

Noetherian rings, Hilbert´s basis theorem.
Modules: Direct sum, finitely generated modules over principal ideal rings with application in linear transformations.
Semi-simple algebra: Maschke´s theorem, Wedderburn´s theorem.
Multilinear algebra: Tensor products, exterior products.

Teaching and learning methods:

Lectures two hours per week

Methods of assessment:

Written or oral examination or written home assignments.

Bibliography:

Stenström, Algebra, Stockholms universitet

Appendix 2

En undersökning om bevis (*An investigation about proof*)

Jag är doktorand i matematik med ämnesdidaktisk inriktning vid Stockholms universitet. Jag är intresserad av bevis och dess roll i undervisningen varför jag nu gör en enkätundersökning bland nybörjarstudenter i hela landet. Jag är tacksam om du vill medverka genom att svara på följande frågor. Om platsen inte räcker till fortsätt gärna på andra sidan av papperet.
(*I am a doctoral student in didactics of mathematics at the University of Stockholm. I am interested in proof and its role in teaching. That is why I now conduct a survey among university entrants in different parts of the country. I am grateful if you can contribute through responding to the following questions. If there is not place enough to write your answers, please use the other side of the paper.*)

Tack på förhand! (*Thank you!*)

Kirsti Nordström

---

1. Varför vill jag studera matematik (*Why do I want to study mathematics*)?

_____

_____

_____

_____

_____

2. När jag får en uppgift som börjar "Visa att…" känner jag mig oftast (*When I get a task starting "Show that…" I most often feel*)

a) nyfiken (*curious*)
b) orolig (*anxious*)
c) ivrig (*eager*)
d) dum (*stupid*)
e) osäker (*insecure*)
f) annat (*some other way*)_____
g) har aldrig fått en sådan uppgift (*I have never got a task like that*)

3. Vilka egenskaper anser du att ett bevis skall ha? (*What do you think is characteristic of a proof?*)

   _____
   _____
   _____
   _____
   _____
   _____

4. Lisa, Tove, Peter, Mattias och Lina försökte visa att följande påstående är sant/inte sant (*Lisa, Tove, Peter, Mattias and Lina tried to show that the following statement is true/not true*):

   Summan av två godtyckliga jämna tal är alltid ett jämt tal. (*The sum of two arbitrary even integers is always an even integer.*)

Lisas svar (*Lisa's answer*)

a är ett godtyckligt heltal

b är ett godtyckligt heltal

$2a$ och $2b$ är godtyckliga jämna heltal

$2a + 2b = 2(a + b)$

(a is an arbitrary integer

b is an arbitrary integer

$2a$ and $2b$ are arbitrary even integers

$2a + 2b = 2(a + b)$)

Så Lisa säger att påståendet är sant.

(*So Lisa says the statement is true.*)

Toves svar (*Tove's answer*)

$2 + 2 = 4$ $\quad 4 + 2 = 6$

$2 + 4 = 6$ $\quad 4 + 4 = 8$

$2 + 6 = 8$ $\quad 4 + 6 = 10$

Så Tove säger att det är sant. (So Tove says the statement is true.)

Peters svar (*Peter's answer*)

Jämna tal kan delas med 2. När man adderar tal med en gemensam faktor, i detta fall 2, har svaret också den samma gemensamma faktorn.

You can divide even integers by 2. When you add integers with the same common factor, in this case 2, the sum has also the same common factor.

Så Peter säger att påståendet är sant. (*So Peter says the statement is true.*)

Mattias svar (Mattias' answer)

Jämna tal slutar med 0, 2, 4, 6 eller 8. När du lägger ihop två sådana tal slutar summan också med 0, 2, 4, 6 eller 8. (*The last number of even integers is 0, 2, 4, 6 or 8. When you add two of these the last number of the sum is also 0, 2, 4, 6 or 8.*)

Så Mattias säger att det är sant. (*So Mattias says the statement is true.*)

Linas svar (*Lina's answer*)

Låt x = ett godtyckligt heltalssvar, y = ett godtyckligt heltal. (*Let x = an arbitrary integer, y = an arbitrary integer.*)

$x + y = z$

$z - x = y$

$z - y = x$

$z + z - (x + y) = x + y + 2z$

Så Lina säger att det är sant. (*So Lina says the statement is true.*)

Välj de svar som bäst motsvarar din bild av ett korrekt bevis och motivera ditt svar. (*Choose the answer(s) which best correspond(s) to your view of a correct proof and give a reason for your choice.*) _____

_____

_____

5. Hur ofta bevisade din gymnasielärare påståenden för klassen (*How often did your upper secondary school teacher prove statements to your class*)?

a) varje lektion (*every lesson*)
b) en gång i veckan (*once a week*)
c) en gång i månaden (*once a month*)
d) ett par gånger i terminen (*about twice a term*)
e) mera sällan (*more seldom*)

6. Hur ofta övade du själv att bevisa matematiska påståenden i gymnasiet (*How often did you practise proving statements yourself in upper secondary school*)?

a) varje lektion (*every lesson*)
b) en gång i veckan (*once a week*)
c) en gång i månaden (*once a month*)
d) ett par gånger i terminen (*about twice a term*)
e) mera sällan (*more seldom*)

7. Hur ofta arbetade du i gymnasiet med egna undersökningar (ensam eller i en grupp) som ledde fram till hypoteser och eventuella bevis (*How often did you work on your own investigations (alone or in a group) that led to conjectures and sometimes to proofs*)?

a) varje lektion (*every lesson*)
b) en gång i veckan (*once a week*)
c) en gång i månaden (*once a month*)
d) ett par gånger i terminen (*about twice a term*)
e) mera sällan (*more seldom*)

8. Hur ofta genomförde du muntligen matematiskt resonemang i gymnasiet (*How often did you reason orally in upper secondary school*)?

f) varje lektion (*every lesson*)
g) en gång i veckan (*once a week*)
h) en gång i månaden (*once a month*)
i) ett par gånger i terminen (*about twice a term*)
j) mera sällan (*more seldom*)

9. Hur ofta fick du muntligt bevisa matematisk påståenden i gymnasiet (*How often could you orally prove mathematical statements in upper secondary school*)?

k) varje lektion (*every lesson*)

l) en gång i veckan (*once a week*)

m) en gång i månaden (*once a month*)

n) ett par gånger i terminen (*about twice a term*)

o) mera sällan (*more seldom*)

Eventuella kommentarer till frågorna 11-15 (*Possible comments about the questions 5-9*):

_____

_____

_____

_____

_____

10. Tag ställning till följande påståenden och välj det svarsalternativ som sammanfaller med din åsikt eller situation. Ringa in lämpligt svarsalternativ (*Choose the alternative which corresponds to your opinion or situation*).

1. Helt av annan åsikt, 2. delvis av annan åsikt, 3. kan inte säga, 4. delvis av samma åsikt, 5. helt av samma åsikt
(*1. totally disagree, 2. partially disagree, 3. cannot say, 4. partially agree, 5. totally agree*)

1) Matematiskt bevis skiljer sig från andra typer av bevis (*Mathematical proofs are different from other kinds of proofs*)	1	2	3	4	5
2) Matematiskt bevis både verifierar och förklarar (*Mathematical proof both verifies and explains*)	1	2	3	4	5
3) Jag har fått tillräckligt med övning i skolan i att konstruera bevis (*I have had exercise enough in constructing proofs in school*)	1	2	3	4	5
4) Exempel övertygar mig om att ett matematiskt resultat är sant (*Examples convince me that a mathematical result is true*)	1	2	3	4	5
5) Bevis är en väsentlig del av matematiken (*Proof is an essential part of mathematics*)	1	2	3	4	5
6) Det är svårt att själv utföra ett bevis (*It is difficult for me to prove statements*)	1	2	3	4	5

7) Bevis bör användas i undervisningen i grundskolan (*Proofs should be used in mathematics education in comprehensive school*)	1	2	3	4	5
8) Jag har fått öva i skolan att muntligt bevisa påståenden (*I have had the possibility to practise proving orally in school*)	1	2	3	4	5
9) Bevis bör användas i undervisningen i gymnasieskolan (*Proofs should be used in mathematics education in upper secondary school*)	1	2	3	4	5
10) Jag tycker att det är roligt att själv försöka visa påståenden i matematiken (*I like to try to show/demonstrate mathematical statements*)	1	2	3	4	5
11) Jag ser ingen vits med bevis: alla resultat har redan visats av kända matematiker (*I see no meaning with proof; famous mathematicians have already proved all the results*)	1	2	3	4	5
12) Om ett resultat i matematik verkar intuitivt rätt finns det inget behov att bevisa det (*If a result seems to be intuitively correct there is no need of proving it*)	1	2	3	4	5
13) Jag vill gärna lära mig mera om matematiskt bevis (*I would like to learn more about mathematical proof*)	1	2	3	4	5
14) Ett kriterium för betyget MVG för alla gymnasiekurser är att eleven genomför såväl muntligt och skriftligt matematiska bevis (*One criterion for the best mark in mathematics in all courses in upper secondary school is that the pupil can prove statements both orally and in writing*)	1	2	3	4	5
15) Jag brukar kontrollera på olika sätt att ett resultat av en räkneuppgift är korrekt (*I usually control the correctness of the result of a mathematical task in different ways*)	1	2	3	4	5
16) Min gymnasielärare brukade ofta bevisa påståenden för klassen (*My teacher in upper secondary school often used to prove statements to us*)	1	2	3	4	5
17) Jag vill alltid förstå vad jag gör i matematik (*I always want to understand what I do in mathematics*)	1	2	3	4	5

18) Jag hade gärna lärt mig mera om bevis i skolan (*I would like to have learned more about proof in school*)	1	2	3	4	5
19) Det är bra att kunna härleda formler (*It is good to be able to derive formulas*)	1	2	3	4	5
20) Det är svårare att bevisa påståenden än att lösa räkneuppgifter (*It is more difficult to prove statements than solve problems*)	1	2	3	4	5
21) Jag har fått övning i skolan att skriftligt formulera bevis (*I have had the possibility to practise proving by writing in school*)	1	2	3	4	5
22) Det räcker att kunna använda formler, det är inte så viktigt att förstå allting (*It is enough to be able to use formulas. It is not so important to understand everything*)	1	2	3	4	5
23) Det är roligt att kunna några matematiska bevis (*It is nice to know some mathematical proofs*)	1	2	3	4	5
24) Det är lättare att förstå att ett matematiskt påstående är sant om man ser ett exempel än om man ser ett bevis (*It is easier for me to understand that a statement is true after seeing an example than after seeing a proof*)	1	2	3	4	5
25) Bevis hjälper mig att förstå matematiska sammanhang (*Proofs help me to understand mathematical connections*)	1	2	3	4	5
26) Det är tråkigare att syssla med bevis än att lösa räkneuppgifter (*It is more boring to prove statements than to solve computational problems*)	1	2	3	4	5
27) Jag tycker att det är roligt att försöka bevisa i matematiken (*It is fun to construct mathematical proofs*)	1	2	3	4	5
28) Ett matematiskt bevis beror på andra matematiska resultat (*A mathematical proof depends on other results in mathematics*)	1	2	3	4	5
29) Jag har fått bekanta mig med olika typer av bevis i skolan (*I have had the possibility to familiarise myself with different kinds of proofs in school*)	1	2	3	4	5
30) Att studera bevis lär mig logiskt tänkande (*Studying proof teaches me logical thinking*)	1	2	3	4	5

Eventuella kommentarer till påståendena (*Possible comments about the statements*):

_____

_____

_____

_____

Till slut önskar jag att du svarar på några frågor angående dig och din bakgrund.
(*Finally, I want you to answer some questions concerning you and your background.*)

1. Ålder (*Age*)_____
2. Kön (*Gender*)_____
3. Året när jag gick ut gymnasiet (*The year I finished upper secondary school*)_____
4. Gymnasielinje/program (*The study programme in upper secondary school*)_____
5. Kunskaper i matematik (*Knowledge in mathematics*)

Kurs/ Nivå (*Course/Level*)	År (*Year*)	Betyg (*Mark*)

6. Utländsk gymnasieexamen, Land (*Foreign upper secondary school background, Country*)_____ Linje (*Course/Programme*)_____

7. Eftergymnasiala studier (*Studies after upper secondary school*)

Kurs (*Course*)	År (*Year*)

8. Jag ställer upp för en fokus-grupp intervju. Du kan nå mig på följande e-postadress/telefonnummer (med riktnummer) (*I agree to be interviewed. You can contact me by the following e-mail address/ telephone number*):

   _____

Tack för din medverkan (*Thank you for your contribution*)!

Appendix 3

Tables about some survey results

Questions 5 – 9

		more seldom	about twice a term	once a month	once a week	every lesson
Swedish upper secondary school background	How often did your upper secondary school teacher prove statements to your class?	15,6%	15,6%	14,3%	38,8%	15,6%
	How often did you practice proving statements yourself in upper secondary school?	40,4%	19,2%	15,9%	17,9%	6,6%
	How often did you work on your own investigations (alone or in groups) that led to hypothesis and probably to proofs?	74,0%	12,3%	5,8%	5,2%	2,6%
	How often did you reason orally in upper secondary school mathematics?	60,8%	17,6%	7,2%	8,5%	5,9%
	How often could you orally prove mathematical statements in upper secondary school?	73,4%	16,2%	3,9%	3,9%	2,6%

Foreign upper secondary school background	How often did your upper secondary school teacher prove statements to your class?				33,3%	66,7%
	How often did you practice to prove statements yourself in upper secondary school?	8,3%	8,3%	25,0%	16,7%	41,7%
	How often did you work with own investigations (alone or in groups) that led to hypothesis and probably to proofs?	33,3%	25,0%	16,7%	16,7%	8,3%
	How often did you reason orally in upper secondary school mathematics?	25,0%	8,3%	33,3%	8,3%	25,0%
	How often could you orally prove mathematical statements in upper secondary school?	33,3%	8,3%	16,7%	33,3%	8,3%

Question 10

	totally disagree	partially disagree	cannot say	partially agree	totally agree
	%	%	%	%	%
1. Mathematical proofs are different from other kinds of proofs	13,9%	17,6%	20,6%	33,9%	13,9%
2. Mathematical proof both verifies and explains	0,6%	8,4%	16,3%	35,5%	39,2%
3. I have had exercise enough in constructing proofs in school	33,9%	30,9%	21,2%	10,3%	3,6%
4. Examples convince me that a mathematical result is true	12,9%	15,2%	11,4%	37,1%	23,5%
5. Proof is an essential part of mathematics	0,0%	2,4%	7,2%	25,1%	65,3%
6. It is difficult for me to prove statements	4,2%	15,6%	21,6%	34,1%	24,6%
7. Proofs should be used in mathematics education in comprehensive school	1,8%	7,8%	21,7%	33,7%	34,9%
8. I have had a possibility to practise proving orally in school	47,9%	29,3%	9,6%	9,6%	3,6%
9. Proofs should be used in mathematics education in upper secondary school	0,6%	1,2%	6,6%	30,7%	60,8%
10. I like to try to show/demonstrate mathematical statements	5,5%	13,4%	16,5%	40,2%	24,4%

11. I see no meaning with proof; famous mathematicians have already proved all the results	65,3%	22,2%	9,6%	1,2%	1,8%
12. If a result seems to be intuitively correct there is no need of proving it	59,0%	25,3%	7,2%	7,2%	1,2%
13. I would like to learn more about mathematical proof	2,4%	6,0%	9,6%	36,1%	45,8%
15. I usually control the correctness of the result of a mathematical task in different ways	3,0%	9,0%	12,0%	48,5%	27,5%
16. My teacher in upper secondary school often used to prove statements to us	15,6%	22,8%	22,2%	25,7%	13,8%
17. I always want to understand what I do in mathematics	0,0%	4,2%	5,4%	22,8%	67,7%
18. I would like to have learned more about proof in school	,0%	6,6%	13,8%	29,9%	49,7%
19. It is good to be able to derive formulas	0,0%	4,2%	4,2%	21,0%	70,7%
20. It is more difficult to prove statements than solve problems	1,2%	4,2%	10,2%	29,3%	55,1%
21. I have had a possibility to practise proving by writing in school	28,3%	25,3%	19,9%	19,3%	7,2%

22. It is enough to be able to use formulas. It is not so important to understand everything	47,0%	34,3%	12,0%	3,6%	3,0%
23. It is nice to know some mathematical proofs	0,6%	4,9%	15,9%	31,1%	47,6%
24. It is easier for me to understand that a statement is true after seeing an example than after seeing a proof	8,4%	19,9%	24,7%	33,7%	13,3%
25. Proofs help me to understand mathematical connections	,0%	4,2%	13,3%	45,5%	37,0%
26. It is more boring to prove mathematical statements than solve computational problems	17,5%	22,3%	26,5%	21,1%	12,7%
27. It is fun to construct mathematical proofs	3,0%	13,9%	28,5%	32,7%	21,8%
28. A mathematical proof depends on other results in mathematics	2,5%	6,1%	41,7%	22,1%	27,6%
29. I have had the possibility to familiarise myself with different kinds of proofs in school	16,9%	23,5%	17,5%	28,3%	13,9%
30. Studying proofs teaches me logical thinking	1,2%	1,2%	19,9%	38,6%	39,2%

Appendix 4

Mathematical Analysis 3, Theory questions for oral examination

Part 1, Functions of one variable

1. Define the limit of a function $f(x)$ when $x \to +\infty$. Formulate and prove the sum-, product-, quotient- and squeeze laws for such limits.

2. Define a limit of a function $f(x)$ when $x \to a$. Formulate and prove the sum-, product-, quotient- and squeeze laws for such limits.

3. Define a limit of a number sequence. Formulate and prove the sum-, product-, quotient- and squeeze laws for such limits.

4. Define supremum and infimum. Define the limit of a number sequence. Formulate the supremum axiom. Formulate and prove a theorem about limits of monotonic number sequences.

5. Account for the definition of the number e as a limit and prove that this limit exists.

6. Define continuity. Formulate and prove the intermediate value theorem. (If the intersection theorem of intervals is used it must be proved.)

7. Define continuity. Formulate and prove the intersection theorem of intervals. Formulate and prove the theorem of the maximum value and the minimum value.

8. Define the derivative of a function. Show that differentiability implies continuity. Show the sum-, product-, and quotient laws for derivatives.

9. Define the derivative of a function. Formulate and prove the theorem of differentiability of a composite function of two differentiable functions (the chain rule).

10. Define the derivative of a function. Formulate and prove the theorem of the derivative of an inverse function.

11. Show that if a function f has a local maximum or a local minimum at a point a, so is the derivate of f at zero under appropriate circumstances. Formulate and prove the mean-value theorem.

12. Formulate the mean-value theorem. Formulate and prove a theorem about the relation between monotonic functions and the derivative.

Part 2, Functions of one variable

1. Formulate and prove Taylor's formula.

2. Formulate and prove the uniqueness theorem of Maclaurin expansions.

3. Let f be a function that is bounded on a closed and bounded interval I. Define the notions of the lower and upper integral of f over I. Show that the lower integral is less than or equal to the upper integral. Define the notions of integrability and the integral of f on I. Show that if f and g are bounded and integrable on $[a, b]$ (where $a < b$) then $f + g$ is bounded and integrable on $[a, b]$ and $\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.

4. Let f be a function that is bounded on an interval $[a, b]$ where $a < b$. Assume that f is integrable on $[a, c]$ and $[c, b]$ where $a < c < b$. Show that f is integrable on $[a, b]$ and that $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$. Formulate and prove also a theorem about integrability of monotonic functions. Explain then how one can combine both of these results and obtain the result that a great collection of functions are integrable.

5. Define the notions of continuity and uniform continuity. Explain the relation between these notions. Formulate and prove the theorem about integrability of continuous functions.

6. Let f and g be functions that are bounded on a closed and bounded interval I. Show that if f is bounded above by g on the interval I, then the integral of f over the interval I is less than or equal to the integral of g over the interval I. Formulate and prove the two mean value theorems for integral calculus.

7. Formulate and prove a theorem about the relation between the derivative and the integral. Formulate and prove the fundamental theorem of calculus.

8. Define convergence, divergence and the sum (if it exists) of an infinite series. Show that the terms of a convergent series approach zero. Define the notions of absolute convergence and conditional convergence. Show that an absolute convergent series is convergent.

9. Define convergence, divergence and the sum of an infinite series (if it exists). Formulate and prove the integral criterion and two comparison criteria for positive series.

10. Define convergence, divergence and the sum of an infinite series (if it exists). Formulate and prove the ratio and the root test for series.

11. Define convergence, divergence and the sum of an infinite series (if it exists). Formulate and prove Leibniz' convergence criterion.

12. Show that if a power series converges at more than one point, then the set of points where it converges is an interval. Define the notion of radius of convergence and give an account of how a power series can be differentiated and integrated.

Part 3, functions of several variables

1. Define the notions of limit, continuity, partial derivative and differentiability for a function of several variables. Show that if a function of several variables is differentiable then it is also continuous and has partial derivatives of first order.

2. Define differentiability. Show that if a function of several variables has continuous partial derivatives of first order then the function is differentiable (only the case of two variables is demanded).

3. Formulate and prove the theorem according to which the both mixed second derivatives for a function in two variables are equal under appropriate conditions.

4. Formulate and prove the chain rule for the case of a composite function of the type $t \mapsto f(g(t), h(t))$. Formulate and outline the proof for the general chain rule for vector-valued functions of several variables.

5. Define the directional derivative and the gradient. Formulate and prove the theorem about the relation between the directional derivative and the gradient. Formulate and prove a theorem about the relation between the increasing of a function of several variables and the direction of the gradient of the function.

6. Show Taylor's formula of second order in two variables.

7. Define a local extreme point of a function of several variables. Define the notions of a positive definite, negative definite and indefinite quadratic form. Formulate and prove a theorem about how the quadratic form in the Taylor expansion decides the character of a local extreme point (only the case of two variables is demanded).

8. Formulate the general theorem about Lagrange multipliers and prove it for the special case of two variables and one constraint.

9. Formulate and prove Green's Theorem for domains in the plane with one lower and one upper and one left and one right part. Then sketch how to obtain Green's Theorem for more general domains in the plane.

10. Let $\mathbb{F}$ be a continuous vector field defined in a pathwise-connected open subset Ω of the plane. Show that line integrals of $\mathbb{F}$ in Ω are independent of the path if and only if $\mathbb{F}$ has a potential in Ω.

11. Let $\mathbb{F} = (P, Q)$ be a continuously differentiable vector field defined in a simply connected open subset of the plane. Show that line integrals of $\mathbb{F}$ in Ω is independent of the path if and only if $D_2P = D_1Q$ in Ω. Give an example which shows that this is not true if simply connected is changed to pathwise-connected.

12. Formulate and prove the Divergence Theorem for domains in space with one lower and one upper surface, one left and one right surface and one back and one front surface. Then sketch how to obtain the Divergence Theorem for more general domains in space.

Appendix 5

	Meaning of dealing with proof	Induction/ Deduction	Intuition/ Formality	Transparency	The reasons for not dealing with proof	The view on students
I Progressive	Invisible tool in the derivation of formulas, "Proof must be needed": given only if they enhance understanding or offer conviction	Start by examples, calculations	Informal, Prefer natural language, Avoid symbols in the beginning, Omit technical and long proofs, Avoid proving evident statements	**Invisible**: No discussions on proof, The word 'proof' avoided, No proofs for "evident" statements **Visible**: the need for proof, the need of the symbols	No intentions because: Most of the students need something else, Calculations more important, Students not interested	Students not capable, Afraid, Most of them do not understand the meaning of proof, Not interested
II Deductive	Real mathematics, Abstract and general, Conviction + explanation connected to critical thinking	Prefer deductive approach	Quite formal with mathematical symbols, Rigorous, because students need to see all the steps in the beginning, Nothing is evident	**Visible**: logical steps, language and symbols, Formal demands, Difference between intuitive and formal, the word 'proof', evident statements **Invisible:** inductive approaches and the origins of proofs	Lack of time	Students interested and capable, but have too little experience
III Classical	Aesthetic, Inspiring, Intellectual challenge, Logical reasoning-transfer, Soul of mathematics	Mostly deductive	Intuitive and informal, not rigorous, Symbols used when needed, Do not bother proving evident statements	**Invisible:** No discussions on proof, Logical steps on elementary level **Visible:** What proofs they like themselves, great ideas, aesthetic	External reasons	Most of them not interested, not capable, Pity for the few others

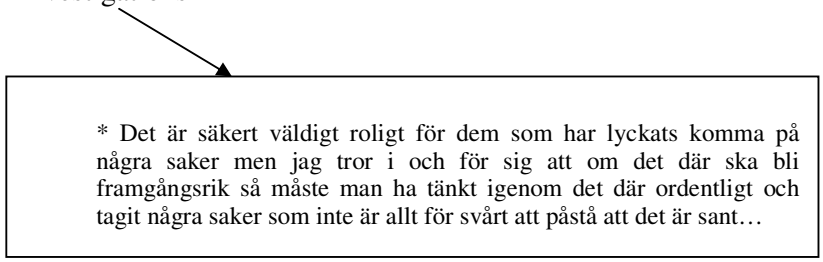
Appendix 6

An example of how I have worked with NVivo. Here the free nodes have been organised into a tree.

Mathematicians' pedagogical perspectives (node)

- Intentions (child node)
  - intentions of dealing with proof
  - no intentions of dealing with proof
    - external reasons
      - students' prior knowledge
      - lack of time
      - examination
      - …
    - internal reasons
      - students do not need proof
      - first intuitive knowledge

  - discussion about proof
  - own investigations



- How see students as learners of proof
  - difficulties
    - idea of proof
  - …

- The practice
  - changes in the practice
  - courses
    - basic course
    - …
  - …

- Pedagogical considerations
  - the dilemma of transparency
  - general

* An example of the utterances coded into the child node "own investigations". In every child node there are a number of utterances from different documents coded into the node.