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MASTER THESIS IN THEORETICAL PHYSICS

Correlators in Matrix Models

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Denna uppsats handlar om att beräkna och analysera korrelatorer i hermitiska matrismodeller huvudsakligen. Matrismodeller kan ses som matematiska leksaksmodeller av fysikaliska system. De har många applikationer inom fysiken, särskilt känd är möjligen den inom strängteorin. Korrelatorer är ofta särskilt intressanta vid studier av fysikaliska system, de har den matematiska strukturen av ett väntevärde och kan under rätt förutsättningar tolkas som fysikaliska storheter. Denna uppsats ger en fristående introduktion till den hermitiska matrismodellen, till vanliga sätt att beräkna dess korrelatorer och till ett särskilt upplysande sätt att beräkna våra korrelatorer med hjälp av en symmetri i den modellen, de så kallade Virasoro-begränsningarna. En förklaring ges till hur man beskriver den hermitiska matrismodellen i form av integraler och hur man explicit beräknar dess korrelatorer. Dessa korrelatorer är av kombinatorisk betydelse och kan tolkas i termer av så kallade 'fat'-grafer. Ett detaljerat recept på hur man ritar och räknar dessa grafer ges. Användning av förståelsen som fås av 'fat'-graferna leder oss till korrelatorernas allmänna struktur. Vidare diskuteras hur man hanterar de oändligheter som uppenbarar sig i gränsvärdet för stora matriser. Virasoro-begränsningarna leder till ett samband mellan korrelatorerna som tillåter oss att hitta godtyckliga korrelatorer utan ytterliggare beräkningar. I gränsvärdet för stora matriser hittas ett liknande samband mellan korrelatorernas kombinatoriska koefficienter och med detta är korrelatorerna helt analyserade. Huvudresultatet i denna uppsats är att metoden som består i att använda virasorobegränsningarna i gränsvärdet av stora matriser för att hitta koefficienter som beskriver korrelatorer också kan användas på en generalisering av matrismodellen, den betadeformerade matrismodellen. Trots att korrelatorenas strukturer initialt är okända, kunde vi här beräkna och analysera dem med denna metod.

Abstract

This master thesis gives an introduction to matrix models, followed by extensive calculations using the Virasoro Constraints obtaining the correlators. The new results found within this master thesis are the solutions, the free energy formulation and the recurrence relation for its coefficients, for β deformed Virasoro constraints at large N.

The focus for most of the thesis lies on the Hermitian Matrix Model. Its description in form of integrals and its eigenvalue formulation using the Vandermonde Determinant are introduced. As the title suggests, the correlators of this model are the main interest of this thesis. The choice of correlators and how to calculate them is explained, as well as their interpretation in the form of so-called "fat" or "ribbon" graphs and their genus expansion using the Euler Characteristic. Recurrence relations for the correlators are found, using the Virasoro Constraints. Using the Free Energy instead of the Generating Function, recurrence relations for connected correlators are found. And finally moving to the large N limit of the hermitian matrix model using the t'Hooft coupling, the Virasoro constraints can be solved for every genus individually and the correlator coefficients, called generalized Catalan numbers, can be obtained. Using this method on the β-deformed matrix model, for which the correlators are a priori not known, a solution for the Virasoro constraints can be found and with it a recurrence relation for integer coefficients, a β deformation of the generalized Catalan numbers. These solutions can be checked using a β transformation invariance. In the end, a final generalization is done, to move away from Gaussian integrals to general integer powers in the exponent. This does not change the prior results, but introduces a need for more initial conditions than given by the Virsoro Constraints. The solution of the β-deformed Virasoro constrains is the main result of this thesis and will be discussed further in [1].

Contents

1 Introduction to Matrix Models

Matrix models are mathematical models comparable in their structure to different areas of physics, like quantum field theory or statistical physics. They can be thought of as a mathematical toy model of a physical system. These models have an integral description, corresponding correlators and symmetries, all depending on matrices. Hermitian matrices in this case. Matrix models could also be formulated in terms of other matrices, like for example real symmetric matrices, but in this thesis we content ourselves investigating only hermitian matrices. A good self-contained introduction to matrix models is found in [2]. Where amongst many other things a short overview of the already widespread applications of matrix models is given, reaching from string theory over integrable systems to biology. They mention the β-deformed matrix models, which are going to be of importance in the last section, section 4, of this thesis. A connection between matrix models and string theory is for example discussed in [3] or [4]. Where we also find many of the introductory concepts discussed in this first section. For finitely large matrices, an interpretation in terms of one dimensional quantum field theory is quite common, whereas for the limit of infinitely large matrices we pass into string theory. In quantum field theory, correlators are understood as graphs representing a world line. In string theory, we pass from a world line to a world sheet. This is reflected by the so-called ribbon graphs, which will be introduced in section 1.5, and which live on Riemann surfaces of varied genus. The different applications and interpretations of matrix models often centre around these ribbon graphs. For a longer and more detailed excursion on the possible graphs and combinatorics of matrix models, have a look at [5]. After this first Introduction section, we will mainly work with the so called Virasoro constraints. A thorough investigation into these constraints can be found in [6]. This "Introduction" section concerns itself with all the things necessary to build and introduce the hermitian matrix model.
First off: a quick reminder of the properties of Hermitian matrices.

1.1 Hermitian Matrices

Hermitian matrices are defined in the following way:

$$M_{ij} = \bar{M}_{ji} \qquad M = M^\dagger \tag{1}$$

Meaning that they come back to their original form after transposing (changing indices) and complex conjugating (denoted by a bar on top, $\bar{M}$) their elements. The diagonal elements of a general hermitian matrix are therefore real numbers, and the upper triangular matrix is the complex conjugate of the lower one. Take a general 2×2 matrix as an example:

$$M = \begin{pmatrix} a & c+id \\ c-id & b \end{pmatrix} \tag{2}$$

with $a, b, c, d \in \mathbf{R}$. Another property of hermitian matrices is that they get diagonalized by unitary matrices.

$$D = UMU^{-1} \tag{3}$$

The diagonal matrix D is again a hermitian matrix. Unitary matrices have the property:

$$UU^{\dagger} = 1 \qquad U^{\dagger} = U^{-1} \tag{4}$$

The trace, which is the sum of the diagonal elements of a matrix, is invariant under cyclic permutations. Because of this the trace of a hermitian matrix is equal to the sum of this eigenvalues.

$$Tr(M) = Tr(MU^{\dagger}U) = Tr(UMU^{\dagger}) = Tr(D) = \sum_{i=1}^{N} \lambda_i \tag{5}$$

With λ_i the eigenvalues of M. This works for any power of M, by putting more unit matrices (4) into the trace:

$$Tr(M^k) = Tr(MU^{\dagger}UMU^{\dagger}U \dots MU^{\dagger}U) = Tr(UMU^{\dagger} \dots UMU^{\dagger}) = Tr(D^k) = \sum_{i=1}^{N} \lambda_i^k \tag{6}$$

$N \times N$ unitary matrices form a group, and because they depend on continuous parameters, they even form a Lie Group, called $U(N)$. The corresponding Lie Algebra $u(N)$, a vector space over $\mathbf{C}$ with a binary operation called the Lie Bracket, can be formed by either hermitian (physics convention) or anti-hermitian (mathematics convention) matrices. Because we want to work with hermitian matrices, we choose the convention usually used by physicists. The Lie bracket in this convention is of the following form:

$$[M_a, M_b] = if^c_{a,b}M_c \tag{7}$$

with M_a, M_b, M_c hermitian matrices and $f^c_{a,b} \in \mathbf{R}$ the so-called structure constant. The adjoint representation of $U(N)$ is defined in the following way:

$$Ad : U(N) \times u(N) \to u(N) \qquad M \mapsto UMU^{-1} \tag{8}$$

Functions invariant under this mapping are called gauge invariant and take the form of traces $Tr(M^k)$, already mentioned above. [3] elaborates on this understanding of a matrix model as a quantum gauge theory, a zero dimensional quantum gauge theory.

Next, we would like to build a path integral in terms of hermitian matrices. Keep in mind the properties of the trace of hermitian matrices as an invariant function under the transformation by unitary matrices.

1.2 Gaussian Integrals

Central to the hermitian matrix models is the Gaussian integral over all $N \times N$ hermitian matrices M.

$$Z_0 = \int_H dM e^{-\frac{1}{2}Tr(M^2)} \tag{9}$$

Let's elaborate on where this expression came from and what it means. First, compare (9) to the well known one dimensional Gaussian integral, which in terms of Hermitian matrices would correspond to a 1×1 matrix:

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi} \tag{10}$$

Integrals of this kind appear in various physics models. In statical physics, we would call them partition function and the function found in the exponent, up to alterations, would be called hamiltonian. In quantum field theory, we call integrals of this form path integrals, and the function in the exponent would be called action.

In matrix models, the goal is to integrate over all possible matrices. As a first intuition, it would therefore make sense to define the Lebesgue measure dM, with M a hermitian matrix, as the measure of the matrix elements $dM = \prod_{i,j=1}^{N} dM_{ij}$. However, it is not immediately clear how to deal with the off diagonal elements, as they are complex. From the section above, we know that the elements in the upper and lower triangle are the same up to complex conjugation. We therefore simply define the measure of the off diagonal elements, by their real and imaginary parts.

$$dM = \prod_{j>i=1}^{N} dM_{ii}\; dRe(M_{ij})\; dIm(M_{ij}) \tag{11}$$

As for the range, because the integral should range over all $N \times N$ hermitian matrices M, H is simply defined to be $H = \{M_{ii}, Re(M_{ij}), Im(M_{ij}) \in \mathbf{R}\}$.

Lastly, the exponent $\exp(-1/2\; Tr(M^2))$. Following the model of a Gaussian integral (10), it should depend on quadratic matrix elements. The way to make sense of a matrix in the exponent in this case is to reduce the matrix to a scalar. The first options that come to mind would be to take the determinant or the trace. In section 1.1 it has already been discussed, that the trace $Tr(M^k)$ has especially nice properties, as it leads to a gauge invariant description of our model, which from a physics point of view is very desirable. It can be further advocated that $Tr(M^2)$ in the exponent is a neat option. Going to an eigenvalue description will leave it invariant, and having a closer look at the proposed expression:

$$Tr(M^2) = \sum_{j>i=1}^{N} M_{ii}^2 + 2Re(M_{ij})^2 + 2Im(M_{ij})^2 \tag{12}$$

This is the sum of all the squared variables found in the measure, without any extra terms of other powers. This leads to a very simple expression, a product of $N \times N$ Gaussian integrals.

$$Z_0 = \int_H dMe^{-\frac{1}{2}Tr(M^2)} = \int_H \prod_{j>i=1}^{N} dM_{ii}\; dRe(M_{ij})\; dIm(M_{ij}) e^{-\frac{1}{2}\sum_{j>i=1}^{N}(M_{ii}^2+2Re(M_{ij})^2+2Im(M_{ij})^2)} = 2^{\frac{N}{2}}\pi^{\frac{N^2}{2}} \tag{13}$$

Before we move on, one might find it rewarding to compare our initial general Gaussian integral (10) with a special case of (13), namely a 1×1 hermitian matrix integral. In this special case, the two integrals are equivalent.

1.3 Eigenvalue Description

Because the Gaussian integral over all hermitian matrices offers so nicely to be written in an eigenvalue description and because it can be very useful to look at it this way, the necessary calculations will be sketched in this section. To calculate the Jacobian Δ^2, the relations $D = UMU^\dagger$ and $UU^\dagger = 1$ from section 1.1 are needed. This is the goal:

$$\int dMe^{-\frac{1}{2}Tr(M^2)} = \int d\phi A(\phi) \prod_{i=1}^{N} d\lambda_i\; \Delta^2\; e^{-\frac{1}{2}\sum_i \lambda_i^2} \tag{14}$$

With Δ^2 the Jacobian and $d\lambda_i$ the measure of the eigenvalues. Temporarily, we also need to consider $d\phi A(\phi)$, which are artefacts of the change of variables. Throughout the thesis we will work in a normalized setting and as neither the exponent nor any other function that will be put under the integral depends on ϕ, the normalization by Z_0 (13) will take care of it. Because there are more matrix elements than eigenvalues, $d\phi$ lets the number of measures on either side of the equation agree.
In order to calculate the Jacobian, we need to express the matrix elements M_{ij} as functions of their eigenvalues. From $D = UMU^\dagger$ we find immediately:

$$M = U^\dagger DU \tag{15}$$

Or written as matrix elements:

$$M_{jk} = \bar{M}_{kj} = \sum_{i=1}^{N} \lambda_i \bar{U}_{ij} U_{ik} \tag{16}$$

Note that this is well-defined. The diagonal elements, M_{jj} which have to be real, correspond to a sum of real terms $\lambda_i |U_{ij}|^2$. And the off diagonal elements $M_{jk} = \sum_i \lambda_i \bar{U}_{ij} U_{ik}$ which need to be equivalent to the complex conjugated and transposed elements $\bar{M}_{jk} = \sum_i \lambda_i \bar{U}_{ij} U_{ik}$ also satisfy this requirement on the right-hand side. Every element in the measure defined above in (11), the diagonal elements M_{ii} and the real and imaginary parts of the off-diagonals $Im(M_{ij})$, $Re(M_{ij})$, can be expressed in terms of the unitary matrix elements and the eigenvalues. In general, a $N \times N$ unitary matrix depends on $2N^2$ real parameters, but using:

$$\sum_{i=1}^{N} U_{ij} \bar{U}_{ik} = \delta_{jk} \tag{17}$$

they can be parameterized by $N^2 - N$ variables. With M_{ii}, $Im(M_{ij})$ and $Re(M_{ij})$ depending on the $N^2 - N$ parameters describing the Unitary matrix elements and on the N eigenvalues, the Jacobian can be calculated. The part of the Jacobian called $A(\phi)$ depending on the parameters of the unitary matrix elements is not of importance for the rest of the thesis and will not be discussed in detail. The part of the Jacobian depending on the eigenvalues, however, is very relevant as our exponent depends on it, as will other functions that will shortly be put into the integral. It is well known to be the square of the so called Vandermonde determinant.

$$\Delta^2 = \prod_{i>j=1}^{N} (\lambda_j - \lambda_i)^2 \tag{18}$$

With it, the Gaussian integral of the hermitian matrix model in its eigenvalue description looks like:

$$Z_0 = \int \prod_{i=1}^{N} d\lambda_i \prod_{i>j=1}^{N} (\lambda_j - \lambda_i)^2 e^{-\frac{1}{2} Tr(M^2)} \tag{19}$$

A thorough general calculation of this can be found in [7]. There the calculations are also shown for other types of matrices, which leads to different powers of the Vandermonde determinant. We will come back to these generalizations of the hermitian matrix model in the last section of the thesis. For this thesis we content ourselves with the calculations of a 2×2 matrix, to convince ourselves of (19): Start with the general description of a hermitian matrix, like we have seen in section 1.1:

$$M = \begin{pmatrix} a & c + id \\ c - id & b \end{pmatrix} \tag{20}$$

with $a, b, c, d \in \mathbf{R}$. For the unitary matrix, we write:

$$U = \begin{pmatrix} r_1 e^{i\theta_1} & r_2 e^{i\theta_2} \\ r_3 e^{i\theta_3} & r_4 e^{i\theta_4} \end{pmatrix} \tag{21}$$

With r_i, $\theta_i \in \mathbf{R}$. Because U is unitary, we know $UU^\dagger = 1$, which in our case means:

$$\begin{aligned} r_1^2 + r_2^2 &= 1 \\ r_1 r_3 e^{i(\theta_1 - \theta_3)} + r_2 r_4 e^{i(\theta_2 - \theta_4)} &= 0 \\ r_1 r_3 e^{i(-\theta_1 + \theta_3)} + r_2 r_4 e^{i(-\theta_2 + \theta_4)} &= 0 \\ r_4^2 + r_3^2 &= 1 \end{aligned} \tag{22}$$

Solving these equations we find:

$$U = \begin{pmatrix} \cos\psi & \sin\psi e^{i\theta} \\ -\sin\psi e^{-i\theta} & \cos\psi \end{pmatrix} \tag{23}$$

Now come to the relation we are actually interested in:

$$M = \begin{pmatrix} a & c + id \\ c - id & b \end{pmatrix} = \begin{pmatrix} \cos^2\psi\lambda_1 + \sin^2\psi\lambda_2 & \cos\psi\sin\psi e^{i\theta}(\lambda_2 - \lambda_1) \\ \cos\psi\sin\psi e^{-i\theta}(\lambda_2 - \lambda_1) & \sin\psi^2\lambda_1 + \cos\psi^2\lambda_2 \end{pmatrix} = UDU^\dagger \tag{24}$$

With this the Jacobian can be calculated:

$$\det \begin{pmatrix} \frac{\partial a}{\partial\lambda_1} & \frac{\partial b}{\partial\lambda_1} & \frac{\partial c}{\partial\lambda_1} & \frac{\partial d}{\partial\lambda_1} \\ \frac{\partial a}{\partial\lambda_2} & \frac{\partial b}{\partial\lambda_2} & \frac{\partial c}{\partial\lambda_2} & \frac{\partial d}{\partial\lambda_2} \\ \frac{\partial a}{\partial\psi} & \frac{\partial b}{\partial\psi} & \frac{\partial c}{\partial\psi} & \frac{\partial d}{\partial\psi} \\ \frac{\partial a}{\partial\theta} & \frac{\partial b}{\partial\theta} & \frac{\partial c}{\partial\theta} & \frac{\partial d}{\partial\theta} \end{pmatrix} \tag{25}$$

$$\begin{vmatrix} \cos\psi^2 & \sin\psi^2 & -\cos\psi\sin\psi\cos\theta & \cos\psi\sin\psi\sin\theta \\ \sin\psi^2 & \cos\psi^2 & \cos\psi\sin\psi\cos\theta & -\cos\psi\sin\psi\sin\theta \\ 2\cos\psi\sin\psi(\lambda_2 - \lambda_1) & -2\cos\psi\sin\psi(\lambda_2 - \lambda_1) & \cos 2\psi\cos\theta(\lambda_2 - \lambda_1) & \cos 2\psi\sin\theta(\lambda_2 - \lambda_1) \\ 0 & 0 & -\cos\psi\sin\psi\sin\theta(\lambda_2 - \lambda_1) & -\cos\psi\sin\psi\cos\theta(\lambda_2 - \lambda_1) \end{vmatrix}$$

Which evaluates to:

$$= A(\psi, \theta)(\lambda_2 - \lambda_1)^2 \tag{26}$$

The first part of this expression $A(\psi, \theta)$ is the discussed uninteresting part. The term depending on the eigenvalues is exactly the Vandermonde determinant discussed before (18). The integral over all 2×2 hermitian matrices, can therefore be denoted:

$$\int d\lambda_1 d\lambda_2 (\lambda_2 - \lambda_1)^2 e^{-\frac{1}{2}(\lambda_1^2 + \lambda_2^2)} \tag{27}$$

1.4 Correlators

In accordance to the integral (9) defined in section 1.2, the correlators of our model are chosen to be of the following form:

$$\langle Tr(M^{k_1}) \dots Tr(M^{k_m}) \rangle = \frac{1}{Z_0} \int dM Tr(M^{k_1}) \dots Tr(M^{k_m}) e^{-\frac{Tr(M^2)}{2}} \tag{28}$$

or equivalently in the eigenvalue description:

$$\langle Tr(M^{k_1}) \dots Tr(M^{k_m}) \rangle = \frac{1}{Z_0} \int \prod_{i=1}^{N} d\lambda_i \prod_{i \neq j}^{N} (\lambda_i - \lambda_j) \sum_{i_1, \dots, i_m = 1}^{N} \lambda_{i_1}^{k_1} \dots \lambda_{i_m}^{k_m} \; e^{-\frac{1}{2}\sum_{i=1}^{N} \lambda_i^2} \tag{29}$$

with the powers k_i being positive integers, and m the number of traces in the correlator. The trace is chosen as our correlator, because it leads to an expression invariant under gauge transformation $M \mapsto U^\dagger M U$. Note that a normalization by Z_0, the integral defining our model, (13) has been chosen. Let's elaborate on this expression and calculate some examples. The general form of correlators like in (28) is comparable to expectation values in statistical physics. The special case of a 1×1 matrix, is a well known integral and might be a familiar place to start:

$$\langle x^i \rangle = \frac{1}{\sqrt{2\pi}} \int dx \; x^i \; e^{-\frac{x^2}{2}} \tag{30}$$

with $x \in \mathbf{R}$. The results of this integral for $i = 0$ and $i = 1$ are 1 and 0 respectively. Any other case can be obtained using the Feynman method of integration, which works as follows:

$$\langle x^{2i} \rangle = \frac{1}{\sqrt{2\pi}} \int dx x^{2i} e^{-\frac{x^2}{2}} = \frac{1}{\sqrt{2\pi}} (-2)^i \frac{\partial^i}{\partial t^i} \int dx e^{-\frac{x^2}{2}t} \Big|_{t=1} = (-2)^i \frac{\partial^i}{\partial t^i} \sqrt{\frac{1}{t}} \Big|_{t=1} = \frac{(2i)!}{2^i i!} = (2i)!! \tag{31}$$

The trick in the first equality is to introduce a dummy variable t, which has to be equal to one, and take its derivative. This trick exists in different versions and can similarly be done by introducing a linear term in the exponent. (31) holds for any even power of x. Any odd power would evaluate to zero,

$$\int dx\, x^{2i+1} e^{-\frac{x^2}{2}} = (-2)^i \frac{\partial^i}{\partial t^i} \int dx\, x\, e^{-\frac{x^2}{2}t}\Big|_{t=1} = 0 \tag{32}$$

The result from above

$$(2i)!! = \frac{(2i)!}{2^i i!} \tag{33}$$

has combinatorical significance, it counts how many ways there are to draw a graph with one vertex of valency 2i. Graph theory has its own section in this thesis, there we will elaborate on this comment.

Now, we would like to formulate an expression similar to (31) for our matrix model. One way of making sense of matrices in our correlator is by reducing them to a scalar. Knowing that the trace is invariant under eigenvalue transformation, motivates us to also pick traces in the correlators. Starting out with the simplest case:

$$\begin{aligned} \langle TrM \rangle &= \frac{1}{Z_0} \sum_{i=1}^{N} \int dM M_{ii} e^{-\frac{1}{2} TrM^2} \\ &= \frac{1}{Z_0} \sum_{k=1}^{N} \int \prod_{i=1}^{N} dM_{ii} M_{kk} e^{-\frac{1}{2}\sum_i^N M_{ii}^2} \int \prod_{i>j} dRe(M_{ij}) dIm(M_{ij}) e^{-\sum_{i>j}^{N}(Re(M_{ij})^2 + Im(M_{ij})^2)} = 0 \end{aligned} \tag{34}$$

We see that it is just a sum of correlators of the form (32), which we know to be zero.

Following the one dimensional example, we are looking for terms with even powers in our correlator. The first two examples of correlators, forming quadratic terms under the integral that come to mind, are correlators with total power of two. $\langle TrMTrM \rangle$, an example of a multi trace correlator and $\langle TrM^2 \rangle$, which is an example of a single trace correlator. Let's calculate them:

$$\langle TrM\; TrM \rangle = \sum_{i,j=1}^{N} \frac{1}{Z_0} \int_H dM M_{ii} M_{jj} e^{-\frac{1}{2} Tr(M^2)} = \sum_{i,j=1}^{N} \delta_{ij}\delta_{ji} = N \tag{35}$$

This is a sum of integrals of the form (31). It is only non-zero if both matrix elements under the integral are the same, and form a quadratic term. There are N possibilities for such a term.

$$\langle TrM^2 \rangle = \sum_{i,j=1}^{N} \langle M_{ji} M_{ij} \rangle = \sum_{i,j=1}^{N} \frac{1}{Z_0} \int_H dM M_{ji} M_{ij} e^{-\frac{1}{2} Tr(M^2)} = \sum_{i,j=1}^{N} \delta_{jj}\delta_{ii} = N^2 \tag{36}$$

What is lovely about this expression, is that every term of the sum $\sum_{i,j=1}^{N} \langle M_{ji} M_{ij} \rangle$ leads to the integral being non-zero. That is why it evaluates to N^2. Writing the integral more detailed, more explicitly using the definitions we established in section 1.2, (11) and (12):

$$\frac{1}{Z_0} \int \prod_{i>j} dM_{ii} dRe(M_{ij}) dIm(M_{ij}) (M_{ii}^2 + 2Re(M_{ij})^2 + 2Im(M_{ij})^2) e^{-\frac{1}{2}\sum_i M_{ii}^2 - \sum_{i>j}(Re(M_{ij})^2 + Im(M_{ij})^2)} \tag{37}$$

It becomes more clear that this long expression for $\langle Tr(M^2)\rangle$ is merely the sum of N^2 terms of the less cluttered one dimensional correlator $\langle x^2\rangle$. It is worth pointing out, that this works so nicely, because we have chosen $Tr(M^2)$ in our exponent.

Moving on to the next higher power. Continuing with our comparison to the more straight forward case (31), and recalling that it was only non-zero for even powers of x, it is fair to assume that this will also hold for our matrix correlators. Having a quick look at $\langle TrM^3\rangle$ shows that it always evaluates to zero, as will $\langle TrM^2TrM\rangle$ and $\langle (TrM)^3\rangle$.

$$\langle TrM^3\rangle = \frac{1}{Z_0}\sum_{i,j,k=1}^{N}\int_H dM M_{ij}M_{jk}M_{ki}e^{-\frac{1}{2}Tr(M^2)} = 0 \tag{38}$$

$$\langle (TrM)^3\rangle = \frac{1}{Z_0}\sum_{i,j,k=1}^{N}\int_H dM M_{ii}M_{jj}M_{kk}e^{-\frac{1}{2}Tr(M^2)} = 0 \tag{39}$$

$$\langle TrM^2TrM\rangle = \frac{1}{Z_0}\sum_{i,j,k=1}^{N}\int_H dM M_{ij}M_{ji}M_{kk}e^{-\frac{1}{2}Tr(M^2)} = 0 \tag{40}$$

Every term of these sums of integrals evaluates to zero, as we are not able to form a term without matrix elements of odd power. This argument holds whenever the total power of a correlator is odd. Including any matrix element of any odd power under the integral, will lead to an expression similar to (32). Note that is only one correlator with total power of one. Two examples were found, for a total power of two. And now for a total power of three, three possible correlators have been stated. The amount of correlators found for any given total power n, is the number of possible integer partitions of n. Next, consider a total power of four. As a first example, consider a single trace correlator:

$$\begin{aligned}\langle Tr(M^4)\rangle = \sum_{i,j,k,l=1}^{N}\langle M_{ij}M_{jk}M_{kl}M_{li}\rangle &= \frac{1}{Z_0}\sum_{i,j,k,l=1}^{N}\int_H dM M_{ij}M_{jk}M_{kl}M_{li}e^{-\frac{1}{2}Tr(M^2)} \\ &= \sum_{i,j,k,l=1}^{N}(\delta_{jj}\delta_{ik}\delta_{ki}\delta_{ll}+\delta_{kk}\delta_{jl}\delta_{lj}\delta_{ii}+\delta_{il}\delta_{kj}\delta_{ji}\delta_{lk}) = 2N^3+N\end{aligned} \tag{41}$$

The integral is only non-zero if $M_{ij}M_{jk}M_{kl}M_{li}$ is a product of two squared terms and there are three ways of pairing them up. Summing over all non-zero terms, leads us to the result of the trace, which can be described in a very handy way:

$$\sum_{i,j,l,k=1}^{N}\langle M_{ij}M_{jk}M_{kl}M_{li}\rangle = \sum_{i,j,l,k=1}^{N}(\langle M_{ij}M_{jk}\rangle\langle M_{kl}M_{li}\rangle + \langle M_{jk}M_{kl}\rangle\langle M_{ij}M_{li}\rangle + \langle M_{ij}M_{kl}\rangle\langle M_{jk}M_{li}\rangle) \tag{42}$$

The fact that we can pair up our correlator like this, is known as Wick's theorem and the number of pairings is given by the same combinatorical number we saw before: $(2i)!!$.

Continuing with correlators of total power 4 and moving to multi trace correlators. Again quadratic terms need to be formed in order for the integral not to be zero, and again a smooth way of doing that is using Wick's theorem.

$$\begin{aligned}\langle TrM^2TrM^2\rangle &= \sum_{i,j,k,l=1}^{N}\langle M_{ij}M_{ji}M_{kl}M_{lk}\rangle = \frac{1}{Z_0}\sum_{i,j,k,l=1}^{N}\int_H dM M_{ij}M_{ji}M_{kl}M_{lk}e^{-\frac{1}{2}Tr(M^2)} \\ &= \sum_{i,j,k,l=1}^{N}(\delta_{jj}\delta_{ii}\delta_{kk}\delta_{ll}+\delta_{kj}\delta_{il}\delta_{li}\delta_{jk}+\delta_{il}\delta_{kj}\delta_{jk}\delta_{li}) = N^4+2N^2\end{aligned} \tag{43}$$

Notice that the first term of this solution is just the square of $\langle Tr(M^2)\rangle$. This should not surprise us, as by Wick's theorem, we are splitting the correlator into all possible pairings of the matrix elements. One of these pairings is bound to give the product of the individual correlators. The second term of the solution is the so-called connected term, and will be denoted with a subscript c.

$$\langle Tr(M^2)Tr(M^2)\rangle = \langle Tr(M^2)\rangle^2 + \langle Tr(M^2)Tr(M^2)\rangle_c \tag{44}$$

We will see shortly why we call it connected, when the interpretation of our integrals in terms of graphs is being discussed.

$$\begin{aligned}\langle (TrM)^4\rangle &= \sum_{i,j,k,l=1}^{N}\langle M_{ii}M_{jj}M_{kk}M_{ll}\rangle = \frac{1}{Z_0}\sum_{i,j,k,l=1}^{N}\int_H dM M_{ii}M_{jj}M_{kk}M_{ll}e^{-\frac{1}{2}Tr(M^2)} \\ &= \sum_{i,j,k,l=1}^{N}(\delta_{ij}\delta_{ji}\delta_{kl}\delta_{lk}+\delta_{ki}\delta_{ik}\delta_{lj}\delta_{jl}+\delta_{il}\delta_{li}\delta_{jk}\delta_{kj}) = 3N^2\end{aligned} \tag{45}$$

$$\begin{aligned}\langle (TrM)^2Tr(M^2)\rangle &= \sum_{i,j,k,l=1}^{N}\langle M_{ij}M_{ji}M_{kk}M_{ll}\rangle = \frac{1}{Z_0}\sum_{i,j,k,l=1}^{N}\int_H dM M_{ij}M_{ji}M_{kk}M_{ll}e^{-\frac{1}{2}Tr(M^2)} \\ &= \sum_{i,j,k,l=1}^{N}(\delta_{jj}\delta_{ii}\delta_{kl}\delta_{lk}+\delta_{kj}\delta_{ik}\delta_{li}\delta_{jl}+\delta_{il}\delta_{lj}\delta_{jk}\delta_{ki}) = N^3+2N\end{aligned} \tag{46}$$

$$\begin{aligned}\langle TrMTrM^3\rangle &= \sum_{i,j,k,l=1}^{N}\langle M_{ii}M_{jk}M_{kl}M_{lj}\rangle = \frac{1}{Z_0}\sum_{i,j,k,l=1}^{N}\int_H dM M_{ii}M_{jk}M_{kl}M_{lj}e^{-\frac{1}{2}Tr(M^2)} \\ &= \sum_{i,j,k,l=1}^{N}(\delta_{ik}\delta_{ji}\delta_{kj}\delta_{ll}+\delta_{il}\delta_{ki}\delta_{jj}\delta_{lk}+\delta_{ij}\delta_{li}\delta_{jl}\delta_{kk}) = 3N^2\end{aligned} \tag{47}$$

This method of solving our correlators readily generalizes to any arbitrary, total, even power. For total power of 6, we find 11 possible integer partitions, therefore 11 correlators could be stated. And Wick's theorem already gives us 15 terms to consider in each of these correlators. The calculations are therefore not spelled out explicitly. Qualitatively, the calculations go along the same lines for any correlator. Going further in this thesis, our goal is to ascend from these long calculations and shortcut them by finding relations between the correlators. In the next section, we are going to consider an interpretation of our correlators in the form of graphs, which will allow us to calculate our correlators by drawing them. The methods used in this section, can be found for example in [8], [2].

1.5 Graph Theory

It has been mentioned shortly that the result of our integrals can be interpreted in form of graphs. Being able to draw these graphs is very useful as it replaces solving a cumbersome integral with counting symmetries and faces. Instructions on how to draw them can be found in [9], [2], [3].
Let's start with a few basic concepts of graph theory: When drawing a graph the most elementary objects we are using, are vertices, edges, and faces.

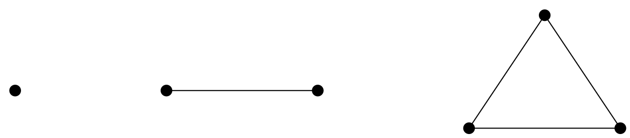

Figure 1: Vertex, Edge, Face

Vertices are points. Edges are lines that always start and end in a vertex. The valency of a vertex is the number of lines going in or out of it. The number of edges in a graph is therefore half of the total valency. The surface on which a graph is drawn, is considered a face. More faces can be found inside of loops. A loop forms when a series of edges, start and end in the same vertex. A graph without loops is called a tree.

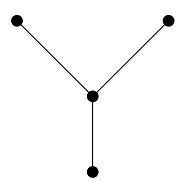

Figure 2: Example for a graph, a tree

This graph has 4 vertices, 3 of valency 1, 1 of valency 3. The edges can simply be counted, but also calculated from the total valency of 6, to be 3. And because the graph is a tree, there are no loops and therefore only one face. Another thing we mentioned above was the notion of general and connected correlators. The general correlator is composed out of connected correlators. This relates quite intuitively to graphs, two vertices are connected if there is an edge starting in one of the vertices and ending in the other. If there are no such edges between two vertices, these vertices are disconnected.

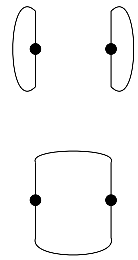

Figure 3: The two vertices on top are disconnected, the two below are connected

When drawing a graph in correspondence to a connected correlator, it needs to be connected. When drawing a general correlator, both connected and disconnected graphs need to be considered. How to draw a graph from a given correlator? The number of vertices is given by the number of traces. A single trace correlator corresponds to one vertex, a multi-trace correlator, with m traces, corresponds to m vertices. The valency of the vertices, corresponds to the power of the matrices. $\langle Tr(M^2)\rangle$ gives us a single vertex of valency two. Part of calculating a correlator is connecting the vertices to each other or themselves in all possible ways.

Figure 4: Drawing a graph from a single vertex with valency 2

One thing to notice is that the only legal way we can form a graph and therefore solve the integral is if the total power $k = \sum_i k_i$ of a correlator $\langle Tr(M^{k_1}) \ldots Tr(M^{k_m})\rangle$ is even. Otherwise, the total valency would be odd, which leads to a non integer number of edges, which makes no sense. This is of no surprise to us, as we have seen above that correlators of odd total power under the Gaussian integral will always evaluate to zero. The number of different ways we can draw a general correlator, is given by $n!!$, with n the total valency. This is the number of pairings given by Wick's theorem, which has been discussed in the previous section. Take as an example Figure 5, a vertex with valency 4. There are $4!! = 3$ ways of drawing the graph.

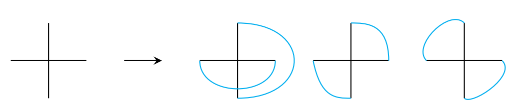

Figure 5: Drawing all possible graphs with a single vertex of valency 4

Note that for multi-trace correlators, this includes connected graphs as well as disconnected ones. The number of different ways we can draw a connected graph is simply $n!!$ minus all the possible ways the disconnected graphs can be drawn. For example, consider the number of ways $\langle Tr(M^2)Tr(M^4)\rangle_c$

can be drawn:

First, find the number of ways the general correlator can be drawn, which in this case is $6!! = 15$.

Then consider the individual traces, $\langle Tr(M^2)\rangle$ and $\langle Tr(M^4)\rangle$. Which can be drawn in $4!! = 3$ and $2!! = 1$ different ways respectively.

Therefore, the number of ways to draw the connected correlator is $6!! - 2!!3!! = 12$.

Everything we have discussed this far also holds for the special 1×1 matrix case. A graph with a given set of vertices (number of traces), with given valencies (respective powers) is drawn by including 1/2 valency edges. What we have been ignoring so far is the whole matrix structure of our correlators. Terms like $\langle Tr(M^4)\rangle = \sum_{i,j,k,l=1}^{N} \langle M_{ji}M_{ik}M_{kl}M_{lj}\rangle$ tell us more than just the number of vertices and edges. A set of indices is given, that is dependent on order, as the hermitian matrices are not symmetric and M_{ij} describes a different element than M_{ji}. This is how we incorporate them into our graphs: Our thin graphs get expanded to so-called "ribbon" or "fat" graphs. The two boundaries of these new fat valencies are labelled going clockwise around the vertex by the indices of our matrices determined by the trace, see figure 6. Because the order of the indices matters, we also arbitrarily assign the first index an outgoing vector and the second index an in going vector.

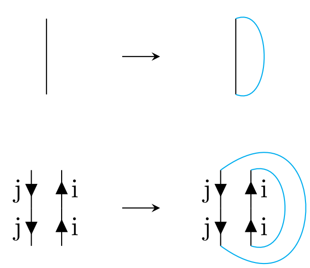

Figure 6: Comparison between a one dimensional correlator $\langle x^2\rangle$ and $\langle Tr(M^2)\rangle = \sum_{i,j}\langle M_{ji}M_{ij}\rangle$

When connecting the edges, orientation needs to be respected. And this is how we connect them: Consider the example:

$$\langle TrM^4\rangle = \sum_{1,j,k,l=1}^{N} \langle M_{ij}M_{jk}M_{kl}M_{li}\rangle = 2N^2 + N$$

First draw the vertex with its valency. Every valency, sticking out of the vertex, corresponds to a matrix element. Going clockwise around the vertex, the first index of a matrix element labels the first boundary of the valency, the second index labels the second boundary:

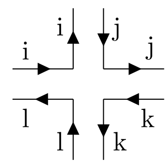

Figure 7: Single vertex of valency 4, of a fat graph, corresponds to $M_{ij}M_{jk}M_{kl}M_{li}$

From above, we know that there are 3 possible graphs. Respecting the chosen orientation, this is how the edges are formed:

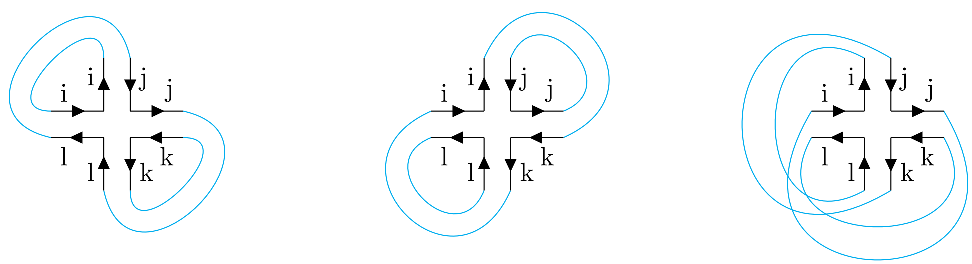

Figure 8: The graphs drawn from $\langle TrM^4 \rangle$

Note that the upper graphs in figure 8 are different to the lower one, in the sense that the edges of the upper one can be drawn without them intersecting each other. The same does not hold for the lower graph. This difference manifests itself in the result of the correlator. The upper two graphs correspond to the two terms, N^3 and the lower one to N. How can this be seen? Remember how faces were defined, by loops, a series of edges that start and end in the same vertex. Working with fat graphs, the boundary of an edge is considered to form a loop, if it starts and ends at the same index. Tracing the boundaries of the edges in figure 8 we find that the upper two graphs, have three faces and the lower one only one. Which agrees with the powers of N, found in our calculations. Compare it to the calculations (41) done in section 1.4. Remember Wick's theorem was used to pair up the matrices:

$$\langle TrM^4 \rangle = \sum_{i,j,k,l=1}^{N} \langle M_{ij}M_{jk}M_{kl}M_{li} \rangle$$

$$= \sum_{i,j,k,l=1}^{N} \left(\langle M_{ij}M_{jk} \rangle \langle M_{kl}M_{li} \rangle + \langle M_{ij}M_{kl} \rangle \langle M_{jk}M_{ik} \rangle + \langle M_{ij}M_{li} \rangle \langle M_{jk}M_{li} \rangle \right)$$

$$= \sum_{i,j,k,l=1}^{N} \left(\delta_{ik}\delta_{jj}\delta_{ll}\delta_{ki} + \delta_{ii}\delta_{jl}\delta_{kk}\delta_{lj} + \delta_{il}\delta_{lk}\delta_{kj}\delta_{ji} \right) = 2N^3 + N$$

Each of the three terms corresponds to one of the graphs. The pairings of matrix elements corresponds to the drawing of edges between two valencies. The edges could therefore be labelled like such:

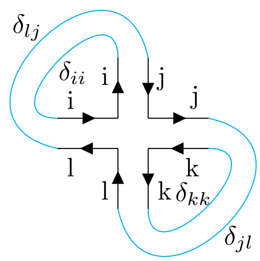

Figure 9: Labelling the edges by the corresponding result $\delta_{ii}\delta_{kk}\delta_{jl}\delta_{lj}$

Summing over the indices in every loop in figure 9 leads to the result N^3. In quantum field theory, we might call these labelled edges our propagators.

How to understand the graph in figure 7 that only had one face? It has been stated in the beginning of the section that the surface on which a graph is drawn is also considered a face. The trick is to draw the graph on a surface where the lines do not intersect. This surface cannot be a plane. The graph needs to be drawn on a different surface, in this case a surface of genus one, a torus. Consider this drawing of a torus, which is common in topology.

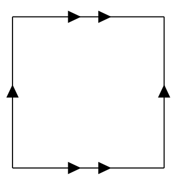

Figure 10: Glue the opposite edges to find a torus

The opposing edges of the square are identified with one another. This process of identifying edges is also referred to as the gluing of edges. Drawing the bottom graph in figure 8 on the torus instead, we find:

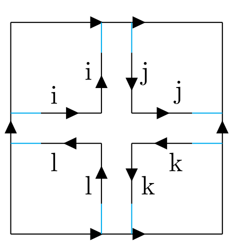

Figure 11: Glue the opposite edges to find a torus

Here we find no additional loops, the only face is the surface of the torus. This interpretation of our graphs living on different surfaces comes in very handy, and we are going to elaborate on it in the next section about the Euler Characteristic.

The methods described in this section mainly by the example of $\langle TrM^4 \rangle$ readily generalize to any other correlator. The recipe goes as follows:

Take a general correlator $\langle TrM^{k_1}...TrM^{k_m} \rangle$. Draw its m vertices with their respective valencies k_i and connect them in all $(\sum_{i=1}^{m} k_i)!!$ possible ways. Then expand your thin graph to a ribbon graph, label the valencies of every vertex, with neighbouring valencies ending with the same index the other one starts with. Then count the loops formed by the boundaries of the fat edges, this gives the number of faces, the power of N.

1.6 Euler Characteristic

We have seen that we can interpret the power of N in our correlator as the number of faces in our graphs. We have also learned that the number of vertices and edges is given by our correlator. Knowing all of these things, suggests having a look at the Euler Characteristic.

$$F + V - E = 2 - 2g \tag{48}$$

With V vertices, E edges, F faces and, what we have not discussed so far, the genus g. All of them positive integers. The genus enumerates the holes of a space. For us this space, is where we draw our graph without it intersecting itself. A planar graph can be drawn on a surface of genus 0, aka a flat plane. A graph with genus 1 can be drawn on a surface with genus 1 without intersecting itself, for example on the torus. Knowing that the power of N enumerates the faces, we write:

$$N^F = N^{2-2g-V+E} \tag{49}$$

We can see the higher g is, the smaller the power of N. There is also a maximum value for g, because every graph needs to have at least one face, the power of N needs to be at least 1.

$$g_{max} < \frac{2 - V + E}{2} \tag{50}$$

Having a minimum and maximum value for g, suggests that we express our correlators in terms of its genus. Ignoring the integer coefficients that count how many possible graphs there are of a given genus, we can approximate a single correlator as an expansion in its genus:

$$\langle TrM^{2k} \rangle = N^{2-1+k} \sum_{g=0}^{g<\frac{1+k}{2}} N^{-2g} c_{2k}^{(g)} \tag{51}$$

with $c_{2k}^{(g)}$ the combinatorical coefficient, enumerating the graphs with 1 vertex of valency $2k$ and genus g. Convince yourself by comparing to results from above that this is true. For example:

$$\langle TrM^2\rangle = N^2 c_2^{(0)} \tag{52}$$

From previous calculations we know $c_2^{(0)} = 1$. Another example:

$$\langle TrM^4\rangle = N^3 c_4^{(0)} + N c_4^{(1)} \tag{53}$$

Again we know from explicit calculations: $c_4^{(0)} = 2$ and $c_4^{(1)} = 1$.

Similarly, for connected multi trace correlators:

$$\langle TrM^{k_1}...TrM^{k_n}\rangle_c = N^{-n+\frac{1}{2}\sum_{i=1}^n k_i} \sum_{g=0}^{g<\frac{1}{2}(2-n+\frac{1}{2}\sum_{i=1}^n k_i)} N^{2-2g} c_{k_1...k_n}^{(g)} \tag{54}$$

with $c_{k_1...k_n}^{(g)}$ a generalisation of what we have seen before, the combinatorical coefficient enumerating the graphs with n vertices, respective valencies $k_1, ..., k_n$ and genus g. An easy example is:

$$\langle TrM^2 TrM^2\rangle_c = N^2 c_{2,2}^{(0)} \tag{55}$$

General correlators as we have seen are just combinations of connected correlators. Writing out their general genus expansion is quite messy and does not give us any new information. As an example, consider the general two trace correlators:

$$\begin{aligned} \langle TrM^k TrM^j\rangle &= \langle TrM^k TrM^j\rangle_c + \langle TrM^k\rangle\langle TrM^j\rangle \\ = N^{-2+(k+j)/2} \sum_{g=0}^{g<(k+j)/4} N^{2-2g} c_{k,j}^{(g)} &+ N^{2+j+k} \sum_{g_j=0}^{g_j<\frac{1+j}{2}} \sum_{g_k=0}^{g_k<\frac{1+k}{2}} N^{-2(g_k+g_j)} c_{2k}^{(g_k)} c_{2j}^{(g_j)} \end{aligned} \tag{56}$$

Stop for a moment and appreciate, how remarkable the findings of this section are. Thanks to the Euler characteristic, we can simply look at a correlator and know its structure up to combinatorical factors. Finding handy equations for these factors on the other hand is going to take a lot more effort. But do not fret, the path there is littered with interesting mathematics. The expansion of the correlators in terms of their genus, that we found in this section, is discussed for example in [8], [2].

1.7 Generating Function and Free Energy

A neat way to summarize all the correlators is in the form of a generating function.

$$Z = \frac{1}{Z_0}\int dM e^{-\frac{1}{2}Tr(M^2)+\sum_k^\infty t_k Tr(M^k)} \tag{57}$$

or equivalently, in its eigenvalue description:

$$Z = \frac{1}{Z_0}\int \prod_{i=1}^{N} d\lambda_i \prod_{i\neq j}^{N}(\lambda_i - \lambda_j) e^{-\frac{1}{2}Tr(M^2)+\sum_k^\infty t_k Tr(M^k)} \tag{58}$$

The time variables $t_1, t_2, t_3, \ldots$ we have introduced, are dummy variables. The correlators can be retrieved by taking the appropriate derivatives and setting all $t_i = 0$

$$\langle Tr(M^{k_1})\ldots Tr(M^{k_m})\rangle = \frac{\partial^m}{\partial t_{k_1}\ldots\partial t_{k_m}} Z\bigg|_{t=0} \tag{59}$$

Note the normalization by $1/Z_0$ that has also been used in section 1.4. With Z_0 the integral defining our model. By encoding all possible correlators of our model, this generating function holds all the information about our model. Naturally, it is of interest to find different ways of expressing and expanding it. Generating functions by definition are a way of encoding an infinite sequence of numbers. Using the expansion of the exponential function:

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \tag{60}$$

the expansion of our generating function in terms of the correlators is found to be:

$$\begin{aligned} Z = \frac{1}{Z_0}\int dM e^{-\frac{1}{2}Tr(M^2)} + \frac{1}{Z_0}\int dM e^{-\frac{1}{2}Tr(M^2)}\sum_{k}^{\infty} t_k Tr(M^k) + \\ + \frac{1}{Z_0}\frac{1}{2}\int dM e^{-\frac{1}{2}Tr(M^2)}\sum_{k,i}^{\infty} t_k t_i Tr(M^k)Tr(M^i) + ... \end{aligned} \tag{61}$$

Which is simply:

$$Z = 1 + \sum_{i=1}^{\infty}\langle Tr(M^i)\rangle t_i + \frac{1}{2!}\sum_{i,j=1}^{\infty}\langle Tr(M^i)Tr(M^j)\rangle t_i t_j + \frac{1}{3!}\sum_{i,j,k=1}^{\infty}\langle Tr(M^i)Tr(M^j)Tr(M^k)\rangle t_i t_j t_k + \ldots \tag{62}$$

This already looks quite neat, so neat in fact that another way of expressing it can immediately be seen. From statistical physics the free energy is known, as the logarithm of the generating function.

$$\ln(Z) = F \tag{63}$$

Let's elaborate on how exactly this can be seen. In section 1.4 we saw that multi-trace correlators can be fully expressed in terms of connected correlators.

$$\langle Tr(M^{k_1})\dots Tr(M^{k_m})\rangle = \langle Tr(M^{k_1})\dots Tr(M^{k_m})\rangle_c + \dots + \langle Tr(M^{k_1})\rangle \dots \langle Tr(M^{k_m})\rangle \tag{64}$$

Take for example $\langle Tr(M^i)Tr(M^j)Tr(M^k)\rangle$:

$$\begin{aligned} \langle Tr(M^i)Tr(M^j)Tr(M^k)\rangle = \langle Tr(M^i)Tr(M^j)Tr(M^k)\rangle_c + \langle Tr(M^i)Tr(M^j)\rangle_c\langle Tr(M^k)\rangle + \\ + \langle Tr(M^k)Tr(M^j)\rangle_c\langle Tr(M^i)\rangle + \langle Tr(M^i)Tr(M^k)\rangle_c\langle Tr(M^j)\rangle + \langle Tr(M^i)\rangle\langle Tr(M^j)\rangle\langle Tr(M^k)\rangle \end{aligned} \tag{65}$$

Plugging this form into (62):

$$\begin{aligned} Z = 1 + \sum_{i=1}^{\infty}\langle Tr(M^i)\rangle t_i + \frac{1}{2!}\sum_{i,j=1}^{\infty}\langle Tr(M^i)\rangle\langle Tr(M^k)\rangle t_i t_k + \frac{1}{3!}\sum_{i,j,k=1}^{\infty}\langle Tr(M^i)\rangle\langle Tr(M^j)\rangle\langle Tr(M^k)\rangle t_i t_j t_k + \dots \\ + \frac{1}{2!}\sum_{i,j=1}^{\infty}\langle Tr(M^i)Tr(M^j)\rangle_c t_i t_j + \frac{4!}{4!2!2!}\sum_{i,j,k,l=1}^{\infty}\langle Tr(M^i)Tr(M^j)\rangle_c\langle Tr(M^k)Tr(M^l)\rangle_c t_i t_j t_k t_l + \dots \\ + \frac{3!}{3!2!}\sum_{i,k,l=1}^{\infty}\langle Tr(M^k)Tr(M^l)\rangle_c\langle Tr(M^i)\rangle t_i t_k t_l + \\ + \frac{1}{3!}\sum_{i,j,k=1}^{\infty}\langle Tr(M^i)Tr(M^j)Tr(M^k)\rangle_c t_i t_j t_k + \dots \end{aligned} \tag{66}$$

One might wonder about the combinatorical factors that appear in this expression. The terms $1/n!$, where n is the total number of traces in a term, are given by the expansion (62). The additional binomial coefficient

$$\begin{pmatrix} n \\ k_1 \dots k_m \end{pmatrix} = \frac{n!}{k_1!\dots k_m!} \qquad \sum_{i=1}^{m} k_i = n \tag{67}$$

appears because all terms are symmetric in their indices. Or, said differently, (67) is the number of possibilities to split an n-trace correlator into m connected correlators. We see that Z is of the form of an expanded exponential function, depending on connected correlators. The first, second and last line are each an expansion of an exponential function. The third line is a mixed term, let's not forget about those!

$$Z = \exp\Big(\sum_{i=1}^{\infty} t_i\langle Tr(M^i)\rangle + \frac{1}{2!}\sum_{i,j=1}^{\infty} t_i t_j\langle Tr(M^i)Tr(M^j)\rangle_c + \frac{1}{3!}\sum_{i,j,k=1}^{\infty} t_i t_j t_k\langle Tr(M^i)Tr(M^j)Tr(M^k)\rangle_c + \dots\Big) \tag{68}$$

And with $Z = e^F$, we finally arrive at our expression for the free energy:

$$F = 1 + t_i\langle Tr(M^i)\rangle + \frac{1}{2!}t_i t_j\langle Tr(M^i)Tr(M^j)\rangle_c + \frac{1}{3!}t_i t_j t_k\langle Tr(M^i)Tr(M^j)Tr(M^k)\rangle_c + \dots \tag{69}$$

Comparing with (62), we see that the expansions look almost the same. Z is an expansion of the general correlators, F an expansion of the connected correlators. Working with the free energy can be very handy, as the connected correlators already hold all the information. Remember that the general correlators can be expressed in terms of only the connected ones.

1.8 Large N

The size of our matrices so far has been arbitrary but finite. Now we are interested in what happens when we let our matrices become infinitely large, $N \to \infty$. In order to handle the infinities, a new parameter g^2 is introduced in our generating function, which has the property:

$$Ng^2 = \lambda \tag{70}$$

with λ some finite constant. This is the so called t'Hooft coupling. It was introduced in [10] by t'Hooft. Do not confuse the parameter g^2 with g the genus or λ with the eigenvalue λ_i, even though they are denoted by the same letters. If it is not explicitly written, it should be clear from context, which one we are referring to. This is how g^2 is implemented:

$$Z = \frac{1}{Z_0} \int \prod_i^N d\lambda_i \prod_{i \neq j}^N (\lambda_i - \lambda_j) e^{-\frac{1}{2g^2} \sum_i^N \lambda_i^2 + \sum_k^\infty t_k \sum_i^N \lambda_i^k} \tag{71}$$

This is the generating function, found in the previous section 1.7, in its eigenvalue description, where an additional g^2 as been introduced in the exponent. How does this help? Consider this bit of trickery: We absorb g into the eigenvalues λ_i. This is a simple substitution and will leave our integral invariant. Additional factors of g, coming from the Jacobian and the measure due to this substitution, also appear in Z_0 and therefore fall victim to the normalization.

$$Z = \frac{1}{Z_0} \int \prod_i^N d\lambda_i \prod_{i \neq j}^N (\lambda_i - \lambda_j) e^{-\frac{1}{2} \sum_i^N \lambda_i^2 + \sum_k^\infty t_k \sum_i^N g^k \lambda_i^k} \tag{72}$$

An example:

$$\begin{aligned} \langle Tr(M^4) \rangle &= \frac{1}{Z_0} \frac{\partial}{\partial t_4} \int \prod_i^N d\lambda_i \prod_{i \neq j}^N (\lambda_i - \lambda_j) e^{-\frac{1}{2} \sum_i^N \lambda_i^2 + \sum_k^\infty t_k \sum_i^N g^k \lambda_i^k} \bigg|_{t=0} \\ &= g^4 \frac{1}{Z_0} \int \prod_i^N d\lambda_i \prod_{i \neq j}^N (\lambda_i - \lambda_j) \sum_i^N \lambda_i^4 e^{-\frac{1}{2} \sum_i^N \lambda_i^2} \end{aligned} \tag{73}$$

In general:

$$\begin{aligned}\langle Tr(M^{k_1})...Tr(M^{k_m})\rangle &= \frac{1}{Z_0}\frac{\partial^m}{\partial t_{k_1}...\partial t_{k_m}}\int\prod_i^N d\lambda_i\prod_{i\neq j}^N(\lambda_i-\lambda_j)e^{-\frac{1}{2}\sum_i^N\lambda_i^2+\sum_k^\infty t_k\sum_i^N g^k\lambda_i^k}\Big|_{t=0}\\ &= g^{\sum_i k_i}\frac{1}{Z_0}\int\prod_i^N d\lambda_i\prod_{i\neq j}^N(\lambda_i-\lambda_j)Tr(M^{k_1})...Tr(M^{k_m})e^{-\frac{1}{2}\sum_i^N\lambda_i^2}\end{aligned} \tag{74}$$

We see that these new correlators for large N are the same ones we have discussed before, multiplied by g to an even power, the total power of the matrices in the correlator.

$$\langle Tr(M^{k_1})...Tr(M^{k_m})\rangle \to g^{\sum_i k_i}\langle Tr(M^{k_1})...Tr(M^{k_m})\rangle \tag{75}$$

This leads to a general N- dependence of this form:

$$\langle TrM^{k_1}...TrM^{k_n}\rangle_c = \lambda^{\frac{1}{2}\sum_{i=1}^n k_i}N^{2-n}\sum_{g=0}^{g<\frac{1}{2}(2-n+\frac{1}{2}\sum_{i=1}^n k_i)}N^{-2g}c^{(g)}_{k_1...k_n} \tag{76}$$

We have used the general N dependence (54) that we discussed in section 1.6 together with (75) and (70). Note that the combinatorical coefficients $c^{(g)}_{k_1...k_n}$ are the same as before. Looking closely, we see that the only connected correlator that still has a positive power of N is the single trace correlator at genus zero. It has a power of one. It is the only term keeping us from a finite expression. Recall now the expansion of Z:

$$Z = 1+\sum_{k=1}^\infty t_k\langle TrM^k\rangle+\frac{1}{2}\sum_{k,j=1}^\infty t_jt_k\langle TrM^jTrM^k\rangle+\frac{1}{3!}\sum_{k,j,l=1}^\infty t_jt_kt_l\langle TrM^jTrM^kTrM^l\rangle+\dots \tag{77}$$

Every m-trace correlator, when written in terms of connected correlators, has at most one term of m single trace correlators and therefore at most a positive power of m. We get rid of these positive powers by the following change of variables:

$$\tilde{t}_k = Nt_k \tag{78}$$

With it Z has no more positive powers of N.

$$Z = 1+\frac{1}{N}\tilde{t}_k\langle TrM^k\rangle+\frac{1}{N^2}\frac{1}{2}\tilde{t}_j\tilde{t}_k\langle TrM^jTrM^k\rangle+\frac{1}{N^3}\frac{1}{3!}\tilde{t}_j\tilde{t}_k\tilde{t}_l\langle TrM^jTrM^kTrM^l\rangle+\dots \tag{79}$$

2 Virasoro Constraints

Another thing that usually accompanies a physical model is some sort of symmetry between the correlators. In quantum field theory, these symmetries are sometimes called Ward Identities. In classical systems, this might be comparable to the Noether Theorem. In our case, symmetries are described by the so called Virasoro Constraints, which are given in the following way:

$$0 = \int \prod_{i}^{N} d\lambda_i \sum_{l=1}^{N} \frac{\partial}{\partial \lambda_l} \Big(\lambda_l^{n+1} \prod_{i \neq j}^{N} (\lambda_i - \lambda_j) e^{-\frac{1}{2}\sum_i^N \lambda_i^2 + \sum_k^\infty t_k \sum_i^N \lambda_i^k} \Big) \tag{80}$$

Note that there are three cases for this equation, $n = -1$, $n = 0$ and $n > 0$. The origin of the Virasoro Constraints is discussed in [11], [12]. For a discussion on how they relate to symmetries consider [13]. Again for better understanding, consider first the 1-dim case:

$$\begin{aligned} 0 &= \sqrt{\frac{1}{2\pi}} \int dx \frac{\partial}{\partial x} \Big(x^{n+1} e^{-\frac{x^2}{2} + \sum_i t_i x^i} \Big) \\ \int dx (n+1) x^n e^{-\frac{x^2}{2} + \sum_i t_i x^i} &= \int dx (x^{n+2} - \sum_i t_i i x^{i+n}) e^{-\frac{x^2}{2} + \sum_i t_i x^i} \\ (n+1)\langle x^n \rangle &= \langle x^{n+2} \rangle \\ (n+1)(n)!! &= (n+2)!! \end{aligned} \tag{81}$$

The individual results on the left and right-hand side are what we have found before in the section on correlators (31). Going from line two to three, we used that the correlators are defined at $t = 0$. Further note that the third line gives us a very neat relation.

$$(n+1)\langle x^n \rangle = \langle x^{n+2} \rangle \tag{82}$$

It is a recursion, that gives us all one dimensional correlators, once we know the $n = 0$ case. Now compare (81) to (80). The main difference is that the Jacobian, the Vandermonde determinate, gets added. Other than this, (80) is just an N-dimensional version of (81). Upon evaluating (80) we hope to find another recurrence relation, similar to (82). And by evaluating we mean, first off, taking the derivative. Following the chain rule, there are three derivatives we have to take, two of which are rather trivial.

$$\sum_{l=1}^{N} \frac{\partial}{\partial \lambda_l} \lambda_l^{n+1} = \sum_{l=1}^{N} (n+1) \lambda^n \tag{83}$$

Keep in mind the special cases. For $n = 0$, this derivative evaluates to $\sum_{l=1}^{N} 1 = N$ and for $n = -1$ to 0.

$$\lambda_l^{n+1} \frac{\partial}{\partial \lambda_l} e^{-\frac{1}{2}\sum_i^N \lambda_i^2 + \sum_k^\infty t_k \sum_i^N \lambda_i^k} = (-\sum_{l}^{N} \lambda_l^{n+2} + \sum_{i}^{N} l \lambda_i^{l+n}) e^{-\frac{1}{2}\sum_i^N \lambda_i^2 + \sum_k^\infty t_k \sum_i^N \lambda_i^k} \tag{84}$$

The special cases are self-explanatory, for $n=0$

$(-\sum_l^N \lambda_l^2 + \sum_i^N l\lambda_i^l)e^{-\frac{1}{2}\sum_i^N \lambda_i^2 + \sum_k^\infty t_k \sum_i^N \lambda_i^k}$

for $n=-1$

$(-\sum_l^N \lambda_l^1 + \sum_i^N l\lambda_i^{l-1})e^{-\frac{1}{2}\sum_i^N \lambda_i^2 + \sum_k^\infty t_k \sum_i^N \lambda_i^k}$

At last, one derivative is left.

$$\sum_l \lambda_l^{n+1} \frac{\partial}{\partial \lambda_l} \prod_{i\neq j}(\lambda_i - \lambda_j) \tag{85}$$

For this one, we need to be a bit clever. The following identity is helpful:

$$a^{n+1} - b^{n+1} = (a-b)(a^n + b^n + ab^{n-1} + a^{n-1}b + \dots) \tag{86}$$

And with it, we get:

$$\begin{aligned}\sum_l \lambda_l^{n+1} \frac{\partial}{\partial \lambda_l} \prod_{i\neq j}(\lambda_i - \lambda_j) &= \sum_l \lambda_l^{n+1} \frac{\partial}{\partial \lambda_l} \exp(\ln(\prod_{i\neq j}(\lambda_i - \lambda_j))) = \prod_{i\neq j}(\lambda_i - \lambda_j) \sum_{l\neq j} \frac{\lambda_l^{n+1} - \lambda_j^{n+1}}{\lambda_l - \lambda_j} \\ &= \prod_{i\neq j}(\lambda_i - \lambda_j)(\sum_{l\neq j}\sum_{s=0}^{n} \lambda_l^{n-s}\lambda_j^s) = \prod_{i\neq j}(\lambda_i - \lambda_j)(\sum_{l,j}\sum_{s=1}^{n-1} \lambda_l^{n-s}\lambda_j^s - (n+1-2N)\sum_l^N \lambda_l^n)\end{aligned} \tag{87}$$

Note that in the last step we changed the range of our sums. These calculations look slightly different for $n=0$ and $n=-1$. For $n=-1$ it is simply 0 and for $n=0$ it reduces to $\prod_{i\neq j}(\lambda_i - \lambda_j)(N^2 - N)$. Putting these results together:

for $n>0$:

$$0 = \int \prod_i^N d\lambda_i (\sum_{l,j}\sum_{s=1}^{n-1} \lambda_l^{n-s}\lambda_j^s + 2N\sum_l^N \lambda_l^n - \sum_l^N \lambda_l^{n+2} + \sum_i^N l\lambda_i^{l+n}) \prod_{i\neq j}^N (\lambda_i - \lambda_j) e^{-\frac{1}{2}\sum_i^N \lambda_i^2 + \sum_k^\infty t_k \sum_i^N \lambda_i^k} \tag{88}$$

for $n=0$:

$$0 = \int \prod_i^N d\lambda_i (N^2 \sum_l^N \lambda_l^n - \sum_l^N \lambda_l^2 + \sum_i^N l\lambda_i^l) \prod_{i\neq j}^N (\lambda_i - \lambda_j) e^{-\frac{1}{2}\sum_i^N \lambda_i^2 + \sum_k^\infty t_k \sum_i^N \lambda_i^k} \tag{89}$$

for $n=-1$:

$$0 = \int \prod_i^N d\lambda_i (-\sum_l^N \lambda_l + \sum_i^N l\lambda_i^{l-1}) \prod_{i\neq j}^N (\lambda_i - \lambda_j) e^{-\frac{1}{2}\sum_i^N \lambda_i^2 + \sum_k^\infty t_k \sum_i^N \lambda_i^k} \tag{90}$$

Now note that we can write:

$$\sum_{l=1}^N \lambda^n e^{-\frac{1}{2}\sum_i^N \lambda_i^2 + \sum_k^\infty t_k \sum_i^N \lambda_i^k} = \frac{\partial}{\partial t_n} e^{-\frac{1}{2}\sum_i^N \lambda_i^2 + \sum_k^\infty t_k \sum_i^N \lambda_i^k} \tag{91}$$

With this we rewrite our result from above and arrive at our final expression for the Virasoro Constraints:

for $n > 0$

$$0 = (2N\frac{\partial}{\partial t_n} - \frac{\partial}{\partial t_{n+2}} + \sum_{k=1}^{\infty} kt_k \frac{\partial}{\partial t_{n+k}} + \sum_{s=1}^{n-1} \frac{\partial}{\partial t_{n-s}} \frac{\partial}{\partial t_s})Z \tag{92}$$

for $n = 0$

$$0 = (N^2 + \sum_{k=1}^{\infty} t_k k \frac{\partial}{\partial t_k} - \frac{\partial}{\partial t_2})Z \tag{93}$$

for $n = -1$

$$0 = (Nt_1 + \sum_{k=2}^{\infty} t_k k \frac{\partial}{\partial t_{k-1}} - \frac{\partial}{\partial t_1})Z \tag{94}$$

As promised, we can find a cute recurrence relation for our correlators from these relations.

2.1 Correlator Relations

Remember the expansion of Z in terms of its correlators:

$$Z = 1 + \sum_{k=1}^{\infty} t_k \langle TrM^k \rangle + \frac{1}{2} \sum_{j,k=1}^{\infty} t_j t_k \langle TrM^j TrM^k \rangle + \frac{1}{3!} \sum_{k,j,l=1}^{\infty} t_j t_k t_l \langle TrM^j TrM^k TrM^l \rangle + \dots \tag{95}$$

Putting this in the Virasoro constraints, we find :

$n > 0$

$$\begin{aligned} 0 = 2N\langle TrM^n \rangle - \langle TrM^{n+2} \rangle + \sum_{s=1}^{n-1} \langle TrM^s TrM^{n-s} \rangle + \sum_{k=1}^{\infty} kt_k \langle TrM^{k+n} \rangle \\ +2N \sum_{k=1}^{\infty} t_k \langle TrM^n TrM^k \rangle - \sum_{k=1}^{\infty} t_k \langle TrM^{n+2} TrM^k \rangle + \dots \end{aligned} \tag{96}$$

$n = 0$

$$0 = N^2 - \langle TrM^2 \rangle + N^2 \sum_{k=1}^{\infty} t_k \langle TrM^k \rangle + \sum_{k=1}^{\infty} t_k k \langle TrM^k \rangle - \sum_{k=1}^{\infty} t_k \langle TrM^k TrM^2 \rangle + \dots \tag{97}$$

$n = -1$

$$0 = -\langle TrM \rangle + Nt_1 + \sum_{k=2}^{\infty} t_k k \langle TrM^{k-1} \rangle - \sum_{k=1}^{\infty} t_k \langle TrM^k TrM \rangle + Nt_1 \sum_{k=1}^{\infty} t_k \langle TrM^k \rangle + \dots \tag{98}$$

The few terms that have been calculated here, are already suggestively ordered by their power of t_i. Indeed, because our equations need to be true independently of t_i, we can solve this equation for every power of t_i individually. Solve for example for the zeroth power of t_i:

$$\langle TrM^{n+2} \rangle = 2N\langle TrM^n \rangle + \sum_{s=1}^{n-1} \langle TrM^s TrM^{n-s} \rangle \tag{99}$$

Convince yourself that this is true, by checking some of our previous results, like:

$$\langle TrM^4 \rangle = 2N\langle TrM^2 \rangle + \langle TrMTrM \rangle = 2N^3 + N \tag{100}$$

This is a very practical relation! Recall how much effort went into calculating the correlators in the last section, now once we know the initial values $\langle TrM \rangle$ and $\langle TrM^2 \rangle$, we can calculate any single trace correlator with this neat recurrence. The initial values we luckily find when evaluating the other two Virasoro constraints, for $n = 0$ and $n = -1$. The same method of solving for every power of t separately also applies here. We are still considering the zeroth power components.

For $n = 0$ we find:

$$\langle TrM^2 \rangle = N^2 \tag{101}$$

For $n = -1$:

$$\langle TrM \rangle = 0 \tag{102}$$

Both results have been seen already multiple times within this thesis.

Now have a look at the first power of t:

$$\langle TrM^{n+2}TrM^k \rangle = 2N\langle TrM^n TrM^k \rangle + k\langle TrM^{k+n} \rangle + \sum_{s=1}^{n-1} \langle TrM^{n-s}TrM^s TrM^k \rangle \tag{103}$$

This already looks a little more complicated, but the takeaway stays the same, we now have one recurrence relation for all two-trace correlators, only dependent on initial values and in this case also the single trace correlators, which we already know due to the recurrence relation found above. Again, consider the $n = 0$ and $n = -1$ constraints for the initial values:

For $n = 0$, we find:

$$\langle TrM^2 TrM^k \rangle = N^2 \langle TrM^k \rangle + k\langle TrM^k \rangle \tag{104}$$

For $n = -1$:

$$\langle TrM^1 TrM^k \rangle = k\langle TrM^{k-1} \rangle + N(k = 1) \tag{105}$$

These two expressions have lovely interpretations in terms of graphs. Let's indulge for a moment. First the $n = 0$ case. Here we see that if we draw a graph with a vertex of valency 2 and a second vertex of any even valency, we can either form two disconnected graphs or connect them to each other. In the first case, we use that we already know how TrM^2 evaluates, namely to N^2. The other part of the graph does whatever it does, described by $\langle TrM^k \rangle$. This is what we find described in the first term. In the second term we see that, if we connect the two vertices, the vertex with valency two cannot have any edges connecting to itself, it basically becomes part of one of an edge of the other graph.

Consider figure 12.

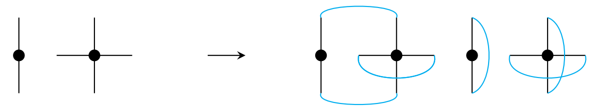

Figure 12: Schematic Visualization of the $n = 0$ Constraint

And of course we could do this to any edge in $\langle TrM^k \rangle$ (twice actually due to symmetry) therefore the factor k in the second term.

Now, for $n = -1$: A correlator only containing odd powers of M, is necessarily a connected correlator. The individual traces, representing vertices, cannot form graphs on their own. A power one trace especially cannot connect to itself at all, all it can do is connect to one of the valencies of another vertex, therefore reducing its valency by one. Especially in the case of a two-trace correlator, as we have seen in the example just now, the valency one vertex has k options to connect to the second vertex and leaves behind a single trace correlator of valency $k - 1$.

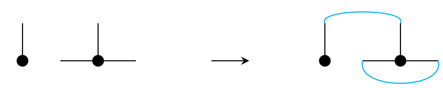

Figure 13: Schematic Visualization of the $n = -1$ Constraint

Enough fun with graphs. The finial thing we want to do in this section is find a general recurrence relation for our correlators. The calculations for any m-trace correlator are along the same lines as what we have seen for single and two trace correlators. We plug the expansion of Z in the Virasoro constraints, and we collect all the terms with power $m - 1$ of t. The principle is exactly the same, but the equations are a bit fiddly, you have to be careful when doing them yourself, because we are not going to spell them out, symmetries of indices are lurking everywhere and the combinatorical factors $1/n!$ are your friend, do not forget about them. If you made it through all the nifty coefficients that cancel out so smoothly, this is what you will find:

for $n > 0$

$$\begin{aligned} \langle TrM^{n+2} TrM^{k_1} \dots TrM^{k_m} \rangle &= 2N \langle TrM^n TrM^{k_1} \dots TrM^{k_m} \rangle \\ + \sum_{j=1}^{m} k_j \langle TrM^{k_1} \dots TrM^{k_j+n} \dots TrM^{k_m} \rangle &+ \sum_{s=1}^{n-1} \langle TrM^{n-s} TrM^s TrM^{k_1} \dots TrM^{k_m} \rangle \end{aligned} \tag{106}$$

Note that the number of indices in every term agrees, as they should, as the number of indices represents the power of t for which we solved. Within a trace, the permutation of indices is of course symmetric. Lastly, consider the equations $n = 0$ and $n = -1$ giving our initial values:

$n = 0$

$$\langle TrM^2 TrM^{k_1}...TrM^{k_m}\rangle = N^2 \langle TrM^{k_1}...TrM^{k_m}\rangle + \sum_{j=1}^{m} k_j \langle TrM^{k_1}...TrM^{k_m}\rangle \quad (107)$$

$n = -1$

$$\begin{aligned}\langle TrM\ TrM^{k_1} \dots TrM^{k_m}\rangle = \sum_{j=1}^{m} k_j \langle TrM^{k_1} \dots TrM^{k_j - 1} \dots TrM^{k_m}\rangle \\ + N \sum_{j=1}^{m} \langle TrM^{k_1} \dots TrM^{k_j} \dots TrM^{k_m}\rangle (k_j = 1)\end{aligned} \quad (108)$$

Now stop and appreciate what we just found. With these relations, we can calculate any m-trace correlator. No more pesky integral, no more graphs, biting our nails, hoping to have considered every possible way of drawing them.

Talking about graphs; the last two relations that were stated have the same interpretation in terms of graphs as the ones we discussed above, when we were merely considering two-trace correlators. This discussion was centred around the fact that one of the traces had either valency one ($n = -1$) or valency two ($n = 0$). The other part of the graph, before a second vertex of some valency k, is more or less a black box, in the sense that we do not care what it does exactly, it could just as well be multiple vertices with total valency k. And this is exactly what we see in the more general case for m-trace correlators.

All this points towards there also being a slick interpretation for the relation 106, and in a way there is, but it is a bit more complicated.

2.2 Connected Correlator Relations

The same procedure as above can be done for connected correlators if we use the free energy instead of the generating function, this alters the Virasoro constraints a bit, but the results are very similar. Because the calculations are analogous to the ones we just did, we are not going to go into too much detail, but simply remark on the differences. We start with reformulating our Virasoro constraints for the free energy. For that we simply take the Virasoro constraints we calculated in the beginning of this section, put in $Z = \exp(F)$ and find:

for $n > 0$

$$0 = (2N\frac{\partial}{\partial t_n} - \frac{\partial}{\partial t_{n+2}} + \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_{n+k}} + \sum_{s=1}^{n-1} \frac{\partial}{\partial t_{n-s}}\frac{\partial}{\partial t_s})F + \sum_{s=1}^{n-1} \frac{\partial}{\partial t_{n-s}} F \frac{\partial}{\partial t_s} F \quad (109)$$

for $n = 0$

$$0 = N^2 + (\sum_{k=1}^{\infty} t_k k \frac{\partial}{\partial t_k} - \frac{\partial}{\partial t_2})F \quad (110)$$

for $n = -1$

$$0 = Nt_1 + (\sum_{k=2}^{\infty} t_k k \frac{\partial}{\partial t_{k-1}} - \frac{\partial}{\partial t_1})F \tag{111}$$

The differences to what we had before are not exactly striking. We gain a non-linear term in the $n > 0$ constraint and in the other two, a term each is independent of F. Now remember the expansion of F in terms of connected correlators:

$$F = 1 + t_i \langle Tr(M^i) \rangle + \frac{1}{2!} t_i t_j \langle Tr(M^i) Tr(M^j) \rangle_c + \frac{1}{3!} t_i t_j t_k \langle Tr(M^i) Tr(M^j) Tr(M^k) \rangle_c + \dots \tag{112}$$

Same as before, we put it into the Virasoro constraints and solve for different powers of t. For the zeroth power of t we find:

for $n > 0$

$$\langle TrM^{n+2} \rangle_c = 2N \langle TrM^n \rangle_c + \sum_{s=1}^{n-1} \langle TrM^{n-s} TrM^s \rangle_c + \sum_{s=1}^{n-1} \langle TrM^{n-s} \rangle_c \langle TrM^s \rangle_c \tag{113}$$

Compare this to what we had before (99). Because single correlators are necessarily connected correlators, these two expressions should be equal. And indeed using the definition of general correlators in terms of connected ones it is easy to see that they are. It figures that their initial values are also the same: for $n = 0$

$$\langle TrM^2 \rangle_c = N^2 \tag{114}$$

for $n = -1$

$$\langle TrM \rangle_c = 0 \tag{115}$$

One more example before we turn to the general case; let's like before solve for t power one:

$$\begin{aligned} \langle TrM^{n+2} TrM^k \rangle_c &= 2N \langle TrM^n TrM^k \rangle_c + k \langle TrM^{n+k} \rangle_c \\ + \sum_{s=1}^{n-1} \langle TrM^{n-s} TrM^s TrM^k \rangle_c &+ 2 \sum_{s=1}^{n-1} \langle TrM^s \rangle_c \langle TrM^{n-s} TrM^k \rangle_c \end{aligned} \tag{116}$$

$n = 0$

$$\langle TrM^2 TrM^k \rangle_c = k \langle TrM^k \rangle_c \tag{117}$$

$n = -1$

$$\langle TrM TrM^k \rangle_c = N(k = 1) + k \langle TrM^{k-1} \rangle_c \tag{118}$$

Compare these to (103)- (105). The first two are clearly different, we cannot simply compose the connected ones into a general one. The last one is coincidentally the same because $\langle TrM TrM^k \rangle =$

$\langle TrMTrM^k \rangle_c$.

Now for the general case; same as before, methods are not changing, but again, for the general case, we have to be very careful considering all the symmetries of the equations.

Here they are: for $n > 0$:

$$\langle TrM^{n+2}TrM^{k_1}\dots TrM^{k_m}\rangle_c = 2N\langle TrM^nTrM^{k_1}\dots TrM^{k_m}\rangle_c + \sum_{j=1}^{m} k_j\langle TrM^{k_1}\dots TrM^{k_j+n}\dots TrM^{k_m}\rangle_c$$
$$+\sum_{s=1}^{n-1}\langle TrM^{n-s}TrM^sTrM^{k_1}\dots TrM^{k_m}\rangle_c + \sum_{\substack{I_1\cap I_2=k_1,\dots,k_m\\ I_1\cup I_2=\emptyset}}\sum_{s=1}^{n-1}\langle TrM^sTrM^{I_1}\rangle_c\langle TrM^{n-s}TrM^{I_2}\rangle_c \tag{119}$$

In the last term in this expression, we use a slightly different notation. This is because between these two correlators we can permute the indices. This is not a symmetry. These permutations must be counted. I_1 and I_2 are sets of indices. $\langle Tr^{I_1}\rangle$ stands for a $|I_1|$-trace correlator. The sum over these sets in (119) restricts them to contain within their intersection the same indices as the other terms in the equation.

Now for $n = 0$ (and $m \neq 0$):

$$\langle TrM^2TrM^{k_1}\dots TrM^{k_m}\rangle_c = \sum_{j=1}^{m} k_j\langle TrM^{k_1}\dots TrM^{k_m}\rangle_c \tag{120}$$

for $n = -1$ (and $m \neq 1$):

$$\langle TrMTrM^{k_1}\dots TrM^{k_m}\rangle_c = \sum_{j=1}^{m} k_j\langle TrM^{k_1}\dots TrM^{k_j-1}\dots TrM^{k_m}\rangle_c \tag{121}$$

The little restrictions ($m \neq 0$) and ($m \neq 1$) come from the terms independent of F, they incidentally only show up in the explicit cases we discussed above.

Now we know all hermitian matrix correlators, even all connected correlators, are we done now? The thesis still seems to continue. That is right, we can analyse our correlators even more, using very similar methods as in this section! Correlators were constructed from connected correlators, and connected correlators in turn can be opened up and described genus by genus. We could for example be interested in describing the coefficients enumerating planar graphs. For that purpose we are moving to the large N limit where we solve basically do the same equations but for every genus separately. Being able to calculate the coefficients individually will come in handy later in section 4, when we will use our developed methods to calculate abstract correlators we a priori do not know the form of.

3 Virasoro for Large-N

We have seen that the Virasoro Constraints introduced above can be used to find recurrence relations describing the correlators. We have also seen in section 1 that our correlators can be expanded by their genus and, especially in the large N case, that our correlators depend only on negative (or zero) powers of N. This perturbative description of the correlators leads with the help of the Virasoro Constraints to a very detailed description of our correlators, down to the combinatorical coefficients enumerating the graphs we have seen before.

We start out with the large N generating function.

$$Z = \int \prod_i^N d\lambda_i \prod_{i \neq j}^N (\lambda_i - \lambda_j) e^{-\frac{1}{2g^2} \sum_i^N \lambda_i^2 + \sum_k^\infty t_k \sum_i^N \lambda_i^k} \tag{122}$$

Note that we will no longer explicitly denote normalization. This is common practice. Because we are only interested in derivates of t, and because Z_0 is by definition independent of t, there is nothing to gain by including Z_0 but clutter. Going forward, we simply assume all of our equations to be normalized.

Recall that we calculated the Virasoro Constraints in dependence on the generating function.

$$0 = \int \prod_i^N d\lambda_i \sum_l \frac{\partial}{\partial \lambda_l} \lambda_l^{n+1} \prod_{i \neq j}^N (\lambda_i - \lambda_j) e^{-\frac{1}{2g^2} \sum_i^N \lambda_i^2 + \sum_k^\infty t_k \sum_i^N \lambda_i^k} \tag{123}$$

Now that we have a slightly different generating function, we will also have slightly different Virasoro constraints. The first part of the derivation is basically equivalent to what we have seen before. We therefore simply state this intermediate result:

for $n > 0$

$$0 = (2N \frac{\partial}{\partial t_n} + \sum_{k=1}^\infty t_k k \frac{\partial}{\partial t_{k+n}} + \sum_{s=1}^{n-1} \frac{\partial^2}{\partial \tilde{t}_{n-s} \partial t_s} - \frac{1}{g^2} \frac{\partial}{\partial t_{n+2}}) Z \tag{124}$$

for $n = 0$

$$0 = (N^2 + \sum_{k=1}^\infty t_k k \frac{\partial}{\partial t_k} - \frac{1}{g^2} \frac{\partial}{\partial t_2}) Z(t) \tag{125}$$

for $n = -1$

$$0 = (N t_1 + \sum_{k=2}^\infty t_k k \frac{\partial}{\partial t_{k-1}} - \frac{1}{g^2} \frac{\partial}{\partial t_1}) Z(t) \tag{126}$$

Now the only thing left to do is to make use of the two relations we found in section 1.9 that made a large N description even possible: the t'Hooft coupling and the large N transformation of the times t:

$$g^2 N = \lambda \qquad \tilde{t} = N t \tag{127}$$

Putting this in the intermediate step stated above and dividing by N^2 leads to our final result:

for $n > 0$

$$0 = (2\frac{\partial}{\partial\tilde{t}_n} + \frac{1}{N^2}\sum_{k=1}^{\infty}\tilde{t}_k k\frac{\partial}{\partial\tilde{t}_{k+n}} + \sum_{s=1}^{n-1}\frac{\partial^2}{\partial\tilde{t}_{n-s}\partial\tilde{t}_s} - \frac{1}{\lambda}\frac{\partial}{\partial\tilde{t}_{n+2}})Z \tag{128}$$

for $n = 0$

$$0 = (1 + \frac{1}{N^2}\sum_{k=1}^{\infty}\tilde{t}_k k\frac{\partial}{\partial\tilde{t}_k} - \frac{1}{\lambda}\frac{\partial}{\partial\tilde{t}_2})Z(t) \tag{129}$$

for $n = -1$

$$0 = (\frac{1}{N^2}\tilde{t}_1 + \frac{1}{N^2}\sum_{k=2}^{\infty}\tilde{t}_k k\frac{\partial}{\partial\tilde{t}_{k-1}} - \frac{1}{\lambda}\frac{\partial}{\partial\tilde{t}_1})Z(t) \tag{130}$$

Even though we already promised a final result it is worth doing a little extra work to retrieve the constraints in terms of the free energy. Going forward, it is going to be easier to work with the free energy, and we will shortly see why. The relation between the generating function and the free energy is as we discussed in section 1.7 $Z = e^F$. Putting this in our "final result":

$n > 0$

$$0 = 2\frac{\partial}{\partial\tilde{t}_n}F + \frac{1}{N^2}\sum_{k=1}^{\infty}\tilde{t}_k k\frac{\partial}{\partial\tilde{t}_{k+n}}F + \sum_{s=1}^{n-1}\frac{\partial^2}{\partial\tilde{t}_{n-s}\partial\tilde{t}_s}F - \frac{1}{\lambda}\frac{\partial}{\partial\tilde{t}_{n+2}}F + \sum_{s=1}^{n-1}\frac{\partial}{\partial\tilde{t}_{n-s}}F\frac{\partial}{\partial\tilde{t}_s}F \tag{131}$$

for $n = 0$

$$0 = 1 + (\frac{1}{N^2}\sum_{k=1}^{\infty}\tilde{t}_k k\frac{\partial}{\partial\tilde{t}_k} - \frac{1}{\lambda}\frac{\partial}{\partial\tilde{t}_2})F \tag{132}$$

for $n = -1$

$$0 = \frac{1}{N^2}\tilde{t}_1 + (\frac{1}{N^2}\sum_{k=2}^{\infty}\tilde{t}_k k\frac{\partial}{\partial\tilde{t}_{k-1}} - \frac{1}{\lambda}\frac{\partial}{\partial\tilde{t}_1})F \tag{133}$$

3.1 Large-N Expansion

In order to find a solution to these equations, we are going to need two pieces of information that have been discussed above. First, remember the expansion of the free energy in terms of connected correlators:

$$\begin{aligned} F &= 1 + \frac{1}{N}\tilde{t}_i\langle Tr(M^i)\rangle + \frac{1}{2!}\frac{1}{N^2}\tilde{t}_i\tilde{t}_j\langle Tr(M^i)Tr(M^j)\rangle_c + \dots \\ &= \sum_{n=0}^{\infty}\frac{N^{-n}}{n!}\sum_{k_1,..k_n=1}^{\infty}\langle Tr(M^{k_1})...Tr(M^{k_n})\rangle_c\prod_{i=1}^{n}\tilde{t}_{k_i} \end{aligned} \tag{134}$$

If the range of the first sum seems troubling, the correlator for $n = 0$, meaning no trace, is simply one, and so is every other term in the expression.

Secondly, remember the general form of the connected correlators:

$$\langle TrM^{k_1}...TrM^{k_n}\rangle_c = N^{-n+2}\lambda^{\sum_i k_i/2}\sum_{g=0}^{g<\frac{1}{2}(2-n+\frac{1}{2}\sum_{i=1}^{n}k_i)}N^{-2g}c^{(g)}_{k_1,\dots,k_n} \tag{135}$$

Recall that we were not able to write the general disconnected correlators in such a nice way. Trying to give their general N dependence was quite cumbersome. This is the reason we prefer to work with the free energy, because their dependence on N is so neat and tidy, and it allows us to do the following; We put our description of the correlators into the expansion of F and find the following expression:

$$F = \sum_{n,g=0}^{\infty} \frac{N^{2-2n-2g}}{n!} \sum_{k_1,\ldots,k_n=1}^{\infty} \lambda^{\sum_i k_i/2} c^{(g)}_{k_1,\ldots,k_n} \prod_{i=1}^{n} \tilde{t}_{k_i} \tag{136}$$

This is what we might call the large N expansion of our free energy, as the highest power of N is 0. Because we are working in the large N limit, this is the leading order. The lower the powers of N in (136), the faster the corresponding terms go to zero. This allows us to finally calculate these pesky coefficients $c^{(g)}_{k_1,\ldots,k_n}$, using the Virasoro constraints. This is how we do it: We order (136) by its power of N. Keeping in mind that in (136) we have seen that the power of N is always even, therefore k always needs to be even.

$$F = \sum_{\substack{k=0 \\ k\ even}}^{\infty} \frac{F_k}{N^k} \tag{137}$$

with:

$$F_k = \sum_{\substack{n,g=0 \\ k=2n+2g-2}} \frac{1}{n!} \sum_{k_1,\ldots,k_n=1}^{\infty} \lambda^{\sum_i k_i/2} c^{(g)}_{k_1,\ldots,k_n} \prod_{i=1}^{n} \tilde{t}_{k_i} \tag{138}$$

Note that as $N \to \infty$, terms with negative powers of N go to zero, the lower the power the faster. (137) describes therefore a perturbation theory, with F_0 as the leading term. Why is it desirable to write F in this fashion? Because this way, we can solve the Virasoro constraints for all powers of N individually. We can see this best by putting (137) into the Virasoro constraints for F:

$n > 0$

$$\begin{aligned} 0 = 2\frac{\partial}{\partial \tilde{t}_n}(F_0 + \frac{F_2}{N^2} + \frac{F_4}{N^4} + \ldots) + \frac{1}{N^2}\sum_{k=1}^{\infty} \tilde{t}_k k \frac{\partial}{\partial \tilde{t}_{k+n}}(F_0 + \frac{F_2}{N^2} + \frac{F_4}{N^4} + \ldots) + \\ \sum_{s=1}^{n-1} \frac{\partial^2}{\partial \tilde{t}_{n-s}\partial \tilde{t}_s}(F_0 + \frac{F_2}{N^2} + \frac{F_4}{N^4} + \ldots) - \frac{1}{\lambda}\frac{\partial}{\partial \tilde{t}_{n+2}}(F_0 + \frac{F_2}{N^2} + \frac{F_4}{N^4} + \ldots) \\ + \sum_{s=1}^{n-1} \frac{\partial}{\partial \tilde{t}_{n-s}}(F_0 + \frac{F_2}{N^2} + \frac{F_4}{N^4} + \ldots)\frac{\partial}{\partial \tilde{t}_s}(F_0 + \frac{F_2}{N^2} + \frac{F_4}{N^4} + \ldots) \end{aligned} \tag{139}$$

for $n = 0$

$$0 = 1 + (\frac{1}{N^2}\sum_{k=1}^{\infty} \tilde{t}_k k \frac{\partial}{\partial \tilde{t}_k} - \frac{1}{\lambda}\frac{\partial}{\partial \tilde{t}_2})(F_0 + \frac{F_2}{N^2} + \frac{F_4}{N^4} + \ldots) \tag{140}$$

for $n = -1$

$$0 = \frac{1}{N^2}\tilde{t}_1 + (\frac{1}{N^2}\sum_{k=2}^{\infty} \tilde{t}_k k \frac{\partial}{\partial \tilde{t}_{k-1}} - \frac{1}{\lambda}\frac{\partial}{\partial \tilde{t}_1})(F_0 + \frac{F_2}{N^2} + \frac{F_4}{N^4} + \ldots) \tag{141}$$

This might seem like a long and frightening equation, but as $N \to \infty$ parts of the equation go to zero at different speeds. We use this to solve this equation order by order. For example, solve it for the leading order N^0:

$n > 0$

$$0 = 2\frac{\partial}{\partial \tilde{t}_n} F_0 + \sum_{s=1}^{n-1} \frac{\partial^2}{\partial \tilde{t}_{n-s} \partial \tilde{t}_s} F_0 - \frac{1}{\lambda} \frac{\partial}{\partial \tilde{t}_{n+2}} F_0 + \sum_{s=1}^{n-1} \frac{\partial}{\partial \tilde{t}_{n-s}} F_0 \frac{\partial}{\partial \tilde{t}_s} F_0 \tag{142}$$

for $n = 0$

$$0 = 1 - \frac{1}{\lambda} \frac{\partial}{\partial \tilde{t}_2} F_0 \tag{143}$$

for $n = -1$

$$0 = \frac{1}{\lambda} \frac{\partial}{\partial \tilde{t}_1} F_0 \tag{144}$$

Next we could collect all the terms with N^{-2} and solve for them:

$n > 0$

$$0 = 2\frac{\partial}{\partial \tilde{t}_n} \frac{F_2}{N^2} + \frac{1}{N^2} \sum_{k=1}^{\infty} \tilde{t}_k k \frac{\partial}{\partial \tilde{t}_{k+n}} F_0 + \sum_{s=1}^{n-1} \frac{\partial^2}{\partial \tilde{t}_{n-s} \partial \tilde{t}_s} \frac{F_2}{N^2} - \frac{1}{\lambda} \frac{\partial}{\partial \tilde{t}_{n+2}} \frac{F_2}{N^2} + 2 \sum_{s=1}^{n-1} \frac{\partial}{\partial \tilde{t}_{n-s}} \frac{F_2}{N^2} \frac{\partial}{\partial \tilde{t}_s} F_0 \tag{145}$$

for $n = 0$

$$0 = \frac{1}{N^2} \sum_{k=1}^{\infty} \tilde{t}_k k \frac{\partial}{\partial \tilde{t}_k} F_0 - \frac{1}{\lambda} \frac{\partial}{\partial \tilde{t}_2} \frac{F_2}{N^2} \tag{146}$$

for $n = -1$

$$0 = \frac{1}{N^2} \tilde{t}_1 + \frac{1}{N^2} \sum_{k=1}^{\infty} \tilde{t}_k k \frac{\partial}{\partial \tilde{t}_{k-1}} F_0 - \frac{1}{\lambda} \frac{\partial}{\partial \tilde{t}_1} \frac{F_2}{N^2} \tag{147}$$

And it goes on like that. The pattern is not hard to spot. The general case for any order F_k goes right along these lines:

for $n > 0$

$$0 = (2\frac{\partial}{\partial \tilde{t}_n} + \sum_{s=1}^{n-1} \frac{\partial^2}{\partial \tilde{t}_{n-s} \partial \tilde{t}_s} - \frac{1}{\lambda} \frac{\partial}{\partial \tilde{t}_{n+2}}) F_k + \sum_{i=1}^{\infty} \tilde{t}_i i \frac{\partial}{\partial \tilde{t}_{i+n}} F_{k-2} + \sum_{l+r=k} \sum_{s=1}^{n-1} \frac{\partial}{\partial \tilde{t}_{n-s}} F_r \frac{\partial}{\partial \tilde{t}_s} F_l \tag{148}$$

for $n = 0$

$$0 = 1(k = 0) + \sum_{i=1}^{\infty} \tilde{t}_i i \frac{\partial}{\partial \tilde{t}_i} F_{k-2} - \frac{1}{\lambda} \frac{\partial}{\partial \tilde{t}_2} F_k \tag{149}$$

for $n = -1$

$$0 = (k = 2)\tilde{t}_1 + \sum_{k=2}^{\infty} \tilde{t}_k k \frac{\partial}{\partial \tilde{t}_{k-1}} F_{k-2} - \frac{1}{\lambda} \frac{\partial}{\partial \tilde{t}_1} F_k \tag{150}$$

Finally we have arrived at this last set of equations (148)-(150) which we set out to build and solve in this section. Luckily, we have already found a solution with (138) and (136). All that is left to do is plug them in and retrieve our prize, the long awaited relations for the coefficients $c^{(g)}_{k_1,\ldots,k_n}$.

3.2 Correlator Coefficients

3.2.1 F_0

Let's start with the easiest case, the leading equation of the Virasoro constraints, which we already stated above: $n > 0$

$$0 = (2\frac{\partial}{\partial \tilde{t}_n} + \sum_{s=1}^{n-1} \frac{\partial^2}{\partial \tilde{t}_{n-s} \partial \tilde{t}_s} - \frac{1}{\lambda}\frac{\partial}{\partial \tilde{t}_{n+2}})F_0 + \sum_{s=1}^{n-1} \frac{\partial}{\partial \tilde{t}_{n-s}} F_0 \frac{\partial}{\partial \tilde{t}_s} F_0 \tag{151}$$

for $n = 0$

$$0 = 1 - \frac{1}{\lambda}\frac{\partial}{\partial \tilde{t}_2} F_0 \tag{152}$$

for $n = -1$

$$0 = \frac{1}{\lambda}\frac{\partial}{\partial \tilde{t}_1} F_0 \tag{153}$$

Our description of F_0 is hiding in here:

$$F = \sum_{n,g=0}^{\infty} \frac{N^{2-2n-2g}}{n!} \sum_{k_1,\ldots,k_n=1}^{\infty} \lambda^{\sum_i k_i/2} c^{(g)}_{k_1,\ldots,k_n} \prod_{i=1}^{n} \tilde{t}_{k_i} \tag{154}$$

Remember, that F_0 is defined to be the collection of terms in F for which the power of N is zero . We see that this can only happen if $n = 1$ and $g = 0$. (Technically also if $n = 0$, but as discussed above, this simply gives one, which in turn evaluates trivially to zero under the Virasoro Constraint.)

$$F_0 = \sum_i \tilde{t}_i \lambda^{i/2} c^{(0)}_i \tag{155}$$

And now putting this into the N^0 Virasoro constraints:

$n > 0$

$$0 = (2\frac{\partial}{\partial \tilde{t}_n} + \sum_{s=1}^{n-1} \frac{\partial^2}{\partial \tilde{t}_{n-s} \partial \tilde{t}_s} - \frac{1}{\lambda}\frac{\partial}{\partial \tilde{t}_{n+2}}) \sum_i \tilde{t}_i \lambda^{i/2} c^{(0)}_i + \sum_{s=1}^{n-1} \frac{\partial}{\partial \tilde{t}_{n-s}} \sum_i \tilde{t}_i \lambda^{i/2} c^{(0)}_i \frac{\partial}{\partial \tilde{t}_s} \sum_i \tilde{t}_i \lambda^{i/2} c^{(0)}_i \tag{156}$$

$$c^{(0)}_{n+2} = 2c^{(0)}_n + \sum_{s=1}^{n-1} c^{(0)}_{n-s} c^{(0)}_s \tag{157}$$

This is the first recurrence relation for correlator coefficients that we have found in this thesis! And it is even a proper famous relation, it describes the so-called Catalan numbers. The first few values of the sequence are $1, 2, 5, 14, 42, 132, \ldots$. The Catalan numbers c_n describe multiple combinatorical problems. Whole books have been written about them, like [14]. They can also be found on OEIS

[15], entry number A000108. They also have a closed description:

$$c_n = \frac{2n!}{(n+1)!n!} \tag{158}$$

In order to find this closed description, or to even be able to call (157) a recurrence relation, we actually also need initial values. Luckily, they are given by the next two equations:
for $n = 0$

$$0 = 1 - \frac{1}{\lambda}\frac{\partial}{\partial \tilde{t}_2}\sum_i \tilde{t}_i \lambda^{i/2} c_i^{(0)} \tag{159}$$

$$1 = c_2^{(0)} \tag{160}$$

We have encountered this specific coefficient a couple of times already in multiple situations. It is for example the number of ways we can draw a graph with one vertex of valency 2.
for $n = -1$

$$0 = \frac{1}{\lambda}\frac{\partial}{\partial \tilde{t}_1}\sum_i \tilde{t}_i \lambda^{i/2} c_i^{(0)} \tag{161}$$

$$0 = c_1^{(0)} \tag{162}$$

This initial value tells us that all coefficients with odd index are going to be zero (because the recurrence relation has a step size of two). This is also in understanding with what we have found many times in this thesis, that we cannot have odd powers of M in single trace correlators. In case it has already been forgotten due to an ever-growing amount of notation, the lower indices of our coefficients $c^{(g)}_{k_1 \ldots k_m}$ correspond to the powers of M our matrices under the individual traces $\langle TrM^{k_1}...TrM^{k_m}\rangle$. Because the ordering of the traces inside the correlator does not matter, our coefficients are also invariant under permutation of their indices.

3.2.2 F_2

Now repeat the calculations for F_2. We first need to find an expression for it. Consult (154) and collect all the terms in F containing N^{-2} . There are two such terms, one at $n = 2$, $g = 0$ the other at $n = 1$, $g = 1$. These two terms form F_2:

$$F_2 = \sum_{i=1}^{\infty} \tilde{t}_i \lambda^{i/2} c_i^{(1)} + \frac{1}{2}\sum_{i,j=1}^{\infty} \tilde{t}_i \tilde{t}_j c_{i,j}^{(0)} \lambda^{(i+j)/2} \tag{163}$$

Now put it in the Virasoro Constraints we found for N^{-2}:

$n > 0$

$$0 = (2\frac{\partial}{\partial \tilde{t}_n} + \sum_{s=1}^{n-1} \frac{\partial^2}{\partial \tilde{t}_{n-s}\partial \tilde{t}_s} - \frac{1}{\lambda}\frac{\partial}{\partial \tilde{t}_{n+2}})(\sum_{i=1}^{\infty} \tilde{t}_i \lambda^{i/2} c_i^{(1)} + \frac{1}{2}\sum_{i,j=1}^{\infty} \tilde{t}_i \tilde{t}_j c_{i,j}^{(0)} \lambda^{(i+j)/2})+$$
$$+2\sum_{s=1}^{n-1} \frac{\partial}{\partial \tilde{t}_{n-s}}(\sum_{i=1}^{\infty} \tilde{t}_i \lambda^{i/2} c_i^{(1)} + \frac{1}{2}\sum_{i,j=1}^{\infty} \tilde{t}_i \tilde{t}_j c_{i,j}^{(0)} \lambda^{(i+j)/2})\frac{\partial}{\partial \tilde{t}_s}(\sum_i \tilde{t}_i \lambda^{i/2} c_i^{(0)}) + \sum_{k=1}^{\infty} \tilde{t}_k k \frac{\partial}{\partial \tilde{t}_{k+n}}(\sum_i \tilde{t}_i \lambda^{i/2} c_i^{(0)}) \tag{164}$$

for $n = 0$

$$0 = \sum_{k=1}^{\infty} \tilde{t}_k k \frac{\partial}{\partial \tilde{t}_k}(\sum_i \tilde{t}_i \lambda^{i/2} c_i^{(0)}) - \frac{1}{\lambda}\frac{\partial}{\partial \tilde{t}_2}(\sum_{i=1}^{\infty} \tilde{t}_i \lambda^{i/2} c_i^{(1)} + \frac{1}{2}\sum_{i,j=1}^{\infty} \tilde{t}_i \tilde{t}_j c_{i,j}^{(0)} \lambda^{(i+j)/2}) \tag{165}$$

for $n = -1$

$$0 = \tilde{t}_1 + \sum_{k=1}^{\infty} \tilde{t}_k k \frac{\partial}{\partial \tilde{t}_{k-1}}(\sum_i \tilde{t}_i \lambda^{i/2} c_i^{(0)}) - \frac{1}{\lambda}\frac{\partial}{\partial \tilde{t}_1}(\sum_{i=1}^{\infty} \tilde{t}_i \lambda^{i/2} c_i^{(1)} + \frac{1}{2}\sum_{i,j=1}^{\infty} \tilde{t}_i \tilde{t}_j c_{i,j}^{(0)} \lambda^{(i+j)/2}) \tag{166}$$

These expressions are already getting pretty long and as we go to higher orders this is only going to get worse. A good thing to notice early on if we want to reduce clutter is that we can and actually have to separate our equations in terms of powers of t. This is because we want our equations to be true independently of t. This has already been discussed in this thesis, at the end of section 2, where we calculated a recurrence relation for correlators. In order to proceed; evaluate equations (164)-(166). This mainly entails taking simple derivatives, which we will not spell out. Then you separate your result in terms of powers of t. For t to the power of one, you'll find:

$$c_{i,n+2}^{(0)} = 2c_{i,n}^{(0)} + 2\sum_{s=1}^{n-1} c_{s/2}^{(0)} c_{i,n-s}^{(0)} + ic_{(n+i)/2}^{(0)} \tag{167}$$

Plus the initial values:

$$c_{2,i}^{(0)} = ic_i^{(0)} \quad c_{1,i}^{(0)} = ic_{i-1}^{(0)} + 1(i=1) \tag{168}$$

This recurrence relation gives us the leading order, the genus 0 coefficients of a two trace correlator. The initial conditions are given in terms of the Catalan numbers, which immediately agrees with what we have seen many times in this thesis, that the total power in a correlator needs to be even.

Here are the first few coefficients, no longer simply represented by a sequence but in this matrix:

$$c_{ij}^{(0)} = \begin{pmatrix} 1 & 0 & 3 & 0 & \dots \\ 0 & 2 & 0 & 8 & \\ 3 & 0 & 12 & 0 & \\ 0 & 8 & 0 & 36 & \\ \vdots & & & & \ddots \end{pmatrix}$$

Solving the equations (164)-(166) for the zeroth power of t you'll find the following relation:

$$c_{n+2}^{(1)} = 2c_n^{(1)} + 2\sum_{s=1}^{n-1} c_{n-s}^{(1)} c_{s/2}^{(0)} + \sum_{s=1}^{n-1} c_{n-s,s}^{(0)}$$

Plus its initial conditions:

$$c_2^{(1)} = 0 \qquad c_1^{(1)} = 0 \tag{169}$$

This recurrence relation gives us the genus 1 coefficient of a single trace correlator, in large N, these are the coefficients of the first correction. Here are the first few of them: $c_i^{(1)} \in \{0, 1, 10, 70, 420, ...\}$. This sequence also has combinatorical significance and can be found in [16].

Note that the first non-zero term of this recursion is $c_4^{(1)}$. Which agrees with what we have seen in examples of single trace correlators so far. $\langle TrM^2\rangle = \lambda N$ had no correction terms, whereas $\langle TrM^4\rangle$ already had one. Further note that the relation is in terms of the Catalan numbers as well as the coefficients $c_{ij}^{(0)}$, which we just found when solving for t to the power of one. This is already telling of the fact that we cannot solve the Virasoro constraints in search for a specific coefficient alone, we have to solve for all coefficients appearing in the same and in the lower orders, otherwise we cannot solve the recurrence relations.

3.2.3 F_k

Let's finally calculate at the general case F_k. Starting out, it might be nice to have another look at the general description of F:

$$F = \sum_{n,g=0}^{\infty} \frac{N^{2-2n-2g}}{n!} \sum_{k_1,\dots,k_n=1}^{\infty} \lambda^{\sum_i k_i/2} c_{k_1,\dots,k_n}^{(g)} \prod_{i=1}^{n} \tilde{t}_{k_i} \tag{170}$$

Above, we have seen we get F_k from F by fixing the power of N to be $-k$. The power of N in turn is given in terms of n and g by the relation $k = 2n + 2g - 2$. We therefore simply restrict the sum over all n and g, to this relation, and there we go, we have got F_k.

$$F_k = \sum_{\substack{n=1,g=0\\k=2n+2g-2}} \frac{1}{n!} \sum_{k_1,\ldots,k_n=1}^{\infty} \lambda^{\sum_i k_i/2} c^{(g)}_{k_1,\ldots,k_n} \prod_{i=1}^{n} \tilde{t}_{k_i} \tag{171}$$

Put it into the Virasoro constraints:

for $n > 0$

$$\begin{aligned}
0 = (2\frac{\partial}{\partial\tilde{t}_n} + \sum_{s=1}^{n-1} \frac{\partial^2}{\partial\tilde{t}_{n-s}\partial\tilde{t}_s} - \frac{1}{\lambda}\frac{\partial}{\partial\tilde{t}_{n+2}}) \sum_{\substack{n=1,g=0\\k=2n+2g-2}} \frac{1}{n!} \sum_{k_1,\ldots,k_n=1}^{\infty} \lambda^{\sum_i k_i/2} c^{(g)}_{k_1,\ldots,k_n} \prod_{i=1}^{n} \tilde{t}_{k_i} \\
+ \sum_{i=1}^{\infty} \tilde{t}_i i \frac{\partial}{\partial\tilde{t}_{i+n}} \sum_{\substack{n=1,g=0\\k=2n+2g}} \frac{1}{n!} \sum_{k_1,\ldots,k_n=1}^{\infty} \lambda^{\sum_i k_i/2} c^{(g)}_{k_1,\ldots,k_n} \prod_{i=1}^{n} \tilde{t}_{k_i} + \\
+ \sum_{l+r=k} \sum_{s=1}^{n-1} \frac{\partial}{\partial\tilde{t}_{n-s}} (\sum_{\substack{n=1,g=0\\r=2n+2g-2}} \frac{1}{n!} \sum_{k_1,\ldots,k_n=1}^{\infty} \lambda^{\sum_i k_i/2} c^{(g)}_{k_1,\ldots,k_n} \prod_{i=1}^{n} \tilde{t}_{k_i}) \\
\frac{\partial}{\partial\tilde{t}_s} (\sum_{\substack{n=1,g=0\\l=2n+2g-2}} \frac{1}{n!} \sum_{k_1,\ldots,k_n=1}^{\infty} \lambda^{\sum_i k_i/2} c^{(g)}_{k_1,\ldots,k_n} \prod_{i=1}^{n} \tilde{t}_{k_i})
\end{aligned} \tag{172}$$

for $n = 0$

$$\begin{aligned}
\sum_{i=1}^{\infty} \tilde{t}_i i \frac{\partial}{\partial\tilde{t}_i} (\sum_{\substack{n=1,g=0\\k=2n+2g}} \frac{1}{n!} \sum_{k_1,\ldots,k_n=1}^{\infty} \lambda^{\sum_i k_i/2} c^{(g)}_{k_1,\ldots,k_n} \prod_{i=1}^{n} \tilde{t}_{k_i}) \\
= \frac{1}{\lambda}\frac{\partial}{\partial\tilde{t}_2} (\sum_{\substack{n=1,g=0\\k=2n+2g-2}} \frac{1}{n!} \sum_{k_1,\ldots,k_n=1}^{\infty} \lambda^{\sum_i k_i/2} c^{(g)}_{k_1,\ldots,k_n} \prod_{i=1}^{n} \tilde{t}_{k_i})
\end{aligned} \tag{173}$$

for $n = -1$

$$\begin{aligned}
\sum_{k=2}^{\infty} \tilde{t}_k k \frac{\partial}{\partial\tilde{t}_{k-1}} (\sum_{\substack{n=1,g=0\\k=2n+2g}} \frac{1}{n!} \sum_{k_1,\ldots,k_n=1}^{\infty} \lambda^{\sum_i k_i/2} c^{(g)}_{k_1,\ldots,k_n} \prod_{i=1}^{n} \tilde{t}_{k_i}) \\
= \frac{1}{\lambda}\frac{\partial}{\partial\tilde{t}_1} (\sum_{\substack{n=1,g=0\\k=2n+2g-2}} \frac{1}{n!} \sum_{k_1,\ldots,k_n=1}^{\infty} \lambda^{\sum_i k_i/2} c^{(g)}_{k_1,\ldots,k_n} \prod_{i=1}^{n} \tilde{t}_{k_i})
\end{aligned} \tag{174}$$

Now this just looks awful, one might think. But remember that we solve this for every power of t independently. We simply evaluate the derivates, which is easy enough, and then solve for an arbitrary power of t. And note that the power of t is equal to n the number of traces. What we end up with is this general recurrence relation, known as the generalized Catalan numbers:

$$c^g_{n+2,i_1\ldots i_p} = 2c^g_{n,i_1\ldots i_p} + \sum_{j=1}^{p} i_j c^g_{i_1\ldots n+i_j,\ldots i_p} + \sum_{a+b=g} \sum_{\substack{I_1\cap I_2=i_1\ldots i_p\\I_1\cup I_2=\emptyset}} \sum_{s=1}^{n-1} c^a_{n-s,I_1} c^b_{s,I_2} + \sum_{s=1}^{n-1} c^{g-1}_{n-s,s,i_1\ldots i_p} \tag{175}$$

This relation has first been found in 1971 by Walsh and Lehman in [16].

We continue to find the initial conditions for $2n+2g-2=k>2$:

$$c^g_{1,i_1\dots i_p}=\sum_{j=1}^{p} i_j c^g_{i_1\dots i_j-1\dots i_p} \qquad c^g_{2,i_1\dots i_p}=\sum_{j=1}^{p} i_j c^g_{i_1\dots i_p} \tag{176}$$

For $2n+2g-2=k\leq 2$:

$$c^{(0)}_1=0 \qquad c^{(0)}_2=1 \qquad c^{(0)}_{1j}=1(j=1)+jc^{(0)}_{j-1} \qquad c^{(0)}_{2j}=jc^{(0)}_j \tag{177}$$

First thing to notice is that this looks very familiar to what we found in the end of section 2, including the special notation with I_1 as the set of indices. Also remember that in the end of section 2 we threatened that we were going to figure out relations for the combinatorical coefficients describing our correlators at each genus, and this is what we just did! For any genus, for any connected correlator.

4 β Deformation

In this last section we are going to stray away a little from our trusted matrix correlators to a deformation, a generalization of them. This is interesting because generalizations are always interesting, and because for this specific deformation we are able to apply the methods from the last section to solve them. More information on β- deformed matrix models can be found in [17], [18], [19]. This is how we generalize our model: by introducing a parameter β in the generating function for the hermitian matrix correlators defined in section 1.8 in (61):

$$Z^\beta=\int\prod_i d\lambda_i\prod_{i\neq j}(\lambda_i-\lambda_j)^\beta e^{-\frac{1}{g^2 2}\sum_i\lambda_i^2+\sum_k t_k\sum_i\lambda_i^k} \tag{178}$$

How this generating function came to be can be found in calculations in [7]. The t-variables, are still our dummy variables. Evaluating

$$\frac{\partial^m}{\partial t_{i_1}\dots\partial t_{i_m}}Z^\beta\Big|_{t=0} \tag{179}$$

still gives the correlators. We can also still expand Z^β by its correlators. We just do not know what these correlators mean. For $\beta=1$, we retrieve the generating function of the hermitian matrix correlators. For general β, however, we have no a priori interpretation in terms of matrix models. We cannot simply draw and count graphs. Knowing that we can use the Virasoro constraints to find the correlator coefficients comes in handy now. Remember that we were able to calculate all the coefficients of the undeformed case in the last section, because we were working in the large N limit, where we could solve the Virasoro constraints for every power of N separately. Following this line of calculations, we choose to again work in the large N limit, and change to the convention:

$$g^2N = \lambda \qquad \tilde{t} = Nt \tag{180}$$

$$Z^\beta = \int \prod_i d\lambda_i \prod_{i\neq j}(\lambda_i - \lambda_j)^\beta e^{-\frac{N}{\lambda 2}\sum_i \lambda_i^2 + \sum_k \frac{1}{N}\tilde{t}_k \sum_i \lambda_i^k} \tag{181}$$

First off. Because the generating function changed, the Virasoro constraints change as well, and we need to recalculate them. But the calculations pose no new obstacles, and we have done them a couple of times already in this thesis, they have therefore been omitted. It has been convenient to work with the free energy so far, so we will keep doing that.

$n > 0$

$$\begin{aligned} 0 = \beta \sum_{s=1}^{n-1} \frac{\partial}{\partial \tilde{t}_{n-s}} F^\beta \frac{\partial}{\partial \tilde{t}_s} F^\beta + \\ +(\frac{1}{N}(n+1)(1-\beta)\frac{\partial}{\partial \tilde{t}_n} + \frac{1}{N^2}\sum_{k=1}^{\infty}\tilde{t}_k k \frac{\partial}{\partial \tilde{t}_{k+n}} + \beta \sum_{s=1}^{n-1}\frac{\partial^2}{\partial \tilde{t}_{n-s}\partial \tilde{t}_s} - \frac{1}{\lambda}\frac{\partial}{\partial \tilde{t}_{n+2}} + 2\beta \frac{\partial}{\partial \tilde{t}_n})F^\beta \end{aligned} \tag{182}$$

$n = 0$

$$0 = \frac{1-\beta}{N} + \beta + (\frac{1}{N^2}\sum_{k=1}^{\infty}\tilde{t}_k \frac{\partial}{\partial \tilde{t}_k} - \frac{1}{\lambda}\frac{\partial}{\partial \tilde{t}_2})F^\beta \tag{183}$$

$n = -1$

$$0 = \frac{\tilde{t}_1}{N^2} + (\frac{1}{N^2}\sum_{k=2}^{\infty}\tilde{t}_k k \frac{\partial}{\partial \tilde{t}_{k-1}} - \frac{1}{\lambda}\frac{\partial}{\partial \tilde{t}_1})F^\beta \tag{184}$$

Like before a large N expansion of the free energy can be found, but note, that as we do not a priori know the form of the correlators this is not given. It is merely an assumption we make. Before we knew that N only ever appeared in even powers. Now, we have no reason to believe that this is still the case. On the contrary, the Virasoro constraint has terms depending on N at odd powers. We therefore let the power of N be any negative integer number (or 0).

$$F^\beta = \sum_{k=0}^{\infty} \frac{F_k^\beta}{N^k} \tag{185}$$

Using this assumption and putting it in the Virasoro constraints we can again, like the last section, solve them for every power of N individually.

$n > 0$:

$$(n+1)(\beta^{-1}-1)\frac{\partial}{\partial\tilde{t}_n}F^\beta_{i-1}+\beta^{-1}\sum_{k=1}^{\infty}\tilde{t}_k k\frac{\partial}{\partial\tilde{t}_{k+n}}F^\beta_{i-2}+\sum_{k+l=i}\sum_{s=1}^{n-1}\frac{\partial}{\partial\tilde{t}_{n-s}}F^\beta_k\frac{\partial}{\partial\tilde{t}_s}F^\beta_l+ \\ +(\sum_{s=1}^{n-1}\frac{\partial^2}{\partial\tilde{t}_{n-s}\partial\tilde{t}_s}-\frac{1}{\lambda\beta}\frac{\partial}{\partial\tilde{t}_{n+2}}+2\frac{\partial}{\partial\tilde{t}_n})F^\beta_i=0 \tag{186}$$

$n = 0$:

$$0=-\frac{1}{\lambda\beta}\frac{\partial}{\partial\tilde{t}_2}F^\beta_i+\beta^{-1}\sum_{k=1}^{\infty}\tilde{t}_k k\frac{\partial}{\partial\tilde{t}_k}F^\beta_{i-2}+1(i=0)+(\beta^{-1}-1)(i=1) \tag{187}$$

$n = -1$:

$$0=-\frac{1}{\lambda\beta}\frac{\partial}{\partial\tilde{t}_1}F^\beta_i+\beta^{-1}\sum_{k=2}^{\infty}k\tilde{t}_k\frac{\partial}{\partial\tilde{t}_{k-1}}F^\beta_{i-2}+\tilde{t}_1(i=2)\beta^{-1} \tag{188}$$

4.1 β deformed Free Energy

In section 3 a detailed description of F in terms of N and in turn a description of F_k was simply given to us by the structure of the correlators. In the present case, as discussed above, we do not have the luxury of this knowledge any more, and we will have to work a little harder to find F^β. A few helpful things about our problem we already know, though. First off, we know that the Virasoro constraints have a unique solution F^β. This is because our generating function Z^β is at its core still a Gaussian integral, which has a unique solution. A counter example, that is generating functions $Z^{\beta,v}$ that do not lead to unique solutions, are discussed in the last part of this section. Secondly, consider the following transformation, which we are going to call β-transformation:

$$(\beta,\lambda,N)\to(\beta^{-1},\lambda\beta^2,-N\beta) \tag{189}$$

and note that the Virasoro constraints are invariant under it. Convince yourself that this is true. Take the β deformed Virasoro Constraints, send β to β^{-1}, λ to $\lambda\beta^2$ and N to $-N\beta$, you will see that you end up with the same expression you started out with. That is, if F^β is equal to $F^{\frac{1}{\beta}}$. If not, there would be two solutions to our Virasoro constraints, which would be a contradiction to F^β being a unique solution. Therefore, F^β has to be invariant under (189). This nifty symmetry is going to help us build F^β.

A few things we can even assume right away. F^β is still going to depend on powers of, $\tilde{t}_i$ and with every such $\tilde{t}_i$ comes a factor $(\lambda)^{i/2}$. We already argued this in section 1.8. The situation was different, we were not working in the β deformed setting, but nonetheless the argument holds. Also in the present case, we can use simple substitution for the eigenvalues $\lambda_i=\lambda^{1/2}\bar{\lambda}_i$ which leaves the integral invariant, and when calculating the correlators by taking derivatives by t_i, we still gain these factors $\lambda^{i/2}$. Now to make use of our knowledge of the β-symmetries; note that λ on its own is obviously not invariant under the β-transformation. The way to make it invariant is by multiplying it by β. To check this, take the term, $(\lambda\beta)$ let it transform and note that it stays the same. We therefore assume that with every $\tilde{t}_i$ in F^β comes a $(\lambda\beta)^{i/2}$. Another thing that we have already assumed is the N-dependency. And of course N^{-k} by itself is not invariant. Rendering it invariant however is a bit more tricky, because there are multiple ways of doing so. Consider for example the following invariant expression:

$$N^{-k}((-\beta)^{-l} + (-\beta)^{l-k}) \qquad l \in \{0, ..., k\} \tag{190}$$

This yields at least $k/2$ ways of rendering N^{-k} invariant. We could however also add them, each with a different constant coefficient in front, this would also lead to invariant expressions. Without further knowledge of the β deformed correlators, the exact form of these functions is difficult to guess. Keeping in mind that F_k^β has to depend on some function $f(\beta, k)$, rendering N^{-k} invariant, we abandon this general discussion. Going forward, we are going to use the scientific method of guessing. Starting with the simplest case F_0^β.

4.1.1 F_0^β

The first thing we have to ask ourselves is which powers of $\tilde{t}$ we should consider in F_0^β. Following the example of the undeformed case F_0, we would guess to only consider a power one monomial, and we would be correct. Remember:

$$F_0 = \sum_{i=1}^{\infty} \tilde{t}_i c_i^{(0)} \lambda^{i/2} \tag{191}$$

Making comparisons like this is helpful as for $\beta = 1$ we need to retrieve $F_0^1 = F_0$. Let's start by writing our ansatz:

$$F_0^\beta = \sum_i \tilde{t}_i c_i^{(0)} (\lambda\beta)^{i/2} \tag{192}$$

It has already been argued, that every $\tilde{t}_i$, comes with a factor $(\lambda\beta)^{i/2}$. $c_i^{(0)}$ are constant coefficients. The upper index, like in the undeformed case, distinguishes coefficients of monomials of the same power. Leaning into that comparison, we call the index $\bar{g}$. Keep in mind that in contrast to g, $\bar{g}$ has no clear interpretation in terms of genus. Note that if $c_i^{(g=0)} = c_i^{(\bar{g}=0)}$ we retrieve F_0 for $\beta = 1$. Consider the Virasoro constraints for N^0:

$n > 0$:

$$\sum_{s=1}^{n-1} \frac{\partial}{\partial \tilde{t}_{n-s}} F_0^\beta \frac{\partial}{\partial \tilde{t}_s} F_0^\beta + (\sum_{s=1}^{n-1} \frac{\partial^2}{\partial \tilde{t}_{n-s} \partial \tilde{t}_s} - \frac{1}{\lambda\beta} \frac{\partial}{\partial \tilde{t}_{n+2}} + 2 \frac{\partial}{\partial \tilde{t}_n}) F_0^\beta = 0 \tag{193}$$

$n = 0$ and $n = -1$:

$$1 = \frac{1}{\lambda\beta}\frac{\partial}{\partial\tilde{t}_2}F_0^\beta \quad 0 = \frac{1}{\lambda\beta}\frac{\partial}{\partial\tilde{t}_1}F_0^\beta \tag{194}$$

Put F_0^β in. The calculations are analogous in its methods to what we have done in section 3. We retrieve:

$$c_{n+2}^{(0)} = 2c_n^{(0)} + \sum_{s=1}^{n-1} c_{n-s}^{(0)}c_s^{(0)} \tag{195}$$

$$1 = c_2^{(0)} \qquad 0 = \bar{c}_1^{(0)} \tag{196}$$

This is again the Catalan numbers! Again, all coefficients of odd index are zero, because of $0 = \bar{c}_1^{(0)}$. This is quite the interesting result. This means the notation did not betray us and the coefficients in F_0 and F_0^β are indeed equal $c_i^{(g=0)} = c_i^{(\bar{g}=0)}$ and for $\beta = 1$ we do retrieve $F_0^1 = F_0$. In fact, all coefficients with upper index (0) will turn out to be equivalent to the undeformed coefficients with index (0). But let's not get ahead of ourselves. Next we want to consider F_1^β.

4.1.2 F_1^β

F_1^β does not exist in the undeformed case, so it will have to be zero for $\beta = 1$. Sadly, this means that we cannot draw inspiration from previous results. What we do know is that F_1^β will have to include some function, rendering N^{-1} invariant under the β-transformation. Coincidentally, there is only one such function, and it is already present in the Virasoro constraints, namely $\frac{(\beta^{-1}-1)}{N}$. It is easily checked, that this is indeed invariant. In order to decide on a monomial structure, have a look at the Virasoro constraints for N^{-1}.

$n > 0$:

$$\begin{aligned}(n+1)(\beta^{-1}-1)\frac{\partial}{\partial\tilde{t}_n}F_0^\beta + 2\sum_{s=1}^{n-1}\frac{\partial}{\partial\tilde{t}_{n-s}}F_1^\beta\frac{\partial}{\partial\tilde{t}_s}F_0^\beta +\\ +(\sum_{s=1}^{n-1}\frac{\partial^2}{\partial\tilde{t}_{n-s}\partial\tilde{t}_s} - \frac{1}{\lambda\beta}\frac{\partial}{\partial\tilde{t}_{n+2}} + 2\frac{\partial}{\partial\tilde{t}_n})F_1^\beta = 0\end{aligned} \tag{197}$$

$n = 0$ and $n = -1$:

$$\frac{1}{\lambda\beta}\frac{\partial}{\partial\tilde{t}_2}F_1^\beta = (\beta^{-1}-1) \qquad 0 = \frac{1}{\lambda\beta}\frac{\partial}{\partial\tilde{t}_1}F_1^\beta \tag{198}$$

Nothing in these constraints motivates us to include monomials of power higher than one, so we choose our ansatz to be:

$$F_1^\beta = \sum_{i=1}^{\infty}\tilde{t}_i c_i^{(1)}(\lambda\beta)^{i/2}(\beta^{-1}-1) \tag{199}$$

Note that this is indeed zero for $\beta = 1$. The upper index of the coefficient $c_i^{(1)}$ tells us that this is the second monomial of power one that we are considering. In the undeformed case, we would call coefficients with $g = 1$ the first correction term. Putting F_1^β in the Virasoro constraints leaves us with:

$$c_{n+2}^{(1)} = 2c_n^{(1)} + 2\sum_{s=1}^{n-1} c_{n-s}^{(1)} c_s^{(0)} + (n+1)c_n^{(0)} \tag{200}$$

$$c_2^{(1)} = 1 \qquad c_1^{(1)} = 0 \tag{201}$$

The sequence, described by this relation, has not come up in this thesis yet, understandably because F_1^β does not exist in our undeformed hermitian matrix model. It is a known sequence, though. It can be found in [16] and also on OEIS [20], sequence A000346. The first few coefficients are: $c_i^{(1,0)} \in \{1, 5, 22, 93, 386, \dots\}$.

As a last special case we are considering F_2^β, then we are going to lay out the general case F_k^β.

4.1.3 F_2^β

When looking for F_2^β, as for F_0^β, we draw inspiration from the $\beta = 1$ case. Remember:

$$F_2 = \sum_{i=1}^{\infty} \tilde{t}_i \lambda^{i/2} c_i^{(1)} + \sum_{i,j=1}^{\infty} \tilde{t}_i \tilde{t}_j \lambda^{(i+j)/2} c_{ij}^{(0)} \tag{202}$$

When going from this to the β deformed case, our strategy is to switch out λ with $\lambda\beta$ and to introduce a function rendering in this case N^{-2} invariant. But, as discussed above, we could find multiple such functions. For now, we will therefore simply denote them $f_1(\beta)$ and $f_2(\beta)$, and write our ansatz function like so:

$$F_2^\beta = \sum_{i=1}^{\infty} \tilde{t}_i (\beta\lambda)^{i/2} c_i^{(2)} f_1(\beta) + \sum_{i,j=1}^{\infty} \tilde{t}_i \tilde{t}_j (\beta\lambda)^{(i+j)/2} c_{ij}^{(0)} f_2(\beta) \tag{203}$$

Note that here we already have $c_i^{(2)}$, while for the undeformed case we have $c_i^{(1)}$. For $\beta = 1$, these two coefficients have to agree. We therefore find the relation $2g = \bar{g}$, at $\beta = 1$. This relation shows that because we are including odd k in the β-deformed setting, $\bar{g}$ is growing twice as fast as g in the undeformed setting. $2g = \bar{g}$ tells us which coefficients of the two settings need to be the same, at $\beta = 1$. As for the other coefficient $c_{ij}^{(0)}$ above, it has already been promised that coefficients with upper index (0), will always coincide with the undeformed case. We will convince ourselves of this shortly. Let's have a look at the Virasoro constraints for N^{-2}:

$n > 0$:

$$(n+1)(\beta^{-1}-1)\frac{\partial}{\partial\tilde{t}_n}F_1^\beta + \beta^{-1}\sum_{k=1}^{\infty}\tilde{t}_k k\frac{\partial}{\partial\tilde{t}_{k+n}}F_0^\beta + 2\sum_{s=1}^{n-1}\frac{\partial}{\partial\tilde{t}_{n-s}}F_0^\beta\frac{\partial}{\partial\tilde{t}_s}F_2^\beta + \\ +\sum_{s=1}^{n-1}\frac{\partial}{\partial\tilde{t}_{n-s}}F_1^\beta\frac{\partial}{\partial\tilde{t}_s}F_1^\beta + (\sum_{s=1}^{n-1}\frac{\partial^2}{\partial\tilde{t}_{n-s}\partial\tilde{t}_s} - \frac{1}{\lambda\beta}\frac{\partial}{\partial\tilde{t}_{n+2}} + 2\frac{\partial}{\partial\tilde{t}_n})F_2^\beta = 0 \tag{204}$$

$n = 0$:

$$0 = -\frac{1}{\lambda\beta}\frac{\partial}{\partial\tilde{t}_2}F_2^\beta + \beta^{-1}\sum_{k=1}^{\infty}\tilde{t}_k k\frac{\partial}{\partial\tilde{t}_k}F_0^\beta \tag{205}$$

$n = -1$:

$$0 = -\frac{1}{\lambda\beta}\frac{\partial}{\partial\tilde{t}_1}F_2^\beta + \beta^{-1}\sum_{k=2}^{\infty}k\tilde{t}_k\frac{\partial}{\partial\tilde{t}_{k-1}}F_0^\beta + \tilde{t}_1\beta^{-1} \tag{206}$$

Remember that the strategy for solving these long expressions is to first evaluate the derivatives and then solve for every power of $\tilde{t}$ independently. For the first power, we find:

$$f_2(\beta)c_{n+2,i}^{(0)} = 2f_2(\beta)c_{n,i}^{(0)} + 2f_2(\beta)\sum_{s=1}^{n-1}c_s^{(0)}c_{n-s,i}^{(0)} + i\beta^{-1}c_{n+i}^{(0)} \tag{207}$$

$$f_2(\beta)c_{2,i}^{(0)} = \beta^{-1}ic_i^{(0)} \qquad f_2(\beta)c_{1,i}^{(0)} = \beta^{-1}(ic_{i-1}^{(0)} + 1(i=1)) \tag{208}$$

Because we want our expressions to be true independently of β, $f_2(\beta)$ needs to be equal to β^{-1}.

$$f_2(\beta) = \beta^{-1} \tag{209}$$

Which is indeed a function rendering N^{-2} invariant. And with it we find the recurrence relation:

$$c_{n+2,i}^{(0)} = 2c_{n,i}^{(0)} + 2\sum_{s=1}^{n-1}c_s^{(0)}c_{n-s,i}^{(0)} + ic_{n+i}^{(0)} \tag{210}$$

with the initial conditions:

$$c_{2,i}^{(0)} = ic_i^{(0)} \qquad c_{1,i}^{(0)} = ic_{i-1}^{(0)} + 1(i=1) \tag{211}$$

Which is indeed the same as what we found for the $\beta = 1$ case F_2.

Solving for the zeroth power of $\tilde{t}$, is going to be a tad more difficult and is going to include the main trick we need when solving for any F_k^β of the β deformed model. We find:

for $n = 0$ and $n = -1$

$$c_2^{(2)} = 0 \qquad c_1^{(2)} = 0 \tag{212}$$

for $n > 0$

$$f_1(\beta)c^{(2)}_{n+2} = 2f_1(\beta)c^{(2)}_n + \beta^{-1}\sum_{s=1}^{n-1} c^{(0)}_{n-s,s} + (\beta^{-1}-1)^2\sum_{s=1}^{n-1} c^{(1)}_{n-s}c^{(1)}_s + 2f_1(\beta)\sum_{s=1}^{n-1} c^{(0)}_s c^{(2)}_{n-s} + (n+1)(\beta^{-1}-1)^2 c^{(1)}_n \tag{213}$$

In the $n > 0$ constraint we find terms depending on β^0, β^{-2} and β^{-1}, but also a term only depending on β^{-1}. For our solution to be independent of β, we need to solve for terms depending on β^{-1} and terms depending on $(\beta^{-2}+\beta^0)$ separately. Now how we do that? We still have an undefined $f_1(\beta)$ lying around, which we are going to use in the following way:

$$f_1(\beta)c^{(2)}_i = (\beta^{-2}+1)c^{(2,0)}_i - \beta^{-1}c^{(1,1)}_i \tag{214}$$

Note the new notation for our coefficients. We have gained a second upper index, which tells us to which β-polynomial the coefficient belongs. In general, we denote these upper indices by $(\bar{g}-l, l)$. The order of the indices is irrelevant, that means: $c^{(a,b)}_i = c^{(b,a)}_i$. Why they are chosen in this specific way will be explained shortly. With this, we find these two recurrence relations:

$$c^{(2,0)}_{n+2} = 2c^{(2,0)}_n + \sum_{s=1}^{n-1} c^{(1,0)}_{n-s}c^{(1,0)}_s + 2c^{(0,0)}_{n-s}c^{(2,0)}_s) + (n+1)c^{(1,0)}_n \tag{215}$$

$$c^{(1,1)}_{n+2} = 2c^{(1,1)}_n + \sum_{s=1}^{n-1}(2c^{(1,0)}_{n-s}c^{(1,0)}_s + 2c^{(0,0)}_{n-s}c^{(1,1)}_s) + (n+1)2c^{(1,0)}_n - \sum_{s=1}^{n-1} c^{(0,0)}_{n-s,s} \tag{216}$$

The initial values are still what we found above.

$$c^{(1,1)}_2 = 0 \quad c^{(1,1)}_1 = 0 \quad c^{(2,0)}_2 = 0 \quad c^{(2,0)}_1 = 0 \tag{217}$$

A few examples of these coefficients are given in (1).

This leads us to our final description of F^β_2:

$$F^\beta_2 = \sum_{i=1}^{\infty}\tilde{t}_i(\beta\lambda)^{i/2}(c^{(2,0)}_i(\beta^{-2}+1) - \beta^{-1}c^{(1,1)}_i) + \sum_{i,j=1}^{\infty}\tilde{t}_i\tilde{t}_j(\beta\lambda)^{(i+j)/2}c^{(0)}_{ij}\beta^{-1} \tag{218}$$

For $\beta = 1$, we find the following relation between the undeformed coefficient, and the two β deformed ones : $c^{(1)}_i = 2c^{(2,0)}_i - c^{(1,1)}_i$.

4.1.4 F^β_k

Using the methods of clever guessing discussed this far, we are able to calculate any F^β_k. As there are no new fundamental obstacles, we will refrain from spelling out the calculations for any more specific cases. When looking for a general, F^β we can draw a lot of inspiration from our previous result for the undeformed case F. Working this way is very fruitful, because we know F very well, and we know that F^β needs to reduce to F for $\beta = 1$.

The first thing we are going to assume is that, for k even, F_k^β has the same monomial structure, that is it contains the same amount of terms depending on the same powers of $\tilde{t}_i$, as F_k. This has already been done in the special cases F_0^β and F_2^β. For odd k, we cannot make such a comparison, as all of F_k^β needs to vanish for $\beta = 1$. We will however guess that F_k^β for odd k has the same monomial structure as F_{k-1}^β. This is what we found for F_1^β. Further motivation for this guess can be found in the labelling of monomials, another feature of the undeformed F from which we would like to draw inspiration from. Remember F:

$$F = \sum_{n,g=0}^{\infty} \frac{N^{2-2n-2g}}{n!} \sum_{k_1,\dots,k_n=1}^{\infty} \lambda^{\sum_i k_i/2} c_{k_1,\dots,k_n}^{(g)} \prod_{i=1}^{n} \tilde{t}_{k_i} \tag{219}$$

This expression shows nicely how N^{-k} depends on n, the power of $\tilde{t}$, and g, the genus. To be explicit: $k = 2n+2g-2$. In the β-deformed setting, we do not a priori have an interpretation in form of graphs, and therefore no interpretation of genus. But remember that the genus also enumerates the corrections of our correlators. Modestly speaking, g distinguishes the coefficients of monomials of equal power. In this sense, we defined a similar quantity $\bar{g}$ for the β-deformed case. And because we are including odd k in the β-deformed setting, $\bar{g}$ is going to grow twice as fast as g in the undeformed setting. When $\beta = 1$ and the two settings need to coincide, and we found $2g = \bar{g}$. For k, we find a similar relation. While above for the undeformed case we found $k = 2n + 2g - 2$, in the β-deformed setting we find $k = 2n + \bar{g} - 2$. Note how these expressions agree for $\beta = 1$ and even k. That is because the definition of n is equal in both settings, and because the monomial structures agree for even k . Also note that odd k happens whenever $\bar{g}$ is odd. Then, as discussed above, the structure of F_k^β (with k odd) is the same as for F_{k-1}^β. If this was not the case, $2g = \bar{g}$ at $\beta = 1$ would not hold and neither would the smooth comparisons we made in the last paragraph.

We almost have everything we need to build our general F_k^β. The only thing we are missing is the exact structure of the polynomials in β, which we have already seen in F_1^β and F_2^β and which we know we need in order to render N^{-k} invariant under the β-transformation. All we can offer within the bounds of this thesis is that upon looking at enough examples, the general form of the polynomials becomes clear. To that end, consider this quick summary of the functions we have found above F_0^β, F_1^β and F_2^β and also F_3^β and F_4^β, for additional reference.

$$F_0^\beta = \sum_i \tilde{t}_i (\lambda\beta)^{i/2} c_i^{(0,0)} \tag{220}$$

$$F_1^\beta = \sum_i \beta^{-1}(1-\beta)(\lambda\beta)^{i/2} c_i^{(1,0)} \tilde{t}_i \tag{221}$$

$$F_2^\beta = \frac{1}{2}\beta^{-1}\sum_{ij}\tilde{t}_i\tilde{t}_j(\lambda\beta)^{(i+j)/2}c_{ij}^{(0,0)} + \sum_i \beta^{-2}\tilde{t}_i(\lambda\beta)^{i/2}(c_i^{(2,0)}(\beta^2+1) - \beta c_i^{(1,1)}) \tag{222}$$

$$F_3^\beta = \frac{1}{2}\sum_{ij}\tilde{t}_i\tilde{t}_j\beta^{-2}(\lambda\beta)^{(i+j)/2}c_{ij}^{(1,0)}(1-\beta) + \sum_i \tilde{t}_i\beta^{-3}(\lambda\beta)^{i/2}(c_i^{(3,0)}(1-\beta^3) + c_i^{(2,1)}(\beta^2-\beta)) \tag{223}$$

$$\begin{aligned} F_4^\beta = \frac{1}{6}\beta^{-2}\sum_{ijk}\tilde{t}_i\tilde{t}_j\tilde{t}_k c_{ijk}^{(0,0)}(\lambda\beta)^{(i+j+k)/2} \\ +\frac{1}{2}\sum_{ij}\tilde{t}_i\tilde{t}_j\beta^{-3}(\lambda\beta)^{(i+j)/2}(c_{ij}^{(2,0)}(1+\beta^2) - \beta c_{(ij)}^{(1,1)}) \\ +\sum_i \tilde{t}_i\beta^{-4}(\lambda\beta)^{i/2}(c_i^{(4,0)}(1+\beta^4) - c_i^{(3,1)}(\beta+\beta^3) + c_i^{(2,2)}\beta^2) \end{aligned} \tag{224}$$

The notation of the coefficients from F_0^β and F_1^β has been changed to match the others, by adding another upper index equal to 0. Staring at these results, we found the following general description:

$$F_k^\beta = \sum_{\substack{n=1,\bar{g}=0 \\ k=2n+\bar{g}-2}} \frac{\beta^{1-n-\bar{g}}}{n!}\sum_{k_1,\dots,k_n=1}^{\infty}(\lambda\beta)^{\sum_i k_i/2}(\sum_{l=0}^{\bar{g}}(-\beta)^l c_{k_1,\dots,k_n}^{(\bar{g}-l,l)})\prod_{i=1}^n \tilde{t}_{k_i} \tag{225}$$

Compare this to the examples given above and find that they are indeed described by this expression. Further we can write:

$$F^\beta = \sum_{n=1,\bar{g}=0}^{\infty}\frac{N^{2-2n-\bar{g}}\beta^{1-n-\bar{g}}}{n!}\sum_{k_1,\dots,k_n=1}^{\infty}(\lambda\beta)^{\sum_i k_i/2}(\sum_{l=0}^{\bar{g}}(-\beta)^l c_{k_1,\dots,k_n}^{(\bar{g}-l,l)})\prod_{i=1}^n \tilde{t}_{k_i} \tag{226}$$

Firstly, convince yourselves that this indeed is equal to the undeformed case (219) for $\beta = 1$. This is a good sign that we found the right solution. Next, we need to check whether (226) is invariant under the β transformation. We send $F^\beta(\beta,\lambda,N)\to F^{\frac{1}{\beta}}(\beta^{-1},\lambda\beta^2,-N\beta)$:

$$\begin{aligned}F^{\frac{1}{\beta}} &= \sum_{n,\bar g=0}^\infty \frac{1}{n!}N^{2-2n-\bar g}(-\beta)^{2-2n-\bar g}\beta^{-1+n+\bar g}\sum_{k_1,\dots,k_n=1}^\infty (\lambda\beta)^{\sum_i k_i/2}(\sum_{l=0}^{\bar g}(-\beta)^{-l}c^{(\bar g-l,l)}_{k_1,\dots,k_n})\prod_{i=1}^n \tilde t_{k_i} = \\ &= \sum_{n,\bar g=0}^\infty \frac{N^{2-2n-\bar g}}{n!}(-1)^{-\bar g}(\beta)^{-\bar g}\beta^{2-2n}\beta^{-1+n+\bar g}\sum_{k_1,\dots,k_n=1}^\infty (\lambda\beta)^{\sum_i k_i/2}(\sum_{l=0}^{\bar g}(-\beta)^{-l}c^{(\bar g-l,l)}_{k_1,\dots,k_n})\prod_{i=1}^n \tilde t_{k_i} = \\ &= \sum_{n,\bar g=0}^\infty \frac{N^{2-2n-\bar g}\beta^{1-n-\bar g}}{n!}(-1)^{\bar g}\beta^{\bar g}\sum_{k_1,\dots,k_n=1}^\infty (\lambda\beta)^{\sum_i k_i/2}(\sum_{l=0}^{\bar g}(-\beta)^{-l}c^{(\bar g-l,l)}_{k_1,\dots,k_n})\prod_{i=1}^n \tilde t_{k_i} = \\ &= \sum_{n,\bar g=0}^\infty \frac{N^{2-2n-\bar g}\beta^{1-n-\bar g}}{n!}\sum_{k_1,\dots,k_n=1}^\infty (\lambda\beta)^{\sum_i k_i/2}(\sum_{l=0}^{\bar g}(-\beta)^{\bar g-l}c^{(\bar g-l,l)}_{k_1,\dots,k_n})\prod_{i=1}^n \tilde t_{k_i} = F^\beta\end{aligned} \tag{227}$$

Indeed, our solution is invariant under this transformation.

At last, we want to check whether (226) actually solves the Virasoro constraints. From above, we know that evaluating these general solutions is quite messy. When doing these calculations, we need to be very aware of symmetries lurking. Essential is of course to be aware that we solve this equation monomial by monomial. These expressions tend to get very massive, and we need to reduce clutter wherever we can. The whole calculation is in fact not going to be spelled out. We simply state the recurrence relation that pops out of the Virasoro constraints:

$n>0$

$$\begin{aligned}c^{\bar g-l,l}_{n+2,i_1\dots i_p} = 2c^{\bar g-l,l}_{n,i_1\dots i_p} + \sum_{j=1}^p i_j c^{\bar g-l,l}_{i_1,\dots i_j+n..i_p} + \sum_{(a,b)+(c,d)=(\bar g-l,l)}\sum_{\substack{I_1\cap I_2=i_1\dots i_p\\ I_1\cup I_2=\emptyset}}\sum_{s=1}^{n-1} c^{a,b}_{n-s,I_1}c^{c,d}_{s,I_2}+ \\ +(n+1)(c^{\bar g-l-1,l}_{n,i_1\dots i_p}+c^{\bar g-l,l-1}_{n,i_1\dots i_p}) - \sum_{s=1}^{n-1}c^{\bar g-l-1,l-1}_{n-s,s,i_1\dots i_p}\end{aligned} \tag{228}$$

$n=0$

$$c^{(\bar g-l,l)}_{2,k_1,\dots,k_m} = \sum_{j=1}^m k_j c^{(\bar g-l,l)}_{k_1,\dots,k_m} \tag{229}$$

$n=-1$

$$c^{(\bar g-l,l)}_{1,k_1,\dots,k_m} = \sum_{j=1}^m k_j c^{(\bar g-l,l)}_{k_1,\dots,k_j-1,\dots k_m} \tag{230}$$

So indeed we were able to solve the β-deformed Virasoro constraints, using (226) and found these β deformed generalized Catalan numbers which are reasonably similar to the undeformed case found in section 3. At $\beta=1$, that is if $\bar g=2g$ we retrieve the undeformed $c^{(g)}_{k_1\dots k_m}$ from the deformed $c^{(\bar g-l,l)}_{k_1\dots k_m}$ like so:

$$c^{(g)}_{k_1\dots k_m} = \sum_{l=0}^{\bar g}(-1)^l c^{(\bar g-l,l)}_{k_1\dots k_m} \tag{231}$$

This is quite remarkable, we completely analysed the correlators of the β-deformed model, down to the integer coefficients, even though we have no interpretation of what its correlators mean. This shows how powerful the Virasoro constraints are and how fruitful it is to work with them. We will leave the calculations at that, summarize and compare our results in the next section.

4.2 Summary

Let's summarize what we have found in this section and compare it to the undeformed case discussed in section 3. The free energy solution to the β-deformed Virasoro constraints at large N was found to be the following:

$$F^{\beta} = \sum_{n,\bar{g}=0}^{\infty} \frac{N^{2-2n-\bar{g}}\beta^{1-n-\bar{g}}}{n!} \sum_{k_1,\ldots,k_n=1}^{\infty} (\lambda\beta)^{\sum_i k_i/2} (\sum_{l=0}^{\bar{g}} (-\beta)^l c^{(\bar{g}-l,l)}_{k_1,\ldots,k_n}) \prod_{i=1}^{n} \tilde{t}_{k_i} \tag{232}$$

For $\beta = 1$ this reduces to

$$F = \sum_{n,g=0}^{\infty} \frac{N^{2-2n-2g}}{n!} \sum_{k_1,\ldots,k_n=1}^{\infty} \lambda^{\sum_i k_i/2} c^{(g)}_{k_1,\ldots,k_n} \prod_{i=1}^{n} \tilde{t}_{k_i} \tag{233}$$

which is the undeformed case, which we have build for most of the thesis and put together in section 3. The two expressions (232) and (233) are very similar, the main difference between them being that (232) is a polynomial in β, whereas (233) clearly is not. The power of N differs, as it also runs over odd integers in the deformed case.

Let's also compare recursion relations for the correlator coefficients in both the deformed and undeformed setting. First the β-deformed coefficients:

$$\begin{aligned} c^{\bar{g}-l,l}_{n+2,i_1\ldots i_p} = 2c^{\bar{g}-l,l}_{n,i_1\ldots i_p} + \sum_{j=1}^{p} i_j c^{\bar{g}-l,l}_{i_1\ldots n+i_j,\ldots i_p} + \sum_{(a,b)+(c,d)=(\bar{g}-l,l)} \sum_{\substack{I_1\cup I_2=i_1\ldots i_p \\ I_1\cap I_2=\emptyset}} \sum_{s=1}^{n-1} c^{a,b}_{n-s,I_1} c^{c,d}_{s,I_2} + \\ +(n+1)(c^{\bar{g}-l-1,l}_{n,i_1\ldots i_p} + c^{\bar{g}-l,l-1}_{n,i_1\ldots i_p}) - \sum_{s=1}^{n-1} c^{\bar{g}-l-1,l-1}_{n-s,s,i_1\ldots i_p} \end{aligned} \tag{234}$$

Summing this expression over all possible $l \in \{0,\ldots\bar{g}\}$, with a negative sign for odd l and setting all coefficients with odd $\bar{g}$ to zero, should lead to the generalized Catalan numbers:

$$c^{g}_{n+2,i_1\ldots i_p} = 2c^{g}_{n,i_1\ldots i_p} + \sum_{j=1}^{p} i_j c^{g}_{i_1\ldots n+i_j,\ldots i_p} + \sum_{a+b=g} \sum_{\substack{I_1\cap I_2=i_1\ldots i_p \\ I_1\cup I_2=\emptyset}} \sum_{s=1}^{n-1} c^{a}_{n-s,i_1\ldots i_r} c^{b}_{s,i_1\ldots i_t} + \sum_{s=1}^{n-1} c^{g-1}_{n-s,s,i_1\ldots i_p} \tag{235}$$

Indeed for even $\bar{g}$ the term in (234) which does not appear in (235), has necessarily an odd $\bar{g}$, which for $\beta = 1$ renders the term to zero. Apart from this term, the two expressions have the same form, up to a sign. Not to forget about the initial conditions. In the β-deformed case, we found for $2n+\bar{g}-2=k>2$:

$$c^{(\bar{g}-l,l)}_{2,i_1,\dots,i_m}=\sum_{j=1}^{m} i_j c^{(\bar{g}-l,l)}_{i_1,\dots,i_m} \tag{236}$$

$$c^{(\bar{g}-l,l)}_{1,i_1,\dots,i_m}=\sum_{j=1}^{m} i_j c^{(\bar{g}-l,l)}_{i_1,\dots,i_j-1,\dots i_m} \tag{237}$$

Very similarly we found in the undeformed setting: initial conditions for $2n+2g-2=k>2$:

$$c^{(g)}_{2,i_1,\dots,i_m}=\sum_{j=1}^{m} i_j c^{(g)}_{i_1,\dots,i_m} \tag{238}$$

$$c^{(g)}_{1,i_1,\dots,i_m}=\sum_{j=1}^{m} i_j c^{(g)}_{i_1,\dots,i_j-1,\dots i_m} \tag{239}$$

And lastly the special cases. In the β-deformed setting: For $2n+\bar{g}-2=k\leq 2$

$$c_1^{(0,0)}=0 \quad c_2^{(0,0)}=1 \quad c_1^{(1,0)}=0 \quad c_2^{(1,0)}=1 \quad c_{1j}^{(0,0)}=1(j=1)+jc_{j-1}^{(0,0)} \quad c_{2j}^{(0,0)}=jc_j^{(0,0)} \tag{240}$$

and in the undeformed setting: For $2n+2g-2=k\leq 2$

$$c_1^{(0)}=0 \quad c_2^{(0)}=1 \qquad c_{1j}^{(0)}=1(j=1)+jc_{j-1}^{(0)} \quad c_{2j}^{(0)}=jc_j^{(0)} \tag{241}$$

For $\beta=1$, where coefficients with odd $\bar{g}$ are set to zero, we find that the initial conditions agree. Which is great, because it allows the two cases to agree at $\beta=1$.
In the following table, a few coefficients have been summarized. It can be seen that for $\bar{g}=g=0$, the coefficients always agree. It can also be checked that for even $\bar{g}=2g$ the following relation between the coefficients holds:

$$\sum_{l=0}^{\bar{g}=2g}(-1)^l c^{(\bar{g}-l,l)}_{i_1\dots i_m}=c^{g}_{i_1\dots i_m} \tag{242}$$

The first table consists of coefficients with only one index. In the undeformed setting, these coefficients correspond to single trace correlators.

	F_0	F_0^β	F_1^β	F_2	F_2^β	F_2^β	F_3^β	F_3^β	F_4
n	$c_n^{(0)}$	$c_n^{(0,0)}$	$c_n^{(1,0)}$	$c_n^{(1)}$	$c_n^{(2,0)}$	$c_n^{(1,1)}$	$c_n^{(3,0)}$	$c_n^{(2,1)}$	$c_n^{(2)}$
2	1	1	1	0	0	0	0	0	0
4	2	2	5	1	3	5	0	0	0
6	5	5	22	10	32	54	15	32	0
8	14	14	93	70	234	466			21
10	42	42	386	420					483

Table 1: Coefficients with one index

The second table summarizes the coefficients depending on two indices, corrsponding to two-trace correlators in the undeformed setting.

		F_2	F_2^β	F_3^β	F_4	F_4^β	F_4^β
i	j	$c_{ij}^{(0)}$	$c_{ij}^{(0,0)}$	$c_{ij}^{(1,0)}$	$c_{ij}^{(1)}$	$c_{ij}^{(2,0)}$	$c_{ij}^{(1,1)}$
1	1	1	1	0	0	0	0
2	2	2	2	2	0	0	0
1	3	3	3	3	0	0	0
3	3	12	12	27	3	15	27
4	2	8	8	38	4	12	20
1	5	10	10	180	5	15	25

Table 2: Coefficients with two indices

Finally consider one last table, for coefficients with three indices.

			F_4	F_4^β
i	j	k	$c_{ijk}^{(0)}$	$c_{ijk}^{(0,0)}$
1	1	2	2	2
2	2	2	8	8
1	2	3	12	12
1	1	4	12	12
3	3	2	72	72
2	2	4	32	32

Table 3: Coefficients with three indices

It might be satisfying to compare some of the results in this list, to correlators calculated throughout the thesis. To do this, remember that the free energy can be expanded by its correlators:

$$F = 1 + \sum_{k=1}^{\infty} \langle TrM^k \rangle \tilde{t_k} + \frac{1}{2!} \sum_{k,j=1}^{\infty} \langle TrM^k TrM^j \rangle \tilde{t_k} \tilde{t_j} + \dots$$

With this, the free energy solution to the Virasoro constraints and coefficients, given to us by the recurrence relation, we can calculate any correlator, without needing any further knowledge about them. Take for example the undeformed correlator $\langle TrM^4 \rangle$ at large N. It is a single trace correlator and therefore has coefficients with one index, namely the index 4. In table 1 the only two nonzero values for $n = 4$ can be read off at $g = 1$ and $g = 0$. Knowing the index, n and g the powers of λ and N can be read off from (233). With this we find:

$$\langle TrM^4 \rangle = \lambda^2 (2 + \frac{1}{N^2})$$

This is what we have found many times in this thesis. Remember that we are working in the large N limit. But even at finite N, the coefficients stay the same. Take another example, $\langle TrM^3 M^3 \rangle$. This time consult table 2, row $3,3$. The only nonzero values can be found for $g = 0$ and $g = 1$. For powers of N consult (233). And we find:

$$\langle TrM^3TrM^3\rangle = \lambda^3(\frac{12}{N^2} + \frac{3}{N^4})$$

Note that we found these results without using any of our prior knowledge of the undeformed correlators. All that has been used was the free energy solution to the Virasoro constraints (233) and the generalized Catalan numbers (235). This can also be done for the β-deformed case. Denote a correlator in the β-deformed setting like so:

$$\langle TrM^{k_1} \dots TrM^{k_m}\rangle^{\beta} \tag{243}$$

The β deformed free energy (232), can also be expanded by its coefficients:

$$F^{\beta} = 1 + \sum_{k=1}^{\infty}\langle TrM^k\rangle^{\beta}\tilde{t}_k + \frac{1}{2!}\sum_{k,j=1}^{\infty}\langle TrM^kTrM^j\rangle^{\beta}\tilde{t}_k\tilde{t}_j + \dots$$

Now, following the same term collecting pattern from before, consider the example $\langle TrM^4\rangle^{\beta}$. Consult the table 2 and (232). The only non-zero values we find are for genus 0, 1 and 2. We find:

$$\langle TrM^4\rangle^{\beta} = (\lambda)^2(2\beta^2 + \frac{5}{N}(\beta - \beta^2) + \frac{3}{N^2}(1+\beta^2) - \frac{5}{N^2}\beta) \tag{244}$$

Note how it reduces to the undeformed $\langle TrM^4\rangle$ for $\beta = 1$. This is the beauty of our result. Without prior knowledge about the β deformed correlators, without a notion of geometry, graphs, or genus, we can calculate them in this manner.

4.3 β deformed Model for a Generalised Power

For the entirety of this thesis, integrals of a Gaussian form, with quadratic terms in the exponent, have been considered. But there are other integrals that can be solved and that might be interesting. This section is about the β-deformed matrix model with a term of general power v in the exponent. For more information about when matrix integrals are solvable consider [21]. This is the corresponding generating function:

$$Z^{\beta,v} = \int\prod_i d\lambda_i\prod_{i\neq j}(\lambda_i - \lambda_j)^{\beta}e^{-\frac{1}{a_v v}\sum_i\lambda_i^v + \sum_k t_k\sum_i\lambda_i^k} \tag{245}$$

In comparison to what we had above in (181), $\frac{1}{2g^2}\lambda_i^2$ has been generalized to $\frac{1}{va_v}\lambda_i^v$. Because we want to work in the large N-limit, we introduce this generalization of the t'Hooft coupling :

$$\tilde{a}_v = N a_v \tag{246}$$

Before, when working in the large N limit we also chose a different dummy variable t, namely: $\tilde{t} = Nt$. Again, we choose to work in terms of $\tilde{t}$. Changing these variables, we find the generating function to be:

$$Z^{\beta,v} = \int \prod_i d\lambda_i \prod_{i\neq j}(\lambda_i - \lambda_j)^\beta e^{-\frac{N}{\tilde{a}_v v}\sum_i \lambda_i^v + \sum_k \frac{1}{N}\tilde{t}_k \sum_i \lambda_i^k} \tag{247}$$

In accordance to before, calculate the Virasoro constraints, which change only slightly. It has been discussed in section 3 that it is easiest to work in terms of free energy, so we will continue in this fashion:

$n > 0$

$$0 = (\frac{1}{N}(n+1)(\beta^{-1}-1)\frac{\partial}{\partial\tilde{t}_n} + \frac{\beta^{-1}}{N^2}\sum_{k=1}^{\infty}\tilde{t}_k k\frac{\partial}{\partial\tilde{t}_{k+n}} + \sum_{s=1}^{n-1}\frac{\partial^2}{\partial\tilde{t}_{n-s}\partial\tilde{t}_s} - \frac{1}{\tilde{a}_v\beta}\frac{\partial}{\partial\tilde{t}_{n+v}} + 2\frac{\partial}{\partial\tilde{t}_n})F^{\beta,v} + \sum_{s=1}^{n-1}\frac{\partial}{\partial\tilde{t}_{n-s}}F^{\beta,v}\frac{\partial}{\partial\tilde{t}_s}F^{\beta,v} \tag{248}$$

$n = 0$

$$0 = \frac{(\beta^{-1}-1)}{N} + 1 + (\frac{\beta^{-1}}{N^2}\sum_{k=1}^{\infty}\tilde{t}_k\frac{\partial}{\partial\tilde{t}_k} - \frac{1}{a_v\beta}\frac{\partial}{\partial\tilde{t}_v})F^{\beta,v} \tag{249}$$

$n = -1$

$$0 = \frac{\tilde{t}_1\beta^{-1}}{N^2} + (\frac{\beta^{-1}}{N^2}\sum_{k=2}^{\infty}\tilde{t}_k k\frac{\partial}{\partial\tilde{t}_{k-1}} - \frac{1}{a_v\beta}\frac{\partial}{\partial\tilde{t}_{v-1}})F^{\beta,v} \tag{250}$$

And again take the ansatz that we can order $F^{\beta,v}$ in terms of its powers of N and like for the quadratic β-deformed model, we let this power be any negative integer or zero:

$$F^{\beta,v} = \sum_{i=0}\frac{F_i^{\beta,v}}{N^i} \tag{251}$$

This way we can solve the Virasoro constraints for every power of N individually, which leads to this expression, very similar to what we had before:

$n > 0$:

$$\begin{aligned}0 = (n+1)(\beta^{-1}-1)\frac{\partial}{\partial\tilde{t}_n}F_{i-1}^{\beta,v} + \frac{1}{\beta}\sum_{k=1}^{\infty}\tilde{t}_k k\frac{\partial}{\partial\tilde{t}_{k+n}}F_{i-2}^{\beta,v} + \sum_{k+l=i}\sum_{s=1}^{n-1}\frac{\partial}{\partial\tilde{t}_{n-s}}F_k^{\beta,v}\frac{\partial}{\partial\tilde{t}_s}F_l^{\beta,v} \\ +(\sum_{s=1}^{n-1}\frac{\partial^2}{\partial\tilde{t}_{n-s}\partial\tilde{t}_s} - \frac{1}{a_v\beta}\frac{\partial}{\partial\tilde{t}_{n+v}} + 2\frac{\partial}{\partial\tilde{t}_n})F_i^{\beta,v}\end{aligned} \tag{252}$$

$n = 0$:

$$0 = -\frac{1}{a_v \beta}\frac{\partial}{\partial \tilde{t}_v} F_i^{\beta,v} + \frac{1}{\beta}\sum_{k=1}^{\infty} \tilde{t}_k k \frac{\partial}{\partial \tilde{t}_k} F_{i-2}^{\beta,v} + 1(i=0) + (\beta^{-1}-1)(i=1) \tag{253}$$

$n = -1$:

$$0 = -\frac{1}{a_v \beta}\frac{\partial}{\partial \tilde{t}_{v-1}} F_i^{\beta,v} + \frac{1}{\beta}\sum_{k=2}^{\infty} k\tilde{t}_k \frac{\partial}{\partial \tilde{t}_{k-1}} F_{i-2}^{\beta,v} + \frac{\tilde{t}_1}{\beta}(i=2) \tag{254}$$

Because the Virasoro constraint barely change, it seems fair to assume a very similar solution, and this is my proposal:

$$F^{\beta,v} = \sum_{n=1,\bar{g}=0}^{\infty} \frac{N^{2-2n-\bar{g}}\beta^{1-n-\bar{g}}}{n!} \sum_{k_1,\ldots,k_n=1}^{\infty} (a_v\beta)^{\sum_i k_i/v} (\sum_{l=0}^{\bar{g}} (-\beta)^l c_{k_1,\ldots,k_n}^{(\bar{g}-l,l)}) \prod_{i=1}^{n} \tilde{t}_{k_i} \tag{255}$$

This leads to recurrence relations similar to what we had before:

$$\begin{aligned} c_{n+v,i_1\ldots i_p}^{\bar{g}-l,l} = 2c_{n,i_1\ldots i_p}^{\bar{g}-l,l} + \sum_{j=1}^{p} i_j c_{i_1\ldots n+i_j,\ldots i_p}^{\bar{g}-l,l} + \sum_{\substack{I_1\cup I_2 = i_1\ldots i_p \\ I_1\cap I_2=\emptyset}} \sum_{(a,b)+(c,d)=(k,j)} \sum_{s=1}^{n-1} c_{n-s,I_1}^{a,b} c_{s,I_2}^{c,d} + \\ +(n+1)(c_{n,i_1\ldots i_p}^{k-1,j} + c_{n,i_1\ldots i_p}^{k,j-1}) - \sum_{s=1}^{n-1} c_{n-s,s,i_1\ldots i_p}^{k-1,j-1} \end{aligned} \tag{256}$$

Comparing this to (234) what we found in the last section, we see that merely the step size between the coefficients has changed, has been generalized. For $v = 2$, we retrieve (234). Of course this is not the whole story the other relations, which give the initial values, are:

for $n = 0$

$$c_{v,i_1,\ldots,i_p}^{(\bar{g}-l,l)} = \sum_{j=1}^{p} i_j c_{i_1,\ldots,i_p}^{(\bar{g}-l,l)} \tag{257}$$

and for $n = -1$

$$c_{v-1,i_1,\ldots,i_p}^{(\bar{g}-l,l)} = \sum_{j=1}^{p} i_j c_{i_1,\ldots,i_j-1,\ldots i_p}^{(\bar{g}-l,l)} \tag{258}$$

What we immediately notice is that for larger v, more initial values would be needed, because v is the step size of the recurrence. The Virasoro constraints however only give us two initial values, this way, our model is not uniquely defined. In order to solve for these cases, more information, more initial values would be needed.

An example: Consider $v = 3$. And for simplicity take $p = 0$ $\bar{g} = 0$, the relations found are the following:

$$c_{n+3}^{0,0} = 2c_n^{0,0} + \sum_{s=1}^{n-1} c_{n-s}^{0,0} c_s^{0,0} \tag{259}$$

This looks just like the Catalan numbers, except for the step size. for $n = 0$ and for $n = -1$

$$c_3^{0,0} = 1 \qquad c_2^{0,0} = 0 \tag{260}$$

Even the initial values found, are the same what we had before. Because the step size is larger, these two values are not enough. There is no initial value for $c_1^{(0,0)}$, and without it this is not really a recurrence relation.

4.3.1 Linear Case

Because the only problem in the last section, was the lack of information, due to the large step size, consider a smaller step size, namely $v = 1$, which leads to a linear model. This model we can also solve uniquely. The calculations are simply the ones from above, with $v = 1$. Note that in this model the Virasoro constraint $n = -1$ does not exist, because t_0 does not exist. We therefore only get one initial value, but we also only need one. Consider a few examples:

For $n = 1$ and $\bar{g} = 0$, we again find the Catalan numbers:

$$2c_n^{(0,0)} + \sum_{s=1}^{n-1} c_{n-s}^{(0,0)} c_s^{(0,0)} = c_{n+1}^{(0,0)} \tag{261}$$

With initial value $n = 0$.

$$c_1^{(0,0)} = 1 \tag{262}$$

In this $v = 1$ case, odd indices do not lead the coefficient to be zero. Remember for $v = 2$ the indices needed to sum up to an even number for the coefficients not to be zero. For $v = 1$ this is not the case. In the undeformed $v = 2$ setting, this property was intuitively understood in terms of valencies of graphs that needed to be connected. This was extensively explained in section 1.5.

For $n = 1$, $\bar{g} = 1$ we find the same relation we found for $v = 2$, $n = 1$, $\bar{g} = 1$, in section 4:

$$(n+1)c_n^{(0,0)} + 2c_n^{(1,0)} + 2\sum_{s=1}^{n-1} c_{n-s}^{(0,0)} c_s^{(1,0)} = c_{n+1}^{(1,0)} \tag{263}$$

With the initial value:

$$c_1^{(1,0)} = 1 \tag{264}$$

Remember this sequence was of the form: $c^{(1,0)} = 1, 5, 22, 93, ...$

Next consider $n = 2$, $\bar{g} = 0$, where we finally find coefficients that differ from the ones found in the quadratic case:

$$c_{k,n+1}^{(0,0)} = 2c_{k,n}^{(0,0)} + kc_{n+k}^{(0,0)} + \sum_{s=1}^{n-1} c_{n-s}^{(0,0)} c_{k,s}^{(0,0)} \tag{265}$$

with the initial value given, by $n = 0$:

$$c_{1,k}^{(0,0)} = kc_k^{(0,0)} = \frac{k}{k+1}\frac{(2k)!}{k!k!} = \begin{pmatrix} 2k \\ k-1 \end{pmatrix} \tag{266}$$

The first few coefficients evaluate to:

$$\begin{aligned} c_{1,1}^{(0,0)} &= 1 \\ c_{1,2}^{(0,0)} &= 4 \\ c_{1,3}^{(0,0)} &= 15 \\ c_{2,2}^{(0,0)} &= 18 \end{aligned} \tag{267}$$

And with this we let the calculations rest.

There are many more interesting things to explore, but this is as far as this master thesis goes. How far the Virasoro Constraints carried us is quite impressive and motivating to continue research in this direction.

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