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A Study of Smooth Functions and Differential Equations on Fractals

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Abstract

Pelander, A. 2007. A Study of Smooth Functions and Differential Equations on Fractals. *Uppsala Dissertations in Mathematics* 47. 39 pp. Uppsala. ISBN 978-91-506-1920-1.

In 1989 Jun Kigami made an analytic construction of a Laplacian on the Sierpiński gasket, a construction that he extended to post critically finite fractals. Since then, this field has evolved into a proper theory of analysis on fractals. The new results obtained in this thesis are all in the setting of Kigami's theory. They are presented in three papers.

Strichartz recently showed that there are first order linear differential equations, based on the Laplacian, that are not solvable on the Sierpiński gasket. In the first paper we give a characterization on the polynomial p so that the differential equation p(Δ)u=f is solvable on any open subset of the Sierpiński gasket for any f continuous on that subset. For general p we find the open subsets on which p(Δ)u=f is solvable for any continuous f.

In the second paper we describe the infinitesimal geometric behavior of a large class of smooth functions on the Sierpiński gasket in terms of the limit distribution of their local eccentricity, a generalized direction of gradient. The distribution of eccentricities is codified as an infinite dimensional perturbation problem for a suitable iterated function system, which has the limit distribution as an invariant measure. We extend results for harmonic functions found by Öberg, Strichartz and Yingst to larger classes of functions.

In the third paper we define and study intrinsic first order derivatives on post critically finite fractals and prove differentiability almost everywhere for certain classes of fractals and functions. We apply our results to extend the geography is destiny principle, and also obtain results on the pointwise behavior of local eccentricities. Our main tool is the Furstenberg-Kesten theory of products of random matrices.

Keywords: Analysis on fractals, p.c.f. fractals, Sierpinski gasket, Laplacian, differential equations on fractals, infinite dimensional i.f.s., invariant measure, harmonic functions, smooth functions, derivatives, products of random matrices

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List of Papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I	Pelander, A. Solvability of differential equations on open subsets of the Sierpiński gasket. To appear in *Journal d'Analyse Mathématique.*
II	Pelander, A., Teplyaev, A. Infinite dimensional i.f.s. and smooth functions on the Sierpński gasket. To appear in *Indiana University Mathematics Journal.*
III	Pelander, A., Teplyaev, A. Products of random matrices and derivatives on p.c.f. fractals. Submitted.

Contents

1. Introduction

In 1989 Jun Kigami presented an analytic construction of a Laplacian on the Sierpiński gasket, a construction that he extended to post critically finite fractals that constitute a large class of so-called finitely ramified fractals [16, 17]. All new results in the thesis are in the setting of Kigami's theory. In this introduction a background and necessary preliminaries are given.

1.1 Background

The first examples of sets with fractal structure were for a long time considered to be of interest to mathematicians only. It was not believed that they could resemble anything in the real world. With the works of Mandelbrot, who actually introduced the term fractal [24], it was recognized that many physical objects should be modeled by sets with fractal properties rather than by smooth sets. This, together with the discovered connections to dynamical systems, lead to an increase in the study of fractal sets, and the emergence of fractal geometry as a proper branch of mathematics.

From a mathematical point of view the mere existence of sets with a different kind of geometric properties is enough motivation to study (classes of) functions defined on them. But to understand physical phenomena on objects modeled by fractals, a theory for the geometry of fractals clearly is not sufficient. It will be necessary to do some kind of analysis on fractals. Therefore it is not surprising that the theory to which this thesis belongs has some of its roots in the works of physicists.

In the early eighties it had become apparent to physicists that porous as well as highly disordered media exhibit anomalous diffusive, conductive and vibrational properties. In the theoretical study of these matters some very interesting mathematical models of diffusion on fractals were made.

Inspired by their work, the first rigorous constructions of Brownian motion as a scaled limit of random walks on approximating graphs of the Sierpiński gasket were made independently by Kusuoka [22] and Goldstein [10]. Other pioneering works in this direction were done by Barlow and Perkins [4] and Lindstrøm [23]. In these probabilistic constructions the Laplacian is obtained via the diffusion process.

The analytic construction [16, 17] appeared only shortly after the works of Kusuoka and Goldstein. Kigami gives two equivalent definitions of the Laplacian; a weak definition via an energy form $\mathcal{E}$, through $\Delta u = f$ if

$$\mathcal{E}(u,v) = -\int_F f v d\mu, \tag{1.1}$$

for all v in an appropriate class of test functions, and a pointwise definition as a renormalized limit of discrete difference operators Δ_m.

Both these approaches towards a theory of analysis on fractals have undergone a strong development since the original works. Roughly speaking, the strength of the probabilistic approach has been the extension to greater generality in terms of the underlying set, while the strength of the analytic approach has been the construction of fractal analogues of a great variety of concepts and results from classical analysis.

Since the probabilistic approach will not be considered here we just mention that already in 1989 Barlow and Bass [3] had extended this construction to the Sierpiński carpet. The Sierpiński carpet is an example of an infinitely ramified fractal not (yet) in the scope of the analytic approach. There is a quick introduction to the probabilistic approach in [8, Section 12.4], whereas a detailed account can be found in [2] and overviews of recent developments in [20, 21].

We will describe the analytic construction in the following sections. The general theory in full detail can be found in Kigami's book [19]. The recent book of Strichartz [32] is also a good introduction to the subject, working with the standard construction on the Sierpiński gasket before the general case. This book also covers most aspects of the rapid development in the last few years.

As mentioned, the advantage of Kigami's theory has been the possibility to do as much analysis as possible on some fractal set, instead of doing some analysis on as many fractal sets as possible, which is the advantage of the probabilistic methods. In this spirit, much work in the analytic theory has been done on the Sierpiński gasket only. The Sierpiński gasket has become the prototype for p.c.f. fractals and the restriction is often not done by necessity, but rather in the aim of clarity. Working on the Sierpiński gasket one will have well-founded reasons on what to expect in general.

There has been other developments towards an analysis on fractal sets, or sets with fractal boundary. In [25] there are references to some of these, as well as an interesting exposition on the connections to the work of physicists mentioned above.

An important feature of Kigami's theory is its intrinsic approach. The derived analytical properties are only dependent of the fractal itself, not its embedding in Euclidean space. For instance the analytic structure of the von Koch curve is not different from that of any closed interval. Functions with some degree of regularity, such as harmonic functions or functions in the domain of the Laplacian, will not necessarily be restrictions of 'nice' functions on Euclidean space. This is in contrast to the approach in the books of Jonsson and Wallin [15] and Triebel [35], who all consider restrictions of function spaces in $\mathbb{R}^n$ to fractal subsets.

In Section 1.2 we introduce self-similar sets, and then give the definition of post critically finite (p.c.f.) fractals. We also give examples of p.c.f. fractals, and self-similar sets that are not p.c.f. fractals. Some notation and other general terminology is also introduced. It should be remarked that notation is fixed separately in each of the subsequent papers. These notations differ in some cases to the one used in the introduction. We have aimed to follow the notation in paper III, since this is the paper where general classes of fractals are considered.

Section 1.3 contains an outline of the construction of the energy form that is used in Section 1.4 to define the Laplacian. Harmonic functions are also defined in this section.

The two equivalent definitions of the Laplacian are stated in Section 1.4. This section also includes the Green's function and operator, the normal derivative and Gauss–Green formula. We end Section 1.4 with two striking results of the theory. The domain of the Laplacian is not a multiplicative domain, and, appropriately chosen, partial sums of Fourier series converge.

1.2 Post critically finite fractals

There is no generally accepted mathematical definition of a fractal set. Although one usually thinks of fractals as sets with topological dimension strictly less than their Hausdorff dimension, this has turned out not to be a satisfactory definition, leaving out many sets with fractal properties. However, for any theory of analysis on fractals it is of course necessary to precisely define what properties the underlying set is assumed to have.

It will turn out that the unit interval is an example of a p.c.f. fractal. But, from a geometric point of view, one would like to distinguish fractal sets from smooth ones. However, from an analytic point of view, this is satisfactory since the Laplacian obtained by Kigami's construction on the unit interval coincides with the usual second order derivative. We will also see that the analytic properties of p.c.f fractals have a 'one dimensional flavor' in many ways. In the first chapters of [32] the details of the construction on the unit interval is included together with that on the Sierpiński gasket.

Self-similar sets are sets with a cellular structure, where each cell is homeomorphic to the entire set. Formally, self-similar sets are quotients of topological Cantor spaces.

Let W_m^S denote the space of finite words $w = w_1, \dots, w_m$ of length m of symbols from a finite set S and let $W_*^S = \cup_{m \geq 1} W_m^S$. For any finite number of mappings ψ_i, $i \in S$ we will for $w \in W_*^S$ denote by ψ_w the composition

$$\psi_w = \psi_{w_1} \circ \dots \circ \psi_{w_m}. \tag{1.2}$$

Let $\Omega^S = S^{\mathbb{N}}$ the space of infinite sequences of elements in S, and for any $\omega = \omega_1\omega_2\ldots \in \Omega^S$ let $[\omega]_m = \omega_1\ldots\omega_m \in W_m^S$.

Define a metric on Ω^S by $\delta(\omega,\tau) = 2^{-s(\omega,\tau)}$, for $\omega \neq \tau$, where $s(\omega,\tau) = \min\{m \mid [\omega]_m \neq [\tau]_m\}$, and $\delta(\omega,\tau) = 0$ if $\omega = \tau$. The space (Ω^S,δ) is compact. There are several different metrics that induces the same topology and Ω^S with this topology is called a topological Cantor space. In part II and III it is implicitly assumed that Ω^S is equipped with this topology. For ease of notation we will omit the superscript S in what follows.

Definition (Self-similar set)**.** Let F be a compact metrizable topological space, and suppose there are continuous injections ψ_i, $i \in S$, from F to itself. For $k \in S$, let σ_k denote the mapping on Ω given by $\sigma_k(\omega_1\omega_2\ldots) = k\omega_1\omega_2\ldots$. Then $(F,\{\psi_i\}_{i\in S})$ is a self-similar structure if there is a continuous surjection $\pi : \Omega \to F$ such that $\psi_i \circ \pi = \pi \circ \sigma_i$. The set F will be called a self-similar set.

The surjection π is uniquely defined, and is given by

$$\pi(\omega) = \cap_{m\geq 1} F_{[\omega]_m}, \tag{1.3}$$

where $F_{[\omega]_m} = \psi_{[\omega]_m}(F)$. The sets F_w, $w \in W_m$ are called *cells of level m*, or *m-cells*.

It is of course natural to assume that the underlying sets in our analysis are connected. A self-similar set is connected if you can 'walk' between any pair of cells of level 1.

Theorem ([13],[19] Theorem 1.6.2)**.** *A self-similar set F is connected if and only if for any $i, j \in S$ there exists $\{i_k\}_{k=1}^n \subseteq S$ such that*

$$F_{i_k} \cap F_{i_{k+1}} \neq \varnothing. \tag{1.4}$$

□

The abstract definition of self-similar sets above is motivated by the following theorem that states that invariant sets of iterated function systems (i.f.s.) of contractions are self-similar. This provides a rich source of examples.

Theorem ([14],[19] Theorem 1.1.4, Theorem 1.2.3)**.** *Let (X,d) be a metric space and ψ_i, $i = 1,\ldots,N$ contractions on X with respect to the metric d. Then there exists a unique non-empty compact set $F \subset X$ such that*

$$F = \psi_1(F) \cup \ldots \cup \psi_N(F). \tag{1.5}$$

Moreover, $(F,\{\psi_i\}_{i=1}^N)$ is a self-similar structure with $\pi : \Omega \to F$ given by (1.3).

□

As mentioned above p.c.f. fractals are *finitely ramified*. This means that they are just barely connected in the sense that they can be disconnected by removing only a finite number of points. Actually, a self-similar set F is finitely ramified if the intersection, $F_i \cap F_j$, $i \neq j$, of any two different cells of level 1 is at most finite. So one can 'cut out' a 1-cell from a finitely ramified fractal by removing the finite number of points intersecting other 1-cells.

Not every finitely ramified fractal is p.c.f. The property that singles out p.c.f. fractals is that $\pi^{-1}\{x\}$ is finite for any $x \in F$. Points will only have a finite number of different 'addresses' in Ω. The geometric interpretation is that for any point x there is a number M such that x will lie in at most M different m-cells for any m. The formal definition is as follows.

Definition (Post critically finite fractal)**.** Let $(F, \{\psi_i\}_{i\in S})$ be a self-similar structure. The critical set is

$$\mathcal{C} = \pi^{-1}(\cup_{i,j\in S, i\neq j} F_i \cap F_j), \tag{1.6}$$

and the post critical set is

$$\mathcal{P} = \cup_{n\geq 1}\sigma^n(\mathcal{C}), \tag{1.7}$$

where σ is the shift map on Ω, i.e. $\sigma(\omega_1\omega_2\omega_3\ldots) = \omega_2\omega_3\omega_4\ldots$. The self-similar structure $(F, \{\psi_i\}_{i\in S})$ is post critically finite, p.c.f., if the post critical set $\mathcal{P}$ is finite. The set F will be called a p.c.f. fractal.

The boundary of F is defined as the set $V_0 = \pi(\mathcal{P})$. If one assumes that every boundary point is the fixed point of one of the mappings ψ_i, then the p.c.f. assumption implies that the boundary points lie in only one 1-cell.

Example 1.2.1 (Interval)**.** Define two mappings on $\mathbb{R}$ by, $\psi_0(x) = \frac{x}{2}$ and $\psi_1(x) = \frac{1}{2}(x+1)$. Then $([0,1], \{\psi_i\}_{i=0}^1)$ is a p.c.f. self-similar structure with boundary $V_0 = \{0,1\}$.

Example 1.2.2 (Sierpiński gasket)**.** Let q_0, q_1, q_2 be the vertices of an equilateral triangle in $\mathbb{R}^2$. Define three contractions on $\mathbb{R}^2$ by $\psi_i(x) = \frac{1}{2}(x-q_i)+q_i$, $i = 0,1,2$. The invariant set of the three homotheties ψ_i is called the Sierpiński gasket (SG), see Figure 1.1, and $(SG, \{\psi_i\}_{i=0}^2)$ is a p.c.f. self-similar structure with $V_0 = \{q_i\}_{i=0}^2$.

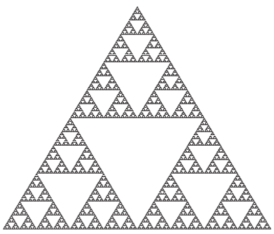

Figure 1.1: Sierpiński gasket

Example 1.2.3 (Level-3 Sierpiński gasket)**.** Let once again q_0, q_1, q_2 be the vertices of an equilateral triangle in $\mathbb{R}^2$. This time define six different contractions on $\mathbb{R}^2$ with contraction ratio $\frac{1}{3}$ and fix points being the boundary points q_i or the midpoints of the edges $\frac{q_i+q_j}{2}$, $i \neq j$. The invariant set of these contractions is called the level-3 Sierpiński gasket (SG_3), see Figure 1.2, and is a p.c.f. self-similar structure with $V_0 = \{q_i\}_{i=0}^2$.

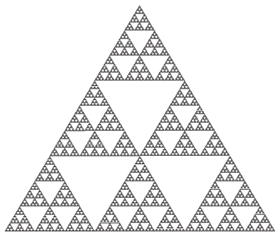

Figure 1.2: Level-3 Sierpiński gasket

Example 1.2.4 (Level-n Sierpiński gasket)**.** The construction in the previous example can naturally be generalized to any level, using $\frac{n(n+1)}{2}$ contractions with contraction ratio $\frac{1}{n}$.

Example 1.2.5 (Hexagasket)**.** Let $q_1, \dots, q_6$ be the vertices of a regular hexagon in $\mathbb{R}^2$ and let ψ_i, $i = 1, \dots, 6$, be the homotheties with contraction ratio $\frac{1}{3}$ and fixed point q_i. The invariant set of $\{\psi_i\}_{i=1}^6$ is called the hexagasket (H), see Figure 1.3, and $(H, \{\psi_i\}_{i=1}^6)$ is a p.c.f. self-similar structure with boundary $V_0 = \{q_i\}_{i=1}^6$.

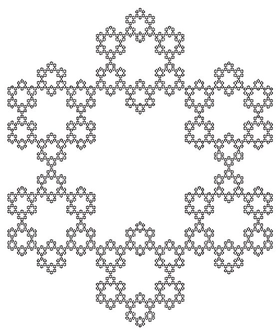

Figure 1.3: Hexagasket

Example 1.2.6 (Polygasket)**.** The construction of the hexagasket can be done for n not divisible by 4 starting from any regular n-gon and with appropriately chosen contraction ratio. The boundary of the polygasket will be the vertices of the regular n-gon. Figure 1.4 shows the pentagasket, that is obtained for $n = 5$ in this construction.

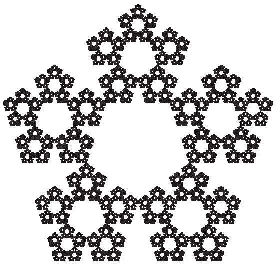

Figure 1.4: Pentagasket

Example 1.2.7 (Polygasket with three boundary points)**.** Composing one of the homotheties in the construction of polygaskets by rotations of the angles $\frac{2\pi j}{n}$, $j = 0, \dots, n-1$, the same invariant set as in the previous example is obtained but the boundary of the p.c.f. self-similar structure consists of only three points.

The next examples of self-similar sets are *not* p.c.f..

Example 1.2.8 (Square)**.** Let p_1, p_2, p_3, p_4, be the vertices of a square Q in $\mathbb{R}^2$. Let ψ_i, $i = 1,2,3,4$ be homotheties with contraction ratio $\frac{1}{2}$ and fix point p_i. Then $(Q, \{\psi_i\}_{i=1}^4)$ is a self-similar structure that is not p.c.f..

Example 1.2.9 (Sierpiński carpet)**.** Let once again p_1, p_2, p_3, p_4, be the vertices of a square Q in $\mathbb{R}^2$. Define eight homotheties ψ_i, $i = 1, \dots, 8$ with contraction ratios $\frac{1}{3}$ and fix point p_i, for $i = 1,2,3,4$ and for $i = 5,6,7,8$ let the fix point be the midpoint of each of the edges of Q. The invariant set of $\{\psi_i\}_{i=1}^8$ is called the Sierpiński carpet (SC), see Figure 1.5, and $(SC, \{\psi_i\}_{i=1}^8)$ is a self-similar structure that is not p.c.f..

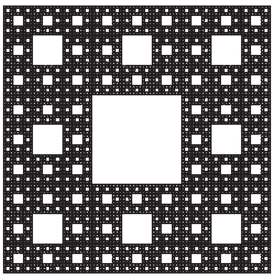

Figure 1.5: Sierpiński carpet

It is clear that both the square and the Sierpiński carpet are infinitely ramified since non-disjoint cells intersect along line segments. It is immediate from the definitions that any p.c.f. fractal is finitely ramified, but the converse is not true as shows the next example.

Example 1.2.10 (Post critically infinite Sierpiński gasket)**.** In [34, Example 8.9] it is given an example of an invariant set of nine contractions that is finitely ramified but not p.c.f., see Figure 1.6.

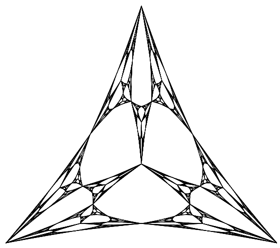

Figure 1.6: The post-critically infinite Sierpiński gasket in harmonic coordinates.

We end this section with a discussion of other types of sets to which Kigami's analytic construction has been extended.

The self-similarity condition can be removed. What is important is rather the cellular structure. Work has been done both for *hierarchical gaskets* [11, 12, 7] (see also [32, Section 4.6]) and for *fractafolds* [28].

Hierarchical gaskets uses the inductive construction of the level-n Sierpiński gaskets, starting from an equilateral triangle and removing in each step $\frac{n(n-1)}{2}$ triangles. Instead of doing the same iteration in each step, one choose one of the possible constructions for each of the cells of the same level. Self-similarity is lost but finite ramification is kept.

Fractafolds are, as the name indicates, fractal manifolds. They are constructed by gluing together p.c.f. fractals at the boundary points. The perhaps simplest example of a fractafold without boundary is the double cover of the Sierpiński gasket which is constructed by pairwise identifying the boundary points of two distinct copies of the Sierpiński gasket. Considering the Sierpiński gasket as the simplest fractal analogue of the interval, its double cover is the simplest fractal analogue of the circle.

The most difficult condition to remove in extending the analytic construction to larger classes of sets has turned out to be the finite ramification. The only progress in this direction is the extension to products of p.c.f. fractals [29].

1.3 Energy and harmonic structures

Throughout this section F is a connected p.c.f. fractal, with self-similar structure induced by mappings ψ_i, $i = 1, \dots N$.

The central idea in Kigami's construction is the definition of an energy form on F as a limit of discrete energy forms on approximating graphs, defined as follows.

Let $V_w = \psi_w(V_0)$ for any $w \in W_*$ and define $V_m = \cup_{|w|=m} V_w$, $V_* = \cup_{m=0}^{\infty} V_m$. Define, inductively, graphs Γ_m with vertices V_m and edge relations $x \sim_m y$ by letting Γ_0 be the complete graph on V_0 and $x \sim_m y$ for $m > 0$ if and only if there exists i such that $x = \psi_i(x')$, $y = \psi_i(y')$ and $x' \sim_{m-1} y'$.

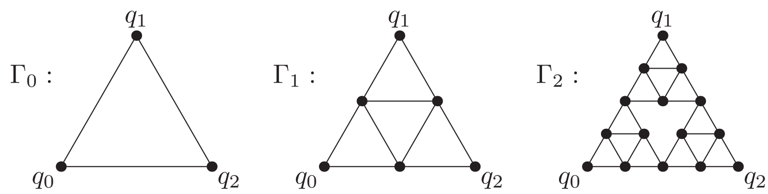

Figure 1.7: The first three graphs approximating the Sierpiński gasket.

The discrete energy forms are given by,

$$\mathcal{E}_m(u,v) = \sum_{x \sim_m y} c(x,y)(u(x) - u(y))(v(x) - v(y)) \tag{1.8}$$

for functions u and v defined on V_m. To obtain the required energy form on F it is necessary that the conductances $c(x,y)$ can be chosen so that $\mathcal{E}_m$ has certain properties.

First of all, the energy forms has to respect the self-similarity in the sense that there are resistance scaling factors $r_i > 0$, $i = 1, \dots, N$ such that

$$\mathcal{E}_m(u,v) = \sum_{i=1}^{N} \frac{1}{r_i} \mathcal{E}_{m-1}(u_i, v_i) = \sum_{w \in W_m} \frac{1}{r_w} \mathcal{E}_0(u_w, v_w), \tag{1.9}$$

for $u_w = u \circ \psi_w$ and $r_w = r_{w_n} \cdots r_{w_1}$. Hence, once r_i and $c(x,y)$ for x and y in V_0 are chosen, $\mathcal{E}_m$ will be defined for all m through (1.9).

The second requirement is that the sequence $\mathcal{E}_m$ must be compatible in the sense that for any function u defined on V_{m-1} and $\tilde{u}$ minimizing $\mathcal{E}_m$ among all extensions of u to V_m, we have $\mathcal{E}_m(\tilde{u}, \tilde{u}) = \mathcal{E}_{m-1}(u,u)$.

The last condition is that all $c(x,y)$ should be positive. Allowing some of the conductances $c(x,y)$ to be zero would yield degenerate energies for which some non-constant functions have zero energy.

Any valid choice of $c(x,y) > 0$ and r_i, $i = 1, \dots N$, is called a *harmonic structure* on F. It is not known whether there always exists a harmonic structure on F, but there are many cases when it is known to do so. Lindstrøm [23] showed existence for a class of *nested fractals* and Kigami [19, Theorem 3.8.10] extended this result to a class of *strongly symmetric* p.c.f. fractals. Examples of a few particular harmonic structures are given below.

We will not be occupied with the issue of existence and uniqueness of harmonic structures. Therefore we only mention that the problem can be reformulated to an eigenvalue problem for a non-linear operator and the condition that all conductances must be positive makes even the existence of solutions a very difficult problem. Important work on this so-called *renormalization problem* has been made by many authors. References can be found in [32, Section 4.7].

Example 1.3.1. Choosing all $c(x,y) = 1$ and all r_i equal, a harmonic structure is obtained in Examples 1.2.1-1.2.3 and 1.2.5 if, for the interval $r_i = \frac{1}{2}$, for the Sierpiński gasket $r_i = \frac{3}{5}$, for the level-3 Sierpiński gasket $r_i = \frac{7}{15}$, and for the hexagasket $r_i = \frac{3}{7}$.

We will make the additional assumption on the harmonic structures that all $r_i < 1$. Such harmonic structures are called *regular*. Note that all harmonic structures in Example 1.3.1 are regular. This is always the case when all r_i are equal [19, Corollary 3.1.9]. In paper I-III we only work with regular harmonic structures. For non-regular harmonic structures some things do not behave as nicely as in the regular case. For instance, continuity is lost for the Green's function and functions in the domain of the energy.

The energy forms $\mathcal{E}_m$ naturally induces an energy form $\mathcal{E}$ on F through

$$\mathcal{E}(u,v) = \lim_{m\to\infty} \mathcal{E}_m(u|_{V_m}, v|_{V_m}), \tag{1.10}$$

since the right hand side of (1.10) form an increasing sequence by the compatibility condition. The *energy* of a function is the extended real number $\mathcal{E}(u) = \mathcal{E}(u,u)$. The domain of the energy form, $\operatorname{Dom}\mathcal{E}$, consists of the functions with finite energy, and functions in $\operatorname{Dom}\mathcal{E}$ are continuous under the regularity assumption. The only functions with zero energy are the constant functions and $\operatorname{Dom}\mathcal{E}$ modulo constants is a Hilbert space with $\mathcal{E}$ as inner product.

Harmonic functions are defined as energy minimizers with respect to boundary values.

Definition 1.3.2. A function $h : F \to \mathbb{R}$ is harmonic if $\mathcal{E}(h) = \mathcal{E}_0(h)$. The space of harmonic functions is denoted by $\mathcal{H}$.

If h is harmonic, then $\Delta_m h = 0$ where Δ_m is the discrete Laplacian defined by

$$\Delta_m u(x) = \sum_{y\sim_m x} c(x,y)(u(y) - u(x)). \tag{1.11}$$

Given any set of boundary values there is a unique harmonic function attaining these and $\mathcal{H}$ forms a linear space. Consequently the dimension of $\mathcal{H}$ is $N_0 = \#V_0$. A function h is a *harmonic spline* of level m if any restriction $h \circ \psi_w$, $w \in W_m$ to a cell of level m is harmonic. A harmonic spline of level m is uniquely defined by its values on V_m.

The restriction of a harmonic function to a cell of level-1 depends linearly on its boundary values. This induces *harmonic extension* mappings A_i, $i = 1,\ldots,N$ on $\mathcal{H}$ defined through

$$A_i h = h \circ \psi_i. \tag{1.12}$$

Restrictions to smaller cells are given by products of these, $A_w h = h \circ \psi_w$, where $A_w = A_{w_n} \dots A_{w_1}$. Notice the order of the matrices in the product.

The harmonic extension mappings are central in papers II and III. In paper III we will also use the analogy between harmonic functions on p.c.f. fractals and affine linear functions on the unit interval, which in that case are exactly the harmonic functions.

1.4 The Laplacian

In this section we continue to let F be a connected p.c.f. fractal, with self-similar structure induced by mappings ψ_i, $i = 1, \dots N$, and assume that F is equipped with a harmonic structure with notation from the previous section. We will not give specific reference to the basic results of the theory, they can all be found in [19, chapter 3] and [32, chapter 2 and 4].

The Laplacian on F is not uniquely defined by a fixed harmonic structure. There is one more degree of freedom, which is the measure chosen on F. Even though $\mathcal{H}$ does not depend on μ it turns out that, as desired, a function h is harmonic if and only if $\Delta_\mu h = 0$.

Definition (Laplacian)**.** Let μ be a finite non-atomic Borel measure on F such that $\mu(O) > 0$ for any open set O. Then we say that $u \in \operatorname{Dom} \Delta_\mu$, with $\Delta_\mu u = f$, if $f \in C(F)$ and

$$\mathcal{E}(u, v) = -\int_F f v d\mu, \tag{1.13}$$

for any $v \in \operatorname{Dom} \mathcal{E}$ vanishing on the boundary V_0.

For convenience we shall assume that μ is a probability measure. Choosing any other value of $\mu(F)$ only give rise to scaled versions.

Clearly there are lots of valid choices of μ. One important natural class of valid measures are the *self-similar measures*. These are measures for which any $m+1$-cell F_{wi} has the same relative weight in the m-cell F_w that the corresponding 1-cell F_i has in F.

Definition (Self-similar measure)**.** A non-atomic measure μ on F is self-similar if there are positive numbers $\mu_1, \dots, \mu_N$ such that $\sum_{i=1}^N \mu_i = 1$ and $\mu(F_w) = \mu_w$ for any $w \in W_*$, where $\mu_w = \mu_{w_1} \cdots \mu_{w_n}$. If all μ_i are equal, μ is called uniform self-similar measure.

In what follows we will assume that the measure μ is self-similar.

If on the unit interval the harmonic structure of Example 1.3.1 is used and μ is uniform self-similar measure, then $\operatorname{Dom}(\Delta_\mu) = C^2(I)$ and $\Delta_\mu u = \frac{d^2u}{dx^2}$. In paper I and II the Laplacian on the Sierpiński gasket obtained from the harmonic structure of Example 1.3.1 and uniform self-similar measure is considered. We call this the standard Laplacian on the Sierpiński gasket and omit the subscript μ.

We will make extensive use of the fact that it is possible to construct a continuous *Green's function* $g(x,y)$ defined on $F\times F$ that gives rise to a *Green's operator*,

$$Gf(y)=\int_F g(x,y)f(y)d\mu(y). \tag{1.14}$$

The Green's operator gives a unique solution to the Dirichlet problem for any continuous f, i.e.,

$$-\Delta_\mu Gf=f \quad \text{and} \quad Gf|_{V_0}=0. \tag{1.15}$$

This means that $\operatorname{Dom}\Delta_\mu$ really contains all the functions one could ask for.

The pointwise definition of the Laplacian uses the graph Laplacians Δ_m on $V_m\setminus V_*$ defined in (1.11). For $x\in V_m$ denote by $h_x^{(m)}$ the harmonic spline of level m such that $h_x^{(m)}(x)=1$ and $h_x^{(m)}(y)=0$ for any other $y\in V_m$. Then for $u\in\operatorname{Dom}\Delta_\mu$ and $x\in V_*\setminus V_0$,

$$\Delta_\mu u(x)=\lim_{m\to\infty}\left(\int_F h_x^{(m)}d\mu\right)^{-1}\Delta_m u(x), \tag{1.16}$$

and the convergence is uniform on $V_*\setminus V_0$. Also, if u is continuous and the right hand side of (1.16) converges uniformly to a continuous function f on $V_*\setminus V_0$, then $u\in\operatorname{Dom}\Delta_\mu$ with $\Delta_\mu u=f$. For the standard Laplacian on the Sierpiński gasket (1.16) becomes

$$\Delta u(x)=\frac{3}{2}\lim_{m\to\infty}5^m\sum_{y\sim_m x}(u(y)-u(x)). \tag{1.17}$$

It follows from (1.13), as well as (1.16), that the Laplacian satisfies the scaling identity $\Delta_\mu(u\circ\psi_i)=r_i\mu_i(\Delta_\mu u)\circ\psi_i$. For the standard Laplacian on the Sierpiński gasket this becomes $\Delta(u\circ\psi_w)=5^{-|w|}(\Delta u)\circ\psi_w$ for any $w\in W_*$, since $r_i=\frac{3}{5}$ and $\mu_i=\frac{1}{3}$. This scaling identity is apparent in the pointwise formula (1.17).

For any function $u\in\operatorname{Dom}\Delta_\mu$ there is a *normal (Neumann) derivative* $\partial_n u(q)$ defined at every boundary point through

$$\partial_n u(q)=\lim_{m\to\infty}\sum_{y\sim_m q}c(x,y)(u(q)-u(y)). \tag{1.18}$$

If every boundary point $q_i\in V_0=\{q_j\}_{j=1}^{N_0}$ is the fixed point of ψ_i then

$$\partial_n u(q_i)=\lim_{m\to\infty}r_i^{-m}\sum_{j\neq i}(u(q_i)-u(\psi_i(q_j))), \tag{1.19}$$

which for the standard Laplacian on the Sierpiński gasket becomes

$$\partial_n u(q_i)=\lim_{m\to\infty}\left(\frac{5}{3}\right)^m\sum_{j\neq i}(u(q_i)-u(\psi_i(q_j))). \tag{1.20}$$

On the unit interval $\partial_n u(0) = -u'(0)$ and $\partial_n u(1) = u'(1)$.

Note that the normal derivative only depends on the harmonic structure. For harmonic functions the right hand side of (1.18) does not depend on m.

There is a *Gauss–Green formula*, involving the normal derivative, that extends (1.13) to any $v \in \operatorname{Dom}\mathcal{E}$. For $u \in \operatorname{Dom}\Delta_\mu$ and $v \in \operatorname{Dom}\mathcal{E}$ we have

$$\mathcal{E}(u,v) = -\int_F v\Delta_\mu u d\mu + \sum_{q\in V_0} v(q)\partial_n u(q). \tag{1.21}$$

If both u and v are in $\operatorname{Dom}\Delta_\mu$ it follows, by using (1.21) twice, that

$$\int_F v\Delta_\mu u d\mu = \sum_{q\in V_0} (v(q)\partial_n u(q) - \partial_n v(q)u(q)) + \int_F u\Delta_\mu v d\mu, \tag{1.22}$$

which can be seen as a nice fractal analogue of iterated use of partial integration on the unit interval.

With the normal derivative one can also speak of Neumann boundary condition $\partial_n u(q) = 0$ for all $q \in V_0$. If $\int_F f d\mu = 0$, then the Neumann problem has a solution u, unique up to an additive constant, i.e.,

$$-\Delta_\mu u = f \quad \text{and} \quad \partial_n u(q) = 0 \quad \text{for every } q \in V_0. \tag{1.23}$$

Building a theory of analysis on fractals one would perhaps expect that things will always be slightly worse compared to classical analysis. We end this section with a discussion on two results. One on a case where things are much worse and then one really surprising result, where things are better than on the interval.

It is clear from the definitions that $\operatorname{Dom}\Delta_\mu$ is a linear space. What about multiplication? Is $\operatorname{Dom}\Delta_\mu$ a multiplicative domain? The answer is no. In fact it is not even close. Ben–Bassat, Strichartz and Teplyaev showed in [5] that, for a large number of fractals, $uv \notin \operatorname{Dom}\Delta_\mu$ for any non-constant $u, v \in \operatorname{Dom}\Delta_\mu$. This causes serious complications in developing a PDE theory.

To indicate what this fact stems from, we can look at the Sierpiński gasket. If $u \in \operatorname{Dom}\Delta_\mu$ and $\partial_n u(q) \neq 0$ one can show that, as is not surprising in view of (1.20), the variation of u on the m-cell neighboring q will decrease as $\left(\frac{3}{5}\right)^m$. If $\partial_n u(q) = 0$, then this variation is bounded by a constant times $\frac{m}{5^m}$. The variation of u^2 will not fit into any of these cases if $\partial_n u(q) \neq 0$, so $u^2 \notin \operatorname{Dom}\Delta_\mu$. If $\partial_n u(q) = 0$ the argument can be localized to a boundary point of one of it cells.

There are orthonormal bases of Dirichlet and Neumann eigenfunctions, so that there are fractal analogues of Fourier series. For instance, if $\{u_j\}_{j=1}^\infty$ is an orthonormal basis of Dirichlet eigenfunctions with non-decreasing eigenvalues λ_j, then for any $f \in L^2(F)$ we have

$$f = \sum_{j=1}^{\infty} c_j u_j, \tag{1.24}$$

where the Fourier coefficients are given by

$$c_j = \int_F f u_j d\mu. \tag{1.25}$$

The perhaps most surprising result of the theory concerns convergence of Fourier Series. In [30] Strichartz showed that, on the Sierpiński gasket, partial sums of Fourier series $\sum_{n=1}^{N_m} c_j u_j$, for an appropriate choice of $\{N_m\}$, converges uniformly to f for any continuous f. Thus, convergence properties of Fourier series are better than on the interval! This result depends on the existence of *large spectral gaps*, i.e., that there are sequences $\{N_m\}$ such that $\lambda_{N_{m+1}}/\lambda_{N_m} > c > 1$. The convergence result for Fourier series is likely to hold for many other fractals.

2. Summary of results

Sections 2.1-2.3 summarizes the results in paper I-III. We set some of the particular notations of the corresponding papers in these summaries. Hopefully this will help for clarity when passing from the summaries to detailed reading of the papers.

2.1 Summary of paper I

In paper I, a problem formulated by Strichartz in [31] concerning the solvability of differential equations on open subsets of the Sierpiński gasket is solved.

As the Laplacian is the basic differential operator of the theory, the term differential equation is used for equations involving the Laplacian. It was shown in [31] that for certain values of λ such simple differential equation as

$$-\Delta u - \lambda u = f \tag{2.1}$$

is not always solvable on the Sierpiński gasket. On the other hand, it was also shown that the equation

$$-\Delta u = f \tag{2.2}$$

is solvable on any open subset for any f continuous there.

Thus, it is natural to ask which restrictions on λ are necessary to have solvability on any open subset for (2.1) and conversely, given an equation of the form (2.1), on what open subsets can we solve it? Paper I contains the answer to both of these questions for arbitrary linear differential equations,

$$p(\Delta)u = f. \tag{2.3}$$

For the sake of clarity the result is first proved for (2.1) and the results for general equations (2.3) are given as corollaries. Before stating the precise formulation of these characterizations we need some further notation and background.

In paper I the Sierpiński gasket is denoted by K, with boundary $\{q_i\}_{i=0}^{2}$ and the three contractions inducing K are denoted by F_i, $i = 0,1,2$. Subscripts are used as in section 1.2 to indicate compositions of mappings, $F_w = F_{w_1} \circ \ldots \circ F_{w_n}$, and cells $K_w = F_w(K)$.

An important and surprising property of the Sierpiński gasket, and many other p.c.f. fractals, is the existence of joint Dirichlet and Neumann eigenfunctions. This property is certainly not shared with the unit interval where, although the Dirichlet and Neumann spectrum are the same, the eigenfunctions ($\sin \pi n x$ respectively $\cos \pi n x$) are always different.

Any joint eigenfunction f with eigenvalue λ gives rise to localized eigenfunctions f^w, with support on the cell K_w and eigenvalue $5^{|w|}\lambda$, through

$$f^w(y) = \begin{cases} f \circ F_w^{-1}(y) & y \in K_w \\ 0 & y \notin K_w. \end{cases} \tag{2.4}$$

As it turns out, the particular case when (2.1) is not solvable occurs when λ is a joint Dirichlet/Neumann eigenvalue and f is a joint Dirichlet/Neumann eigenfunction. Strichartz showed this by a careful analysis of these eigenfunctions, using their construction through *spectral decimation* [9]. Spectral decimation means, essentially, that restrictions of eigenfunctions to V_m are eigenfunctions of the discrete Laplacians Δ_m and that the eigenvalues can be obtained as renormalized limits of the corresponding eigenvalues of the discrete Laplacians. Spectral decimation is only valid for a restricted class of fractals [27].

It follows from the above that if $\lambda/5$ is a joint Dirichlet/Neumann eigenvalue then (2.1) cannot always be solved on an open subset Ω containing a 1-cell K_i. Let f be a joint eigenfunction with eigenvalue $\lambda/5$. Then, if (2.1), with the localized eigenfunction f^i on the right hand side, had a solution v on Ω, one get, using the scaling property of the Laplacian, the contradiction that $5v_i$ solves $-\Delta u - \lambda/5u = f$ on K. Thus, the following result is not surprising.

Theorem (Theorem 3 paper 1)**.** *If $\lambda/5$ is not a joint Dirichlet/Neumann eigenvalue, then* (2.1) *is solvable on any proper open subset of the Sierpiński gasket.*

From this a characterization of the polynomials for which (2.3) is solvable on open subsets follows.

Corollary (Corollary 3 paper I)**.** *The equation* (2.3) *is solvable on any proper open subset of the Sierpiński gasket if and only if $p(-5\lambda) \neq 0$ whenever λ is a joint Dirichlet/Neumann eigenvalue.*

We can also characterize the open subsets on which, given any polynomial p, the equation (2.3) is solvable.

Corollary (Corollary 4 paper I)**.** *Suppose p does not satisfy the hypothesis of the previous Corollary. Let $n \geq 1$ be the largest n such that $p(-5^n\lambda) = 0$ for λ a joint Dirichlet/Neumann eigenvalue. Then* (2.3) *is solvable on any open subset of the Sierpiński gasket only containing m-cells, for $m > n$.*

The proofs use local solutions on maximal cells contained in the open set. The crucial step is to show that there exists what we call a λ-eigenfunction spline that, added to these local solutions, gives the desired solution. What must be taken care of is the *matching condition* that says that for functions in $\operatorname{Dom} \mathcal{E}$, the sum of normal derivatives at boundary points of neighboring cells is zero. In the proof we use some detailed properties of the spectrum of the Laplacian that follows from spectral decimation.

2.2 Summary of paper II

The second paper concerns the limit distribution of *eccentricities*, a kind of generalized direction of gradients, of restrictions to small cells of fixed level on the Sierpiński gasket. Some results by Öberg, Strichartz and Yingst [26] on local properties of harmonic functions are extended to functions with Hölder continuous Laplacian. We will call a function with Hölder continuous Laplacian *smooth*.

The same notation as in paper I is used for the Sierpiński gasket and the contractions. For restrictions of functions defined on K to cells notation $f_w = f \circ F_w$ is used. The uniform self-similar measure on K is denoted by m.

The eccentricity $e(f)$ of a function f defined on K and non-constant on the boundary is defined as

$$e(f) = \frac{f(q_1) - f(q_0)}{f(q_2) - f(q_0)}, \tag{2.5}$$

where the boundary points has been (re)labelled so that $f(q_0) \leq f(q_1) \leq f(q_2)$. Note that $0 \leq e(f) \leq 1$ and that $e(f)$ is invariant under affine transformations and under the symmetries of K.

Non-constant harmonic functions on the Sierpiński gasket are non-constant on every cell. Therefore the eccentricity of the restriction of a non-constant harmonic function to any cell will also be defined. The harmonic extension algorithm induces mappings ψ_i, $i = 0,1,2$, on $[0,1]$, through $\psi_i(e(h)) = e(h_i)$ for any non-constant harmonic function h so that $\psi_i(e)$ gives the eccentricity on the cell K_i for any harmonic function with eccentricity e. Hence, the eccentricities of restrictions to cells is governed by the i.f.s. $\{\psi_i\}_{i=0}^{2}$.

In [26] it was shown that the limit distribution of eccentricities on n-cells, $\{e(h_w)\}_{w \in W_n}$, is independent of the (non-constant) harmonic function h. More precisely, there is a measure μ on $[0,1]$ so that, for any (non-constant) $h \in \mathcal{H}$ the discrete measures

$$\sum_{w \in W_n} \frac{1}{3^n} \delta(e(h_w)) \tag{2.6}$$

converges to μ in the *Wasserstein metric*.

Take as representative of all harmonic functions with eccentricity e, the harmonic function h^e with boundary values $0, e, 1$. Then, considering a function as built up by $h^{e(h_w)}$, $w \in W_m$, any two (non-constant) harmonic functions are built up by statistically the same functions on small scales. This is a property similar to the 'geography is destiny' principle that says that restrictions to small cells depend on the cell, and its location in the fractal, rather than the function.

The same convergence result, but to another measure $\mu_{\mathcal{E}}$, was also shown in [26] for energy weights instead of uniform weights. This means that each cell is weighted with its contribution to the energy of the function. Thus the uniform weights $\frac{1}{3^n}$ in (2.6) is replaced by $\frac{5^n \mathcal{E}(h_w)}{3^n \mathcal{E}(h)}$.

The main results of paper II are Theorem 4 and Theorem 5. Instead of restating them here we give their interpretation in terms of limit distribution of eccentricities. Theorem 4 extends the convergence to μ in the Wasserstein metric of (2.6) to a class of smooth functions called *nearly harmonic functions*. Theorem 5 extends the corresponding result with respect to energy weights to arbitrary non-constant smooth functions.

The restriction in Theorem 4 is necessary since the functions under consideration can be constant on entire cells, for instance localized eigenfunctions. However, these cells do not contribute to the energy of the function, so they are neglected by the energy weights. That is the reason why Theorem 5 is valid for any non-constant smooth function.

There is no simple rule, equivalent to the functions ψ_i, for the eccentricities of restrictions to smaller cells of smooth functions. As a matter of fact, we already have the complication that the function can be constant on the boundary of cells, and even on entire cells. Still, we will use an extension of the original i.f.s. that describes the distribution of eccentricities for any smooth function to obtain these results.

To deal with functions constant on the boundary of cells define $e(f) = -1$ if $f|_{V_0}$ is constant. Then note that any smooth function f can be written

$$f = Hf - Gu, \tag{2.7}$$

where Hf is the harmonic function that coincide with f on the boundary, $u = \Delta f$, and G is the Green's operator (1.14) and after composition of a symmetry of the Sierpiński gasket and an affine transformation $f = h^e(f) - Gu$. Since these operations do not change eccentricities the distribution of eccentricities is completely determined by $e(f)$ and u.

We use this to extend the i.f.s. $\{\psi_i\}_{i=0}^2$ to an i.f.s. $\{\Psi_i\}_{i=0}^2$ on the infinite dimensional space $(\{-1\} \cup [0,1]\}) \times H^\alpha$, where H^α is the space of Hölder continuous functions on K. This new i.f.s. is defined so that $\Psi_i(e,0) = \psi_i(e,0)$ and for a smooth function $f = h^e - Gu$ the distribution of eccentricities of restrictions to n-cells is given by the first coordinate of $\{\Psi_w(e,u)\}_{w \in W_n}$.

What makes it possible to arrive at Theorem 4 and Theorem 5 is that the second coordinate in the new i.f.s. tends to zero. Thus, there is no need to worry about exactly how the eccentricities of restrictions are obtained for general f. We can concentrate on showing that the perturbation of the original i.f.s. that leads to Ψ_i is in a sense continuous with respect to the second coordinate. These results are Lemma 8 and Lemma 10.

We conclude with a discussion on nearly harmonic functions, mentioned in connection to Theorem 4. Theorem 2 is used for the definition of this class of functions. This theorem gives a lower bound of the energy of Hf_w under the assumption that the quotient of the Hölder norm of Δf and the energy norm of f is smaller than some real number $\varepsilon_0 > 0$. We say that a function is nearly harmonic if it satisfies the hypothesis of Theorem 2. In particular, this means that for any nearly harmonic function f, $e(f_w) \in [0,1]$ for every $w \in W_*$.

The proof of Theorem 2 uses a gradient, $\mathrm{Grad}_{\omega} f$ for $\omega \in \Omega$ and f a function on K defined by Teplyaev in [33]. In particular we use a theorem in that paper which says that for smooth functions the Gradient always exists and the energy norm of $\mathrm{Grad}_{\omega} f - Hf$ is estimated by the Hölder norm of the Laplacian. We restate this as Theorem 1 of paper II, where we have determined a numerical value of the constant in this estimate not present in [33]. This constant can be used to determine the value of ε_0 in the definition of nearly harmonic functions (Proposition 3).

The term nearly harmonic comes from the fact that most of the energy comes from the harmonic part in the decomposition (2.7). In fact, $\mathcal{E}(f) = \mathcal{E}(Hf) + \mathcal{E}(Gu)$ and for f nearly harmonic,

$$\mathcal{E}(Hf) \geq (1 - \|g\|_{\infty}\varepsilon_0^2)\mathcal{E}(f), \tag{2.8}$$

where g is the Green's function.

It may seem as a sever restriction that Theorem 4 is valid only for nearly harmonic functions. However, we show in Theorem 3, also using the gradient, that (except on a closed nowhere dense set) essentially any smooth function f is either nearly harmonic or constant on small enough cells.

2.3 Summary of paper III

In the third paper we also consider local behavior of functions but the approach is different from that of paper II. Instead of the distributive properties of all restrictions to cells of fixed level, f_w, $w \in W_n$, we investigate the pointwise local behavior of functions. This is done by exploiting the connection between local behavior of harmonic functions on p.c.f. fractals and product of random matrices, that stems from the harmonic extension mappings. In particular, we extend the geography is destiny principle to larger classes of functions and fractals.

Recall that each infinite sequence $\omega \in \Omega$ correspond to a unique point $x = \pi(\omega) \in F$ where π is defined through (1.3). The set $\pi^{-1}\{x\}$ is finite by the p.c.f. assumption, and for *non-junction points*, points that do not lie in the intersection of two different cells of the same level, this set consists of one point. Thus for a non-junction point $x = \pi(\omega)$ we can define $[x]_n = [\omega]_n$. Note that non-junction points are a set of full measure and that every such point has a canonical basis of neighborhoods $F_{[x]_n}$.

The harmonic extension mappings A_i contain some information that is superfluous with respect to local behavior. Since any harmonic function constant on the boundary is constant, all A_i has the constant functions as eigenvectors with eigenvalue 1. We therefore factor out the constant functions and denote in paper III by $\mathcal{H}$ the space of harmonic functions such that $\sum_{q\in V_0} h(q) = 0$, and define mappings $M_i = P_{\mathcal{H}} A_i P_{\mathcal{H}}^*$, where $P_{\mathcal{H}} h = h - \sum_{q\in V_0} h(q)$.

The results of paper III regards the generic local behavior, with respect to any self-similar measure μ, at non-junction points of functions on p.c.f. fractals. For harmonic functions this is given by the product of i.i.d. random matrices $M_{[\omega]_n}$, with $P[\omega_n = i] = \mu_i$, where by abuse of notation, we have written M_i for the matrices corresponding to M_i, in some basis of $\mathcal{H}$. Properties of products of random matrices can be interpreted as local properties of harmonic functions on fractals.

To use results on products of random matrices we make two additional assumptions on the harmonic structure. The first is that it should be *non-degenerate*, which means that the matrices M_i all are invertible. This implies that the restriction to any cell of a non-constant harmonic function will be non-constant, which is not always the case. For instance the harmonic structure on the Hexagasket in Example 1.3.1 is degenerate. Our second condition, which we call the SC-assumption, says that the semigroup generated by M_i is strongly irreducible and contracting, Definition 2.2-2.4. Proposition 2.6 gives sufficient conditions for the SC-assumption if V_0 consists of three points.

Inspired by the analogy between harmonic functions and affine linear mappings, we take as starting point for paper III the definition of a derivative $\frac{df}{dh}$ with respect to a harmonic function h at a non-junction point x by

$$f(y) = f(x) + \frac{df}{dh}(x)(h(y) - h(x)) + o\big(\|M_{[x]_n}h\|\big)_{y \to x}, \tag{2.9}$$

for $y \in F_{[x]_n}$.

Section 2 of paper III is devoted to the properties of this derivative. The main results of the section are Theorem 1 and Theorem 2, where we prove almost everywhere differentiability with respect to arbitrary harmonic functions for large classes of functions.

Theorem 1 is valid under the SC-assumption and is stated for a class of functions $C^k(\mathcal{H})$, a multiplicative domain containing $\mathcal{H}$. For Theorem 2 we add what we call the *weak main assumption*, Definition 2.11, that involves the measure μ. In return, we get almost everywhere differentiability for a larger class $C^k(\mathrm{Dom}\,\Delta_\mu)$, a multiplicative domain including $\mathrm{Dom}\,\Delta_\mu$. The proof of Theorem 1 is straightforward while the proof of Theorem 2 is quite involved and makes use of several detailed properties of products of random matrices.

The weak main assumption is an inequality that involves the *upper Lyapunov exponent* of the matrices M_i with respect to μ and the scaling factor of the Laplacian. The upper Lyapunov exponent, the almost sure limit of $\lim_{n\to\infty} \frac{1}{n} \log \|M_{[x]_n}\|$, is very difficult to determine precisely but we can nevertheless assert that the weak main assumption is valid for the harmonic structures in Example 1.3.1 on the Sierpiński gasket and Level-3 Sierpiński gasket with uniform self-similar measure.

In Theorem 3 we state a differentiability result for *periodic points*. A point $x \in F$ is periodic if it is a fixed point of some ψ_w, $w \in W_*$. We also discuss the relation between our derivative at periodic points and the local derivatives previously studied at periodic points [1, 6].

The section on the derivative also include two analogs of Fermat's theorem on stationary points, Corollary 2.15-16.

In the third section we obtain the promised extension of geography is destiny. The results follow more or less directly from the differentiability theorems.

Under the SC-assumption the geography is destiny principle for harmonic functions is a direct consequence of results on products of random matrices. The principle was stated for the first time in [26], where it was noted that the standard harmonic structure on the Sierpiński gasket satisfies the SC-assumption. Essentially, it says that restrictions to the canonical neighborhoods of a point will, for most harmonic functions, line up in the same direction, in the sense that $h_{[x]_n} \approx c_1 h_n + c_2$, for some harmonic functions h_n.

We give a precise formulation of geography is destiny for harmonic functions in Proposition 3.1 and in Theorem 4 we show that the same property is valid, under the weak main assumption, for functions in $C^k(\operatorname{Dom}\Delta_\mu)$ where the derivative is nonzero. Corollary 3.4 is a similar result on the behavior of eccentricities on neighborhoods of points.

The last section relates the derivative to the gradient from [33] that was used in paper II. We work under the *strong main assumption* that involves the *lower Lyapunov exponent*.

In Theorem 6 we show that at a non-junction point the gradient $\operatorname{Grad}_x f$ is the unique function in $\mathcal{H}$ that best approximates f on the canonical neighborhoods of x. Proposition 4.5 gives the explicit relation between the gradient and the derivative. In particular $\operatorname{Grad}_x f = 0$ implies $\frac{df}{dh}(x) = 0$. Theorem 7 says that the converse can only fail on a null set and geography is destiny with conditions on the gradient is formulated in Corollary 4.7.

Summary in Swedish

En fraktal är en mängd med en starkt sönderbruten, cellulär, struktur. Exempel på sådana finns givna i Figur 1.1-1.6. Temat i den här avhandlingen är analys på fraktaler, studiet av differentialekvationer och funktioner definierade på fraktaler.

Ur en matematisk synvinkel är själva existensen av mängder med annorlunda geometriska egenskaper tillräcklig för att studera (klasser av) funktioner definierade på dessa. Men för att studera fysikaliska fenomen på objekt som modelleras med fraktaler räcker det inte att studera fraktalers geometriska egenskaper. Det är nödvändigt att göra någon slags analys på fraktaler. Den matematiska teori som vi använder här har också sina rötter i fysikers matematiska modeller för att förstå de oväntade egenskaper för exempelvis värmeledning som uppvisas av vissa oordnade material.

Jun Kigami gav 1989 en analytisk konstruktion av en Laplaceoperator på Sierpińskitriangeln (Figur 1.1), en konstruktion som han senare utvidgade till postkritiskt ändliga fraktaler, en klass av så kallade ändligt förgrenade fraktaler [16, 17]. Alla nya resultat i avhandlingen tillhör Kigamis teori.

Styrkan hos Kigamis konstruktion ligger i att en rik analytisk teori av fraktala motsvarighet till objekt och resultat från klassisk analys har uppnåtts, dock till priset av en lägre generalitet med avseende på de underliggande mängderna. Flertalet arbeten har gjorts enbart på Sierpińskitriangeln. En restriktion som ofta har gjorts mer för tydlighets skull än av nödvändighet. Sierpińskitriangeln har blivit en modell för klassen av postkritiskt ändliga fraktaler.

Postkritiskt ändliga fraktaler är självlikformiga, dvs de har en cellulär struktur, i oändligt många skalor, där varje cell är en kopia av fraktalen. En viktig typ av självlikformiga mängder är invarianta mängder till iterativa funktionssystem av kontraktioner. De exempel vi nämnt ovan är sådana invarianta mängder. Exempelvis är Sierpińskitriangeln invariant mängd till tre kontraktioner med kontraktionsfaktor 1/3 och fixpunkter i hörnen på en likformig triangel.

Ändligt förgrenade fraktaler har egenskapen att två olika celler av samma storleksordning bara kan skära varandra i ett ändligt antal punkter. Postkritiskt ändliga fraktaler uppfyller det ytterligare villkoret att varje punkt har en övre gräns för antalet celler av samma storlek som innehåller punkten.

Kigami ger två ekvivalenta definitioner av Laplaceoperatorn. Dels en svag definition via en energiform $\mathcal{E}$, genom att $\Delta_\mu u = f$ får betyda

$$\mathcal{E}(u,v) = -\int_F fv d\mu,$$

för varje funktion v i en viss klass av testfunktioner, dels en punktvis definition som ett renormaliserat gränsvärde av diskreta differensoperatorer Δ_m.

Centralt i bägge definitionerna är att fraktalen approximeras av en följd av grafer. De första graferna i denna approximation för Sierpińskitriangeln är givna i Figur 1.7. Energiformen $\mathcal{E}$ som används i den svaga definitionen konstrueras med hjälp av en kompatibel följd av energiformer på de approximerande graferna. Differensoperatorerna Δ_m är diskreta Laplaceoperatorer på dessa grafer.

Laplaceoperatorn beror som synes på vilket mått μ som används på fraktalen. Däremot är inte klassen av harmoniska funktioner beroende av μ. En funktion h är harmonisk om $\Delta_\mu h = 0$. De definieras via energiformen genom att de minimerar energin för funktioner med givet randvärde.

Då Laplace operatorn är den grundläggande differentialoperatorn i teorin används termen differentialekvation för ekvationer innehållande denna. Strichartz visade i [31] att första ordningens linjära differentialekvation

$$-\Delta u - \lambda u = f$$

inte alltid är lösbar på Sierpińskitriangeln. I det första arbetet i avhandlingen ges en karaktärisering av de polynom p för vilka den allmänna linjära differentialekvationen

$$p(\Delta)u = f \tag{2.10}$$

alltid är lösbar på öppna äkta delmängder av Sierpińskitriangeln. Omvänt så ges också, för givet polynom p, en karaktärisering av de öppna delmängder där (2.10) alltid är lösbar.

I det andra arbetet betraktas distributionen av eccentriciteter, ett slags generaliserad gradientriktning, för restriktioner till celler av samma storlek. Resultat för harmoniska funktioner på Sierpińskitriangeln som visades i [26] utvidgas till funktioner med Hölder kontinuerlig Laplace. Resultaten uppnås genom att studera ett iterativt funktionssystem (i.f.s.) på ett oändligt dimensionellt rum. Detta i.f.s. är en perturbation av det i.f.s. på $[0,1]$ som karaktäriserar distributionen av eccentriciteter för harmoniska funktioner.

I det tredje arbetet studeras lokala egenskaper hos allmänna klasser av funktioner med vissa regularitets egenskaper. Detta görs genom att utnyttja sambandet mellan restriktioner av harmoniska funktioner till celler och produkter av slumpmatriser. Speciellt så utvidgas ”geografin är ödet” principen, formulerad i [26] för harmoniska funktioner på Sierpińskitriangeln, till större klasser av fraktaler och funktioner. Denna princip säger att restriktioner till kanoniska cellomgivningar kommer, för de flesta harmoniska funktioner, ha samma riktning, i betydelsen att $h_{[x]_n} \approx c_1 h_n + c_2$, för harmoniska funktioner h_n.

En stor del av arbetet kretsar kring en derivata, som vi introducerar, $\frac{df}{dh}$ med avseende på harmoniska funktioner h. Två satser angående deriverbarhet μ-nästan överallt, där μ är ett självlikformigt mått, visas. Den första gäller under vissa förutsättningar på energiformen, medan den andra, som gäller för en större klass av funktioner, också inkluderar förutsättningar på måttet μ.

Dessa deriverbarhetssatser används till utvidgningen av ”geografin är ödet” och ett resultat angående de punktvisa egenskaper för eccentriciteter. Avslutningsvis studeras sambandet mellan den nämnda derivatan och gradienten definierad i [33].

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Paper I

SOLVABILITY OF DIFFERENTIAL EQUATIONS ON OPEN SUBSETS OF THE SIERPIŃSKI GASKET

ANDERS PELANDER

ABSTRACT. We give necessary and sufficient conditions on the polynomial p so that the differential equation $p(\Delta)u = f$, based on the Laplacian, is solvable on any open subset of the Sierpiński gasket for any f continuous on that subset. For general p we find the open subsets on which $p(\Delta)u = f$ is solvable for any continuous f.

The basic differential operator in the theory of analysis on post-critically finite (p.c.f.) fractals constructed by Kigami [5, 6, 7, 11] is the Laplacian. Therefore when speaking of differential equations on fractals, or fractal differential equations, one means equations involving the Laplacian.

In this article we answer a question asked by Strichartz in [9] concerning solvability of linear differential equations on open subsets of the Sierpiński gasket. We prove that the equation $p(\Delta)u = f$ is solvable on any open subset for any f continuous there, if -5λ is not a root of p when λ is a joint Dirichlet/Neumann eigenvalue. For arbitrary p the equation is always solvable on open subsets that only contain copies of the Sierpiński gasket of size less than a particular value, determined by the roots of p . This is in contrast to the convexity conditions in the Malgrange–Ehrenpreis Theorem [4, chapter 3].

Non-linear differential equations on the Sierpiński gasket were studied in [2].

Let $V_0 = \{q_i\}_{i=0}^2$ be the set of vertices of an equilateral triangle. The Sierpiński gasket K is the attractor of the iterated function system consisting of the contractions

$$F_i x = \frac{1}{2}(x - q_i) + q_i, \qquad i = 0, 1, 2.$$

Thus K satisfies the self-similar identity

$$K = \bigcup_{i=0}^{2} F_i K,$$

and the condition

$$F_i K \cap F_j K \subseteq F_i V_0 \cap F_j V_0 \quad \text{for} \quad i \neq j.$$

The Sierpiński gasket is one of the basic examples of a p.c.f. fractal and the standard one used in establishing as many fractal analogs as possible of results and objects from analysis on smooth sets

For any word $w = w_1 w_2 \cdots w_n,\ w_j \in \{0,1,2\}$ of finite length $|w| = n$ we denote $F_w = F_{w_1} \circ \cdots \circ F_{w_n}$. The sets $K_w = F_w K$ are called *cells* of level n and V_0 respectively $F_w V_0$ are the *boundaries* of K respectively K_w.

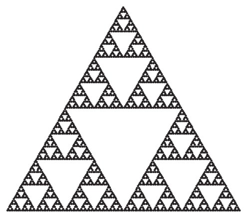

FIGURE 1. Sierpiński gasket.

We give a brief introduction to the basic concepts of Kigami's analytic theory in the case of the Sierpiński gasket. Complete expositions in the general setting can be found in the books of Kigami [7] and Strichartz [11].

We treat K as a limit of graphs Γ_m with vertices V_m and edge relations $x \sim_m y$ defined inductively as follows. Γ_0 is the complete graph on $V_0 = \{q_0, q_1, q_2\}$. Then $V_m = \bigcup_i F_i V_{m-1}$ with $x \sim_m y$ if and only if there exists i such that $x = F_i x'$, $y = F_i y'$ and $x' \sim_{m-1} y'$. Points in $V_m \setminus V_0$ are called *junction points*.

The bilinear graph energy forms

$$E_{\Gamma_m}(u,v) = \sum_{x \sim_m y} (u(x) - u(y))(v(x) - v(y)),$$

where the sum extends over the edges of Γ_m, are used to define the energy form

$$\mathcal{E}(u,v) = \lim_{m\to\infty} \mathcal{E}_m(u,v) = \lim_{m\to\infty} \left(\frac{5}{3}\right)^m E_{\Gamma_m}(u,v)$$

on functions defined on K. The sequence $\mathcal{E}_m(u,u)$ is non-decreasing so the *energy* $\mathcal{E}(u) = \lim_{m\to\infty} \mathcal{E}_m(u,u)$ of a function is a well-defined extended real number. The class of functions with finite energy is denoted by $\mathrm{dom}\mathcal{E}$ and form a dense linear subspace of $C(K)$.

Let μ be the standard self-similar probability measure on K, i.e.,

$$\mu(A) = \sum_{i=0}^{2} \frac{1}{3} \mu(F_i^{-1} A).$$

A function $u \in \mathrm{dom}\mathcal{E}$ is in the domain of the Laplacian, $\mathrm{dom}\Delta$, and $\Delta u = f$, if there exists a continuous function f such that

$$\mathcal{E}(u,v) = -\int_K f v \, d\mu,$$

for any function $v \in \mathrm{dom}\mathcal{E}$ that vanishes on the boundary.

In junction points the Laplacian can be calculated as a limit of graph Laplacians on Γ_m

$$\Delta u(x) = \frac{3}{2}\lim_{m\to\infty} 5^m \Delta_m u(x) = \frac{3}{2}\lim_{m\to\infty} 5^m \sum_{y\sim_m x} (u(y) - u(x)).$$

This is similar to the difference quotient formula for the second derivative and justifies that we speak of the Laplacian as a differential operator. Note that the Laplacian satisfies the scaling property

$$\Delta(u \circ F_w) = 5^{-|w|}\Delta u \circ F_w.$$

If $v \in \mathrm{dom}\mathcal{E}$ does not vanish on the boundary there is a Gauss–Green formula that relates energy with the Laplacian

$$\mathcal{E}(u,v) = -\int_K (\Delta u) v d\mu + \sum_{q\in V_0} v(q)\partial_n u(q),$$

where the normal derivative $\partial_n u$ is defined by

$$\partial_n u(q_0) = \lim_{m\to\infty}\left(\frac{5}{3}\right)^m (2u(q_0) - u(F_0^m q_1) - u(F_0^m q_2))$$

and likewise at q_1 and q_2.

The normal derivative can be localized to a junction point $x = F_w q_i$ through the formula

$$\partial_n u(F_w q_i) = \left(\frac{5}{3}\right)^{|w|} \partial_n (u\circ F_w)(q_i).$$

Since junction points lie on the boundary of two different cells K_w and $K_{w'}$ of high enough level there are two local normal derivatives defined at junction points. If $u \in \mathrm{dom}\mathcal{E}$ then the normal derivatives sum up to zero at any junction point, they satisfy the *matching condition*. The following well-known proposition [9] will be an important tool for proving our results.

Proposition 1. *Suppose u and f are continuous functions on K such that $u \circ F_w \in dom\Delta$ for all w of length m, and $\Delta(u\circ F_w) = 5^{-|w|} f \circ F_w$. Then $u \in dom\Delta$ with $\Delta u = f$ if and only if the matching condition holds at every junction point in V_m.*

With Proposition 1 one can define a Laplacian on any union of cells in the obvious way. There is also a local version of the Gauss–Green formula [11].

Proposition 2. *Let $A = \cup_{w\in W} K_w$ be a finite union of cells such that $K_w \cap K_{w'}$ consists of at most one point if $w, w' \in W$. If u and v are in $dom\Delta$ on A then*

$$\int_A u\Delta v d\mu - \int_A \Delta u v d\mu = \sum_{\partial A} u\partial_n v - \partial_n u v,$$

where ∂A are the points that lie on the boundary of exactly one of the cells K_w, $w \in W$.

A function $u \in \mathrm{dom}\Delta$ is called a Dirichlet, respectively Neumann, eigenfunction with eigenvalue λ if

$$\begin{cases} -\Delta u = \lambda u \\ u|_{V_0} = 0, \end{cases}$$

respectively

$$\begin{cases} -\Delta u = \lambda u \\ \partial_n u|_{V_0} = 0. \end{cases}$$

The Dirichlet and Neumann eigenfunctions and spectrum has been completely described through the method of *spectral decimation* [1, 3, 8]. It relates the spectra of the discrete Laplacians Δ_m and the spectrum of Δ through backwards iteration of a quadratic polynomial. We refer to [11, chapter 3] for a transparent and detailed account.

In [9] Strichartz proved a variety of results on differential equations on p.c.f. fractals. We are particularly interested in the following two theorems.

Theorem 1. *Let Ω be an open set of K not containing any point of V_0. Then there exists a solution of*

$$-\Delta u = f \quad \textit{on} \quad \Omega \tag{1}$$

for any function f continuous on Ω.

This theorem was actually proved for any p.c.f. fractal. Also note that nothing is assumed about the behavior of f near the boundary. Theorem 1 is also true if Ω is a set that contains points in V_0, interpreted in the sense that there is a solution of (1) that is continuous at the boundary points contained in Ω.

Theorem 2. *The equation*

$$-\Delta u = \lambda u + f \tag{2}$$

is solvable on K for every continuous f if and only if λ is not a joint Dirichlet/Neumann eigenvalue.

The obstructive case in Theorem 2 occurs when f is a joint Dirichlet/Neumann eigenfunction. The existence of such functions is a peculiar feature of analysis on fractals. They are also called pre-localized eigenfunctions, since they give rise to localized eigenfunctions; if f is a pre-localized eigenfunction then

$$g(x) = \begin{cases} f \circ F_w^{-1}(x), & x \in K_w \\ 0, & x \notin K_w \end{cases}$$

is a $5^{|w|}\lambda$-eigenfunction with support in K_w.

Solvability in the case that λ is not a Dirichlet (Neumann) eigenvalue is shown using the fact that there is a complete orthonormal basis $\{u_j\}$ in

$L^2(K)$ of Dirichlet (Neumann) eigenfunctions with eigenvalue λ_j. If

$$f = \sum_{j=1}^{\infty} c_j u_j, \tag{3}$$

then

$$u = \sum_{j=1}^{\infty} \frac{c_j u_j}{\lambda_j - \lambda} \tag{4}$$

is a continuous function that solves (2).

We will generalize Theorem 1 to linear differential equations with constant coefficients,

$$p(\Delta)u = f, \tag{5}$$

where p is a polynomial. In view of Theorem 2 it is not possible, without restrictions on p, to have solvability of (5) on any open subset. We give in Theorem 3 necessary and sufficient conditions on λ so that (2) is solvable on arbitrary open subsets and in Corollary 3 conditions on p so that the same is true for equation (5). For arbitrary p we give in Corollary 4 a complete description of those open subsets on which (5) is solvable.

The proof uses ideas from that of Theorem 1 in [9]. The equation is solved locally on maximal cells contained in Ω and then it is shown that for certain choices of local solutions one can glue together a global solution. The main difference is that we will, naturally, have to use 'eigenfunction splines' instead of harmonic splines to glue together the local solutions and rely on Proposition 2 to prove that this construction is always possible.

For transparency the result is first proved for equation (2).

Theorem 3. *Suppose $\lambda/5$ is not a joint Dirichlet/Neumann eigenvalue of the Laplacian on K and that $\Omega \subsetneq K$ is an open subset of K. Then there exists a solution to* (2) *for any function f continuous on Ω.*

Proof. Suppose $\lambda/5$ is a Neumann eigenvalue. This assures that $\lambda/5^n$ is not a Dirichlet eigenvalue for any $n \geq 1$. If we only assume that $\lambda/5$ is not a Dirichlet eigenvalue it might occur that $\lambda/5^n$ is a Dirichlet eigenvalue from the so-called 2-series. To be able to use the Dirichlet expansion of $f \circ F_w$ for local solutions we do not want this to happen.

Without loss of generality we assume that Ω is connected. Write

$$\Omega = \cup_{w \in W} K_w, \tag{6}$$

as an infinite union of cells K_w, maximal with the property that $K_w \subset \Omega$. If

$$f \circ F_w = \sum_{j=1}^{\infty} c_{j,w} u_j$$

is the Dirichlet expansion of $f \circ F_w$ then

$$v_w = \sum_{j=1}^{\infty} \frac{c_{j,w} u_j \circ F_w^{-1}}{(5^{|w|}\lambda_j - \lambda)}, \tag{7}$$

is a solution to (2) on K_w. The denominator in (7) cannot be zero by our assumptions on λ. Define a function v on Ω through $v|_{K_w}(x) = v_w(x)$. Since $v_w|_{F_w V_0} = 0$ it is clear that v is continuous. However, to be a solution to (2), it must also satisfy the matching condition at every junction point $F_w q_i$, $w \in W$.

As in the proof of Theorem 1 in [9] define $\tilde{V} = \cup_{w \in W} F_w V_0$ and the graph $\tilde{\Gamma}$, with set of vertices $\tilde{V}$, in the obvious way. Approximate $\tilde{\Gamma}$ by an increasing sequence of connected finite graphs $\{\tilde{\Gamma}_n\}$. The boundary $\partial\tilde{\Gamma}_n$ and interior $\mathrm{int}(\tilde{\Gamma}_n)$ are defined in the usual way ($x \in \mathrm{int}(\tilde{\Gamma}_n)$ if x is a vertex connected to four other vertices in $\tilde{\Gamma}_n$). We define A_n to be the finite union of the cells whose boundary points are vertices in $\tilde{\Gamma}_n$.

We say that a function g, on Ω respectively A_n, is a continuous λ-*eigenfunction spline* if it is continuous and satisfies $-\Delta g = \lambda g$ in the interior of every K_w, $w \in W$ respectively K_w, $w \in W$ and $K_w \subset A_n$. The set of continuous λ-eigenfunction splines can be identified with $l(\tilde{V})$, the set of functions on $\tilde{V}$, since $g_w = g \circ F_w$ is a $5^{-|w|}\lambda$-eigenfunction, and $5^{-|w|}\lambda$ is not a Dirichlet eigenvalue. Likewise we identify continuous λ-eigenfunction splines on A_n with $l(\tilde{\Gamma}_n)$.

The proof is completed by showing that there is a continuous λ-eigenfunction spline g such that $v + g$ satisfies the matching condition at every junction point $F_w q_i$, $w \in W$. The crucial step is to show that for any values of the normal derivatives of v at the points in $\mathrm{int}(\tilde{\Gamma}_\mathrm{n})$, it is possible to define values of g on $\tilde{\Gamma}_n$ so that $v + g$ satisfies the matching condition at every point in $\mathrm{int}(\tilde{\Gamma}_\mathrm{n})$.

Define linear operators $\tilde{N}_n$ on $l(\tilde{\Gamma}_n)$ through

$$\tilde{N}_n g(q) = \begin{cases} \partial_N g(q), & \text{if } q \in \partial\tilde{\Gamma}_n \\ \sum_{F_w q_i = q} \partial_N g(F_w q_i), & \text{if } q \in \mathrm{int}(\tilde{\Gamma}_n), \end{cases}$$

where the sum is taken over the two normal derivatives at q. Clearly, the existence of a spline g such that $v+g$ satisfy the matching condition at every point in $\mathrm{int}(\tilde{\Gamma}_n)$ is equivalent to surjectivety of $P_{l(\mathrm{int}(\tilde{\Gamma}_n))}\tilde{N}_n$, where $P_{l(\mathrm{int}(\tilde{\Gamma}_n))}$ denotes projection on $l(\mathrm{int}(\tilde{\Gamma}_n))$.

Define the following subspaces of $l(\tilde{\Gamma}_n)$. The 'Dirichlet splines'

$$D = \{g \in l(\tilde{\Gamma}_n) \mid g|_{\partial\tilde{\Gamma}_n} = 0\} = \mathrm{Ker} P_{l(\partial\tilde{\Gamma}_n)},$$

the 'Neumann splines'

$$N = \{g \in l(\tilde{\Gamma}_n) \mid \tilde{N}_n g|_{\partial\tilde{\Gamma}_n} = 0\} = \mathrm{Ker} P_{l(\partial\tilde{\Gamma}_n)}\tilde{N}_n,$$

and the 'smooth splines'

$$L = \{g \in l(\tilde{\Gamma}_n) \mid \tilde{N}_n g|_{\mathrm{int}(\tilde{\Gamma}_n)} = 0\} = \mathrm{Ker} P_{l(\mathrm{int}(\tilde{\Gamma}_n))} \tilde{N}_n,$$

which are those splines that satisfies the matching condition in every interior point and thus represent λ-eigenfunctions on A_n.

Let $m = \#\mathrm{int}(\tilde{\Gamma}_n)$ and $m' = \#\partial\Gamma_n$. It is clear that $\mathrm{Dim}D = m$, $\mathrm{Dim}N \geq m$, $\mathrm{Dim}L \geq m'$ and we need to show $\mathrm{Dim}L = m'$.

Suppose $\mathrm{Dim}L > m'$. Then there are both non-trivial Dirichlet and non-trivial Neumann eigenfunctions on A_n. However, there cannot be a function that is both. Such a function could trivially be extended to a joint λ-Dirichlet/Neumann eigenfunction on all of K with support strictly included in K, which means that $\lambda/5$ is a joint Dirichlet/Neumann eigenvalue.

Thus, if $v \in D \cap L$ is a nontrivial Dirichlet eigenfunction on A_n, then $\partial_N v$ cannot be identically zero on $\partial A_n = \partial\tilde{\Gamma}_n$, so

$$\mathrm{Dim}(D \cap L) = \mathrm{Dim} P_{l(\partial\tilde{\Gamma}_n)} \tilde{N}_n (D \cap L) = k > 0.$$

Let $u \in L$ be any spline corresponding to a λ-eigenfunction on A_n. The Gauss–Green's formula on A_n tells us that

$$\begin{aligned}\sum_{\partial\tilde{\Gamma}_n} (u\partial_N v - v\partial_N u) &= \sum_{q \in \partial\tilde{\Gamma}_n} u(q)\partial_N v(q) \\ &= \int_A u\Delta v d\mu - \int_A v\Delta u d\mu = 0,\end{aligned}$$

i.e., $\mathrm{Dim}P_{l(\partial\tilde{\Gamma}_n)} L \leq m' - k$. So $k = \mathrm{Dim}(D \cap L) = \mathrm{Dim}L - \mathrm{Dim}P_{l(\partial\tilde{\Gamma}_n)} L \geq \mathrm{Dim}L - (m' - k) > k$ and we have a contradiction.

From here the proof is completed in the same way as the proof of Theorem 1 in [9].

We conclude with the case when $\lambda/5$ is not a Neumann eigenvalue. The equation is then solved locally on the cells K_w, $w \in W$ using the Neumann expansion of f_w.

Gluing together local solutions v_w given by (7) to a function v on Ω there is a new complication since v is not necessarily continuous. It is necessary to prove existence of a λ-eigenfunction spline g, such that $v+g$ not only satisfies the matching condition at every point of $\mathrm{int}(\tilde{\Gamma}_n)$, but also is continuous there. However, this complication also gives us the freedom to use discontinuous λ-eigenfunction splines, i.e., functions g such that $-\Delta g = \lambda g$ in the interior of every K_w but g is allowed to have discontinuities at the junction points $F_w q_i$, $w \in W$.

With discontinuous splines on $\tilde{\Gamma}_n$ we can, for every $w \in W$ such that $K_w \subseteq A_n$, fill the inside with any λ-eigenfunction regardless of what λ-eigenfunctions we have chosen on neighboring cells. Let S be the linear space of such splines. Then $\mathrm{Dim}S = 2m + m'$, where $m = \#\mathrm{int}(\tilde{\Gamma}_n)$ and $m' = \#\partial\tilde{\Gamma}_n$, since vertices in $\partial\tilde{\Gamma}_n$ only belong to one cell such that $F_w V_0 \subseteq \tilde{\Gamma}_n$ and vertices in $\mathrm{int}(\tilde{\Gamma}_n)$ belong to two cells such that $F_w V_0 \subseteq \tilde{\Gamma}_n$.

Note that if we want to identify S with $l(\sqcup_{K_w\subseteq A_n}F_wV_0)$ then we need to do it through the values of $\partial_N g(F_wq_i)$ and not through the values of $g(F_wq_i)$, since there is no longer a 1–1 correspondence there in case g_w is a Dirichlet eigenfunction.

Define linear operators

$$\begin{aligned}
&\tilde{N}_n : S \to l(\tilde{\Gamma}_n) \oplus l(\tilde{\Gamma}_n)\\
&\tilde{N}_n(g) = \sum_{F_wq_i=q} \partial_N g(F_wq_i)\delta_q \oplus \sum_{F_wq_i=q} p(q,q_i)g(F_wq_i)\delta_q
\end{aligned}$$

where $p(q,q_j)=1$ if $q\in\partial\tilde{\Gamma}_n$ and if $q\in\mathrm{int}(\tilde{\Gamma}_n)$ we define $p(q,q_0)=1$, $p(q,q_2)=-1$ and $p(q,q_1)=-1$ when $q=F_{w'}q_0=F_{w''}q_1$ and $p(q,q_1)=1$ when $q=F_{w'}q_2=F_{w''}q_1$. The factors $p(q,q_i)$ are chosen to assure that the splines in $\mathrm{Ker}P_{\{0\}\oplus l(\mathrm{int}(\tilde{\Gamma}_n))}$ are exactly the continuous ones.

It is necessary to show that $P_{l(\mathrm{int}(\tilde{\Gamma}_n))\oplus l(\mathrm{int}(\tilde{\Gamma}_n))}\tilde{N}_n$ is surjective, since then there is always a spline g on $\tilde{\Gamma}_n$ such that $v+g$ is continuous and satisfies the matching condition at every point in $\mathrm{int}(\tilde{\Gamma}_n)$. This time the 'Dirichlet splines'

$$D = \mathrm{Ker}P_{\{0\}\oplus l(\partial\tilde{\Gamma}_n)}\tilde{N}_n,$$

the 'Neumann splines'

$$N = \mathrm{Ker}P_{l(\partial\tilde{\Gamma}_n)\oplus\{0\}}\tilde{N}_n,$$

and the 'smooth splines'

$$L = \mathrm{Ker}P_{l(\mathrm{int}(\tilde{\Gamma}_n))\oplus l(\mathrm{int}(\tilde{\Gamma}_n))}\tilde{N}_n,$$

are those splines that satisfies both the matching condition and continuity condition in every point in $\mathrm{int}(\tilde{\Gamma}_n)$ and thus represent λ-eigenfunctions on A_n. The proof is completed as in the first case using the Gauss–Green's formula on A_n. □

Concerning global solutions of general linear differential equations with constant coefficients (5), note that if there is no Dirichlet (Neumann) eigenvalue λ such that $p(-\lambda)=0$, and f can be written as

$$f = \sum_{j=1}^{\infty} c_j u_j$$

where $\{u_j\}$ is an orthonormal basis of Dirichlet (Neumann) eigenfunctions with eigenvalue λ_j then

$$u = \sum_{j=1}^{\infty} \frac{c_j u_j}{p(-\lambda_j)} \tag{8}$$

solves (5) on K. If there are both Dirichlet and Neumann eigenvalues λ_D and λ_N such that $p(-\lambda_D)=p(-\lambda_N)=0$ but no joint Dirichlet/Neumann eigenvalue with this property, write $p(\Delta)=p_D(\Delta)p_N(\Delta)$ where $p_D(-\lambda_D)\neq 0$ and $p_N(-\lambda_N)\neq 0$. Then find u_D that solves $p_D(\Delta)u_D=f$ and then solve

$p_N(\Delta)u = u_D$. If there is a joint Dirichlet/Neumann eigenvalue λ such that $p(-\lambda) = 0$, Theorem 2 implies that there are continuous functions for which $p(\Delta)u = f$ is not solvable on K

Corollary 3. *The equation* (5) *is solvable on any open subset* $\Omega \subsetneq K$ *for any* f *continuous in* Ω *if and only if* $p(-5\lambda) \neq 0$ *whenever* λ *is a joint Dirichlet/Neumann eigenvalue of the Laplacian on* K*.*

Proof. Factorize the polynomial as $p(\Delta) = p_D(\Delta)p_N(\Delta)$, where the roots of p_D are exactly the roots of p of the form -5λ for λ a Neumann eigenvalue. As in the discussion preceding the corollary it is enough to show solvability for p_D and p_N.

Write Ω as an infinite union (6) of maximal cells contained in Ω. If

$$f \circ F_w = \sum_{j=1}^{\infty} c_{j,w} u_j,$$

where u_j are Dirichlet eigenfunctions then

$$v_w = \sum_{j=1}^{\infty} \frac{c_{j,w} u_j \circ F_w^{-1}}{p_D(-5^{|w|}\lambda_j)}, \tag{9}$$

is a solution to $p_D(\Delta)u = f$ on K_w, where the denominator in (9) is nonzero by our assumptions on the roots of p_D. The local solutions can be glued together to a continuous function v on Ω. To assure that the matching condition is satisfied at every junction point in $\tilde{\Gamma}_n$ we can add continuous splines g such that $p_D(\Delta)g = 0$ in the interior of the cells K_w, $w \in W$. We thus have a larger family of splines than in the proof of Theorem 3 so this obviously is possible and the proof can be completed in the same way. To solve $p_N(\Delta)u = f$ we find local solutions using Neumann expansions and follow the same path to obtain a solution on all of Ω.

In case $p(-5\lambda) = 0$ for some joint Dirichlet/Neumann eigenvalue, Theorem 2 shows that the equation is not solvable for every f whenever Ω contains a 1-cell. □

The open sets where (5) is solvable can be characterized as follows.

Corollary 4. *Suppose* p *does not satisfy the hypothesis of Corollary 3. Let* $n \geq 1$ *be the largest* n *such that* $p(-5^n\lambda) = 0$ *for* λ *a joint Dirichlet/Neumann eigenvalue. Then* (5) *is solvable on the open set* Ω *for any* f *continuous on* Ω *if and only if* Ω *only contains* m*-cells for* $m > n$*.*

Proof. If Ω contains a cell of level n then (5) is not always solvable by Theorem 2. If $K_w \subset \Omega$ implies $|w| \geq n+1$ the same idea as in the proof of Theorem 3 can be used to solve the equation. The only thing that is new is that one has to use the fact that if f is a joint Dirichlet/Neumann λ-eigenfunction and no cell of level n is included in the support of f then $\lambda/5^{n+1}$ also is a joint eigenvalue. □

Note that if $n = 1$ in Corollary 4, then Ω can be any open set without points on the boundary. In the obstructive case of Theorem 2 that f is a λ-eigenfunction and λ a joint eigenvalue, there is, if $\lambda/5$ is not a joint eigenvalue, a solution to equation (2) on the interior of K that extends continuously to two points on the boundary. If $\lambda/5$ is a joint eigenvalue but $\lambda/5^2$ is not, and f is such that f_i, $i = 0, 1, 2$ is a $\lambda/5$-eigenfunction, then there are still solutions on the interior of K, but no one can be extended continuously at any boundary point.

As noticed in [9] concerning Theorem 1, our results also generalize to fractafolds based on the Sierpiński gasket [10].

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Paper II

INFINITE DIMENSIONAL I.F.S. AND SMOOTH FUNCTIONS ON THE SIERPIŃSKI GASKET

ANDERS PELANDER AND ALEXANDER TEPLYAEV

ABSTRACT. We describe the infinitesimal geometric behavior of a large class of intrinsically smooth functions on the Sierpiński gasket in terms of the limit distribution of their local *eccentricity*, which is essentially the direction of the gradient. The distribution of eccentricities is codified as an infinite dimensional perturbation problem for a suitable iterated function system, which has the limit distribution as an invariant measure. Continuity properties of the gradient are used to define a class of *nearly harmonic* functions which are well approximated by harmonic functions. The gradient is also used to identify the part of the Sierpiński gasket where a smooth function is nearly harmonic locally. We prove that for nearly harmonic functions the limit distribution is the same as that for harmonic functions found by Öberg, Strichartz and Yingst. In particular, we prove convergence in the Wasserstein metric. We consider uniform as well as energy weights.

1. INTRODUCTION AND NOTATION

There is an extensive theory of analysis on fractals, see for example the books by Kigami [3] and Strichartz [9], and the survey article [7]. For the most part of the analytic theory (there is also a probabilistic theory) one is concerned with fractals which are not too complicated. In the present paper we consider the Sierpiński gasket, which is the example of two-dimensional fractal theory which is best understood from an analytic point of view.

In classical analysis the study of the local structure of smooth functions is fundamental and has many important consequences. For instance it gives rise to such a basic notion as the tangent space. In analysis on fractals the local structure of smooth functions is not yet understood well enough to make it clear what conclusions can be drawn. In this paper we will address some questions concerning the local structure of smooth functions on the Sierpiński gasket. We actually show that they inherit a property of the local structure of harmonic functions, which could be seen as the analogues of linear functions on an interval, proven in [6].

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The Sierpiński gasket K is the invariant set for the iterated function system (i.f.s.) in the plane given by

$$F_i x = \frac{1}{2}(x - q_i) + q_i \qquad i = 0, 1, 2,$$

where q_i are the vertices of an equilateral triangle. More specifically, K is the unique compact subset of $\mathbb{R}^2$ such that $K = F_0(K) \cup F_1(K) \cup F_2(K)$.

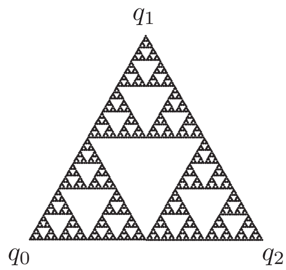

FIGURE 1. Sierpiński gasket.

One reason why the Sierpiński gasket is not 'too complicated', is that it is an example of a fractal which is post-critically finite (p.c.f.). In this particular case, the p.c.f. condition says that for the boundary $V_0 := \{q_0, q_1, q_2\}$ of K we have

$$F_i K \cap F_j K \subseteq F_i V_0 \cap F_j V_0,$$

for $i \neq j$. The general definition can be found in [3].

We regard K as the limit of graphs Γ_n with vertices V_n and edge relations $x \sim_n y$ defined inductively as follows. Let Γ_0 be the complete graph on $V_0 = \{q_0, q_1, q_2\}$. Then $V_n = \bigcup_i F_i V_{n-1}$ with $x \sim_n y$ if and only if there exists i such that $x = F_i x'$, $y = F_i y'$ and $x' \sim_{n-1} y'$. Note that $V_{n-1} \subseteq V_n$. We regard $V_0 = \partial K = \{q_0, q_1, q_2\}$ as the boundary of each of the graphs Γ_n, so that $V_n \setminus V_0$ consists of all non-boundary vertices in Γ_n. Note that every such vertex has exactly four neighbors in V_n. Points in $V_n \setminus V_0$ are called *junction points*.

We define W_n as the space of finite sequences, or words, $w = w_1 \cdots w_n$ of length $|w| = n$, $W_* = \bigcup_{n \geqslant 0} W_n$ as the space of finite words of all lengths, and Ω as the space of infinite sequences $\omega = w_1 w_2 \cdots,\ w_j \in W_1 = \{0, 1, 2\}$. For $\omega = w_1 w_2 \cdots \in \Omega$, let $[\omega]_k = w_1 \cdots w_k \in W_k$ and likewise for $w \in W_*$ and $k < |w|$. We denote

$$F_w = F_{w_1} \circ \cdots \circ F_{w_n} \qquad \text{and} \qquad K_w = F_w(K).$$

For any function f on K and $w \in W_*$ we will use notation f_w for the function $f_w = f \circ F_w$ defined on K.

We will denote by m the standard self-similar measure on K defined by

$$m(K_w) = \frac{1}{3^{|w|}}.$$

Note that there is a natural continuous projection $\pi : \Omega \to K$ defined by

$$\pi(\omega) = \bigcap_{n \geqslant 0} K_{[\omega]_n}.$$

We will abuse notation and define a measure m on Ω as the pullback of the measure m on K under the projection map π, that is $m(\pi^{-1}(\cdot)) = m(\cdot)$. Then m is the product Bernoulli measure.

A continuous function h on K is said to be harmonic if for all n its restriction to V_n is graph-harmonic: its value at every non-boundary vertex $x \in V_n$ is equal to the average of its values at the four neighboring points in V_n,

$$h(x) = \frac{1}{4} \sum_{y \sim_n x} h(y). \tag{1.1}$$

We say that f is n-harmonic if all restrictions f_w, $w \in W_n$ are harmonic.

We will need the concept of energy for functions defined on K. Define graph energy forms

$$\mathcal{E}_n(u, v) = \left(\frac{5}{3}\right)^n \sum_{y \sim_n x} (u(x) - u(y))(v(x) - v(y)).$$

Then the sequence of graph energies $\mathcal{E}_n(u) = \mathcal{E}_n(u, u)$ is nondecreasing for every u and the harmonic functions are the only ones for which the sequence is constant. The *energy* of a continuous function u can thus be defined as

$$\mathcal{E}(u) = \lim_{n \to \infty} \mathcal{E}_n(u),$$

and we will say that $u \in \mathrm{Dom}\mathcal{E}$ if and only if u has finite energy. The energy form is defined on $\mathrm{Dom}\mathcal{E}$ through

$$\mathcal{E}(u, v) = \lim_{n \to \infty} \mathcal{E}_n(u, v).$$

Constant functions are the only ones with zero energy and $\mathrm{Dom}\mathcal{E}$ modulo constants is a Hilbert space with the energy form as inner product. Functions with finite energy are continuous and form a dense subspace of $C(K)$. To every function $f \in \mathrm{Dom}\mathcal{E}$ we associate its energy measure ν_f through

$$\nu_f(K_w) = \left(\frac{3}{5}\right)^{-|w|} \mathcal{E}(f_w),\ w \in W_*,$$

and, as with m, we denote also by ν_f the measure on Ω that is the pullback under π of ν_f.

There is an unbounded Laplacian denoted Δ for which the domain of definition, $\mathrm{Dom}\Delta$, is a dense subset of $C(K)$, and such that the harmonic functions are exactly those for which $\Delta f = 0$. The Laplacian Δf can be defined as a pointwise limit of difference operators $\Delta_n f|_{V_n}$ but also by means of a Green's operator G (see [2, 3, 9]). We will say that $\Delta f = u$ if f and u are continuous and

$$f = -Gu + Hf$$

where Hf is the unique harmonic function that coincides with f on the boundary and

$$Gu(x) = \int_K u(y)g(x,y)dm(y). \tag{1.2}$$

Here $g(x,y)$ is a Green's function, which is nonnegative, symmetric and $g(x,y) = 0$ if x or y is a boundary point. Since the Sierpiński gasket is a *regular* harmonic structure, $g(x,y)$ is continuous on $K \times K$ [2, Proposition 5.4]. The relation between the Laplacian and the energy form is given by the Gauss–Green's formula

$$\mathcal{E}(u,v) = -\int_K u\Delta v dm + \sum_{p\in V_0} u(p)dv(p),$$

where $dv(p)$ is a certain normal (Neumann) derivative of v at p (see [2, Proposition 7.3]). The Laplacian satisfies the following scaling identity

$$\Delta(f_w) = 5^{-|w|}(\Delta f)_w.$$

The functions we will consider in this paper are those for which Δf is Hölder continuous. We will call such functions *smooth*.

It is proved in [8] that any function in the domain of the Laplacian is Hölder continuous with Hölder exponent $\alpha = -\frac{\log \frac{3}{5}}{\log 2}$. Thus, the important eigenfunctions of Δ and multiharmonic functions, i.e. functions for which $\Delta^n f = 0$ for some n, are smooth.

A central notion in this paper is the concept of *eccentricity* of a function defined on the Sierpiński gasket

Definition 1. For a function f defined on the Sierpiński gasket K with boundary points q_0, q_1, q_2, ordered so that $f(q_0) \le f(q_1) \le f(q_2)$, we define the *eccentricity* $e(f)$ by

$$e(f) = \begin{cases} \frac{f(q_1)-f(q_0)}{f(q_2)-f(q_0)} & \text{provided } f(q_0) < f(q_2), \\ -1 & \text{if } f(q_0) = f(q_1) = f(q_2). \end{cases}$$

For every n the Sierpiński gasket is naturally decomposed into 3^n copies K_w, $w \in W_n$ of itself. Our objective is to study how eccentricities are distributed among the restrictions f_w, $w \in W_n$, of a smooth function f to these copies (cells), generalizing results obtained in [6] for harmonic functions.

Note that the eccentricity is invariant under the symmetries of the Sierpiński gasket, and also is invariant under any affine transformation $f \mapsto af + b$, $a \ne 0$. So we may assume, without loss of generality, that if f is not constant on the boundary then $f(q_0) = 0$, $f(q_1) = e$, $f(q_2) = 1$ and if f is constant on the boundary then $f(q_0) = f(q_1) = f(q_2) = 0$

The distribution of eccentricities of harmonic functions is governed by an i.f.s. $\{\psi_i\}_{i=0}^2$ acting on $(\{-1\} \cup [0,1])$ that produces the new eccentricities on each of the three smaller copies K_i, given an eccentricity on K for a harmonic function. The i.f.s. is derived from the *harmonic extension algorithm*:

$$h(x) = \frac{2}{5}h(y) + \frac{2}{5}h(z) + \frac{1}{5}h(v) \tag{1.3}$$

where $x \in V_n \setminus V_{n-1}$, where y and z are the two neighbors of x in V_n that belong to V_{n-1}, and v is the third vertex of the triangle in V_{n-1} that contains y and z.

The maps of the i.f.s. are computed by letting the maps ψ_i be defined as

$$\psi_i(e(h)) = e(h \circ F_i),$$

where $e(h)$ is the eccentricity on K for the harmonic function h. If h is constant on the boundary, then h is a constant function, thus $\psi_i(-1) = -1$ for $i = 0, 1, 2$. If h is not constant on the boundary, we let $h(q_0) = 0$, $h(q_1) = e$ and $h(q_2) = 1$. The harmonic extension algorithm gives the new values for the blow-up $h \circ F_0$:

$$e(h \circ F_0) = (2e + 1)/(e + 2) = \psi_0(e),$$

since $h(F_0(q_0)) = 0$, $h(F_0(q_1)) = (2e + 1)/5$ and $h(F_0(q_2)) = (e + 2)/5$. The other maps, ψ_1 and ψ_2 are calculated analogously and one obtains the full iterated function system for $x \in [0,1]$:

$$\begin{cases} \psi_0(x) = \dfrac{2x+1}{x+2}, \\ \psi_1(x) = \begin{cases} \dfrac{1-3x}{2-3x}, & \text{if} \quad 0 \le x \le \dfrac{1}{3} \\ 3x - 1, & \text{if} \quad \dfrac{1}{3} \le x \le \dfrac{2}{3} \\ \dfrac{1}{3x-1}, & \text{if} \quad \dfrac{2}{3} \le x \le 1, \end{cases} \\ \psi_2(x) = \dfrac{x}{3-x}. \end{cases} \tag{1.4}$$

Since the only harmonic functions for which any restriction h_w is constant on V_0 actually are the constant functions, the arbitrary definition of eccentricity for functions constant on V_0 does not give any extra information in the harmonic case. However, when working in the larger class of smooth functions it may happen that some f_w are constant on V_0 even though f is not constant. To describe the distribution of eccentricities for our larger class it is therefore necessary to define the eccentricity of functions constant on V_0.

In [6] the i.f.s. $\{\psi_i\}, i = 0, 1, 2$ acting on $[0,1]$ were studied. It was shown that, with respect to *uniform weights* $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$, there exists a unique probability measure μ_0 which is a weak limit, as $n \to \infty$, of the distribution of the eccentricities at level n, the discrete measure $3^{-n}\sum_{|w|=n}\delta(\psi_w(e))$. This limit distribution μ does not depend on the non-constant harmonic function, that is, the starting point of the iterations.

The case when each map in the i.f.s. is given the same weight as the restriction of the function to the corresponding subcell contributes to the energy of the whole function was also considered in [6]. Let h be the harmonic function with boundary values $h(q_0) = 0$, $h(q_1) = e$ and $h(q_2) = 1$. These *energy weights* will be

$$p_i(e) = \frac{5\mathcal{E}(h_i)}{3\mathcal{E}(h)},$$

which equals

$$\begin{cases} p_0(e) = \frac{1}{5}\frac{e^2+e+1}{e^2-e+1}, \\ p_1(e) = \frac{1}{5}\frac{3e^2-3e+1}{e^2-e+1}, \\ p_2(e) = \frac{1}{5}\frac{e^2-3e+3}{e^2-e+1}. \end{cases} \tag{1.5}$$

The same type of convergence result as for uniform weights holds in the energy case. There exists a unique probability measure $\mu_{\mathcal{E}}$, different from μ_0, that is the weak limit of the discrete measures $\sum_{|w|=m} p_w(e)\delta(\psi_w(e))$. Here $p_w(e) = \prod_{i=1}^m p_{w_i}(\psi_{w_{i-1}\dots w_1}(e))$.

In Section 2 we show that for a certain class of *nearly harmonic functions*, eccentricities are in $[0, 1]$ on all scales. Using the gradient defined in [10], we identify the part of the Sierpiński gasket where a smooth function is nearly harmonic locally.

In Section 3 we define an i.f.s. $\{\Psi_i\}_{i=0}^2$ that governs the distribution of eccentricities of smooth functions. This i.f.s. will be a perturbed version of the original i.f.s. (1.4), and it will act on an infinite dimensional space, since the space of smooth functions is not finite dimensional. We prove convergence of the perturbed i.f.s. to the same measures μ_0 resp. $\mu_{\mathcal{E}}$, as in [6] with uniform weights (Theorem 4) and energy weights (Theorem 5) respectively. But with uniform weights we have the restriction that the starting point must correspond to a nearly harmonic function. This restriction is not necessary in the energy case since the subset of the Sierpiński gasket where a smooth function is nearly harmonic locally has full energy measure.

The same measures μ_0 and $\mu_{\mathcal{E}}$ occurs as limit distribution of eccentricities, because the perturbation of the original i.f.s. collapses fast enough on smaller scales. This could be interpreted that every function with Hölder continuous Laplacian in the limit satisfies the $\frac{1}{5} - \frac{2}{5}$ extension algorithm.

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2. Gradient and local eccentricities

2.1. **Nearly harmonic functions.** In this section we define a class of functions for which the local eccentricities are in $[0, 1]$ on all levels. These are functions for which most of the energy comes from the harmonic part, i.e, the harmonic function with the same boundary values. We rely to a great extent on the theory of gradients developed in [10], in particular on Theorem 3 of that paper.

Let $\|f\|_\alpha$ be the Hölder norm with Hölder exponent α (with respect to the Euclidean norm in $\mathbb{R}^2$) of a function f on K. This norm is equivalent to an intrinsic norm

$$\|f\|_\rho = \|f\|_\infty + \sup_{n\geq 0} \sup_{w\in W_n} \sup_{x,y\in K_w} \rho^{-n}|f(x) - f(y)| \tag{2.1}$$

where $\alpha = -\frac{\log \rho}{\log 2}$. We will be using this intrinsic norm on the space H^α of Hölder continuous functions on K in the rest of the paper.

Following the notation in [10] we equip the space of harmonic functions $\mathcal{H}$ with the norm $\|h\|^2_{\mathcal{H}} = \mathcal{E}(h,h) + (\sum_{x\in V_0} h(x))^2$. Let $\tilde{\mathcal{H}}$ be the orthogonal complement to constant functions and $\tilde{P}$ the orthogonal projection from $\mathcal{H}$ onto $\tilde{\mathcal{H}}$. On $\tilde{\mathcal{H}}$, as well as on $\mathrm{Dom}\mathcal{E}$ modulo constants, we will use the norm $\|f\|^2 = \mathcal{E}(f,f)$.

If $\{h_1, h_2\}$ is an orthonormal basis of $\tilde{\mathcal{H}}$ then the *Kusuoka measure*, $\nu = \nu_{h_1} + \nu_{h_1}$, is independent of the choice of orthonormal basis. The Kusuoka measure is non-atomic and ν_f is absolutely continuous with respect to ν for any $f \in \mathrm{Dom}\mathcal{E}$, see [1, 5, 10]. Again, we denote by ν its pullback on Ω under π.

For $i = 0, 1, 2$ let the linear map $M_i : \mathcal{H} \to \mathcal{H}$ be defined by $M_i h = h \circ F_i$ and define $\tilde{M}_i : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$, by $\tilde{M}_i = \tilde{P} M_i \tilde{P}^*$. The Sierpiński gasket is a *non-degenerate* harmonic structure, i.e., the restriction of any non-constant harmonic function to any K_w, $w \in W_*$ is non-constant, since the matrices $\tilde{M}_i$, $i = 0, 1, 2$ are invertible. For any continuous f we denote by Hf the unique harmonic function that coincides with f on V_0 and let $\tilde{H} = \tilde{P}H$. In [10] $\mathrm{Grad}_w f$ for $w \in W_n$ is defined as

$$\mathrm{Grad}_w f = \tilde{M}_w^{-1} \tilde{H}(f_w),$$

where $\tilde{M}_w = \tilde{M}_{w_n} \ldots \tilde{M}_{w_1}$ and

$$\mathrm{Grad}_\omega f = \lim_{n\to\infty} \mathrm{Grad}_{[\omega]_n} f,$$

for $\omega \in \Omega$ whenever the limit exists.

Hölder continuity of Δf gives the following estimate of $\|\mathrm{Grad}_\omega f\|$, which is a refinement of Theorem 3 in [10].

Theorem 1 ([10] Theorem 3)**.** *Suppose that Δf is Hölder continuous on the Sierpiński gasket, that is $|\Delta f(x) - \Delta f(y)| \leqslant c\rho^n$ if $x, y \in K_w$, $w \in W_n$. Then $\mathrm{Grad}_\omega f$ is defined for every $\omega \in \Omega$ and*

$$\|\mathrm{Grad}_\omega f - \tilde{H} f\| \leqslant 8\left(\frac{c}{1-\rho} + \|\Delta f(x)\|_\infty\right). \tag{2.2}$$

The estimate (2.2) *also holds for $\mathrm{Grad}_w f$, $w \in W_*$.*

The map $\omega \mapsto \mathrm{Grad}_\omega f$ is continuous at ω in the standard topology of Ω if ω is not constant after a finite segment or $\Delta f(\pi(\omega)) = 0$.

For the convenience of the reader we mention that the proof in [10] consists of writing $\mathrm{Grad}_w f$ as a telescoping sum of terms $\mathrm{Grad}_{[w]_{n+1}} f - \mathrm{Grad}_{[w]_n} f$, and carefully estimating these terms using the Green's formula (1.2), and properties of the matrix $\tilde{M}^{-1}_{[w]_n}$. The constant 8 in (2.2) is not explicitly found in [10], however it follows from the argument there by inserting elementary estimates of the terms h_a and h_s into the proof. The estimates we use are

$$\|h_a\| \leq \frac{3\sqrt{6}}{5} c\rho^n$$

and

$$\|h_s\| \leq \frac{1}{\sqrt{2}} \|\Delta f\|_\infty.$$

Remark 1. If x is a point in K, then the definition of the gradient of f at x is more delicate. If $x \in K$ is not a junction point, then there is a unique $\omega \in \Omega$ such that $x = \pi(\omega)$. Then one can see that the map $x = \pi(\omega) \mapsto \mathrm{Grad}_\omega f$ is well defined, and is continuous at x in the topology of K.

However, $x = \pi(\omega)$ is a boundary point if and only if ω is constant, and $x = \pi(\omega)$ is a junction point if and only if ω is constant after a finite segment. If $x \in K$ is a junction point, then there are two different $\omega_1, \omega_2 \in \Omega$ such that $x = \pi(\omega_1) = \pi(\omega_2)$. Then there can be two different gradients, $\mathrm{Grad}_{\omega_1} f$ and $\mathrm{Grad}_{\omega_2} f$, of f at x. It is easy to construct examples of such a situation, for example, every localized eigenfunction of the Laplacian has points with this property.

Remark 2. In [10] there was an obvious typo that $-\tilde{H}f$ was omitted in (2.2)

The following theorem gives a criterion to have all local eccentricities in $[0,1]$, i.e. for the function to have a non-constant harmonic part on all cells. It will also be a key for uniqueness of the distribution of eccentricities of such functions.

Theorem 2. *There exists a real number $\epsilon_0 > 0$, such that if f is smooth and*

$$\frac{\|\Delta f\|_\rho}{\|f\|} < \epsilon_0, \tag{2.3}$$

then $f|_{V_0}$ is not constant and

$$\|Hf_w\| \geqslant \frac{1}{2} \cdot \frac{\|f\|}{\|\tilde{M}_w^{-1}\|} > 0, \tag{2.4}$$

for any finite word w.

Proof. We write $f = Hf - Gu$, where $u = \Delta f$. If $Hf = 0$ then since

$$\mathcal{E}(Gu) = -\int_K Gu \cdot \Delta Gu dm = \int_K Gu \cdot u dm$$

$$= \int_{K\times K} g(x,y)u(x)u(y)d(m\times m)(x,y)$$

$$\leq \|g\|_\infty \|u\|_\infty^2,$$

we have

$$\frac{\|\Delta f\|_\rho}{\|f\|} \geq \frac{1}{\sqrt{\|g\|_\infty}},$$

and $f|_{V_0}$ is not constant for appropriate ϵ_o in (2.3). In the sense of the energy norm, f is a slightly perturbed harmonic function, since $\mathcal{E}(f) = \mathcal{E}(Hf) + \mathcal{E}(Gu)$ and $\mathcal{E}(Gu) \leq \|g\|_\infty \epsilon_0^2 \mathcal{E}(f)$ implies

$$\|Hf\| \geq \sqrt{(1 - \|g\|_\infty \epsilon_0^2)} \|f\|. \tag{2.5}$$

Then for $\epsilon_0 > 0$ small enough we have

$$\|Hf_w\| = \|\tilde{H}f_w\| = \left\|\tilde{H}(Hf)_w + \tilde{H}(Gu)_w\right\| = \left\|\tilde{M}_w \tilde{P}Hf + \tilde{M}_w \tilde{M}_w^{-1} \tilde{H}(Gu)_w\right\|$$

$$= \left\|\tilde{M}_w \left(\tilde{P}Hf + \tilde{M}_w^{-1}\tilde{H}(Gu)_w\right)\right\| \frac{\|\tilde{M}_w^{-1}\|}{\|\tilde{M}_w^{-1}\|} \geq \frac{1}{\|\tilde{M}_w^{-1}\|} \left\|\tilde{P}Hf + \tilde{M}_w^{-1}\tilde{H}(Gu)_w\right\|$$

$$= \frac{1}{\|\tilde{M}_w^{-1}\|} \left\|\tilde{P}Hf + \mathrm{Grad}_w Gu\right\| \geq \frac{\|Hf\| - \|\mathrm{Grad}_w Gu\|}{\|\tilde{M}_w^{-1}\|} \geq \frac{1}{2} \cdot \frac{\|f\|}{\|\tilde{M}_w^{-1}\|}.$$

The last inequality follows from Theorem 1. □

Definition 2. A smooth function f defined on K is *nearly harmonic* if f satisfies (2.3) with ϵ_0 small enough that the conclusions of Theorem 2 hold.

The term nearly harmonic stems from inequality (2.5). Note that if h is a non-constant harmonic function and u is any Hölder continuous function on K with $\|u\|_\rho = 1$, then $h + tGu$ is nearly harmonic whenever $0 \leq |t| \leq \epsilon_0 \|h\|$.

Proposition 3. *If $\rho \leq 1 - \frac{3}{20}\sqrt{\frac{3}{2}} \approx 0.816288\ldots$ then ϵ_0 is independent of ρ, and can be put to $\epsilon_0 = 0.06$*

Proof. To give a numerical value of ϵ_0 it is necessary to estimate the supremum norm of the Green's function g, which is defined by (see [2] and [3]),

$$g(x,y) = \sum_{w \in W_* \cup \varnothing} r_w \Psi_w(x,y), \tag{2.6}$$

where

$$\Psi_w(x,y) = \begin{cases} \Psi((F_w)^{-1}(x)), (F_w)^{-1}(y)) & \text{if } x, y \in K_w \\ 0 & \text{otherwise} \end{cases}$$

and

$$\Psi(x,y) = \sum_{p,q \in V_1 \setminus V_0} X_{p,q} \psi_p(x) \psi_q(y). \tag{2.7}$$

Since the functions ψ_p are 1-harmonic the maximum of Ψ will be obtained for x and y in V_1, which gives $\|\Psi\|_\infty = \frac{9}{50}$.

For any pair of points x and y, it is clear that $\Psi_w(x,y)$ can be non-zero for more than one $w \in W_k$, only if x and y lie in V_k, but for such points $\Psi_w(x,y) = 0$. Thus, for every k, there can only be at most one non-zero term $\Psi_w(x,y)$, $w \in W_k$, and

$$\|g\|_\infty \le \sum_{k=0}^{\infty} \left(\frac{3}{5}\right)^k \|\Psi\|_\infty = \frac{9}{20}.$$

From the proof of Theorem 1 it follows that if $\rho \le 1 - \frac{3}{20}\sqrt{\frac{3}{2}}$ the sum of the asymmetric parts are bounded by $8c$ and thus the right hand side of (2.2) can be replaced by $8\|\Delta f\|_\rho$. In the last step of the proof of Theorem 2 we choose ϵ_0 small enough that

$$\|Hf\| - \|\mathrm{Grad}_w Gu\| \ge \left(\sqrt{(1 - \|g\|_\infty \epsilon_0^2)} - 8\epsilon_0\right)\|f\| \ge \frac{1}{2}\|f\|.$$

which holds for $\epsilon_0 = 0.06$. This value is also small enough to assure that $f|_{V_0}$ is not constant. □

Remark 3. In [4] it is conjectured that $\|g\|_\infty = 178839/902500$.

Remark 4. Note that in the important case $\rho = \frac{3}{5}$, which includes all functions whose Laplacian is itself in DomΔ, the hypothesis of Proposition 3 is satisfied.

Remark 5. The value $\frac{1}{2}$ in (2.5) is of course arbitrarily chosen from $(0,1)$. Replacing it with a number close to 0, it is possible to obtain a value of ϵ_0 arbitrarily close to $\frac{1}{\sqrt{64+\|g\|_\infty}}$ in Proposition 3. Also note that we can change $\frac{1}{2}$ to a factor $b(\rho) \in (0,1)$ depending on ρ to have Proposition 3 valid for more values of ρ. For $\rho < 1 - \frac{3\sqrt{6}\epsilon_0}{5\sqrt{1-\|g\|_\infty \epsilon_o^2}}$ it is possible to choose $b(\rho)$ so that Proposition 3 is valid but it seems impossible to have a value ϵ_0 valid for all ρ.

2.2. **Eccentricities of restrictions of smooth functions.** In this section we show that the value of eccentricities of restrictions of smooth functions depend on whether the gradient vanishes or not. In particular we prove that restrictions of smooth functions are nearly harmonic on small enough cells where the gradient does not vanish.

Proposition 4. *Suppose f is a smooth function. Let O be the subset of Ω where* $\mathrm{Grad}_\omega f \neq 0$. *Then for any $\epsilon > 0$ there exists an open set $O_\epsilon \subseteq O$ with the following property. For any $\omega \in O_\epsilon$ there is n such that*

$$\frac{\|\Delta f_{[\omega]_m}\|_\rho}{\|f_{[\omega]_m}\|} < \epsilon \tag{2.8}$$

for all $m \geqslant n$. Moreover, $O \backslash O_\epsilon$ consists only of sequences which are constant after a finite segment. In particular, $O \backslash O_\epsilon$ is at most countable.

Proof. We have,

$$\|\Delta f_w\|_\rho \leqslant 5^{-|w|}\|\Delta f\|_\rho \tag{2.9}$$

and

$$\|f_{[\omega]_m}\| \geqslant \|\tilde{H} f_{[\omega]_m}\| = \|\tilde{M}_{[\omega]_m}\mathrm{Grad}_{[\omega]_m} f\| \geqslant \frac{1}{\|\tilde{M}^{-1}_{[\omega]_m}\|}\|\mathrm{Grad}_{[\omega]_m} f\|. \tag{2.10}$$

Suppose $\mathrm{Grad}_{\omega_0} f \neq 0$ and ω_0 is not constant after a finite segment. Then $\liminf_{m\to\infty}\|\mathrm{Grad}_{[\omega]_m} f\| > 0$ uniformly in a neighborhood of ω_0, we even have for some n that $\|\mathrm{Grad}_{[\omega_0]_n w} f\| \geq c_{\omega_0}\|\mathrm{Grad}_{\omega_0} f\|$ for every $w \in W_*$. In addition, $\|\tilde{M}_j^{-1}\| = 5$ and $\lim_{n\to\infty} 5^{-n}\|\tilde{M}^{-1}_{[\omega_0]_n}\| = 0$ by the estimate in Theorem 2 in [10] (see also Lemma 8 below). We thus have

$$\frac{\|\Delta f_{[\omega_0]_n w}\|_\rho}{\|f_{[\omega_0]_n w}\|} \leq \frac{5^{-n}5^{-|w|}\|\tilde{M}^{-1}_{[\omega_0]_n w}\|\|\Delta f\|_\rho}{\|\mathrm{Grad}_{[\omega_0]_n w} f\|} \leq \frac{5^{-n}\|\tilde{M}^{-1}_{[\omega_0]_n}\|\|\Delta f\|_\rho}{c_{\omega_0}\|\mathrm{Grad}_{\omega_0} f\|},$$

for every $w \in W_*$. This completes the proof.

□

Corollary 5. *Suppose f is a non-constant smooth function. Then for any $\epsilon > 0$ there exists $W'_\epsilon \subseteq W_*$ such that*

$$\frac{\|\Delta f_w\|_\rho}{\|f_w\|} < \epsilon \tag{2.11}$$

for all w that can be written as $w = w'w_*$ where $w' \in W'_\epsilon$ and $w_* \in W_*$. Moreover, if O is the subset of Ω where $\mathrm{Grad}_\omega f \neq 0$, then $\pi(O)\backslash(\bigcup_{w\in W'_\epsilon} K_w)$ consists only of boundary and junction points. In particular, this set is at most countable.

Proof. As W'_ϵ take the set of all $[\omega]_n$ with $\omega \in O_\epsilon$ not constant after a finite segment, where n is the least possible value for which (2.8) holds. Then apply the projection π to the objects in the previous corollary. □

This corollary tells us that any restriction f_w, $w \in W'_{\epsilon_0}$ is nearly harmonic. We want to show that f is constant on cells whose intersection with $\cup_{w\in W'_{\epsilon_0}} K_w$ is at most finite. This does not follow directly from Theorem 1 since the set $\pi(O)\backslash(\bigcup_{w\in W'_\epsilon} K_w)$ might intersect such cells. We will need the following result.

Proposition 6. *Suppose f is a smooth function and that*

$$\nu\left(\{\omega \in \Omega \mid \mathrm{Grad}_\omega f = 0\}\right) = 1,$$

where ν is the Kusuoka measure. Then f is constant.

Proof. We prove that $\mathcal{E}(f) = 0$. Let f_n be the n-harmonic function that coincides with f on V_n. Then $\mathcal{E}(f) = \lim_n \mathcal{E}(f_n)$. Let

$$g_n = \sum_{w\in W_n} < \mathrm{Grad}_w f, Z_n(w)\mathrm{Grad}_w f > 1_{K_w}$$

where 1_{K_w} denotes the characteristic function of K_w and

$$Z_n(w) = \frac{\tilde{M}_w^* \tilde{M}_w}{Tr \tilde{M}_w^* \tilde{M}_w}$$

It is noted in [10, section 4] that

$$\mathcal{E}(f_n) = \int_K g_n d\nu.$$

Theorem 1 implies g_n is uniformly bounded and $\mathrm{Grad}_\omega f = 0$ for ν a.e. ω gives $\lim_{n\to\infty} g_n(x) = 0$ for ν a.e. x. Dominated convergence completes the proof.

□

Remark 6. If the set

$$K_z \cap \left(\cup_{w \in W'_{\epsilon_0}} K_w\right), \ z \in W_*$$

is finite or empty, then f_z is constant, since by Corollary 5, $\mathrm{Grad}_\omega f_z \neq 0$ for at most a countable number of ω, and ν has no atoms, so Proposition 6 applies. The converse is trivially true.

For smooth functions, depending on where in K a point x lies, restrictions to small enough neighborhoods of x will exhibit one of three possible behaviors. Either they will be constant, nearly harmonic or exhibit what we will call exceptional behavior.

Theorem 3. *Let f be a smooth function on K. Then there are sets K_f^H, K_f^C and K_f^E such that*

$$K = K_f^H \cup K_f^C \cup K_f^E$$

where pairwise intersections between the sets in the union are at most countable and such that f is nearly harmonic locally on K_f^H, in the sense that the restriction to any cell contained in K_f^H is nearly harmonic. Also f is constant locally on K_f^C in the same sense. The set K_f^E is closed and nowhere dense.

Proof. The different parts of K can be constructed as follows. Partition W_n into

$$W_{n,f}^H = \{w \mid [w]_k \in W'_{\epsilon_0} \text{ for some } k \leq n\}$$

$$W_{n,f}^C = \{w \mid f|_{K_w} = \text{const}\}$$

and $W_{n,f}^E$ what is left. Then define three sequences of subsets of K

$$K_{n,f}^H = \cup_{w \in W_{n,f}^H} K_w, \ K_{n,f}^C = \cup_{w \in W_{n,f}^C} K_w, \text{ and } K_{n,f}^E = \cup_{w \in W_{n,f}^E} K_w,$$

with the property that

$$K = K_{n,f}^H \cup K_{n,f}^C \cup K_{n,f}^E.$$

Note that $K_{n,f}^H$ and $K_{n,f}^C$ are increasing and $K_{n,f}^E$ decreasing. Define

$$K_f^H = \cup_{n \geq 1} K_{n,f}^H, \ K_f^C = \cup_{n \geq 1} K_{n,f}^C \text{ and } K_f^E = \cap_{n \geq 1} K_{n,f}^E.$$

Then $K = K_f^H \cup K_f^C \cup K_f^E$ with pairwise intersections at most countable, f is nearly harmonic (constant) locally in K_f^H (K_f^C) and the closed set K_f^E has empty interior (Remark 6). $\square$

On the exceptional set K_f^E we can not say anything about the local behavior of f. If $x = \pi(\omega) \in K_f^E$ is not a junction or boundary point then $\mathrm{Grad}_\omega f = 0$ but we don't have $\mathrm{Grad}_{[\omega]_n} f = 0$ for n big enough. Thus the eccentricity of $f_{[\omega]_n}$ might very well jump between -1 and $[0,1]$.

This partition shows that in the case of uniform weights we can not hope for convergence of the perturbed i.f.s. for arbitrary starting points since possibly $m(K_f^E) > 0$. But in the energy case this is true because of the following fact.

Proposition 7. *If f is a function with Hölder continuous Laplacian then $\nu_f(K_f^C) = \nu_f(K_f^E) = 0$.*

Proof. It is trivial that $\nu_f(K_f^C) = 0$, so we can suppose that f is not constant on any subcell of K.

Let f_n be the n-harmonic function that coincides with f on V_n. From [10, section 4], we know that

$$\nu_{f_n}(K_{m,f}^E) = \sum_{w \in W_m^E} \sum_{w' \in W_{n-m}} < \mathrm{Grad}_{ww'} f, Z_n(ww')\mathrm{Grad}_{ww'} f > \nu(K_{ww'}),$$

for $n \geq m$.

Then, because $K_{m,f}^E$ is a finite union of cells, we have

$$\nu_f(K_{m,f}^E) = \lim_{n\to\infty} \nu_{f_n}(K_{m,f}^E)$$

$$= \lim_{n\to\infty} \sum_{w \in W_m^E} \sum_{w' \in W_{n-m}} < \mathrm{Grad}_{ww'} f, Z_n(ww')\mathrm{Grad}_{ww'} f > \nu(K_{ww'})$$

$$= \int_{\pi^{-1}(K_{m,f}^E)\setminus\pi^{-1}(K_f^E)} < \mathrm{Grad}_\omega f, Z(\omega)\mathrm{Grad}_\omega f > d\nu(\omega),$$

where we have used that $\mathrm{Grad}_\omega f = 0$, ν a.e. on $\pi^{-1}(K_f^E)$. Since Hölder continuity of Δf implies that $< \mathrm{Grad}_\omega f, Z(\omega)\mathrm{Grad}_\omega f >$ is uniformly bounded, we see that

$$\nu_f(K_f^E) = \lim_{m\to\infty} \nu_f(K_{m,f}^E) = 0.$$

$\square$

3. Distribution of eccentricities

3.1. **Perturbation of the iterated function system.** To study the limit distribution of eccentricities of smooth functions it is necessary to extend the original i.f.s. on $\{-1\} \cup [0,1]$ describing the harmonic case to $(\{-1\} \cup [0,1]) \times H^\alpha$, where H^α is the space of Hölder continuous functions on K. For this purpose we make the following identification, the notation for which will be used throughout this section.

Let $(e, u) \in (\{-1\} \cup [0,1]) \times H^\alpha$ correspond to a function with Hölder continuous Laplacian through the following identification. If $e \in [0,1]$ let $f = h - Gu$ where h is the unique harmonic function such that $h(q_0) = 0$, $h(q_1) = e$ and $h(q_2) = 1$, and if $e = -1$ let $f = -Gu$. After composition with a symmetry of the Sierpiński gasket and an affine transformation any function with Hölder continuous Laplacian is of this form so it is sufficient to study such functions.

The i.f.s. that describes the distribution of eccentricities on this larger class of functions is of course the same as the original i.f.s. on $(\{-1\} \cup [0,1]) \times \{0\}$ but for non-zero second coordinate the maps are perturbed to

$$\Psi_j(e,u) = \begin{cases} \left(e(f_j), \frac{u'_j}{5(\max_{V_0} f_j - \min_{V_0} f_j)}\right) & \text{if } f_j|_{V_0} \text{ is not constant} \\ \\ \left(-1, \frac{u_j}{5}\right) & \text{if } f_j|_{V_0} \text{ is constant} \end{cases} \tag{3.1}$$

with $u'_j = u_j \circ R$ where R is a symmetry of K such that $f'_j = f_j \circ R$ has the property that $\max_{V_0} f'_j$ is achieved at the vertex q_2 of K and $\min_{V_0} f'_j$ is achieved at the vertex q_0 of K. Thus, in the above identification $\Psi_j(e,u)$ corresponds to f_j if $f_j|_{V_0}$ is constant and to $\frac{f'_j}{(\max_{V_0} f_j - \min_{V_0} f_j)}$ if $f_j|_{V_0}$ is not constant.

In the case of energy weights there will also be new weights $p_i(e,u)$ that depend on the second coordinate.

For ease of notation we will let $\Psi_w = \Psi_{w'_1} \circ \cdots \circ \Psi_{w'_n}$ where $w \mapsto w'$ is the permutation of W_n such that

$$\Psi_{w'}(e,u) = \begin{cases} \left(e(f_w), \frac{u'_w}{5^n(\max_{V_0} f_w - \min_{V_0} f_w)}\right) & \text{if } f_w|_{V_0} \text{ is not constant} \\ \\ \left(-1, \frac{u_w}{5^n}\right) & \text{if } f_w|_{V_0} \text{ is constant.} \end{cases}$$

Since we will only be interested in estimating the norm of the second coordinate we will skip the prime notation.

Lemma 8. *The second component in the perturbed i.f.s.* Ψ_w *tends to* 0 *for every orbit* $\omega \in \Omega$ *that is not constant after a finite segment, from any starting point* $(e,u) \in (\{-1\} \cup [0,1]) \times H^\alpha$ *corresponding to a function* f *such that* $\mathrm{Grad}_\omega f \neq 0$.

Proof. We know from Corollary 5 that $[\omega]_m = w$ for some $w \in W'_{\epsilon_0}$. Then according to Theorem 2

$$(\max_{V_0} f_{[\omega]_n} - \min_{V_0} f_{[\omega]_n})^2 \geqslant \frac{1}{3}\mathcal{E}(Hf_{[\omega]_n}) \geqslant \mathrm{Const}\frac{\mathcal{E}(f_{[\omega]_m})}{\|\tilde{M}^{-1}_{[\sigma^m(\omega)]_{n-m}}\|^2}.$$

With the estimate

$$\|\tilde{M}^{-1}_{[\omega]_n}\| \leq 5^n \beta^{C(\omega,n)}, \tag{3.2}$$

where $\beta < 1$ and $C(\omega, n)$ is the number of changes in $[\omega]_n$, from the proof of Theorem 2 in [10], it follows that for any $\omega \in \Omega$ we have

$$5^n(\max_{V_0} f_{[\omega]_n} - \min_{V_0} f_{[\omega]_n}) \geqslant \text{Const} \frac{\|f_{[\omega]_m}\|}{\beta^{C(\sigma^m(\omega), n-m)}} \to \infty. \tag{3.3}$$

We conclude that the second term of the iterates

$$\frac{u_{[\omega]_n}}{5^n(\max_{V_0} f_{[\omega]_n} - \min_{V_0} f_{[\omega]_n})} \to 0$$

in Hölder norm. $\square$

Remark 7. Note that if (e, u) corresponds to a nearly harmonic function f then Lemma 8 is true without any assumption on $\text{Grad}_\omega f$. However, for nearly harmonic functions $\text{Grad}_\omega f \neq 0$ for every $\omega \in \Omega$ anyway, because of (2.2).

3.2. **Limit distribution with uniform weights.** It was shown in [6] that the i.f.s. (1.4) on $[0, 1]$ with uniform weights has a unique invariant measure μ_0 in the sense that

$$\mu_0 = \sum_{j=1}^{3} \frac{1}{3} \mu_0 \circ \psi_j^{-1}. \tag{3.4}$$

Our extension of this i.f.s. to $\{-1\} \cup [0, 1]$ trivially gives rise to some new invariant measures that satisfy (3.4), namely

$$\mu_t = t\delta_{-1} + (1 - t)\mu_0, \ t \in [0, 1].$$

Since $\Psi_j(x, 0) = (\psi_j(x), 0)$, it is obvious that $\mu_t \times \delta_0$ are invariant measures of the perturbed i.f.s. (3.1) in the sense that

$$\mu_t \times \delta_0 = \sum_{j=1}^{3} \frac{1}{3} (\mu_t \times \delta_0) \circ \Psi_j^{-1}.$$

We define the action of an operator A on a probability measure λ on $(\{-1\} \cup [0, 1]) \times H^\alpha$ by

$$A\lambda(B) = \sum_{j=1}^{3} \frac{1}{3} \lambda(\Psi_j^{-1}(B)) = \int_{(\{-1\}\cup[0,1])\times H^\alpha} P((e, u), B) d\lambda(e, u),$$

where B is any Borel subset of $(\{-1\} \cup [0, 1]) \times H^\alpha$ and

$$P((e, u), B) = \sum_{j=1}^{3} \frac{1}{3} \delta_{\Psi_j(e,u)}(B) = m(\omega \mid \psi_{[\omega]_1}(e, u) \in B)$$

is the probability, with respect to uniform weights, of ending up in B when starting from (e, u). Then the invariant measures $\mu_t \times \delta_0$ are exactly the fixed points of A.

To state our main result we need the following definition.

Definition 9. The *Wasserstein metric* for probability measures μ and ν on a measurable set X is defined as

$$d_W(\mu,\nu) = \sup_{\|f\|_{Lip}\leq 1} \left| \int_X f d\mu - \int_X f d\nu \right|.$$

In [6] it was proven that $A^n\delta_e \to \mu_0$ in the Wasserstein metric, regardless of the starting point e. Next, we prove that the limit distribution of eccentricities for nearly harmonic functions is the same as for harmonic functions.

Theorem 4. *For any* $(e,u) \in (\{-1\}\cup[0,1])\times H^\alpha$ *corresponding to a nearly harmonic function* f*,*

$$A^n\delta_{(e,u)} \to \mu_0 \times \delta_0$$

in the Wasserstein metric.

Theorem 4 does not follow immediately from Lemma 8. That Lemma only tells us that if $A^n\delta_{(e,u)}$ converges in the Wasserstein metric it must converge to a measure with support in $(\{-1\}\cup[0,1])\times\{0\}$. However, to prove Theorem 4 it is necessary to show that the perturbation of the original i.f.s. is, in some sense, continuous in the second coordinate; if a function is close enough to harmonic, eccentricities distribute almost like in the harmonic case.

Lemma 10. *Suppose* $f|_{V_0}$ *and* $f_i|_{V_0}$*,* $i = 0,1,2$ *are not constant. If* $\|u\|_\infty \leq \frac{1}{20\|g\|_\infty}$ *then*

$$|e(f_i) - \psi_i(e)| \leq \text{Const}\|u\|_\infty \quad i = 0,1,2. \tag{3.5}$$

Proof. Let $V_1 \setminus V_0 = \{p_0,p_1,p_2\}$ where $p_0 = F_1(q_2)$, $p_1 = F_2(q_0)$, $p_2 = F_0(q_1)$ and $f = Hf - Gu$ with $f(q_0) = 0 \leq f(q_1) = e \leq f(q_2) = 1$. The harmonic extension algorithm (1.1) gives that $Hf(p_0) = \frac{2}{5}+\frac{2e}{5}$, $Hf(p_1) = \frac{2}{5}+\frac{e}{5}$, and $Hf(p_2) = \frac{1}{5}+\frac{2e}{5}$.

Under the hypothesis of the lemma it is clear from (1.2) that $\|Gu\|_\infty \leq \frac{1}{20}$ and this is enough to control in what point of $F_i(V_0)$ either $\max_{V_0} f_i$ or $\min_{V_0} f_i$ will occur. In the case $i = 0$ it is clear that $\min_{V_0} f_0 = f(q_0) = 0$ and independently of e we have

$$e(f_1) = \min\left(\frac{\frac{2}{5}+\frac{e}{5}+Gu(p_1)}{\frac{1}{5}+\frac{2e}{5}+Gu(p_2)}, \frac{\frac{1}{5}+\frac{2e}{5}+Gu(p_2)}{\frac{2}{5}+\frac{e}{5}+Gu(p_1)} \right).$$

Define

$$\text{ecc}_0 : I \times [-\frac{1}{20},\frac{1}{20}] \times [-\frac{1}{20},\frac{1}{20}] \to R$$

$$(e,x,y) \mapsto \min\left(\frac{\frac{2}{5}+\frac{e}{5}+x}{\frac{1}{5}+\frac{2e}{5}+y}, \frac{\frac{1}{5}+\frac{2e}{5}+y}{\frac{2}{5}+\frac{e}{5}+x} \right).$$

Note that ecc_0 is Lipschitz continuous and that $ecc_0(e,0,0) = \psi_0(e)$ and $ecc_0(e,Gu(p_1),Gu(p_2)) = e(f_0)$, hence

$$|e(f_0) - \psi_0(e)| \leq \text{Const}\|(e,Gu(p_1),Gu(p_2)) - (e,0,0)\|$$

$$\leq \mathrm{Const}\max(|Gu(p_1)|, |Gu(p_2)|) \leq \mathrm{Const}\|Gu\|_\infty \leq \mathrm{Const}\|u\|_\infty.$$

For $i = 2$ we know that $\max_{V_0} f_2 = f(q_2) = 1$ so

$$e(f_2) = \min\left(\frac{\left|\frac{e}{5} + Gu(p_0) - Gu(p_1)\right|}{\frac{3}{5} - \frac{e}{5} - Gu(p_1)}, \frac{\left|\frac{e}{5} + Gu(p_0) - Gu(p_1)\right|}{\frac{3}{5} - \frac{2e}{5} - Gu(p_0)}\right)$$

and a similar proof as for $i = 0$ can be done with

$$ecc_2(e, x, y) = \min\left(\frac{\left|\frac{e}{5} + x - y\right|}{\frac{3}{5} - \frac{e}{5} - y}, \frac{\left|\frac{e}{5} + x - y\right|}{\frac{3}{5} - \frac{2e}{5} - x}\right).$$

The case $i = 2$ is a mixture of the two previous cases and is treated similarly.

□

Proof of Theorem 4. We must estimate

$$d_W(A^N\delta_{(e,u)}, \mu_0 \times \delta_0)$$

$$= \sup_{\|h\|_{\mathrm{Lip}}\leq 1}\left|\frac{1}{3^N}\sum_{w\in W_N} h(\Psi_w(e, u)) - \int h(x, 0)d\mu_0(x)\right|$$

$$= \sup_{\|h\|_{\mathrm{Lip}}\leq 1}\left|\frac{1}{3^N}\sum_{w\in W_N} h(e(f_w), \frac{u_w}{5^N(\max_{V_0} f_w - \min_{V_0} f_w)}) - \int h(x, 0)d\mu(x)\right|.$$

For this it is necessary to first iterate a certain number of steps so that the norm of the second coordinate is small enough on most subcells of K, and then use the result obtained in [6, Thm 5.6] together with Lemma 10 on those subcells.

Inequality (3.3) from the proof of Lemma 8 tells us that

$$\left\|\frac{u_w}{5^{|w|}(\max_{V_0} f_w - \min_{V_0} f_w)}\right\|_\infty \leq \mathrm{Const}\frac{\|u\|_\infty \beta^{C(w,n)}}{\|f\|} \tag{3.6}$$

for any $w \in W_n$. In the rest of the proof we will always assume that M is big enough that

$$\mathrm{Const}\frac{\|u\|_\infty\beta^M}{\|f\|} < \frac{1}{20\|g\|_\infty},$$

so that Lemma 10 applies whenever $C(w, n) \geqslant M$.

Let m, m' and M be such that $m+m' = N$ and $M \leq m$. Then for any $\|h\|_{\mathrm{Lip}} \leq 1$ we have

$$\left|\frac{1}{3^N}\sum_{w\in W_N} h(e(f_w), \frac{u_w}{5^{|w|}(\max_{V_0} f_w - \min_{V_0} f_w)}) - \int h(x, 0)d\mu_0(x)\right|$$

$$\leq \frac{1}{3^N}\sum_{w\in W_N, C(w,m)<M}\left|h(e(f_w), \frac{u_w}{5^{|w|}(\max_{V_0} f_w - \min_{V_0} f_w)}) - \int h(x, 0)d\mu_0(x)\right|$$

$$+\frac{1}{3^N}\sum_{w\in W_N, C(w,m)\geq M}\left|h(e(f_w),\frac{u_w}{5^{|w|}(\max_{V_0}f_w-\min_{V_0}f_w)})-h(e(f_w),0)\right|$$

$$+\frac{1}{3^m}\sum_{w_0\in W_m, C(w_0,m)\geqslant M}\frac{1}{3^{m'}}\sum_{z\in W_{m'}}|h(e(f_{w_0z}),0)-h(\psi_z(e(f_{w_0})),0)|$$

$$+\frac{1}{3^m}\sum_{w_0\in W_m, C(w_0,m)\geqslant M}\left|\frac{1}{3^{m'}}\sum_{z\in W_{m'}}h(\psi_z(e(f_{w_0}),0)-\int h(x,0)d\mu_0(x)\right|.$$

The last term in the previous inequality is in [6, Thm 5.6] shown to be bounded by Const$a^{m'}$, with $a<1$. To estimate the third term, let $B=\max_{i=1,2,3}\|\psi_i\|_{\mathrm{Lip}}$ and use that if $C(w_0,m)\geqslant M$ we obtain by using (3.2) and Lemma 10 that for every $z\in W_{m'}$

$$\begin{aligned}
&|e(f_{w_0z})-\psi_z(e(f_{w_0}))|\\
&\leq |e(f_{w_0z})-\psi_{z_{m'}}(e(f_{w_0z_1\dots z_{m'-1}}))|\\
&+|\psi_{z_{m'}}(e(f_{w_0z_1\dots z_{m'-1}}))-\psi_{z_{m'}z_{m'-1}}(e(f_{w_0z_1\dots z_{m'-2}}))|\\
&+\dots\\
&+|\psi_{z_{m'}\dots z_2}(e(f_{w_0z_1}))-\psi_z(e(f_{w_0}))|\\
&\leq \mathrm{Const}\sum_{k=0}^{m'-1}B^k\frac{\|u\|_\infty\beta^M}{\|f\|}.
\end{aligned}$$

We can conclude that

$$\begin{aligned}
&\left|\frac{1}{3^N}\sum_{w\in W_N}h(e(f_w),\frac{u_w}{5^{|w|}(\max_{V_0}f_w-\min_{V_0}f_w)})-\int h(x,0)d\mu_0(x)\right|\\
&\leq 2m[C(\omega,m)<M]+\frac{1}{3^N}\sum_{w\in W_N, C(w,m)\geqslant M}\left\|\frac{u_w}{5^{|w|}(\max_{V_0}f_w-\min_{V_0}f_w)}\right\|\\
&+\mathrm{Const}\frac{B^{m'}-1}{B-1}\beta^M+\mathrm{Const}a^{m'}\\
&\leq 2m[C(\omega,m)<M]+\beta^M(\mathrm{Const}+\mathrm{Const}\frac{B^{m'}-1}{B-1})+\mathrm{Const}a^{m'}
\end{aligned}$$

where $a<1$, $\beta<1$ and $B>1$. This completes the proof.

□

With Theorem 4 we know that eccentricities of smooth functions locally has the same limit distribution of eccentricities as harmonic functions in the set K_f^H. In particular Theorem 4 remains true if $\mathrm{Grad}_\omega f\neq 0$ for every $\omega\in\Omega$. Without control on the behavior of the perturbed i.f.s. on $\pi^{-1}(K_f^E)$ it is not possible to have convergence for arbitrary starting points. But in case K_f^E is negligible we still have convergence.

Corollary 11. *If* $(e,u) \in (\{-1\} \cup [0,1]) \times H^\alpha$ *corresponds to a function* f *for which* $m(K_f^E) = 0$. *Then*

$$A^n \delta_{(e,u)} \to \mu_t \times \delta_0,$$

with $t = m(K_f^C)$, *in the Wasserstein metric.*

Proof. Partition W_n into $W_{n,f}^H$, $W_{n,f}^C$ and $W_{n,f}^E$ as in section 2. We estimate

$$d_W(A^N \delta_{(e,u)}, \mu_t \times \delta_0)$$

$$= \sup_{\|h\|_{\mathrm{Lip}} \le 1} \left| \frac{1}{3^N} \sum_{w \in W_N} h(\Psi_w(e,u)) - (1-t) \int h(x,0) d\mu_0(x) - th(-1,0) \right|$$

$$\le \sup_{\|h\|_{\mathrm{Lip}} \le 1} \left| \frac{1}{3^N} \sum_{w \in W_{N,f}^C} h(\Psi_w(e,u)) - th(-1,0) \right|$$

$$+ \sup_{\|h\|_{\mathrm{Lip}} \le 1} \left| \frac{1}{3^N} \sum_{w \in W_{N,f}^H} h(\Psi_w(e,u)) - (1-t) \int h(x,0) d\mu_0(x) \right|$$

$$+ \sup_{\|h\|_{\mathrm{Lip}} \le 1} \left| \frac{1}{3^N} \sum_{w \in W_{N,f}^E} h(\Psi_w(e,u)) \right|.$$

Since $\Psi_w(e,u) = (-1,0)$ if $w \in W_{N,f}^C$, we have

$$\sup_{\|h\|_{\mathrm{Lip}} \le 1} \left| \frac{1}{3^N} \sum_{w \in W_{N,f}^C} h(\Psi_w(e,u)) - th(-1,0) \right| \le \left| m\left(K_{N,f}^C\right) - t \right| \to 0$$

and for the last term note that

$$\sup_{\|h\|_{\mathrm{Lip}} \le 1} \left| \frac{1}{3^N} \sum_{w \in W_{N,f}^E} h(\Psi_w(e,u)) \right| \le m(K_{N,f}^E) \to 0.$$

Given $\epsilon > 0$, take n such that $m(K_f^H \setminus K_{n,f}^H) < \epsilon$. Then for any $N \ge n$ we can estimate the mid term by,

$$\sup_{\|h\|_{\mathrm{Lip}} \le 1} \left| \frac{1}{3^N} \sum_{w \in W_{N,f}^H} h(\Psi_w(e,u)) - (1-t) \int h(x,0) d\mu_0(x) \right|$$

$$\le \sup_{\|h\|_{\mathrm{Lip}}\le 1} \left| \frac{1}{3^N} \sum_{w\in W^H_{N,f}} h(\Psi_w(e,u)) - m(K^H_{n,f}) \int h(x,0)d\mu_0(x) \right|$$
$$+ \sup_{\|h\|_{\mathrm{Lip}}\le 1} \left| \big(m(K^H_{n,f}) - (1-t)\big) \int h(x,0)d\mu_0 \right|$$

$$\le \sum_{w'\in W'_{\epsilon_0}\ |w'|\le n} \frac{1}{3^{|w'|}} \sup_{\|h\|_{\mathrm{Lip}}\le 1} \left| \frac{1}{3^{N-|w'|}} \sum_{w\in W_{N-|w'|}} h(\Psi_{w'w}(e,u)) - \int h(x,0)d\mu_0(x) \right|$$
$$+2\epsilon.$$

and for each $w' \in W'_{\epsilon_0}$ with $|w'| \le n$ this supremum goes to 0 by Theorem 4.

□

3.3. **Limit distribution with energy weights.** Energy weights are naturally expressed as normalized energy measure of the subcells of level one. If f is the function corresponding to the point $(e,u) \ne (-1,0)$, then $p_i(e,u) = \bar\nu_f(K_i)$ where

$$\bar\nu_f = \frac{\nu_f}{\mathcal{E}(f)}.$$

Cells on which a function is constant do not matter since they give no contribution to the energy. Thus we can arbitrarily define $p_i(-1,0) = \frac{1}{3}$.

It was shown in [6] that there is a unique invariant measure $\mu_{\mathcal{E}}$ to the i.f.s. (1.4) on $[0,1]$ with energy weights $p_i(e) = p_i(e,0)$ satisfying

$$\mu_{\mathcal{E}} = \sum_{j=1}^{3} p_i(e)\mu_{\mathcal{E}} \circ \psi_j^{-1}. \tag{3.7}$$

The extension of the original i.f.s. with energy weights to $(\{-1\} \cup [0,1])$ will then have invariant measures

$$\mu_{\mathcal{E},t} = t\delta_{-1} + (1-t)\mu_{\mathcal{E}},$$

and clearly $\mu_{\mathcal{E},t} \times \delta_0$ are invariant measures to the perturbed i.f.s. (3.1) with energy weights, and in fact there are no others.

Proposition 12. *$\mu_{\mathcal{E},t} \times \delta_0$ are the only invariant measure for the perturbed i.f.s.* (3.1) *with energy weights.*

Proof. The result follows from Lemma 8 and Proposition 7. Suppose λ is an invariant measure. Then λ is a fixed point of the operator

$$A_{\mathcal{E}}\lambda(B) = \sum_{j=1}^{3} \int_{\Psi_j^{-1}(B)} p_j(e,u)d\lambda(e,u) = \int_{(\{-1\}\cup[0,1])\times H^\alpha} P_{\mathcal{E}}[(e,u),B]d\lambda(e,u)$$

acting on the probability measures on $(\{-1\} \cup [0,1]) \times H^\alpha$. Here $P_{\mathcal{E}}[(e,u),B] = \bar{\nu}_f(\omega \mid \Psi_{[\omega]_1}(e,u) \in B)$ is the probability, with respect to energy weights, of ending up in the Borel set B starting from (e,u). So

$$\lambda(B) = A_{\mathcal{E}}^n \lambda(B) = \int P_{\mathcal{E}}^{(n)}((e,u),B)\, d\lambda(e,u), \tag{3.8}$$

where

$$P_{\mathcal{E}}^{(n)}((e,u),B) = \bar{\nu}_f(\omega \mid \Psi_{[\omega]_n}(e,u) \in B).$$

Let $B = (\{-1\} \cup [0,1]) \times B_m$ in equality (3.8), with $B_m = \{u \in H^\alpha \mid \|u\|_\rho > \frac{1}{m}\}$. The second coordinate of $\Psi_{[\omega]_n}(e,u)$ tends to zero in Hölder norm for every $\omega \in O_\epsilon$ that is not constant after a finite segment. This is a set of full $\bar{\nu}_f$ measure thus $P_{\mathcal{E}}^{(n)}((e,u),(\{-1\} \cup [0,1]) \times B_m) \to 0$ for every (e,u).

Dominated convergence gives that $\lambda((\{-1\} \cup [0,1]) \times B_m) = 0$ and thus $\lambda((\{-1\} \cup [0,1]) \times \{0\}^c) = 0$ and the support of λ must be included in $(\{-1\} \cup [0,1]) \times \{0\}$. The only possibilities are $\lambda = \mu_{\mathcal{E},t} \times \delta_0$. □

With energy weights we have a nicer convergence result than for uniform weights since we have convergence, to the same measure, no matter what starting point. With respect to energy, the limit distribution of eccentricities is the same for all non-constant smooth functions. This is a consequence of the fact that the set K_f^H where a non-constant smooth function f is nearly harmonic locally has full energy measure.

Theorem 5. *For any $(e,u) \in (\{-1\} \cup [0,1]) \times H^\alpha$, with $(e,u) \neq (-1,0)$*

$$A_{\mathcal{E}}^n \delta_{(e,u)} \to \mu_{\mathcal{E}} \times \delta_0$$

in the Wasserstein metric.

Proof. The proof follows the same path as the proofs of Theorem 4 and Corollary 11, only some more attention to the weights has to be paid. In view of Proposition 7, one can mimic the proof of Corollary 11 to see that it is enough to consider starting points (e,u) corresponding to a nearly harmonic function f.

We must show

$$d_W(A_{\mathcal{E}}^N \delta_{(e,u)}, \mu_{\mathcal{E}} \times \delta_0)$$

$$= \sup_{\|h\|_{\mathrm{Lip}} \leq 1} \left| \sum_{w \in W_n} \bar{\nu}_f(K_w) h(\Psi_w(e,u)) - \int h(x,0)\, d\mu_{\mathcal{E}}(x) \right|$$

$$= \sup_{\|h\|_{\mathrm{Lip}} \leq 1} \left| \sum_{w \in W_n} \bar{\nu}_f(K_w) h\left(\left(e(f_w), \frac{u_w}{5^{|w|}(\max_{V_0} f_w - \min_{V_0} f_w)}\right)\right) \right.$$

$$\left. - \int h(x,0)\, d\mu_{\mathcal{E}}(x) \right| \to 0.$$

As in the proof of Theorem 4 we will always assume that M is big enough that

$$\mathrm{Const}\frac{\|u\|_\infty \beta^M}{\|f\|} < \frac{1}{20\|g\|_\infty},$$

so that Lemma 10 applies whenever $C(w,n) \geqslant M$.

Let m, m' and M be such that $m + m' = n$ and $M \leq m$. Then for any $\|h\|_{\mathrm{Lip}} \leq 1$ we have

$$\left| \sum_{w\in W_n} \bar{\nu}_f(K_w) h(e(f_w), \frac{u_w}{5^{|w|}(\max f_w - \min f_w)}) - \int h(x,0) d\mu_{\mathcal{E}}(x) \right| \tag{3.9}$$

$$\leq \sum_{w\in W_n, C(w,m)<M} \bar{\nu}_f(K_w) \left| h(e(f_w), \frac{u_w}{5^{|w|}(\max f_w - \min f_w)}) - \int h(x,0) d\mu_{\mathcal{E}}(x) \right|$$

$$+ \sum_{w\in W_n, C(w,m)\geq M} \bar{\nu}_f(K_w) \left| h(e(f_w), \frac{u_w}{5^{|w|}(\max f_w - \min f_w)}) - h(e(f_w),0) \right|$$

$$+ \sum_{w_0\in W_m, C(w_0,m)\geqslant M} \bar{\nu}_f(K_{w_0}) \sum_{z\in W_{m'}} \bar{\nu}_{f_{w_0}}(K_z) \left| h(e(f_{w_0 z}),0) - h(\psi_z(e(f_{w_0})),0) \right|$$

$$+ \sum_{w_0\in W_m, C(w_0,m)\geqslant M} \bar{\nu}_f(K_{w_0}) \sum_{z\in W_{m'}} |\bar{\nu}_{f_{w_0}}(K_z) - \bar{\nu}_{Hf_{w_0}}(K_z)| |h(\psi_z(e(f_{w_0})),0)|$$

$$+ \sum_{w_0\in W_m, C(w_0,m)\geqslant M} \bar{\nu}_f(K_{w_0}) \left| \sum_{z\in W_{m'}} \bar{\nu}_{Hf_{w_0}}(K_z) h(\psi_z(e(f_{w_0}),0) - \int h(x,0) d\mu_{\mathcal{E}}(x) \right|.$$

The three first terms can be handled as in the proof of Theorem 4. The last term in the previous inequality is in [6, Thm 5.9] shown to be bounded by $\mathrm{Const}a^{m'}$, with $a < 1$. So what is new in this proof is the fourth term.

To estimate it note that

$$\bar{\nu}_{f_{w_0}}(K_z) = \prod_{j=1}^{m'} \bar{\nu}_{f_{w_0 z_1 \dots z_{j-1}}}(K_{z_j})$$

and

$$\bar{\nu}_{Hf_{w_0}}(K_z) = \prod_{j=1}^{m'} \bar{\nu}_{(Hf_{w_0})_{z_1 \dots z_{j-1}}}(K_{z_j})$$

so using the fact that all terms in the product are bounded by 1

$$|\bar{\nu}_{f_{w_0}}(K_z) - \bar{\nu}_{Hf_{w_0}}(K_z)| \leq \sum_{j=1}^{m'} |\bar{\nu}_{f_{w_0 z_1 \dots z_{j-1}}}(K_{z_j}) - \bar{\nu}_{(Hf_{w_0})_{z_1 \dots z_{j-1}}}(K_{z_j})|, \tag{3.10}$$

and each term can be estimated by

$$|\bar{\nu}_{f_{w_0 z_1 \dots z_{j-1}}}(K_{z_j}) - \bar{\nu}_{(Hf_{w_0})_{z_1 \dots z_{j-1}}}(K_{z_j})| \tag{3.11}$$
$$\leq |\bar{\nu}_{f_{w_0 z_1 \dots z_{j-1}}}(K_{z_j}) - \bar{\nu}_{Hf_{w_0 z_1 \dots z_{j-1}}}(K_{z_j})|$$
$$+|\bar{\nu}_{Hf_{w_0 z_1 \dots z_{j-1}}}(K_{z_j}) - \bar{\nu}_{(Hf_{w_0})_{z_1 \dots z_{j-1}}}(K_{z_j})|.$$

To bound the first term of (3.11) we show that if $f = Hf - Gu$ with $\max_{V_0} f = 1$ and $\min_{V_0} f = 0$, then for $\|u\|_\infty$ small enough

$$|\bar{\nu}_f(K_i) - \bar{\nu}_{Hf}(K_i)| \leq \text{Const}\|u\|_\infty. \tag{3.12}$$

Since the difference in the first term does not change if we rescale $f_{w_0 z_1 \dots z_{j-1}}$ as in the i.f.s. (3.1) and that u for this function is bounded by (3.6) inequality (3.12) will hold for large enough M.

Note the estimates $\mathcal{E}(Gu) \leq \text{Const}\|u\|_\infty^2$ and $\mathcal{E}((Gu)_i) \leq \text{Const}\|u\|_\infty^2$ that follows by the same reasoning as in the proof of Theorem 2 and $\mathcal{E}((Hf)_i, (Gu)_i) = \mathcal{E}_0((Hf)_i, (Gu)_i) \leq \text{Const}\|u\|_\infty$, where the equality holds since Hf is harmonic.

Thus (3.12) for small enough $\|u\|_\infty$ is a consequence of the equalities

$$\frac{3}{5}\bar{\nu}_f(K_i) = \frac{\mathcal{E}(f_i)}{\mathcal{E}(f)} = \frac{\mathcal{E}((Hf)_i) + \mathcal{E}((Gu)_i) + 2\mathcal{E}((Hf)_i, (Gu)_i)}{\mathcal{E}(Hf) + \mathcal{E}(Gu)}$$

and

$$\frac{3}{5}\bar{\nu}_{Hf}(K_i) = \frac{\mathcal{E}((Hf)_i)}{\mathcal{E}(Hf)}$$

together with $\mathcal{E}(Hf) \geq \frac{3}{2}$. Using (3.6) once more gives

$$\left|\bar{\nu}_{f_{w_0 z_1 \dots z_{j-1}}}(K_{z_j}) - \bar{\nu}_{Hf_{w_0 z_1 \dots z_{j-1}}}(K_{z_j})\right| \leq \text{Const}\beta^M.$$

Using Lemma 10 and once again that the second coordinate of the iterates satisfies (3.6) we estimate the second term of inequality (3.11) by

$$\left|\bar{\nu}_{Hf_{w_0 z_1 \dots z_{j-1}}}(K_{z_j}) - \bar{\nu}_{(Hf_{w_0})_{z_1 \dots z_{j-1}}}(K_{z_j})\right|$$
$$\leq C|e(f_{w_0 z_1 \dots z_{j-1}}) - \psi_{z_1 \dots z_{j-1}}(e(f_{w_0}))|$$
$$\leq C\sum_{k=0}^{j-1} \text{Const}B^k\beta^M \leq \text{Const}\frac{B^j}{B-1}\beta^M,$$

where $B = \max_{i=1,2,3}\|\psi_i\|_{\text{Lip}}$ and $C = \max_{i=1,2,3}\|p_i\|_{\text{Lip}}$, where p_i are the energy weights (1.5) for the original i.f.s.

Summing up all terms on the right hand side of (3.10) we have

$$\left|\bar{\nu}_{f_{w_0}}(K_z) - \bar{\nu}_{Hf_{w_0}}(K_z)\right|$$

$$\leq m'\beta^M(1+\frac{B^{m'}}{B-1}).$$

It follows from (3.9) that

$$\left|\sum_{w\in W_n}\bar{\nu}_f(K_w)h(e(f_w),\frac{u_w}{5^{|w|}(\max_{V_0} f_w-\min_{V_0} f_w)})-\int h(x,0)d\mu_{\mathcal{E}}(x)\right|$$

$$\leq 2\bar{\nu}_f[C(\omega,m)<M]$$

$$+\beta^M\left(\text{Const}+\text{Const}\frac{B^{m'}-1}{B-1}+m'(1+\frac{B^{m'}}{B-1})\right)$$

$$+\text{Const}a^{m'}$$

where $a<1$, $\beta<1$ and $B>1$. Note that $\bar{\nu}_f[C(\omega,m)<M]\to 0$ since $\bar{\nu}_f$ does not have atoms. This completes the proof. □

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Paper III

PRODUCTS OF RANDOM MATRICES AND DERIVATIVES ON P.C.F. FRACTALS

ANDERS PELANDER AND ALEXANDER TEPLYAEV

Abstract. We define and study intrinsic first order derivatives on post critically finite fractals and prove differentiability almost everywhere with respect to self-similar measures for certain classes of fractals and functions. We apply our results to extend the geography is destiny principle to these cases, and also obtain results on the pointwise behavior of local eccentricities, previously studied by Öberg, Strichartz and Yingst, and the authors. We also establish the relation of the derivatives to the tangents and gradients previously studied by Strichartz and the authors. Our main tool is the Furstenberg-Kesten theory of products of random matrices.

Contents

1. Introduction

For the last twenty years a theory of analysis on fractals has evolved, with the construction of Laplacians and Dirichlet forms as cornerstones. There is both a probabilistic approach, where the Laplacian is constructed as an infinitesimal generator of a diffusion process, and an analytic approach where the Laplacian can be defined as a limit of difference operators. In this article we will work in the context of post critically finite (p.c.f.) fractals, for which Kigami laid the foundations of an analytic theory [7, 8, 9, 10].

We consider one of the most fundamental topics in analysis; the local structure of smooth functions. This is not only an interesting matter as

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such, it also shed light on an important phenomenon that does not occur when the underlying set is smooth.

In classical analysis any two points in the interior of the considered set have homeomorphic neighborhoods. This is not the case in analysis on fractals. Some points, called junction points, are boundary points of several copies of the self-similar set and neighborhoods of such points are different from those at non-junction points that have a canonical basis of neighborhoods consisting of copies of the self-similar set. However, although two non-junction points x, x' have bases of homeomorphic neighborhoods, the homeomorphisms do not in general map x onto x'.

It turns out that, as a consequence of the above, the local behavior of functions depend on the point under consideration. This *geography is destiny* principle, that has no analog whatsoever in analysis on smooth sets, were proven for harmonic functions on the Sierpiński gasket by Öberg, Strichartz and Yingst in [14]. Restrictions to the canonical neighborhoods will, for most harmonic functions, line up in the same direction, a direction that depends on the point, or rather the neighborhood. This property follows from theorems on products of random matrices since the restrictions to the canonical neighborhoods are given by linear mappings.

We will show that the geography is destiny principle extends to other fractals and to larger classes of functions with certain smoothness properties.

Generally speaking, the notion of smoothness of functions addresses the degree of differentiability of the function and its derivatives. Since the basic differential operator in analysis of fractals is the Laplacian, the term smooth has mostly been used to point out that a function f, sometimes together with $\Delta^k f$, are in the domain of the Laplacian.

On the other hand, in the classical calculus a differentiable function locally behaves like an affine linear mapping. In fractal analysis the analogs of such mappings are the harmonic functions, and from this point of view we make a natural definition of a derivative, and thus a concept of differentiability, of a function with respect to a harmonic function. This give us wider classes of functions with some degree of smoothness for which we can prove geography is destiny. We also relate this derivative to the gradient defined by the second author [20].

Our results concerns generic, with respect to a self-similar measure, properties of the local behavior of smooth functions at non-junction points. It would be interesting to know if the same properties hold generically with respect to the Kusuoka energy measure [12, 20]. Local behavior at junction points were studied in [18].

It is probable that our results can be extended to the category of self-similar finitely ramified fractals defined in [21].

We need to fix some notation, and at the same time recall some of the basic results of the theory. We refer to the books by Kigami [11] and Strichartz [19] for the whole story.

Throughout this paper, F will denote a p.c.f. self-similar fractal, by which we mean a compact connected metric space F equipped with a post critically finite self-similar structure as defined in [11]. Thus, there are continuous injections

$$\psi_1, ..., \psi_m : F \to F \tag{1.1}$$

such that

$$F = \bigcup_{i=1}^{m} \psi_i(F), \tag{1.2}$$

and a finite set $V_0 \subset F$ such that for any n and for any two distinct words $w, w' \in W_n = \{1, ..., m\}^n$ we have

$$F_w \cap F_{w'} = V_w \cap V_{w'}, \tag{1.3}$$

where $F_w = \psi_w(F)$ and $V_w = \psi_w(V_0)$. Here for a finite word $w = w_1...w_n \in W_n$ we denote

$$\psi_w = \psi_{w_1} \circ ... \circ \psi_{w_n}. \tag{1.4}$$

We call F_w, $w \in W_n$ a *cell* of level n. The set V_0 is called the *boundary* of F and consequently points in V_0 are referred to as *boundary points*. The fractal F is p.c.f. self-similar fractal if every boundary point is contained in only one 1-cell. We denote the number of boundary points by N_0 and will assume that $N_0 \geqslant 2$. A point $x \in F$ is called a *junction point* if $x \in F_w \cap F_{w'}$, for two distinct w, $w' \in W_n$.

Define $V_n = \bigcup_{w \in W_n} V_w$, $V_* = \bigcup_{n \geqslant 1} V_n$ and $W_* = \bigcup_{n \geqslant 1} W_n$. If $w = w_1 \dots w_k \in W_*$, we say that $|w| = k$ is the *length* of w. It is easy to see that V_* is dense in F. Note that, by definition, each ψ_i maps V_* into itself injectively.

Let $\Omega = \{1, \dots, m\}^{\mathbb{N}}$ be the space of infinite sequences $\omega = w_1 w_2 \dots$, $w_j \in W_1 = \{1, \dots, m\}$. For any $\omega \in \Omega$ let $[\omega]_n = w_1 \cdots w_n \in W_n$, and likewise for $w \in W_*$ and $n \leqslant |w|$. There is a natural continuous projection $\pi : \Omega \to F$ defined by

$$\pi(\omega) = \bigcap_{n \geqslant 0} F_{[\omega]_n}, \tag{1.5}$$

and $\pi^{-1}\{x\}$ is finite for any x by the p.c.f. assumption. Moreover, $\pi^{-1}\{x\}$ consists of more than one element if and only if x is a junction point. In case x is not a junction point we can therefore define $[x]_n = [\omega]_n$ if $x = \pi(\omega)$. In particular, $[x]_n$ is well defined for any $x \notin V_*$.

We assume that a harmonic structure, as defined in [11], is fixed on the p.c.f. self-similar structure. This will give rise to a self-similar Dirichlet (resistance, energy) form

$$\mathcal{E}(f, f) = \sum_{i=1}^{m} \rho_i \mathcal{E}(f_i, f_i) = \sum_{w \in W_n} \rho_w \mathcal{E}(f_w, f_w). \tag{1.6}$$

Here $f_w = f \circ \psi_w$ and $\rho_w = \rho_{w_1} \dots \rho_{w_n}$, where $\rho = (\rho_1, ..., \rho_m)$ are the energy renormalization factors. The energy renormalization factors, or weights, are often called conductance scaling factors because of the relation of resistance forms and electrical networks. They are reciprocals of the resistance scaling factors $r_j = 1/\rho_j$. We will always assume that the resistance form is *regular*, i.e. $\rho_j > 1$, $j = 1, \dots, m$.

The domain, $\operatorname{Dom} \mathcal{E}$, of $\mathcal{E}$ consists of continuous functions such that the *energy*, $\mathcal{E}(f) = \mathcal{E}(f, f) < \infty$. A function on F is *harmonic* if it minimizes the energy for the given set of boundary values.

Harmonic functions are uniquely defined by their restrictions to V_0 and we often, for convenience, identify the space of harmonic functions with the N_0-dimensional space $l(V_0)$ of functions on V_0. The restrictions of a harmonic function to cells of level 1 give rise to linear mappings A_i, $i = 1, \dots, m$ on $l(V_0)$ through $A_i h = h \circ \psi_i$. The restrictions to smaller cells are given by products of these matrices since $h|_{F_w} = A_w h$, where $A_w = A_{w_n} \dots A_{w_1}$ for $w \in W_n$.

Constant functions are harmonic so constant functions on $l(V_0)$ will be eigenvectors of all the mappings A_i, $i = 1, \dots, m$ with the corresponding eigenvalue equal to 1. To study the local behavior of harmonic functions it is therefore usable to factor out the constant functions. Denote by $\mathcal{H}$ the space of harmonic functions such that $\sum_{q \in V_0} h(q) = 0$ and define operators M_i, $i = 1, \dots, m$ on $\mathcal{H}$ by $M_i = P_{\mathcal{H}} A_i P_{\mathcal{H}}^*$, where $P_{\mathcal{H}}$ is the projection of $l(V_0)$ onto $\mathcal{H}$ given by $P_{\mathcal{H}} h = h - \sum_{q \in V_0} h(q)$. Note that each A_j commutes with $P_{\mathcal{H}}$.

For any function f defined on F we will denote by Hf the unique harmonic function that coincides with f on the boundary.

Given a finite non-atomic measure μ on F with the property that $\mu(O) > 0$ for any nonempty open set O there is a Laplacian Δ_μ that is an unbounded operator defined on a dense set of continuous functions defined by

$$\mathcal{E}(u, v) = -\int_F u \Delta_\mu v d\mu \tag{1.7}$$

for any $u \in \operatorname{Dom} \mathcal{E}$ with $u|_{V_0} = 0$. In this paper we will always assume that $\Delta_\mu v \in L^\infty(F)$. Functions with this property is denoted $\operatorname{Dom}_{L^\infty} \Delta_\mu$ but we will in what follows omit the index L^∞. We will also always assume that μ is self-similar, i.e. that there are real numbers μ_i, $i = 1, \dots, m$ such that $\mu(F_w) = \mu_w$ for any $w \in W_*$. For convenience we will assume that $\mu(F) = 1$.

Harmonic functions are exactly those for which $\Delta_\mu h = 0$. It should be noted that even though the Laplacian depends on the measure μ, the set of harmonic functions only depend on the harmonic structure.

There is a *Green's operator*

$$Gu(x) = \int_F g(x, y) u(y) d\mu(y) \tag{1.8}$$

acting on $L^\infty(F)$ such that $-\Delta G u = u$, and $Gu|_{V_0} = 0$. Thus, any function $f \in \operatorname{Dom}\Delta_\mu$ can be written $f = Hf - Gu$. The *Green's function* $g(x,y)$ is continuous for regular harmonic structures.

We next define some regularity classes of functions on F.

Definition 1.1. We say that $f \in C^k(\mathcal{H})$ if there are harmonic functions $h_1, ..., h_l \in \mathcal{H}$ and $u \in C^k(\mathbb{R}^l)$ such that $f = u(h_1, ..., h_l)$. We say that $f \in C^k(\operatorname{Dom}\Delta_\mu)$, if there are $g_1, ..., g_l \in \operatorname{Dom}\Delta_\mu$ and $u \in C^k(\mathbb{R}^l)$ such that $f = u(g_1, ..., g_l)$.

Note that whereas $C^k(\operatorname{Dom}\Delta_\mu)$ and $C^k(\mathcal{H})$ are multiplication domains, in general $\operatorname{Dom}\Delta_\mu$ is not by [2, 5, 6]. Also note that by definition

$$C^k(\mathcal{H}) \cup \operatorname{Dom}\Delta_\mu \subset C^k(\operatorname{Dom}\Delta_\mu). \tag{1.9}$$

There are several approaches to define derivatives on a p.c.f. fractal F. A weak gradient was studied by Kusuoka in [12, 13]. A stronger notion of gradients and tangents was considered in [18, 20] by Strichartz and the second author. In this we paper introduce the following definition.

Definition 1.2. Let f and h be real valued functions on a p.c.f. fractal F, and suppose h is continuous at $x \in F$. For $S \subseteq F$ let $Osc_S h = \sup_{x,y\in S} |h(y) - h(x)|$. Then we say that f is *differentiable* with respect to h at a non-junction point x if there is a real number $\frac{df}{dh}(x)$ such that

$$f(y) = f(x) + \frac{df}{dh}(x)(h(y) - h(x)) + o\big(Osc_{F_{[x]_n}} h\big)_{y\to x}, \tag{1.10}$$

where n is such that $y \in F_{[x]_n}$, and at a junction point x if

$$f(y) = f(x) + \frac{df}{dh}(x)(h(y) - h(x)) + o\big(Osc_{U_n(x)} h\big)_{y\to x}, \tag{1.11}$$

where $U_n(x)$ is a canonical basis of neighborhoods and n is such that $y \in U_n(x)$. Naturally, $\frac{df}{dh}(x)$ is called the *derivative* of f at x with respect to h.

It is easy to show usual properties of the derivative $\frac{df}{dh}(x)$, such as sum, product, ratio and chain rules. Also if f is differentiable with respect to h at x, then f is continuous at x. For later use we formulate the following version of the chain rule.

Proposition 1.3. *Suppose $f_j : F \to \mathbb{R}$, $j = 1, \ldots, l$ are differentiable with respect to h at x and that $g : \mathbb{R}^l \to \mathbb{R}$ is in $C^1(\mathbb{R}^l)$. Then $g(f_1, \ldots, f_l)$ is differentiable with respect to h at x and*

$$\frac{d(g(f_1, \ldots, f_l))}{dh}(x) = \sum_{j=1}^{l} \frac{\partial g}{\partial f_j}(f_1, \ldots, f_l)\frac{df_j}{dh}(x) \tag{1.12}$$

We will only use Definition 1.2 for h harmonic. Harmonic functions are the natural choice with respect to which one should differentiate since they are, in a sense, the analogues of linear functions on the interval. In fact, we will only differentiate with respect to $h \in \mathcal{H}$ since $\frac{df}{d(h+c)} = \frac{df}{dh}$ for any constant c. The maximum and minimum of a harmonic function is always attained on the boundary and we can therefore replace $Osc_{F_{[x]_n}} h_{[x]_n}$ by $\|M_{[x]_n} h\|$ in (1.10).

In section 2 we prove in Theorem 1, under certain conditions on the harmonic structure on F, that given any non-constant harmonic function $h \in \mathcal{H}$, a function $f \in C^1(\mathcal{H})$ is differentiable with respect to h at generic points. Then, according to Definition 1.2, the function f behaves as a function of one variable up to smaller order terms. This means, in a sense, that the space F is essentially one dimensional. Under some additional hypotheses, that we call the weak main assumption, on the measure μ, we prove the same result for any function $f \in C^1(\mathrm{Dom}\,\Delta_\mu)$ in Theorem 2. We also discuss the relationship between our derivative and the local derivatives defined at periodic points in [1, 3].

In section 3 we prove the "geography is destiny" principle for smooth functions on the set where the derivative is different from zero and then use this to prove a result on the local behavior of the eccentricity for functions defined on fractals with three boundary points. The concept of eccentricity was introduced and studied for harmonic functions on the Sierpiński gasket in [14] and were studied for larger classes of functions in [15].

In section 4 we relate the derivative to the gradient defined in [18, 20] under a stronger assumption on μ. Using this relation and technical results from the theory of products of random matrices we are also able to show geography is destiny on the set where the gradient is different from zero.

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2. Derivatives on p.c.f. fractals

Since our aim is to describe the local behavior of functions with certain smoothness properties with that of harmonic functions it is essential to understand their local structure. We therefore first state conditions on the harmonic structure under which we can use the theory of products of random matrices, developed in the 60s and 70s by Furstenberg, Kesten, Guivarch, Le Page, Raugi, Osseledec et al., to draw some immediate conclusions on the properties of the local behavior of harmonic functions. It was noted in [14, 18] that these conditions hold for the standard harmonic structure on the Sierpiński gasket. We refer to [4] when any result on products of random matrices is used. The reader will find references to the original sources there.

If $x \in F$ is a non-junction point it is contained in a unique sequence of cells $F_{[x]_n}$, and the local behavior of harmonic functions at x is given by the properties of the products $M_{[x]_n}$. The generic local behavior of harmonic functions with respect to a self-similar measure μ will thus be governed by the product of iid random matrices $M_{[\omega]_n}$, where $P[\omega_n = i] = \mu_i$. In the rest of this section we will only consider non-junction points.

From now on we will always assume that the matrices M_i are invertible. This is equivalent to the property that the restriction of a non-constant harmonic function to any cell is itself non-constant. Harmonic structures with this property are called *non-degenerate*. To see what the local behavior of harmonic functions on a degenerate harmonic structure might be like, there is an interesting study in [14, Section 7] on the case of the hexagasket.

It follows from a theorem by Furstenberg and Kesten [4, Theorem I.4.1] that there is $\alpha_+ > 0$ such that $\lim_{n\to\infty} \frac{1}{n} \log M_{[x]_n} = \log \alpha_+$ for μ a.e. x. The number $\log \alpha_+$ is called the *upper Lyapunov exponent* of the matrices M_j, $j = i, \ldots, m$ with respect to the measure μ.

Notation 2.1. We use notation $c_n = \text{Ø}(a^n)$ if $\lim\limits_{n\to\infty} \frac{1}{n} \log c_n = \log a$.

Note that $c_n = \text{Ø}(a^n)$ is equivalent to $c_n = o\big((a+\varepsilon)^n\big)_{n\to\infty}$ and $(a-\varepsilon)^n = o(c_n)_{n\to\infty}$, for any $\varepsilon > 0$ but does not imply that $c_n = O(a^n)_{n\to\infty}$.

Under additional assumptions on the harmonic structure it turns out that, for a fixed harmonic function h, $h_{[x]_n}$ will decrease as $\text{Ø}(\alpha_+^n)$ for μ a.e. x, and for a fixed x every h outside a $N_0 - 1$ dimensional subspace will exhibit this rate of decrease at x.

Definition 2.2. A subset S of $Gl(d, \mathbb{R})$ is *strongly irreducible* if there does not exist a finite family $\{L_1, \ldots, L_k\}$ of proper linear subspaces of $\mathbb{R}^d$ such that

$$M(L_1 \cup L_2 \cup \ldots \cup L_k) = L_1 \cup L_2 \cup \ldots \cup L_k, \tag{2.1}$$

for any $M \in S$.

Definition 2.3. The *index* of a subset T of $Gl(d, \mathbb{R})$ is the least integer p such that there exists a sequence M_n in T for which $\|M\|_n^{-1} M_n$ converges to a rank p matrix. T is *contracting* if its index is one.

Definition 2.4. We say that F satisfies the *SC-assumption* if the semigroup generated by M_i, $i = 1, \ldots, m$ is strongly irreducible and contracting.

The index of a set is in general difficult to determine, however in the case of semigroups there is a useful result in [4, Corollary IV.2.2]. Recall that an eigenvalue λ of a matrix M is *simple* if $\operatorname{Ker}(M - \lambda Id)$ has dimension one and equals
$\operatorname{Ker}(M - \lambda Id)^2$ and it is *dominating* if $|\lambda| > |\lambda'|$ for any other eigenvalue λ'.

Proposition 2.5. *A semigroup T in $Gl(d, \mathbb{R})$ which contains a matrix with a simple dominating eigenvalue is contracting.*

If a matrix $M \in Gl(2, \mathbb{R})$ has two distinct real eigenvalues it is clear that the lines in a finite union of lines invariant under M are the eigenspaces, so we have the following.

Proposition 2.6. *If the boundary V_0 consists of three points, then F satisfies the SC-assumption if there is some M_v with a simple dominating eigenvalue and there are two matrices M_w, $M_{w'}$ both with two distinct real eigenvalues and no eigenvector in common.*

It is readily verified that for instance the standard harmonic structures on the Sierpiński gasket and the level 3 Sierpiński gasket satisfies the SC-assumption. In fact, any non-degenerate structure with D_3 symmetry considered in [18, Section 5] satisfies the SC-assumption if $a \neq b$ where

$$\begin{pmatrix} 1 & 0 & 0 \\ 1-a-b & a & b \\ 1-a-b & b & a \end{pmatrix} \tag{2.2}$$

is the matrix corresponding to the restriction to a level 1 cell containing one of the boundary points.

It is clear from the remark following [2, Theorem 5.1] that if F satisfies the SC-assumption then $\operatorname{Dom}\Delta_\mu$ is not a multiplication domain.

Definition 2.7. We say that $x \in F$ is weakly generic if there is a subspace $\mathcal{H}_x^- \subset \mathcal{H}$ of co-dimension one such that

$$\|M_{[x]_n} h^-\| = o\|M_{[x]_n}\|_{n\to\infty} \tag{2.3}$$

for any $h^- \in \mathcal{H}_x^-$.

Denote by $\mathcal{H}_x^+$ the orthogonal complement of $\mathcal{H}_x^-$ and by P_x^- and P_x^+ the orthogonal projections onto $\mathcal{H}_x^-$ and $\mathcal{H}_x^+$ respectively. Also denote by h_x^+ an element of $\mathcal{H}_x^+$ of norm one.

Proposition 2.8. *$x \in F$ is weakly generic if and only if there is a subspace $\mathcal{H}_x^- \subset \mathcal{H}$ of co-dimension one such that*

$$\|M_{[x]_n} h^-\| = o\|M_{[x]_n} h\|_{n\to\infty} \tag{2.4}$$

for any $h^- \in \mathcal{H}_x^-$ and $h \notin \mathcal{H}_x^-$.

Proof. Necessarily $\|M_{[x]_n} h_x^+\| = O\|M_{[x]_n}\|_{n\to\infty}$, since if not $\|M_{[x]_n} h\| = o(\|M_{[x]_n}\|)$ for any $h \in \mathcal{H}$. The proposition follows immediately since if $h \notin \mathcal{H}_x^-$ then $P_x^+ h \neq 0$.

□

Proposition 2.9. *If $x \in F$ is weakly generic and $f = u(h_1, \dots, h_l) \in C^1(\mathcal{H})$ then $\frac{df}{dh}$ exists for any $h \notin \mathcal{H}_x^-$ with*

$$\frac{df}{dh} = \sum_{j=1}^{l} \frac{\partial u}{\partial h_j} \frac{dh_j}{dh}. \tag{2.5}$$

If $h' \in \mathcal{H}$ then

$$\frac{dh'}{dh} = \frac{< h', h_x^+ >}{< h, h_x^+ >}, \tag{2.6}$$

and in particular $h' \in \mathcal{H}_x^-$ if and only if $\frac{dh'}{dh_x^+} = 0$.

Proof. Because of Proposition 1.3 it is enough to show that $\frac{dh'}{dh}$ exists for any $h' \in \mathcal{H}$. Write $h' = a_x h + h^-$ with $h^- \in \mathcal{H}_x^-$. Then since

$$(h'(y) - h'(x))|_{F_{[x]_n}} = a_x(h(y) - h(x)) + (M_{[x]_n} h^-(\psi_{[x]_n}^{-1} y) - M_{[x]_n} h^-(\psi_{[x]_n}^{-1} x)), \tag{2.7}$$

it is clear from Proposition 2.8 that $\frac{dh'}{dh}(x) = a_x = \frac{<h', h_x^+>}{<h, h_x^+>}$ and (2.6) follows. □

Lemma 2.10. *Suppose F satisfies the SC-assumption. Then μ-almost every x is weakly generic and moreover, at μ a.e. weakly generic x we have $\|M_{[x]_n}\| = \emptyset(\alpha_+^n)$.*

Proof. This follows from [4, Corollary VI.1.7]. □

Thus, under the SC-assumption Proposition 2.9 hold at μ a.e. x. The following result shows that also for given harmonic h and $f \in C^1(\mathcal{H})$, $\frac{df}{dh}$ exists for μ a.e. x.

Theorem 1. *Suppose F satisfies the SC-assumption. Then for any nonzero $h \in \mathcal{H}$ and any $f = u(h_1, \ldots, h_l) \in C^1(\mathcal{H})$ we have that $\frac{df}{dh}(x)$ exists for μ a.e. x and is given by* (2.5)*.*

Proof. This is a direct consequence of the theory of products of random matrices [4, Theorem III.3.1]. Note, in particular, that $h \notin \mathcal{H}_x^-$ for μ a.e. x. □

One of the main results of our paper is the extension of this theorem to functions in $C^1(\mathrm{Dom}\,\Delta_\mu)$ under some additional hypotheses on the measure μ. To this end, we define γ by

$$\log \gamma = \sum_{j=1}^{m} \mu_j \log(r_j \mu_j). \tag{2.8}$$

Then

$$r_{[x]_n} \mu_{[x]_n} = \emptyset(\gamma^n) \tag{2.9}$$

for μ a.e. x. One can see that γ is the analog of the Lyapunov exponent for the Laplacian scaling factor $r_{[x]_n} \mu_{[x]_n}$, which in turn is the product of energy and measure scaling factors.

Definition 2.11. We will say that (F, μ) satisfies the *weak main assumption* if F satisfies the SC-assumption and

$$\gamma < \alpha_+. \tag{2.10}$$

Example 2.12. It is known that the Sierpiński gasket with the standard harmonic structure and uniform self-similar measure satisfies the weak main assumption. It also holds for the level 3 Sierpiński gasket with the uniform self-similar measure and standard harmonic structure, which is discussed in detail in [18, 19]. In this case $\gamma = 7/90$ and of the six restriction matrices three has determinant $7/15^2$ and three has determinant $8/15^2$. It is known that if all determinants equals one, then $\alpha_+ > 1$. It follows that for the level 3 Sierpiński gasket $\alpha_+ > \sqrt{7}/15 > \gamma$.

Theorem 2. *Suppose (F, μ) satisfies the weak main assumption and h is a non-constant harmonic function. Then for μ-almost every x the derivative $\frac{df}{dh}(x)$ exists for any function $f = u(g_1, \ldots, g_l) \in C^1(\operatorname{Dom}\Delta_\mu)$ and is given by*

$$\frac{df}{dh} = \sum_{j=1}^{l} \frac{\partial u}{\partial g_j} \frac{dg_j}{dh}. \tag{2.11}$$

Moreover, there exists C such that if $f \in \operatorname{Dom}\Delta_\mu$, then for μ a.e. x

$$\left|\frac{df}{dh}\right| \leqslant \left|\frac{d(Hf)}{dh}\right| + C\frac{\|\Delta f\|_\infty}{|< h, h_x^+ >|} \sum_{n=0}^{\infty} r_{[x]_n} \mu_{[x]_n} \|M_{[x]_n}^{-1*} h_x^+\|. \tag{2.12}$$

We first state and prove two Lemmas.

Lemma 2.13. *Suppose $u \in L^\infty(F)$ has support in a cell F_w. Then*

$$Osc_{F_{[w]_k}} Gu \leqslant Const\, r_{[w]_k} \mu_w \|u\|_\infty$$

for $k = 0, 1, ..., n = |w|$.

Proof. It will be enough to show that

$$|Gu(x) - Gu(x_0)| \leqslant Const\, r_{[w]_k} \mu_w \|u\|_\infty \tag{2.13}$$

for $x \in F_{[w]_k}$ and $x_0 \in V_{[w]_k}$. This can be done by using properties of the Green's function

$$g(x, y) = \sum_{v \in \varnothing \cup W^*} r_v \Psi(\psi_v^{-1}(x), \psi_v^{-1}(y)). \tag{2.14}$$

For the exact definition of Ψ, see [11].

Since we consider points in $F_{[w]_k}$ and u has support in F_w we only bother about x and y in $F_{[w]_k}$. For those, $\Psi(\psi_v^{-1}(x), \psi_v^{-1}(y)) = 0$ in case $|v| \geqslant k$ and $[v]_k \neq [w]_k$, and in case $|v| < k$ and $[w]_{|v|} \neq v$. The properties of Ψ also makes $\Psi(\psi_v^{-1}(x_0), \psi_v^{-1}(y)) = 0$ for all $|v| \geqslant k$. In all

(2.15)

$$|g(x_0, y) - g(x, y)| \leqslant \sum_{m=1}^{k-1} r_{[w]_m} |\Psi(\psi_{[w]_m}^{-1}(x_0), \psi_{[w]_m}^{-1}(y)) - \Psi(\psi_{[w]_m}^{-1}(x), \psi_{[w]_m}^{-1}(y))|$$

$$+ \left| \sum_{v \in \phi \cup W^*} r_v r_{[w]_k} \Psi(\psi_{vw}^{-1}(x), \psi_{vw}^{-1}(y)) \right|.$$

The difference in the first term is, by the definition of Ψ, bounded by a constant times the difference of the value of 1-harmonic functions at the points $\psi_{[w]_m}^{-1}(x_0)$ and $\psi_{[w]_m}^{-1}(x)$. Both points lie in the cell $F_{[w]_{m,k}}$, where $[w]_{m,k}$ is the word of length $k-m$ defined by $[w]_m [w]_{m,k} = |w]_k$ for $k \geqslant m$ and the difference is thus bounded by a constant times $r_{[w]_{m,k}}$ since the largest eigenvalue of M_i is less or equal to r_i , see [11, Appendix A], and the first term is bounded by $Const\, r_{[w]_k}$. The second term is $r_{[w]_k} g(\psi^{-1}_{[w]_k} x, \psi^{-1}_{[w]_k} y) \leqslant r_{[w]_k} \|g\|_\infty$ and we conclude that

$$|Gu(x) - Gu(x_0)| \leqslant \int_F |g(x,y) - g(x_0,y)||u(y)|d\mu(y) \tag{2.16}$$

$$\leqslant Const\, r_{[w]_k} \int_{F_w} |u(y)|d\mu(y) \leqslant Const\, r_{[w]_k} \mu_w \|u\|_\infty.$$

□

Lemma 2.14. *Suppose F satisfies the SC-assumption. Given any non-constant $h, h' \in \mathcal{H}$, we have for μ a.e. $x \in F$ that*

$$\sup_{y \in F_{[x]_n}} \left| h'(y) - h'(x) - \frac{dh'}{dh}(x)(h(y) - h(x)) \right| \leqslant c_{n,x} \frac{\|h\|\|h'\|}{|< h, h_x^+ >|}, \tag{2.17}$$

where

$$\limsup \frac{1}{n} \log c_{n,x} \leqslant \log \alpha_2, \tag{2.18}$$

with $\log \alpha_2 < \log \alpha_+$ being the second Lyapunov exponent.

Proof. Since, in the proof of Proposition 2.9, $h^- = P_x^- h' - \frac{<h',h_x^+>}{<h,h_x^+>} P_x^- h$, it follows from (2.7) that for $y \in F_{[x]_n}$

$$\left| h'(y) - h'(x) - \frac{dh'}{dh}(h(y) - h(x)) \right| \leqslant \|M_{[x]_n}(P_x^- h' - \frac{< h', h_x^+ >}{< h, h_x^+ >} P_x^- h)\|. \tag{2.19}$$

Now [4, Corollary VI.1.7] says that

$$\limsup_n \frac{1}{n} \log \|M_{[x]_n} h_-\| \leqslant \log \alpha_2 \tag{2.20}$$

for any $h_- \in \mathcal{H}_x^-$. Let $h_1, \dots h_l$ be an ON-basis of $\mathcal{H}_x^-$ and $b_{n,i} = \|M_{[x]_n} h_i\|$. Then $c_{n,x} = 2\sum_{i=1}^l b_{n,i}$ satisfies (2.18) and for any $h_- \in \mathcal{H}_x^-$ with $\|h_-\| = 1$ we have $\|M_{[x]_n} h_-\| \leqslant c_{n,x}$. This gives (2.17). □

Proof of Theorem 2. In view of Proposition 1.3 it is enough to suppose $f \in \mathrm{Dom}\, \Delta_\mu$. It is clear from Theorem 1 that we can suppose $f = Gu$. We also assume $x \in F$ is weakly generic, $r_{[x]_n} \mu_{[x]_n} = \emptyset(\gamma^n)$ and $h \notin \mathcal{H}_x^-$ with $\|M_{[x]_n} h\| = \emptyset(\alpha_+^n)$.

Denote $B_{[x]_n} = F_{[x]_{n-1}} \setminus F_{[x]_n}$ and let $u^{[x]_n}$ be the restriction of u to $B_{[x]_n}$ so that

$$f = \sum_{n=1}^\infty Gu^{[x]_n}. \tag{2.21}$$

Since $u^{[x]_n} = 0$ on $F_{[x]_n}$, $Gu^{[x]_n}$ is harmonic on $F_{[x]_n}$ and thus $\frac{d(Gu^{[x]_n})}{dh}$ exists and our aim is to show that

$$\frac{df}{dh} = \sum_{n=1}^{\infty} \frac{d(Gu^{[x]_n})}{dh}. \tag{2.22}$$

To prove convergence of the right hand side of (2.22) let $v^{[x]_n}$ be the function in $\mathcal{H}$ that corresponds to $(Gu^{[x]_n})_{[x]_n}$ and note that

$$\frac{d(Gu^{[x]_n})}{dh} = \frac{d(v^{[x]_n})}{d(M_{[x]_n}h)}(\psi^{-1}_{[x]_n}(x)) = \frac{\left\langle v^{[x]_n}, h^+_{\psi^{-1}_{[x]_n}(x)}\right\rangle}{\left\langle M_{[x]_n}h, h^+_{\psi^{-1}_{[x]_n}(x)}\right\rangle}, \tag{2.23}$$

where the last equality follows from (2.6). We show that the absolute value of the denominator of the right hand side of (2.23) is $\O(\alpha_+^n)$ and that the absolute value of the nominator is bounded by $\O(\gamma^n)$.

From [4, Theorem VI.3.1] it follows that there is $\tilde{h} \in \mathcal{H}$ such that

$$h^+_x = \lim_{n\to\infty} \frac{M^*_{[x]_n}\tilde{h}}{\|M^*_{[x]_n}\tilde{h}\|} \tag{2.24}$$

and

$$h^+_{\psi_w(x)} = \lim_{n\to\infty} \frac{M^*_{w[x]_n}\tilde{h}}{\|M^*_{w[x]_n}\tilde{h}\|}, \tag{2.25}$$

consequently

$$h^+_{\psi^{-1}_{[x]_n}(x)} = \frac{M^{-1*}_{[x]_n}h^+_x}{\|M^{-1*}_{[x]_n}h^+_x\|}. \tag{2.26}$$

Another result of the theory of products of random matrices [4, Corollary VI.1.8] says that $\|M^{-1*}_{[x]_n}h^+_x\| = \O((1/\alpha_+)^n)$, and it follows that

$$\left|\left\langle M_{[x]_n}h, h^+_{\psi^{-1}_{[x]_n}(x)}\right\rangle\right| = \frac{|< h, h^+_x >|}{\|M^{-1*}_{[x]_n}h^+_x\|} = \O(\alpha^n_+). \tag{2.27}$$

The nominator has the bound

$$|< v^{[x]_n}, h^+_{\psi^{-1}_{[x]_n}(x)} >| \leqslant \|v^{[x]_n}\| \leqslant Const\, Osc(v^{[x]_n}) \leqslant Const\, r_{[x]_n}\mu_{[x]_n}\|u\|_\infty, \tag{2.28}$$

where the last inequality follows from Lemma 2.13. Thus, the right hand side of (2.22) converges and (2.12) follows from (2.27) and (2.28) as soon as we have shown (2.22).

For $y \in F_{[x]_k}$ we must show

$$\left|Gu(y) - Gu(x) - \sum_{n=1}^{\infty} \frac{d(Gu^{[x]_n})}{dh}(h(y) - h(x))\right| = o(\|M_{[x]_k}h\|). \tag{2.29}$$

We write

$$\left|Gu(y) - Gu(x) - \sum_{n=1}^{\infty} \frac{d(Gu^{[x]_n})}{dh}(h(y) - h(x))\right| \tag{2.30}$$

$$\leqslant \left|\sum_{n=1}^{k}(Gu^{[x]_n}(y) - Gu^{[x]_n}(x)) - \sum_{n=1}^{k} \frac{d(Gu^{[x]_n})}{dh}(h(y) - h(x))\right|$$

$$+ \left|\sum_{n=k+1}^{\infty} (Gu^{[x]_n}(y) - Gu^{[x]_n}(x))\right|$$

$$+ \left|\sum_{n=k+1}^{\infty} \frac{d(Gu^{[x]_n})}{dh}(h(y) - h(x))\right|.$$

Lemma 2.13 implies that the second term is estimated from above by $\emptyset(\gamma^k)$. The third term is also is estimated from above by $\emptyset(\gamma^k)$ as a product of something that is at most $\emptyset((\gamma/\alpha_+)^k)$ and something that is $\emptyset(\alpha_+^k)$. Remains the first term which we write

$$\left|\sum_{n=1}^{k} Gu^{[x]_n}(y) - Gu^{[x]_n}(x) - \frac{d(Gu^{[x]_n})}{dh}(h(y) - h(x))\right|. \tag{2.31}$$

Suppose that we fix a (large) constant M, which is to be chosen later, and that the integers from 1 to k are divided into M subintervals $[jk/M, (j+1)k/M]$. From the arguments below it is evident that without loss of generality we can assume that k is an integer multiple of M, say $k = Mm$. So we write the sum in (2.31) as M sums of $m = k/M$ addends each, and have to show that for each $j = 1, ..., M$ we have

$$\left|\sum_{n=m(j-1)+1}^{jm} Gu^{[x]_n}(y) - Gu^{[x]_n}(x) - \frac{d(Gu^{[x]_n})}{dh}(h(y) - h(x))\right| = o(\|M_{[x]_k} h\|). \tag{2.32}$$

If we denote

$$h_j = \sum_{n=m(j-1)+1}^{jm} Gu^{[x]_n} \tag{2.33}$$

then we have to show

$$\left|\sum_{n=m(j-1)+1}^{jm} h_j(y) - h_j(x) - \frac{dh_j}{dh}(h(y) - h(x))\right| = o(\|M_{[x]_k} h\|). \tag{2.34}$$

Note that h_j is harmonic on $F_{[x]_{jm}}$. By Lemma 2.13 we have $\|h_j\| = \emptyset(\gamma^{m(j-1)})$ and Lemma 2.14 then implies that the left hand side of (2.34) is bounded by $\emptyset\left(\gamma^{m(j-1)}\alpha_2^{m(M-j)}\right)$. Let $\widetilde{\alpha} = \max\{\gamma, \alpha_2\}$ and $\varepsilon = \frac{1}{2}(\alpha_+ - \widetilde{\alpha}) > 0$. If we have that

$$M > \frac{\log \gamma}{\log \widetilde{\alpha} - \log(\widetilde{\alpha} + \varepsilon)} \tag{2.35}$$

then

$$\gamma^{j-1}{\alpha_2}^{M-j} \leqslant \widetilde{\alpha}^M \gamma^{-1} < (\widetilde{\alpha} + \varepsilon)^M = (\alpha_+ - \varepsilon)^M \tag{2.36}$$

which implies

$$\emptyset\left(\gamma^{m(j-1)}\alpha_2^{m(M-j)}\right) = o\big((\alpha_+ - \varepsilon)^k\big)_{k\to\infty} \tag{2.37}$$

and this completes the proof. □

The next corollary is an analog of Fermat's theorem about stationary points in our context.

Corollary 2.15. *Suppose (F, μ) satisfies the weak main assumption. Then for any non-constant harmonic function h there exists a set F' of full μ-measure such that if $f = u(g_1, \ldots, g_l) \in C^1(\mathrm{Dom}\,\Delta_\mu)$ has a local maximum at $x \in F'$, then $\frac{df}{dh}(x) = 0$.*

Proof. Let F'' be the set of full μ-measure such that, according to Theorem 2, the derivative $\frac{df}{dh}(x)$ exists for any $f \in C^1(\mathrm{Dom}\,\Delta_\mu)$. There exists $w \in W_*$ such that the cell F_w does not contain any boundary points. We define F' as the set of all x such that $x \in F''$ and there are infinitely many n such that $[x]_{n,k} = w$, $|w| = n - k$. Obviously F' is a set of full μ-measure. The result follows from Theorem 2 by standard arguments because, by an elementary compactness argument and the Harnack inequality [11, Proposition 3.2.7], there is $c > 0$ such that

$$\max_{y\in F} h'(y) \geqslant c\|h'\| \tag{2.38}$$

for any harmonic function h' with zero in F_w. □

For the next theorem recall that a point $x \in F$ is called *periodic* if it is a fixed point of some ψ_w, $w \in W_*$.

Theorem 3. *Let $x = \psi_w(x) \in F$ be a periodic point. Suppose M_w has a dominating eigenvalue λ and the corresponding eigenvector is denoted by h_λ. If $|\lambda| > r_w\mu_w$ then the local derivative $\dfrac{df}{dh_\lambda}(x)$ exists for any $f \in C^1(\mathrm{Dom}\,\Delta_\mu)$. In particular, if x is a boundary fixed point then the normal derivative $\partial_N f(x)$ exists for any $f \in C^1(\mathrm{Dom}\,\Delta_\mu)$.*

Proof. In order to prove this one can adapt the proof of Theorem 2 defining $B_{w^n} = F_{w^{n-1}} \setminus F_{w^n}$, where $w^n = \underbrace{w \ldots w}_{n \text{ times}}$ and use

$$f = \sum_{n=1}^{\infty} G u^{w^n}. \tag{2.39}$$

The condition $|\lambda| > r_w\mu_w$ is necessary to have convergence of $\sum_{n=1}^{\infty} \frac{d(Gu^{w^n})}{dh_\lambda}$.

For a boundary fixed point $x = \psi_i(x)$ this condition is always fulfilled since $\lambda = \lambda_2 = r_i$ in this case. □

The next corollary is another analog of Fermat's theorem.

Corollary 2.16. *If x is a non-boundary periodic point, assumptions of Theorem 3 hold, and $f = u(g_1, \ldots, g_l) \in C^1(\mathrm{Dom}\,\Delta_\mu)$ has a local maximum at x, then $\dfrac{df}{dh_\lambda}(x) = 0$.*

Proof. The proof is the same as that of Corollary 2.15 and uses Theorem 2 and Theorem 3. □

The result of Theorem 3 partially improves Theorem 3.2 in [3] where it was shown in the case of the Sierpiński gasket that $\partial_2 f$ and $\partial_3 f$ exists for any $f \in \mathrm{Dom}\,\Delta$. Namely, under the assumption that M_w has two real eigenvalues $\lambda_2 > \lambda_3$, two local derivatives at periodic points of the Sierpiński gasket were defined in [3]. If $h_2, h_3 \in \mathcal{H}$ are any harmonic functions corresponding to these eigenvalues and

$$Hf_{[x]_n} = a_{1n} + a_{2n}h_{2,[x]_n} + a_{3n}h_{3,[x]_n} \tag{2.40}$$

then

$$\partial_2 f(x) = \lim_{n\to\infty} a_{2n} \text{ and } \partial_3 f(x) = \lim_{n\to\infty} a_{3n} \tag{2.41}$$

if the limits exists, and $\partial_2 f(x) = \frac{df}{dh_2}(x)$. Note that the notation λ_2 for the leading eigenvalue is used in [3] because $\lambda_1 = 1$ denotes the leading eigenvalue of the matrix A_w.

For arbitrary p.c.f. fractals, local derivatives $\partial_2, \ldots, \partial_{N_0}$ can be defined analogously to (2.41) at any periodic point $x = \psi_w(x)$ such that M_w has distinct real eigenvalues $|\lambda_2| > \ldots > |\lambda_{N_0}|$ with corresponding harmonic functions $h_2, \ldots, h_{N_0}$. Periodic points of this type are weakly generic and $\mathcal{H}_x^-$ is spanned by $h_3, \ldots, h_{N_0}$, but the rate of decrease for $h \notin \mathcal{H}_x^-$ is $\|M_{[x]_n}h\| = \text{Ø}(\sigma^n)$ for $\sigma = \lambda_2^{1/|w|}$ instead of $\text{Ø}(\alpha_+^n)$.

It should be noted that if $x = \psi_i(x)$ is a boundary point then ∂_2 equals, for an appropriate choice of h_2, the normal derivative ∂_N. For the Sierpiński gasket, ∂_3 equals the tangential derivative ∂_T, for an appropriate choice of h_3. For periodic points on the Sierpiński gasket where M_w has two complex conjugate eigenvalues local derivatives ∂^+ and ∂^- were defined in [1] using the eigenvectors. It was also shown that there are infinitely many periodic points with this property. Such periodic points are not weakly generic. Actually for any non-constant $h \in \mathcal{H}$, $\|M_{[x]_n}h\| = O((\sqrt{3}/5)^n)$ and h is only differentiable with respect to harmonic functions that are proportional to h. The local behavior at such points is thus truly different from the generic behavior.

3. Directions on p.c.f. fractals

In this section we prove the geography is destiny principle for large classes of functions and use it to obtain a result on the pointwise behavior of eccentricities. We begin by giving a precise formulation of the principle. It was formulated for the first time in [14] for harmonic functions on the Sierpiński gasket. For harmonic functions it holds under the SC-assumption.

For any $h \in l(V_0)$, $h \neq 0$ we define the *direction* $\mathrm{Dir}h$ as the element in the projective space $\mathbb{P}(\mathcal{H})$ corresponding to $P_{\mathcal{H}}h$. This definition extends to any function f defined on F, and non-constant on the boundary, through $\mathrm{Dir}f = \mathrm{Dir}f|_{V_0}$. We denote by ρ the standard angular distance on $\mathbb{P}(\mathcal{H})$.

Proposition 3.1. *Suppose F satisfies the SC-assumption. Then for any non-constant harmonic functions $h_1, h_2 \in \mathcal{H}$*

$$\lim_{n\to\infty} \rho(\mathrm{Dir}h_1|_{F_{[x]_n}}, \mathrm{Dir}h_2|_{F_{[x]_n}}) = 0 \tag{3.1}$$

for μ a.e. x.

Proof. This follows from [4, Theorem III.4.3]. □

In fact, the convergence in (3.1) is even exponential [4, Proposition III.6.4].

If f is differentiable with respect to h with nonzero derivative at a point x, then the difference in direction of $f_{[x]_n}$ and $h_{[x]_n}$ will tend to zero. Note that by definition of the derivative, $\mathrm{Dir}f_{[x]_n}$ exists for n large enough if $\frac{df}{dh}(x) \neq 0$.

Proposition 3.2. *Suppose $\frac{df}{dh}(x)$ exists and is different from zero. Then*

$$\lim_{n\to\infty} \rho(\mathrm{Dir}f_{[x]_n}, \mathrm{Dir}h_{[x]_n}) = 0 \tag{3.2}$$

Proof. This is clear since $f(y) - f(x) = c(h(y) - h(x)) + o(\|M_{[x]_n}h\|)$ implies

$$\rho(\mathrm{Dir}f_{[x]_n}, \mathrm{Dir}h_{[x]_n}) = \rho(\mathrm{Dir}(ch_{[x]_n} + o(\|M_{[x]_n}h\|)), \mathrm{Dir}h_{[x]_n}) \to 0 \tag{3.3}$$

□

The above Proposition together with Theorem 2 immediately gives the following broad extension of the geography is destiny principle.

Theorem 4. *Suppose (F, μ) satisfies the weak main assumption and that $f \in C^1(\mathrm{Dom}\,\Delta_\mu)$ and $h \in \mathcal{H}$ is a non-constant harmonic function. Then*

$$\lim_{n\to\infty} \rho(\mathrm{Dir}f_{[x]_n}, \mathrm{Dir}h_{[x]_n}) = 0 \tag{3.4}$$

for μ a.e. x outside the set where $\frac{df}{dh}(x) = 0$.

Remark 3.3. From the estimate (2.12) it follows that given any $Hf \neq 0$ and $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ with $\lim_{\varepsilon\to 0}\delta(\varepsilon) = 0$, such that

$$\mu\{x \,:\, \frac{df}{dh}(x) \neq 0\} > 1 - \delta(\varepsilon) \tag{3.5}$$

for any $f = Hf + G\Delta f$ with $\|\Delta f\|_\infty < \varepsilon$ and $\|h\| = 1$.

In [14] the *eccentricity* $e(h)$ of a non-constant harmonic function h on the Sierpiński gasket were defined as

$$e(h) = \frac{h(q_1) - h(q_0)}{h(q_2) - h(q_0)}, \tag{3.6}$$

where q_i, $i = 0, 1, 2$ are the boundary points labeled so that $h(q_0) \leqslant h(q_1) \leqslant h(q_2)$. Note that the eccentricity is the same for harmonic functions corresponding to the same element in $\mathcal{H}$. The concept of eccentricity extend to any F with three boundary points and any function defined on F and non-constant on the boundary.

It was shown in [14] that there is a measure on $[0, 1]$ such that for any non-constant harmonic function, the distribution of eccentricities of the restrictions h_w to cells of a fixed level $|w| = n$ converges in the Wasserstein metric to this measure. This result was extended to functions with Hölder continuous Laplacian in [15].

If, instead of the global distribution of local eccentricities, we look at the generic behavior of the eccentricities on neighborhoods of a point, the geography is destiny principle applies. Since $e(-f) = 1 - e(f)$ we define an equivalence relation on $[0, 1]$ by $e \sim e'$ if and only if $e = e'$ or $e = 1 - e'$. We denote by $\bar{e}$ the equivalence class of e and let $d(\bar{e}, \bar{e}') = \min_{x \sim e, x' \sim e'} |x - x'|$ be the natural distance on $[0, 1]/\sim$.

Corollary 3.4. *If F satisfies the SC-assumption then for any non-constant harmonic functions h, h'*

$$\lim_{n\to\infty} d(\bar{e}(h_{[x]_n}), \bar{e}(h'_{[x]_n})) = 0, \tag{3.7}$$

for μ a.e. x. If (F, μ) satisfies the weak main assumption then for any $f, f' \in C^1(\mathrm{Dom}\,\Delta_\mu)$ and non-constant $h \in \mathcal{H}$ we have

$$\lim_{n\to\infty} d(\bar{e}(f_{[x]_n}), \bar{e}(f'_{[x]_n})) = 0 \tag{3.8}$$

for μ a.e. x outside the set where $\frac{df}{dh}$ or $\frac{df'}{dh}$ are zero.

Proof. Since $\bar{e}$ depends continuously on the direction this follows immediately from Theorem 4. □

4. Derivatives and gradients

In this section we clarify the relation between the derivative and the *gradient* of a function on F defined in [20]. We will restrict attention to cases where (F, μ) satisfies the *strong main assumption* that F satisfies the SC-assumption and

$$\gamma < \alpha_-. \tag{4.1}$$

Here α_- is the lower Lyapunov exponent of the matrices M_j with respect to μ.

It has been shown [22, 18] that the Sierpiński gasket with standard harmonic structure and uniform self-similar measure satisfies the even stronger inequality,

$$\gamma\alpha_+ < \alpha_-^2. \tag{4.2}$$

For the standard harmonic structure on the Sierpiński gasket the resistance scaling factors are all 3/5. Sabot showed in [16] that for small perturbations of these factors there is a unique harmonic structure on the Sierpiński gasket, see also [17]. Since the harmonic restriction mappings depend continuously on the resistances, (4.2) implies that for small enough perturbations of the harmonic structure the Sierpiński gasket, with a self-similar measure not far from being uniform, will still satisfy the strong main assumption.

For a non-junction point $x \in F$, let $\mathrm{Grad}_{[x]_n} f = M_{[x]_n}^{-1} P_{\mathcal{H}} H f_{[x]_n}$. The gradient of f at x is defined as

$$\mathrm{Grad}_x f = \lim_{n\to\infty} \mathrm{Grad}_{[x]_n} f, \tag{4.3}$$

if the limit exists. In [20] the gradient were defined for sequences $\omega \in \Omega$, so at junction points there are several "directional" gradients defined, but for non-junction points $\mathrm{Grad}_x f$ is defined unambiguously.

Immediately from the definition we have

Proposition 4.1. *If $h \in \mathcal{H}$ then $\mathrm{Grad}_x h$ exists for all x and $\mathrm{Grad}_x h = h$.*

In [20, Theorem 1] the following estimate was proved for any harmonic structure on a p.c.f. fractal.

$$\|\mathrm{Grad}_{[x]_{n+1}} f - \mathrm{Grad}_{[x]_n} f\| \leqslant C\|\Delta f\|_\infty r_{[x]_n}\mu_{[x]_n}\|M_{[x]_n}^{-1}\|. \tag{4.4}$$

It implies the following theorem.

Theorem 5. *There exists a constant C such that for any $f \in Dom\Delta$ with $\|\Delta f\|_\infty < \infty$ and any $x \in F \setminus V_*$ with*

$$\sum_{n\geqslant 1} r_{[x]_n}\mu_{[x]_n}\|M_{[x]_n}^{-1}\| < \infty, \tag{4.5}$$

$\mathrm{Grad}_x f$ exists and

$$\|P_{\mathcal{H}} H f - \mathrm{Grad}_x f\| \leqslant C\|\Delta f\|_\infty \sum_{n\geqslant 1} r_{[x]_n}\mu_{[x]_n}\|M_{[x]_n}^{-1}\|. \tag{4.6}$$

Also, for any $n > 0$

$$\|P_{\mathcal{H}} H f - \mathrm{Grad}_{[x]_n} f\| \leqslant C\|\Delta f\|_\infty \sum_{k=1}^{n} r_{[x]_k}\mu_{[x]_k}\|M_{[x]_k}^{-1}\|. \tag{4.7}$$

From Theorem 5 we can immediately deduce the following lemma.

Lemma 4.2. *If (F,μ) satisfies the strong main assumption, then for any function $f \in \mathrm{Dom}\,\Delta_\mu$, $\mathrm{Grad}_x f$ exists for μ-almost all $x \in F$.*

Proof. The upper Lyapunov exponent of the matrices M_j^{-1} with respect to the measure μ is $1/\alpha_-$ and so the series (4.5) converges exponentially μ-almost everywhere. $\square$

The next lemma uses central limit theorem and large deviations results for products of random matrices. We will use it to show that $\mathrm{Grad}_x f$ is a unique function in $\mathcal{H}$ that best approximates f in neighborhoods of x.

Lemma 4.3. *Suppose (F,μ) satisfies the strong main assumption. Then for any $\varepsilon > 0$*

$$\sum_{k\geqslant n} r_{[x]_k}\mu_{[x]_k}\|M^{-1}_{[x]_{n,k}}\| o\big((\gamma+\varepsilon)^n\big)_{n\to\infty} \tag{4.8}$$

for μ a.e. x.

Proof. By the Borel-Cantelli lemma it is enough to show that for any $\delta > 0$

$$\sum_{n=1}^{\infty} \mu\Big\{x : (\gamma+\varepsilon)^{-n}\sum_{k\geqslant n} r_{[x]_k}\mu_{[x]_k}\|M^{-1}_{[x]_{n,k}}\| > \delta\Big\} < \infty. \tag{4.9}$$

Since $\lim\limits_{n\to\infty}\frac{1}{n}\log(r_{[x]_n}\mu_{[x]_n}) = \log\gamma$ for μ a.e. x, it is enough to show that

$$\sum_{n=1}^{\infty}\mu\left\{x : \left(\frac{\gamma}{\gamma+\varepsilon}\right)^n \sum_{k\geqslant n} r_{[x]_{n,k}}\mu_{[x]_{n,k}}\|M^{-1}_{[x]_{n,k}}\| > \delta\right\} \tag{4.10}$$

$$= \sum_{n=1}^{\infty}\mu\left\{x : \left(\frac{\gamma}{\gamma+\varepsilon}\right)^n \sum_{k=1}^{\infty} r_{[x]_k}\mu_{[x]_k}\|M^{-1}_{[x]_k}\| > \delta\right\}$$

$$= \sum_{n=1}^{\infty}\mu\left\{x : \sum_{k=1}^{\infty} r_{[x]_k}\mu_{[x]_k}\|M^{-1}_{[x]_k}\| > \delta\left(\frac{\gamma+\varepsilon}{\gamma}\right)^n\left(\frac{1-\beta}{\beta}\right)\sum_{k=1}^{\infty}\beta^k\right\} < \infty,$$

where we assume that $1 > \beta > \frac{\gamma}{\alpha_-}$ is a fixed number. Thus, it is enough to show that

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\mu\left\{x : r_{[x]_k}\mu_{[x]_k}\|M^{-1}_{[x]_k}\| > \delta\left(\frac{\gamma+\varepsilon}{\gamma}\right)^n\left(\frac{1-\beta}{\beta}\right)\beta^k\right\} \tag{4.11}$$

$$= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\mu\left\{x : \log\left(r_{[x]_k}\mu_{[x]_k}\|M^{-1}_{[x]_k}\|\right) - k\log\left(\frac{\gamma}{\alpha_-}\right) > c_0 + nc_1 + kc_2\right\} < \infty,$$

where $c_1, c_2 > 0$. The last inner sum can be estimated from above by

$$\frac{1}{c_1}\int_{A_k} a_k(x)\,d\mu(x) \leqslant \frac{1}{c_1}\sqrt{\mu(A_k)}\|a_k(x)\|_{L^2_\mu} \tag{4.12}$$

where

$$a_k(x) = \log\left(r_{[x]_k}\mu_{[x]_k}\|M^{-1}_{[x]_k}\|\right) - k\log\left(\frac{\gamma}{\alpha_-}\right) \tag{4.13}$$

and

$$A_k = \left\{ x : \log\left(r_{[x]_k}\mu_{[x]_k}\|M_{[x]_k}^{-1}\| \right) - k\log\left(\frac{\gamma}{\alpha_-}\right) > c_0 + kc_2 \right\}. \tag{4.14}$$

The L^2_μ-norm of $a_k(x)$ grows polynomially by [4, Lemma V.5.2], while $\mu(A_k)$ decreases exponentially according to [4, Theorem V.6.2], which completes the proof. □

Theorem 6. *Suppose* (F,μ) *satisfies the strong main assumption and* $f \in \operatorname{Dom}\Delta_\mu$*. Then for any* $\varepsilon > 0$ *and* μ *a.e.* x

$$f(y) = f(x) + \operatorname{Grad}_x f(y) - \operatorname{Grad}_x f(x) + o\big((\gamma+\varepsilon)^n\big)_{y\to x}, \tag{4.15}$$

where $y \in F_{[x]_n}$*.*

Proof. The proof follows the same ideas as the proof of Theorem 2, but is actually simpler. We assume that $f = Gu$ and let u_n be u multiplied by the indicator function of $F_{[x]_n}$. For $y \in F_{[x]_n}$ we have that

$$G(u-u_n)(y) - G(u-u_n)(x) - (\operatorname{Grad}_x G(u-u_n)(y) - \operatorname{Grad}_x G(u-u_n)(x)) = 0 \tag{4.16}$$

since $G(u-u_n)$ is harmonic on $F_{[x]_n}$. Thus, we have to show that, for $y \in F_{[x]_n}$,

$$Gu_n(y) - Gu_n(x) - (\operatorname{Grad}_x Gu_n(y) - \operatorname{Grad}_x Gu_n(x)) = o((\gamma+\varepsilon)^n). \tag{4.17}$$

Lemma 2.13 implies that for $y \in F_{[x]_n}$,

$$\|Gu_n(y) - Gu_n(x)\| \leqslant Const\, \mu_{[x]_n} r_{[x]_n}\|u\|_\infty = o((\gamma+\varepsilon)^n). \tag{4.18}$$

Since, in general, $\operatorname{Grad}_x f_{[x]_n} = M_{[x]_n}\operatorname{Grad}_{\psi_{[x]n}(x)} f$, we have

$$\begin{aligned} &\|(\operatorname{Grad}_x Gu_n(y) - \operatorname{Grad}_x Gu_n(x))_{F_{[x]_n}}\|_\infty \\ &= \|\operatorname{Grad}_{\psi^{-1}_{[x]_n}}(Gu_n)_{[x]_n}(y) - \operatorname{Grad}_{\psi^{-1}_{[x]_n}}(Gu_n)_{[x]_n}(x)\|_\infty, \end{aligned} \tag{4.19}$$

which by Theorem 5 is bounded by

$$\begin{aligned} &Const\,\|\Delta(Gu_n)_{[x]_n}\|_\infty \sum_{k>n} r_{[x]_{n,k}}\mu_{[x]_{n,k}}\|M^{-1}_{[x]_{n,k}}\| \\ &\leqslant Const\,\|u\|_\infty r_{[x]_n}\mu_{[x]_n}\sum_{k>n} r_{[x]_{n,k}}\mu_{[x]_{n,k}}\|M^{-1}_{[x]_{n,k}}\|, \end{aligned} \tag{4.20}$$

where $[x]_{n,k}$ is the word of length $k-n$ defined by $[x]_n[x]_{n,k} = [x]_k$ for $k \geqslant n$. The left hand side of (4.17) is thus estimated by

$$Const\,\|u\|_\infty \sum_{k\geqslant n} r_{[x]_k}\mu_{[x]_k}\|M^{-1}_{[x]_{n,k}}\|, \tag{4.21}$$

which is $o((\gamma+\varepsilon)^n)$ by Lemma 4.3. □

As an immediate consequence we obtain the following Corollary, which make it straightforward to prove generic differentiability at points where $\operatorname{Grad}_x f$ exists.

Corollary 4.4. *Suppose (F,μ) satisfies the strong main assumption and $f \in \operatorname{Dom}\Delta_\mu$. Then for μ a.e. x*

$$f(y) = f(x) + \operatorname{Grad}_x f(y) - \operatorname{Grad}_x f(x) + o\big(\|M_{[x]_n} h\|\big)_{y\to x}, \tag{4.22}$$

for any non-constant $h \in \mathcal{H}$.

The same result for $\operatorname{Grad}_x f$, or rather the tangent $T_1(f)$, on the Sierpiński gasket was proved in [18, Section 7] under a stronger assumption (4.2).

We can now state the relations between the derivative and the gradient.

Proposition 4.5. *Suppose (F,μ) satisfies the strong main assumption, $f \in \operatorname{Dom}\Delta_\mu$ and h is a non constant harmonic function. Then the following assertions hold.*

(1) *For μ a.e. x such that $\operatorname{Grad}_x f = 0$, we have that $\dfrac{df}{dh}(x) = 0$.*
(2) *For μ a.e. x such that $\operatorname{Grad}_x f \neq 0$, we have that $\dfrac{df}{d\operatorname{Grad}_x f}(x) = 1$.*
(3) *For μ a.e. x*

$$\frac{df}{dh}(x) = \frac{< \operatorname{Grad}_x f, h_x^+ >}{< h, h_x^+ >}. \tag{4.23}$$

In particular for μ a.e. x we have

$$\frac{df}{dh_x^+}(x) =< \operatorname{Grad}_x f, h_x^+ >, \tag{4.24}$$

$$\left|\frac{df}{dh}(x)\right| = \frac{\|P_x^+ \operatorname{Grad}_x f\|}{\|P_x^+ h\|} \tag{4.25}$$

and $\frac{df}{dh}(x) = 0$ if and only if $\operatorname{Grad}_x f \in \mathcal{H}_x^-$.

Proof. The first two statements are obvious from Corollary 4.4. For the third, we know $h \notin \mathcal{H}_x^-$ for μ a.e. x, and in that case

$$\begin{aligned} f(y) - f(x) &= \operatorname{Grad}_x f(y) - \operatorname{Grad}_x f(x) + o\big(\|M_{[x]_n} h\|\big)_{y\to x} \\ &= \frac{< \operatorname{Grad}_x f, h_x^+ >}{< h, h_x^+ >}(h(y) - h(x)) + o\big(\|M_{[x]_n} h\|\big)_{y\to x}. \end{aligned} \tag{4.26}$$

□

As formulated, Theorem 4 on geography is destiny, raises the question about where the derivative is different from zero. Our next results relates this to the same question on the gradient.

Lemma 4.6. *Suppose (F,μ) satisfies the strong main assumption. Then for any $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ with $\lim\limits_{\varepsilon\to 0} \delta(\varepsilon) = 0$ such that if*

$$\frac{\|\Delta f\|_\infty}{\|P_{\mathcal{H}} H f\|} < \varepsilon, \tag{4.27}$$

then

$$\mu\{x : \mathrm{Grad}_x f \in \mathcal{H}_x^-\} < \delta(\varepsilon). \tag{4.28}$$

In particular, $\mu\{x : \mathrm{Grad}_x f \neq 0\} > 1 - \delta(\varepsilon)$.

Proof. For simplicity assume $\|P_{\mathcal{H}} H f\| = 1$ and $\|\Delta f\|_\infty < \varepsilon < \frac{1}{4}$. Define F_ε as the set of x such that

$$C \sum_{n \geqslant 1} r_{[x]_n} \mu_{[x]_n} \|M_{[x]_n}^{-1}\| < \varepsilon^{-\frac{1}{2}}, \tag{4.29}$$

where C is the constant in the estimate (4.4). Then by (4.6) for any $x \in F_\varepsilon$ we have

$$\|P_{\mathcal{H}} H f - \mathrm{Grad}_x f\| \leqslant \sqrt{\varepsilon}, \tag{4.30}$$

so $\mathrm{Grad}_x f \neq 0$ and

$$\rho(\mathrm{Dir} P_{\mathcal{H}} H f, \mathrm{Dir}\mathrm{Grad}_x f) < 2\sqrt{\varepsilon} \tag{4.31}$$

for all $x \in F_\varepsilon$. Let $V \subset \mathbb{P}(\mathcal{H})$ be the set of directions orthogonal to $P_{\mathcal{H}} H f$, and let $V_\varepsilon = \{v_0 \in \mathbb{P}(\mathcal{H}) : \inf_{v \in V} \rho(v_0, v) < \varepsilon\}$. We then have the estimate

$$\begin{aligned} \mu\{x : \mathrm{Grad}_x f \in \mathcal{H}_x^-\} &\leqslant \mu\{x \in F_\varepsilon : \mathrm{Grad}_x f \in \mathcal{H}_x^-\} + 1 - \mu(F_\varepsilon) \\ &\leqslant \mu\{x : \mathrm{Dir} h_x^+ \in V_{2\sqrt{\varepsilon}}\} + 1 - \mu(F_\varepsilon) \\ &= \nu(V_{2\sqrt{\varepsilon}}) + 1 - \mu(F_\varepsilon) = \delta(\varepsilon) \end{aligned} \tag{4.32}$$

where the measure ν is a μ-invariant measure on $\mathbb{P}(\mathcal{H})$ and $\lim_{\varepsilon \to 0} \nu(V_{2\sqrt{\varepsilon}}) = 0$ since $\nu(V) = 0$ [4, Proposition III.2.3]. □

Theorem 7. *If* (F, μ) *satisfies the strong main assumption, then for any* $f \in \mathrm{Dom}\,\Delta_\mu$,

$$\mathrm{Grad}_x f \notin \mathcal{H}_x^- \tag{4.33}$$

for μ *a.e.* x *with* $\mathrm{Grad}_x f \neq 0$.

Proof. For simplicity assume $\|\Delta f\|_\infty < 1$. Define F_ε as the set of x such that

$$\|\mathrm{Grad}_x f\| > \varepsilon. \tag{4.34}$$

Then define $F_{n,\varepsilon}$ as the set of x such that

$$\|\mathrm{Grad}_{[x]_n} f\| > \varepsilon \tag{4.35}$$

and

$$r_{[x]_n} \mu_{[x]_n} \|M_{[x]_n}^{-1}\| < \varepsilon^2. \tag{4.36}$$

Clearly

$$\lim_{n \to \infty} \mu(F_\varepsilon \setminus F_{n,\varepsilon}) = 0 \tag{4.37}$$

and

$$\lim_{\varepsilon \to 0} \mu(F_0 \setminus F_\varepsilon) = 0. \tag{4.38}$$

Then for any $x \in F_{n,\varepsilon}$ we have

$$\frac{\|\Delta f_{[x]_n}\|_\infty}{\|P_{\mathcal{H}} H f_{[x]_n}\|} = \frac{\|M^{-1}_{[x]_n}\| \|\Delta f_{[x]_n}\|_\infty}{\|M^{-1}_{[x]_n}\| \|M_{[x]_n} \mathrm{Grad}_{[x]_n} f\|} \leqslant \frac{r_{[x]_n} \mu_{[x]_n} \|M^{-1}_{[x]_n}\|}{\|\mathrm{Grad}_{[x]_n} f\|} < \varepsilon. \tag{4.39}$$

Here we can use Lemma 4.6 for each $f_{[x]_n}$ together with $\mathrm{Grad}_x f_{[x]_n} = M_{[x]_n} \mathrm{Grad}_{\psi_{[x]_n}(x)} f$ and $M^{-1}_{[x]_n} \mathcal{H}^-_x = \mathcal{H}^-_{\psi_{[x]_n}(x)}$, to obtain that

$$\begin{aligned} \delta(\varepsilon) &> \mu\{x : \mathrm{Grad}_x f_{[x]_n} \in \mathcal{H}^-_x\} \\ &= \mu\{x : M_{[x]_n} \mathrm{Grad}_{\psi_{[x]_n}(x)} f \in \mathcal{H}^-_x\} \\ &= \mu\{x : \mathrm{Grad}_{\psi_{[x]_n}(x)} f \in M^{-1}_{[x]_n} \mathcal{H}^-_x\} \\ &= \mu\{x : \mathrm{Grad}_{\psi_{[x]_n}(x)} f \in \mathcal{H}^-_{\psi_{[x]_n}(x)}\} \\ &= \mu_w^{-1} \mu\{y \in F_w : \mathrm{Grad}_y f \in \mathcal{H}^-_y\}. \end{aligned} \tag{4.40}$$

Therefore,

$$\begin{aligned} &\mu\{x \in F_{n,\varepsilon} : \mathrm{Grad}_x f \in \mathcal{H}^-_x\} \\ &= \sum \mu\{x \in F_w : \mathrm{Grad}_x f \in \mathcal{H}^-_x\} < \sum \mu_w \delta(\varepsilon) = \mu(F_{n,\varepsilon})\delta(\varepsilon), \end{aligned} \tag{4.41}$$

where the sum is over all $w \in W_n$ such that $F_w \subset F_{n,\varepsilon}$. Thus,

$$\mu\{x \in F_\varepsilon : \mathrm{Grad}_x f \in \mathcal{H}^-_x\} < \limsup \mu(F_\varepsilon \backslash F_{n,\varepsilon}) + \mu(F_{n,\varepsilon})\delta(\varepsilon) < \delta(\varepsilon) \tag{4.42}$$

and

$$\mu\{x \in F_0 : \mathrm{Grad}_x f \in \mathcal{H}^-_x\} = 0. \tag{4.43}$$

□

We can now formulate geography is destiny with conditions on the gradient.

Corollary 4.7. *Suppose (F, μ) satisfies the strong main assumption, $f \in \mathrm{Dom}\,\Delta_\mu$ and h is a non-constant harmonic function. Then*

$$\lim_{n\to\infty} \rho(\mathrm{Dir} f_{[x]_n}, \mathrm{Dir} h_{[x]_n}) = 0 \tag{4.44}$$

for μ a.e. x where $\mathrm{Grad}_x f \neq 0$

Proof. Theorem 7, Proposition 4.5 and Theorem 4. □

The next corollary is one more analog of Fermat's theorem.

Corollary 4.8. *Suppose (F, μ) satisfies the strong main assumption. Then there exists a set F' of full μ-measure such that if $f = u(g_1, \dots, g_l) \in C^1(\mathrm{Dom}\,\Delta_\mu)$ has a local maximum at $x \in F'$, then* $\mathrm{Grad}_x f = 0$.

Proof. The proof is the same as that of Corollary 2.15 and uses Theorem 6. □

Similarly to Corollary 2.16, we can obtain an analogous corollary for non-boundary periodic points under the assumption $r_w \mu_w \|M_w^{-1}\| < 1$. The existence of the gradient in such a case is guaranteed by Theorem 5.

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