Unwrapping a conic section

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Abstract

To enhance understanding when analysing the intersection between a cone and a plane, it is helpful to have a simple 3-dimensional paper model of the intersection. In this note, it is shown how the conic sections appear when the cone is unwrapped (flattened). Equations for the flattened conic intersections together with a printable figure and some LaTeX code for experiments are included.

1. Introduction

Curves and surfaces generated from the conic sections ([3, Chapt. 1 Sect. 2]), that is from the intersection between a cone and a plane, are commonly used. Although these intersections can be sketched by hand or generated in a computer program, having a real 3-dimensional model is helpful for learning. In particular, when discussing parametrizations and projections. Such a model is of course also useful when in geometry the conic sections themselves are discussed.

A naive approach to produce a real world model is to start from a circular sector, fold it into a cone and try with a pair of scissors to cut the cone. This seldom works since the paper tend to buckle and the cut does not become straight. Easier would be to draw the unwrapped intersection directly on the circular sector before folding it into cone. But what does such a curve look like on the circular sector, that is on the flattened cone? There is a derivation presented in the work [1]. However, it is a geometric description, and no concrete example suitable for printing is given.

Therefore, we present some algebraic derivations for the conic sections. Using the obtained formulas, a figure of the circular sector together with the unwrapped conic sections can be generated and is

The document was finished on 30th January 2022 included, which the reader can print and use to fold back into a cone. Moreover, a LaTeX code for this figure is given at the end (see Appendix on p. 6, Section 3), in which parameters can be changed for experiments.

2. Equation for the unwrapped conics

Choose a circular cone with vertex at the origin in a rectangular coordinate system, such that the equation for the cone is $z^2 = c^2(x^2 + y^2)$, $z \geq 0$ (with $c > 0$). This choice makes

$$\sin\alpha = \frac{1}{\sqrt{1+c^2}}, \tag{1}$$

where α is half of the angle at the vertex of the cone, see Fig. 1 for notation.

Let the plane $z = ax + by + d$ intersect the cone. We only consider at the moment $a > 0$, $b > 0$ and $d > 0$ chosen such that the intersection is an ellipse. The polar representation of this ellipse in the xy-plane is sought. Putting $x = r\cos\varphi$ and $y = r\sin\varphi$, and using the equality of the intersection

$$(ax + by + d)^2 = c^2(x^2 + y^2),$$

it follows that

$$d^2 + 2d(a\cos\varphi + b\sin\varphi)r - (c^2 - (a\cos\varphi + b\sin\varphi)^2)r^2 = 0.$$

This can be written

$$d^2 + 2d\sqrt{a^2+b^2}\sin(\varphi+\varphi_0)r - (c^2 - (\sqrt{a^2+b^2}\sin(\varphi+\varphi_0))^2)r^2 = 0, \tag{2}$$

where

$$\varphi_0 = \arctan\frac{a}{b}.$$

We put $\varphi_0 = \frac{\pi}{2}$ when $b = 0$. It is straightforward to verify that a solution to (2) is

$$r(\varphi) = \frac{\frac{d}{c}}{1 - \frac{\sqrt{a^2+b^2}}{c}\sin(\varphi+\varphi_0)}. \tag{3}$$

We assume that the parameters and angles are restricted such that (3) is positive, it is then the polar form of the projection onto the xy-plane of the intersection between the cone and the plane.

To find the corresponding polar form of the unwrapped intersection consider Fig. 1. As shown there, we first tilt the cone to lie along an axis in the xy-plane. The projected ellipse with polar equation $r(\varphi)$ given by (3) is shown at the base of the tilted cone. The length of the corresponding polar radius $R(\varphi)$, describing the elliptic intersection on the cone itself, can be found using similarity applying (1) to render

$$R(\varphi) = \sqrt{1+c^2}\, r(\varphi).$$

The cone is unwrapped (flattened) by rolling its surface along the xy-plane. The curve traced out in this process by the ellipse on the cone is described by the radius $R(\varphi)$. Note that as φ revolves one full lap (angle from say 0 to 2π), the angle θ of the circular sector is only increased by $2\pi/\sqrt{1+c^2}$, since the ratio between the radius r of the cone and slant height s is $\frac{r}{s} = \sin\alpha = \frac{1}{\sqrt{1+c^2}}$. We get the following.

Proposition 2.1 *The unwrapped trace of an elliptic intersection of the cone $z^2 = c^2(x^2+y^2)$, $z \geq 0$, and the plane $z = ax + by + d$, has polar equation*

$$R(\theta) = \frac{\sqrt{1+c^2}\,\frac{d}{c}}{1 - \frac{\sqrt{a^2+b^2}}{c}\sin(\sqrt{1+c^2}\,\theta + \varphi_0)}, \quad \text{for } 0 \leq \theta < \frac{2\pi}{\sqrt{1+c^2}}. \tag{4}$$

A formal geometrical proof of this relation when $a = 0$, and hence $\varphi_0 = 0$, is given in [1, Thm. 7].

The above includes the case of a parabola as well as hyperbola. In the case of a parabola, the equation of the plane is parallel with a generatrix of the cone (rendering $\sqrt{a^2+b^2} = c$) and the polar angle is not one full lap but has to be suitably restricted (to avoid division by zero), and similar for a hyperbola ($\sqrt{a^2+b^2} > c$).

In Fig. 2 is a circular sector together with unwrapped conic sections marked out, which the reader can print and use. In this figure, $b = 0$ and $c = \sqrt{8}$, and for the respective curve the parameters are, for the circle: $a = 0$, $d = 6$, ellipse: $a = 0.5$, $d = 8$, parabola: $a = \sqrt{8}$, $d = 24$ and for the hyperbola: $a = 8$, $d = 28$. The unwrapped intersection in Fig. 1 is generated with $a = 0$ and $b > 0$.

It is advisable to put some sticky tape on the dashed tab after printing the model in Fig. 2, to be able to make the cone flat again. We have let students make their own cones from Fig. 2., and their model (flattened) can easily be carried to lectures and tutorials. In terms of size, Fig. 2. can be enlarged 2.5 times to fit nicely on A3 size paper (landscape). It is also good to produce models where a part of the cone is removed by cutting along one of the flattened conics in Fig. 2., before folding it to demonstrate a cone cut by a plane.

It is a good exercise to adjust parameters and print the unwrapped conic intersections in the case when $\alpha = \pi/6$, rendering the circular sector to be a semi-circle; a figure of this case is given in [2, p. 52] (no formulas presented there).

3-D printing of the conics can be an alternative, however, they are more resource demanding to produce to a large class and are not that easy to carry (see [4] for a 3D-printable machine for conics).

We point out that to generate a paper model of the intersection of a plane and a cylinder is easier, since the trace of the flattened curve of intersection is then a sine-curve, see [1].

References

[1] Apostol, T. M. and Mnatsakanian, M. A., Unwrapping curves from cylinders and cones, *Amer. Math. Monthly* **114** (2007), 388–416.

[2] Bardos, L. C., Amazing Math Projects You Can Build Yourself, Nomad Press, Ann Arbor, MI, 2010.

[3] Hilbert, D. and Cohn-Vossen, S. Geometry and the Imagination, Chelsea Publishing Co., New York, N. Y., 1952.

[4] Milici, P., Plantevin, F. and Salvi, M., A 3D-printable machine for conics and oblique trajectories, *Internat. J. Math. Ed. Sci. Tech.*, (2021), 1–17.

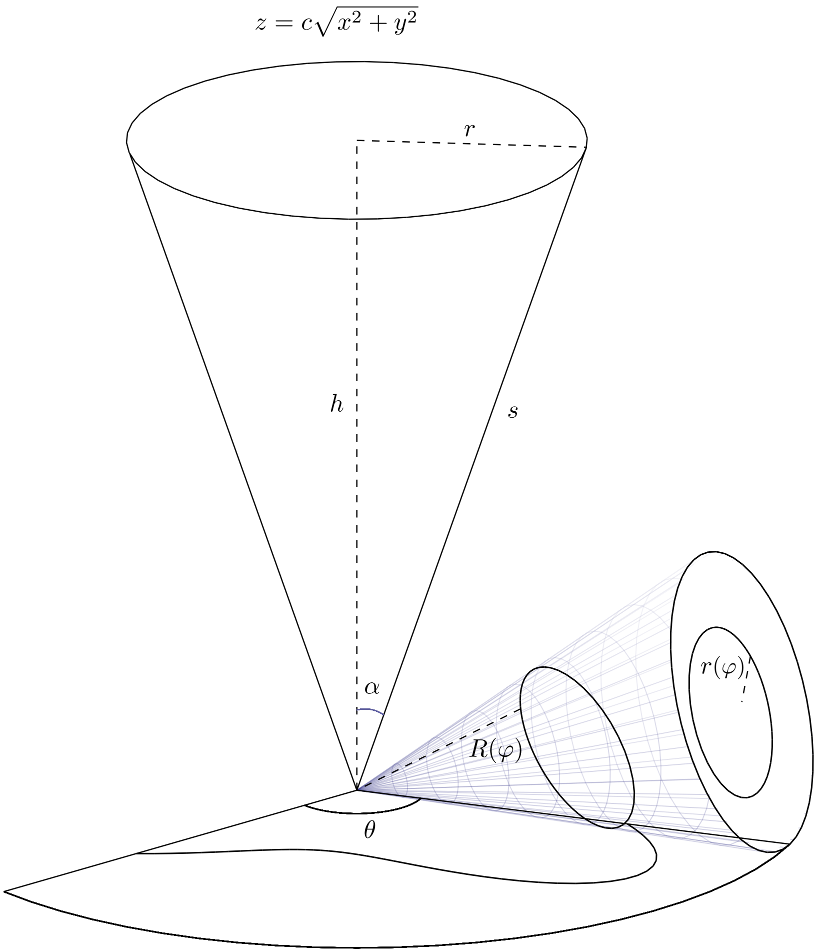

Figure 1: A tilted cone, its unwrapped circular sector and corresponding planar trace of a conic section

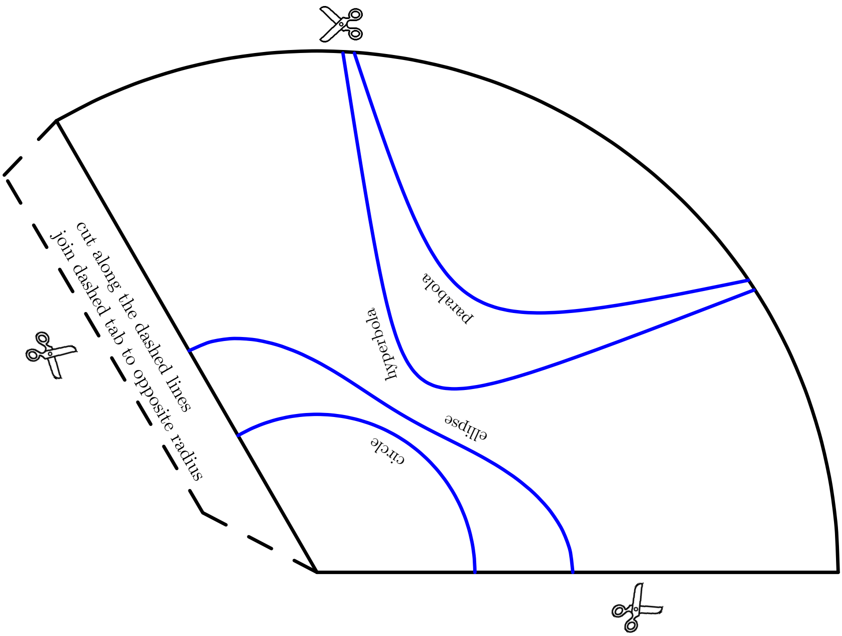

Figure 2: Unwrapped conic sections

Appendix: LaTeX code for the unwrapped conics

The code implements formula (4) for the polar radius R, and then draw a figure of the flattened conic intersection, similar to Fig 2., using a parametric plot in the plane. Parameters are set for the case of the cone being $z^2 = 8(x^2 + y^2)$ and plane $z = x + 8$, that is $c = \sqrt{8}, a = 1, b = 0$ and $d = 8$, rendering the intersection between the cone and plane to be an ellipse.

The reader is encouraged to change b into $b = 1$ in the code to generate a figure when the plane is instead $z = x + y + 8$. Looking at the resulting figure, it is not obvious that the flattened curve will fold back into an ellipse.

```
\documentclass[11pt,a4paper]{article}
\usepackage{tikz}
\usetikzlibrary{arrows}

\begin{document}
\begin{tikzpicture}[x=0.25cm,y=0.25cm]
% Parameters defined, cone z^2=c^2(x^2+y^2), plane z=ax+by+d
\def\a{1}
\def\b{0}
\def\d{8}
\def\c{(8)^(1/2)}
% These parameters are used in quantites found in formula (4)
\def\l1{((\a^2+\b^2)^(1/2)/\c)}
\def\R0{((1+\c^2)^(1/2))*\d/\c}
\def\k{((1+\c^2)^(1/2))}
\pgfmathsetmacro{\fiphase}{ifthenelse(\b==0,"pi/2","atan(\a/\b)")}
% Polar radius function from formula (4)
\tikzset{declare function
={R(\t,\R0,\l1,\k,\fiphase)=(\R0/(1-\l1*sin(((\k*\t)+\fiphase) r)));}}
% The unwrapped conic section is drawn as (R(t)cos(t),R(t)sin(t))
% with R(t) given by formula (4)
\draw [blue] plot[domain=0:2*pi/\k,variable=\t,smooth]
({R(\t,\R0,\l1,\k,\fiphase)*cos(\t r)},
{(R(\t,\R0,\l1,\k,\fiphase)*sin(\t r)});
% The circular sector is drawn
\draw plot[domain=0:2*pi/\k,variable=\t,smooth]
({21*cos(\t r)},{21*sin(\t r)});
% Supporting radial lines are drawn
\draw (0,0) -- (21,0);
\draw (0,0) -- ({21*cos(2*pi/\k r)},{21*sin(2*pi/\k r});
\end{tikzpicture}
\end{document}
```