UPPSALA
UNIVERSITET

U.U.D.M. Project Report 2021:54

Hilbert spaces and the Spectral theorem

Felicia Lapinski

Examensarbete i matematik, 15 hp
Handledare: Kaj Nyström
Examinator: Jörgen Östensson
September 2021

Department of Mathematics
Uppsala University

Contents

1 Introduction **1**

2 Hilbert space **1**

2.1 Definition of Hilbert space . . . 1

2.2 Properties of the inner product and the norm . . . 3

2.3 Subspaces of Hilbert space and direct sum . . . 5

2.4 Orthonormal sets and sequences . . . 8

3 Linear operators **11**

3.1 Definition and basic properties . . . 12

3.2 Bounded operators . . . 15

3.3 Self-Adjoint operators . . . 17

3.4 A special case: the projection operator . . . 20

4 Spectral theory **22**

4.1 Finite dimensional case . . . 23

4.2 Infinite dimensional case: basic concepts . . . 23

4.3 Spectral theory of bounded self-adjoint operators . . . 25

4.3.1 The Spectral theorem . . . 27

5 Discussion **36**

1 Introduction

This thesis considers several aspects of functional analysis; linear operators, Hilbert spaces, and spectral theory of operators. The development of functional analysis started in the beginning of the 20^{th} century in order to create a unified approach to several different fields of mathematics such as linear algebra, Fourier analysis, calculus of variations and linear ordinary and partial differential equations. [1]

The strength of functional analysis is that it considers *abstract spaces*, consisting of a set of elements whose nature is left unspecified. We only require these spaces to satisfy a few necessary axioms, such as having a norm or inner product defined on them, or the requirement that the space is complete. Building on the chosen axioms, we then develop a theory which can be applied to a broad variety of spaces and which connects various branches of mathematics.

The idea to consider abstract spaces was introduced by M. Frechét in 1906 when he studied the class of metric spaces. [1] Other important contributors to the field of functional analysis are D. Hilbert, S. Banach and F. Riesz. [2]

Functional analysis has numerous applications in physics; in fact, quantum mechanics motivated much of the theory of Hilbert spaces in order to establish a solid mathematical foundation for quantum theory.

The main aim of this thesis is to prove the Spectral theorem for bounded, self-adjoint operators. In order to do this, some theory regarding Hilbert spaces and linear operators is needed. Section 2 examines the structure of Hilbert spaces and leads up to a proof of the existence of orthonormal bases in infinite-dimensional Hilbert space. Section 3 studies the properties of linear operators in normed spaces, and in particular bounded, self-adjoint operators in Hilbert spaces for which the Spectral theorem holds. Finally, section 4 considers the spectral theory of bounded, self-adjoint operators and ends with a proof of the Spectral theorem.

It is assumed that the reader is familiar with some concepts from real analysis such as completeness, convergence, Riemann-Stieltjes integrals, normed and Banach spaces, as well as linear algebra.

2 Hilbert space

2.1 Definition of Hilbert space

The concept of Hilbert spaces arises frequently in both theory and applications. These spaces generalize the notions of length and orthogonality from Euclidean space to vector spaces of finitely or infinitely many dimensions, for example infinitely-dimensional function spaces.

In Euclidean space $\mathbb{R}^n$, the dot product of two vectors $x, y \in \mathbb{R}^n$ is defined as

$$x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \tag{1}$$

We would like to extend this operation to a general vector space, where the vectors aren't necessarily numbers (or n-tuples of numbers), but virtually any kind of element that can be added or multiplied by a scalar, such as sequences, matrices, or functions. This is done by introducing the *inner product* in the following definition.

Definition 2.1 (**Inner product space, Hilbert space**). An *inner product space* is a normed vector space X on which an inner product is defined; that is, a mapping from $X \times X$ into its associated scalar field K that satisfies properties (IP1)-(IP4). An inner product space that is also complete is called a *Hilbert space* and is often denoted H.

The inner product of two elements x, y in X is denoted

$$\langle x, y \rangle.$$

For all vectors $x, y, z \in X$ and scalars $a \in K$, we have

$$\langle x + y, z \rangle = \langle x, z \rangle + \langle y + z \rangle \tag{IP1}$$
$$\langle ax, y \rangle = a \langle x, y \rangle \tag{IP2}$$
$$\langle x, y \rangle = \overline{\langle y, x \rangle} \tag{IP3}$$
$$\langle x, x \rangle \geq 0, \qquad \langle x, x \rangle = 0 \iff x = 0. \tag{IP4}$$

Note that (IP2) and (IP3) yield that $\langle x, by \rangle = \overline{\langle by, x \rangle} = \overline{b} \overline{\langle y, x \rangle} = \overline{b} \langle x, y \rangle$. The inner product is said to be *sesquilinear*, meaning that it is linear in its first argument and conjugate-linear in its second argument.

As a consequence of defining an inner product on a vector space X, we obtain a norm on X given by

$$\sqrt{\langle x, x \rangle} = \|x\| \tag{2}$$

(called the *norm induced by the inner product*) and a metric, given by

$$\sqrt{\langle x - y, x - y \rangle} = \|x - y\| = d(x, y). \tag{3}$$

This implies that inner product spaces are also normed metric spaces, and that Hilbert spaces are Banach spaces. Note that the norm is a function that maps a real or complex space onto the non-negative real numbers, so that $\|x\| \in \mathbb{R}$ even though x may be complex.

Example 2.1 Euclidean space $\mathbb{R}^2$. The two-dimensional Euclidean space is a Hilbert space equipped with an inner product as given by (1) with $n = 2$, and the norm induced by the inner product is

$$\|x\| = |x| = \sqrt{x_1^2 + x_2^2}. \tag{4}$$

The inner product may also be expressed as

$$\langle x, y \rangle = |x||y| \cos \theta \tag{5}$$

where θ is the angle between x and y.

Example 2.2 Function space $L^2[a, b]$. The function space $L^2[a, b]$ consists of real- or complex-valued functions that are square-integrable on the interval $[a, b]$, that is, functions $x : [a, b] \to \mathbb{C}$ such that

$$\int_a^b |x(t)|^2 \, dt < \infty \, .$$

In this space, the inner product is defined as

$$\langle x, y \rangle = \int_a^b x(t) \overline{y(t)} \, dt$$

and the resulting norm is

$$\|x\| = \langle x, x \rangle^{1/2} = \left(\int_a^b |x(t)|^2 \, dt \right)^{1/2}$$

since $x(t)\overline{x(t)} = |x(t)|^2$. The condition that the functions be square-integrable ensures that the norm of every element is well-defined. The space $L^2(\mathbb{R})$ is of particular importance in physics since it is inhabited by the wave functions of quantum particles. These functions represent a particle's quantum state and can be regarded as probability functions, and as such they are required to be square integrable on $\mathbb{R}$ and to have norm 1.

2.2 Properties of the inner product and the norm

By defining the inner product, we can expand the meaning of two objects being *orthogonal* to each other. In $\mathbb{R}^2$, the notion of orthogonality means that two vectors are perpendicular to each other, i.e that the angle θ between them is $\pi/2$ and thus the inner product of as defined in (5) is zero. It follows from (5) that the inner product between two vectors in $\mathbb{R}^2$ cannot be larger than the product of each of their lengths,

$$\langle x, y\rangle \leq |x||y| \max\cos\theta = |x||y|$$

and that equality holds only when the vectors are parallel, i.e when $\theta = 0$.

In the more abstract setting that is Hilbert spaces, the inner product can be viewed as a measure of how much two elements have in common. Just as with Euclidean space, the inner product of two elements of fixed lengths is maximum when the two elements are linearly dependent, and zero if they are linearly independent. This gives a natural definition for orthogonality in Hilbert space.

Definition 2.2 (**Orthogonality**)**.** If, for two elements x, y in a Hilbert space H, we have

$$\langle x, y\rangle = 0$$

then x and y are said to be orthogonal.

The fact that the inner product of two vectors of fixed length in Euclidean space cannot exceed the product of their lengths can also be generalized to Hilbert space, and this is stated in the next theorem.

Theorem 2.1 (**Cauchy-Schwarz inequality**)**.** *If an inner product satisfies (IP1)-(IP4), then the inner product and the corresponding norm satisfy*

$$|\langle x, y\rangle| \leq \|x\|\,\|y\| \tag{6}$$

where equality holds if and only if x and y are linearly dependent.

Proof. Let $y \neq 0$ and let a be a scalar. Then we have

$$\begin{aligned} 0 \leq \|x + ay\|^2 &= \langle x + ay, x + ay\rangle \\ &= \langle x, x\rangle + \langle x, ay\rangle + \langle ay, x\rangle + \langle ay, ay\rangle \\ &= \|x\|^2 + \overline{a}\langle x, y\rangle + a\left[\langle y, x\rangle + \overline{a}\,\|y\|^2\right] \end{aligned}$$

where we have used (IP1)-(IP4). Let $\overline{a} = -\frac{\langle y,x\rangle}{\|y\|^2}$ so that the expression inside the brackets is zero, resulting in the inequality

$$
\begin{aligned}
0 &\le \|x\|^2 + \overline{a}\langle x, y\rangle \\
&= \|x\|^2 - \frac{\langle y, x\rangle\langle x, y\rangle}{\|y\|^2} \\
&= \|x\|^2 - \frac{\overline{\langle x, y\rangle}\langle x, y\rangle}{\|y\|^2} \\
&= \|x\|^2 - \frac{|\langle x, y\rangle|^2}{\|y\|^2}.
\end{aligned}
$$

Shifting one term to the other side and multiplying by $\|y\|^2$ gives the Cauchy-Schwarz inequality. If y is instead equal to zero, we have $|\langle x, 0\rangle| = 0$. If x, y are linearly dependent, there is a constant b such that $x = by$ which gives us $|\langle by, y\rangle| = |b| \, \|y\|^2$. Both cases are in accordance with the Cauchy-Schwarz inequality, and these two cases are the only ones where equality holds. $\square$

Other useful properties of the norm are the triangle inequality for the norm and the parallelogram equality, stated below.

Theorem 2.2 (**Triangle inequality**). *A norm satisfies*

$$
\|x + y\| \le \|x\| + \|y\| . \tag{7}
$$

Proof. We begin with

$$
\begin{aligned}
\|x + y\|^2 &= \langle x + y, x + y\rangle \\
&= \|x\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|y\|^2 \\
&\le \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2
\end{aligned}
$$

where the triangle inequality for numbers was applied at the last step. The Cauchy-Schwarz inequality applied to the second term yields

$$
\begin{aligned}
\|x + y\|^2 &\le \|x\|^2 + 2 \|x\| \, \|y\| + \|y\|^2 \\
&= (\|x\| + \|y\|)^2.
\end{aligned}
$$

Taking the square root on both sides results in the triangle inequality. $\square$

Theorem 2.3 (**Parallelogram equality**). *Let X be an inner product space. Then it holds for all $x, y \in X$ that*

$$
\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2. \tag{8}
$$

Proof. Since for the norm in an inner product space we have $\|\cdot\|^2 = \langle\cdot,\cdot\rangle$, we can write

$$
\|x + y\|^2 + \|x - y\|^2 = \langle x + y, x + y\rangle + \langle x - y, x - y\rangle
$$

and with the sesquilinearity of the inner product and the fact that $\overline{-1} = -1$ we get

$$
\begin{aligned}
&= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle + \langle x, x\rangle + \langle x, -y\rangle + \langle -y, x\rangle + \langle -y, -y\rangle \\
&= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle + \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle \\
&= 2\langle x, x\rangle + 2\langle y, y\rangle \\
&= 2\|x\|^2 + 2\|y\|^2.
\end{aligned}
$$

$\square$

Theorem 2.4 (Equality). *Let X be an inner product space. If $\langle x, y\rangle = \langle z, y\rangle$ for all $y \in X$, then $x = z$. Also, if $\langle x, y\rangle = 0$ $\forall y$ then $x = 0$.*

Proof. Assume that $\langle x, y\rangle = \langle z, y\rangle$. Then we have

$$\langle x - z, y\rangle = \langle x, y\rangle - \langle z, y\rangle = 0$$

for all y. Take $y = x - z$. Substituting into the above equation yields

$$\langle x - z, x - z\rangle = \|x - z\|^2 = 0$$

which implies that $x - z = 0$ so that $x = z$. If we have $\langle x, y\rangle = 0$ for all y, then putting $y = x$ gives $\|x\|^2 = 0$ so that $x = 0$. □

2.3 Subspaces of Hilbert space and direct sum

In this section we will examine properties of subspaces of Hilbert spaces. By a subspace Y of a Hilbert space H we mean a subset that is a vector space, with the same inner product as H but restricted to $Y \times Y$. Therefore a subspace of a Hilbert space is an inner product space, but not necessarily a Hilbert space since it might not be complete.

The following theorem considers subspaces of Banach spaces, and hence it applies also to Hilbert spaces (here, a subspace of a Banach space means a subset that is a vector space equipped with the same norm as the Banach space. Similarly to the case of the Hilbert space, a subspace of a Banach space may not be a Banach space itself).

Theorem 2.5 (Subspace of a Banach space). *Suppose that Y is a subspace of a Banach space X. Then, Y is complete (using the norm of X) if and only if it is closed.*

Proof. Assume that Y is closed. Let $\{y_n\}$ be a Cauchy sequence in Y; then it has a limit y in X (using the norm of X) since X is complete. Since $y_n \to y$, y is a limit point of Y. But Y is closed, i.e contains all its limit points. Thus all Cauchy sequences in Y converge in Y, and Y is complete. □

The following two theorems apply to inner product spaces, and will be useful in deriving further properties of Hilbert spaces.

Theorem 2.6 (Minimizing vector). *Let Y be a closed and convex subset of an inner product space X. Then there exists a unique $y \in Y$ such that*

$$\delta = \inf_{\tilde{y} \in Y} \|x - \tilde{y}\| = \|x - y\| \tag{9}$$

for all $x \in X$.

Proof. We define a minimizing sequence $\{y_i\}$ in Y, such that

$$y_n \longrightarrow y, \quad \delta_n = \|x - y_n\| \longrightarrow \delta.$$

By construction, it holds that $\|x - y_n\| \geq \delta \quad \forall n$. We claim that $\{y_i\}$ is Cauchy.

Define another sequence $\{u_i\}$: $u_n = y_n - x$, so that $\|u_n\| = \delta_n$. Then we have that $y_n - y_m = u_n - u_m$. First we note that

$$\|u_n + u_m\| = \|y_n + y_m - 2x\| = 2\left\|\frac{y_n + y_m}{2} - x\right\| \geq 2\delta. \tag{10}$$

The last step holds since Y is convex, so that $\frac{y_n+y_m}{2} \in Y$ and thus $\left\|\frac{y_n+y_m}{2} - x\right\| \geq \delta$. Next, we take

$$\|y_n - y_m\|^2 = \|u_n - u_m\|^2$$

and applying the parallelogram equality to the right hand side we obtain

$$\begin{aligned} \|y_n - y_m\|^2 &= - \|u_n + u_m\|^2 + 2\|u_n\|^2 + 2\|u_m\|^2 \\ &= - \|u_n + u_m\|^2 + 2\delta_n^2 + 2\delta_m^2 \\ &\leq -(2\delta)^2 + 2\delta_n^2 + 2\delta_m^2 \end{aligned} \tag{11}$$

where the last step follows from (10). By letting $m, n \longrightarrow \infty$ the right hand side of (11) becomes zero, so $\|y_n - y_m\| \leq 0$. But by the definition of the norm, $\|y_n - y_m\| \geq 0$. Then we must have that

$$\|y_n - y_m\| \longrightarrow 0 \qquad \text{as} \qquad n, m \longrightarrow \infty$$

and thus $\{y_i\}$ is Cauchy.

Since Y is complete, $\{y_i\}$ converges to some element $y \in Y$ and (as for all elements of Y) it holds that $\|x - y\| \geq \delta$. We also have that

$$\begin{aligned} \|x - y\| &= \|x - y_n + y_n - y\| \\ &\leq \|x - y_n\| + \|y_n - y\| \\ &= \delta_n + \|y_n - y\| \xrightarrow[n\to\infty]{} \delta. \end{aligned}$$

Thus we have $\|x - y\| \geq \delta$ and $\|x - y\| \leq \delta$ which implies $\|x - y\| = \delta$.

In order to prove uniqueness, we let y and $\hat{y}$ both be such that

$$\|x - y\| = \delta \quad , \qquad \|x - \hat{y}\| = \delta$$

and show that $y = \hat{y}$,

$$\begin{aligned} \|y - \hat{y}\|^2 &= \|(y - x) - (\hat{y} - x)\|^2 \\ &= - \|(y - x) + (\hat{y} - x)\|^2 + 2\|y - x\|^2 + 2\|\hat{y} - x\|^2 \\ &= -4\left\|\frac{y + \hat{y}}{2} - x\right\| + 2\delta^2 + 2\delta^2 \\ &\leq -4\delta^2 + 2\delta^2 + 2\delta^2 = 0 \end{aligned}$$

where we have again used the parallelogram equality as well as the fact that Y is convex, so that $\frac{y+\hat{y}}{2} \in Y$ and thus $\left\|\frac{y+\hat{y}}{2} - x\right\| \geq \delta$. As before, the norm is required to be non-negative which yields

$$\|y - \hat{y}\| = 0$$

$$\Longrightarrow \; y = \hat{y}$$

which concludes the proof. □

Theorem 2.7 (**Orthogonality with respect to a subspace**). *Let Y be a complete subspace of an inner product space X, let x be an element of X, and define $y \in Y$ as the vector that minimizes δ in 2.6. Then $z = x - y$ is orthogonal to Y.*

Proof. First we note that since a subspace of an inner product space is a vector space in its own right, it is a convex subset: if for any $x_1, x_2 \in X$ we have that $ax_1 + bx_2$ lies in X for all scalars a, b, the same holds for the segment $S = \{x : x = px_1 + (1-p)x_2,\ 0 < p < 1\}$.

Let us assume that the theorem doesn't hold; z is not orthogonal to Y. Then there exists a $y_1 \in Y$ such that

$$\langle z, y_1 \rangle = b \neq 0.$$

For a scalar a, we have

$$\begin{aligned}\|z - ay_1\|^2 &= \langle z - ay_1, z - ay_1 \rangle \\ &= \langle z, z \rangle - \bar{a}\langle z, y_1 \rangle - a\langle y_1, z \rangle + a\bar{a}\langle y_1, y_1 \rangle \\ &= \|z\|^2 - \bar{a}b - a\Big(\bar{b} - \bar{a}\,\|y_1\|^2\Big).\end{aligned}$$

Choose $\bar{a} = \bar{b}/\,\|y_1\|^2$ so that the expression inside the parenthesis is zero, yielding

$$\|z - ay_1\|^2 = \|z\|^2 - \frac{|b|^2}{\|y_1\|^2} < \|z\|^2 = \delta^2. \tag{12}$$

But at the same time,

$$z - ay_1 = x - y - a_y = x - y_2$$

for $y_2 = y + ay_1 \in Y$, and thus $\|z - ay_1\|^2 = \|x - y_2\|^2 \geq \delta^2$. This contradicts (12), so that there can't exist a $y_1 \in Y$ that isn't orthogonal to z. ☐

Definition 2.3 (Direct sum). Let X be a vector space, and let Y and Z be subspaces of X. If every $x \in X$ has a unique representation

$$x = y + z$$

where $y \in Y$, $z \in Z$, we say that X is a *direct sum* of Y and Z and we write

$$X = Y \oplus Z.$$

In that case, we call Y and Z a *complimentary pair* of subspaces, and we say that Y is the *algebraic complement* of Z in X and vice versa.

Note that for a given subspace Y, its algebraic complement isn't unique. For example, the real line $\mathbb{R}$ is a subspace of the real plane $\mathbb{R}^2$, and its algebraic complement can be any straight line with a non-zero slope.

Considering Hilbert spaces, we are most interested in complimentary pairs of subsets which are also orthogonal. For a subspace Y of a Hilbert space H, its *orthogonal complement* is defined as

$$Y^\perp = \{z \in H \mid z \perp Y\}.$$

Theorem 2.8 (Hilbert space as a direct sum). *Let H be a Hilbert space and Y be any closed subspace of H. Then*

$$H = Y \oplus Z \tag{13}$$

where $Z = Y^\perp$.

Proof. Since Y is closed, it is complete by 2.5. It is also convex, and thus 2.6 and 2.7 imply that for all $x \in H$ there exists a $z \perp Y$ such that

$$z = x - y \iff x = y + z.$$

This shows that x can be represented by two elements $y \in Y$ and $z \in Z$ that are orthogonal. To show that this representation is unique, assume that

$$\begin{aligned} x &= y + z = \tilde{y} + \tilde{z} \\ \implies y - \tilde{y} &= z - \tilde{z}. \end{aligned}$$

We know that $y - \tilde{y} \in Y$ and $z - \tilde{z} \in Z$, so that $y - \tilde{y} = z - \tilde{z} \in (Y \cap Z)$. But the intersection of two orthogonal sets is $\{0\}$, and thus we have that

$$\begin{aligned} y - \tilde{y} &= z - \tilde{z} = 0 \\ \implies y = \tilde{y}, &\quad z = \tilde{z} \end{aligned}$$

which proves uniqueness. □

Theorem 2.9 (**Dense set**). *Let H be a Hilbert space and $M \neq \emptyset$ a subset of H. Then $M^{\perp} = 0$ if and only if the span of M is dense in H.*

The proof of 2.9 is stated in section 3.3 of [1].

The role played by y and z in theorem 2.8 is called the *orthogonal projection* of x onto Y and Z, respectively. We will see in the next section that the inner product is a useful tool for expressing an element in Hilbert space by projecting it onto an orthonormal set.

2.4 Orthonormal sets and sequences

Of particular interest in Hilbert spaces are orthonormal sets and sequences, since these allow us to easily express an arbitrary element in a Hilbert space H as a linear combinations of orthonormal elements in H. The following definition specifies these concepts.

Definition 2.4 (**Orthogonal and orthonormal sets and sequences**). An *orthogonal set* in an inner product space X is defined as a subset $M \subset X$ such that all elements of M are pairwise orthogonal. If each element of M also has norm 1, the set is said to be *orthonormal.* In that case we have

$$\langle x, y \rangle = \begin{cases} 1, & x = y \\ 0, & x \neq y \end{cases}$$

for all $x, y \in M$. If M is an orthogonal or orthonormal set that is also countable (so that the elements of M can be enumerated) we call M an orthogonal/orthonormal *sequence.*

We have already alluded to the fact that the inner product of two elements in a Hilbert space is a measure of how linearly dependent two elements are. The next theorem states that orthogonality in fact implies linear independence.

Theorem 2.10 (**Linear independence**). *Orthonormal sets are linearly independent.*

Proof. Let $\{x_1, \ldots, x_n\}$ be an orthonormal set, and suppose we have

$$a_1 x_1 + \ldots + a_n x_n = 0.$$

Taking the inner product of the above equation with a fixed $x_k \in \{x_1, \dots x_n\}$, we obtain

$$\Big\langle \sum_i a_i x_i, x_k \Big\rangle = \sum_i a_i \langle x_i, x_k \rangle = a_k \langle x_k, x_k \rangle = a_k = 0$$

which shows that every a_i needs to be zero in order for the equation to hold, and thus $\{x_1, \dots x_n\}$ is linearly independent. The theorem holds for infinite orthonormal sets as well, since an infinite set M is said to be linearly independent if every non-empty finite subset of M is linearly independent. □

Note: The above holds also for orthogonal sets that aren't normalized, in which case it is required that all elements in the set are non-zero.

Orthonormal sequences can be exploited in order to easily express an element of an inner product space as a linear combination of elements from the orthogonal sequence. Again using Euclidean space as a concrete example, an arbitrary element $x \in \mathbb{R}^3$ can be expressed as

$$x = a_1 e_1 + a_2 e_2 + a_3 e_3$$

where $\{e_1, e_2, e_3\}$ are the unit vectors in $\mathbb{R}^3$: $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$. The coefficients a_i can be obtained by taking the inner product of x with each unit vector, for example a_1 is given by

$$\langle x, e_1 \rangle = a_1 \langle e_1, e_1 \rangle + a_2 \langle e_2, e_1 \rangle + a_3 \langle e_3, e_1 \rangle = a_1.$$

Similarly, let $\{e_i\}$ be an orthonormal sequence in an inner product space X. Then, for any $x \in X$ that lies in span$\{e_1, \dots e_n\}$ for some fixed n, we can write

$$x = \sum_{i=1}^{n} a_i e_i$$

by the definition of the span. The coefficients a_i are determined by taking the inner product of x with each fixed e_k,

$$\langle x, e_k \rangle = \Big\langle \sum_{i=1}^{n} a_i e_i, e_k \Big\rangle = a_k \langle e_k, e_k \rangle = a_k$$

so that the representation of x becomes

$$x = \sum_{i=1}^{n} \langle x, e_i \rangle e_i.$$

If x doesn't lie in span$\{e_1, \dots, e_n\}$ on the other hand, we can still relate x to its projection onto span$\{e_1, \dots, e_n\}$ by Bessel's inequality. In order to derive it, we will need the Pythagorean relation.

Theorem 2.11 (**Pythagorean relation**). *Let $\{x_1, \dots, x_n\}$ be an orthogonal set. Then it holds that*

$$\big\| x_1 + \dots + x_n \big\|^2 = \|x_1\| + \dots + \|x_n\|^2. \tag{14}$$

Proof. Since $\langle x_i, x_j \rangle = 0$ for $i \neq j$, we have

$$\Big\| \sum_i x_i \Big\|^2 = \Big\langle \sum_i x_i, \sum_j x_j \Big\rangle = \sum_i \sum_j \langle x_i, x_j \rangle = \sum_i \langle x_i, x_i \rangle = \sum_i \|x_i\|^2.$$

□

Theorem 2.12 (Bessel's inequality). *Let X be an inner product space and let $\{e_1, e_2, \dots\}$ be an orthonormal sequence in X. Then for any $x \in X$ we have*

$$\sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2 \leq \|x\|^2 \tag{15}$$

Proof. Let x be an element of X that isn't necessarily in $Y_n = \operatorname{span}\{e_1, \dots, e_n\}$. We define an element

$$y = \sum_{i=1}^{n} \langle x, e_i\rangle e_i$$

which lies in Y_n since it is a linear combination of $\{e_i\}$. We will show that $z = x - y$ is orthogonal to y. First, we note that

$$\|y\|^2 = \left\|\sum_{i=1}^{n} \langle x, e_i\rangle e_i\right\|^2 = \sum_{i=1}^{n} \|\langle x, e_i\rangle e_i\|^2 = \sum_{i=1}^{n} |\langle x, e_i\rangle|^2$$

due to the orthogonality of $\{e_i\}$. Taking the inner product of z with y yields

$$\begin{aligned}
\langle z, y\rangle &= \langle x - y, y\rangle \\
&= \langle x, y\rangle - \langle y, y\rangle \\
&= \left\langle x, \sum_{i=1}^{n} \langle x, e_i\rangle e_i \right\rangle - \|y\|^2 \\
&= \sum_{i=1}^{n} \left\langle x, \langle x, e_i\rangle e_i \right\rangle - \|y\|^2 \\
&= \sum_{i=1}^{n} \overline{\langle x, e_i\rangle} \langle x, e_i\rangle - \|y\|^2 \\
&= \sum_{i=1}^{n} |\langle x, e_i\rangle|^2 - \|y\|^2 \\
&= 0
\end{aligned}$$

which shows that $z \perp y$. Then we can apply the Pythagorean relation to $x = y + z$,

$$\begin{aligned}
\|x\|^2 &= \|y\|^2 + \|z\|^2 \\
&= \sum_{i=1}^{n} |\langle x, e_i\rangle|^2 + \|z\|^2
\end{aligned}$$

and since $\|z\| \geq 0$, we arrive at

$$\|x\|^2 \geq \sum_{i=1}^{n} |\langle x, e_i\rangle|^2.$$

The inequality holds for every n, so we can let n go towards infinity and obtain

$$\|x\|^2 \geq \sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2$$

which is Bessel's inequality. The inner products $\langle x, e_i\rangle$ are called the *Fourier coefficients* of x. $\square$

Compare the proof of the Bessel inequality with the proof of 2.8: y is again the orthogonal projection of x onto a subset of Hilbert space.

A question of interest that remains is: can we always find an orthonormal set such that every element of Hilbert space can be fully represented by its projection onto the set, i.e, does every Hilbert space have an orthonormal basis? It turns out that this is indeed true, and the proof of the existence of an orthonormal basis makes use of a result from set theory called Zorn's lemma.

Theorem 2.13 (Zorn's lemma). *If $A \neq \emptyset$ is a partially ordered set such that every totally ordered subset $B \subset A$ has an upper bound, then A has at least one maximal element.*

The proof is omitted since it involves knowledge of set theory that is beyond the scope of this thesis, the interested reader can find the proof and more discussion on Zorn's lemma in [3].

Theorem 2.14 (Orthonormal basis). *Every Hilbert space $H \neq \emptyset$ has an orthonormal basis.*

Proof. Define Y as the set of all orthonormal subsets of H. Y is not empty, since $H \neq \emptyset$ implies that there is an element $y \neq 0 \in H$ and thus $\{e_0\} = \{y/\|y\|\}$ constitutes an orthonormal set in H. Y is partially ordered since set inclusion is a partial order, and each totally ordered subset C in Y has an upper bound; namely, the union of all elements of C. Thus, according to Zorn's lemma, there exists a set that is a maximal element of Y. We call this set F and will show that it is an orthonormal basis in H.

To prove this, suppose the opposite: F is not an orthonormal basis, so that there exists an element $z \neq 0 \in H$ such that $z \perp F$. Then we can define a set $F_1 = F \cup e_1$ where $e_1 = z/\|z\|$. But this implies that F is a proper subset of F_1, which contradicts maximality. Thus F is an orthonormal basis in H. □

When an infinite set $\{e_i\}$ is an orthonormal basis for a Hilbert space H, Bessel's inequality becomes the *Parseval relation* [1]:

$$\|x\|^2 = \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2. \tag{16}$$

Note: The existence of a basis can be proved for an arbitrary vector space as well, and the proof is very similar to the one above. However, the two cases don't consider the same kind of basis. In an arbitrary vector space of finitely or infinitely many dimensions, we consider an algebraic (or Hamel) basis, that is, a set of linearly independent elements such that every element of the vector space can be expressed as a *finite* linear combination of elements of the basis. In Banach and Hilbert space we consider a Schauder basis, which makes it possible to express an element of the space using an *infinite* linear combination of elements of the basis. Proof of the existence of an algebraic basis in vector spaces and more discussion on the existence of a Schauder basis in Banach spaces can be found in [1].

The theorems in section 2 are taken from [1], and the proofs are strongly influenced by the proofs there.

3 Linear operators

In calculus, we consider mappings in real or complex Euclidean space and call the mappings functions. In general vector spaces on the other hand, mappings are called *operators* and an operator T applied to a vector x is usually denoted Tx. A class of operators that is of particular importance in functional analysis are *linear operators*, which we consider in the following section.

3.1 Definition and basic properties

Definition 3.1 (Linear operator). A *linear operator* is a mapping $T : X \to U$ such that its domain $\mathscr{D}(T) \subseteq X$ is a vector space, its range $\mathscr{R}(T)$ lies in a vector space U over the same scalar field K, and such that

$$T(ax + by) = aTx + bTy \tag{17}$$

for all $x, y \in \mathscr{D}(T)$ and all scalars $a, b \in K$.

An important consequence of (17) is that, for all linear operators, $T0 = 0$. This is since

$$T(0x) = 0Tx = 0. \tag{18}$$

Example 3.1 (Zero operator). The zero operator $0 : X \to Y$ is such that is maps all $x \in X$ to the zero element 0 in Y:

$$0x = 0$$

Example 3.2 (Identity operator). The identity operator $I : X \to X$ is such that $Ix = x$ $\forall x \in X$.

Example 3.3 (Differentiation operator). On a function space X consisting of all polynomials $x(t)$ on [0,1], we can define a differential operator $D : X \to X$ defined by

$$Dx = \frac{dx}{dt}$$

$\forall x \in X$.

Theorem 3.1 (Range and null space of a linear operator). *If T is a linear operator, then*

(i) The range $\mathscr{R}(T)$ is a vector space.
(ii) The null space $\mathscr{N}(T)$ is a vector space.
(iii) If $\dim \mathscr{D}(T) < \infty$, then $\dim \mathscr{R}(T) \leq \dim \mathscr{D}(T)$.

Proof. (i) Suppose that $u_1, u_2 \in \mathscr{R}(T)$. Then there exist $x_1, x_2 \in \mathscr{D}(T)$ such that $Tx_1 = u_1$ and $Tx_2 = u_2$. Since $\mathscr{D}(T)$ is a vector space, $ax_1 + bx_2 \in \mathscr{D}(T)$ for arbitrary scalars a, b. Using the linearity of T, we get

$$T(ax_1 + bx_2) = aTx_1 + bTx_2 = au_1 + bu_2.$$

Since $T(ax_1 + bx_2)$ lies in $\mathscr{R}(T)$, so does $au_1 + bu_2$. This shows that $\mathscr{R}(T)$ is a vector space.

(ii) We take x_1, x_2 such that $Tx_11 = Tx_2 = 0$, so that $x_1, x_2 \in \mathscr{N}(T)$. Then, for any scalars a, b, we have

$$T(ax_1 + bx_2) = aTx_1 + bTx_2 = a \cdot 0 + b \cdot 0 = 0$$

which proves that $ax_1 + bx_2$ lie in $\mathscr{N}(T)$. Thus, $\mathscr{N}(T)$ is a vector space.

(iii) Suppose that $\dim \mathscr{D}(T) = n < \infty$, and choose $n + 1$ arbitrary elements from $\mathscr{R}(T)$: $u_1, \ldots, u_{n+1}$. Then there is a set of elements $\{x_1, \ldots, x_{n+1}\}$ such that $Tx_i = u_i$ for $i = 1, \ldots, n+1$. Since the dimension of $\mathscr{D}(T)$ is n, the set $\{x_1, \ldots, x_{n+1}\}$ must be linearly dependent. That means that there exist scalars $a_1, \ldots, a_{n+1}$ such that

$$a_1x_1 + \ldots + a_{n+1}x_{n+1} = 0.$$

$T0 = 0$, and applying T to both sides yields

$$T(a_1x_1 + \ldots + a_{n+1}x_{n+1}) = a_1Tx_1 + \ldots + a_{n+1}Tx_{n+1} = a_1u_1 + \ldots + a_{n+1}u_{n+1} = 0$$

which shows that $\{u_1, \ldots, u_{n+1}\}$ is a linearly dependent set. Since the elements of this set were chosen arbitrarily, this means that no subset of $\mathscr{R}(T)$ with $n+1$ elements is linearly dependent. Hence $\dim \mathscr{R}(T) \leq n$. □

Note that a consequence of 3.1 is that linear dependence is preserved by linear operators.

We are also interested in knowing whether an operator $T : X \to U$ has an inverse operator T^{-1}. Similarly to one-variable calculus, in order for T^{-1} to exist we require that T be one-to-one, i.e that

$$x_1 \neq x_2 \implies Tx_1 \neq Tx_2 \tag{19}$$

which is equivalent to

$$Tx_1 = Tx_2 \implies x_1 = x_2 \tag{20}$$

for any $x_1, x_2 \in \mathscr{D}(T)$. Then the inverse operator T^{-1} is a mapping from $\mathscr{R}(T)$ to $\mathscr{D}(T)$, and we have that

$$TT^{-1}u = u$$
$$T^{-1}Tx = x$$

for any $u \in \mathscr{R}(T)$ and for any $x \in \mathscr{D}(T)$.

In fact, it is sufficient to say that $Tx = 0$ implies $x = 0$ in order for T to have an inverse. This requirement and some further properties of the inverse operator are stated in the next theorem.

Theorem 3.2 (Inverse of a linear operator). *Let $T : X \to U$ be a linear operator with domain $\mathscr{D}(T) \subset X$ and range $\mathscr{R}(T) \subset U$. Then*

(i) There exists an inverse operator $T^{-1} : \mathscr{R}(T) \to \mathscr{D}(T)$ if and only if

$$Tx = 0 \implies x = 0.$$

(ii) If T^{-1} exists, then it is a linear operator.
(iii) If $\dim \mathscr{D}(T) < \infty$ and if T^{-1} exists, then $\dim \mathscr{R}(T) = \dim \mathscr{D}(T)$.

Proof. (i) Suppose that T is such that $Tx = 0 \implies x = 0$, and that we have x_1, x_2 such that $Tx_1 = Tx_2$. Then we have

$$T(x_1 - x_2) = Tx_1 - Tx_2 = 0$$

so according to our first assumption, $x_1 = x_2$. Thus $Tx_1 - Tx_2$ implies $x_1 = x_2$, i.e T is one-to-one, so that the inverse $T^{-1} : \mathscr{R}(T) \to \mathscr{D}(T)$ exists.

(ii) Let $Tx_1 = u_1$ and $Tx_2 = u_2$, an let a, b be arbitrary scalars. Then, since T is linear, we have

$$au_1 + bu_2 = aTx_1 + bTx_2 = T(ax_1 + bx_2).$$

Applying T^{-1} to the equation above yields

$$\begin{aligned} T^{-1}(au_1 + bu_2) &= T^{-1}T(ax_1 + bx_2) \\ &= ax_1 + bx_2 \\ &= aT^{-1}u_1 + bT^{-1}u_2 \end{aligned}$$

which proves that T^{-1} is a linear mapping.

(iii) We know from 3.1 (iii) that $\dim \mathscr{R}(T) \leq \dim \mathscr{D}(T)$, and applying the same theorem to T^{-1} we get that $\dim \mathscr{D}(T) \leq \dim \mathscr{R}(T)$. Combining these results gives us $\dim \mathscr{R}(T) = \dim \mathscr{D}(T)$. □

Theorem 3.3 (Matrix representation of an operator). *Let $T : X \to Y$ be a linear operator between two finite dimensional vector spaces X and Y. Let $\dim X = n$ and $\dim Y = r$, and let $E = \{e_1, \ldots, e_n\}$ be a basis for X and $B = \{b_1, \ldots, b_r\}$ a basis for Y. Then T can be represented by a matrix T_{EB}.*

Proof. We investigate the image $y = Tx$ of some arbitrary $x \in X$. Since E is a basis in X, x can be represented by a linear combination of elements in E:

$$x = \sum_{k=1}^{n} \xi_k e_k. \tag{21}$$

Then, applying T yields

$$y = Tx = T \sum_{k=1}^{n} \xi_k e_k = \sum_{k=1}^{n} \xi_k T e_k. \tag{22}$$

Let y_k denote the image of each basis vector e_k under T, $y_k = Te_k$. Since y and y_k are in Y, they can be represented using the basis B:

$$y = \sum_{j=1}^{r} \eta_j b_j \tag{23}$$

$$y_k = \sum_{j=1}^{r} \tau_{jk} b_j. \tag{24}$$

Combining (23) and (22), we obtain

$$y = \sum_{j=1}^{r} \eta_j b_j = \sum_{k=1}^{n} \xi_k T e_k \tag{25}$$

and substituting (24) into the right hand side of (25) yields

$$y = \sum_{k=1}^{n} \xi_k \sum_{j=1}^{r} \tau_{jk} b_j = \sum_{j=1}^{r} \Big(\sum_{k=1}^{n} \xi_k \tau_{jk} \Big) b_j. \tag{26}$$

Comparing the middle term of (25) to the last term of (26), we realize that it must hold that

$$\eta_j = \sum_{k=1}^{n} \tau_{jk} \xi_k, \qquad j = 1, \ldots, r.$$

The coefficients τ_{jk} can now be taken as elements of a matrix T_{EB} with r rows an n columns. Putting x and y in vector form, denoted as $\tilde{x}$ and $\tilde{y}$

$$\tilde{x} = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}, \qquad \tilde{y} = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_r \end{pmatrix}$$

we can now express the equation $y = Tx$ in matrix form:

$$\tilde{y} = T_{EB}\tilde{x}$$

$$\iff \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_r \end{pmatrix} = \begin{pmatrix} \tau_{11} & \dots & \tau_{1n} \\ \vdots & & \vdots \\ \tau_{r1} & \dots & \tau_{rn} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}.$$

The matrix T_{EB} is uniquely determined by fixing bases E and B, and so choosing a different base will yield a different matrix representation of T. Such two matrices representing a linear map with respect to different bases are called *similar* matrices. □

3.2 Bounded operators

This section considers properties of bounded operators, an important class of linear operators. The theory of bounded operators is extensive since, as will become evident in the following sections, boundedness makes it possible to define a variety of other properties and theorems.

Definition 3.2 (**Bounded operator, operator norm**)**.** Let $T : \mathscr{D}(T) \to Y$ be a linear operator and X, Y normed spaces where $\mathscr{D}(T) \subset X$. T is said to be *bounded* if we have

$$\frac{\|Tx\|}{\|x\|} \leq c$$

for all $x \in \mathscr{D}(T)$ and for some real number c. In that case, we define the norm of T as

$$\|T\| = \sup_{\substack{x \in \mathscr{D}(T) \\ x \neq 0}} \frac{\|Tx\|}{\|x\|}.$$

Note that this definition of boundedness differs from the one in calculus, where a function is said to be bounded if its range is a bounded set.

Example 3.4 (Identity operator). The identity operator I defined in ex. 3.2 is bounded and has norm 1, since for all $x \in X$ we have

$$\|I\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Ix\|}{\|x\|} = \frac{\|x\|}{\|x\|} = 1.$$

Example 3.5 (Differentiation operator). The differentiation operator D defined in ex. 3.3 (with the norm on X taken as $\|x\| = \max |x(t)|, t \in [0, 1]$) is not bounded. To show this, let $x_n(t) = t^n$ for some $n \in \mathrm{N}$ and compute the norm of D:

$$\frac{\|Dx_n(t)\|}{\|x(t)\|} = \frac{\left\|nt^{n-1}\right\|}{\|t^n\|} = n\frac{\left\|t^{n-1}\right\|}{\|t^n\|} = n.$$

Since n was arbitrary, we cannot choose a fixed c that bounds the norm for all polynomials and thus D is unbounded.

Theorem 3.4 (**Finite dimension**). *Every linear operators on a finite dimensional normed space X is bounded.*

Proof. Let X be a normed vector space of dimension n, and let $\{e_1, \ldots, e_n\}$ be an basis in X. Then every element $x \in X$ can be represented as $x = \sum_i a_i e_i$ for some scalars a_i. Let T be a linear operator. Then we have

$$\begin{aligned} \|Tx\| &= \left\| T \sum_i a_i e_i \right\| \\ &= \left\| \sum_i a_i T e_i \right\| \\ &\leq \sum_i |a_i| \, \|Te_i\| \\ &\leq \max_i \|Te_i\| \sum_i |a_i| \end{aligned} \tag{27}$$

where we have used the linearity of T and the triangle inequality for the norm. From Lemma 2.4-1 in [1], we know that there exists a number $c > 0$ such that

$$\|a_1 e_1 + \ldots + a_n e_n\| \geq c(|a_1| + \ldots + |a_n|) \tag{28}$$

where, in our case, $\|a_1 e_1 + \ldots + a_n e_n\| = \|x\|$. Applying this result to the right hand side of (27), we obtain

$$\|Tx\| \leq b \, \|x\| \, , \qquad b = \max_i \|Te_i\| \, \frac{1}{c}$$

which shows that T is bounded. □

Theorem 3.5 (**Boundedness and continuity**). *Let X, Y be normed spaces and $T : \mathscr{D}(T) \to Y$ a linear operator where $\mathscr{D}(T) \subset X$. Then T is bounded if and only if it is continuous.*

Proof. If $T = 0$, boundedness and continuity are trivial. Let instead $T \neq 0$, so that $\|T\| \neq 0$.

We first assume that T is bounded, and take an arbitrary $x_0 \in \mathscr{D}(T)$. Define $\delta = \varepsilon / \|T\|$. Then for every $x \in \mathscr{D}(T)$ such that $\|x - x_0\| < \delta$, using the linearity and boundedness of T we have

$$\|Tx - Tx_0\| = \|T(x - x_0)\| \leq \|T\| \, \|x - x_0\| < \|T\| \, \delta = \|T\| \, \frac{\varepsilon}{\|T\|} = \varepsilon$$

which shows that T is continuous.

Now, assume instead that T is continuous at some point x_0. Then there exist $\delta, \varepsilon > 0$ such that $\|Tx - Tx_0\| \leq \varepsilon$ for all $x \in \mathscr{D}(T)$ for which $\|x - x_0\| \leq \delta$. Now, take an arbitrary $y \neq 0$ in $\mathscr{D}(T)$ and define x as

$$x = x_0 + \frac{\delta}{\|y\|} y$$

so that we have $\|x - x_0\| = \delta$. Then, due to the linearity and continuity of T we have

$$\begin{aligned} \varepsilon \geq \|Tx - Tx_0\| = \|T(x - x_0)\| = \left\| T\Big(\frac{\delta}{\|y\|} y \Big) \right\| = \frac{\delta}{\|y\|} \, \|Ty\| \\ \implies \frac{\|Ty\|}{\|y\|} \leq \frac{\varepsilon}{\delta} \end{aligned}$$

which can be written $\|Ty\| / \|y\| \leq c$ for $c = \varepsilon/\delta$ and thus T is bounded.

□

Theorem 3.6 (Zero operator). *Let $Q : X \to Y$ be a bounded linear operator between two inner product spaces X and Y. Then*

(i) $Q = 0 \iff \langle Qx, y \rangle = 0$ for all $x \in X$ and for all $y \in Y$.
(ii) If X is a complex inner product space, $Q : X \to X$ and $\langle Qx, x \rangle = 0 \ \forall x \in X$, then Q=0.

Proof. (i) If $Q = 0$, then $Qx = 0$ for all x which yields

$$\langle Qx, y \rangle = \langle 0, y \rangle = 0.$$

And if $\langle Qx, y \rangle = 0$ for all x and y, 2.4 implies that $Qx = 0$ for all x so that Q is the zero operator.
(ii) Assume that $\langle Qx, x \rangle = 0$ for all $x = ay + z \in X$. Then

$$\begin{aligned} 0 &= \langle Q(ay + z), ay + x \rangle \\ &= |a|^2 \langle Qy, y \rangle + \langle Qz, z \rangle + a\langle Qy, z \rangle + \bar{a}\langle Qz, y \rangle. \end{aligned}$$

By assumption, the first two terms are zero. This yields

$$a\langle Qy, z \rangle + \bar{a}\langle Qz, y \rangle = 0$$

which needs to hold for an arbitrary a. Putting for example $a = 1$ yields

$$\langle Qy, z \rangle + \langle Qz, y \rangle = 0 \tag{29}$$

while $a = i$ (so that $\bar{a} = -i$) yields

$$\langle Qy, z \rangle - \langle Qz, y \rangle = 0. \tag{30}$$

Adding together (29) and (30) yields

$$\langle Qy, z \rangle = 0 \tag{31}$$

so that Q is the zero operator by part (i) of the theorem. □

Defining an operator norm suggests that bounded linear operators themselves can be treated as elements of a vector space. For any two normed spaces X and Y, we consider the set of all bounded operators that map X into Y and denote this set $B(X, Y)$. Indeed, it is straight-forward to check that a sum of two bounded operators is bounded and that a bounded operator multiplied by a scalar is bounded, which shows that $B(X, Y)$ is a vector space. The operator norm defined in the beginning of this section satisfies the properties (IP1)-(IP4), so that $B(X, Y)$ is a normed vector space.

3.3 Self-Adjoint operators

Self-adjoint linear operators arise naturally in quantum mechanics and are interesting to study from both a practical and a theoretical point of view. While general quantum operators defined on an unbounded domain are typically unbounded, this section considers bounded self-adjoint operators in order to make use of the operator norm.

Definition 3.3 (Hilbert-adjoint operator). Let H_1 and H_2 be Hilbert spaces, and let $T : H_1 \to H_2$ be a bounded linear operator. The operator $T^* : H_2 \to H_1$ is called the *Hilbert-adjoint operator* of T if we have that

$$\langle Tx, y \rangle = \langle x, T^*y \rangle$$

for all $x \in H_1$, $y \in H_2$. On the left hand side of the equation, the inner product is the one defined on H_2 and on the right hand side is the one defined on H_1.

It can be shown that for every bounded linear operator $T : H \to H$ there exists a unique Hilbert-adjoint operator T^* and that $\|T^*\| = \|T\|$ (theorem 3.9-2 in [1]).

Definition 3.4 (Self-adjoint operator). Let $T : H \to H$ be a bounded linear operator. T is called *self-adjoint* if it is its own Hilbert-adjoint, i.e if

$$\langle Tx, y \rangle = \langle x, Ty \rangle$$

for all $x, y \in H$.

Theorem 3.7 (Self-adjointness). *If $T : H \to H$ is a bounded self-adjoint linear operator on a Hilbert space H, then it holds that*

(i) $\langle Tx, x \rangle \in \mathbb{R}$ for all $x \in H$.
(ii) If $\langle Tx, x \rangle \in \mathbb{R}$ for all x and H is complex, then T is self-adjoint.

Proof. (i) Let T be self-adjoint. Then we have

$$\overline{\langle Tx, x \rangle} = \langle x, Tx \rangle = \langle Tx, x \rangle$$

where we have used the sesquilinearity of the inner product and the fact that T is self-adjoint. Thus, since $\langle Tx, x \rangle$ is equal to its complex conjugate, it is real.

(ii) If $\langle Tx, x \rangle \in \mathbb{R}$ for all x, we have that

$$\langle Tx, x \rangle = \overline{\langle Tx, x \rangle} = \overline{\langle x, T^*x \rangle} = \langle T^*x, x \rangle$$

$$\begin{aligned} \implies 0 &= \langle Tx, x \rangle - \langle T^*x, x \rangle \\ &= \langle (T - T^*)x, x \rangle. \end{aligned}$$

Thus by 3.6, $(T - T^*) = 0$ so that $T = T^*$. □

Theorem 3.8 (Self-adjointness of sum). *Let S and T be two bounded, self-adjoint operators on a Hilbert space H and let $a, b \in \mathbb{R}$. Then $aS + bT$ is self-adjoint.*

Proof. Using the linearity of the inner product and the self-adjointness of S and T (individually), we have

$$\begin{aligned} \langle (aS + bT)x, y \rangle &= a\langle Sx, y \rangle + b\langle Tx, y \rangle \\ &= \langle x, aSy \rangle + \langle x, bTy \rangle \\ &= \langle x, (aS + bT)y \rangle \end{aligned}$$

so that $aS + bT$ is self-adjoint. □

Theorem 3.9 (**Self-adjointness of product**). *Let S and T be two self-adjoint operators on a Hilbert space H. Their product ST is self-adjoint if and only if*

$$ST = TS \tag{32}$$

and in that case we say that the operators commute.

Proof. Assuming that ST is self-adjoint and using the fact that $(ST)^* = T^*S^*$, we have

$$(ST)^* = T^*S^* = TS.$$

In order for the above to equal ST, we thus require that $ST = TS$. □

It is useful to introduce a partial ordering on self-adjoint operators. For two operators T_1, T_2 in a Hilbert space H we write

$$T_1 \leq T_2$$

if and only if $\langle T_1x, x\rangle \leq \langle T_2x, x\rangle$ for all $x \in H$. As a consequence of the partial ordering, we can define a *positive* operator:

Definition 3.5 (**Positive operator**). A self-adjoint operator T in a Hilbert space H is said to be positive if we have

$$\langle Tx, x\rangle \geq 0$$

for all $x \in H$, and in that case we write

$$T \geq 0. \tag{33}$$

Theorem 3.10 (**Product of positive operators**). *Let S and T be two self-adjoint positive operators on a Hilbert space H. Then if S and T commute, their product ST is positive.*

Another concept that will be needed in derivations are *positive square roots* of operators. We know that for a self-adjoint operator T, T^2 is positive since $\langle T^2x, x\rangle = \langle Tx, Tx\rangle \geq 0$. The converse question is, for a given positive operator T, does there always exist a self-adjoint operator A such that $A^2 = T$? The definition and proof of existence of a positive square root operator is stated below.

Definition 3.6 (**Positive square root**). If $T : H \to H$ is a positive, self-adjoint operator on a complex Hilbert space H and A is a self-adjoint operator, then A is called the *square root* of T if

$$A^2 = T.$$

If A is also positive, we call it the *positive square root* of T and denote it $A = T^{1/2}$.

Theorem 3.11 (**Positive square root**). *Let $T : H \to H$ be a positive self-adjoint operator on a complex Hilbert space H. Then T has a unique positive square root A, and A commutes with all the same operators as T.*

The proofs of 3.10 and 3.11 are stated in sections 9.3 and 9.4 of [1], respectively.

3.4 A special case: the projection operator

Projection operators are a class of bounded, linear operators that are of fundamental importance since they allow us to express other operators in a way that makes them easier to study.

Recall from theorem 2.8 that if Y is a closed subspace in a Hilbert space H, every $x \in H$ can be represented uniquely as

$$x = z + y$$

where $y \in Y$ and $z \in Y^{\perp}$. We mentioned that y is the orthogonal projection of x onto Y, and since y is unique, we can define a linear operator $P : H \to H$ that maps $x \longmapsto y = Px$. P is a projection operator, and we say that P is the orthogonal projection of H onto Y.

Note also, that since

$$\begin{aligned} x &= y + z = Px + z \\ \implies z &= x - Px = (I - P)x \end{aligned}$$

the projection of H onto $Y^{\perp}$ is $I - P$.

Since P maps x into Y, if x is already in Y then the projection of x is simply $Px = x$. We say that a projection operator is *idempotent*, i.e

$$P^2 = P$$

since we have that

$$P^2 x = P(Px) = Px.$$

Taking P again as the projection of H onto Y, with $x = y + z$ and $Px = y$, it is straight forward to show that P is bounded and has norm 1. Since $Px = y \perp z$ we can use the Pythagorean theorem,

$$\begin{aligned} \|x\|^2 &= \|y\|^2 + \|z\|^2 = \|Px\|^2 + \|z\|^2 \\ \iff \|Px\|^2 &= \|x\|^2 - \|z\|^2 \le \|x\| \\ \iff \frac{\|Px\|^2}{\|x\|^2} &\le 1 \end{aligned} \tag{34}$$

and since $\frac{\|Px\|^2}{\|x\|^2}$ is bounded from above by 1, then so is $\frac{\|Px\|}{\|x\|}$.

Another property of a projection operator that maps H onto Y is that it maps $Y^{\perp}$ onto $\{0\}$, since all $z \in Y^{\perp}$ are orthogonal to all $y \in Y$.

An alternative definition of a projection is

Theorem 3.12 (**Projection operator**). *Let $P : H \to H$ be a bounded linear operator on a Hilbert space H. Then P is a projection if and only if it is idempotent and self-adjoint.*

The proof of this theorem can be found in section 9.5 of [1].

Theorem 3.13 (**Positivity and norm of a projection**). *If P is a projection on a Hilbert space H, it holds that*

$$\langle Px, x \rangle = \|Px\|^2 \tag{35}$$

$$P \ge 0 \tag{36}$$

$$\|P\| \le 1, \qquad P(H) \ne \{0\} \implies \|P\| = 1. \tag{37}$$

Proof. In order to prove (35) and (36), we use the fact that a projection is idempotent and self-adjoint and write

$$\langle Px, x\rangle = \langle P^2x, x\rangle = \langle Px, Px\rangle = \|Px\|^2 \geq 0$$

by the properties of the norm.

Using (35) and the Cauchy-Schwarz inequality, we see that

$$\begin{aligned} \|Px\|^2 = \langle Px, x\rangle \leq \|Px\| \, \|x\| \\ \iff \|Px\| \leq \|x\| \, . \end{aligned}$$

Thus the norm of P is smaller than 1,

$$\|P\| = \sup_{\substack{x\in H\\ x\neq 0}} \frac{\|Px\|}{\|x\|} \leq 1.$$

In addition, if there exists an $x \in Y = P(H)$ such that $x \neq 0$, we have that $Px = x$ so that

$$\|P\| = \sup_{\substack{x\in H\\ x\neq 0}} \frac{\|Px\|}{\|x\|} = \frac{\|x\|}{\|x\|} = 1$$

which concludes the proof of (37). □

Theorem 3.14 (Partial order). *Let P_1, P_2 be projections on a Hilbert space H which map H onto Y_1 and Y_2 and have null spaces $\mathcal{N}(P_1), \mathcal{N}(P_2)$, respectively. Then the following statements are equivalent.*

(a) $P_1P_2 = P_2P_1 = P_1$
(b) $Y_1 \subset Y_2$
(c) $\mathcal{N}(P_1) \supset \mathcal{N}(P_2)$
(d) $\|P_1x\| \leq \|P_2x\| \; \forall x \in H$
(e) $P_1 \leq P_2$

Proof. We will prove that each of the statements implies another one.

(a) $\implies$ (d): From (a) we have that

$$\|P_1x\| = \|P_1P_2x\| \leq \|P_1\| \, \|P_2x\| \leq \|P_2x\|$$

where we have used the fact that $\|P_1\| \leq 1$.

(d) $\implies$ (e): Starting from (d), we have that

$$\begin{aligned} \|P_1x\| \leq \|P_2x\| \\ \iff \|P_1x\|^2 \leq \|P_2x\|^2 \, . \end{aligned}$$

By (35), this is equivalent to

$$\langle P_1x, x\rangle \leq \langle P_2x, x\rangle$$

so that $P_1 \leq P_2$.

(e) $\implies$ (c): Let $x \in \mathcal{N}(P_2)$ so that $P_2x = 0$. Then we have by (35) that

$$\|P_1x\|^2 = \langle P_1x, x\rangle \leq \langle P_2x, x\rangle = 0$$

so that $P_1x = 0$. Thus $P_1 \leq P_2$ implies $\mathscr{N}(P_2) \subset \mathscr{N}(P_1)$.

(c) $\implies$ (b): Since the null space of a projection P is the orthogonal complement of the subspace Y it projects H onto, we have that

$$\mathscr{N}(P_1) \supset \mathscr{N}(P_2) \iff Y_1^\perp \supset Y_2^\perp \iff Y_1 \subset Y_2.$$

(b) $\implies$ (a): Let $P_1x \in Y_1$, which implies that $P_1x \in Y_2$ since $Y_1 \subset Y_2$. Thus $P_2P_1x = P_1x$. This implies that P_2P_1 is a projection i.e self-adjoint, so by 3.9 the operators P_1, P_2 commute. This yields that

$$P_1P_2 = P_2P_1 = P_1.$$

□

Theorem 3.15 (Difference of projections). *If P_1, P_2 are projections which map a Hilbert space H onto subsets Y_1, Y_2 respectively, then $P = P_2 - P_1$ is a projection if and only if $Y_1 \subset Y_2$.*

Proof. If $P = P_2 - P_1$ is a projection it must be idempotent, i.e

$$\begin{aligned} P_2 - P_1 = (P_2 - P_1)^2 = P_2^2 - P_2P_1 - P_1P_2 + P_1^2 \\ \iff 2P_1 = P_2P_1 + P_1P_2 \end{aligned} \tag{38}$$

Multiplying by P_2 from the left and from the right, we obtain the two equations

$$\begin{aligned} 2P_2P_1 = P_2^2P_1 + P_1P_1P_2 = P_2P_1 + P_2P_1P_2 \\ 2P_1P_2 = P_2P_1P_2 + P_1^2P_2^2 = P_2P_1P_2 + P_1^2P_2 \end{aligned}$$

where we have used that $P_2^2 = P_2$ and $P_1^2 = P_1$. This yields that

$$\begin{aligned} P_2P_1P_2 = P_2P_1 \\ P_2P_1P_2 = P_1P_2 \\ \implies P_2P_1 = P_1P_2 \end{aligned}$$

and by (38) we have that $P_2P_1 = P_1P_2 = P_1$, which is equivalent to $Y_1 \subset Y_2$ by 3.14.

If $Y_1 \subset Y_2$, then 3.14 implies

$$\begin{aligned} P_2P_1 = P_1P_2 = P_1 \\ \iff P_2P_1 + P_1P_2 = 2P_1 \end{aligned}$$

so that $P = P_2 - P_1$ is idempotent. In addition, P_1 and P_2 are self-adjoint so that P is self-adjoint. Thus P is a projection. □

The theorems in section 3 are taken from [1], and the proofs are strongly influenced by the proofs there.

4 Spectral theory

Spectral theory is a branch of functional analysis that studies certain inverse operators which arise naturally e.g when solving systems of linear equations or differential equations. In short, spectral theory can be said to generalize eigenvalue theory of matrices to operators in abstract spaces. In particular, the spectral theory of bounded, self-adjoint operators leads up to a result called the Spectral theorem which shows how a self-adjoint operator can be represented using projection operators. This theorem is the final result of this thesis.

4.1 Finite dimensional case

In finite dimensional vector spaces, every linear operator $T : X \to Y$ can be represented by a matrix as was shown in 3.3. Thus, spectral theory in finite dimensions is essentially matrix eigenvalue theory and familiar results from linear algebra can be applied. A brief review of eigenvalue theory and some basic concepts connected to spectral theory are stated below.

Let $T : X \to X$ be a linear operator in a finite dimensional normed space X and let A be a matrix representation of T with respect to some fixed basis in X. As a starting point, consider the equation

$$Ax = \lambda x \tag{39}$$

where A is a $n \times n$ matrix, x a column vector with n elements and λ a real or complex number. If for a given λ there exists a vector $x \neq 0$ such that (39) holds, we say that λ is an eigenvalue of A and that x is the eigenvector of A corresponding to that eigenvalue. Eigenvalues can be determined using the equation

$$det(A - \lambda I) = 0 \tag{40}$$

which is the characteristic equation of A. This is a polynomial equation in λ, and its roots correspond to the eigenvalues of A. The eigenvectors corresponding to an eigenvalue λ span a subspace of X, called the *eigenspace* of A corresponding to λ. The set of all eigenvalues of A is called the *spectrum* of A and is denoted $\sigma(A)$. Its complement in the complex plane, $\mathbb{C} - \sigma(A) = \rho(A)$, is called the *resolvent set* of A.

Below follow two theorems regarding eigenvalues. The proofs of these theorems will not be stated here, but can be found in section 7.1 of [1].

Theorem 4.1 (**Eigenvalues of a matrix**)**.** *An $n \times n$ matrix A has at least one and at most n (numerically different) eigenvalues.*

It should also be noted that while A may be a real matrix, its eigenvalues aren't necessarily real.

Theorem 4.2 (**Similar matrices, eigenvalues of an operator**)**.** *Let $T : X \to X$ be a linear operator on a finite dimensional normed space X. Then all matrices A representing T with respect to different bases in X have the same eigenvalues.*

Theorem 4.2 is a consequence of the fact that two matrices representing the same operator are *similar*, i.e that we can write

$$B = P^{-1}AP$$

where A, B are similar matrices and P is a coordinate transformation matrix. It can be shown that the determinant of a matrix is invariant under coordinate transformations from one basis to another, and thus two similar matrices have the same eigenvalues. [4]

Due to this fact, we say that the eigenvalues of a matrix A representing a linear operator T are the eigenvalues of T, and correspondingly for the spectrum and the resolvent set. From 4.1 and 4.2 it also follows that a linear operator in an n-dimensional vector space has at least one eigenvalue and at most n numerically different eigenvalues.

4.2 Infinite dimensional case: basic concepts

In section 4.1 we considered spectral theory in finite dimensional normed spaces and reviewed some basic concepts and results from matrix eigenvalue theory. In the infinite dimensional case however, the situation becomes more complicated. We begin by defining some basic concepts in spectral theory in general normed and Banach spaces in this section, and continue by describing spectral theory of self-adjoint operators in Hilbert spaces in section 4.3 .

We start by defining the central notions which are considered in spectral theory. Let $T : \mathscr{D}(T) \to X$ be a linear operator in a complex normed space X and $\mathscr{D}(T) \subset X$. We begin with an equation similar to (39) from the finite-dimensional case,

$$
\begin{aligned}
&Tx = \lambda x \\
\iff &(T - \lambda I)x = 0
\end{aligned}
$$

where λ is a complex number and I the identity operator on X. From this we define another operator T_λ as

$$T_\lambda = T - \lambda I \tag{41}$$

and its inverse, if it exists,

$$R_\lambda(T) = T_\lambda^{-1} = (T - \lambda I)^{-1}. \tag{42}$$

We call $R_\lambda(T)$ the *resolvent operator* (or simply *resolvent*) of T, and may sometimes denote it as just R_λ if it is evident from context which operator it is connected to.

It is clear that if there exists a $x \neq 0$ such that $T_\lambda x = 0$, then R_λ does not exist by 3.2(i). By part (ii) of the same theorem, we know also that if R_λ exists, it is a linear operator.

It is properties of T_λ and R_λ that spectral theory investigates, and these properties depend of course on the operator T but also on the value of λ. We are also concerned with the values of λ for which R_λ does or doesn't exist. As with the finite-dimensional case, we can sort all values of λ into the resolvent set and the spectrum. The following definition specifies these notions for the infinite-dimensional case.

Definition 4.1 (**Resolvent set, regular value, spectrum, eigenvalue**). Let $T : \mathscr{D}(T) \to X$ be a linear operator in a complex normed space X with $\mathscr{D}(T) \subset X$. λ is called a *regular value* of T if

(R1) R_λ exists,
(R2) R_λ is bounded,
(R3) the domain of R_λ is dense in X.

The set of all regular values of T is called the *resolvent set* of T and is denoted $\rho(T)$. Its complement in the complex plane $\sigma(T) = \mathbb{C} - \rho(T)$ is called the *spectrum* of T, and a λ belonging to the spectrum is called a *spectral value* of T. The spectrum can be divided into three disjoint sets:

The *point spectrum* of T is the set of all spectral values such that R_λ doesn't exist. It is denoted $\sigma_p(T)$ and a λ belonging to $\sigma_p(T)$ is an eigenvalue of T.

The *continuous spectrum* $\sigma_c(T)$ consists of all λ such that R_λ exists and has a domain which is dense in X but isn't bounded.

The *residual spectrum* of T is the set of all λ such that R_λ exists and may or may not be bounded, but its domain is not dense in X. It is denoted $\sigma_r(T)$.

It is worth noting that some of these three sets may be empty for a given operator T. For example, in the finite dimensional case we have that $\sigma_c(T)$ and $\sigma_r(T)$ are empty for every linear operator T so that the spectrum is a pure point spectrum and every spectral value is an eigenvalue.

The following lemma considers bounded linear operators in complex Banach spaces and will be useful in deriving properties of self-adjoint operators in Hilbert spaces in the next section. Its proof can be found in section 7.2 of [1].

Lemma 4.3 (**The domain of** R_λ). *Let $T : X \to X$ be a bounded linear operator in a complex Banach space X. Then R_λ is bounded and defined on the whole space X.*

4.3 Spectral theory of bounded self-adjoint operators

Before proving the Spectral Theorem in section 4.3.1, a few properties and theorems regarding the spectrum of a self-adjoint operator are needed.

Theorem 4.4 (**Eigenvalues, eigenvectors**). *Let $T : H \to H$ be a self-adjoint operator in a complex Hilbert space H. Then*

(i) All eigenvalues of T (if they exist) are real.
(ii) For two numerically different eigenvalues λ, μ, their corresponding eigenvectors are orthogonal.

Proof. (i) Let λ be an eigenvalue of T and let x be the corresponding eigenvector. By the definition of an eigenvector, we know that $x \neq 0$. Using the self-adjointness of T, we have

$$\lambda\langle x, x\rangle = \langle \lambda x, x\rangle = \langle Tx, x\rangle = \langle x, Tx\rangle = \langle x, \lambda x\rangle = \overline{\lambda}\langle x, x\rangle.$$

Since $\langle x, x\rangle = \|x\|^2 \neq 0$, we can divide by $\langle x, x\rangle$ and obtain

$$\lambda = \overline{\lambda}$$

which shows that λ is real.

(ii) Let λ and μ be two numerically different eigenvalues and let x and y be their corresponding eigenvectors so that $Tx = \lambda x$ and $Ty = \mu y$. Using the self-adjointness of T and the fact that μ is real, we obtain

$$\lambda\langle x, y\rangle = \langle \lambda x, y\rangle = \langle Tx, y\rangle = \langle x, Ty\rangle = \langle x, \mu y\rangle = \overline{\mu}\langle x, y\rangle = \mu\langle x, y\rangle.$$

We have already stated that $\lambda \neq \mu$, so the only way the above equation can hold is if $\langle x, y\rangle = 0$. This proves that x, y are orthogonal. □

In fact, it can be shown that not only the eigenvalues but the whole spectrum of a self-adjoint operator is real. In order to prove this, we need a theorem regarding the resolvent set of self-adjoint operators. The proof can be found in section 9.1 of [1].

Theorem 4.5 (**Resolvent set**). *Let H be a complex Hilbert space and $T : H \to H$ a self-adjoint operator. Then a number λ is in $\rho(T)$ if and only if for every $x \in H$ there exists a number $c > 0$ such that*

$$\|T_\lambda x\| \geq c\,\|x\| \tag{43}$$

where $T_\lambda = T - \lambda I$.

Theorem 4.6 (**Spectrum**). *Let $T : X \to X$ be a self-adjoint operator on a complex Hilbert space H. Then the spectrum $\sigma(T)$ of T is real.*

Proof. Define a number $\lambda = a + ib$ where $a, b \in \mathbb{R}$ and $b \neq 0$. We will use Theorem 4.5 to show that λ must be a member of the resolvent set $\rho(T)$, so that all $\lambda \in R$ belong to $\sigma(T)$. For all $x \neq 0$ in H we have

$$\langle T_\lambda x, x\rangle = \langle (T - \lambda I)x, x\rangle = \langle Tx, x\rangle - \lambda\langle x, x\rangle \tag{44}$$

We know that $\langle x, x\rangle$ is real, as is $\langle Tx, x\rangle$ by 3.7. The complex conjugate of the equation above is

$$\overline{\langle T_\lambda x, x\rangle} = \langle Tx, x\rangle - \overline{\lambda}\langle x, x\rangle \tag{45}$$

Subtracting (45) from (44) yields

$$\begin{aligned}\overline{\langle T_\lambda x, x\rangle} - \langle T_\lambda x, x\rangle &= (\lambda - \overline{\lambda})\, \|x\|^2 \\ &= 2ib\, \|x\|^2\end{aligned}$$

where the left hand side is $-2i\,\mathrm{Im}\langle T_\lambda x, x\rangle$,

$$-2i\,\mathrm{Im}\langle T_\lambda x, x\rangle = 2ib\, \|x\|^2 .$$

Diving by 2 and taking the absolute value on both sides yields

$$|\,\mathrm{Im}\langle T_\lambda x, x\rangle| = |b|\, \|x\|^2 . \tag{46}$$

We know that $|\,\mathrm{Im}\langle T_\lambda x, x\rangle|$ cannot be larger than $|\langle T_\lambda x, x\rangle|$. Using this and the Cauchy-Schwarz inequality, (46) becomes

$$|b|\, \|x\|^2 = |\,\mathrm{Im}\langle T_\lambda x, x\rangle| \leq |\langle T_\lambda x, x\rangle| \leq \|T_\lambda x\|\, \|x\|$$

Since $x \neq 0$ we can divide by $\|x\|$ and obtain

$$|b|\, \|x\| \leq \|T_\lambda x\|$$

which shows that $\lambda \in \rho(T)$ according to 4.5. Since λ was an arbitrary complex number, $\sigma(T)$ must be real. □

Theorem 4.7 (**Spectrum**). *Let $T : H \to H$ be a bounded self-adjoint operator on a complex Hilbert space H. Then the spectrum $\sigma(T)$ lies in a closed interval $[m, M]$ on the real axis, with*

$$m = \inf_{\|x\|=1} \langle Tx, x\rangle, \qquad M = \sup_{\|x\|=1} \langle Tx, x\rangle \tag{47}$$

Proof. We will consider a real number $\lambda = m - c$ where $c > 0$, and show that λ belongs to the resolvent set $\rho(T)$. Taking any $x \neq 0$ in H and $v = x/\|x\|$ so that $\|v\| = 1$, we have $x = \|x\|\, v$ so that

$$\begin{aligned}\langle Tx, x\rangle &= \|x\|^2 \langle Tv, v\rangle \\ &\geq \|x\|^2 \inf_{\|\tilde{v}\|=1} \langle T\tilde{v}, \tilde{v}\rangle \\ &= m\, \|x\|^2 .\end{aligned} \tag{48}$$

From the definition of T_λ, we have

$$\langle T_\lambda x, x\rangle = \langle Tx, x\rangle - \lambda\langle x, x\rangle = \langle Tx, x\rangle - \lambda\, \|x\|^2$$

and applying (48) to the right hand side yields

$$\langle T_\lambda x, x\rangle \geq m\, \|x\|^2 - \lambda\, \|x\|^2 = (m - \lambda)\, \|x\|^2 .$$

The left hand side can be related to $\|x\|$ using the Cauchy-Schwarz inequality, resulting in

$$\|T_\lambda x\| \, \|x\| \geq \langle T_\lambda x, x\rangle \geq (m-\lambda)\, \|x\|^2 .$$

Dividing by $\|x\|$ and recalling that $m - \lambda = c$, we have that

$$\|T_\lambda x\| \geq c\, \|x\| \tag{49}$$

so that λ belongs to the resolvent set by 4.5. In a similar way it can be shown that any $\lambda = M + c$ with $c > 0$ belongs to the resolvent set as well; thus the spectrum $\sigma(T)$ lies in the interval $[m, M]$. □

It can also be shown that $m = \inf_{\|x\|=1}\langle Tx, x\rangle$ and $M = \sup_{\|x\|=1}\langle Tx, x\rangle$ are spectral values themselves, so that the interval $[m, M]$ containing the spectrum of T cannot be shrinked. [1]

Theorem 4.8 (**Residual spectrum**). *Let $T : H \to H$ be a bounded self-adjoint operator on a complex Hilbert space H. Then the residual spectrum $\sigma_r(T)$ is empty.*

Proof. We assume that $\sigma_r(T) \neq \emptyset$. Then there exists a $\lambda \in \sigma_r(T)$, i.e a λ such that T_λ^{-1} exists but its domain $\mathscr{D}(T_\lambda^{-1})$ is not dense in H. By 2.9 there exists a $y \neq 0$ in H such that $y \perp \mathscr{D}(T_\lambda^{-1})$. The domain of T_λ^{-1} is the range of T_λ, so that $y \perp \mathscr{R}(T_\lambda)$ and we have

$$\langle T_\lambda x, y\rangle = 0$$

for all $x \in H$. Since T is self-adjoint and λ is real, the above equation is equivalent to

$$\langle x, T_\lambda y\rangle = 0.$$

Letting $x = T_\lambda y$, this yields

$$\begin{aligned} \langle T_\lambda y, T_\lambda y\rangle &= \|T_\lambda y\|^2 = 0 \\ \iff T_\lambda y &= Ty - \lambda y = 0 \end{aligned}$$

Since y is non-zero, λ is an eigenvalue. This contradicts the assumption that $\lambda \in \sigma_r(T)$, proving that $\sigma_r(T)$ must be empty. □

4.3.1 The Spectral theorem

The Spectral theorem is a result which allows us to represent self-adjoint operators in terms of projection operators, which are much simpler to work with than an arbitrary self-adjoint operator.

In order to introduce some of the concepts needed to prove the spectral theorem in infinite-dimensional spaces later, let's again consider the finite dimensional case: let $T : H \to H$ be a bounded, self-adjoint operator in a n-dimensional Hilbert space H. Then, using some fixed basis in H, we can express T with a self-adjoint (or *Hermitian*) matrix which we will denote as T as well. As before, we assume for simplicity that T has n numerically different eigenvalues $\lambda_1, \ldots, \lambda_n$ and that the eigenvalues are ordered from smallest to largest. The eigenvalues are real since T is self-adjoint, and make up the whole spectrum of T since H is finite-dimensional. By 4.4 the corresponding eigenvectors $x_1, \ldots, x_n$ are all pairwise orthogonal, so that there exists an orthonormal set of eigenvectors of T which constitutes a basis for H. Then every x in H can be uniquely represented as

$$x = \sum_{j=1}^{n} \gamma_j x_j, \tag{50}$$

$$\gamma_j = \langle x, x_j\rangle \tag{51}$$

where (51) follows from taking the inner product of x with the sum in (50) and using the orthogonality of the eigenvectors. Applying T to (50) yields

$$Tx = \sum_{j=1}^{n} \gamma_j T x_j = \sum_{j=1}^{n} \gamma_j \lambda_j x_j \tag{52}$$

where we have used that $Tx_j = \lambda_j x_j$ since x_j is an eigenvector. Taking the inner product of some x with any basis vector x_j projects x onto the space spanned by x_j, so that we can define a projection operator $P_j : H \to H$ as

$$P_j x = \langle x, x_j \rangle x_j = \gamma_j x_j.$$

Then (50) becomes

$$x = \sum_{j=1}^{n} P_j x$$

which implies that $\sum_{j=1}^{n} P_j = I$, the identity operator. Using this, (52) becomes

$$Tx = \sum_{j=1}^{n} \lambda_j P_j x \tag{53}$$

so that T equals a sum of projections onto its eigenvectors,

$$T = \sum_{j=1}^{n} \lambda_j P_j. \tag{54}$$

Instead of each individual projection P_j, we can shift our perspective and consider sums of projections. For any $\lambda \in \mathbb{R}$, we define

$$E_\lambda = \sum_{\lambda_j \leq \lambda}^{n} P_j. \tag{55}$$

The collection $\{E_\lambda\}$ is a one-parameter family of projections, where λ is the parameter. When λ is less than the smallest eigenvalue λ_1, the sum in (55) contains no projections P_j and E_λ is the zero operator:

$$E_{\lambda < \lambda_1} = 0. \tag{56}$$

When $\lambda = \lambda_1$, E_λ includes exactly one projection operator: P_1, the projection onto the eigenvector corresponding to λ_1. Then, as the parameter λ grows and traverses the real line in the positive sense, E_λ grows to include another projection every time λ passes an eigenvalue of T, and stays constant in intervals which contain no eigenvalues. The range of E_λ is the subspace V_λ spanned by the corresponding eigenvectors, and as λ grows, V_λ spans more and more of the space H. For two different parameters λ and μ, we have that

$$V_\lambda \subset V_\mu$$

when $\lambda < \mu$.

When the parameter λ is greater than the largest eigenvalue λ_n, the sum in (55) contains the projections onto all eigenvectors of T so that E_λ is the identity operator:

$$E_{\lambda \geq \lambda_n} = I. \tag{57}$$

The properties of E_λ motivate the definition of a *spectral family*, stated below.

Definition 4.2 (**Spectral family or decomposition of unity**). Let $\varepsilon = \{E_\lambda\}_{\lambda\in\mathbb{R}}$ be a one-parameter family of projections E_λ on a Hilbert space H. ε is called a *spectral family* (or a *decomposition of unity*) if the following holds:

$$\begin{aligned}
\lim_{\lambda\to-\infty} E_\lambda &= 0\\
\lim_{\lambda\to\infty} E_\lambda &= I\\
E_\lambda E_\mu = E_\mu E_\lambda &= E_\lambda, \qquad \lambda < \mu\\
E_{\lambda+0}x = \lim_{\mu\to\lambda+0} E_\mu x &= E_\lambda x, \qquad x \in H.
\end{aligned}$$

The notation $\mu \to \lambda + 0$ means that μ approaches λ from the right, and the last equation means that ε is continuous from the right. Furthermore, ε is called a *spectral family on an interval* $[a, b]$ if

$$\begin{aligned}
E_\lambda &= 0, \qquad \lambda < a\\
E_\lambda &= I, \qquad \lambda \geq b.
\end{aligned}$$

This definition will be useful in connection with bounded self-adjoint operators in infinite-dimensional spaces, since the spectrum of such an operator lies in a finite interval on the real line.

A spectral family can be seen as a mapping from $\mathbb{R}$ to $B(H, H)$, the space of all bounded linear operators that map H into itself.

From (55), we can shift perspective back to each individual projection P_j by writing

$$\begin{aligned}
E_{\lambda_1} &= P_1\\
E_{\lambda_2} &= P_1 + P_2\\
&\vdots\\
E_{\lambda_n} &= P_1 + \ldots + P_n
\end{aligned}$$

so that each projection P_j can be expressed as

$$P_j = E_{\lambda_j} - E_{\lambda_{j-1}}.$$

Recalling that E_λ stays constant in the interval $[\lambda_{j-1}, \lambda_j)$, we can equivalently write

$$P_j = E_{\lambda_j} - E_{\lambda_j - 0}$$

where the notation $\lambda_j - 0$ means that the parameter λ approaches λ_j from the left and is strictly smaller than λ_j. We can denote the right hand side as

$$\delta E_{\lambda_j} = E_{\lambda_j} - E_{\lambda_j - 0}$$

so that (54) becomes

$$T = \sum_{j=1}^{n} \lambda_j (E_{\lambda_j} - E_{\lambda_j - 0}) = \sum_{j=1}^{n} \lambda_j \delta E_{\lambda_j} \tag{58}$$

which is the spectral representation of T. From this result we see that for all $x, y \in H$ we have

$$\langle Tx, y\rangle = \sum_{j=1}^{n} \lambda_j \langle \delta E_{\lambda_j} x, y\rangle \tag{59}$$

which is equivalent to the Riemann-Stieltjes integral

$$\langle Tx, y\rangle = \int_{-\infty}^{\infty} \lambda dw(\lambda) \tag{60}$$

with $w(\lambda) = \langle E_\lambda x, y\rangle$.

The spectral representation in (58) holds for the bounded, self-adjoint operator T, where the spectrum of T is a pure point spectrum due to H being finite-dimensional. In an infinite-dimensional Hilbert space however, the spectrum of a self-adjoint operator T might be both continuous and a point spectrum, or purely continuous so that T has no eigenvalues. In order to express such an operator using a spectral representation, the spectral family needs to be defined in a bit more complicated manner than in the finite-dimensional case.

Let T be a bounded, self-adjoint operator on a complex Hilbert space H, and let $T_\lambda = T - \lambda I$. Denote the positive square root of T_λ^2 as B_λ,

$$B_\lambda = (T_\lambda^2)^{1/2} \tag{61}$$

and define the *positive part of* T_λ as

$$T_\lambda^+ = \frac{1}{2}(B_\lambda + T_\lambda). \tag{62}$$

Let E_λ denote the projection onto $\mathcal{N}(T_\lambda^+)$, the null space of T_λ^+ (compare this to the finite-dimensional case, where P_j was the projection onto the eigenvector x_j which spanned the null space of T_λ). Then the spectral family of T is defined as $\varepsilon = (E_\lambda)_{\lambda\in\mathbb{R}}$.

We need to first show that ε is indeed a spectral family according to Def. 4.2, and then prove that the operator T can be represented using a Riemann-Stieltjes integral of $\varepsilon = (E_\lambda)_{\lambda\in\mathbb{R}}$. In order to do this, we need some properties of operators related to T_λ and T which are stated in the lemma below.

Lemma 4.9 (Operators related to T). *Let T be a bounded, self-adjoint operator on a complex Hilbert space H, and let B be the positive square root of T^2. We define the positive and negative part of T as*

$$T^+ = \frac{1}{2}(B + T) \qquad\qquad T^- = \frac{1}{2}(B - T) \tag{63}$$

and let E be the projection of H onto $\mathcal{N}(T^+)$. Then it holds that

(i) T^+, T^- and B are bounded and self-adjoint.

(ii) If T commutes with some bounded linear operator S, then T^+, T^- and B commute with S. If S is also self-adjoint then E commutes with S as well. In particular, it holds that

$$BT = TB \qquad\qquad T^+T = TT^+ \tag{64}$$

$$T^-T = TT^- \qquad\qquad T^+T^- = T^-T^+ \tag{65}$$

$$ET = TE \qquad\qquad EB = BE \tag{66}$$

(iii) In addition,

$$T^+T^- = 0 \qquad T^-T^+ = 0 \tag{67}$$
$$T^+E = ET^+ = 0 \qquad T^-E = ET^- = T^- \tag{68}$$
$$TE = -T^- \qquad T(I-E) = T^+ \tag{69}$$
$$T^+ \geq 0 \qquad T^- \geq 0. \tag{70}$$

Lemma 4.10 (Operators related to T_λ). *Lemma 4.9 holds also if T, T^+, T^-, B and E are replaced with T_λ, T_λ^+, T_λ^-, B_λ and E_λ, where $\lambda \in \mathbb{R}$. Furthermore, for any $\kappa, \lambda, \mu, \nu, \tau \in \mathbb{R}$, the operators*

$$T_\kappa, \quad B_\lambda, \quad T_\mu^+, \quad T_\nu^-, \quad E_\tau \tag{71}$$

all commute.

The proofs of 4.9 and 4.10 are stated in section 9.8 of [1], and with these lemmas we have the tools needed in order to show that $\varepsilon = (E_\lambda)_{\lambda\in\mathbb{R}}$ is indeed a spectral family.

Note also that by addition and subtraction we have that

$$T = T^+ - T^- \qquad T_\lambda = T_\lambda^+ - T_\lambda^-. \tag{72}$$

Before beginning proving that ε is a spectral family, a couple of theorems regarding convergence of sequences of operators are needed.

Definition 4.3 (Convergence of sequences of operators). Let $T_n : X \to Y$ be bounded operators between two normed spaces X and Y, so that all $T_n \in B(X,Y)$. Then the sequence (T_n) is said to be *uniformly operator convergent* if

$$\|T_n - T\| \to 0 \tag{73}$$

and *strongly operator convergent* if

$$\|T_n x - Tx\| \to 0 \quad \forall x \in X \tag{74}$$

for some $T : X \to Y$. Note that in (73) the norm is the one on $B(X,Y)$, while the norm in (74) is the one on Y.

Theorem 4.11 (Uniform and strong operator convergence). *Uniform operator convergence implies strong operator convergence.*

Proof. Let $T_n : X \to Y$ be uniformly operator convergent with limit T. Then by the definition of the norm we have

$$\|T_n x - Tx\| \leq \|T_n - T\| \, \|x\| \tag{75}$$

which goes to zero as $n \to \infty$ since T_n is uniformly operator convergent. Thus uniform operator convergence implies strong operator convergence. □

Theorem 4.12 (Monotone sequence). *Let (T_n) be a monotone decreasing sequence of operators (i.e $T_1 \geq T_2 \geq \ldots$) on a Hilbert space H such that every T_j is bounded and self-adjoint. Furthermore, (T_n) is such that*

$$T_1 \geq T_2 \geq \ldots \geq T_n \geq \ldots \geq K \tag{76}$$

where K is bounded and self-adjoint, and every T_j commutes with K and with every T_m. Then (T_n) is strongly operator convergent, and the limit operator T is bounded, self-adjoint and it holds that $T \geq K$.

The proof of Theorem 4.12 is stated in section 9.3 of [1].

Theorem 4.13 (**Spectral family associated with an operator**). *Let T be a bounded, self-adjoint operator on a complex Hilbert space H, and let E_λ be the projection onto $\mathscr{N}(T_\lambda^+)$ where T_λ^+ is the positive part of $T_\lambda = T - \lambda I$. Then $\varepsilon = (E_\lambda)_{\lambda \in \mathbb{R}}$ is a spectral family on $[m, M] \subset \mathbb{R}$, whith m, M are defined as in 4.7.*

Proof. We need to show that

$$\lambda < \mu \implies E_\lambda \leq E_\mu \tag{77}$$
$$\lambda < m \implies E_\lambda = 0 \tag{78}$$
$$\lambda \geq M \implies E_\lambda = I \tag{79}$$
$$\mu \to \lambda + 0 \implies E_\mu x \to E_\lambda x. \tag{80}$$

Proof of (77): Let $\lambda < \mu$. We have that

$$T_\lambda = T_\lambda^+ - T_\lambda^- \leq T_\lambda^+ \tag{81}$$

where the inequality holds since $T_\lambda^- \geq 0$ by 4.10. This yields that

$$T_\lambda^+ - T_\mu \geq T_\lambda - T_\mu = T - \lambda I - T + \mu I = (\mu - \lambda) I \geq 0.$$

In addition, we know that $T_\lambda^+ - T_\mu$ is self-adjoint since each of the operators is self-adjoint. $T_\lambda^+ - T_\mu$ commutes with T_μ^+ since both T_λ^+ and T_μ commute with T_μ^+ by 4.10. Since $T_\lambda^+ - T_\mu$ is positive by the equation above and T_μ^+ is positive by 4.10, and they commute, theorem 3.10 implies that their product is positive:

$$\begin{aligned} T_\mu^+(T_\lambda^+ - T_\mu) &= T_\mu^+(T_\lambda^+ - T_\mu^+ + T_\mu^-) \\ &= T_\mu^+ T_\lambda^+ - (T_\mu^+)^2 + T_\mu^+ T_\mu^- \\ &= T_\mu^+ T_\lambda^+ - (T_\mu^+)^2 \\ &\geq 0 \end{aligned}$$

where the last term in the second row is zero by 4.10. This yields that

$$\begin{aligned} & T_\mu^+ T_\lambda^+ \geq (T_\mu^+)^2 \\ \iff & \langle T_\mu^+ T_\lambda^+ x, x \rangle \geq \langle (T_\mu^+)^2 x, x \rangle = \langle T_\mu^+ x, T_\mu^+ x \rangle = \left\| T_\mu^+ x \right\|^2 \geq 0. \end{aligned} \tag{82}$$

From (82), we see that $T_\lambda^+ x = 0$ implies $T_\mu^+ x = 0$ so that $\mathscr{N}(T_\lambda^+) \subset \mathscr{N}(T_\mu^+)$. Thus $E_\lambda \leq E_\mu$ by 3.14 and we have proved (77).

Proof of (78): We let $\lambda < m$ and assume that $E_\lambda \neq 0$. Then there exists a $z \in H$ such that $E_\lambda z \neq 0$, say $E_\lambda z = x$ for some $x \in H$. In that case, $E_\lambda x = E_\lambda^2 z = E_\lambda z = x$. Without loss of generality we can set $\|x\| = 1$. Then we have that

$$\begin{aligned} \langle T_\lambda E_\lambda x, x \rangle &= \langle T_\lambda x, x \rangle \\ &= \langle (T - \lambda I) x, x \rangle \\ &= \langle T x, x \rangle - \lambda \|x\|^2 \end{aligned}$$

where $\|x\|^2 = 1$ and $\langle Tx, x \rangle \geq \inf_{\|x\|=1} \langle Tx, x \rangle = m$, yielding

$$\langle T_\lambda E_\lambda x, x \rangle \geq m - \lambda > 0.$$

But according to 4.10, $T_\lambda E_\lambda = -T_\lambda^- \leq 0$ which is a contradiction, proving (78).

Proof of (79): Using a similar approach as in the previous proof, we let $\lambda > M$ but assume that $E_\lambda \neq I$. Then there exists some $x \in H$ such that $(I - E_\lambda)x = x$, and we can again assume that $\|x\| = 1$. Then we have that

$$\begin{aligned} \langle T_\lambda(I - E_\lambda)x, x\rangle &= \langle T_\lambda x, x\rangle \\ &= \langle (T - \lambda I)x, x\rangle \\ &= \langle Tx, x\rangle - \lambda \|x\| \\ &\leq M - \lambda < 0. \end{aligned} \tag{83}$$

But according to 4.10, $T_\lambda(I - E_\lambda) = T_\lambda^+ \geq 0$ which is a contradiction.

Proof of (80): We define an operator $E(\Delta) = E_\mu - E_\lambda$ which is associated with an interval $\Delta = (\lambda, \mu]$. Since $\lambda < \mu$, $E_\lambda \leq E_\mu$ by (77). Then $E_\lambda(H) \subset E_\mu(H)$ by 3.14 so that $E(\Delta)$ is a projection by 3.15. It holds that

$$\begin{aligned} E_\mu E(\Delta) &= E_\mu(E_\mu - E_\lambda) \\ &= E_\mu^2 - E_\mu E_\lambda \\ &= E_\mu - E_\lambda \\ &= E(\Delta) \end{aligned} \tag{84}$$

where we have used that projection operators are idempotent and that $P_j P_k = P_k \iff P_k \leq P_j$ for two projection operators P_j, P_k. Similarly, we have that

$$\begin{aligned} (I - E_\lambda)E(\Delta) &= E(\Delta) - E_\lambda(E_\mu - E_\lambda) \\ &= E(\Delta) - E_\lambda E_\mu + E_\lambda^2 \\ &= E(\Delta) - E_\lambda + E_\lambda \\ &= E(\Delta). \end{aligned} \tag{85}$$

$E(\Delta), T_\mu^-, T_\lambda^+$ commute and are all positive by 4.10, so that the products $T_\mu^- E(\Delta)$ and $T_\lambda^+ E(\Delta)$ are positive by 3.10. Using this fact and the two equations above, we have that

$$T_\mu E(\Delta) = T_\mu E_\mu E(\Delta) = -T_\mu^- E(\Delta) \leq 0$$

since $T_\mu E_\mu = -T_\mu^- \leq 0$ and $E(\Delta) \geq 0$, and also

$$T_\lambda E(\Delta) = T_\lambda(I - E_\lambda)E(\Delta) = T_\lambda^+ E(\Delta) \geq 0.$$

From these two equations we obtain

$$\begin{aligned} (T - \mu I)E(\Delta) \leq 0 &\iff TE(\Delta) \leq \mu E(\Delta) \\ (T - \lambda I)E(\Delta) \geq 0 &\iff TE(\Delta) \geq \lambda E(\Delta) \end{aligned}$$

and combining the two inequalities yields

$$\lambda E(\Delta) \leq TE(\Delta) \leq \mu E(\Delta). \tag{86}$$

This inequality will also be useful in the proof of the spectral theorem.

Let $\mu \to \lambda$ from the right in a monotonous manner. Then Theorem 4.12 applies, so that $E(\Delta)x \to P(\lambda)x$ where $P(\lambda)$ is bounded, self-adjoint and also idempotent since $E(\Delta)$ is idempotent. Thus $P(\lambda)$ is a projection. With $\mu \to \lambda + 0$, (86) yields

$$\begin{aligned} \lambda P(\lambda) &= TP(\lambda) \\ \iff (T - \lambda I)P(\lambda) &= T_\lambda P(\lambda) = 0. \end{aligned}$$

so that we get

$$T_\lambda^+ P(\lambda) = T_\lambda (I - E_\lambda)P(\lambda) = (I - E_\lambda)T_\lambda P(\lambda) = 0.$$

Thus $T_\lambda^+ P(\lambda)x = 0$ for all $x \in H$, so that $P(\lambda)x \in \mathscr{N}(T_\lambda^+)$. Since E_λ is a projection onto $\mathscr{N}(T_\lambda^+)$, we have that

$$\begin{aligned} E_\lambda P(\lambda)x &= P(\lambda)x, \quad x \in H \\ \implies E_\lambda P(\lambda) &= P(\lambda) \end{aligned}$$

Moreover, if we let $\mu \to \lambda + 0$ in (85) so that $E(\Delta) \to P(\lambda)$, we obtain

$$\begin{aligned} P(\lambda) &= (I - E_\lambda)P(\lambda) \\ &= P(\lambda) - E_\lambda P(\lambda) \\ &= P(\lambda) - P(\lambda) \\ &= 0 \end{aligned}$$

Recalling that $\lim_{\mu \to \lambda + 0} E(\Delta) = \lim_{\mu \to \lambda + 0} E_\mu - E_\lambda = P(\lambda) = 0$, this shows that ε is continuous from the right. Thus we have proven that ε has all the properties required of a spectral family on the interval $[m, M]$. □

Theorem 4.14 (The Spectral Theorem). *Let $T : H \to H$ be a bounded, self-adjoint operator on a complex Hilbert space H. Then T can be expressed using its spectral representation*

$$T = \int_{m-0}^{M} \lambda dE_\lambda \tag{87}$$

where $\varepsilon = (E_\lambda)_{\lambda \in R}$ and E_λ is the projection of H onto $\mathscr{N}(T_\lambda^+)$.

Proof. Define the interval $(a, b]$ where $a < m$ and $b > M$, and let $(\mathscr{P}_n)$ be a sequence of partitions of $(a, b]$ into n intervals $\Delta_{nj} = (\lambda_{nj}, \mu_{nj}]$, $j = 1, \dots, n$. Each Δ_{nj} has length $l(\Delta_{nj}) = \mu_{nj} - \lambda_{nj}$, and we assume that $(\mathscr{P}_n)$ is such that

$$\eta(\mathscr{P}_n) = \max_j l(\Delta_{nj}) \xrightarrow[n \to \infty]{} 0 \tag{88}$$

With $\Delta = \Delta_{nj}$, (86) becomes

$$\lambda_{nj} E(\Delta_{nj}) \leq TE(\Delta_{nj}) \leq \mu_{nj} E(\Delta_{nj}) \tag{89}$$

and since this holds for every $j = 1, \dots, n$, we can sum over j to obtain

$$\sum_{j=1}^{n} \lambda_{nj} E(\Delta_{nj}) \leq T \sum_{j=1}^{n} E(\Delta_{nj}) \leq \sum_{j=1}^{n} \mu_{nj} E(\Delta_{nj}). \tag{90}$$

Noting that $\mu_{nj} = \lambda_{n,j+1}$ for $j = 1, \ldots, n-1$, the middle expression can be rewritten as

$$T\sum_{j=1}^{n} E(\Delta_{nj}) = T\Big(E_{\mu_{n1}} - E_{\lambda_{n1}} + E_{\mu_{n2}} - E_{\lambda_{n2}} + \ldots + E_{\mu_{nn}} - E_{\lambda_{nn}}\Big) \tag{91}$$

$$= T(-E_{\lambda_{n1}} + E_{\mu_{nn}}) \tag{92}$$

since all other terms in the sum cancel out. The fact that $\lambda_{n1} = a < m$ and $\mu_{nn} = b > M$ yields that $E_{\lambda_{n1}} = 0$ and $E_{\mu_{nn}} = I$, so that the expression above becomes

$$T\sum_{j=1}^{n} E(\Delta_{nj}) = T(0 + I) = T. \tag{93}$$

Eq. (88) implies that for all $\varepsilon > 0$ there exists a n such that

$$\eta(\mathscr{P}_n) = \max_j l(\Delta_{nj}) < \varepsilon \tag{94}$$

so that, for any $\hat{\lambda}_{nj} \in \Delta_{nj}$ we have

$$\begin{aligned}\sum_{j=1}^{n} \mu_{nj} E(\Delta_{nj}) - \sum_{j=1}^{n} \hat{\lambda}_{nj} E(\Delta_{nj}) &= \sum_{j=1}^{n} (\mu_{nj} - \hat{\lambda}_{nj}) E(\Delta_{nj}) \\ &\leq \sum_{j=1}^{n} (\mu_{nj} - \lambda_{nj}) E(\Delta_{nj}) \\ &\leq \sum_{j=1}^{n} \varepsilon E(\Delta_{nj}) \\ &= \varepsilon I.\end{aligned} \tag{95}$$

Similarly, we have that

$$\begin{aligned}\sum_{j=1}^{n} \lambda_{nj} E(\Delta_{nj}) - \sum_{j=1}^{n} \hat{\lambda}_{nj} E(\Delta_{nj}) &= \sum_{j=1}^{n} (\lambda_{nj} - \hat{\lambda}_{nj}) E(\Delta_{nj}) \\ &\geq \sum_{j=1}^{n} (\lambda_{nj} - \mu_{nj}) E(\Delta_{nj}) \\ &\geq \sum_{j=1}^{n} (-\varepsilon) E(\Delta_{nj}) \\ &= -\varepsilon I.\end{aligned} \tag{96}$$

Subtracting $\sum \hat{\lambda}_{nj} E(\Delta_{nj})$ from (90), equations (95) and (96) yield that

$$-\varepsilon I \leq T - \sum_{j=1}^{n} \hat{\lambda}_{nj} E(\Delta_{nj}) \leq \varepsilon I.$$

Taking the norm of this equation, we obtain

$$\left\| T - \sum_{j=1}^{n} \hat{\lambda}_{nj} E(\Delta_{nj}) \right\| \leq \varepsilon \tag{97}$$

which shows that $\sum \hat{\lambda}_{nj} E(\Delta_{nj})$ converges uniformly to T as n goes to infinity. Thus T can be expressed as a Riemann-Stieltjes integral of $\hat{\lambda}$ over $(a, b]$ with respect to E_λ, since $E(\Delta_{nj}) \to E_\lambda$ as $n \to \infty$. Moreover, E_λ is constant for $\lambda < m$ and $\lambda > M$ so that we can just take the integration limits as m, M. In conclusion, T has the spectral representation

$$T = \int_{m-0}^{M} \lambda dE_\lambda \tag{98}$$

where $m - 0$ indicates that a possible contribution at $\lambda = m$ has to be taken into account. □

It is important to note that 4.3.1 holds for bounded, self-adjoint operators, while important classes of operators such as quantum and differential operators are typically unbounded. However, an analog of the Spectral theorem above can be proven also for unbounded operators using a Cayley transform. [1]

The Spectral theorem can also be extended to functions of operators in order to introduce a functional calculus, resulting in the equation

$$f(T) = \int_{m-0}^{M} f(\lambda) dE_\lambda \tag{99}$$

where f is a real-valued function defined on $[m, M]$. This theorem can also be extended to the larger class of operators that are normal, i.e operators T such that $TT^* = T^*T$. [5]

The theorems in section 4 are taken from [1], and the proofs are strongly influenced by the proofs there.

5 Discussion

This thesis examined the basic properties of Hilbert space and proved the Spectral theorem for bounded, self-adjoint operators. The aim was that the thesis should be self-contained, i.e that every theorem needed in the derivation of the Spectral theorem should be proven as well. This, however, proved to be difficult since the theory of linear operators and Hilbert spaces cannot be separated from the more general theory of normed and Banach spaces. Proving the Spectral theorem 'from scratch' would require more knowledge of advanced functional analysis and would result in a much longer text, and this is the main reason that the proofs of some theorems are not stated in the thesis but left for the interested reader to delve into on their own.

A proposition for a longer text would be to write about the Spectral theorem in more detail and investigate its consequences more fully, or to extend the theorem to unbounded self-adjoint operators. Another possibility is to study the mathematical aspects of the unbounded operators that arise in quantum mechanics, such as Schrödinger differential operators.

It is also possible to study the spectra of operators numerically. Computing the spectra of Schrödinger operators numerically is an active field of research, an example of such research is the paper 'Computing spectra without solving eigenvalue problems' [6] by Mayboroda et al.

References

[1] Erwin Kreyszig. *Introductory Functional Analysis with Applications*. Wiley, 1989.

[2] Jean Dieudonné. *History of Functional Analysis*. North-Holland, 1981.

[3] Paul R. Halmos. *Naive Set Theory*. Springer, 1974.

[4] Chris Rorres Howard Anton. *Elementary Linear Algebra with Supplemental Applications*. Wiley, 2014.

[5] Haim Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer, 2011.

[6] Mayboroda et al. “Computing Spectra Without Solving Eigenvalue Problems”. In: *SIAM Journal on Scientific Computing* 41.1 (2019).