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School of Education and Communication

Doctoral Thesis

Teaching for the Learning of Additive Part-whole Relations

The Power of Variation and Connections

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Jönköping University
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Abstract

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In this thesis, results from four empirical studies and a re-analysis are synthesized with what can constitute a structural approach to teaching and learning additive part-whole relations among learners aged four to eight years. In line with a structural approach to additive relations, the relations of parts and whole are in focus from the outset and are seen as the basis for addition and subtraction (Davydov 1982; Neuman, 1987). This approach was introduced by the researches in two intervention studies across different contexts. The researches collaborated with teachers in planning part-whole activities, teachers teaching them in their own settings, and then reflecting on them together with the research team. The empirical material consists of video-recorded lessons (Grade 3), small-group teaching (preschool) and individual video-recorded task-based learner interviews (with preschoolers). The teaching episodes and interviews were analyzed on a micro-level, using analytical tools and concepts from variation theory (Marton, 2015). To deepen the knowledge, a re-analysis was also conducted with the purpose of identifying qualitative differences in teachers' enactments of mathematical ideas and principles associated with a structural approach to additive relations.

Looking at the articles and the re-analysis, the results suggest that, for learning, it matters which representations are offered to the children. Some representations seem to facilitate the discernment of the parts and whole, and their relations. The results suggest that it matters *which* examples are offered. A systematic sequence of examples has the potential to bring to the fore relations between different part-whole examples, which offer the children opportunity to learn mathematical principles such as commutativity. Furthermore, the results indicate that what is made possible to learn about additive part-whole relations is associated with what aspects are opened up as dimensions of variation (Marton, 2015). Foremost, though, the results reveal the importance of making *connections* to highlight number relations and key features associated with the structural approach to additive relations. The results suggest that how *variation* is offered, and whether and how the teacher explicitly (verbally and gesturally) draws attention to relations, ideas and aspects, is crucial for the learning of additive part-whole relations. Moreover, through the separate articles and the re-analysis, the outcomes indicate that the structural approach to additive part-whole relations and conjectures from variation theory are possible to implement in different contexts and for different ages of children.

Preface

Working on this thesis has been a challenging process, and now I'm at the end of it. The thesis is framed in two research projects from two different contexts, and reports on selected findings from these projects. Being involved in these projects from start to finish – designing, planning, analyzing and discussing research findings in close collaboration with teachers and research colleagues – has been a worthwhile way for me as a doctoral student to develop various abilities, I think. Being close to practice has been inspiring and has suited me fine. Even though the projects are now complete and the thesis is written, I still have an interest in further exploring the teaching and learning of mathematics.

However, my curiosity for teaching and learning started a long time ago, when I got my first job as a primary teacher at a school in Nynäshamn. From the first day, I was encouraged by experienced colleagues to put into practice my new ideas from the teaching training program. In this open and friendly atmosphere, pedagogical questions were constantly on the agenda. Thank you, *Sunnerby girls*, for all the trust you had in me from the start, and for inspiring me to continuously reflect on the children's learning and my teaching. When I first met Anita Sandahl at a course at Högskolan för Lärande och Kommunikation, some of the key ideas I had developed in regard to teaching mathematics were turned upside-down. Thank you, Anita – you made me frustrated and inspired me to further study and explore the teaching and learning of mathematics. Also, thank you for entrusting me with the task of lecturing at HLK and thereby starting my Master's studies. Anita, you believed in me!

A decade later I met you, Professor Ulla Runesson! You became my supervisor, and have supported me on this journey towards a PhD degree. In 2013, when you invited me to a join a research project in Johannesburg, an intensive and fantastic research journey started. Thank you, Ulla, for letting me participate in the project from the start and for trusting in my skills! Your knowledge of variation theory, experience working with teachers in practice, and engagement in trying to improve teaching, inspired me from the start. We have traveled together to Johannesburg eight times, I think, and have analyzed thousands of worksheets and discussed questions related to teaching and learning, sometimes accomplished along with a glass of South African red wine. Ulla, you've always believed in my ideas and appreciated my practice-based experience as a primary school teacher. Even though it's sometimes been quite challenging working with you, your sharp-

ness and constructive criticism have led to a qualitative improvement in my work. Ulla, thank you for always believing in my competence!

Professor Hamsa Venkat, my second supervisor, I'm so grateful for getting to know you and collaborating with you. I'm also grateful for your generosity in sharing your vast knowledge as an expert in the research field of mathematical educations. I admire your having such great patience with me, trying to understand my English, my ideas, and my arguments. Thank you also for all your valuable feedback on my writing! You've always believed in my competence, you as well, and you've become my friend. Professor Camilla Björklund, my third supervisor, thank you for letting me participate in the FASETT project from the start, and for sharing your knowledge in early childhood mathematics. You've also helped me become a better writer, and have guided and supported me even though I sometimes doubted myself. You've also become a dear friend. And you, too, have always believed in me.

My gratitude goes to all the teachers who participated in the two projects: Thank you for opening your classrooms to us! It has been so fascinating to be a part of your practice and listen to your reflections on teaching and learning! Thank you to all the children I interviewed and met in the classrooms: You've taught me so much and deepened my understanding of how children of your age can reason about numbers and number relations!

During my time as a researcher I've been involved in different research groups. Thank you, MER-gruppen (the Mathematics Education Research group) – Ulla, Robert, Pernilla, Jesper, Birgitta, Klara, Per, Andreas, and new and old members! – for recurring fruitful discussions and for providing me with constructive feedback. Also, to the research group at Wits University, thank you for the meaningful sessions and for your kindness and hospitality. The research team of the Wits Maths Connect Primary project – Hamsa, Mike, Corin, Sam, Lawan, Marie, Herman, Samira – you're amazing! And I love your craziness! Also, thanks for driving me around and for letting me visit suburban and township schools and be involved in ongoing projects! The time I've spent in Joburg has given me perspective on my situation as a researcher, mother and teacher, as well as on the world around me; thank you! Special thanks to Mike Askew, who wanted to be a coauthor of the "weaving article". Mike, you and Hamsa have inspired me! Wits University, I look forward to further collaborations – keeping my fingers crossed!

I also wish to thank Ference Marton! It's been stimulating and inspiring working with you in the FASETT project – I loved those Gothenburg meetings! And of course, thanks also to the other members of the FASETT group: Angelika, Maria, Maria, Ulla and Camilla. I really enjoyed working and discussing things with you as well.

Angelika Kullberg, Eva Björk-Åkesson, Martin Hugo, Åke Ingerman, Jesper Boesen, Gunvie Möllås: Thank you for your valuable viewpoints and constructive criticism at my 50% and 90% seminars. Joakim Öberg, thank you for the final assistance with the formatting!

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I also want to thank Rebecka, Pia, Birgitta, and Maria for being great friends and listeners, Bo for fixing the "af-bas", ALL my lovely colleagues and friends "inside and outside HLK", and my relatives who have followed my research journey, offered kind thoughts, and asked interesting questions. THANK YOU!

Foremost, I'm most grateful to my husband, Mikael. Thank you for supporting me! You've always been there for me when I've been frustrated, happy, or "on the move". When I've sometimes despaired, you've constantly encouraged me and believed in my competence to complete this work. To our children, Lisa and William: I'm so grateful that you've always been there supporting me! At the same time, you've reminded me about other more important things in life. Mikael, Lisa, and William, you're the most important of all to me! All my love to all three of you!

Kvarnängen, Ölmstad, October 2019

Anna-Lena Ekdahl

Original studies

Article I

Ekdahl, A-L, Venkat, H., Runesson, U. (2016). Coding teaching for simultaneity and connections - Examining teachers' part-whole additive relations instruction. *Educational Studies in Mathematics*, *93*(3), 293–313.

The author was responsible for designing intervention lesson content, the data collection, development of the coding framework, and the analysis process.

Article II

Ekdahl, A-L, Venkat, H. Runesson, U. & Askew, M. (2018). Weaving in Connections: Studying changes in early grades additive relations teaching. *South African Journal of Childhood Education.*

The author was responsible for designing intervention lesson content, the data collection, development of the coding framework, and coding of lessons.

Article III

Ekdahl, A-L. (in review). Different Learning Possibilities from the same Activity - Swedish Preschool Teachers' Enactment of a Number Relation Activity.

Article IV

Björklund, C. Ekdahl, A-L, & Runesson Kempe, U. (in review). Implementing a Structural Approach in Preschool Number Activities. Effects of an Intervention Program.

The author was responsible for the data collection and the analysis of data ii. Data (i) and (iii) were analyzed in collaboration with other authors.

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Contents

1 Introduction

This thesis explores the teaching of additive part-whole relations, grounded in a structural approach. Also studied is how this teaching can bring to the fore mathematical ideas and principles that can facilitate the learning of part-whole additive relations among young learners. This structural approach to additive part-whole relations stands in contrast to more common descriptions of children's development of arithmetic skills. Many researchers within the field of mathematics education have studied children's strategies for solving additive relation problems, frequently describing them in terms of stages of development (e.g. Carpenter, Hiebert & Moser, 1981; Fuson, 1982; Nesher, 1982; Steffe, Thompson & Richard, 1982; Vergnaud, 1982). This research largely suggests that sequential counting procedures form the basis of a natural pathway to solving additive relation problems in a larger number range (Fuson, 1992).

However, some researchers have raised questions about the children who do not develop efficient counting strategies and instead rely on single-unit counting (Cheng, 2012; Ellemor-Collins & Wright, 2009; Hoadley, 2007; Schollar, 2008). Neuman (1987), whose work was an inspiration for this thesis, infers that those children who get stuck counting by ones forward and backward run the risk of **not** developing effective strategies for solving addition and subtraction problems. She suggests that children are to be taught strategies based on numbers' part-whole relations. This kind of teaching emphasizes the part-whole relation from the outset, with the relation of the parts and the whole seen as the basis for addition and subtraction, characterizing the *structural approach* to additive relations. Furthermore, a key feature associated with a structural approach to additive relations entailing the ability to see items as composite sets of units, also involves how some representations and organizations of items might support this ability. Therefore, the idea of teaching to see numbers as composite sets of units, is further explored in this thesis.

The conjecture that a structural approach to teaching and learning additive relations[1], whereby an awareness of numbers' part-whole relations[2] is essential for children's development of arithmetic skills (Marton, 2015; Neuman,

[1] In this thesis, including all four articles, *part-whole relation* concerns the additive, not the multiplicative, relation.
[2] The part-whole relations of numbers are the focus of this thesis. While the term *part-whole relation* is most often used to describe this, *part-part-whole relation* is also used. Thus, even if the terms are not used consistently, they have the same meaning.

1987, 2013), is the point of departure in this thesis. The structural approach to numbers also stems from Davydov's (1982) curriculum, in which numbers are introduced as labels for measured quantities and then part-whole relations are introduced as representations of the sizes of quantities, *without* a trajectory beginning in counting single units.

Researchers in the field of mathematics education (Baroody, 1987; Ching & Nunes, 2017; Clements & Sarama, 2009) further argue that it is important that children learn to master certain *mathematical principles* in order to develop flexible strategies for solving additive relation problems, for instance the commutativity of addition (irrelevant order of the addends) and the complement principle (the inverse relation between addition and subtraction). A consequence of taking the structural approach to part-whole relations, with the focus on additive part-whole relations of numbers is that it offers the possibility to introduce these principles earlier in the instructional sequence than the more common counting operation-based approaches typically do. As regards learning on the side of the trajectory from counting operations to additive relations, some have argued that it is challenging for preschoolers to understand the part-whole relations of numbers (Piaget, 1965; Resnick, 1983). In contrast, other researchers argue that children of preschool age can achieve knowledge about the part-whole relations of numbers before lower primary school (Fischer, 1990; Jung, Hartman, Smith & Wallace, 2013).

The facts that it is possible to learn the part-whole relations of numbers in preschool, that children can learn to see part-whole relations as a composite set of units, that important mathematical principles are linked to part-whole relations, and that relational reasoning might support young learners' development of arithmetic skills support the further exploration of "teaching from a structural approach" – and, more specifically, what can constitute a structural approach to teaching additive part-whole relations that facilitates learning among young learners. The intent in the four empirical studies (Articles I-IV) associated with this thesis is to answer the framing question. Results from these studies are re-analyzed and synthesized to further enhance the knowledge regarding how it is possible to introduce mathematical principles and ideas in additive relation teaching.

Variation theory (Marton, 2015; Marton & Booth, 1997) forms the theoretical basis of this thesis. According to the theory, learning something means discerning aspects of what is to be learned (the object of learning). When new aspects are discerned, the way the object of learning is experienced changes. As the theory sees the discernment of difference as a necessary condition for learning, the teacher can help the learners experience the object of learning in a new way by offering variation within the aspects. In this thesis, variation theoretical principles enable me to study "part-whole relation teaching", with a focus on how the teachers handled this content and what learning opportunities were offered. Principles from the theory were also used in the design of activities and examples.

From this theoretical basis, in this thesis *teaching* relates specifically to how the mathematical content, additive part-whole relations, is handled by the teachers. The term *enactment* is also used here to describe the teaching on a detailed level, especially in Article III and in the re-analysis. There is no essential difference between these two terms or how they are used. However, in the detailed re-analysis, *enactment* is used with the purpose of drawing distinctions between enactment that includes the teacher's linking actions and that which does not.

The data that form the basis of this thesis have been collected from two projects, both using a structural approach to part-whole teaching: the Wits Maths Connect Primary project and the FASETT project[3]. Wits Maths Connect aimed to develop intervention programs to improve teaching and learning mathematics in the Foundation Phase[4] at ten government primary schools in South Africa. FASETT aimed to generate knowledge about Swedish preschoolers' ability to learn addition and subtraction, and how designed pedagogical activities can contribute to preschool children's development of these arithmetic skills. The children who participated in the two projects were four to eight years old. A structural approach to additive relations forms the basis of both projects, with part-whole relations of numbers seen as central to early number learning. The projects had in common that the participating teachers were relatively unfamiliar with the idea of structural approaches to additive relations. In both interventions, the research group and the teachers worked in close collaboration. Finally, principles from variation theory (Marton, 2015) were used in the interventions as well in the data analysis.

In summary, given the importance of the part-whole relations of numbers in early number education, my intention is to deepen the understanding regarding the teaching and learning of additive part-whole relations, and to contribute knowledge to the field of mathematics education research with an orientation towards early childhood and early primary mathematics, with

[3] *The ability to discern the first ten numbers as a necessary ground for arithmetic skills*. In Swedish *Förmågan Att Sinnligt Erfara de Tio första Talen som nödvändig grund för aritmetiska färdigheter*. The project was funded by the Swedish Research Council (grant no. 721-2014-1791).
[4] Foundation Phase consists of Reception class to Grade 3, with students aged five to eight years.

pedagogical implications for practice. Variation theoretical ideas are used to expand the knowledge regarding the teaching of additive part-whole relations and to study the learning opportunities offered. The key data sources for this thesis consisted of video-recorded lessons (Grade 3), video documentation of small-group teaching (preschool), and individual video-recorded task-based learner interviews.

2 Literature review

This section provides background on the structural approach to additive relations. The literature review looks at the research from a learning and a teaching perspective.

Structure in (early) mathematics teaching and learning

In mathematics education research, *structure* has been an object of attention for several decades. Increasing focus has been on preschoolers' and young students' understanding of mathematical structure (Lüken & Tiedenman, 2019; Mulligan & Mitchelmore, 2009, Warren, 2003). Several studies indicate that an understanding of the structure in arithmetic and spatial patterns is important in early mathematics learning (Mulligan & Vergnaud, 2006; Papic, Mulligan & Mitchelmore; 2011). For example, in a longitudinal study with five- to eight-year-olds, Lüken (2012) assessed children's *structure sense* using pattern and structure-oriented tasks (e.g. spatial dot patterns and ten chain) as well as standardized tests. The results pointed to children's structure sense as a predictor of arithmetical competences in Grade 2. Other mathematical education research asserts that structure in mathematics is to be emphasized in teaching, arguing that it supports learners' understanding of, for instance, relations between quantities, the partitioning of numbers, composite units, relations between operations, and multiplicative reasoning (Mulligan, Mitchelmore, English & Crevensten, 2013). Many researchers see mathematical structure is as a bridge between arithmetic knowledge and algebraic reasoning (Carraher, Schliemann, Brizuela & Ernst, 2006; Kieran, 2007; Warren, 2003).

Even though structure is described as important in the teaching and learning of mathematics, structure in mathematics is defined, discussed, and argued for in slightly different ways. For example, Freudenthal (1991) considers structure in mathematics in a wide sense: "Structuring is a means of organising phenomena, physical and mathematical, and even mathematics as a whole" (p. 20). Doing mathematics, he proposes, is a way to organize concepts, numbers, shapes, etc. into more formal or abstract structures, preferably in realistic learning situations. In his didactical work, he discusses structure within different areas of mathematics as well as in materials, environment, and instruction. Mulligan and Mitchelmore (2009) give a narrower definition of the term: an *awareness of mathematical patterns and structure*. This definition builds on their empirical work with preschoolers' and young students' performances across number, measurement, and space tasks. They argue that this awareness of structure can be measured and is correlated with general mathematical understanding. In Mason, Stephen and Watson's (2009) work, structure in mathematics is defined as "… the identification of general properties which are instantiated in particular situations as relationships between elements or subsets of elements of set" (p. 10). They suggest that mathematical structure is an awareness, which is to be emphasized in teaching by drawing attention to adequate relations and properties and opening up for generalizing to other cases. This kind of instruction, they argue, will develop students' mathematical thinking.

In recent years, Venkat, Askew, Watson and Mason (2019) have discussed the term *structure* and made the clarification that, in a wider sense, structure consists of mathematical relations between elements. Further, they draw a distinction between *emergent* and *mathematical* structure. Emergent structure is an arrangement of "…elements – symbols or images – in some particular organisation that serves to stress a mathematically appropriate relationship" (p. 14), based on the local relation that arises within a specific case. Mathematical structure[5], in contrast, involves more general relations – those that go beyond one case, are linked to each other (generic), and can include general properties that are applicable across classes of examples. In teaching, examples can be worked with as a particular case (the even number 6), a generic case (an even number like 6), or a general case (any even number) (Mason & Pimm, 1984).[6]

In summary, there is an increasing body of research indicating that structure is important in early mathematics learning. It is also known that preschoolers' and young students' ability to discern structure differs, implying that it is important to emphasize structure in teaching. Whereas Freudenthal (1991) considers structure in a wider sense, the definition of structure in the focal studies associated with this thesis is much narrower. Mulligan and Mitchelmore's (2009) definition associates mostly with spatial patterns, but also with number relations. Since the focus in this thesis is on number – and specifically additive – part-whole relations, their definition would have corresponded with that of a structural approach in this thesis. However, the closest alignment is with the definition by Venkat et al. (2019), in terms of structure as both local relations (emergent structure, seen in the formatting of quantities in particular examples into a mathematically appropriate relationship) and more general ones (mathematical structure, seen when particular relationships start to be seen as generic or general in nature). In the

[5] Mason, Stephen and Watson's (2009) definition of mathematical structure aligns with that of Venkat et al. (2019).
[6] Venkat et al. (2019, p. 15).

context of additive situations, structure is therefore associated with additive part-whole relations specifically.

Structural and counting operations approaches to additive relations

Given the attention to structure in mathematics and its central point in this thesis, structural approaches will be further emphasized, with a narrower focus on additive relations. A way to distinguish the approaches from each other might be to say that in a structured approach to additive part-whole relations, the parts and whole of quantities are focused on simultaneously, compared to the operational counting operations approach in which the quantities are handled in sequence.

Structuring as basis for additive relations

A structural approach to additive relations is taken in Davydov's work (Schmittau, 2004). Davydov (1982) criticized the dominant research in the 80s that focused on young children's counting strategies, instead arguing that measurement serves as an introduction to number knowledge. Initially, children are to be given the opportunity to "recognize the multiple relationship that can exist between a continuous or discrete object (as expressed by its numerical measure) and some part of that object that has been used as the unit of measure" (pp. 226–227). Then, by starting to compare quantities (physical and abstract) rather than emphasizing counting single units, children are taught to express relations between quantities using literal symbols (e.g. A>B). Thereafter, they are encouraged to explore the part-whole relations of numbers and the algebraic structure of a "…composition by which the relation between two elements determines a unique third element as a function" (p. 229). The plus and minus symbols were introduced in this way in order to describe relations between elements and quantities. Polotskaia and her colleagues (Polotskaia, 2017; Polotskaia & Savard, 2018) have developed Davydov's (1982) theoretical ideas of relation reasoning between quantities focusing on additive word problems in early primary years. They argue for what they call a *relational paradigm*[7], whereby the underlying additive relations in the problem first need to be understood in order to then be able to identify the operation for calculating the unknown quantity (part or whole) in the part-whole relation.

In Swedish research, the structural approach to additive relations takes its departure in Neuman's (1987) and Ahlberg's (1997) studies on children's different ways of solving additive relation context problems. Children who

[7] The authors describe in detail what characterize the relational paradigm and how it contrasts with the *operational paradigm* (cf. counting operational approach, used in this thesis).

successfully solved the problems structured the numbers in the problems in parts and whole, without counting on the number sequence. Neuman (1987) found that seven-year-olds who failed to solve additive problems used *single-unit counting*, with or without their fingers. They had to keep track of how many units they had counted or when to stop counting, in contrast to the children who succeeded in solving the tasks by structuring their fingers into numbers as patterned part-whole relations. Hence, Neuman (1987; 2013) and Björklund, Kullberg and Runesson Kempe (2018), in contrast to most researchers in early mathematics, draw a distinction between different ways of using one's fingers, not simply whether or not to use them, for solving addition and subtraction problems. They argue that *structured finger patterns* unify the ordinal aspect (the order of the fingers) and the cardinal aspect (each number refers to a certain group of fingers) of numbers. When structured finger patterns are used successfully, for instance in solving a subtraction problem, the cardinal and ordinal aspects are discerned simultaneously and the finger pattern provides an understanding of the numbers' part-whole relations. This argumentation supports the promotion of structured finger patterns in early childhood mathematics.

Ellemor-Collins and Wright (2009) also argue for structuring numbers based on Freudenthal's (1991) work, as previously mentioned. Focusing on the number range 1–20, they define structuring numbers as "…organizing numbers more formally, establishing regularities in numbers, relating numbers to other numbers and constructing symmetries and pattern in numbers" (Ellemor-Collins and Wright, 2009, p. 53). They present a program including ten topics, for instance *partitioning of small numbers*, *complements to 10*, *automatizing the doubles*, *adding two numbers*…, in order to develop students' skills in structuring numbers and solving tasks without counting by ones. They also argue that seeing numbers in relation to multiples of 10 and 5 as *key benchmarks* is particularly useful for developing number structure. In early number learning, this begins with seeing representations of 10 constituted by two 5s. Compared to Ellemor-Collins and Wright's (2009) program focusing on number structure by learning to organize the numbers, the focus in this thesis is primarily on the within relations of numbers, which include the semi-decimal structure of 10 as two 5s, associating with the part-whole relations of numbers (Davydov, 1982), the relation reasoning between three quantities (e.g. Baroody, 2016), and the connection between addition and subtraction.

Counting as basis for additive relations

Most often, the dialogue about young children's development of whole-number arithmetic (addition and subtraction) takes a cognitive perspective on learning (Carpenter & Moser, 1984; Fuson, 1982; 1992; Steffe, Cobb & von Glasersfeld, 1988). Steffe et al. (1988) give a detailed description of the development stages of counting types, whereby children initially rely on concrete objects for counting, moving towards *Figural Unit Items*, when objects can be counted even if they are not within the child's range of immediate perception, by for instance using fingers as a replacement for the counting of the perceptually inaccessible collections. In the steps that follow, children start using more abstract counting types, and are not dependent on perceived items. This is in line with Carpenter and Moser's study (1984) in which first-graders most commonly solved addition and subtraction word problems through direct modelling with their fingers or physical objects, or by counting on the number line. Most often, the mathematics literature takes its departure in Fuson's (1992) definitions outlining the progression of young children's ability to solve addition and subtraction situations in terms of developmental levels. The first level is called *counting all,* for example solving the task 3 + 5 = _ by counting the first addend (1, 2, 3), then the second addend (1, 2, 3, 4, 5), and then counting all together in a final sum count (1, 2, 3, 4, 5, 6, 7, 8). The second level is called *counting on.* On this level, the child counts forward by ones from the first addend (from the example above 3 …4, 5, 6, 7, 8), or in subtraction backward by ones, solving 8 – 5 = _ by counting backward from the starting number by ones (8…7, 6, 5, 4, 3). *Counting on* is a more sophisticated strategy than counting all. On both levels, though, children may use manipulatives such as counters or fingers. Using increasingly abbreviated counting procedures, the children, according to the operational approach, will develop their ability to use more efficient counting procedures for solving addition and subtraction problems, eventually truncating into knowing number combinations as recalled facts or by using derived number facts (Fuson, 1992), for instance knowing that for 3 + 4 "3 + 3 = 6 and one more is 7". By retrieving various combinations of numbers, children may solve different problems within a larger number range.

Counting versus structuring

A way of further comparing the counting operational approach with the structural approach to additive relations can be done through an illustration of a girl solving an additive relation problem within the number range of 10 (Neuman, 1987; 2013). The girl was asked to solve the following task:

> You have two things and need nine. How many more things do you need?

The girl started by putting both her hands on the table and folding her little finger. She thereby knew that 9 is one less than 10, without using single-unit counting. Then, she immediately looked at her hands and said "seven", still without counting any of her fingers. By creating a finger pattern, making the part-whole relations visible with her fingers, the girl *saw* on her finger pattern that 7 is two less than 9 and identified the missing number (7). The parts and the whole were visible simultaneously, and the relation between seven fingers and nine fingers was discerned. This way of using finger patterns for structuring number relations enabled the girl to see how the parts and the whole in the problem were related to each other. Through this, the structured part-whole relations of numbers were foregrounded. Even though the problem's semantic formulation itself was ambiguous as to which of the numbers was missing and whether it was a subtraction or addition problem (See Nesher, Greeno & Riley, 1982; Vergnaud, 1982), the girl experienced the part-whole relation 9, 2, and 7 (number triple) as a structure, e.g. enabling her to transform a subtraction problem to an addition problem). Solving the problem by attending to the part-whole relation, for instance using finger patterns, stands in contrast to using a *counting on strategy* (Fuson, 1992), which could be illustrated solving the same task by raising one finger at a time, saying or thinking the number words: (2; 3 [is 1 more]; 4 [is 2 more]; 5 [is 3 more]; 6 [is 4 more]; 7 [is 5 more]; 8 [is 6 more]; 9 [is 7 more]. Children who use this strategy have to keep track of how many units they have counted and when to stop. In the literature, this counting strategy is sometimes called *double counting* (Fuson, 1992). Some researchers argue that children who get stuck counting by ones forward and backward with or without their fingers, and rely on these strategies for solving addition and subtraction problems, might be hampered in developing an understanding of the part-whole relations of numbers (Baroody & Gannon, 1984). These naïve counting strategies become ineffective and time-consuming in smaller as well as larger number ranges (Cheng, 2012; Laski, Ermakova & Vasilyeva, 2014; Schollar, 2008; Svenson & Sjöberg, 1982; Zhou & Peverly, 2005). Studies conducted in the early primary years, for instance (Canobi, 2005; Ding & Auxter, 2017), have seen students' understanding of part-whole number relations as an indicator of how they managed to solve different addition and subtraction problems. Furthermore, other studies have found that low-attaining students (Grades 2 to 4) who used ineffective counting strategies (single-unit counting) made progress and learned to use more structured number strategies, for instance involving a knowledge of part-whole relations and using 10 and 5 as key benchmarks (Ellemor-Collins & Wright, 2009; Morrison, 2018).

Part-whole relations of numbers

The terms *whole* and *parts* for describing the set-subset relation (cf. Piaget, inclusion relationship) were introduced by Payne and Rathmell (1975). Fischer (1990) defines the relation as involving "a set of objects associated with a number can be separated into subsets, each of which may also be represented by a number" (p. 207). In the literature on early numeracy the part-whole concept is often emphasized, for instance in children's development of arithmetic skills and in learning trajectories (e.g. Baroody, Lai & Mix, 2006; Clements & Sarama, 2009; Resnick, 1983).

Advocates of the *counting operations approach* to addition and subtraction do not disagree in regard to the importance of part-whole relations of numbers in the development of arithmetic skills. However, they do argue that counting precedes part-whole relation thinking (Baroody, 1985; Nesher, Greeno and Riley, 1982; Resnick, 1983; Verschaffel, Greer, & De Corte, 2007). According to Resnick's (1983) psychological theory of number understanding, children develop an informal knowledge (protoquantitative schemas) about numbers during the preschool years. The children are able to compare quantities, reason about changes in quantities, and make judgements about the parts and the whole without exact definitions. Even if they are able to use counting to determine quantities of given sets, they might not use counting as a strategy for solving an additive relation problem. In the preschool years the counting and protoquantitative schemas exist, albeit separate from each other, and the children are therefore to be pushed to use counting to make an exact numerical quantification. It is not until the protoquantitative part-whole schemas become integrated with children's knowledge of counting (Quantification of the schemas) that they will understand numbers as compositions of other numbers (part-whole relations) and understand, for instance, the complementary principle of addition and subtraction and the commutative principle of addition (Resnick, 1983; Resnick, Lesgold & Bill, 1990). Following this theoretical approach, children's understanding of part-whole relations of numbers goes *through counting*. This is opposite to a structural approach, which does not neglect counting but suggests that it is possible to understand the part-whole relation without pushing towards single-unit counting (Davydov, 1982).

It has been further discussed at what age children are capable of understanding the part-whole relations of numbers. Piaget (1965) claimed that children might not understand this before the age of seven, an assertion built on clinical interviews with four- and five-year-olds. The children were able to compare quantities (less, more, most), but when asked to decompose a whole collection into two parts (subsets) and re-compose those parts, it was not obvious to them that this was the same whole. It seemed as if they disregarded the whole or forgot one of the parts. Some studies on preschoolers' part-whole reasoning indicate that preschoolers are able to reason about numbers' part-whole relations before the primary grades. Sophian and her colleagues (Sophian & McCorgray, 1994; Sophian & Vong, 1995) examined how four- to six-year-olds identified missing addends and initial values in contextual story problems. It was found that many preschoolers could reason about part-whole relations and identify missing addends and initial values in contextual story problems; however, they most often did this without exact numerical precision.

Other researchers advocate that preschoolers can understand the part-whole relations of numbers using exact numerical quantification. Hunting (2003), using a task similar to that in Piaget (1965), found that three- and four-year-olds could reason about part-whole relations in a task with a set of five items, in which one subset was hidden (these findings do not correspond with Piaget's findings). Hunting also found that the children changed focus from counting single items in the subsets to seeing the items as composite sets. Hunting argues that the ability to see groups of items was fundamental to the development of their number knowledge. Also, studies by Ekdahl, Björklund and Runesson Kempe (2019), Cheng (2012), Fisher (1990) and Jung et al. (2013) suggest that children are able to learn part-whole relations of numbers earlier than primary school.

Key mathematical ideas of learning part-whole relations

Within the structural approach to teaching and learning additive part-whole relations there are key ideas that, while they may not be seen as mathematical principles, are nonetheless inherent in relational reasoning, which does not emphasize single-unit counting.

Numbers as composite sets of units

When children are to determine the quantity in different collections of items, they may estimate, count the items as separate units, or see the items as composite sets of units. Within a structural approach to number relations, there is an intention to promote the composite sets of units approach, in which the arrangement of elements is crucial (cf. local relation, described above). The idea of spatial/perceptual structuring consists of the ability to recognize a structured pattern, such as dots on a die or finger patterns, either as a wholeness and/or as part of a whole pattern (e.g. Benz, 2013). Recognizing patterns as composite units also relates to the intuitive cognitive process of *subitizing*, (Wynn, 1992*),* or what Clements (1999) calls *perceptual subitizing*, "…the direct perceptual apprehension of the numerosity of a group (p. 400)", the ability to discern three/four items simultaneously without needing to count each one. The number of subitized items may be extended, for example perceiving six dots organized in two rows as two subsets of three, or seeing eight fingers – four fingers on each hand and both thumbs folded in – as "one eight" without having to count each finger.

Through such grouping, the range of items is extended (more than three/four items) and the pattern is recognized as subitized parts put together into a whole, an ability called *conceptual subitizing* (Clements & Sarama, 2009). Clements., Sarama and MacDonald (2019) argue that the idea of subitizing supports children's learning of numbers and development of arithmetic skills. Especially conceptual subitizing, whereby children's ability to mentally decompose numbers into parts and compose them back together facilitates them in operating on numbers (solving arithmetic tasks). They state that "… subitizing is a critical competence in children's number development" (Clements et al., 2019, p. 39).

Structured finger patterns, for instance, support children's ability to discern their fingers as composite sets without counting (Brissiaud, 1992) and see structured finger patterns as a composition of parts and the whole (Ahlberg, 1997; Neuman, 2013; Björklund et al., 2018). Young children's numerical understanding can be developed if they are encouraged to use their fingers as "symbol sets" or structured patterns representing quantities (Brissiaud, 1992; Hunting, 2003). In addition to this, as mentioned earlier, Neuman (1987) asserts that structured fingers are useful for extending the number of items being subitized, without needing to count them as single units. The two hands, with five fingers on each hand, facilitate for the children to see numbers in the range of 6–10 as composite sets of units, using the *undivided 5* as a benchmark. The bead string[8], with ten beads on a string grouped in five of one color and five of another (van den Heuvel-Panhuizen, 2008), is another example of a structural representation, which foregrounds the semi-decimal structure and is associated with two hands and ten fingers (Ekdahl, 2019).

Decomposing numbers

A whole number can be decomposed in different ways, with the various part-part combinations all relating to the same whole number (Payne & Rathmell, 1975). In the early mathematics education literature decomposition tasks are described as, for instance, starting with a concrete situation in which a whole quantity is to be decomposed into two parts (Cobb, Boufi, McClain & Whitenack 1997; Ekdahl, Venkat & Runesson, 2016), or with a *hidden* quantity (a whole) decomposed into two parts in different ways (Neuman, 1987) or with one part hidden (Hunting, 2003; Tsamir, Tirosh, Levensson, Tabach & Barkai, 2015). These kinds of tasks enable children to see how different possible parts are related to each other and to the whole, see different ways to split a whole number, and explore all the possible ways to decompose a given whole number (completeness by systematicity).

[8] Sometimes called the *ten chain* in the educational literature.

Knowing that a specific whole number can be split in many different ways is important pre-knowledge to bridging over ten (Payne & Rathmell, 1975), and contributes to what is called *the base-10 decomposition* strategy, which can be seen as an alternative to the counting on strategy for solving addition and subtraction problems over 10 (Cheng, 2012; Murata & Fuson, 2006; Laski et al., 2014).

Mathematical principles linked to the part-whole relation

In Piaget's (1965) argumentation, knowledge of the various additive compositions of parts within a whole number is a prerequisite for understanding the addition concept. Knowing the "within relation" (triple of numbers)[9] provides the basis for solving different missing number problems with these numbers. Based on one specific part-whole relation (e.g. the relation of the whole number 8 and the two parts 5 and 3), it is possible to solve informal (verbally or with manipulatives) or formal (with symbols) problems, when one of the numbers is missing. The part-whole relations of numbers also provide openings for learning *equality* and "arithmetic-algebraic ideas" (Carraher et al., 2006; Schmittau, 2004); for instance, in a missing number problem the missing number could be a missing start number (_ − 5 = 3), a missing addend (5 + _ = 8), or a missing subtrahend (8 − _ = 5). Likewise, the positions of the missing number or the equal sign within the problem could be positioned differently (e.g. 3 = _− 5 or _ − 5 = 3). The part-whole relations of numbers also provide children opportunities to experience other mathematical properties such as the *commutative principle* and the *complement principle* (the inverse relation between addition and subtraction).

The commutative principle for addition

The commutative principle for addition (the irrelevance of addend order to the sum) is a mathematical principle that it is possible to learn when the part-whole relations of numbers are emphasized (e.g. Baroody, 1987; 2016; Ching & Nunes, 2017; Clements & Sarama, 2009). In Canobi, Reeve and Pattison's (2002) study, four- to six-year-olds were able to see commutativity in practical situations (no formal symbols used) in terms of seeing that when two groups of objects (parts) can be combined, the order of the group does not matter; the total amount (whole) is still the same. However, studies within the same age group in which symbols were added, as an alternative or a complementary representation, showed that children continued counting the objects included in both tasks presented (Baroody, 1987; Baroody & Gannon, 1984). When presented two symbolic examples simultaneously

[9] Slightly different vocabulary is used in the literature to describe the additive relation of three numbers (family/fact family/triple of numbers).

(e.g. "Is 3 + 5 the same as 5 + 3?"), most children used a counting strategy to "check" the answers. Baroody (1987) also argues that even if children are able to express the similarities between two numeric expressions, it cannot be taken for granted that they understand that "the commuted combinations have the same sum" (p. 146) and will spontaneously apply the commutative principle when solving arithmetic tasks in other learning situations (Haider et al., 2014).

The complement principle

Part-whole relations of numbers form the basis of the connection between addition and subtraction (e.g. Baroody, 1999; 2016; Carpenter, Franke & Levi, 2003; Davydov, 1982; Ding & Auxter, 2017; Neuman, 1987; Resnick, 1983). Knowledge of the *complement principle* of one part-whole relation (if Part A and Part B = Whole C, then Whole C – Part B =Part A or Whole C – Part A = Part B) facilitates the solving of arithmetic problems. For instance, 9 – 7 = _ can be solved by converting the problem into addition: 7 + _ = 9 ("What must be added to 7 to make 9?")[10]. This complement principle is not obvious to all children. In Baroody's (1999) study, six- and seven-year-olds were interviewed on pair of tasks (e.g. 5 + 3 = _; 8 – 5 = _), to investigate how they could make use of the first task to solve the second one. Few of the children saw the complementary relation, with one exception: the easy addition doubles (e.g. 3 + 3 = _; 6 – 3 = _). Therefore, in his later work, Baroody (2016) stresses even more the importance of "part-whole number relation knowledge" and how it provides flexibility in problem solving. He argues that when the children use "subtraction as addition strategy" automatically, knowing the three numbers as a combination, addition and subtraction problems can be solved with fluency. Ding and Auxter's (2017) study in Kindergarten to Grade 3 showed what characterized the strategies of children who successfully solved different additive inverse problems. It was found that the children who showed full or partial understanding used their previous knowledge of additive part-whole relations, for instance by drawing a part-whole picture of the presented problem.

Furthermore, a study examining five- and six-year-olds' understanding of both the *commutativity* and *complement principles*, in the context of concrete quantities and abstract symbols, was conducted by Ching and Nunes (2017). They found that some children succeeded in solving only the commutative tasks with concrete material (cf. Canobi et al., 2002), while other children managed to solve all the commutative tasks (both concrete and abstract) but not the complement tasks. Another group of children solved the commutative tasks (both concrete and abstract) and the complement tasks with con-

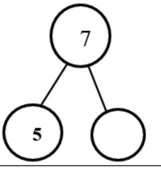
[10] See also the example discussed on pp. 19–20.

crete material, whereas yet another group of children solved tasks with both principles, in both concrete and abstract settings. Building on the results, the authors argued that the commutative principle may be easier to understand than the complement principle, and that the understanding seems to be developed through using concrete material into a more abstract understanding.

In summary, there seems to be agreement regarding the importance of understanding mathematical principles for developing arithmetic skills. Results from earlier studies indicate that it is challenging for children in preschool and early primary grades to understand the commutative and complement principles. However, it seems as if the commutative principle is easier to understand than the complement principle, particularly in practical situations.

Teaching part-whole relations - taking a structural approach

There is agreement that knowledge of the part-whole relations of numbers is important for children's development of arithmetic skills. Thus, if the part-whole concept is developed through counting or if it is seen from a structural approach has different implications for teaching. As mentioned previously, according to Davydov (1982) counting is avoided by starting in measurement and comparing relations (See also p. 17). Number symbols and algorithms are to be carefully introduced, allowing the children to see connections between the concrete objects and more abstract representations (literal symbols and numerals). In order to retain the structural approach to numbers, it is suggested that the transition to number use and algorithms be handled through schematic representations: "There is one extremely important modality in making this connection and bridging actions with objects to their expression in symbols" (Schmittau, 2004; p. 27). A part-whole relation can be illustrated through a schematic representation (See Figure 1), the top representing the whole and the two "legs" the parts (cf. triad diagram, Articles I and II). In Figure 1, the part-whole diagrams are presented.

Discrete quantity	Fact families	Missing number problem (triad diagram)	Missing addend word problem (1)
000 000 / \ 00 0000	7 / \ 5 2 10 / \ 4 6		(1) $7 + x = 15$ 15 / \ 7 x $x = 15 - 7$ $x = 8$

(1) John had 7 baseball cards. Tom gave him some more baseball cards and now John has 15 cards. How many cards did Tom give him?

Figure 1: Instances of part-whole structure (revised from Schmittau, 2004, p. 28).

In the schema, literal symbols or numerals as well as *x* can be used in an empty circle (See Figure 1), marking a missing part/whole. It is argued that the schematic representation enables children's analysis of the structure of word problems, supporting how the quantities are related to each other and whether a whole or a part is missing[11]. In the example in Figure 1, it also makes sense that the position of the unknown number can vary, translating the missing number into an equation (Schmittau, 2004; Schmittau & Morris, 2004).

The Chinese curriculum (Zhou & Peverly, 2005) emphasizes a structural approach to numbers. It gives detailed instructions for part-whole relation teaching from concrete and semi-concrete representations to a more abstract one, before moving to the solving of addition and subtraction problems. It is explicitly pronounced that counting up and down is not recommended, instead the teaching should focus on decomposition and composition of numbers. In teaching, children are to be encouraged to explore the commutative and reversible properties of addition and subtraction, being able to use these to solve new problems. Another curriculum that does not focus solely on the part-whole relation but rather takes a structural approach is the ACE (Arithmetic and Comprehension at Primary School) curriculum (Programmation et Progression ACE, n. d.). In a design-based study, in collaboration with teachers and their six-year-old pupils, a group of French researchers developed this curriculum. First, the children are to become familiar with small numbers, and then explore numbers as relations within numbers and the equal sign as relational, not as producing an operation. Another principle is to help the children understand various properties

[11] Sometimes the Bar Model is used for representing part-whole relations (e.g. Carpenter et al., 1999). In later lessons in the South African focal study, the Bar Model was introduced as an alternative to the triad diagram.

of numbers by, for instance, comparing different representations. Games and activities are to be carefully introduced by the teachers (Programmation et Progression ACE, n. d.; Sensevy, Quilio & Mercier, 2015).

The curricula presented above have implications for teaching part-whole relations using a structural approach, which is to be taken under consideration in the design and planning of the activities and the implementation in practice, for instance the choice of adequate part- whole representations that push towards structural relations, emphasizing translation between representations, and the careful introduction of number symbols. Also, children are to be encouraged to explore numbers as relations within numbers in teaching, foregrounding the meaning of the equal sign as relational and focusing on commutative and reversible properties in additive relation teaching.

Intervention programs promoting part-whole relations

Only a small number of studies in early childhood and early primary years have investigated “part-whole relation teaching”. In one study conducted in Kindergarten in China, a program was introduced by Fischer (1990). He saw that children receiving part-whole teaching (relations of sets and subsets with few objects were used) developed a more concise knowledge of numbers compared to the group who received more common instruction in a “Count/Say/Write program”. Fischer’s results indicate that the part-whole teaching also facilitated the children in solving addition and subtraction word problems, even though these were not included in the instruction. The results from another intervention program (Cheng, 2012) with children aged five to six years show that children can learn to use effective decomposing strategies building on their knowledge of part-whole relations within 1–10 in solving additive relation problems, instead of relying on single-unit counting. The results further suggest that children’s pre-knowledge of numbers’ part-whole relations was shown to be crucial in their use of systematic decomposition.

In the context of early primary grades (Grades 1 and 2) in Canada, Polotskaia and her research group implemented an intervention program, building on Davydov’s ideas about relational reasoning (See p. 17). In the instruction on additive word problems, the teachers were to discuss the additive relation within the presented word problems and draw the learners’ attention to the relation between three quantities, encouraging them to model the problems using a bar model (part-part-whole) before identifying the arithmetic operations linked to the problem. After the program, the target group (Grade 2) performed significantly better than the control group on different types of additive relation problems. Also, the target group succeeded to a greater extent than the control group in solving more complex problems that required relational thinking, for instance those in which the verb could not directly be transformed into an operation (e.g. missing start/missing addend problems[12]) (Polotskaia, 2017; Polotskaia & Savard, 2018; Savard, Polotskaia, Freiman, Gervais, 2013).

Interventions promoting patterns and numbers as composite sets

Given the importance of a structural approach to patterns and numbers, Mulligan and her colleagues (Mulligan & Mitchelmore, 2009; Papic et al., 2011; Mulligan et al., 2013) implemented spatial structured patterns in preschool and primary grades (4- to 8-year-olds) in Australia. The tasks promoted the children's structural development of number, measurement, and space. The results indicated that children showing a higher awareness of structure more easily learned about properties of early numbers and tended to look at and explore new patterns in successful ways. This was especially the case with the low attainers.

Results from other intervention studies show that it is possible for teachers to promote preschoolers' subitizing ability by offering plays, games, and materials in the children's learning environment. Schöner and Benz (2017), for instance, examined five- and six-year-olds' ways of determining number of items (cardinality) before and after having participated in an intervention program. They found that many of the children developed an ability to perceive structure in order to determine the cardinality of sets. For instance, before the intervention program they counted all dots one by one, whereas after the intervention they saw a collection of dots as a subset (conceptual subitizing). Similarly, the intervention by Jung et al. (2013) with four- and five-year-olds focused on subitizing ability, but in this case in combination with the part-whole relations of numbers and relations between numbers. It was argued that these three features are so closely connected to one another that they can be developed concurrently:

> For example, eight can be conceptually subitized as two groups of four; represented in more/less relations, such as two less than ten; and viewed as a whole made of sets of five and three. (Jung, 2011, p. 555)

Therefore, the implemented preschool tasks focused on all these features. Even though the sample was small (73 children), positive effects were found in the group who received instruction emphasizing number relations, compared to the control group. It can be concluded from these studies that

[12] For instance, an additive relation problem with the same semantic structure as the problem discussed on pp. 19–20.

seeing numbers as composite sets of units rather than single units might promote children's understanding of numbers and the part-whole relations of numbers (Clements et al., 2019; Jung et al., 2013; Neuman, 1987). These studies' results encourage the further examination of how part-whole relation tasks might be designed and implemented to enable children to perceive the items as composite sets and numbers decomposed as composite parts.

Teaching for structure

Several researchers accentuate that teachers' actions in the teaching situation play a crucial role. Venkat et al. (2019), for instance, argue that teachers' pedagogical actions should highlight structure in the instruction, involving actions that allow the learners to see both local and general relations (See also p. 16). Payne and Rathmell (1975) and Baroody (1999) argue specifically for the reinforcement of the part-whole relations by, for instance, underlining and verbally talking about "parts" and their relations to the "whole" and "other parts" in instruction with younger learners. In the context of teaching mathematical principles, Zhou and Peverly (2005) recommend emphasizing the complementary relation between addition and subtraction in instruction, proposing that teachers juxtapose two number sentences, with the relation to be discerned by the learners. Another example is found in Ching and Nunes (2017), who conclude that when children are to be taught the commutative and complement principles, they need to be offered representations that specifically visualize the respective principles. As mentioned earlier, some tasks and representations facilitate the discernment of relations. However, teachers need to draw the learners' attention to the target relation in the learning situation. For instance, an abacus, having ten rows of groups of ten beads, five of each color on each row, has the potential to be used for visualizing place value and/or bridging through 10. However, whether the material is to be seen as structured or not will depend on the teacher's handling of it. Consider a teacher presenting the example $4 + 8 = \_$ on the abacus by counting four beads on the top row, followed by counting eight beads on the second row, and then enumerating the beads by counting them one by one (one, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve). Here, the abacus is used as an *unstructured* collection of counting objects and the potential of the structured representation is lost (Venkat & Naidoo, 2012).

Connections in instruction

There is agreement that connections in instruction enhance teaching (e.g. Askew, Brown, Rhodes, Wiliam & Johnson, 1997; Hiebert, Stiegler & Manaster, 1999; Rowland, 2008; Venkat & Askew, 2018). For instance, Askew and his colleagues (1997) identified what characterized effective primary numeracy teaching in England. This effective teaching was connection-oriented, with connections between different mathematical ideas within the same area as well as different areas constantly being made. One of the reasons for analyzing connection in instruction is the evidence of disconnected teaching in South African primary grades' mathematics instruction. The disconnections were identified within the handling of examples (Venkat & Adler, 2012), and where individual examples were treated separately even though they had the potential to be connected (Venkat & Naidoo, 2012). With regard to connections in instruction, connections have been studied at different levels of analysis, for instance on a macro-level with the argument that connections between different areas in mathematics matter (Askew et al., 1997) or comparing teaching in different countries (Hiebert et al., 1999), while connections between examples within a sequence of examples refer to connection on a micro-level (Venkat & Naidoo, 2012).

Towards the aim of this thesis

Building on previous research, the part-whole relation is seen as critical for children's development of arithmetic skills (e.g. Neuman, 1987). There is also an acknowledgement in the early mathematical field that number triples (part/part/whole) provide a foundation for learners to solve additive relation problems with one missing value in a flexible way (e.g. Baroody, 2016; Baroody & Purpura, 2017). In mathematics, structure is considered important in the development of children's arithmetic skills and therefore also important to introduce in teaching. In this thesis, structure is associated with mathematical relations between elements (Venkat et al., 2019), in terms of both emergent structure and mathematical structure, as the relation between the parts and the whole is essential, as are representations in which the whole and the parts are possible to discern simultaneously, especially concrete representations in which the organization of items/sets of items enabling items to be seen without counting them as single units. Further, in this thesis, the examples and how they are linked together might generate mathematical structure by emphasizing relations, mathematical principles, and ideas. In line with a structural approach to teaching and learning additive relations, this thesis also builds on Davydov's theoretical ideas (Davydov, 1982; Schmittau, 2004) on the relational reasoning of numbers and on Neuman's (1987) findings concerning differences in young students' performance on early addition and subtraction problems.

It is mostly the learning perspective of part-whole relations of numbers that has been studied. Few studies in early childhood and the early primary years have investigated "part-whole relation teaching" grounded in a structural approach, and this is especially the case at a more detailed level. Additionally, there are few intervention studies that clearly describe how the activities are designed and enacted in practice, and the theoretical assumption on which they build their interventions. This thesis seeks to address these deficits, studying the part-whole structure of additive relations from both a teaching and a learning perspective, using principles from the variation theory of learning (Marton, 2015).

Furthermore, connections in mathematics instruction have been widely examined. It has been described on a more general level that connections seem to provide better learning opportunities. However, the literature also describes instruction with a lack of connections (e.g. Venkat & Naidoo, 2012). This led to my interest in further exploring how connections that emphasize relations and mathematical ideas and principles are made in teaching, on a micro-level within my area of interest, additive part-whole relations, and thereby add to previous research the nature of making connections in early arithmetic teaching. In conclusion, in this thesis I aim to explore what can constitute teaching and learning additive part-whole relations grounded in a structural approach among young learners, with support from previous research on additive relations and using tools and terminology from variation theory.

3 Aim and framing question

The overall aim of this thesis is to deepen the understanding of the teaching and learning of additive part-whole relations, grounded in a structural approach. Variation theoretical ideas are used to expand the knowledge of part-whole teaching and the learning opportunities offered.

The framing question is:

- What can constitute a structural approach to teaching additive part-whole relations that facilitates learning among young learners?

The thesis is framed within the context of two intervention studies. The participated children were four-to-eight-year-olds. All four articles are related to the overall aim of the thesis. Each of the four articles has its own specific aims and research questions (See pp. 99–101 for short summaries). The first two examine teachers' handling of additive relation problems in a South African context. In Article I, a coding framework was developed with the purpose of identifying and describing fine-grained differences in the teaching of additive relations in early primary grades. In Article II, changes in teaching the same topic over time were studied, using an expanded version of the same framework. The other two articles are connected to the Swedish FASETT project. In Article III, I studied how the same part-whole activity was taught differently. An examination of how dimensions of variation were opened up in teachers' enactment of the activity revealed differences in the children's opportunities to learn about number relations. In Article IV, the aim was to examine the relation between what was taught and what was learned. Analyses were conducted on how the children's ways of experiencing numbers changed after participating in the intervention, and how the ideas in the program were associated with the development of the children's arithmetic skills. However, in order to synthesize the results of the four articles and describe differences in teaching additive relations from a structural approach, with key tenets from variation theory (Marton, 2015), a re-analysis of the data from the four articles has been conducted.

4 Theoretical basis

As the aim of this thesis is to deepen the understanding of the teaching and learning of additive part-whole relations, grounded in a structural approach, the concern is to explore how the mathematical content (additive part-whole relations) was handled rather than studying other aspects of teaching. For this reason, variation theory was considered appropriate, since according to this theory learning is always related to *something* that is to be learned – the object of learning (Marton, 2015; Marton & Tsui, 2004; Runesson, 2005). Furthermore, variation theory asserts that learning means discerning aspects of the object of learning that have not been previously discerned, and that these aspects can only be discerned if they are experienced as dimensions of variation (Marton & Tsui, 2004; Runesson, 2006). These theoretical principles have enabled me to study "part-whole relation teaching" with a focus on how the teachers handled this content in terms of relations within and between examples, relations between representations and how mathematical principles and ideas associated with the structural approach were brought to the fore in the teaching. Variation theory also offers concepts and tools for analyzing and giving detailed descriptions of the teaching and the learning opportunities offered.

Ways of experiencing

Variation theory has its roots in a research approach called phenomenography (Bowden & Marton, 1998; Marton & Booth, 1997). This research approach was developed in the 70s and 80s by a group of researchers at the University of Gothenburg. In phenomenographic research, the interest is on qualitatively different ways in which people experience the same thing, taking a second-order perspective (Marton, 1981). This means that the researcher should try to grasp the phenomenon through another person's eyes (Marton & Booth, 1997). Phenomenography builds on a non-dualistic ontology, meaning that the person and the world are not separated from each other. Experience is described "…as an internal relationship between person and world" (Marton & Booth, 1997, p. 122). This suggests that a way of experiencing a phenomenon is constituted in the relation between the person and the experienced phenomenon. When a person draws attention to this "something", it takes on meaning for that person. This meaning arises precisely in the meeting, and is an internal relation between the subject (the person who experiences) and the object (what is experienced) Another person may draw attention to the same phenomenon in another way, at which point it takes on a different meaning for that person (Marton & Booth, 1997). An example of different people's meaning of the same phenomenon arose in an interview study with nine-year-olds (Ekdahl, 2012). Each student was asked to solve and reason about various number sequence problems, for instance 1, 4, 8, _, _, _. Some students focused their attention on familiar relations between some of the numbers within the sequences, for instance talking about "4-jumps", "plus 4" or "doubles". Other students focused on what was between the numbers in the sequence. There were also some students who focused their attention on the whole sequence, looking at several relations between numbers and the numbers' relation to the whole sequence. Additionally, some students reasoned about the regularity of the sequence and held that by knowing this "rule" they could identify a number later in the sequence. Hence, according to phenomenography, how people (in this case, 9-year-olds) handle a situation and act on problems, as well as what they focus their attention on, reveals the differences in the ways they experience a certain phenomenon (in this case number sequences), and hence that the same phenomenon can be experienced in various ways.

In a series of empirical studies, the phenomenographic research group studied learners' ways of experiencing different phenomena, a situation, or problems within and outside school (See for instance Bowden & Marton, 1998). Generally, these studies showed that there were a limited number of different ways of experiencing a specific phenomenon, that some ways of experiencing were more comprehensive than others, and that the different ways of experiencing most often had a logical relation to each other.

Later, Marton and Booth (1997) re-analyzed numerous phenomenographic studies. In addition to describing the various ways of experiencing a specific phenomenon, they focused their analysis on the nature of each way of experiencing in terms of aspects discerned and what constituted the differences *between* the various ways of experiencing the same phenomenon, noting that some aspects are critical for discerning something specific. For instance, in the re-analysis of Neuman's (1987) results, different ways of experiencing numbers among seven-year-olds were described in terms of which aspects of numbers and number relations were focused on simultaneously in each way of experiencing. In the most comprehensive way of experiencing numbers and number relations, the ordinal (each number refers to a place in an order) and cardinal aspects (each number refers to a certain group of items) of numbers as well as the numbers' part-whole relations were discerned simultaneously. In contrast, in another way of experiencing only the cardinal aspect was discerned, and not the ordinality or part-whole relations (a less developed way of experiencing)[13]. Hence, the nature of different ways of

[13] Marton (2015) makes a further theorization of Neuman's results.

experiencing a phenomenon, in terms of aspects being discerned (or not) and coming to the foreground simultaneously, makes it possible to also describe differences between different ways of experiencing the same object of learning (Runesson, 2006). Pang (2003) describes the shift from the phenomenographic methodological research approach to a theory of learning, variation theory, as a move "… from questions about how to describe different ways of experiencing something to questions concerning what is the nature of the different ways of experiencing something described" (p. 146). Further, it was argued that the development towards variation theory could have implications on how to make learning possible, by relating teaching and learning to each other (Marton & Booth, 1997; Marton, 2015; Runesson, 1999).

Variation theory – a theory of learning

From a variation theory perspective, there is always something to be learned: a situation, content, a phenomenon, a skill or a capability, often called the *object of learning* (Runesson, 2005). How a person experiences or "sees" an object of learning depends on what aspects are discerned and are in the person's awareness, how the aspects are related to each other, and whether they are discerned *simultaneously* (Marton & Booth, 1997; Marton & Tsui, 2004). Thus, learning is experiencing a specific *object of learning* in a new, different way. In order to be able to experience in this "new way", one must discern certain aspects of the object of learning. Some aspects are necessary to discern while others are not, depending on what is to be learned and what aspects the learner has already discerned. The term *critical aspects* is used for aspects that are seen as critical for experiencing an object of learning in a certain way (e.g. Marton & Tsui, 2004). The necessary aspects for experiencing an object could presumably be the critical aspects; however, as the critical aspects are related to the individual learner, the same ones will not be critical for all learners.

As mentioned earlier, according to the variation theory, learning is due to *seeing* the object of learning in a new way by discerning certain aspects that have not previously been discerned, and discerning them at the same time. Variation theory is influenced by the theory of perceptual learning as differentiation (Gibson & Gibson, 1955). Furthermore, Marton (2015) describes learning as differentiation: when someone discerns relevant features and details of the surrounding world, they will experience it in an increasingly differentiated way. When someone learns something, an initially vague (undifferentiated) perception becomes increasingly differentiated, both within the actual learning situation and in relation to earlier perceptions (Marton & Pang, 2006). Expressed in another way, "…learning and development proceeds from the undivided wholes to more and more differentiated and integrated wholes" (Marton, 2015, p. 37). Thus, learning to see something means being able to make more differentiations and discern nuances of that "something" that is to be learned, thereby discerning how aspects are related to each other and the whole.

Variation a necessary condition for learning

Based on variation theoretical assumptions, learning emerges from discerning differences and similarities, starting with *differences*. Marton (2015) addresses that this experience of differences is a necessary condition for learning something specific. We learn by experiencing how instances vary: "New meanings are appropriated through contrast or differences" (p. 64). As stated before, when a person has not yet discerned an aspect which is necessary to discern in order to understand something specific, according to the theory it can only be discerned if it is experienced as a dimension of variation. "A discerned aspect or a feature of an experienced object is thus discerned as a dimension of variation. So what is discerned are actually dimensions of variation" (Runesson, 2006, p. 402). Consequently, if no variation of the aspect is provided, it cannot be discerned. For instance, it would not be possible to experience "height" or what a tall person looks like if one had not met people of different heights. To be able to discern the meaning of height, one has to experience a variation of heights; the aspect of height is thereby experienced as a dimension of variation. Runesson (2006) illustrates how dimensions of variation can be opened up in a learning situation by the learner herself. An analysis was done on an experimental situation with a girl exploring how her body's movements were depicted as a graph on a monitor screen. When the girl, for instance, noticed that her position affected what happened to the graph on the screen, she changed positions (e.g. moving to standing still), thereby herself opening up the aspect "position" as a dimension of variation.

An often-used illustration of the significance of variation is the example of color (Lo, 2014; Marton, 2015; Marton & Tsui, 2004). If a child is to experience what the color red is, it is necessary to have the experience of other colors as well (a variation of colors). Children who already know what distinguishes red from other colors have previously experienced other colors, and have thus experienced color as a dimension of variation.

Elucidating from a teaching perspective, if we want children to learn something specific, their attention should be drawn to what is to be learned by foregrounding an aspect and opening it up as a dimension of variation. So, depending on what is opened up as a dimension of variation, different aspects are made possible to learn within an episode of teaching. For example, if we want to draw the children's attention to the part-whole relation and to how a specific whole number (e.g. 7) can be decomposed into two parts[14], we must offer different splits (e.g. 4/3; 5/2; 6/1) of that whole number. Thereby, a dimension of variation for "part-whole relations" is opened up and different part-whole relations are possible to discern simultaneously. This means that something *varies* within the aspect focused on while something else is kept constant (*invariant*) (Runesson, 2006; Marton, 2015). Since what varies against a stable background will most likely be discerned, not everything can vary at the same time; something needs to be kept invariant. In this example it is the sizes of the parts that are to be varied, while the whole number (7) is to be invariant.

Connections – drawing attention to relations in mathematics

As stated earlier, from a variation theory perspective, what is afforded to be learned is explained by what dimensions of variation are opened up in the learning situation (Marton , 2015; Marton & Tsui, 2004). How the children experience the object of learning in a learning situation depends on what they have already experienced and what aspects are opened up as dimensions of variation in the interaction between teacher and students. Marton (2015) asserts that the teacher needs to handle the object of learning in a powerful way by offering variation, pointing out similarities and differences between tasks and examples. He also states, in more general terms, that the teacher should also draw attention to and explain relations associated with the aspect in focus. In a variation theoretical analysis of teaching division in Grade 8, Kullberg, Runesson and Mårtensson (2014) describe more specifically what this kind of teaching could look like. They found differences in how the object of learning was handled, depending on how the teacher drew the students' attention to the object of learning by comparing certain examples, pointing out specific numbers within the example space.

More specifically, selected tasks, examples and sequences of examples, even with sufficient variation, are to be seen as "raw material". There is also a need for pedagogical support to foreground the relations or mathematical ideas embedded in the tasks (Watson and Chick, 2011). Similar to Marton (2015), Watson and Mason and their colleagues (Mason et al., 2009; Watson & Mason, 2006a) argue that the teacher needs to direct the learners' attention to this relation or idea by, for example, making connections between terms in different equations or asking questions that explicitly support the discernment of something the teacher wants the learners to discern (Mason

[14] See Ekdahl et al. (2016).

& Watson, 2006a). This could entail, for instance, an exercise with carefully planned variation and invariance, such as:

$17 - 9 = \_$ $\quad$ $27 - 9 = \_$ $\quad$ $37 - 9 = \_$ (Ibid., p. 106)

Some students may notice the structure and generality, whereas others may see the exercise as three isolated subtraction problems. Mason & Watson (ibid.) suggest that if the teacher in the exercise above draws the learners' attention to what is similar and different between the subtraction problems, the concept of "nineness" is likely to be discerned. Venkat and Askew (2018) follow the argument that students' attention needs to be drawn to mathematical ideas and relations in instruction; especially in the early primary years, when it is likely that young learners have not been sufficiently introduced to looking at patterns and relations. The variation theoretical argument regarding *connections*, expressed and discussed above, as *drawing attention to aspects and various relations in mathematics*, corresponds with how "connection" is described and used in this thesis.

Connections – the role of gestures

Given the importance of *making connections*, the role of gestures in mathematics teaching and learning has been studied within different research approaches, for instance within a semiotic approach (e.g. Arzarello, Paola, Robutti & Sabena, 2009; Duval, 2006; Radford, 2009). Hostetter's (2011) meta-analysis of the effects of gestures in instruction (not just mathematics) points out that students younger than twelve benefit more in their understanding when gestures are used as a complement to talk, compared to older students and adults. These results correspond with those in Flevares and Perry's study, (2001) which shows that teachers' linking actions supported first graders in learning about place value. A combination of teacher speech and/or gestures in instruction, especially when the mathematical topic is new, also seems to enhance learning (Alibali et al., 2013; Richland, 2015).

In early childhood, gestures used by teachers and by children themselves are seen as an important source of developing the children's mathematical thinking. For instance, in a case study on geometry it was found that a child spontaneously used gestures that helped her explain different space and shape aspects in a construction. The child was also helped by observing the teacher's gestures in the interaction (Elia, Gagatsis & van den Heuvel-Panhuizen, 2014). Instead of taking departure in a semiotic theoretical perspective, the way gesture is used in this thesis emerges from McNeill's

(1992) *deictic gestures*[15], in which pointing or movement is used to indicate a link. Gestures can be expressed through arm, hand, or finger movements[16], in order to draw attention to relations, ideas, and mathematical principles.

Hence, connection in teaching has been studied from different perspectives and different levels (See also pp. 30–31 in the literature section) and has been thoroughly argued for from a variation theoretical perspective on teaching and opportunities for learning (Watson & Mason, 2006a). A narrower focus on making connections leads to the interest in studying teachers' linking actions (gestures and speech) on a micro-level, and which relations and mathematical ideas and principles can be made visible for learners in the four- to eight-year age group. Variation theoretical principles have been used for analyzing teaching in different ways (e.g. Huang, Zhang, Chang & Kimmins, 2018; Kullberg, 2010). Using the concept of dimensions of variation as an analytical tool makes it possible to examine and compare learning opportunities offered in teaching (Häggström, 2008; Maunula, 2018; Runesson, 1999). In this thesis, especially in the re-analysis, I do not solely analyze what dimensions of variation are opened up; I also examine *how* they are opened up in the teacher's enactment – thus, how the teachers direct the learners' attention to the aspect opened up as a dimension of variation. In this analysis, *enactment* comprises not only the teacher's handling of the mathematical content but also the teacher's linking actions (both gestures and speech) that facilitate for learning.

[15] Alibali et al. (2013) and Elia et al. (2014) also build on McNeill's definitions of gestures.

[16] For examples of linking actions, see Ekdahl et al. (2016, p. 301) and Ekdahl et al. (2018, p. 4).

5 Methods

The methods in this thesis are associated with the overall aim and framing question, the research questions in the four articles, and the re-analysis. The focal studies are framed within an *educational design research* approach (Gravemeijer & Cobb, 2013), conducted in collaboration with teachers with an iterative design and driven by theory. The focal studies are based on variation theory, and some variation theoretical principles were used in designing activities and as analytical tools in all analyses. Furthermore, the empirical data for analyzing different part-whole teaching emerge mainly from video-recorded teaching episodes (whole-class and small-group teaching) and learner interviews. Detailed theoretical analyses of various data sets enabled me to understand and describe differences in teaching part-whole additive relations within a structural approach. Another central issue concerning methods relates to ethical considerations, for instance issues related to interviewing young children.

The contexts of the projects

This thesis is connected to two different research projects: the South African Wits Maths Connect Primary project and the Swedish FASETT project. In Wits Maths Connect my thesis is linked to one focal study, a small-scale intervention study. Within the FASETT project, two focal studies are associated with this thesis.

Wits Maths Connect project

In the overall Wits Maths Connect Primary project, a research group worked with ten government primary schools in one district in Johannesburg, developing, trialing, and researching interventions that sought to improve mathematics teaching and learning in the Foundation Phase (Grades R-3). At five of those schools English is the language of teaching, and at the other five there is a range of African languages (Zulu, Sepedi, Tsonga or Xhosa) as the language of teaching. The project aimed to add to a research foundation for intervening for mathematical development in a South African context, as well as to contribute more broadly to the international field of primary mathematics education. A Swedish team from Jönköping University, including myself, became involved in this project in 2013. In collaboration with the South African research team, we developed and conducted a small-scale intervention at one of the English-medium schools. In the context of South African primary mathematics, studies point to lessons as collections of isolated facts and procedures, particularly in the case of early work in addition and subtraction, underpinned by the use of naïve counting methods through the primary years (Hoadley, 2007; Venkat, 2013). The research team implemented a structural approach to teaching additive relations, which contrasts with the counting operational approach foregrounded in the South African curriculum (Department for Basic Education, 2011)[17] . Articles I and II are based on the data from the focal study carried out in one of the English-medium schools in Wits Maths Connect.

The FASETT project

The Swedish *FASETT* project (The ability to discern the first ten numbers as a necessary ground for arithmetic skills) is the second research project connected to this thesis (Articles III and IV). Its overall aim is to generate knowledge about preschoolers' ability to learn addition and subtraction, building on Neuman's findings (1987; 2013) and Marton's (2015) theoretical and methodological basis rather than those that have dominated the research field in whole-number arithmetic (counting-operational approach). In this intervention, activities based on theoretical principles (Neuman, 1987; Marton, 2015) were designed collectively and implemented by teachers in their preschool groups in order to determine whether and how the tasks could contribute to preschool children's development of arithmetic skills. Similar to the South African project, the Swedish preschool teachers were unfamiliar with the structured approach to early numbers. Two focal studies within the FASETT project are linked to this thesis. Article III is based on data from the preschool teachers' video documentation of one specific part-whole activity. In Article IV, data from eight learners' interviews before and after the program, as well as video documentation of different activities, formed the basis for analyzing what was taught and learned in the intervention program.

Designs of the focal studies within the two projects

Even though the two projects (Math Wits Connect and FASETT) were conducted in different countries and cultures, their designs have a great deal in common. For instance, both involve educational design research. Building on research in the design research field, van den Akker, Gravemeijer, McKenney & Nieveen (2006, p. 5) summarized what characterizes design research: *Interventionist* – the research is aimed at designing an intervention in the real world; *Iterative* – the research incorporates cycles of analysis, design and development, evolutions, and revisions; *Process-oriented* – (…) the focus is on understanding and improving interventions; *Utility-oriented* – the merit

[17] *Curriculum and Assessment Policy Statement (CAPS): Foundation Phase Mathematics Grades R-3.*

of a design is measured, in part, by its practicality for users in real contexts; and *Theory-oriented* – the design is (at least partly) based upon theoretical propositions, and field testing of the design contributes to theory building. In addition, Plomp (2013) argues that the *involvement of practitioners* will increase the relevance of the intervention for practice.

In the focal studies, the purpose was to understand and develop the teaching and learning of additive relations from a structural approach. The researchers introduced theoretical ideas linked to this structural approach that the participating teachers, working in cooperation with the research team, implemented in their practice. Both projects were conducted within the ordinary classroom teaching or in the teachers' preschool practice, giving the studies ecological value (Cobb, Confrey, di Sessa, Lehrer & Schauble, 2003). The iteration process was a salient feature. The design consists of cycles of analyzing, planning, evaluating, and revising the conjecture (Gravemeijer & Cobb, 2013), taking into account learners' responses and teachers' reflections[18]. The researchers were involved in the interventions and kept close to the practice (observations, regular meetings with teachers, analysis of video observations) during the whole process.

To a certain extent, differences can be identified between the designs in the two contexts. In the focal study in Wits Maths Connect, the research team worked intensively for three-week periods (February 2013, October 2013 and February 2014). Each cycle was comprised of three lessons taught by each of three Grade 3 teachers, with all lessons observed and video-recorded by the research team. A written test was administered in the classes before and after each cycle, and students' worksheets following each section of teaching were analyzed. In planning meetings, video-recorded episodes from the lessons, teachers' reflections, and worksheet results formed the basis for the discussion and planning. The design was repeated in all three cycles.

The design of the FASETT project differed in some ways from that of the South African study. Besides having a collective process-oriented approach, the study used a combination of a design research (Cobb et al., 2003) and a quasi-experimental design (Salvin, 2010), including a target group and a control group. Both groups had regular meetings with the research team during a period of two semesters in 2015–2016 (target group 12 meetings; control group 6 meetings). In the target group meetings, the researchers guided the discussion towards a structural approach to additive relations

[18] cf. the iterative design in the learning study model (e.g. Pang & Marton, 2003).

and children's different ways of experiencing numbers. Certain activities[19] based on these ideas were implemented (See Table 1). Thereafter, the teachers in the target group enacted the collectively planned activities in their preschool groups. As a starting point for the teachers' reflections on their instruction and the children's reactions, short episodes of the recorded videos were selected for discussion. In the planning meetings, reflections around the theoretical and mathematical ideas included in the activities were in focus. As described in Articles III and IV, the meetings usually resulted in decisions about refining the same activities and reenacting them in their preschools, producing an iterative and theory-driven approach. This contrasted with the control group, for which documentation of the preschoolers' incorporated numbers and counting formed the basis of the meetings with the researchers[20]. The discussions were in line with the preschool teachers' national curriculum praxis[21].

Designed activities associated with the studies

All activities used in the focal studies were designed to encourage a structured approach to additive relations. The part-whole relation was to be foregrounded in the instruction as well as in the work with representations. Therefore, the activities and examples were designed and selected in a way that made it possible to emphasize in the teaching the part-whole relations of numbers as well as mathematical ideas and principles associated with "part-whole teaching". Also, the results from written pre-tests, worksheets, and learner interviews in the Wits Maths Connect focal study as well as the individual learner interviews in the FASETT project were used as a starting point for the design of the activities. For instance, in the written pre-test we noticed that missing number problems were challenging for the students, especially if the missing number was a subtrahend or start number. Therefore, this aspect was discussed in the planning meetings with the South African teachers. In the Swedish project, a few children used their fingers as a tool in the pre-interviews (Björklund et al., 2018), this was discussed in the planning meetings with the preschool teachers. Also, the teachers' reflections on their teaching during the project and the discussions about the children's learning opportunities in the collaborative meetings (between teachers and researchers) were considered in the refining process and instruction of activities.

[19] No distinction is made between tasks and activities. Most often, the term *activity* is used in the articles.

[20] Data from the control group are not analyzed in this thesis. I was not involved in the meetings with the control group.

[21] Swedish National Agency for Education (2010).

In the process of designing activities, principles from variation theory were used (Watson & Mason, 2006a; 2006b; Mason et al., 2009). The aspects in focus that were to be experienced in the tasks were planned carefully, so that variation allowed the learners to experience variation of the aspect in focus. As an illustration, in the context problems (See p. 48 for a full description of the activity), the examples of number stories that were planned kept the whole number of animals invariant (e.g. 8) and the animal in the story was the same (e.g. bears), but the number of bears that ran away or had no place to sleep in a sequence of examples differed (8 – 3; 8 – 4; 8 – 5; 8 – 6). Another illustration is selected from the design of one of the missing number problems in Grade 3. The learners were to discern that the missing number could be a missing subtrahend, addend or start number, and the position of the missing number could be placed differently; therefore, two number sentences (11 - _ = 5 and 5 = 11 - _) were written on the board. The numbers were related to one specific part-whole relation (11/5/6 triple). The whole numbers and the operation were the same in this sequence, while the position of the missing subtrahend varied. In another teaching sequence, two missing number problems were presented: one with a missing addend and one with a missing subtrahend (11 = 6 + _ and 11 - _= 6). The whole number (11) and one part (6) were the same, whereas the missing number varied (missing subtrahend and missing addend). Table 1 presents a summary of the activities and problems from both projects referred to in this thesis.

Table 1: Overview of activities and problems used in this thesis[22]

Wits Maths Connect
Partitioning of numbers
Splitting a whole into two parts, different versions: seven monkeys into two trees, seven/nine balls into two bags. Different representations are used: concrete, triad diagram, table, number sentence. The teachers encourage learners to find all combinations and strive for completeness.
Missing number problems
Missing subtrahend, addend or start number, a specific number triple combination is discussed. Different representations are used: triad diagram, double bar, number sentence.
FASETT
Statement game
Before throwing a die, the children decide what number they think it will show. When they have agreed on a number, they show this number with their fingers using both hands. The teachers encourage them to show the number in different ways.
Snake game: 5-snake and 10-snake
Two resources are used: a string with five beads of the same color or one with ten beads, grouped as five of one color and five of another. The teachers encourage the children to represent the whole number (five or ten) with their fingers. Then, some beads are hidden and the children show the number of beads they see on the string. By looking at their unfolded/folded fingers, they are then able to "see" the missing part (number of hidden beads).
Finger patterns
The children identify finger patterns shown by the teacher with a number word. The "undivided five" is emphasized, so numbers >5 are chosen. The teacher asks how many fingers need to be added/taken down to make a different finger pattern (>5), and then follows the children's suggestions. The teacher reverses the task and asks how many are to be added/taken down to give the first number again.
Context problems
Short number stories of different types within the number range 1–10 are presented. The children model the problems on their fingers. A systematic order of the presented problems emphasizes the number relations.

[22] See Ekdahl et al. (2018, p. 4) for a more detailed descriptions of activities and problems used in the South African study.

As can be seen in Table 1, a limited number of activities and problems were implemented. Instead of introducing a large battery of tasks (cf. Dyson, Jordan & Gluttin, 2013), the intention of the studies was to "enter deeply into the activities and missing number problems" by revising and refining the activities and, moreover, by focusing on the mathematical principles that were possible to bring to the fore in the part-whole teaching within a structural approach and how the learners experienced the activities being conducted (iterative process). Also, the teachers usually wanted to repeat the activities after having reflected on the videos and their own teaching with colleagues and researchers. Occasionally, teachers felt that a specific activity was too great a challenge for the children. Instead of following the setup for an activity, which had been planned together, some teachers simplified it. For instance, in the Snake game (See Table 1), instead of encouraging the children to identify the *hidden* part by looking at their structured finger patterns (unfolded/folded fingers), the teacher repeatedly asked the children to simply show as many fingers as the number of visible beads on the snake. On the South African side, a teacher felt that the partitioning activity (seven monkeys playing in two trees) was too elementary for her class, and instead chose the whole value 26 to decompose into two parts (See Article I, p. 303).

The chosen representations (concrete, triad diagram, bar model, table, number sentence, bead string, and fingers) made it possible to discern the whole and its parts simultaneously, promoting structure. Payne and Rathmell (1975) and Baroody (1999) argue for the reinforcement of the part-whole relations by underlining and verbally talking about *parts*, and their relations to the *whole* and *other parts* in instruction with children. Therefore, the teachers were encouraged to use "parts and whole" in the instruction of the partitioning number activity and various missing number problems (Wits Maths Connect project). In the FASETT project, the instruction emphasized the connection between decomposing a whole into two parts and composing the same parts into the original whole (without verbally talking about "parts and whole"). However, in both projects, relations within numbers were to be emphasized in the instruction.

Participants, samples, and data collection

Data for this thesis were sampled from three Grade 3 classes of 35+ students each at a suburban government English-medium primary school in South Africa during 2013–2014, and from nine Swedish preschool teachers and their 65 preschoolers at five Swedish preschool units during 2015–2017. The total data collection from the two projects was extensive, which is preferable in educational design research (Brown, 1992; Cobb et al., 2003). Data included individual learner and teacher interviews, video-recorded lessons, video-recorded small-group teaching in preschool, audio-recording from planning meetings in both projects, pre- and post-tests and worksheets from three South African intervention cycles, and the preschool teachers' logbooks of conducted activities. However, this thesis presents a selection of the total data collection. Table 2 provides an overview of the contexts, participant samples, and data types used in the **four articles**.

Table 2: Overview of contexts, participant samples, and data types used in the thesis

	Article I	**Article II**	**Article III**	**Article IV**
Context	South Africa	South Africa	Sweden	Sweden
School Grade	Primary Grade 3	Primary Grade 3	Preschool	Preschool
Year	2013	2013-2014	2016	2015-2016
Participant samples				
N Teachers	3	3	9	3
N Classes/groups and units	3 classes 1 school unit	3 classes 1 school unit	8 preschool groups 5 units	1 preschool group 1 unit
N Children	132	132 + 110	65	8
Age of children	7-8 years	7-8 years	5 years	4-6 years
Data types				
Video-recorded lessons/ observations	6 lessons whole-class teaching (+40 students)	18 lessons whole-class teaching (+35 students)	67 observations small group teaching (2-8 children)	23 observations small group teaching (2-8 children)
Total time	120 minutes	440 minutes	450 minutes	210 minutes
Video-recorded task-based interviews	-	-	-	16 individual learner interviews (pre & post)
Total time	-	-	-	114 minutes
Written documentation	396 Worksheets	-	-	Teacher's Logbook

As can be seen in the overview (Table 2), three Grade 3 teachers and 11 preschool teachers[23] participated in the four studies. In the South African intervention, Grade 2 teachers often joined the planning meetings and conducted the planned lessons in their own classrooms. However, it was three Grade 3 teachers who regularly attended the meetings during all three periods and taught all the planned lessons who were sampled for analysis in the first and second articles. Similarly, in the FASETT project, it often happened that more than nine preschool teachers participated in the planning meetings, but their video-recorded activities were not included in the data set in Article III. In addition to the 132 participation children in Article I another 110 children participated in Article II. In the focal studies in the FASETT project, a total of 65 children participated in the video-recorded Snake game activity. The eight children in the target group (Article IV) were part of the group of 65 children in Article III.

In Table 2, it can be seen that the data sampled for analysis in this thesis mainly emerged from video recordings of whole-class teaching, small-group teaching, and learner interviews. Video recording provides a rich set of material that can capture details of the teaching in natural settings (Heath, Hindmarsh & Luff, 2010). Using video recordings enables researchers to record in real time, and they do not need to be present (ibid.). Recordings also show talk, gestures, and visible resources, making it possible to observe on a highly detailed level. Since the aim was to study part-whole teaching on a micro-level and the learning opportunities offered in classrooms and in preschool settings, video recording was a relevant method. Video-recorded data enabled me to make scrutinize in detail the teachers' handling of the content and the teacher-child interaction, and to return to recorded sequences several times in the analysis process. The video-recordings also facilitated sharing with other researchers and with a wider public (ibid.). In the intervention processes as well as the analysis processes connected to this thesis, video recordings were continuously shared within the research group. During the process, videos were analyzed and discussed in the research group in order to carry the projects forward. Also, videos formed the basis for discussions at the collaborative planning meetings (between teachers and researchers) in both projects. Recorded data were also shared with researchers outside the research group as well as at conferences, communicating ongoing analyses and results (Ekdahl & Runesson, 2015; Ekdahl & Björklund, 2017; Ekdahl et al., 2019).

[23] In Article IV, one of the preschool teachers was the same as in the sample in Article III. However, in Article IV all video-recorded activities conducted at the target preschool were sampled and the teaching of another two preschool teachers was analyzed.

A total of 1,220 minutes of video-recorded teaching and 114 minutes of learner interviews have been analyzed. The data were collected over a period of four years. The key data sources for Article I are three video-recorded sections of teaching from the first and second lessons in the first cycle in February 2013. The whole-class instruction related to partitioning taught by the three teachers was sampled for analysis of the part-whole teaching (See Table 2). The data sources for Article II consisted of nine video-recorded lessons (three from each teacher) from the first cycle in 2013 and nine lessons from the third cycle (February 2014) (*N*=18 lessons, whole-class teaching). Thus, in 2014 the same three Grade 3 teachers were teaching three new classes on the same topic. The lessons were recorded by me or another member of the research team. Since the interest was mainly in the teachers' handling of the content, the camera was set up at the back of the classroom, focused towards the teacher and the board. It had a fixed position, but it was possible to zoom and follow the teacher's movement. As backup, audio recording with a Dictaphone was used. I myself observed all the lessons. Only the recorded whole-class teaching was sampled for analysis; the introduction of the lesson as well as the students' individual work were excluded. The small-group video recordings in FASETT were handled by the preschool teachers themselves. Most often an iPad was used for recording, placed on a table in front of the group of children and their teacher, while sometimes a colleague or a child did the recording. The preschool teachers continuously uploaded their videos onto a server to which the research group had access. I myself did not observe any of these episodes in real life. In order to determine the different ways in which the teachers enacted the same part-whole activity, 67 uploaded recorded videos from the Snake game, conducted over a three-month period, were sampled for analysis in the third paper (Article III). In Article IV, 23 uploaded videos of four activities from one preschool unit were analyzed.

Individual task-based learner interviews were conducted in the FASETT project[24]. In the fourth article, eight children from a target preschool were sampled for analysis. This group was chosen due to their low results on the pre-interview and the substantial data collection of recorded videos and logbook notes from this site. From the video-recorded learner interviews (pre and post), eight tasks (See Appendix) were chosen for deeper analysis of how their way of experiencing numbers had developed since their participation in the intervention program. All interviews were conducted by me or my colleague at the children's preschools, in a room close to the indoor play area. A video camera was set up beside the table where the child and inter-

[24] In the FASETT project, a total of 103 children (target and control group) were interviewed on three occasions (309 interviews in total).

viewer sat side by side. The interviews focused on how the children counted and solved oral numerical and word problems. After having conducted pilot interviews, we agreed that if the child failed twice at the guessing game task, we would only present two examples. Further, if the child seemed to be troubled at the end of the interview, the last two questions would not be asked; alternatively, we could use a lower number range. Starting the interview, I explained to the children the purpose of the interview in language they understood. Also, I told them that the interest was in hearing their reasoning about numbers, and they were furthermore encouraged to use their fingers. Sometimes a child wanted to play the guessing game with me and s/he was allowed to do this. In some cases, after having finished the interview, children wanted to watch themselves in the camera, which they were allowed to do.

Also, learning interviews in the initial phase of a project provide useful information for the design of interventions[25] (Brown, 1992). For a reason similar to that described above, video-recorded interviews (Heath et al., 2010) provided me and the research team with detailed information about the children's verbal utterances and how they used their fingers, counted, and used gestures when handling the tasks.

Analytical tools

Concepts from variation theory (Marton, 2015; Marton & Tsui, 2004; Häggström, 2008) were used as analytical tools in this thesis. To some extent, the same concepts and analytical tools were used across the articles and in the re-analysis, but it also differed in terms of which ones were chosen and how they were used. *Simultaneity* was one concept that formed the categories of the coding framework developed in Articles I and II. Within a structural approach to part-whole relations, the simultaneous presence of parts and whole within examples is a necessary condition for making the relations discernable (Marton & Tsui, 2004). This was also the case with the simultaneous presence of different representations and part-whole examples. The other concept used for analyzing was *connections,* referring to a teacher's linking action (McNeill, 1992), which helped to draw learners' attention to something specific (e.g. Mason et al., 2009). These two concepts formed the criteria for coding the teaching of the same partitioning number activity in three different South African classrooms. In order to further explore other aspects of part-whole relation teaching (missing number problems), and

[25] Individual learner interviews were conducted in the South African project as well. They formed the basis for planned lessons and discussions, but are not used in the thesis.

changes in teaching over time, the same analytical tool (the combination of *simultaneity and connection*) was employed in Article II.

The concept *dimension of variation* was used as an analytical tool for examining and comparing the learning opportunities offered in teachers' teaching. Following the variation theoretical assumptions, the learner will most likely discern an aspect if it is opened up as a dimension of variation (Marton & Pang, 2006); thus, the dimensions of variation open up is what is made possible to learn. In Article III, I identified the aspects that were opened up as dimensions of variation in the teachers' enactment of a part-whole relation activity. This analysis provided me with tools to describe the different *enacted objects of learning* constituted of certain dimensions of variation that were opened up, and thereby the different *learning opportunities* that were offered (Häggström, 2008). In a similar way in Article IV, the concept of dimension of variation was the analytical tool used for analyzing what aspects the teaching (all four activities) afforded the five-year-olds to discern.

Way of experiencing was another analytical tool used in Article IV, for analyzing the children's way of experiencing numbers before and after an intervention program. According to variation theory (Marton and Booth, 1997), how people handle a situation or act on a problem, and what they focus their attention on, reveals how they experience that situation or problem. Also, how something is experienced depends on what aspects are discerned. The nature of each way of experiencing something specific can be described in terms of *discerned aspects*[26]. Therefore, analyzing children's ways of acting on additive relation problems in the interviews made it possible to describe their way of experiencing numbers, in terms of aspects being discerned or not. The teachers' handling of the part-whole activities and emphasizing aspects by opening them up as a dimension of variation made it possible to analyze what was afforded in the teaching. Then, using the concepts *discerned aspects* (the children's way of experiencing) and *afforded aspects* (what the children were offered in the activities) and relating them to each other enabled us to describe how the discerned aspects reflect the afforded aspects. Since teaching and learning are described in commensurable terms (as afforded and experienced aspects, respectively) what is taught and what is learned can be related.

In order to further deepen the understanding of the structural approach to part-whole teaching, in the re-analysis I approached the empirical data in a slightly different way. The analytical tools were refined, resulting in a combination of the concepts *dimension of variation* (from Articles III and IV) and

[26] See also p. 37.

connections (from the coding framework in Articles I and II), in order to identify qualitative differences in the teachers' enactments of certain mathematical ideas and principles, and to describe the *learning opportunities* offered.

Analysis processes – the four articles

Variation theory has offered appropriate concepts and analytical tools in relation to the research questions. Also, all analyses associated with the different data sets have been conducted on a micro-level, in order to understand the teaching and learning of additive part-whole relations grounded in a structural approach. Even though the theoretical principles have been taken as a point of departure, it has been a challenge to approach the data in an appropriate way according to the research questions. Therefore, the analysis methods have been tried out, rejected, revised, and refined.

Development of the coding framework: Articles I and II

The first article is a methodological paper, in which the development of the coding framework is described and applied to three sections of teaching related to partitioning in three South African Grade 3 classrooms. A preliminary analysis of the same data set in 2014 suggested that differences in learning outcomes on worksheets following each section of teaching could be associated with differences in teachers' teaching (Venkat, Ekdahl & Runesson, 2014). In explaining these differences, *connection* was applied as "an indicator" for describing the teaching. Each section of teaching for each teacher was categorized as "weak or strong connections" (in some cases "mostly weak")[27]. Questions were raised according to the distinction between weak, mostly weak, and strong connections. Therefore, in the following analytical process the definition of connections became more distinct and the criteria for coding the teaching were refined. In order to describe the fine-grained difference in the teachers' instruction relating to partitioning problems, the coding framework was developed, considering the nature of simultaneity and connections at three levels: between representations (SCBR), within examples (SCWE), and between examples (SCBE).

One issue that needed to be considered in the coding process was how to demarcate the teaching episodes in smaller units. The definition of the unit of analysis depends on what is to be examined (Herbst & Chazan, 2009). In this case, the purpose was to analyze the teachers' connecting work (linking gestures and talk) of a specific topic (part-whole relations) across representations and examples, looking at every single example presented[28]. There-

[27] See for instance Venkat, Ekdahl & Runesson (2014, p. 343, Table 3).
[28] One example is an instance of a part-whole relation.

fore, the analysis was on the micro-level, with the "micro units" was defined as *segments*. A description of when a segment started, what it included, and when it was considered complete was formulated[29]. Demarcating the teaching episodes into segments enabled the coding of simultaneity and connections. In the same segment, at least one instance was to fulfil the criteria for each SC category (SCBR, SCWE, SCBE).

In Article II, a new process of defining criteria for different SC categories, applicable to missing number problems, started. The expanded coding framework[30] was used for analyzing shifts in part-whole teaching over time. The results of the number and proportion of segments meeting the criteria for each SC category in 2013 and 2014 for partitioning activities and missing number problems were summarized for each teacher, and qualitative changes in teaching related to connections within examples and between examples were illustrated.

Analysis process - Article III

Videos selected for discussion in the planning meetings during the FASETT project had indicated that, even if the activity had been planned collectively and handled several times, it was conducted differently. Therefore, my interest was to go deeper into the data of one specific part-whole activity, the Snake game. The video-recorded observations were analyzed in several steps. First, an overall analysis of 39 observations was made in order to identify how the game was handled. It was demonstrated that how mathematical ideas became visible depended on the teacher's way of handling the activity, described in more general terms (Ekdahl & Björklund, 2017). Thereafter, a more detailed analysis of all video-recorded observations was conducted, focusing on differences and similarities in the teachers' enactment (Ekdahl, 2019). Until this moment, the analysis had focused on teachers' different ways of *handling* the Snake game activity.

But now, a need for theoretical clarification and further distinction between different enactments emerged. As variation theory assumes that an aspect is considered possible for learners to discern if the corresponding dimension of variation is opened up as a dimension of variation (Häggström, 2008), I revisited the data and focused on what was made possible for the five-year-olds to learn (the enacted object of learning). By analyzing all 67 observations of the Snake game activity again, I identified aspects related to number relations that were opened up as dimensions of variation in each observation. Here, I used questions that would help me maintain focus on the en-

[29] For a more detailed definition av *segment*, see Article I (p. 299).

[30] For the criteria for missing number problems, see Article II, (p. 4, Table 2).

acted object of learning. For each dimension of variation identified as being opened, I described in detail what varied and what was kept invariant. Then, for each observation I analyzed which dimension of variation was opened up. Those with the same sets of dimensions of variation were grouped together, constituting one enacted object of learning.

Analyzing teaching and learning - Article IV
The purpose of Article IV was to examine the relation between what was taught and what was learned through an analysis of how aspects of numbers were afforded in the intervention program and how these aspects were reflected in five-year-olds' learning. Three analyses were conducted.

In the first analysis, the focus was on changes in the children's ways of experiencing numbers expressed when they solved different additive relation tasks in the pre- and post-assessment. On a detailed level, the children's ways of acting on each task, both reasoning verbally and their use of their fingers, were analyzed. This resulted in six different ways of experiencing numbers (See Björklund & Runesson Kempe, *in press*). Then, a coding was made of how these ways of experiencing numbers changed from one category to another during the intervention.

The second analysis focused on how the structural approach to part-whole relations of numbers was implemented in the intervention program. The video-recorded observations of the activities were analyzed. Similar to the analysis conducted in Article III, each observation was analyzed in terms of what dimensions of variations were opened up in the teachers' teaching. This analysis made it possible to identify what aspects the children were afforded to experience in the teaching of the four different activities (See also Ekdahl et al., 2019).

The third analysis enabled us to synthesize the two other analyses, by identifying what aspects the children had discerned and what aspects had been afforded in the activities (See also p. 54). This analysis made it possible to identify certain aspects that the children had not discerned before the intervention but did discern after it.

Revisiting the data - a re-analysis
To further explore how mathematical ideas and principles linked to additive part-whole relations can be brought to the fore following a structural approach to teaching, a re-analysis of the data set associated with the articles was conducted. This provided me with a way to further answer the question of what can constitute part-whole relation teaching grounded in a structural

approach, and the learning opportunities that are afforded using principles from the variation theory of learning.

In the re-analysis, the teachers' different enactment was scrutinized. In this re-analysis *enactment* comprises the teachers' handling of the content, but also their linking actions – their gestures and talk (McNeill, 1992) that might facilitate the learning of additive part-whole relations. This analysis involved attention to mathematical ideas and principles associated with a structural approach to additive part-whole relations, which are important for children (aged 4 to 8) in developing arithmetic skills. These ideas and principles have been noted in earlier research within the field of early numbers, and are, to various extents, described in the literature section[31]. Among the mathematical ideas and principles are: *number relations as composite sets*, *decompositions of whole numbers, the commutative principles for addition, the complement principle (the inverse relation between addition and subtraction), equality, and completeness by systematicity*. Teachers have the possibility to bring these ideas and principles to the fore within structural part-whole teaching. Some ideas were possible to emphasize in certain activities in the interventions, whereas others were more suitable to emphasize in other activities. For instance, using finger patterns for structuring number relations as composite sets was an object of attention in FASETT (but not in the focal study in Wits Maths Connect), whereas the decomposition of a whole number into different parts was emphasized in both projects. Moreover, commutativity and the complement principle, and completeness, were possible to emphasize within structured part-whole teaching in all the focal studies.

In order to discern how the same mathematical idea/principle was handled, excerpts from the articles, transcripts, and video observations from the whole data sets associated with the four articles were sampled for analysis. The intention was to find short teaching episodes in which the mathematical ideas/principles were focused on in the teaching. As the activities had been planned cooperatively, it was plausible to find episodes with similar examples in the data set. Moreover, in some case comparisons, teaching of the same activity had been analyzed and described in the articles.

The first analytical step was to identify, in each teaching episode sampled, what dimensions of variation were opened up (or not) that were associated with the mathematical idea (cf. analysis of the Snake game p. 56). In most enactments, more than one dimension of variation was opened up. In certain cases, another dimension of variation that was **not** critical for the discernment of the specific mathematical principle was opened up. For in-

[31] See pp. 20–26.

stance, the aspect *representations of numbers* was opened up as a dimension of variation when the same example was represented in number sentence form and in a triad diagram. This aspect would have been necessary if number relations represented in different representations were to be foregrounded; however, it was not critical for discerning the commutativity principle. It was found that in some of the sampled teaching episodes it should have been possible to open up the dimension of variation associated with the mathematical principle, but this was not the case. For instance, if examples presented on the board were erased, they were not possible to simultaneously discern and it was not made clear to the learners how an "erased example" was related to the next example produced.

In the analytical process, it was also found that a certain dimension of variation associated with the mathematical idea/principle can be opened more or less explicitly in the enactment. Having repeatedly analyzed the teaching episodes of the same idea, I noticed differences in whether and how the teacher directed the children's attention to the mathematical idea through their actions. For instance, in one enactment of the Snake game activity associated with *number relations as composite sets*, I saw that the teacher explicitly pointed to the children's finger patterns and compared their different ways of showing them (folded/unfolded fingers), whereas in another enactment different finger patterns of the same number relation were possible to discern, **but were not** specially brought to the fore by the teacher. However, in both these enactments the part-whole relation as composite sets of units was opened up as a dimension of variation.

Therefore, being able to distinguish between a more or less explicit enactment, the notation *connections* was used. The definition of connection corresponds with the definition in the coding framework in Articles I and II[32]. Connections consist of linking actions in a teacher's speech and/or gestures emphasizing, in this analysis, the dimension of variation opened up within relations of part-whole and the mathematical idea in the context of structured additive relation teaching. Since connections were not a component of the analysis process in Articles III and IV, I revisited the video-recorded observations in certain cases. The teacher's connecting action was supposed to explicitly direct the attention to focused ideas. Several comparisons of the selected excerpts within the same mathematical idea or principle were made. Each enactment was described based on dimensions of variation that were opened up (or not), connections made by the teacher (or not), and what learning opportunities were offered within each of them.

[32] See for instance Article I (p. 300) and Article II (p. 4).

This micro-level analysis enabled me to identify and describe subtle differences in the teaching of the same mathematical idea or principle that was made possible to bring to the fore within a specific activity. With this in view, different enactments of the same idea were described. The excerpts were condensed and focused mainly on the teacher's instruction. Three mathematical ideas/principles – *number relations as composite sets, the commutative principle, and completeness by systematicity* – were analyzed in the same way.

Ethical considerations

Ethical approval for the Wits Maths Connect project was granted by the University of the Witwatersrand within the Discussing Lessons initiative in the Wits Maths Connect Primary project. The data collection and research procedures were consistent with the principles of research ethics published in South Africa. The principal and the three teachers at the primary school had indicated their willingness, and voluntarily gave informed consent to participate in the planning and development of the lessons. Parents and students also gave their informed written consent for the students' participation. The teachers' names (Teachers A, B, and C) were not associated with their class notations. As the teachers' board work was important in the analysis process, photos (though not showing their faces) were used to illustrate the coding framework and the teachers' enactments in the results sections of the articles. Students' names involved in the interaction with the teacher were changed to pseudonyms as well (Articles I and II).

Before the video recordings were made in starting up the FASETT project, parents as well as teachers were informed about the purpose and design of the project, both in a written document and at an information meeting at each preschool unit. All participating teachers gave their written consent. The parents of the children involved in the project also gave their written consent for their children's participation, according to the ethical guidelines from the Swedish Research Council (2011). They were also informed that participation was voluntary and they had the possibility to withdraw from the study at any time. Interviews and samples of group activities were video recorded. To ensure confidentiality the participants were coded, and the coding list was stored separately from the recordings. Ethical approval was granted by the Ethical Review Board in Region Västra Götaland. Some parents gave their permission for their children's participation in the project, but not to video-record their children in the individual interviews or in the group activities. Therefore, in these cases we audio-recorded the interviews, with one researcher interviewing and another researcher making field notes on the child's use of their fingers or gestures. In the group activities, the camera was placed so that the children's fingers were in the focus of the camera. The teachers' and children's names were changed to pseudonyms in the written papers and conference presentations.

One ethical dilemma relates to the confidentiality. The number of participating teachers was quite small in both projects, and the analyses were done on a highly detailed level. Even though no photos have been used and the teachers' names have been changed, it could be the case that a participating teacher might recognize herself when reading this thesis or the articles. It was difficult to entirely overcome this anonymity problem, especially in the South African focal study, where the results revealed differences between the three teachers' teaching. However, several sections of teaching as well as teaching over time were coded, and there were shifts and differences that emerged, implying that it was not the same teacher who always performed the most linking actions. The sections of teaching were compared in order to identify differences in teaching the same content. It is also important to emphasize that the focus of this thesis is on the *teaching* of additive relations, and the differences in teaching that are possible to identify by using the coding framework. Even though sections of teaching (Articles I and II) and teaching the preschool activities (Articles III and IV) were compared, the teachers themselves were not my research interest as the main focus was on their teaching.

Another ethical dilemma in one of the focal studies concerns the task-based learner interviews. Even though the research group had made outlines for when to stop, as an interviewer it was sometimes hard to determine how far to go and how much pressure to put on the child. Especially in the pre-interviews, some tasks were a very challenging for the children. Therefore, I had to handle each interview uniquely and consider when to stop, how long to wait for an answer, and when it was appropriate to stop asking follow-up questions. When the child seemed troubled, I sometimes decided to not ask all the questions. In such cases, the answers were handled as fallout.

6 Results

This thesis aims to deepen the understanding of the teaching and learning of additive part-whole relations grounded in a structural approach, using variation theoretical ideas to expand the knowledge of the teaching and learning opportunities offered. The framing question of the thesis – *What can constitute a structural approach to teaching additive part-whole relations that facilitates learning among young learners?* – is to be answered in the results of the four articles and in the re-analysis. The results emerged from focal studies associated with two intervention programs, among children four-eight years old. In this chapter, first, the results of each articles are summarized. Then the results of the re-analysis are provided, followed by conclusions based on the overall results.

Article I

In *Article I*, the structural approach to teaching additive part-whole relations was explored through a coding framework based on variation theoretical assumption. The aim of Article I was to describe the development of this coding framework, in terms of criteria for coding viewed as important within additive relations. However, using the framework also allowed me to see fine-grained differences in three Grade 3 South African teachers' handling of additive part-whole relations related to the simultaneity of, and connections between, representations (SCBR) and examples (SCBE) as well as within examples (SCWE). Summarizing the total number of segments meeting the criteria for each SC category in the teaching indicates differences and similarities in the three teachers' enactment of the jointly planned activities. The results show that, in some teaching episodes, poor attention to connections between representations was identified. For instance, one teacher merely focused on the symbol representation when splitting a number into parts, whereas another teacher constantly verbally and gesturally linked concrete, triad diagram and symbol representation[33]. In some episodes, the teacher's handling of part-whole relations could be restricted to one single split of one specific whole number, whereas in other episodes the teacher ended up with complete partition sets of 7 or 9, but with different simultaneity and connection pathways. In another episode, the teacher erased each example (part-whole relation) being handled, compared to another teacher who recorded several examples but made no linking actions between the different examples produced, thus not meeting the criterion for SCBE[34].

[33] See Article I (p. 303, Table 1).

[34] See Article I (p. 305, Table 2).

However, in the same episode one teacher presented several examples, and explaining and pointing to the relations between different part-whole examples provided the opportunity to discern commutativity and completeness in the example space. Furthermore, in Article I, the students' performance on worksheets was tentatively reflected in the way they dealt with partitioning in the different teaching in the three classrooms. The shifts in the worksheet results, on the class level, followed the analyzed teaching episodes in terms of the number of segments meeting the criteria for linking examples[35].

Article II

The structured approach to additive relations was further explored in *Article II*. The same coding framework as in Article I was used to examine how relations and mathematical ideas associated with the structural approach were taught. The aim was to examine differences in three Grade 3 South African teachers' teaching, with respect to connections in teaching within and between examples. The application of the coding framework comparing the teaching of part-whole relations over time (Year 1 and Year 2) points to differences in teaching and a progression in teaching the same kind of partitioning and missing number problems. Improvement was identified at both the individual teacher level and the collective level (all three teachers). The numbers and proportion of segments meeting the criteria for SCWE and SCBE had increased by the second year, particularly proportions of segments coded SCBE[36]. Thereby, a richer range of structural relations within part-whole examples and more connecting work between examples were identified in the second year compared to the first. For instance, in Year 2 one teacher foregrounded the equivalent and commutative ideas of additive relations by juxtaposing and comparing examples, while these ideas were not discussed in the same way in Year 1. With reference to changes in teaching, the results also show that examples were erased from the board after being treated more often in the first year than in the second. In Year 2, examples were most often simultaneously visible on the board, which provided opportunities for the students, to a greater extent, to discern that part-whole examples can be related to each other.

Article III

In *Article III*, the structural approach to additive part-whole relations was explored by studying how an activity was implemented in a Swedish preschool context. The aim of this article was to contribute knowledge about

[35] See Article I (pp. 310–311).

[36] See Article II (p. 6, Table 4).

how the same part-whole activity, designed collectively, can be enacted differently, using principles from variation theory. The results suggest differences in teaching the same part-whole activity, and that these different ways of teaching provide different learning opportunities. Four enacted objects of learning were constituted, one of which did not offer any variation of representations or alternative to counting single units. Thereby, the children simply had the possibility to learn that collections of beads can be determined by counting them as single units. This can be compared to another enacted object of learning in which the teacher contrasted counting single units with composite sets of units, or decomposed a part into smaller parts. In some of teachers' enactment, the children were given the opportunity to learn how to use finger patterns as a tool for structuring part-whole relations without having to count their fingers as single units. In another enactment, the children also had the opportunity to experience that the undivided 5 (one whole hand) can be used as a benchmark for determining collections slightly below 5. One enacted object of learning also provided a systematic variation of examples, giving the children the opportunity to experience the commutative principle.

Article IV

Article IV reports on a preschool intervention program, grounded in a structural approach to additive relations and essential principles from variation theory (Marton, 2015). The aim was to explore how the teaching of additive relations was reflected in the children's learning of numbers. The results suggest that the five-year-olds following the program made great progress. After the intervention program, they were able to discern aspects related to numbers that they had not previously discerned. This new way of experiencing enabled them to act more flexibly on the arithmetic tasks in the post-assessment than was the case in the pre-assessment. For instance, the results show that before the intervention, in most observations the cardinal and ordinal aspects were not discerned at the same time, and the part-whole aspect was undiscerned. Meanwhile, in the post-assessment the children were able to discern the cardinal and ordinal aspects simultaneously as the part-whole relations of numbers. The results show that in the cases in which the children did not experience numbers as known facts after the intervention, they used their fingers to create finger patterns to solve the additive relation problems. In the program, the children were afforded finger patterns as a tool for structuring different relations of parts and whole, when for instance missing parts in different context problems[37] were to be identified and a systematic variation of examples was provided. This suggests that

[37] See p. 48 for a description of activities.

the aspect *finger patterns representing numbers* that had been afforded in the program had also been made possible for the children to discern.

To further summarize Article IV, the results point to other necessary aspects of structured additive part-whole relations inherent in the activities that the participating children were afforded to experience, for instance *numbers as composite sets*. The analyses showed how aspects were opened up as dimensions of variation, and were thereby made possible for the children to discern. For example, the children were afforded the opportunity to *discern numbers as composite sets* when the teacher offered variation by directing their attention to the undivided 5, offering an alternative to counting the beads or their fingers as single units. In the analysis of the post-assessment, there were no observations of children counting single units. This suggests that they had developed the ability to experience numbers as composite sets; and consequently, what they were afforded to experience of numbers, as associated with the structural approach to additive relation, mirrored and explained their progress in solving arithmetic problems[38].

The re-analysis - Teaching mathematical ideas and principles

To further explore the structural approach to teaching and learning additive part-whole relations, using principles from variation theory, a detailed analysis of *how mathematical ideas and principles were brought to the fore in teaching* was conducted. In order to elucidate qualitative differences in the teaching and the learning opportunities that were offered, data associated with the four articles were re-analyzed. From the analysis, evidence emerges supporting the enactment of *number relations as composite sets, the commutative principle*, and *completeness by systematicity*.

Number relations as composite sets

Results of different enactments of the mathematical idea of *number relations as composite sets* are sampled from Articles III and IV. Structured finger patterns were used to support the ability to see number relations as composite sets of units rather than merely as single units, and to facilitate the discernment of how the parts are related to each other and to the whole. Paying attention to a structural approach to additive relation problems, children in the FASETT intervention were introduced to finger patterns for structuring part-whole relations. Also, the bead string with ten beads grouped together, five of each color, associating with five fingers on each hand, was promoted in the intervention. In the Snake game and the context problems, as well as

[38] For more detailed results, see Article IV (pp. 25–27).

in other activities (See p. 48 for descriptions of activities), *number relations as composite sets* is possible to focus on in the teaching.

Four different ways of enacting were identified. The results show that the main difference in the enactments is associated with the teacher's affordance of finger patterns for structuring number relations and whether the teacher draws the children's attention to seeing numbers as groups of units as an alternative to counting them as single units.

Table 3 presents the different enactments (A-D)[39] of the mathematical idea of *number relations as composite sets*. The table includes examples and excerpts from the teacher-child dialogues, and what dimensions of variation, significant in relation to the idea, were opened up[40]. The table also includes the nature of connections manifested by the teachers as identified in the analysis.

[39] Enactment A (p. 12) in Article III; Enactments B and C are sampled from the video-recorded observations analyzed in Articles III and IV (Observations 21 and 44); and Enactment D is sampled from Article IV (p. 20).
[40] In Table 3–5: DoV.

Table 3: Enactments of *number relations as composite sets*

	Enactment A	Enactment B	Enactment C	Enactment D
Examples	Five beads of the 10-snake hidden, three beads of the 10-snake hidden.	Six beads of the 10-snake hidden, three beads of the 10-snake hidden.	Two beads of the 10-snake hidden, four beads of the 10-snake hidden.	Seven beads of the 10-snake hidden, eight beads of the 10-snake hidden.
Instruction	The teacher asked: "How many on this snake? How many are hidden?" After having identified five, she asked them to count the ten beads again. She hid three beads. The children counted the seven beads as single units, and she said: "How many in my hand?". After the children had answered, she said "You think it's four, and you think it's three; how do you know?" (No one answered.) "Let's check!" (Opened her hand).	The teacher presented a sequence of examples. She said: "There are ten from the beginning". How many beads do you see? How many hidden? How do you know?" The children showed their fingers by raising one at a time or by structuring finger patterns. Some showed the visible part, and some did not identify the hidden part. Different finger patterns (folded/unfolded fingers) were shown.	The teacher said: "There are ten beads, two hands. Show me, how many do you see?" The children modelled finger patterns in different ways. Noticing this, the teacher said: "Yes, you show it like this, and you did it another way (pointing)." "Look at your fingers, how many are hidden?" A child said: "Five", and the teacher said: "If you look at your hands again (touching the child's eight unfolded fingers) – how many folded?" The next time, four were hidden. The teacher compared the finger patterns. She said: "Some counted five red beads, just like one hand…".	The teacher noticed that one child counted the beads as single units, having identified the hidden number by looking at his finger pattern. She said: "Remember, there are five here (making a circle with her hand around the five red beads of the 10-snake) … you don't have to count those…6 and 7 (Points to the two white beads) just like your fingers." (Points to a child's finger pattern) Having identified eight, she said: "Eight, exactly. Do you remember… five (circling the five red beads, and pointing to the three white beads) … six, seven, eight". So, eight and two, altogether, make ten (making two circles, around the two parts).
DoV opened	-	Part-whole relations as composite sets of units.	Part-whole relations as composite sets of units.	Part-whole relations as composite sets of units. The undivided 5 as a composite unit (within one part).
Connections	-	-	Directing attention to different structured finger pattern; Contrasting counting single units and composite set.	Contrasting composite sets with unit counting; Directing attention to the undivided 5 within a collection.

In Table 3 presents four different ways of enacting the 10-snake task. The results reveal that in Enactment A, the children were not afforded any alternative to counting the beads as single units. As can be seen in Enactment A, the teacher asked the children to figure out how many beads there were on the whole bead string and when five were hidden. After they had identified the seven beads by counting, and disagreed about the hidden part, the teacher asked them how they knew this but they could not answer. In this enactment, the use of their fingers was not offered as an alternative for structuring number relations. No attention was paid to the fact that the five beads of different colors constituted a composite set of beads, or to the relation of the visible part to the whole number, which would have facilitated for the learners to identify the hidden part. Therefore, *number relations as composite sets* was **not** brought to the fore by the teacher. In Enactment A, the children were not offered any alternative to counting one by one to determine collections of items larger than three. Thus, no variation of how to figure out the missing part was provided. Still, the children had the opportunity to learn that groups of items can be determined by counting them as single units or by estimating.

In Enactment B, the aspect *part-whole relations as composite sets of units* was opened up as a dimension of variation. In order to identify the hidden part, in a sequence of examples the teacher encouraged the children to show on their fingers how many beads they saw on the string. One variation was provided when some of them counted their fingers as single units, while others structured them as patterns (folded/unfolded fingers) without counting them. Another variation was provided when different ways of structuring the same part-whole relation with finger patterns among the learners were possible to discern (See Table 3). However, neither connections in talk and gestures pointing out differences and similarities between the ways of showing fingers, nor contrasting counting with seeing items as composite sets, were identified in the enactment. Since the teacher, with no discussion, simply asked "How do you know?" when a child had found the hidden part, some children failed to identify the hidden part. It was made possible for the children to see number relations as composite sets of units (without counting) (e.g. by looking at folded/unfolded fingers or the group of beads on the string) and perceptually discern various ways of showing numbers using their fingers (their own as well as their friends' finger patterns). However, the teacher **did not** explicitly direct the learners' attention to finger patterns for structuring number relations as composite sets. Since the teacher did not pay attention to these aspects in her enactment (no linking talk or gestures), the learners could perceptually discern the variations by looking at their own and their friends' ways of showing their fingers. This enactment enabled them to experience that the part-whole relation can be structured on one's fingers as composite sets or as single counted units, in different ways.

In Enactment C, *part-whole relations as composite sets of units* was opened up as a dimension of variation. When eight fingers were to be shown, the way of unfolding eight fingers varied, and the same variation occurred when six fingers were to be shown. Variation also occurred when the same numbers of items (beads/fingers) could be discerned as separate single units or as composite sets (See Table 3). In Enactment C, opposite to Enactment B, the teacher noticed that the children structured their finger patterns differently. She directed their attention to this variation by pointing to and comparing the different patterns, saying for instance "Yes, you show it like this, and you did it in another way." When a child answered incorrectly (5 as the hidden part), she underlined the power of structured finger patterns by talking to the child and touching the child's correct finger pattern, which supported the child in seeing the hidden part without having to use one-by-one counting. In the end she contrasted counting single units with *5 as a composite set of units*, pointing out the similarity between five fingers on one hand and five beads of the same color. In Enactment C, the learners were offered finger patterns as a tool for structuring number relations. They also had the opportunity to experience that a specific number relation can be structured using different fingers and that finger patterns allow them to identify the hidden part by "seeing it" (not counting one by one). Since the teacher also connected five fingers with five beads, it enabled the children to see 5 as a composite unit. The way the teacher explicitly pointed out the features of the aspect may have facilitated for the learners to discern number relations as composite sets.

In Enactment D, two dimensions of variation were opened: Firstly, *part-whole relations as composite sets of units* was opened up as a dimension of variation. Various examples of hidden parts (7 and 8) of the same 10-snake, which were to be structured on the children's fingers, were offered. Secondly *undivided 5 as a composite unit* was opened up as a dimension of variation, through the variation of selected numbers of the hidden parts slightly below 5 (invariant).

The teacher's way of handling the Snake game differed from Enactments A and B as well as Enactment C. In Enactment D, the teacher noticed that the children were able to structure number relations with their fingers and identify the hidden part by looking at their patterns (without repeated single counting). However, some of the children did not see the number relations on the bead string as composite sets, instead starting to count all the beads as single units. At that point, the teacher directed the learners' attention to *number relations as composite sets*, underlining the undivided 5 and saying "Remember, there are five there (making a circle with her hand around the five red beads of the 10-snake) … you don't have to count them" (7 = 5 + 2). Furthermore, she linked to the undivided 5 on the finger of one hand, saying "…just like your fingers" and linking to a child's finger pattern of seven unfolded and three folded fingers (7 + 3 = 10). The teacher then connected the "undivided 5 and three more" on the bead string to the finger pattern of one whole hand and three more fingers in her talk and with her gestures. She circled the five red beads and pointed to the three white beads, saying "…six, seven, eight" (5 + 3 = 8). Also, she emphasized that the two parts "8 and 2", make "10" by making two circles around the two parts (8 + 2 = 10). In Enactment D, the principle of *number relations as composite sets* was foregrounded, in relation not only to finger patterns for structuring number relations as composite sets but also to the beads grouped together in two colors.

Enactment D allowed the children to experience different ways of structuring the same part-whole relation as composite sets, and the undivided 5 as a composite unit as a variation to single counting. When the teacher contrasted *repeated single counting* with *seeing numbers as composite sets* and explicitly directed the children's attention to the undivided 5, the learners were given the opportunity to extend their subitizing range (conceptual subitizing). They were also given the opportunity to experience that five beads of the same color, just like five fingers on one hand, support the idea of the undivided 5 and are helpful when collections (parts) of slightly less than five are to be identified. Furthermore, they were afforded the possibility to experience that a part within a part-whole relation can be decomposed into two smaller parts or as a combination of a decomposed set of units and single counted units (10 = 2 + (5 + 3); 10 = 3 + (5 + 2)[41]. Since the teacher explicitly pointed out the composite sets, the teaching afforded the children richer learning possibilities compared to Enactment B (in which no connections were identified) and Enactment C, where only the similarities between five fingers and a group of five beads of the same color served as the object of attention.

The commutative principle

The commutative principle for addition (a + b = b + a) is possible to bring to the fore in structural part-whole teaching. In both projects, implemented activities and problems provided opportunities for learning commutativity. For example, in the context problems, some of the stories encouraged the chil-

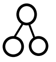
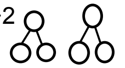
[41] See Article III (p. 15 Figure 3).

dren to solve the problem by starting with the larger number (part) instead of following the story's semantic structure. For instance, "Two bears were playing, then six more bears came. How many were there altogether? (2 + 6 = _)" could be solved by starting with six and adding two (6 + 2 = _)[42]. In the partitioning number activity, in which the same whole number was to be decomposed into two parts in different ways, pairs of combinations were possible to present as a commutative pattern. For instance, 7 (whole number) can be decomposed into 3 and 4; 4 and 3; 2 and 5; 5 and 2; and so on. In the instruction of additive relation problems, combinations within a specific part-whole relation (e.g. 2/7/9) provided discussions about commutativity as well. Four different enactments (A-D) (See Table 4) of the mathematical principle of commutativity were identified in the analysis[43]. The results suggest that in three of these enactments *the commutative principle* was offered by a sequence of two additive relation problems. The main differences between the enactments are associated with how the teacher directed the learners' attention to the principle.

Table 4 is a summary of the four enactments, with examples, the teachers' instruction, the dimensions of variation opened up relative to *the commutative principle*. The table also includes connections, made by the teacher both verbally and through gestures, associated with the property of commutativity. The selected data set is sampled from Articles I and II[44].

[42] See also the results section in Article IV.
[43] Selected from the Wits Maths Connect project.
[44] Enactment A is sampled from the second section of teaching (Article I, p. 304–306) and is further discussed in Article II (p.7); Enactment B is sampled from section 13.3.2 (Overview of teaching p. 4, in Article II). This enactment is taken from the data set analyzed in Article II. Enactments C and D are sampled from Article II (pp. 6–7).

Table 4: Enactments of *the commutative principle*

	Enactment A	Enactment B	Enactment C	Enactment D
Examples	~~7= 4+3~~ ~~7= 0+7~~ ~~7= 6+1~~ 7= 2+5 	9=2+7 9=7+2 	2+7=9 7+2=9	2+7=9 7+2=9 5=1-6 5=11-6
Instruction	Four different ways of splitting the whole number 7 (triad diagram and number sentence) were produced. The teacher recorded learners' suggested splits, then erased each split after it had been handled.	The teacher wrote 9 in the top circle, then 2 and 7 in the bottom circles in the triad diagram, and asked for the whole and the parts, saying: "We can also write it like this: 9=2+7." She then drew a new triad diagram on the right side of the other one, and asked: "What are the whole and the parts?". The teach-er recorded the split and said: "9 equals 7 and 2. I'm going to give you another one."	The teacher recorded 2+7 and 7+2 on the board and asked: "Are these the same?" When some students hesitat-ed, she said: "Give me a reason why they're not the same." (pointing to the '2s and 7s' in the two num-ber sentences) She then wrote the sums (9). "No matter if I take 2 plus 7 or 7 plus 2 (pointing to the two number sentences) I'll have the same answer (pointing to the 9s).	The teacher recorded 2+7 and 7+2 and com-pared them. She wrote 5=1-6 and 5=11-6 and said: "These; are they the same (pointing to 5=1-6 and 5=11-6)? Which one is correct? Why?" The teacher continued discussing the differences between 5=1-6 and 5=11-6, and which was correct. She said: "You can't subtract 6 from 1 (finger moving from right to left). We're not reading it from this side (finger moving from right (6) to left in the first number sentence again). Listen, when you're adding, you can say 7 plus 2 (moving her finger from right to left in the first number sentence) or 2 plus 7 (movement right to left). You can do it in addition, but in subtraction you can't start with 6 (points to the '6' in 5=1-6) going backward (moving her finger to the left)."
DoV opened	-	The irrelevant order of the ad-dends.	The irrelevant order of the addends.	The irrelevant order of the addends.
Connections	-	-	Directing atten-tion to the same addends and to different positions and same sums in the number sentences.	Directing attention to the same addends, position, and sums; Emphasizing differences between addition and subtraction; Juxtaposing correct/incorrect an-swers.

The results suggest that in Enactments B, C and D *the commutative principle* of addition was emphasized by presenting the examples in a systematic order. However, in Enactment A (See Table 4) this was not the case. Firstly, when the students offered different ways of partitioning a whole number in two parts, the order of the combinations was randomly produced. Secondly, after each partition was recorded on the board, the teacher erased the triad diagram and the number sentence. Each partition was hence handled separately, and they were therefore not possible to discern simultaneously. As a consequence, and although four different ways of partitioning the same whole number were offered, it was not possible to discern the commutative pattern (or any other pattern) as the students offered combinations randomly and the teacher erased each produced combination. If the examples were produced randomly without systematicity, and examples were erased from the board (Enactment A), it would be hard for the students to visually discern the relations between the examples. In this enactment, the students were afforded the opportunity to experience that a specific whole number can be decomposed into two parts in different ways. However, *the commutative principle* for addition **was not** opened up as a dimension of variation and was not made possible to learn.

In Enactment B, *the irrelevant order of addends* was opened up as a dimension of variation when two examples (9 = 2 + 7; 9 = 7 + 2) with the same whole number and same parts/addends but in different order were presented in a sequence. The teacher recorded the two part-part-whole relations in a triad diagram and in a number sentence format. In this way, two examples containing the same sum and addends, but with the addends in different positions, were simultaneously possible for the learners to discern. From Table 4, it can be seen that in Enactment B the teacher emphasized the part-part-whole relation 9/7/2 and talked about parts and whole. However, no attention was directed to *the commutative principle* in the teacher's instruction, since she made no connections between the '2s', '7s' or '9s' and their positions in the triad diagram or the number sentences were made. In other words, when analyzing this enactment, *the commutative principle* was offered within the example space but no connections, either in talk or with gestures pointing out the principle, were identified. Since the teacher selected these two part-whole relation problems and varied the order of the addends/parts, *the irrelevant order of addends* was opened up as a dimension of variation. However, *the commutative principle* was only made perceptually visible for the learners. In terms of dimensions of variation being opened up, the examples as such provided this variation. In Enactment B, the learners were given the opportunity to visually discern the commutative principle of two examples, and that regardless of the order of the addends the sum is the same.

In Enactment C, similar to Enactment B, the aspect *the irrelevant order of the addends* was opened up as a dimension of variation. The addends' order differed, while the numbers (2 and 7) and the sum (9) were the same. However, like in Enactment B, in Enactment C it emerged that, apart from making a pair of commutative examples perceptually visible simultaneously (See Table 4), the teacher also directed the learners' attention to *the commutative principle* by comparing the two examples. For instance, she said "Are these the same?", and noticing some students' hesitation then continued to emphasize the similarities and differences by underlining the '2' (first addend) and '7' (second addend) in the first number sentence and the '7' (first addend) and '2' (second addend) in the second example. She also emphasized that the sums in the two examples were the same, by pointing to the 9s. In Enactment C, besides offering the examples as a commutative pattern, the teacher's enactment included connecting work that underlined *the commutative principle*. So, beyond the experience of variation provided by the visual format of the number sentences presented on the board, *the commutative principle* for addition was made explicit by the teacher. This enactment, in which the properties of commutativity were also reinforced in the teacher's connecting work, may facilitate the students' discernment that, regardless of the order of the addends, the sum is the same.

The results suggest that in Enactment D as well, *the irrelevant order of the addends* remains as a dimension of variation that was opened up, when a pair of examples (2 + 7 = 9; 7 + 2 = 9) with the same addends and sums, and in various positions, were simultaneously visible on the board. The relations between the examples were emphasized in the teacher's connecting work when she asked for differences and similarities between the number sentences and explicitly linked the addend '2s' and '7s' and the same sums (similar to Enactment C). However, there was an attempt at further elaboration of this dimension in Enactment D when two subtraction problems, a correct and an incorrect example (5 = 1 - 6; 5= 11 - 6), were written on the board below the two addition examples. So, similarities and differences between the four number sentences were possible to visually discern. When the teacher juxtaposed the correct and incorrect examples, saying "Are they the same (pointing to 5 = 1 - 6 and 5 = 11 - 6)? "Which one is correct?... Why?", the discussion started.

Then, in the enactment there was a reference explicitly directing the learners' attention to the irrelevant order of the addends in 2 + 7 and 7 + 2, with the teacher saying "…when you're adding, you can say 7 plus 2 (moving her finger from right to left in the first number sentence) or 2 plus 7 (movement right to left) along with the fact that in the 5=1-6 instance, "You can do it in addition, but in subtraction (…) you can't start with 6 (points to the

'6' in 5=1-6) going backward (moving her finger to the left)." Here, the teacher offered a contrast and stated as a generalization that irrelevant order cannot be applied to subtraction[45].

In summary, in Enactment D, four additive relation problems were simultaneously visible, and the teacher explicitly directed the learners' attention to *the commutative principle* and explored the principle in interaction with the learners. This enactment afforded the students the opportunity to learn that, following *the commutative principle*, the order of the addends in addition does not matter as the sums are still the same, and that commutativity applies to addition but not subtraction. The teacher's enactment, with the attention to comparing the number sentences, served as a way to elaborate the mathematical principle into noting that irrelevant order holds for addition but not subtraction. The teacher's explorative actions point to enriched learning coming from the students being able to distinguish examples and nonexamples[46] of a particular idea, in this case irrelevant order. The enactment might also allow the students to learn that the direction of the number sentences does not matter in addition but must be considered in subtraction.

Completeness by systematicity

A third example of differences in teaching was chosen from the South African Grade 3 teachers' ways of handling the partitioning task ("seven monkeys playing in two trees" or "nine balls in two bags") (See p. 48). This activity offers students the opportunity to represent part-whole relations in a concrete situation and learn how these relations can be represented in symbolic form, for instance in a table. One principle that is possible to focus on in this activity is *completeness by systematicity*. Completeness in the context of the part-whole relations of numbers can be used to highlight the relations between the produced partitions when recording the different ways of decomposing a whole number, for instance producing a compensational pattern (0/7; 1/6; 2/5…) or a commutative pattern (7/0; 0/7; 1/6; 6/1…) Also, systematicity supports the production of all possible partitions and enables the discernment of completeness.

When analyzing the data, with regard to *completeness by systematicity*, four different enactments were found (A-D). The main difference relates to whether and how completeness was brought to the fore by systematicity in the enactment of the partitioning task. In Enactment A, the part-whole rela-

[45] It would likely have been better to use examples in which the relation holds with one order but not the reverse order. Working with 5 = 11 - 6 might therefore have been more useful for discussing the irrelevant order more clearly than 5 = 1 - 6, where neither works.

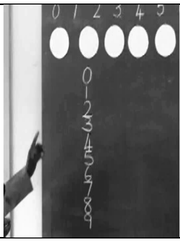
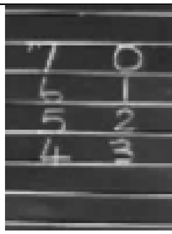
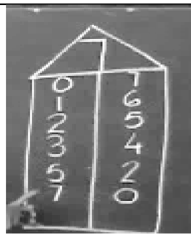
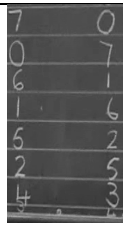
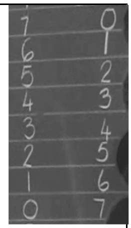
[46] See also Ekdahl & Runesson (2015).

tions were not possible to discern simultaneously. In Enactment B, systematicity was inherent in the partitions visible on the board, but the teacher did not explicitly point it out. This enactment differs from Enactments C and D, in which the teachers directed the learners' attention to completeness by systematicity, using connecting talk and gestures. Differences in the learning opportunities offered depend on whether *completeness by systematicity* is inherent in the example space (perceptually visible), and whether and how completeness is explicitly pointed out by the teacher.

Table 5 presents a summary of the different enactments (examples and instructions)[47] of *completeness by systematicity*. The table includes the dimensions of variation opened up in regard to *completeness by systematicity*. Also included in the table is the nature of the connections associated with the mathematical idea identified in each enactment.

[47] Enactment A is sampled from third section on teaching in Article I (pp. 307–308) and discussed in Article II, p. 6; Enactment B is sampled from the coding framework in Article I (p. 302); Enactment C is found in third section on teaching in Article I, illustrated in p. 308; and Enactment D is taken from the section on teaching, 14.1.1 (Article II, pp. 4 and 6) and the transcriptions of the episode.

Table 5: Enactments of *completeness by systematicity*

	Enactment A	Enactment B	Enactment C	Enactment D
Examples				 
Instruction	Teacher and students produced an increasing number sequence in one column, then a decreasing sequence starting with 9, 8, 7… for the second parts in the right column. She said: "Counting from smallest to biggest", here we're subtracting." (pointing vertically).	The teacher produced the children's offers of splitting 7 into two parts. After the '4 and 3 split', she said: "4 and 3, how many monkeys? We're still counting…?" (pointing vertically to the first column) Then she asked: "And this side?" (pointing vertically to the second column).	Teacher said: "I think we have something here, can you see?" (Moving her hand horizontally, twice over the 5/2 and 7/0 partitions, pointed vertically from the top to the bottom of the table, down each column) "A pattern?" After the learners had suggested the missing combinations, she rewrote the table and said: "What's happening?"	After having produced a complete set of 7, the teacher said: "You can do it in another way. I can start with my 7 and 0; 6 and 1; what are the next numbers?" A complete compensational pattern was produced. Then she said: "Have we finished?" (pointing horizontally and vertically to the compensational pattern) Pointing to each combination, she said: "These parts, you can put them together to get 7, like a pattern. "What if I put 8 here?" (pointing to right column) "How many combinations? What if I give you 9, how many? … or 12…24?"
DoV opened	-	Decomposing one whole into two parts.	Decomposing one whole into two parts. Systematicity as one complete set of partitions.	Decomposing one whole into two parts. Systematicity as two complete sets of partitions. Number of partitions of various whole values.
Connections	-	-	Pointing to relations between combinations; and the complete compensational pattern.	Directing attention to relations between combinations and to completeness; possible (incorrect) partitions and total number relations for whole numbers 7 and 9.

As can be seen from Table 5, four different ways of enactment (A-D) associated with *systematicity* were identified. In one of them, Enactment A, the first part in each combination was initially produced as an increasing number sequence (0, 1, 2, 3, …), and thereafter the second part in each combination as a decreasing number sequence (9, 8, 7, 6, …). As each part was produced separately, the different part-part-whole combinations were not possible to discern simultaneously. Instead, the teacher pointed vertically to the left column, talking about "counting from smallest…" and "subtracting", drawing the students' attention to the decreasing number sequence in the right column and emphasizing only one of the parts within each part-part-whole combination. The connections made by the teacher were **not** in accordance with the principle in focus, even if the episode resulted in a complete set of splitting the whole number 9. In Enactment A, in which the focus of attention consisted of counting the separate parts backward and forward, the learners were afforded openings to learn a pattern of adding and subtracting numbers by ones. Thus, the dimension of variation, *decomposing one whole into two parts*, was not opened up in this enactment.

In Enactment B, the partitioning task "seven monkeys playing in two trees" was handled (See p. 48). *Decomposing one whole into two parts* was opened up as a dimension of variation by various partitionings of the same whole number (seven monkeys) being recorded in table format, as a compensational pattern (7/0; 6/1; 5/2; 4/3). However, in a way similar to that in Enactment A, the teacher emphasized the vertical pattern, pointing to each column one at a time and saying "counting backward and counting forward" with separate attention to the increasing (7, 6, 5, 4) and decreasing (0, 1, 2, 3) patterns of the parts. Even if several partitions were simultaneously visible on the board, no attention was paid to *how* the different part-whole relations were connected. For instance, no attention was paid to how the '4/3 partition' was connected to the previous partitions (7/0; 6/1; 5/2). There was indeed a pattern produced, but not a complete systematic pattern. Several partitions were possible to discern simultaneously, which implies that *systematicity* was made possible for the learners to discern perceptually, but without any verbal or other designation. However, this dimension of variation was opened up in Enactment B when a variation of partitions of the same whole value was presented. In this way, it was possible for the learners to discern a pattern when the compensational sequence of four partitions of 7 were visible on the board. Still, the compensational pattern was not made explicit because the teacher's attention (both in talk and with gestures) was directed at each column separately. Therefore, it might have been challenging for the learners to understand how the part-whole combinations were connected to each other. Thus, if the learners only focused on the partitions written on the board, it would have been possible to experience that a whole number can be decomposed in different ways and the partitions can be produced as a pattern. Completeness was not afforded in this enactment, as only four examples were perceptually visible.

Also, in Enactment C, different partitions of the whole number 7 were recorded in table format. Several ways of decomposing the whole number 7 were simultaneously visible on the board. The teacher created variation by recording the first four examples in systematic order (0/7; 1/6; 2/5; 3/4) (See Table 5), thereby opening up *decomposing one whole into two parts* as a dimension of variation. When the teacher focused on the production of all possible partitions of the same whole number, *systematicity as one complete set of partitions* was opened up as a dimension of variation. The teacher explicitly directed the learners' attention to the systematicity in the relation between the partitions. Initially, the teacher recorded the students' suggestions, even if they did not follow the properties of a compensational pattern. The partitioning of 4 and 3 was missing in the sequence, as was the combination of 6 and 1 (See Table 5). Noticing this, the teacher wanted the students to find the missing combinations. She further directed their attention to systematicity, using linking gestures (gesturing vertically and horizontally), which emphasized the change in part-whole relations from one example to the next. After the systematic compensational pattern (including all partitions) had been produced, the teacher asked the students to explain the pattern. Variation was provided when various partitions of the same whole number were simultaneously visible on the board, and the sequence of recorded different partitions opened up for producing a complete set of partitions of that whole number. In addition, the teacher explicitly directed the learners' attention to the missing combinations by pointing to relations between combinations and discussing the complete produced compensational pattern. This attention afforded the learners the opportunity to discern *completeness by systematicity*, and to experience how many ways in which a whole number can be decomposed into two parts and how to figure out all the combinations (completeness).

Still more attention was directed to systematicity and completeness in Enactment D. Similar to Enactment C, the aspect *decomposing one whole into two parts* was opened up as a dimension of variation. Instead of producing one complete set of partitions, though, in Enactment D *systematicity as two complete sets of partitions of the whole number 7* – a compensational and a commutative one – was produced in the interaction with the students (See Table 5). Thus, this aspect was opened up as a dimension of variation.

Attention was directed to systematicity when the teacher had produced a commutative pattern and offered the first combinations (7 and 0; 6 and 1) with the purpose of producing another systematic pattern. Having finished this pattern, the teacher explicitly pointed out systematicity, underlining each combination from top to bottom in the table and referring to them as parts that can compose the same whole. Even if the pattern was complete, her verifying question 'Have we finished?" (pointing horizontally and vertically to the compensational pattern of 7) and her incorrect suggestion "What if I put 8 here?" (pointing below 7 in the left column) show how the principle of systematicity was made explicit. Ultimately, *number of partitions of various whole values* was opened up as a dimension of variation. The teacher offered a variation by asking for the total number of combinations of the number 7, and made a generalization, elaborating on the total number of combinations of other whole numbers (9, 12, and 24).

In Enactment D, the learners were offered the chance to experience how systematicity is useful in determining how many different ways a whole number can be decomposed into two parts; that there are different ways of producing completeness. They were also afforded the opportunity to experience the pattern of possible partition options for 7 as a whole value and that could be generalized to other whole values (n+1). This connecting teaching' might enrich students' opportunities to learn *completeness by systematicity*.

Learning opportunities enhanced by variation and connection
In the findings from the re-analysis, the enactments of the same mathematical idea point to differences that are sometimes subtle, but that provide different learning opportunities. The various ways of teaching additive part-whole relations relate, on the one hand, to the potential of the task or activity as such, and on the other to how the teacher directs the learners' attention to important mathematical ideas and principles concerning additive relations. As argued before, when the activities, example spaces and enactments were similar, what the teacher explicitly pointed out – alongside the dimensions of variation opened up – can explain the differences in learning opportunities offered. The linking action, pointing out what is to be connected with what, is of importance in what the learners need to discern in order to be able to understand the specific idea, principle, or relation. It is the feature(s) that are crucial within this mathematical idea, relation or focus aspect that are to be explicitly pointed out by the teacher.

The results of the re-analysis indicate that it matters whether or not aspects associated with the mathematical principle or idea in focus are opened up as a dimension of variation. In some enactments within a certain idea or principle, more than one aspect was opened up as dimension of variation as well. The provided variation afforded the learners the opportunity to discern the object of attention: *number relations as composite sets, commutativity, completeness by systematicity*. Written commutative examples recorded on the board, or visible finger patters showing composite sets, may allow learners to *perceptually* discern the aspect in focus (Enactment B). However, the results suggest that, even if a mathematical principle or idea is present and variations of this aspect are provided, this principle may not be visible to all learners. There seems to be a need for teachers to support the learners' discernment of the principle or idea in focus. According to the analysis, it might be necessary to direct children's attention to the principle gesturally and/or verbally, rather than taking for granted that every child has discerned the principle in focus.

Besides which aspects were opened up as dimensions of variation and made perceptually visible, the analysis suggests that making connections in instruction can enhance learning opportunities. The results reveal that aspects that were necessary to discern in order to experience a specific mathematical idea or principle were in some cases explicitly pointed out (both verbally and gesturally). However, it appears that *making connections* does not merely entail pointing out something (not just anything); it is rather about what is to be connected to what, and how the connections are made. An illustration of these differences in linking connections can be seen in a comparison of Enactments B and C regarding *completeness by systematicity* (See Table 5). In Enactment B connections were made to the vertical pattern of each column separately, when the teacher pointed to the increasing and decreasing patterns of separate parts, whereas in Enactment C the connections made were directed towards systematicity, pointing out both the horizontal pattern of part-whole relations and the vertical pattern (systematicity) (cf. Watson & Mason, 2006b).

It seems that in some enactments, the teacher directed the learners' attention to other mathematical properties of the principle than was planned. For instance, in Enactment D (See Table 4) the part-whole reasoning of generalization had not been planned in advance. When the teacher discussed completeness by also making connections to the completeness of other whole numbers, the mathematical discussion seems to have become more advanced, and may have empowered the teaching mathematically.

In addition to the data sets analyzed in the re-analysis above, the power of variation and connection can be illustrated with two empirical examples from Articles II and IV, respectively. In the first example, selected from the South African focal study, two missing number problems (_- 7 = 10 and 10 = _ - 7) were recorded on the board, one under the other[48]. The teacher created a variation by offering the two missing number problems simultaneously. Following variation theoretical principles, the numbers and operations were invariant and the position of the unknown number varied.

[48] See Article II (p. 7 Figure 3).

Hence, it could be argued that there is a difference depending on whether the instruction was focused on producing the answers to the two missing number problems or if the teacher explicitly directed the learners' attention to the structural relations of parts and whole and the position of the unknown number. In this episode the teacher made connections by juxtaposing the examples, asking for differences and similarities. She also drew lines between the missing whole numbers and the 10s, and used circling gestures to emphasize the same quantities across the two examples. The example itself (_- 7 = 10 and 10 = _ - 7) provides learners the opportunity to experience variation perceptually. However, if the mathematical ideas embedded in the task are also explicitly pointed out and discussed, this may expand the learning space even more.

The power of variation and connections can be further described with reference to the second empirical example of the Statement game (description on p. 48) in the FASETT project (See Article IV). Within this game, the children were to reach agreement on what number they thought the die would show. When the children had shown the same number on the fingers of two hands in various ways, the teacher could verify their finger patterns without making any comments and then move on to the next example, which sometimes happened. In this case, different ways of showing the same number on two hands were possible to discern in the enactment, however only implicitly since the teacher did not direct the children's attention to various ways of showing the same number. This is different from an enactment[49] in which the teacher explicitly directed the children's attention to the different ways of showing the same number on two hands, for instance asking a child "How many do you have?" and turning to another child, asking "… and you?" (at the same time touching each child's hands), and directing the children's attention to another child's fingers while saying "How many does Kim have there?" She then juxtaposed the children's ways of showing the same number, saying "Did you show it in the same way?" and continued, saying "How many different ways do we have?" Besides manifesting different ways of showing the same numbers on the fingers of two hands, the teacher made connections between finger patterns, by asking questions and using linking gestures emphasizing the similarities and differences between the finger patterns shown. In these empirical examples (missing number problems and the Statement game), as well as in the three principles described in detail in the re-analysis, the theoretical concepts of variation and connections are central in the argumentation concerning how to enhance part-whole teaching within a structural approach.

[49] See Article IV (p. 19).

Conclusion

The results from the four empirical studies and the re-analysis contribute a deeper understanding of what can constitute a structural approach to additive relations. Looking across the articles and the re-analysis, the results suggest that it matters for learning which representations, examples, and problems are offered to the children. Some representations and resources (e.g. finger patterns and bead strings) seem to facilitate the discernment of the parts and whole and their relations, which is seen as essential for a structural approach to additive relations.

Structural part-whole relation teaching is also associated with the simultaneous presence of more than one example at a time. The results suggest that it matters if several examples are offered by the teacher, but also *which* examples are offered in a sequence. A systematic sequence of examples has the potential to bring to the fore the relations between different part-whole examples, which provides the opportunity to learn mathematical principles such as commutativity or completeness by systematicity.

The overall results indicate that ideas and principles from variation theory made it possible to expand the knowledge of part-whole relation teaching and the learning opportunities offered. In the articles as well as the re-analysis, the results indicate that what is made possible to learn about additive part-whole relations is associated with what aspects are opened up as dimensions of variation.

Foremost, however, the results reveal the importance of *making connections* to highlight number relations and key features associated with the structural approach to additive relations. Whether and how the teacher explicitly (verbally and gesturally) draws attention to relations, ideas and aspects seems to be crucial for the structural approach to additive part-whole relation teaching. The results suggest that teachers' connecting work has implications for the learning opportunities. It seems to matter not only *what* the teacher explicitly draws the learners' attention to within a part-whole relation by opening dimension of variation (between different part-whole relations, between various representations and ideas and within an aspect) but also *how* this is accomplished.

To conclude, looking across the articles and the re-analysis, the micro-analyses suggest differences in terms of provided representations, examples and problems; the variation in the aspects offered; and particularly whether and how relations and ideas are connected. These sometimes subtle differences identified in the teaching offer the children different learning opportunities for. The results suggest that how variation is offered (dimensions of variation opened up) and whether or not relations and mathematic ideas are specifically pointed out (by making connections) seem to be critical in teaching for the learning of additive part-whole relations. In addition, through the separate articles and the re-analysis, the results imply that it is possible to implement the structural approach to additive part-whole relations and assumptions from variation theory in the two contexts chosen for this thesis.

7 Discussion

The overall aim of this thesis is to contribute deeper knowledge about the structural approach to teaching and learning additive relations, in the context of two intervention studies with children aged five to eight years. One point of departure is Neuman's (1987) findings and her argument that an awareness of numbers' part-whole relations is essential for children's development of arithmetic skills. Another point of departure is the structural approach to relations and the 'relational reasoning' argued for by Davydov (1982), and the importance of offering children the opportunity to explore numbers as relations within numbers in teaching, instead of focusing on counting (Schmittau, 2004). These ideas (structure and relations) and their implications for teaching, as well as principles from variation theory, have been the main thread throughout the collaborative work in the intervention studies.

Discussion of the results

The overall findings indicate how examples, representations, resources, and problems offered in teaching might support learners in discerning part-whole relations. It seems to be a question of how teachers direct children's attention to the relations between the parts and the whole, regardless of which representations come into play[50]. This might enhance children's ability to see how quantities are related to each other and whether a whole or a part is missing (cf. Schmittau, 2004; Sensevy et al. 2015). In teaching, the teacher needs to direct the learners' attention to representations that push towards structure (van den Heuvel-Panhuizen, 2008; Venkat et al., 2019). In the focal studies and the re-analyses associated with this thesis, the teacher directed the learners' attention to structured finger patterns, or alternatively the bead string grouped in two colors on the string, and the organization of the items/sets of items. This seemed to facilitate for the children to see items without counting them as single units. The results also reveal the teachers' different ways of highlighting the undivided 5, and 10 as constituted of two 5s, for instance circling the undivided 5, and thereby directing the children's attention to the semi-decimal structure. This provided the children with an alternative to single-unit counting, and gave them the opportunity to experience numbers as composite sets. These results add to previous early number research (e.g. Ellimore-Collin & Wright, 2009), regarding how teachers can draw attention to 5 as a benchmark[51], allowing learners to

[50] See for instance Article I (p. 301, Fig. 2).
[51] See for instance Enactments C and D (pp. 70–71).

extend their subitizing range: conceptual subitizing (Clements, 1999; Clements et al., 2019). In earlier intervention studies, for instance, it has been concluded that seeing numbers as composite sets of units – subitizing and conceptual subitizing – promotes children's understanding of part-whole relations of numbers (Jung et al., 2013, Benz, 2013; Schöner and Benz, 2017). Whereas these studies did not explicitly examine the role of finger patters related to subitizing, their findings related to Articles III and IV as well as the re-analysis can further contribute knowledge about how structured finger patterns can be emphasized in teaching. When children's attention is drawn to numbers as composite sets and highlighting how these sets (parts) are related to each other and to the whole, they are offered an alternative to single-unit counting (cf. Carpenter & Moser, 1984; Fuson, 1992). Furthermore, children's ability to see a part-whole relation as composite sets, by creating finger patterns and learning how to decompose/compose numbers (conceptual subitizing), might develop their ability to solve additive relation problems (cf. Clements et al., 2019). The results associated with this thesis imply that the relation of the parts and the whole is seen as the basis for addition and subtraction. Even though the structural approach emphasizes the part-whole relation from the outset, counting is not seen as unimportant. However, following the results, single-unit counting might not necessarily be the foundation for the development of arithmetic skills.

As mentioned previously, finger counting is strongly rooted in South African classrooms (Venkat, 2013). Therefore, finger patterns were not explicitly promoted in teaching this intervention, even though the students used them spontaneously, counting them as single units. Nevertheless, since it is known that some representations associated with a structural approach support the discernment of part-whole relations (Carpenter et al., 1999; Schmittau, 2004), other representations were introduced in the South African context. These were the part-whole schema (triad) and the bar diagram, combined with other more familiar representations as well as symbol representations. The results do not specifically reveal whether or not the triad diagram and the bar model supported the students' learning of part-whole relations. However, these representations seem to have afforded the students the possibility to discern the part and the whole at the same time, and helped the teachers to focus on relational reasoning (structure) instead of emphasizing single-unit counting. The results from the coding suggest that, in addition to the linking gestures, the verbal linking using the terms "parts and wholes" (cf. Payne and Rathmell, 1975) seems to have drawn the learners' attention to the within relations of numbers (Articles I and II)[52].

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[52] See for instance Article II (p. 5, Figure 1).

Paying attention to the within relations of numbers is fundamental in a structural approach to additive relations (Neuman, 1987; Schmittau, 2004); additionally, the "between examples relation" needs to be emphasized in teaching (Watson & Mason, 2006b), in order to be able to generate mathematical structure (Venkat et al., 2019). The results of this thesis suggest many different ways in which principles and ideas can be brought to the fore in teaching, by emphasizing relations between examples. For instance, the results revealed that more mathematical structure was brought to the fore in teaching additive relation problems in the second year than the first (Article II). These improvements in teaching are important in the light of South African evidence of disconnected teaching (Mathew, 2016; Venkat & Naidoo, 2012).

Earlier research emphasizes the importance of acquiring knowledge of the commutativity and complement principles for developing arithmetic skills (Baroody, 1999, 2016; Canobi et al., 2002); however, the research base does not tell us much about teaching that helps young learners to learn these principles. Some researchers deal with this deficit by suggesting curricula that emphasize relational reasoning (e.g. Zhou & Peverly, 2005; Schmittau, 2004; Sensevy et al., 2015). The results of the focal studies associated with this thesis, as well as those of the re-analysis, also address this deficit. By taking a structural approach to additive part-whole relations, it has been possible to offer several concrete illustrations of how, for instance, the commutative principle can be taught in different ways, by means of both concrete representations and abstract symbols[53]. For example, it was found that teachers directed learners' attention to the interchangeability of the parts/terms that composed the whole/sum by juxtaposing two examples. This enactment gave the children the opportunity to experience the commutative principle. One interpretation of these findings might be that, in addition to the teachers' enactment, the commutative principles had been discussed to various extents in planning meetings, the activities and examples were designed based on principles of variation and invariance to make it easier to discern the commutative principle, and the representations supported the discernment of part-whole relations from the outset.

However, the results also show ways of teaching in which relations and mathematical ideas and principles were **not** brought to the fore, and were hence not possible for the learners to discern. For instance, if examples are erased, it may be difficult to discern a specific relation, idea, or principle that was the object of attention[54]. Alternatively, if the sequence of examples of a

[53] See for instance Table 4, Enactment C (p. 73) and Article IV (pp. 24–25).

[54] Compare to Enactment A (p. 73) in the re-analysis.

hidden number of beads on the 10-snake is presented in a more or less randomly chosen order, it may be difficult to discern commutativity[55]. The results also suggest that, even though teachers in both projects had been involved in the design and planning of the activities and the underlying mathematical principles and ideas had been discussed, there were differences in the extent to which learning opportunities related to the mathematical ideas and principles were offered.

Furthermore, in the South African focal study, examining teaching missing number problems (Article II) revealed how the part-whole reasoning within a structural approach was brought to the fore by the teachers[56]. These "arithmetic-algebraic ideas" are seen as a challenging task for many learners (e.g. Carraher et al., 2006). The results can contribute knowledge about how principles from variation theory – variation in the midst of invariance – can be used for a carefully planned example space, in order to emphasize these ideas. In one of the lessons, two examples were recorded on the same piece of paper (so that they could not be erased) if, for instance, the position of the unknown number was to be focused on, the positions of the missing number varied, and the given numbers were the same[57]. Thereby, the children's attention was drawn to the aspect of position and they were given the opportunity to experience that the position of a missing number is not always the same. These results, involving how missing number problems were handled through symbol representations, offer implications regarding how to implement these ideas in primary teaching[58]. These findings would not have emerged if merely concrete representations and resources had been the object of attention in teaching, such as in the preschool intervention. Therefore, I would argue that the different contexts and age groups offer complementary knowledge about what can constitute the structural approach to additive part-whole teaching.

Making connections in teaching

The results from the first two articles and the re-analyses indicate that making connections in teaching can enhance children's learning opportunities. In mathematics education research, connections in mathematics are seen as an essential part of the instruction (Askew et al., 1997; Hiebert et al., 1999; Rowland, 2008). The results in this thesis support these findings: connections in teaching do seem to matter. Nevertheless, the sort of connections emphasized in this thesis refer to *making connections* and consist of the teach-

[55] See for instance Enacted Objects of Learning A and B in Article III.
[56] See for instance the example discussed on pp. 82–83.
[57] See for instance p. 47 and Article II (p. 7).
[58] See also Ekdahl & Runesson (2015).

ers' linking actions (verbal and gestural), studied on a fine-grained level. This kind of *making connections* on the micro-level, specifically the additive part-whole relation teaching, is reported on only to a limited extent in the research literature. There is empirical support for the importance of teachers' linking gestures in instruction (Alibali et al., 2013; Flevares & Perry, 2001), and preschoolers' and teachers' gestures are seen as an important source of developing children's mathematical skills in one-to-one interaction (Elia et al., 2014). Nevertheless, the findings associated with this thesis extend these studies' results by focusing on what "connecting work" (gestures and speech) looks like on a micro-level, related to the teaching of a structural approach to additive relations and how this connecting work might facilitate learning among young learners.

The criteria for defining a linking action were developed in the coding framework in Article I. Here, it was illustrated what teachers' linking actions looked like when connections were made within an example as well as between examples and representations. Therefore, it was possible to identify what additive relation teaching without linking actions or with few linking actions looked like, compared to teaching including several linking actions. These fine-grained differences in teaching the same topic enabled me to identify differences in teaching.

Given the importance of using various representations in mathematics and learning how to move between them (Carpenter et al.,1999; Lesh, Post & Behr, 1987), it cannot be taken for granted that all learners are able to discern how different representations are related to each other. As an illustration from Articles III and IV, it might be the case that the preschoolers do not get the chance to experience that the same collections of items can be represented with beads **and** with fingers, even though these representations are familiar to them. If this relation is pointed out by the teacher, by for instance circling the beads on the string and then grasping the raised fingers on a child's hand, saying "There are five, just like the fingers on one hand", it might facilitate for the learners to experience that the same number can be represented with both beads and fingers. Another illustration is selected from the Grade 3 classroom, where missing number problems were handled (Article III) and the number sentence representation and the double bar diagram[59] were presented at the same time. Even though the two representations were visible on the board and the same example as well as different examples were recorded in both representations, it cannot be taken for granted that all the children understand that the marking for the missing number was associated with the empty square in the double bar. However,

[59] See p. 27.

if the teacher, for instance, directs the learners' attention to these similarities by drawing lines or making a movement with her finger from the "empty line" in the number sentence to the empty square in the double bar, and discusses the positions of the missing number, it might happen that more learners are able to discern the similarities between the representations.

To some extent, there are overlaps between the way connections are applied in the coding framework (Articles I and II) as well as in the re-analysis and the MPM (Mediating Primary Mathematics) framework developed by Venkat and Askew (2018). However, the MPM framework takes its point of departure in sociocultural theory. Some principles from variation theory (e.g. Watson & Mason, 2006b) are used as a complement in the MPM framework within the mediation with examples, talk, and gestures. Connections used for linking between mathematical ideas are, for instance, seen as indicators of a higher qualitative level of instruction. However, given that the MPM framework is broader and used generically across topics and takes its departure in sociocultural theory, the coding in this thesis is conducted on an even more detailed level and linked to additive relations specifically, and variation theory is the solid theoretical base.

The re-analysis – a variation theoretical contribution?
In the re-analysis, the teachers' enactments were in focus. The enactments that included linking actions were analyzed on a highly detailed level. Also, in the analyses a distinction was made between dimensions of variation opened up and connections (linking actions) made by the teacher. Previous variation theoretical studies (e.g. Häggström, 2008; Runesson, 1999) have analyzed and compared teaching, using the concept of dimensions of variation as an analytical tool. However, in this thesis, both dimensions of variation and connections were used. This way of "expanding the analysis" might make a contribution to further discussions on how to analyze teaching using tools and principles from variation theory.

My findings align with the interpretation that learners' attention should be drawn to relevant relations and patterns (Kullberg et al., 2014; Marton, 2015; Watson & Chick, 2011). However, the re-analysis of the data shows what this could look like in practice; as its findings reveal qualitative differences in the enactment of the same mathematical idea or principle. These findings indicate that it is not only the example space provided and the dimensions of variation opened up (or not) in the teaching that imply what is made possible to learn, but also how the relations, ideas, or principles are explicitly pointed out by the teacher[60]. Through this analysis I may have found something that, while not completely new, is important to discuss from a pedagogical as well as a variation theoretical point of view.

According to variation theory, *making connections* might be seen as a way to offer variation within the specific dimension, or offer new values in one dimension that has already been opened up. However, building on the results from the micro-level re-analysis, I would argue that *making connections* is something that explicitly points out the features associated with the mathematical principle or idea focused on in the teaching in the dimension of variation that has been opened up; a way of extending the teaching, and expanding the learning that is offered, so to say. I would argue that, besides offering variation, connections can facilitate for children who do not see the relation, idea, or mathematical principle in focus even if it is perceptually visible in the learning situation.

Hence, the results of the re-analysis suggest that it is a combination of the dimensions of variation opened up and the connections made explicitly that seems to be powerful in what is made available for the children to learn[61]. Even though there is no robust data on learner performance associated with this thesis, the classroom learner data (Article I) tend to say that *making connections* matters. More connecting in order to work on partitioning a number in different ways was seen in the class in which more actions directing the attention to *completeness by systematicity* were identified compared to other classes. The empirical example of Enactment C (See Table 5, p. 78), in which the teacher emphasized systematicity, relates to the results on the worksheet following this teaching episode[62]. This class performed better than the two other classes. Out of 44 students, 26 produced all the possible partitions of whole number 9, and 14 of these 26 students showed systematic working; this can be compared to other classes, in which 2 out of 13 and 3 out of 20 students, respectively, produced a systematic pattern. These results may indicate that variation and connections matter. However, much more work is needed if any sort of claims is to be made about linking actions on the micro-level and the relation between "connecting teaching" and learner performances. Nevertheless, I have found a phenomenon – *variation and connections* – that is important to further explore from a variation

[60] The results in the articles also show that mathematical ideas and principles linked to additive relations were brought to the fore in the teaching. However, the combination of *dimensions of variation* and *connections* as analytical concepts was only applied in the re-analysis.

[61] Article II reveals similar results, but dimensions of variation were not used as a concept in the analysis.

[62] See Article I (p. 311) for a detailed description.

theoretical as well as an educational perspective, especially for early childhood and primary mathematics. Therefore, further intervention studies could be designed to investigate where the impact of connections might be tested.

Critical reflections on methods and findings

There are several questions concerning the validity and reliability of this thesis that need to be considered. During the whole research process, to strengthen the validity of this study, my ambition has been to provide a transparent description of how the data were sampled, considerations that were made, and how the analyses were carried out. The validity and reliability were strengthened by, for instance, continuously discussing method issues and analyses of video-recorded sequences and transcriptions at collaborative research meetings, both within and outside the project. The reliability of the coding associated with the analytical framework in the first two articles was improved when the co-authors coded a limited number of segments (agreed/disagreed) with the coding I had done. However, to establish even higher reliability in the agreement regarding the coding, it would have been desirable for the co-authors to have coded all lessons separately.

Using *segments* as the smallest unit of analysis (Articles I and II) has enabled me to describe differences in part-whole additive relation teaching. It might be the case that another unit of analysis would have given slightly different results. For instance, in Article II, according to the coding one of the teachers used fewer linking actions in 2014 than in 2013 (Article II, p. 5). However, within some segments (2014) multiple linking actions were identified, suggesting that one example was treated more mathematically in terms of emphasizing the within relations, compared to other examples in the sequence in which no linking action occurred. Consequently, if the smallest unit of analyses had been "even smaller", there might have been slightly different numbers of linking actions (coded units); however, this would likely not have revealed any great differences in the results of comparing the teaching over time.

There might be other shortcomings in the methods that had an influence on the results, for instance concerning the samples of video documentation that formed the basis of analysis of the conducted activities in Articles III and IV. The preschool teachers themselves decided which learning situations to upload to the server. Even if the large number (N=67) of uploaded videos likely provided me with sufficient variation in the enactments of the Snake game activity, other enactments that were not uploaded might have constituted another enacted object of learning. Likewise, the videos for discussion in the "teacher-researcher meetings" were selected in order to discuss mathematical ideas related to additive part-whole relations. It is not possible to know whether other choices of videos might have stimulated the mathematical discussion even more (or less), or supported the reflections on how to conduct the activities. Another issue that might have influenced the outcome is the number of teachers who participated. It might be the case that if other teachers or a greater number of teachers had participated, other aspects of "part-whole-teaching" could have been identified. However, a greater number of participating teachers would have generated more whole-class lessons and small-group teaching to analyze, which would have been too time-consuming. Furthermore, it might have made the results more robust if analyses had been conducted on all 65 participating children's ways of experiencing numbers in the intervention program. Nevertheless, following the phenomenographic tradition (Marton, 1981), the main purpose of this analysis was to understand different ways of experiencing numbers and collect further knowledge about preschoolers' development of arithmetic skills rather than to make generalizations. Finally, a re-analysis of another mathematical principle or idea (e.g. the complement principle), or a re-analysis of qualitative differences in the enactment of commutativity in preschool settings, might have deepened the results, and might have strengthened the argument for teaching with "variation and connection" even more.

Reflections on the design

The studies associated with thesis had the characteristics of educational design research (van den Akker, et al., 2006; Cobb et al., 2003), whereby teaching part-whole relations grounded in a structural approach was implemented in natural settings and in close collaboration between the practitioners and researchers. It could be concluded that the process orientation approach allowed the involved teachers and researchers to continuously discuss underlying mathematical principles related to numbers from different perspectives. I saw the advantage of having a common interest in teaching part-whole relations. It was possible to link theory to practice by continuously referring back to lessons, reflecting on teaching episodes, and discussing the teachers' experiences. Furthermore, the iterative design of these focal studies, with few activities being focused on and reenacted, enabled the teachers to reflect on their teaching and become more aware of the children's ways of reasoning about number relations. Also, the iterative design focusing on one specific topic over a longer period may contribute to teachers' deeper understanding of the mathematical ideas that are possible to bring to the fore within teaching about number relations.

Implementing activities that unpack mathematical ideas can be a challenge, especially when dealing with unfamiliar ideas such as alternative approaches to numbers. For example, it sometimes seemed to be challenging for the Grade 3 teachers, who were not familiar with the structural approach, to change their focus. Instead of focusing on producing answers to a sequence of tasks, they were to encourage the students to, for instance, reason about the relation between three numbers (number triples), comparing two missing number problems without being asked to produce the correct answers. Similar to other studies (Mulligan et al., 2013; Savard et al., 2013), it seems as if the participating teachers in the current studies needed to become confident with the structural approach and mathematical ideas themselves in order to be able to implement them in practice.

In conclusion, several questions and shortcomings related to the validity and reliability of the methods, and other issues concerning the designs and how these issues might have affected the results, have been discussed above, and there are likely others. Based on these focal studies and the re-analysis it is not possible to generalize the results, for instance that this is what differences in part-whole teaching grounded in a structural approach look like in all classrooms and preschool groups, or to make claims regarding what structured part-whole relation teaching "should" look like. This has not been my intention with the thesis. Nevertheless, following Larsson's (2005) argumentation, the nuances and detailed descriptions as well as the clear structure, I will argue, strengthen the interpretation of the results. By using video observations and making comprehensive analysis that capture the essential aspects of teaching additive part-whole relations, I have, on a clear empirical basis, captured what can constitute part-whole relation teaching. Furthermore, I have strengthened the quality of the results by focusing on *a specific* mathematical content. According to variation theory, which forms the basis of this thesis, if several different mathematical contents had been studied, too great a variation in teaching would have been demonstrated and the analyses would have revealed other results related to teaching. Instead, keeping the content invariant made it possible to describe different ways of handling this specific mathematical content (additive part-whole relations). For instance, in the analyses I examined how the *same* part-whole activities and additive relation problems were handled by *different* teachers, and in the re-analysis, I analyzed *various* enactments of the *same* mathematical idea/principle. I was thus able to describe subtle differences in the ways the same activity, or the same idea, was taught. In order to enhance the descriptions of part-whole teaching and highlight subtle differences in the teaching, I have provided concrete illustrations of it.

Generalization of research results is sometimes discussed by focusing on similarities between the researched context and other contexts, and the extent to which a study's results can be transferred to other contexts (Cohen, Manion & Morrison, 2011; Larsson, 2009). This thesis is framed within two quite different contexts. A South African classroom is completely different from a Swedish classroom. However, all Swedish and South African classrooms do not have the same settings; there are also contextual nuances within the same country. In this thesis, such differences and nuances related to contexts and settings are not decisive for the results as the focus is on examining the teaching of a specific mathematical content while taking a specific approach. Hence, the result of this thesis can contribute deeper knowledge about additive part-whole relation teaching and learning grounded in a structural approach. The detailed results' descriptions of teaching and learning opportunities and the "content focus" have the potential to be used in other contexts and settings. Thereby, other researchers and teachers might discover the relevance of the results in their contexts (Larsson, 2005). Thus, the results in this thesis can indeed be useful outside its own context.

Pedagogical implications and future research

Supported by the results of this thesis, implications for teaching have been discussed in previous sections. Nevertheless, some noteworthy implications for teaching practices deserve further mention. For instance, the results imply that taking additive part-whole relations of numbers as a starting point in teaching (in preschool and the first grades of lower primary school) enables learners to reason and explore the part-whole relations of numbers. This is the case in concrete and play-based activities (without written numerals) as well as in tasks with different kinds of representations, including symbols.

The activities developed in the two contexts are not in themselves a prerequisite for teaching part-whole relations. However, teachers need to consider what activities, examples, resources, and representations facilitate for their learners to discern numbers' relations and numbers as composite sets. Teachers should also reflect on what mathematical ideas and principles, essential for developing arithmetic skills, are important and possible to bring to the fore in their teaching. The results indicate that the teacher's pedagogical role also consists of directing learners' attention to the relation, idea, or principle and not taking for granted that all children are able to discern the necessary aspect of the object of learning on their own. Following this idea of making connections, it is important for the teacher to reflect on: what sort of questions should I use; what do I compare with what; how can I explicitly point out the relation with fingers, hands, drawing, etc. This kind of pedagogical support might provide qualitative reasoning about relations across examples and make the mathematics available to the children, offering a way to generate mathematical structure (Venkat et al., 2019).

Another pedagogical implication of the results is related to structured finger patterns. The results suggest that a teacher ought to reflect on the differences between fingers used as a tool for counting single units and fingers used as structuring part-whole relations, and see the potential of introducing and encouraging structured finger patterns as a tool for solving additive relation problems in early childhood mathematics. This might help the children to focus on numbers' part-whole relations instead of getting stuck counting forward and backward, and use single-unit counting as their main strategy for solving addition and subtraction problems.

The results of this research have generated new questions that are potentially of interest to gather more knowledge about. One suggestion for further research that has already been mentioned is to further explore the concepts of *variation and connections* in teaching by analyzing other teaching episodes and larger samples. Results from one of the focal studies revealed that structured finger patterns offered a powerful way for five-year-olds to solve additive relation problems within the number range of 1–10. Therefore, a second suggestion for future research would be to conduct an intervention study, with an educational design research, in a South African Reception class (the first year of formal schooling). Here it would be possible to implement the results from the focal studies of the FASETT project, developing and adjusting the design to large group sizes, and implement structured finger patterns to see units as composite sets, and examine how this could be offered as an alternative to the strongly rooted single-unit counting seen in South African classrooms. A third suggestion for future research would be to investigate how mathematical ideas and principles, which have been found to be essential for developing flexibility in additive relation problem solving, can be implemented in practice by preschool teachers – not only examining how these ideas and principles can be implemented as collectively planned activities in an iterative process, but also discussing how the same mathematical ideas and principles can be brought to the fore in more play-based learning situations.

8 The four articles: a summary

Short summaries of the four articles associated with this thesis are provided here. Article I is published in *Educational Studies in Mathematics*; Article II is published in *South African Journal of Childhood Education*; Article III was re-submitted in August 2019 and is under revision; Article IV was submitted in September 2019.

Article I

Ekdahl, A-L, Venkat, H., Runesson, U. (2016). Coding teaching for simultaneity and connections Examining teachers' part-whole additive relations instruction. *Educational Studies in Mathematics*, *93*(3), 293–313.

This article presents a coding framework based on simultaneity and connections. The coding focuses on micro-level attention to three aspects of simultaneity and connections: between representations; within examples; and between examples. Criteria were developed for coding what was viewed as mathematically important within part-whole additive relations instruction. Teachers' use of multiple representations within an example, attention to part-whole relations within examples, and relations between multiple examples were identified, with teachers' linking actions in speech or gestures pointing to connections between them. In this article, the coding framework is detailed and exemplified in the context of a structural approach to part-whole teaching in South African Grade 3 lessons. The coding framework made it possible to identify fine-grained differences in the teachers' handling of part-whole relations related to the simultaneity of, and connections between, representations and examples as well as within examples. Further, the associations between the simultaneity and connections, seen through the coding framework in sections of teaching and students' responses on worksheets following each teaching section, were explored.

Article II

Ekdahl, A-L; Venkat, H. Runesson, U & Askew, M. (2018). Weaving in Connections: Studying changes in early grades additive relations teaching. *South African Journal of Childhood Education.*

This article presents aspects of teaching that draw attention to connections both within and between examples, in order to explore the potential objects of learning that are brought into being in the classroom space and thus what is made available to learn. The article's focus is on exploring differences in teaching over time, in the context of learning a study style development activity of additive relation problems in three Grade 3 classes in South Africa. In a context where highly localized and fragmented instruction has been noted, this study reports on the nature and extent of changes in connections in instruction over time. The application of a coding framework focused on simultaneity and connections in teaching points to a richer range of structural relations within examples, and more connecting work between examples, in the second year than in the first.

Article III

Ekdahl, A-L. (in review). Different Learning Possibilities from the Same Activity – Swedish Preschool Teachers' Enactment of a Number Relation Activity.

This paper studies differences in the implementation of a number activity called the Snake game. Nine Swedish preschool teachers worked in collaboration with a research team, enacting the same activity with their groups of five-year-olds over a three-month period. Variation theory forms the basis of the analysis of 67 video-recorded enactments. The results suggest that an activity such as the Snake game can bring to the fore various aspects of numbers through differences in enactment. The activity became mathematically richer when the teachers compared the children's different finger patterns and used systematically varied examples of number relations. This study's results contribute knowledge about characteristics of teaching that foreground numbers' part-whole relations.

Article IV

Björklund, C., Ekdahl, A-L. & Runesson Kempe, U. (in review). Implementing a Structural Approach in Preschool Number Activities. Effects of an Intervention Program.

This article reports on results from an intervention program implementing a structural approach to arithmetic problem solving in relation to learning outcomes among preschoolers. Using the fundamental principles of the variation theory of learning to develop the intervention and as an analytical framework, we discuss teaching and learning in commensurable terms. The research question is how teaching grounded in a structural approach and designed based on principles of variation theory is reflected in children's learning of numbers. To answer this, three analyses were conducted: i) addressing how the children's ways of experiencing numbers changed after participating in the intervention; ii) addressing how the theoretical ideas were afforded in the intervention program; and iii) synthesizing how the affordance was associated with the children's arithmetic learning. One group of eight children participating in the intervention program was chosen for thorough analysis. Progression was observed in how the children changed their ways of experiencing numbers during the intervention to allow them to enact more advanced arithmetic strategies, which was associated with the structural approach in teaching. The results also show how analysis focusing on aspects discerned in learning and aspects afforded in teaching provides an alternative way of describing arithmetic learning, with significant implications for teaching practice.

9 Summary in Swedish

I den här avhandlingen har resultatet från fyra empiriska studier och en reanalys sammanförts i syfte att fördjupa förståelsen av vad som kan utgöra en strukturell ansats till undervisning och lärande av tals additiva delhelhetsrelationer. Med tals additiva del-helhetsrelation menas att exempelvis talet 7 kan delas upp i 5 och 2. En strukturell ansats innebär i det här sammanhanget att relationerna mellan tals delar och helhet står i fokus från början och ses som grunden för att lära sig addition och subtraktion (Davydov 1982; Neuman, 1987). En mer vanligt förekommande beskrivning av barns kunskapsutveckling i aritmetik (ex. Carpenter, Hiebert & Moser, 1981; Fuson, 1992) innebär att barn till en början räknar med hjälp av konkreta föremål och använder ett-till-ett räkning, för att sedan utveckla strategier som att räkna vidare från ett givet tal och därefter utveckla mer sofistikerade räknestrategier, för att slutligen lära sig kombinationer som talfakta. Vissa forskare (ex. Ellemor-Collins & Wright, 2009; Neuman, 1997) ifrågasätter vad som händer med de barn som fastnar i ett-till-ett räkning, och inte utvecklar mer hållbara strategier för att lösa addition och subtraktionsuppgifter. Istället föreslår de att barnen ska erbjudas strategier som baseras på tals relationer. Denna avhandling tar sin utgångspunkt i Neumans (1987) argument om att tals del-helhetsrelationer är avgörande för barns utveckling av aritmetiska färdigheter och Davydovs (1982) argument om vikten av att i undervisningen erbjuda barn möjligheten att utforska tal som relationer, istället för att fokusera på ett-till-ett räknande. I sådant utforskande av relationer förefaller det vara av betydelse att barn får möjlighet att se delarna och helheten samtidigt samt att se antal som sammansatta enheter. Det har visat sig att vissa representationer och formationer av föremål möjliggör denna förmåga, exempelvis fingermönster (Neuman, 2013).

Många forskare inom det matematikdidaktiska fältet lyfter fram vikten av att barn/elever behöver lära sig att se relationer mellan tre tal i en additiv talrelation (ex. 7/5/2) (Baroody, 2016; Clements & Sarama, 2009) för att därmed kunna utveckla ett flexibelt sätt att lösa additions- och subtraktionsproblem (ex. 5 + _ = 7; 7 - _ = 5). I sammanhanget förespråkas även vikten av att barn och yngre elever lär sig matematiska principer såsom kommutativitet för addition och reversibilitet mellan addition och subtraktion. Dessa principer är möjliga att introducera tidigt i undervisningen utifrån en strukturell ansats till tals del-helhetsrelationer (Schmittau, 2004).

En teori som fokuserar på hur ämnesinnehållet hanteras i undervisningen är variationsteorin (Marton, 2015). Eftersom lärandet, utifrån teorin, alltid är relaterat till *något* som ska läras – lärandeobjektet (Marton & Tsui, 2004; Runesson, 2005) ansågs variationsteorin vara lämplig för att undersöka hur tals additiva del-helhetsrelationer hanteras i undervisningen. Utifrån variationsteorin beskrivs lärande som att urskilja aspekter av lärandeobjektet som inte tidigare har urskiljts, alternativt urskilja aspekterna på ett mer differentierat sätt (Marton, 2015). Enligt teorin kan dessa aspekter endast urskiljas om de öppnas upp som dimensioner av variation (Häggström, 2008). Variationsteorin erbjuder relevanta begrepp och verktyg för att analysera undervisning och lärande, varför denna valdes i avhandlingen.

Resultat från tidigare interventionsstudier visar att det är möjligt för förskolebarn att lära sig om tals del-helhetsrelationer (ex. Fischer, 1990; Jung, m.fl., 2013). Det finns ett begränsat antal interventioner som i mer detalj har undersökt lärares del-helhetsundervisning grundad i en strukturell ansats. Det finns få interventionsstudier som tydligt har beskrivit hur aktiviteterna designats och implementeras i praktiken, och vilket teoretiskt antagande dessa studier grundar sig i. Denna avhandling syftar till att bidra med kunskap inom detta fält, genom att studera en strukturell ansats till tals additiva del-helhetsrelationer från både ett undervisnings- och lärandeperspektiv, med hjälp av principer från variationsteorin (Marton, 2015).

Utifrån såväl ett pedagogiskt perspektiv (ex. Askew, 1999) som ett variationsteoretiskt perspektiv (ex. Kullberg m.fl., 2015) lyfts i den här avhandlingen betydelsen av lärarens agerande fram, och vikten av att rikta barnens och elevernas uppmärksamhet mot det som ska läras. Det kan exempelvis handla om hur läraren påvisar samband och relationer mellan begrepp, exempel, matematiska idéer och områden. Tidigare studier (ex. Alibali m.fl. 2013; Flevares & Perry, 2001) visar att lärarens gester och verbala kommunikation kan stödja elevers lärande. Venkat och Askew (2018) hävdar dock att uppmärksammande av samband och relationer inte alltid är så vanligt förekommande i undervisning med yngre elever.

Syfte och frågeställning

Det övergripande syftet med avhandlingen är att fördjupa förståelsen av undervisning och lärande av tals additiva del-helhetsrelationer, utifrån en strukturell ansats. I avhandlingen används variationsteoretiska idéer för att utvidga kunskapen om undervisning i additiva del-helhetsrelationer och för att studera de lärandemöjligheter som erbjuds.

Följande frågeställning avses att besvaras:

- Vad kan konstituera en strukturell ansats till undervisning och lärande av tals additiva del-helhetsrelationer, som främjar barns och yngre elevers lärande?

De fyra artiklarna har specifika syften och forskningsfrågor, som besvaras i respektive artikel. Utöver dessa resultat och i syfte att mer i detalj utforska vad som kan konstituera undervisning och lärandemöjligheter av tals additiva del-helhetsrelationer, har en re-analys av data från dessa artiklar genomförts. Skillnader i hur matematiska principer och idéer lyfts fram i undervisning är i fokus i denna analys.

Metod

Avhandlingen är inramad i två forskningsprojekt: *Wits Maths Connect-Primary* projektet och FASETT-projektet[63]. Wits Maths Connect-Primary syftar till att förbättra matematikundervisningen från förskoleklass till årskurs 3 i tio skolor i Johannesburg. Inom detta project designades en mindre interventionsstudie där jag och några andra forskare arbetade tillsammans med tre lärare från årskurs 3 på en av dessa skolor (Artikel I och II). Det svenska FASETT-projektet syftar till att implementera del-helhetsaktiviteter som främjar förskolebarns utveckling av aritmetikfärdigheter. Två av avhandlingens delstudier (Artikel III och IV) baseras på data från FASETT-projektet.

I båda dessa delprojekt arbetade lärare och forskare i ett tätt samarbete och utgångspunkten togs i den strukturella ansatsen till tals additiva del-helhetsrelation. Aktiviteter och uppgifter planerades gemensamt, lärarna genomförde dem i sina praktiker och tillsammans diskuterade sedan forskare och lärare hur dessa skulle kunna förfinas för att möjliggöra för barn och elever att lära sig det som avsågs. Endast ett fåtal aktiviteter och uppgifter användes, men prövades och förfinades under processen. Samtliga aktiviteter tog sin utgångspunkt i en strukturell ansats till tals delhelhetsrelationer. Det empiriska materialet som har samlats in och analyserats i denna avhandling består av 18 videofilmade lektioner i årskurs 3 i Sydafrika (Wits Maths Connect-Primary) och 80 videofilmade undervisningsmoment i barngrupper som går sitt sista år i svensk förskola (FASETT-projektet). Därutöver valdes 16 individuella intervjuer där barn fick lösa aritmetikuppgifter i uppstarten av FASETT projektet och efter det att projektet var avslutat för analys.

[63] Förmågan Att Sinnligt Erfara de Tio första Talen som nödvändig grund för aritmetiska färdigheter.

Datamaterialet har i samtliga fall analyserats på en detaljerad nivå, utifrån ett variationsteoretiskt perspektiv (Marton, 2015; Marton & Tsui, 2004). I Artikel I and II utvecklades ett ramverk för att identifiera relationer inom del-helhets exempel, mellan del-helhetsexempel och mellan representationer i tre lärares undervisning om additiva del-helhetsrelationer. De teoretiska begreppen *simultaniety* och *connections* utgjorde de centrala analysverktygen i dessa artiklar. *Connections,* åsyftar lärares handlingar med icke-verbala gester och verbala uttryck. Definitionen av gester utgår från det McNeill (1992) kallar ”utpekande gester”, vilket kan vara lärarens hand- eller fingerrörelser i syfte att påvisa exempelvis en relation mellan två representationer eller sambandet mellan två del-helhetsrelationer. *Connections* kan även handla om lärarens verbala uttryck, såsom frågor som explicit pekar ut denna relation eller detta samband (Se gärna sidan 83 för ett exempel).

I Artikel III och IV användes begreppet *dimensioner av variation* (Marton, 2015) för att undersöka och jämföra lärandemöjligheter som erbjöds i lärares undervisning. Därutöver analyserades i Artikel IV barns uppfattningar av tal före och efter interventionsprogrammet, samt vilka aspekter som var möjliga att urskilja i undervisningen och vilka aspekter som barnen urskilde. I syfte att mer i detalj utforska vad som kan konstituera undervisning av tals additiva del-helhetsrelationer, genomfördes en ny analys av de olika sätt på vilka samma matematiska princip eller idé behandlades i undervisningen. De tre matematiska idéer/principer som analyserades var; *tal som sammansatta enheter, kommutativitet* och *systematik*. I den här analysen användes begreppen *dimensioner av variation* och *connections,* i syfte att beskriva kvalitativa skillnader i lärares undervisning och vilka olika lärandemöjligheter som därmed erbjöds.

Resultat och konklusion

En sammanfattning av resultatet från re-analysen visar kvalitativa skillnader i undervisningen av samma matematiska idé. I ett sätt att undervisa synliggjordes inte den specifika idéen, även om så var möjligt. I ett annat sätt att undervisa öppnades aspekten upp som en dimension av variation genom att exempelvis en systematisk sekvens av exempel dokumenterades på tavlan, vilket gjorde det möjligt för barnen att visuellt urskilja den matematiska idéen. Ytterligare ett sätt att undervisa, kännetecknas av att förutom att en eller flera dimensioner av variation öppnades upp, använde läraren *connections* (icke verbalt och verbalt) då läraren pekade ut, och riktade barnens och elevernas uppmärksamhet mot denna matematiska idé. Dessutom identifierades sätt där idéen mer explicit synliggjordes i undervisningen. Sammantaget bidrog dessa, ibland subtila skillnader som identifieras i analysen av undervisningen av samma innehåll, till olika lärandemöjligheter för barnen/eleverna.

Resultat från de fyra empiriska studierna och re-analysen bidrar med en djupare förståelse för vad som kan utgöra en strukturell ansats till tals additiva del-helhetsrelationer. Om man ser till artiklarna och re-analysen visar resultaten att specifika representationer, exempel och problem verkar gynna barns/elevers förmåga att urskilja delarna och helheten samtidigt. Det övergripande resultatet indikerar att det är viktigt att flera del-helhetsexempel erbjuds av läraren, men det är också viktigt *vilka* exempel som erbjuds. En systematisk sekvens av exempel har potential att lyfta fram relationerna mellan olika exempel, vilket i sin tur ger barnen eller eleverna möjlighet att urskilja en systematik i de olika sätt på vilket exempelvis ett tal kan delas upp eller urskilja en matematisk princip som kommutativitet.

De övergripande resultaten tyder på att principer från variationsteorin har gjort det möjligt att fördjupa kunskapen om ”del-helhetsundervisningen” och lärandemöjligheter som erbjuds. Resultatet visar att det som görs möjligt att lära om additiva del-helhetsrelationer relaterar till vilka aspekter som öppnas som dimensioner av variation i undervisningen (Marton, 2015). Men, framförallt visar resultaten hur viktigt det är med *connections* , det vill säga hur läraren med sina handlingar (muntligt och med gester) pekar ut relationer och samband. Om och hur läraren explicit uppmärksammar relationer, matematiska idéer och principer, förefaller vara väsentligt för vad barnen erbjuds att lära. De ibland subtila skillnader som har identifierats i analysen av undervisningen av samma innehåll, erbjuder olika lärandemöjligheter för barnen och eleverna. Hur variation erbjuds (dimensioner av variation öppnas) och om och hur relationer och matematiska idéer explicit pekas ut av läraren eller inte, verkar vara kritiskt för elevernas lärande av additiva del-helhetsrelationer. Vidare framgår det av de separata artiklarna och re-analysen att den strukturella ansatsen och antaganden från variationsteorin är möjlig att implementera i de två olika kontexterna, valda i den här avhandlingen.

Diskussion

Yngre elevers förståelse av tal och utveckling av aritmetiska färdigheter är ett välbeforskat område (ex. Carpenter, Hiebert & Moser, 1981; Fuson, 1992). Vissa delar av resultatet i avhandlingen stämmer överens med tidigare forskningsresultat. Emellertid tillför resultatet från avhandlingen kunskap om en teoretiskt underbyggd undervisning där idéer, matematiska principer, mönster och relationer kopplade till en strukturell ansats till additiva relationer står i fokus. Även om tidigare forskning lyfter fram vikten av att erbjuda mer än ett exempel, att påvisa samband mellan exempel (ex. Watsson & Mason, 2006b) och representationer (ex. Lesh m.fl., 1987), samt att lärares gester är viktiga i matematikundervisningen (ex. Alibali m.fl., 2013) ger avhandlingens resultat ett bidrag till hur *connections* i undervisningen kan se ut inom ett specifikt avgränsat ämnesområde. De detaljerade och empirinära analyser som har genomförts och beskrivits i så väl artiklarna som i reanalysen, bidrar med fördjupad kunskap om vad som kännetecknar en undervisning där tals del-helhetsrelationer lyfts fram i den här åldersgruppen. Exempelvis visar resultaten hur lärare kan synliggöra matematiska principer, såsom kommutativitet i såväl en förskole- som i en grundskolekontext. Resultaten visar även att barn i undervisningen erbjuds alternativ till att alltid räkna föremål ett och ett, och istället se gruppen av föremål som sammansatta enheter (jmf. Clement m.fl. 2019). Vidare ger resultatet implikationer om att användandet av strukturerade fingermönster (jmf. Neuman, 1987; 2013) underlättar för fem- till sexåringar att urskilja relationer mellan tals delar och helhet och kan därmed vara ett verktyg för barn i denna åldersgrupp att lösa additions- och subtraktionsproblem.

Det är viktigt att betona att avhandlingens resultat inte är menade att vara en lösning på hur undervisning utifrån en strukturell ansats *ska* se ut, snarare kan resultaten vara ett stöd i lärares reflektion över vad i undervisningen som kan gynna barns och elevers lärande av tals del-helhetsrelationer. Det handlar inte om att replikera de specifika aktiviteterna eller uppgifterna som designats och används i projekten. Det är inte uppgifter och aktiviteter i sig som är avgörande, utan i vilket syfte de används, hur de kan gynna barns och elevers möjligheter att se samband, relationer mellan tal, exempel och representationer, samt möjligheter att utforska matematiska idéer och principer. Dock föreslår resultatet att *connections*; lärarens pedagogiska handlingar genom gester och verbala uttryck, det vill säga vad uppmärksamhet riktas mot, verkar främja vad barnen och eleverna ges möjlighet att lära sig.

Sättet på vilket re-analysen i kappan gjorts skiljer sig till viss del från andra variationsteoretiska analyser av undervisning av samma innehåll (ex. Häggström, 2008; Runesson, 1999). I analysen använde jag de teoretiska begreppen *dimension av variation* och *connections* och kunde därmed identifiera subtila skillnader i undervisningen, skillnader som kanske inte hade identifierats om endast ett av begreppen hade använts. Min förhoppning är att sättet på vilket jag har utvidgat analysen kan bidra till fortsatta diskussioner om hur undervisning och lärandemöjligheter kan analyseras med hjälp av principer och begrepp från variationsteorin.

Förutom de resultat som har presenterats och diskuterats ovan, ger de separata artiklarnas resultat ytterligare implikationer för praktiken. Värt att notera är att ämnesinnehållet står i förgrunden i samtliga av dessa implikationer. Min förhoppning är att dessa resultat kan ses som en källa till reflektion över vad som kan känneteckna en strukturell ansats till additiva del-helhetsrelationer och den viktiga roll läraren i förskolan och tidigare årskurser har, i mötet med barn och elever, för att *matematiken* ska bli synliggjord.

References

Ahlberg, A. (1997). *Children's ways of handling and experiencing numbers* (No. 113). Acta Universitatis Gothoburgensis.

Alibali, M. W., Nathan, M. J., Wolgram, M. S., Church, R. B., Jacobs, S. A., Martinez, C. J., & Knuth, E. J. (2013). How teachers link ideas in mathematics instruction using speech and gesture: A corpus analysis. *Cognition and Instruction, 32*(1), 65–100.

Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics, 70*(2)*,* 97–109.

Askew, M., Brown, M., Rhodes, V., Wiliam, D., & Johnson, D. (1997). *Effective Teachers of Numeracy: Report of a study carried out for the Teacher Training Agency*. London: King's College, University of London.

Baroody, A. J. (1985). Mastery of basic number combinations: Internalization of relationships or facts? *Journal for Research in Mathematics Education*, 83–98.

Baroody, A. J., & Gannon, K. E. (1984). The development of the commutativity principle and economical addition strategies. *Cognition and Instruction, 1*(3), 321–339.

Baroody, A. J. (1987). The development of counting strategies for single-digit addition. *Journal for Research in Mathematics Education,* 141–157.

Baroody, A. J. (1999). Children's relational knowledge of addition and subtraction. *Cognition and Instruction*, *17*(2), 137–175.

Baroody, A. J., Lai, M. L., & Mix, K. S. (2006). The development of young children’s early number and operation sense and its implications for early childhood education. *Handbook of research on the education of young children, 2,* 187–221.

Baroody, A. J. (2016). Curricular approaches to connecting subtraction to addition and fostering fluency with basic differences in grade 1. *PNA, 10*(3), 161–190.

Baroody, A., & Purpura, D. (2017). Early number and operations: Whole numbers. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 308–354). Reston, VA: National Council of Teachers of Mathematics.

Benz, C. (2013). Identifying quantities of representations—Children using structures to compose collections from parts or decompose collections into parts. In U. Kortenkamp, B. Brandt, C. Benz, G. K. S. Ladel, R. Vogel (Eds). *Early Mathematics Learning Selected Papers of the POEM 2012 Conference* (pp. 189–203). New York: Springer.

Björklund, C., & Runesson Kempe, U. (in press). Framework for analysing children’s ways of experiencing numbers. *Proceedings from CERME11*, 2019, Utrecht, the Netherlands.

REFERENCES

Björklund, C., Kullberg, A., & Kempe, U. R. (2018). Structuring versus counting: critical ways of using fingers in subtraction. *ZDM*, 1–12.

Bowden, J., & Marton, F. (1998). *The university of learning: Beyond quality and competence*. London: Routledge.

Brissiaud, R., (1992). A tool for number construction: Finger symbol sets. (Trans.) In J. Bideaud, C. Meljac, & J.-P. Fischer (Eds.), *Pathways to number: Children's developing numerical abilities* (pp. 41–65). Hillsdale, NJ: Lawrence Erlbaum.

Brown, A. L. (1992). Design experiments: Theoretical and methodological challenges in creating complex interventions in classroom settings. *The journal of the learning sciences*, *2*(2), 141–178.

Canobi, K. H. (2005). Children's profiles of addition and subtraction understanding. *Journal of Experimental Child Psychology*, *92*(3), 220–246.

Canobi, K. H., Reeve, R. A., & Pattison, P. E. (2002). Young children's understanding of addition concepts. *Educational Psychology*, *22*(5), 513–532.

Carpenter, T. P., Hiebert, J., & Moser, J. M. (1981). Problem structure and first-grade children's initial solution processes for simple addition and subtraction problems. *Journal for Research in Mathematics education*, 27–39.

Carpenter, T. P., & Moser, J. M. (1984). The acquisition of addition and subtraction concepts in grades one through three. *Journal for research in Mathematics Education*, 179–202.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999*). Children's Mathematics: Cognitively Guided Instruction.* Portsmouth. NH: Heinemann.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking Mathematically: Integrating Arithmetic & Algebra in Elementary School.* Portsmouth, NH: Heinemann.

Carraher, D. W., Schliemann, A. D., Brizuela, B. M., & Earnest, D. (2006). Arithmetic and algebra in early mathematics education. *Journal for Research in Mathematics education*, 87–115.

Cheng, Z.-J. (2012). Teaching young children decomposition strategies to solve addition problems: An experimental study. *The Journal of Mathematical Behavior, 31*(1), 29–47.

Ching, B. H. H., & Nunes, T. (2017). Children's understanding of the commutativity and complement principles: A latent profile analysis. *Learning and Instruction*, *47*, 65–79.

Clements, D. H. (1999). Subitizing: What is it? Why teach it? *Teaching Children Mathematics, 5*, 400–405.

Clements, D. H., & Sarama, J. (2009). *Learning and teaching early math: The learning trajectories approach*. New York: Routledge.

Clements, D. H., Sarama, J., & MacDonald, B. L. (2019). Subitizing: The neglected quantifier. In A. Norton & M. W. Alibali (Eds.) *Constructing Number, Merging Perspectives from Psychology and Mathematics Education* (pp. 13–45). Switzerland: Springer, Nature.

Cobb, P., Boufi, A., McClain, K., & Whitenack, J. (1997). Reflective discourse and collective reflection. *Journal for Research in Mathematics Education, 28*(3), 258–277.

Cobb, P., Confrey, J., DiSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational researcher, 32*(1), 9–13.

Cohen, L., Manion, L., & Morrison, K. (2011). *Research Methods in Education* (7th ed.). London: Routledge.

Davydov, V. V. (1982). The psychological characteristics of the formation of elementary mathematical operations in children. In T.P. Carpenter, J.M. Moser & T.A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 225–238.). Hillsdale, NY: Lawrence Erlbaum Associates.

Ding, M., & Auxter, A. E. (2017). Children's strategies to solving additive inverse problems: a preliminary analysis. *Mathematics Education Research Journal, 29*(1), 73–92.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*, 103–131.

Dyson, N. I., Jordan, N. C., & Gluttin, J. (2013). A number sense intervention for low-income kindergartners at risk for mathematics difficulties. *Journal of Learning Disabilities*, *46*(2), 166–181.

Ekdahl, A-L. (2012). *Elevers skilda sätt att erfara talmönster – en studie av elever i årskurs 3 och 4.* [Pupils different ways of discerning mathematical patterns – a study of pupils in grade 3 and 4]. (Master's thesis). Department of Mathematics and Science Education, Stockholm University.

Ekdahl, A-L. (2019). Differences in pre-school teachers' ways of handling a part-part-whole activity. In J. Häggström, Y. Liljekvist, J. B. Ärlebäck, M. Fahlgren, O. Olande (Eds.). Proceedings of MADIF 11, The eleventh Research Seminar Swedish Mathematics Education (pp. 71–80). Karlstad.

Ekdahl, A-L., & Runesson, U. (2015). Teachers' responses to incorrect answers on missing number problems in South Africa. In Sun, X., Kaur, B., Novotna, J. (Eds.) *Proceedings of ICMI STUDY 23: Primary mathematics study on whole number.* June 2015, Macao, China.

Ekdahl, A. L., Venkat, H., & Runesson, U. (2016). Coding teaching for simultaneity and connections. *Educational Studies in Mathematics, 93*(3), 293–313.

Ekdahl, A-L., & Björklund, C. (2017). A structural approach to the ten first natural numbers: Pre-school teachers' different ways of handling the artefact; 'Snake game'. *In 17th Biennal EARLI conference for Research on Learning and Instruction.* Tampere, Finland.

Ekdahl, A-L., Venkat, H., Runesson, U., & Askew, M. (2018). Weaving in connections: Studying changes in early grades additive realations teaching. *South African Journal of Childhood Education, 8*(1), 1–9.

Ekdahl, A-L., Björklund, C., & Runesson Kempe, U. (2019). Teaching to change ways of experiencing numbers – An intervention program for arithmetic learning in preschool. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psy-*

chology of Mathematics Education (Vol. 2, pp. 209–216). Pretoria, South Africa: PME.

Elia, I., Gagatsis, A., & van den Heuvel-Panhuizen, M. (2014). The role of gestures in making connections between space and shape aspects and their verbal representations in the early years: findings from a case study. *Mathematics Education Research Journal, 26*(4), 735–761.

Ellemor-Collins, D., & Wright, R. B. (2009). Structuring numbers 1 to 20: Developing facile addition and subtraction. *Mathematics Education Research Journal, 21*(2), 50–75.

Fischer, F. E. (1990). A part–part–whole curriculum for teaching number in the kindergarten. *Journal for Research in Mathematics Education, 21*(3), 207–215.

Flevares, L. M., & Perry, M. (2001). How many do you see? The use of nonspoken representations in first-grade mathematics lessons. *Journal of Educational Psychology, 93*(2), 330–345.

Freudenthal, H. (1991). *Revisiting mathematics education.* Dordrecht, The Netherlands: Kluwer.

Fuson, K. (1982). An analysis of the counting-on solution procedure in addition. In T.P. Carpenter, J.M. Moser & T.A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 67–81). Hillsdale, NY: Lawrence Erlbaum Associates.

Fuson, K. (1992). Research on whole number addition and subtraction. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 243–275). New York: Macmillan.

Gibson, J. J., & Gibson, E. J. (1955). Perceptual learning: Differentiation or enrichment? *Psychological review, 62*(1), 32.

Gravemeijer, K., & Cobb, P. (2013). Design Research from the Learning Design Perspective. In T., Plompt & N., Nieveen, (Eds.) *Educational Design Research.* SLO Netherlands institute for curriculum development.

Haider, H., Eichler, A., Hansen, S., Vaterrodt, B., Gaschler, R., & Frensch, P. A. (2014). How We Use What We Learn in Math: An Integrative Account of the Development of Commutativity. *Frontline Learning Research, 2*(1), 1–21.

Heath, C., Hindmarsh, J., & Luff, P. (2010). *Video in qualitative research.* Sage Publications.

Herbst, P., & Chazan, D. (2009). Methodologies for the study of instruction in mathematics classrooms. *Recherches en Didactique des Mathématiques*, 29(1), 11–32.

Hiebert, J., Stigler, J., & Manaster, A. (1999). Mathematical features of lessons in the TIMSS Video Study. *Zentralblatt für Didaktik der Mathematik, 31*(6), 196–201. https://doi.org/10.1007/BF02652695.

Hoadley, U. (2007). The reproduction of social class inequalities through mathematics pedagogies in South African primary schools. *Journal of Curriculum Studies*, 39(6), 679–706.

Hostetter, A. B. (2011). When do gestures communicate? A meta-analysis. *Psycological bulletin, 137*(2), 297–315.

Huang, R., Zhang, Q., Chang, Y. P., & Kimmins, D. (2019). Developing students' ability to solve word problems through learning trajectory-based and variation task-informed instruction. *ZDM*, *51*(1), 169–181.

Hunting, R. P. (2003). Part-whole number knowledge in preschool children. *The Journal of Mathematical Behaviour, 22*(3), 217–235.

Häggström, J. (2008). *Teaching systems of linear equations in Sweden and China: What is made possible to learn?* Gothenburg: Acta Universitatis Gothoburgensis.

Jung, M. (2011). Number Relationships in Preschool. *Teaching Children Mathematics, 17*(9), 550 –55.

Jung, M., Hartman, P., Smith, T., & Wallace, S. (2013). The Effectiveness of Teaching Number Relationships in Preschool. *International Journal of Instruction, 6*(1), 165–178.

Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels: Building meaning for symbols and their manipulation. *Second handbook of research on mathematics teaching and learning*, *2*, (pp. 707–762). Charlotte, NC: Information Age Publishing.

Kullberg, A. (2010). *What is taught and what is learned. Professional insights gained and shared by teachers of mathematics*. Gothenburg: Acta Universitatis Gothoburgensis.

Kullberg, A., Runesson, U., & Mårtensson, P. (2014). Different possibilities to learn from the same task. *PNA, 8*(4), 139–150.

Larsson, S. (2005). Om kvalitet i kvalitativa studier. *Nordisk pedagogik*, *25*(1), 16–35.

Larsson, S. (2009). A pluralist view of generalization in qualitative research. *International journal of research & method in education*, *32*(1), 25–38.

Laski, E. V., Ermakova, A., & Vasilyeva, M. (2014). Early use of decomposition for addition and its relation to base-10 knowledge. *Journal of Applied Developmental Psychology, 35*(5), 444–454.

Lesh, R., Post, T., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.) *Problems of representation in teaching and learning of mathematics* (pp. 33–40). Hilldale, NJ: Erlbaum.

Lo, M. L. (2014). *Variationsteori: för bättre undervisning och lärande*. Lund: Studentlitteratur.

Lüken, M. M. (2012). Young children's structure sense. *Journal für Mathematik-Didaktik, 33*(2), 263–285.

Lüken, M., & Tiedemann, K. (2019). Young children explain repeating patterns. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 49–56). Pretoria, South Africa: PME.

Marton, F. (1981). Phenomenography—describing conceptions of the world around us. *Instructional science, 10*(2), 177–200.

Marton, F. (2015). *Necessary conditions of learning*. New York, NY: Routledge.

Marton, F., & Booth, S. A. (1997). *Learning and awareness*. Mawah, NJ: Lawrence Erlbaum.

Marton, F., & Pang, M. F. (2006). On some necessary conditions for learning. *The Journal of the Learning Sciences, 15*(2), 193–220.

Marton, F., & Tsui, A. B. (2004). *Classroom discourse and the space of learning*. New York, NY: Routledge.

Mason, J., & Pimm, D. (1984). Generic examples: Seeing the general in the particular. *Educational Studies in Mathematics*, *15*(3), 277–289.

Mason, J., Stephens, M., & Watson, A. (2009). Appreciating mathematical structure for all. *Mathematics Education Research Journal, 21*(2), 10–32.

Mathews, C. (2016). Division means less … Chains of signification in a South African classroom. Extended paper presented at the *13th International Congress on Mathematical Education*, Hamburg, 24–31 July 2016.

Maunula, T. (2018). *Students' and teachers' jointly constituted learning opportunities—The case of linear equations*. Gothenburg: Acta Universitatis Gothoburgensis.

McNeill, D. (1992). *Hand and mind: What gesture reveals about thought*. Chicago, IL: University of Chicago Press.

Morrison, S. (2018) *Developing Early Number Learning using Math Recovery Principles*. Doctoral Thesis. University of the Witwatersrand. Johannesburg.

Mulligan, J., & Mitchelmore, M. (2009). Awareness of pattern and structure in early mathematical development. *Mathematics Education Research Journal, 21*(2), 33–49.

Mulligan, J. T., Mitchelmore, M. C., English, L. D., & Crevensten, N. (2013). Reconceptualizing early mathematics learning: The fundamental role of pattern and structure. In L. D. English & J. T. Mulligan (Eds.), *Reconceptualizing early mathematics learning* (pp. 47–66). Dordrecht, The Netherlands: Springer.

Mulligan, J. T. & Vergnaud, G. (2006). Research on children's early mathematical development: Towards Integrated perspectives. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 261–276). London: Sense Publishers.

Murata, A., & Fuson, K. (2006). Teaching as assisting individual constructive paths within an interdependent class learning zone: Japanese first graders learning to add using 10. *Journal for Research in Mathematics Education, 37*(5), 421–456.

Nesher, P. (1982). Levels of description in the Analysis of Addition and Subtraction Word Problems. In T.P. Carpenter, J.M. Moser & T.A. Romberg (Eds.), Addition and subtraction: *A cognitive perspective* (pp. 67–81). Hillsdale, NY: Lawrence Erlbaum Associates.

Nesher, P., Greeno, J. G., & Riley, M. S. (1982). The development of semantic categories for addition and subtraction. *Educational Studies in Mathematics, 13*(4), 373–394.

Neuman, D. (1987). *The origin of arithmetic skills: A phenomenographic approach.* Gothenburg: Acta Universitatis Gothoburgensis.

Neuman, D. (2013). Att ändra arbetssätt och kultur inom den inledande aritmetikundervisningen [Changing approach and culture within the introduction of arithmetic]. *Swedish Nordic Studies in Mathematics Education*, *18*(2), 3–46.

Pang, M. F. (2003). Two faces of variation: On continuity in the phenomenographic movement. *Scandinavian journal of educational research*, *47*(2), 145–156.

Pang, M. F., & Marton, F. (2003). Beyond``lesson study'': Comparing two ways of facilitating the grasp of some economic concepts. *Instructional Science*, *31*(3), 175–194.

Papic, M. M., Mulligan, J. T., & Mitchelmore, M. C. (2011). Assessing the development of preschoolers' mathematical patterning. *Journal for Research in Mathematics Education, 42*(3), 237–269.

Payne, J. N., & Rathmell, E. C. (1975). Mathematics Learning in Early Childhood: Number and Numeration. *National Council of Teachers of Mathematics Yearbook.*

Piaget, J. (1965). *The child's conception of number.* New York: Routledge.

Plomp, T. (2013). Educational design research: An introduction. In T., Plomp & N., Nieveen, (Eds.) *Educational Design Research.* SLO Netherlands institute for curriculum development.

Polotskaia, E. (2017). Who's wrong? Tasks fostering understanding of mathematical relationships in word problems in elementary students. *ZDM, 49*(6), 823–833.

Polotskaia, E., & Savard, A. (2018). Using the Relational Paradigm: effects on pupils' reasoning in solving additive word problems. *Research in Mathematics Education*, *20*(1), 70–90.

Programmation et Progression ACE. (n. d.) *Progression Modules, SIT, (2012–2013),* (pp. 72–112).

Radford, L. (2009). Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings. *Educational Studies in Mathematics, 70*, 111–126.

Resnick, L. B. (1983). A developmental theory of number understanding. In H. Ginsburg (ed.), *The development of mathematical thinking* (pp. 109–151). New York: Academic Press.

Resnick, L. B., Bill, V., & Lesgold, S. (1990). From Protoquantities to Number Sense. Paper presented at the *Psychology of Mathematics Education Conference.* Mexico, July, 1990.

Richland, L.E., (2015) Linking Gestures: Cross-Cultural Variation During Instructional Analogies. *Cognition and Instruction*, 33:4, 295–321, DOI:10.1080/07370008.2015.1091459

Rowland, T. (2008). The purpose, design, and use of examples in teaching of elementary mathematics. *Educational Studies in Mathematics, 69*(2), 143–163.

Runesson, U. (1999). Variationens pedagogik. Skilda sätt att behandla ett matematiskt innehåll [The pedagogy of variation. Different ways of handling a mathematical content]. Gothenburg: Acta Universitatis Gothoburgensis.

Runesson, U. (2005). Beyond discourse and interaction. Variation: A critical aspect for teaching and learning mathematics. *Cambridge Journal of Education*, *35*(1), 69–87.

Runesson, U. (2006). What is it possible to learn? On variation as a Necessary Condition for Learning. *Scandinavian Journal of Educational research*, *50*(4), 397–410.

Salvin, R. E. (2010) Experimental studies in education. In Creemers, B., P., M., Kyriakides, L. Sammons, P. (Eds.) *Methodological advances in educational effectiveness research, Quantitative Methodology Series*. New York: Routledge.

Savard, A., Polotskaia, E., Freiman, V., & Gervais, C. (2013). Tasks to promote holistic flexible reasoning about simple additive structures. *Proceedings of ICMI Study*, 22, 271–280.

Schollar, E. (2008). *Final Report: The primary mathematics research project 2004–2007 - Towards evidence-based educational development in South Africa*. Johannesburg: Eric Schollar & Associates.

Schmittau, J. (2004). Vygotskian theory and mathematics education: Resolving the conceptual-procedural dichotomy. *European Journal of Psychology of Education, 19*(1), 19–43.

Schmittau, J., & Morris, A. (2004). The development of algebra in the elementary mathematics curriculum of VV Davydov. *The Mathematics Educator, 8*(1), 60–87.

Schöner, P., & Benz, C. (2017). "Two, three and two more equals seven"- Preschoolers' perception and use of structures in sets. *CERME 10,* Feb 2017, Dublin, Ireland.

Sensevy, G., Quilio, S., & Mercier, A. (2015). Arithmetic and Comprehension at Primary School. In Sun, X., Kaur, B., Novotna, J. (Eds.) *Proceedings of ICMI STUDY 23: Primary mathematics study on whole number*. June 2015, Macao, China.

Sophian, C., & McCorgray, P. (1994). Part-whole knowledge and early arithmetic problem solving. *Cognition and Instruction*, *12*(1), 3–33.

Sophian, C., & Vong, K. I. (1995). The parts and wholes of arithmetic story problems: Developing knowledge in the preschool years. *Cognition and Instruction*, *13*(3), 469–477.

Steffe, Thompson & Richard, (1982). Children's Counting in Arithmetical Problem Solving**.** In T.P. Carpenter, J.M. Moser & T.A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective*. (pp. 83–89). Hillsdale, NY: Lawrence Erlbaum Associates.

Steffe, L. P., Cobb, P., & von Glasersfeld, E. (1988). *Construction of arithmetical meanings and strategies*. New York, NY: Springer.

Svenson, O., & Sjöberg, K., (1982). Solving simple Subtractions during the first three school years. *Journal of Experiment Education*. *50* (91–100).

Swedish Research Council. (2011). *Good research practice* (Report No. 3.2011). Stockholm: Vetenskapsrådet.

Tsamir, P., Tirosh, D., Levenson, E., Tabach, M., & Barkai, R. (2015). Analyzing number composition and decomposition activities in kindergarten from a numeracy perspective. *ZDM*, *47*(4), 639–651.

Van den Akker, J., Gravemeijer, K., McKenney, S., & Nieveen, N. (Eds.). (2006). *Educational design research*. London: Routledge.

Van den Heuvel-Panhuizen, M. (2008). *Children learn mathematics: A learning-teaching trajectory with intermediate attainment targets for calculation with whole numbers in primary school.* The Netherlands: Sense Publisher.

Venkat, H. (2013). Using temporal range to theorize early number teaching in South Africa. *For the learning of mathematics*, *33*(2), 31–37.

Venkat, H., & Adler, J. (2012). Coherence and connections in teachers' mathematical discourses in instruction. Pythagoras, *33*(3), 25–32.

Venkat, H., & Naidoo, D. (2012). Analyzing coherence for conceptual learning in a Grade 2 numeracy lesson. *Education as Change, 16*(1), 21–33.

Venkat, H., Ekdahl, A-L., & Runesson, U. (2014). Connections and simultaneity: Analysing South African G3 Part-part-whole teaching. In C. Nicol, S. Oesterle, P. Liljedahl & D. Allan (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36*, (Vol. 5, pp. 337–344). Vancouver, Canada: PME.

Venkat, H., & Askew, M. (2018). Mediating primary mathematics: theory, concepts, and a framework for studying practice. *Educational Studies in Mathematics, 97*(1), 71–92.

Venkat, H., Askew, M., Watson, A., & Mason, J. (2019). Architecture of Mathematical Structure. *For the Learning of Mathematics*, *39*(1), 13–17.

Vergnaud, G., (1982). A classification of Cognitive Tasks and Operations of Thought Involved in Addition and Subtraction Problems. In T.P. Carpenter, J.M. Moser & T.A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 35–59). Hillsdale, NY: Lawrence Erlbaum Associates.

Verschaffel, L., Greer, B., & De Corte, E. (2007). Whole number concepts and operations. In F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 557–628). Charlotte, NC: Information Age Publishing Inc.

Warren, E. (2003). The role of arithmetic structure in the transition from arithmetic to algebra. *Mathematics Education Research Journal*, *15*(2), 122–137.

Watson, A., & Chick, H. (2011). Qualities of examples in learning and teaching. *ZDM, 43*(2), 283–294.

Watson, A., & Mason, J. (2006a). Seeing an exercise as a single mathematical object: Using variation to structure sense-making. *Mathematical Thinking and Learning, 8*(2), 91–111.

Watson, A., & Mason, J. (2006b). Variation and mathematical structure. *Mathematics Teaching (Incorporating Micromath), 194*, 3–5.

Wynn, K. (1992). Children's acquisition of the number words and the counting system. *Cognitive Psychology, 24*(2), 220–251.

REFERENCES

Zhou, Z., & Peverly, S. T. (2005). Teaching addition and subtraction to first graders: A Chinese perspective. *Psychology in the Schools, 42*(3), 259–272.

Appendix

Tasks given in the interviews:

A. The Guessing-game

First:

7=_+_	Seven marbles are counted and then hidden in the interviewer's closed hands. How many can there be in the one hand and how many in the other hand? (Multiple options possible).
7=4+_	The interviewer opens one of her hands, showing four marbles. If one knows there are seven altogether, how many are there in the closed hand?

Second:

7=_+_	The same seven marbles are put back on the table. Then hidden in the interviewers closed hands. How many …
7=2+_	The interviewer opens one of her hands, showing two marbles and ask the same question.

Third: The same seven marbles and the same procedure.

7=_+_

7=1+_

B. Context tasks

5=4+_	You and your friend collected five shells together, you collected four of them, how many did your friend collect?
9-7=_	If you have nine shells and your friend has seven shells, how many more shells do you have?
2+5=_	You have two shells and receive five more, how many do you have then?
10-6=_	If you have ten candies and eat six of them, how many are left?
3+_=8	You have three glasses, but are going to set the table for eight people, how many more glasses do you need?
_-3=6	On the morning of your birthday party, you blew up balloons. At the party, three balloons broke, and there were only six balloons left. How many balloons did you blew up that morning?

In Article IV, the first two tasks (7=_+_; 7=4+_) in the Guessing game and the Context tasks are sampled.