Developing a classification scheme of definitions of Fermi problems in education from a modelling perspective

Jonas Bergman Ärlebäck[1] and Lluís Albarracín[2]

[1]Linköping University, Department of Mathematics, Sweden; jonas.bergman.arleback@liu.se

[2]Serra Húnter Fellow, Universitat Autònoma de Barcelona, Spain; lluis.albarracin@uab.cat

In this paper we use a modelling perspective to analyse three descriptions and definitions of so-called Fermi problems found in the literature. We discuss how the three definitions align with, and what they potentially have to offer to, realistic or applied modelling, contextual modelling, educational modelling (either a didactical or conceptual), socio-critical modelling, epistemological or theoretical modelling, and cognitive modelling. Our findings show that the definitions share some similarities, but for the most part are formulated in lose terms. From a modelling perspective, we found that the conceptualisation of Fermi problem we studied foremost and directly align with contextual modelling and both strands of educational modelling. We also discuss the seemly incompatibility between Fermi problems and the other modelling perspectives, and suggest new lines of research on Fermi problems in particular, and on conceptualizing modelling in general.

Keywords: Fermi problem, modelling, modelling perspectives.

Introduction

The notion *Fermi problem* is tributed to the Italian Enrico Fermi (1901-1954), the 1938 Nobel Prize winner in physics, who had a special liking for posing and solving problems like *How many shopping malls are there in the United States?* (Anderson & Sherman, 2010). Fermi's philosophy was that any thinking and reasonably educated person should be able to solve problems of this type by just combing one's capabilities of making quantitatively accurate realistic and intelligent order of magnitude estimates, reasoning, and doing simple calculations (Efthimiou & Llewellyn, 2007). The perhaps most famous and classic *Fermi problems* is *How many piano tuners are there in Chicago?* Allegedly Fermi repeatedly gave this problem to his physics students at the University of Chicago many times over the years, and illustrated of the power of such reasoning by quickly calculating an astoundingly accurate and reasonable answer based on just a few sensible assumptions and estimates. Besides going under the name *Fermi problems*, these types of problems are also called *back-of-envelope calculation problems* or *order of magnitude (estimation) problems*.

Much due to the influence of Fermi, *Fermi problems* have been widely used in physics and engineering college courses in the US. Indeed, one can find many "shout-out" advocating and claiming various beneficiary effects for using *Fermi problems* in teaching, often exemplifying the assumptions and calculations involved in an explicit example as well as listing *Fermi problems* to try out in the classroom (see for example Carlson (1997)). However, it seems that systematic science- and engineering education research focusing on *Fermi problem* is sparse or at best marginalized. In recent years however, a number of studies in mathematics education have focused on the use of *Fermi problems* in the teaching and learning of mathematical modelling. Peter-Koop (2004) used *Fermi problems* to investigate third and fourth graders' problem solving strategies and among other things found that students' solutions "revealed multi-cyclic modelling processes" (p.

461). At the upper secondary level Ärlebäck (2009) investigated the potential of using *Fermi problems* as 'miniature modelling problems' to introduce modelling. Using so-called MADs (Modelling Activity Diagrams) the result showed the complexity of the modelling process involved when students at the high school level engaged in solving *Fermi problems*, something which recently also have been documented for college students (Czocher, 2016). *Fermi problems* have also been used to study students' reasoning involved in solving so called *Big numbers estimation problems*, such as *How many persons can fit in the playground of our high school to attend a concert there?* Albarracín and Gorgorió (2013) showed that problems requiring equivalent mathematical solving approaches, but formulated using different context-specific wording, resulted in the students using differing solving strategies. Building and furthering this study, Albarracín and Gorgorió (2014) showed that some of the solving strategies the students used normally not would been considered valid as mathematics classrooms activities. For example, one such strategy found was *the exhaustive recounting of objects*, which requires excessive effort and/or time, or input from external sources which would eliminate the need to solve the problem altogether. However, it was concluded that 47% of the students' strategies were based on mathematical models.

Sriraman and Lesh (2006) have argued for the introduction of *Fermi problems* as interdisciplinary tasks which potentially bridge and connect mathematics and other school subjects. In addition, due to the directness aspect of *Fermi problem*, one can also easily incorporate different social issues of interest within the task, such as estimating the amount drinking water consumed, the consumption of gasoline or other fuels, the amount of discarded food or other ecological types of problems (Sriraman & Knott, 2009).

In this paper we present our on-going work aimed at doing an exhaustive and systematically review of the literature on *Fermi problems* from all educational fields. As part of this endeavour, we in this paper analyse three different definitions and descriptions of *Fermi problems* in the literature from a modelling perspective. We use the classification of perspectives on modelling by Kaiser and Sriraman (2006), and map the key features of *Fermi problems* in the definitions and description onto the different perspectives and discuss the potential of using *Fermi problems* in a modelling setting from different viewpoints. Our aim is that this preliminary analysis will point out areas and directions that are worth to further explore in the larger study.

The research question that guided our work in this paper was: *How does the definitions and descriptions of Fermi problems in the literature align with different perspective on modelling?*

Methodology and method

Three of our goals with doing a systematic review of *Fermi problem* is to i) elaborate a research grounded coherent definition that characterize *Fermi problems* as completely as possible; ii) find and describe the connection between *Fermi problems* and modelling in general and connections between modelling perspectives in particular; and iii) create a research agenda for future research (Ärlebäck & Albarracín, in preparation).

The literature for the exhaustive review was identified using a) search engines such as Academic Primer, ERIC, Google Scholar, and Scopus, and key word searches on *Fermi problem/question/ estimate*, *back-of-envelope problem*, *order of magnitude estimate*, *"how many piano tuners"*, b) snowballing (using literature already found and concluded relevant for the research to identify further literature; cf. Petticrew & Roberts, 2006), and c) asking colleagues with other mother toughs than our own for papers in their native language. It should be noted that there are similar notions and concepts in chemistry and physics, and hence the searches will result in large numbers of hits. However, the majorities of these can be dismissed since they not are about education. The papers that did have and educational focus was skimmed and paper that only mentioned Fermi problems in the passing was excluded from the final selection. This resulted in a list of 59 papers from mathematics education and other educational subjects (such as science, economics and engineering), written in English, Spanish, German Japanese and Dutch. All 59 papers were read and three representative definitions and descriptions were selected. We then used the characterisation of perspectives on modelling by Kaiser and Sriraman (2006) as an analytic lens to compare the three definitions as well as contrast them relative the different modelling perspectives. We chose this high-level framework to structure the analysis rather than a more specialized and "derived" framework (such a framework classifying modelling tasks) for two reasons. Firstly we wanted do use the existing definitions and descriptions of Fermi problems in the literature as the point of departure for the analysis, and secondly we wanted to use a neural framework not based on too specific cultural or epistemological stances.

The three definitions and descriptions of Fermi problems

Although the number of papers related to *Fermi problem* found is numerous, many of them do not offer any explicit definitions of the notion, but are rather based on shared knowledge and often provide some elaborated examples to characterize how *Fermi problems* are conceptualized and understood.

For the analysis and discussion in this paper we have chosen to focus on the following three different definitions and characterisations of *Fermi problems* in the literature: Ärlebäck (2009), Goodchild and Fuglestad (2008), and Sriraman and Knott (2009). All three sources are selected from the mathematics education research literature and use and discuss characteristics of *Fermi problems* and how students work with these. Ärlebäck (2009) is included since the characterizing of *Fermi problem* in this paper is one of the most cited and used definition in the more recent literature (in 9 of the 59 papers in our list of research paper on *Fermi problem*). Goodchild and Fuglestad (2008) and Sriraman and Knott (2009) are both included since their papers are representative for much of the other papers in literature. One can discuss whether the expressed conceptualizations of *Fermi problems* in the three papers are definitions in strict sense or mere characterizations or descriptions, but to avoid ambiguity and awkward formulations in the paper we will from now on refer to the three simply as definitions.

The first quote, from now on referred to as (Ärlebäck), comes from Ärlebäck (2009) who suggested and adapted so-called *Realistic Fermi problems* defined by:

- their *accessibility*, meaning that they can be approached by all individual students or groups of students, and solved on both different educational levels and on different levels of complexity. A realistic Fermi problem does not necessarily demand any specific pre-mathematical knowledge;
- their clear real-world connection, to be *realistic*. As a consequence a Realistic Fermi problem is more than just an intellectual exercise, and I fully agree with Sriraman and Lesh (2006)

when they argue that "Fermi problems which are directly related to the daily environment are more meaningful and offer more pedagogical possibilities" (p. 248);

- the *specifying and structuring of the relevant information and relationships* needed to tackle the problem. This characteristic prescribes the problem formulation to be open, not immediately associated with a know strategy or procedure to solve the problem, and hence urging the problem solvers to invoke prior constructs, conceptions, experiences, strategies and other cognitive skills in approaching the problem;
- the absence of numerical data, that is the *need to make reasonable estimates* of relevant quantities. An implication of this characteristic is that the context of the problem must be familiar, relevant and interesting for the subject(s) working in it;
- (in connection with the last two points above) their inner momentum to *promote discussion*, that as a group activity they invite to discussion on different matters such as what is relevant for the problem and how to estimate physical entities. (Ärlebäck, 2009, pp. 339-340, italics in original)

The second definition of *Fermi problem* is by Goodchild and Fuglestad (2008), who draw on (Swan & Ridgway, n.d.). Their definitions will be referenced as (Goodchild & Fuglestad):

> These [*Fermi problems*] are 'plausible estimation' tasks, which consist of one or two easily-stated questions which at first glance seem impossible to answer without reference material, but which can be reasonably estimated by following a series of simple steps that use only common sense and numbers that are generally known or amenable to estimation (Goodchild & Fuglestad, 2008, p. 52).

The third and last definition, from this point referred to as (Sriraman & Knott), is from Sriraman and Knott (2009):

> Fermi problems are estimation problems used with the pedagogical purpose of clearly identifying starting conditions or assumptions and making educated guesses about various quantities or variables which arise within a problem with the added requirement that the end computation be feasible or computable by hand. (p. 220)

Analysing and situating Fermi problems from different perspectives on modelling

We now briefly summarise the main characteristics of the different perspectives in Kaiser and Sriraman (2006) and discuss how the three definitions of *Fermi problems* above "fits" with the respective perspective and why. The brief characterization presented of *realistic or applied modelling, contextual modelling*, *educational modelling* (either a *didactical* or *conceptual*), *socio-critical modelling*, *epistemological or theoretical modelling*, and *cognitive modelling* are based on Kaiser and Sriraman (2006) and Blomhøj (2009).

The *realistic or applied perspective of modelling* stresses the importance of using authentic problems from science and industry as well as for the students to engage in the whole modelling process rather than fragmented parts thereof. Although none of the definitions explicitly excludes authentic contexts from science and industry, they all tend to suggest and promote more mundane and everyday problem contexts: “the context of the problem must be familiar, relevant and interesting for the subject(s)” (Ärlebäck); “reasonably estimated by following a series of simple steps that use only common sense” (Goodchild & Fuglestad); “making educated guesses” (Sriraman & Knott). It could be noted that the use of the word ‘realistic’ in Ärlebäck’s definition might be misleading with respect to the realistic and applied perspective of modelling. This wording merely stresses that the Ferm*i problem* should have a meaningful real-world connection and not be purely intellectual in nature. However, in the sense that *Fermi problems* that focus on issues like the number of piano tuners in a city, or the number of grains of sand in a glass, are not normally relevant questions for students. On the other hand, problems that ask students to estimate the amount of trash produced, or the volume of fresh water consumption, connect with the students’ physical and social environment and have meanings by themselves. The meaning of ‘realistic’ in the realistic or applied perspective on modelling is much stronger. This suggests that Femi problems, at least as portrayed in the definitions discussed here, have little to offer to the realistic and applied perspective on modelling.

Contextual modelling, having its roots in the word problem solving tradition, is centred around the design of carefully structured and meaningful situations, were the students develop, refine, and extend their own mathematical constructs as well as apply these in different contexts. The emphasis on meaning-making in the contextual modelling perspective can be seen echoed in (Goodchild & Fuglestad) and (Ärlebäck) but not evidently in (Sriraman & Knott). In (Goodchild & Fuglestad) the students have to meaningfully understand and come to grips with the context of the *Fermi problem* at hand to overcome the “easily-stated questions which at first glance seem impossible to answer”, whereas (Ärlebäck) stresses the problem formulation to “be open, not immediately associated with a know strategy or procedure to solve the problem, and hence urging the problem solvers to invoke prior constructs, conceptions, experiences, strategies and other cognitive skills in approaching the problem“, which resonates with the ‘traditional’ problem solving tradition that historically has been strongly associated with the contextual perspective on modelling. (Sriraman & Knott) on the other hand describe *Fermi problems* as intentionally designed with the explicit “pedagogical purpose of clearly identifying starting conditions or assumptions and making educated guesses about various quantities or variables which arise within a problem”. This focuses more on solving (meta-) strategies than stressing meaning-making or for the students to develop, refine, and extend their own mathematical constructs.

Both ‘flavours’ of *educational modelling* (didactical and conceptual) are so-called integrative perspectives in that they seek to combine modelling as a learning goal in its own right as well as modelling as a vehicle for learning other content matter. The two strands within this perspective forefront pedagogical goals such as using modelling as a didactical tool for structure learning processes and modelling as a mean to introduce concepts and promote concept development. Within this perspective, the cyclic view of modelling (aka the modelling cycle) has a prominent role. Looking at the three definitions, we argue that (Ärlebäck) and (Sriraman & Knott) both put forward *Fermi problems* as vehicles for learning other curricula objectives as well as have explicit didactical considerations as central features. On the one hand the two characteristics of *accessibility* and *discussion promoting* in (Ärlebäck) address classroom dynamics and classroom norms as innate components of the *Fermi problems* themselves. (Sriraman & Knott) on the other hand explicitly describe the use of *Fermi problems* as having a “pedagogical purpose”. Looking at the definition in (Goodchild & Fuglestad) however, these educational aspects are not emphasised.

Central from the *socio-critical perspective on modelling* is critical reflection and critique of mathematics role and function in society as manifested in the use of mathematical models and modelling. Although it is an innate feature of *Fermi problems* to engage the problem solver in making reasonable, and arguable critically realistic, assumptions and estimates, these need not inherently nor explicitly focus on or be connected to the social dimensions involved in the context of the problem. Similarly as for the realistic and applied perspective on modelling, there are nothing in the definitions that explicitly stresses the fundamental core characteristics of the respective perspective. That is, with regards to the socio-critical perspective on modelling, neither of the definitions analysed forefronts the social aspects and implications of the use of models and modelling in society. However, it is worth noticing that (Goodchild & Fuglestad) use a formulation that indicates that *Fermi problem* can be used to get students to appreciate the potential and power of mathematics to address and make sense of real problems in the world, namely “questions which at first glance seem impossible to answer without reference materials”.

Epistemological modelling focuses on theory building and uses modelling as a mean to re-construct topics and branches of mathematics as a discipline. Neither of the three definitions (Ärlebäck), (Goodchild & Fuglestad) and (Sriraman & Knott) express the ambition to draw on *Fermi problems* to derive theory in terms of re-building and constructing mathematical (sub-)topics or (sub-)areas. Indeed, as pointed out in Ärlebäck (2009), *Fermi problem* can be experienced as limited with respect to various mathematical content, and given a particular learning goal within mathematics, it might be very challenging to design and formulate a *Fermi problem* that focuses on eliciting this content in a natural way.

The *cognitive modelling perspective* is sometime described as meta-perspective in the sense that it focuses on fundamental research questions related to various aspects of modelling from a cognitive perspective. From the point of view of the cognitive modelling perspective being a meta-perspective that guides research into the practices of mathematical modelling and all that goes around and into the modelling process, we find it difficult to elaborate on what the different definitions might offer in this respect. We fear that such a discussion would be far too speculative to be constructive or productive and not inform our aim about how to classify definitions of *Fermi problems*.

Discussion, conclusions, and future research

The limited analysis we have presented in this paper points to some of the challenges in developing a classification scheme of definitions of *Fermi problems* from a modelling perspective. Having engaged in this exercise, we conclude that the level of interpretation needed to apply the different perspectives on modelling as analytical lens introduces uncertainty in the results. Partly we believe this has to do with the fact that the definitions and characterizations of *Fermi problems* in the papers found in literature are vague and ambiguous. However, we also contribute some of this difficulty to the used perspectives on modelling in Kaiser and Sriraman (2006), which describes the modelling debate from evolutionary viewpoint, connecting todays trends and approaches with their historical traditions and roots. This suggests on the one hand, that an overarching and general characterisation and definition of *Fermi problem* could make the research on *Fermi problem* more connected and coherent, rather than scattered and compartmentalized. On the other hand, is also suggests that alternative ways of thinking about and characterize different aspects of the on-going modelling debate might provide new insight into the growing literature on the teaching and learning models and modelling – ranging from basic ontological and epistemological considerations to different aspects of both general and particular designs and practices involved in the teaching and learning of mathematical models and modelling.

In going through the papers in our list of research on *Fermi problem* and looking at the definitions, we found that most definitions adapted in the different papers are of a local and pragmatic nature in the sense that they are relevant and work fine in the particular setting and study described and reported on the paper. We also identified patterns of linkages between the work of some authors who draws and build on each other's work, whereas some pieces of research are more like isolated islands. To us this is a second indication motivating the need for a more coherent view and characterisation of *Fermi problems*, in order to coordinate the various research finding in the literature and advance our collective experiences and knowledge with respect to *Fermi problems*.

As we mentioned before, there is no consensus of what the characteristics of *Fermi problems* are in the research literature. This is perhaps not surprising since this type of problems have been part of everyday mathematics and science teaching in various degrees and in various forms for decades, but only in recent time been subject for more systematic investigations. Doing the analysis of the three definitions have pointed to some communalities and difference in general and from a modelling perspective in particular. We are of the opinion that *Fermi problems* have much to offer from a modelling perspective, both as a tool to promote modelling (cf. (Ärlebäck, 2009)) and as a research tool. Hence we would like to promote the use of *Fermi problems* in schools, and through our systematic literature review (Ärlebäck & Albarracín, in preparation) we hope to lay the foundation for finding a common ground for promoting these types of problems in education and research. Our next step is to build on the initial ideas and results presented in this paper to make a more carful analysis of our sought out literature, with the ambition, to among other things, come up with a tentative and coherent definition of *Fermi problems* together with a rationale for how, when and why to used then in connection to mathematical modelling.

References

Albarracín, L., & Gorgorió, N. (2013). Problemas de estimación de grandes cantidades: Modelización e influencia del contexto. *Revista Latinoamericana de Investigación en Matemática Educativa*, *16*(3), 289–315.

Albarracín, L., & Gorgorió, N. (2014). Devising a plan to solve Fermi problems involving large numbers. *Educational Studies in Mathematics*, *86*(1), 79–96.

Anderson, P. M., & Sherman, C. A. (2010). Applying the Fermi estimation technique to business problems. *The Journal of Applied Business and Economics*, *10*(5), 33–42.

Ärlebäck, J. B. (2009). On the use of realistic Fermi problems for introducing mathematical modelling in school. *The Mathematics Enthusiast*, *6*(3), 331–364.

Ärlebäck, J. B., & Albarracín, L. (in preparation). Fermi problems in education: A literature review seeking common ground.

Blomhøj, M. (2009). Different perspectives in research on the teaching and learning mathematical modelling - Categorising the TSG21 papers. In M. Blomhøj & S. Carreira (Eds.), *Mathematical applications and modelling in the teaching and learning of mathematics. Proceedings from Topic Study Group at the 12th International Congress on Mathematical Education* (pp. 1–17). Monterrey, Mexico.

Carlson, J. E (1997) Fermi problems on gasoline consumption. *The Physics Teacher*, *35*, 308–309.

Czocher, J. A. (2016). Introducing Modeling Transition Diagrams as a tool to connect mathematical modeling to mathematical thinking. *Mathematical Thinking and Learning*, *18*(2), 77–106.

Efthimiou, C. J., & Llewellyn, R. A. (2007). Cinema, Fermi problems and general education. *Physics Education*, *42*(3), 253–261. Retrieved from http://iopscience.iop.org/0031-9120/42/3/003

Goodchild, S., & Fuglestad, A. B. (2008). Affordances of inquiry: The case of one teacher. In J. L. Cortina, S. Alarorre, O. Figueras, T. Rojano, & A. Sepúlveda (Eds.), *Proceedings of the joint meeting of PME and PME-NA XXX* (Vol. 3, 39–56). Mexico: Cinvestav - UMSNH.

Kaiser, G., & Sriraman, B. (2006). A global survey of international perspectives on modelling in mathematics education. *ZDM*, *38*(3), 302–310.

Peter-Koop, A. (2004). Fermi problems in primary mathematics classrooms: Pupils' interactive modelling processes. In I. Putt, R. Faragher, & M. McLean (Eds.),. *Proceedings of the 27th Annual Conference of the Mathematics Education Research Group of Australasia*, (pp. 454–461). Townsville, QLD: MERGA.

Petticrew, M., & Roberts, H. (2006). *Systematic reviews in the social sciences*. Oxford, UK: Blackwell.

Sriraman, B., & Knott, L. (2009). The mathematics of estimation: Possibilities for interdisciplinary pedagogy and social consciousness. *Interchange*, *40*(2), 205–223.

Sriraman, B., & Lesh, R. A. (2006). Modeling conceptions revisited. *ZDM*, *38*(3), 247–254.

Swan, M., & Ridgway, J. (n.d.). Plausible estimation' tasks. Retrieved September 14, 2016, from http://www.flaguide.org/extra/download/cat/math/estimation/estimation.pdf