Comparative Study of Several Bases in Functional Analysis

Department of Mathematics, Linköping University

Maria Miranda Navarro

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Supervisor: **Jana Björn**,
Department of Mathematics, Linköping University

Examiner: **Jana Björn**,
Department of Mathematics, Linköping University

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Abstract

From the beginning of the study of spaces in functional analysis, bases have been an indispensable tool for operating with vectors and functions over a concrete space. Bases can be organized by types, depending on their properties. This thesis is intended to give an overview of some bases and their relations. We study Hamel basis, Schauder basis and Orthonormal basis; we give some properties and compare them in different spaces, explaining the results. For example, an infinite dimensional Hilbert space will never have a basis which is a Schauder basis and a Hamel basis at the same time, but if this space is separable it has an orthonormal basis, which is also a Schauder basis. The project deals mainly with Banach spaces, but we also talk about the case when the space is a pre Hilbert space.

Keywords:

Banach space, Hilbert space, Hamel basis, Schauder basis, Orthonormal basis

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Contents

Introduction

The use of bases makes it possible to encode vectors and their properties into sequences of their coefficients, which are easier to study. As an introductory example, in the space $\mathbb{R}^3$ we can express any element $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ as $x_1 \cdot (1,0,0) + x_2 \cdot (0,1,0) + x_3 \cdot (0,0,1)$. We call the collection of vectors $((1,0,0),(0,1,0),(0,0,1))$ a basis of the space $\mathbb{R}^3$. There are several different types of bases, with distinct properties and defined in different spaces. Bases are not only very useful in many analytic calculations and constructions, they can also be used to classify spaces and also to prove theorems.

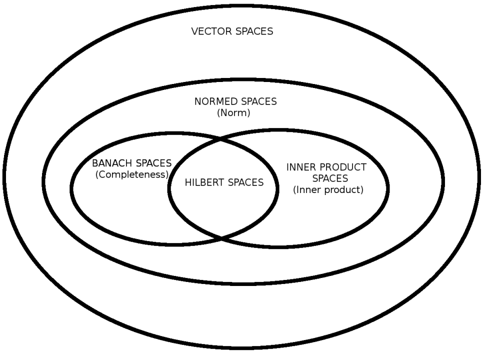
Bases are fundamental in the study of Banach spaces but also are important in other branches of mathematics, such as Fourier analysis and classical and applied harmonic analysis; physics and engineering. Bases are also essential in wavelets for signal processing.

As examples of types of bases we have Hamel bases, Schauder bases, Orthonormal bases, Boundedly complete bases, Biorthogonal bases, Dual bases and Symmetric bases.

As already said, each kind of basis has different properties and hence different applications. Even so, a basis can have properties of more than one type. This fact makes some bases very useful for concrete spaces. For that reason, we found it interesting to create a network with some relations between types of bases.

In this project we are going to talk about Hamel basis, Schauder basis and Orthonormal basis.

Figure below shows the spaces we are going to talk about and where are they located with respect to the other spaces.



The aim is to study bases in each region of the graph.

The project is divided into 4 chapters: First chapter is intended to give some background knowledge that may be important during the study. Chapter 2 is divided in three sections: Hamel basis, Schauder basis and Orthonormal basis. It gives a deep study of each type of basis and some theorems which will help us to compare it with our other bases. Hence, the first section explains the Hamel basis and some of its properties; the second section explains the Schauder basis, some of its properties and a comparison with Hamel basis; and the third section explains the Orthonormal basis, some of its properties and a comparison with the Hamel and the Schauder bases.
The third chapter summarizes the results, i.e. it gives a small graph with all the possible scenarios.
Finally, Chapter 4 gives an overview and suggests a possible continuation of the project.

Chapter 1

Preliminaries

This chapter is a compilation of the basic concepts about spaces in functional analysis. Also theorems and propositions that will be needed later are announced.

Definition 1.1. A non-negative function $||\cdot||$ on a vector space X is called a *norm* on X if

1. $||x|| \geq 0$ for every $x \in X$,
2. $||x|| = 0$ if and only if $x = 0$,
3. $||\lambda x|| = |\lambda|\ ||x||$ for every $x \in X$ and every scalar λ,
4. $||x + y|| \leq ||x|| + ||y||$ for every $x, y \in X$ (the "triangle inequality").

Definition 1.2. A vector space X is said to be *finite dimensional* if there is a positive integer n such that X contains a linearly independent set of n vectors whereas any set of $n+1$ or more vectors of X is linearly dependent. n is called the *dimension* of X, written $n = \dim X$. By definition, $X = \{0\}$ is finite dimensional and $\dim X = 0$. If X is not finite dimensional, it is said to be *infinite dimensional*.

Definition 1.3. A *normed space* X is a vector space with a norm defined on it.

Definition 1.4. A sequence $(x_n)_{n=1}^{\infty}$ in a metric space $X = (X, d)$ is said to be *Cauchy* if for every $\varepsilon > 0$ there is an N_ε such that

$$d(x_m, x_n) < \varepsilon \quad \text{for every } m, n > N_\varepsilon.$$

Definition 1.5. A sequence $(x_n)_{n=1}^{\infty}$ in a metric space $X = (X, d)$ is said to *converge* if there is an $x \in X$ such that

$$\lim_{n\to\infty} x_n = x$$

or, simply,

$$x_n \to x.$$

We say that $(x_n)_{n=1}^{\infty}$ *converges to* x. If $(x_n)_{n=1}^{\infty}$ is not convergent, it is said to be *divergent.*

Definition 1.6. Let $a_1, a_2, \ldots$ be an infinite sequence of real numbers. The *infinite series* $\sum_{i\geq 1} a_i$ is defined to be

$$\sum_{i\geq 1} a_i = \lim_{N\to\infty} \sum_{i=1}^{N} a_n.$$

If the limit exists in $\mathbb{R}$ then $\sum_{n\geq 1} a_n$ is *convergent.*

Recall that a space is complete if every Cauchy sequence in X converges to some point in X.

Definition 1.7. A *Banach space* is a complete normed space.

Example 1.8. The following vector spaces are Banach spaces.

(i) Euclidean space $\mathbb{R}^n$ and unitary space $\mathbb{C}^n$ with norm defined by

$$||x|| = \left(\sum_{j=1}^{n} |x_j|^2\right)^{1/2} = \sqrt{|x_1|^2 + \cdots + |x_n|^2},$$

where $x = (x_j)_{j=1}^{n}$.

(ii) Space of all the continuous functions defined on $[a, b]$, $C[a, b]$, with norm given by

$$||x|| = \max_{t\in[a,b]} |x(t)|.$$

(iii) Space $\ell^p = \{(x_j)_{j=1}^{\infty};\ x_j \in \mathbb{C},\ \sum_{j=1}^{\infty} |x_j|^p < \infty\}$, $1 \leq p < \infty$ with norm defined by

$$||x|| = \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{1/p}.$$

(iv) Space $\ell^\infty = \{(x_j)_{j=1}^\infty;\ x_j \in \mathbb{C},\ \sup_j |x_j| < \infty\}$ with norm given by

$$||x|| = \sup_j |x_j|.$$

Theorem 1.9 ([8, Theorem 1.4-7])**.** *A subspace Y of a Banach space X is complete if and only if the set Y is closed in X.*

Definition 1.10. A *linear operator* T is an operator such that

1. the domain $\mathcal{D}(T)$ of T is a vector space and the range $\mathcal{R}(T)$ lies in a vector space over the same field,

2. for all $x, y \in \mathcal{D}(T)$ and scalars α,

$$T(x + y) = Tx + Ty$$
$$T(\alpha x) = \alpha Tx.$$

Definition 1.11. Let $T : \mathcal{D}(T) \to \mathcal{R}(T)$ be a linear operator, where $\mathcal{D}(T)$ and $\mathcal{R}(T)$ are normed spaces. The operator T is said to be *bounded* if there is a real number c such that for all $x \in \mathcal{D}(T)$,

$$||Tx|| \leq c||x||.$$

It is important to mention that the smallest c which satisfies the inequality is the norm of T, $||T||$.

Definition 1.12. A *linear functional* f is a linear operator with domain in a vector space X and range in the scalar field K of X; thus,

$$f : X \to K,$$

where $K = \mathbb{R}$ if X is real and $K = \mathbb{C}$ if X is complex.

Definition 1.13. A *bounded linear functional* f is a bounded linear operator with range in the scalar field of the normed space X in which the domain $\mathcal{D}(f)$ lies.
Thus there exists a real number c such that for all $x \in \mathcal{D}(f)$,

$$|f(x)| \leq c||x||.$$

Furthermore, the *norm* of f is

$$||f|| = \sup_{\substack{x \in \mathcal{D}(f) \\ x \neq 0}} \frac{|f(x)|}{||x||} = \sup_{\substack{x \in \mathcal{D}(f) \\ ||x|| = 1}} |f(x)|.$$

Definition 1.14. Let X and Y be metric spaces. Then $T : \mathcal{D}(T) \to Y$ with domain $\mathcal{D}(T) \subset X$ is called an *open mapping* if for every open set in $\mathcal{D}(T)$ the image is an open set in Y.

Open Mapping Theorem 1.15 ([8, Theorem 4.12-2])**.** *A bounded linear operator T from a Banach space X onto a Banach space Y is an open mapping. Hence if T is bijective, T^{-1} is continuous and thus bounded.*

Definition 1.16. Let X be a normed space. Then the set of all bounded linear functionals on X is called the *dual space* of X and is denoted by X'.

Using the same notation as in the previous definition we state the next proposition

Proposition 1.17. The dual space X' constitutes a normed space, with norm defined by

$$||f|| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{|f(x)|}{||x||} = \sup_{\substack{x \in X \\ ||x||=1}} |f(x)|.$$

We can easily see that this is true.

Proof. We want to show that $||f||$ is a norm in X'. Then, let us see if it fulfills all the properties:

1. $||f|| \geq 0 \ \forall f \in X'$.

$$||f|| = \sup_{\substack{x \in \mathcal{D}(f) \\ ||x||=1}} |f(x)| \geq [\text{since we are taking absolute values}] \geq 0.$$

2. $||f|| = 0$ if and only if $f = 0$

$$||f|| = 0 \Leftrightarrow \sup_{\substack{x \in \mathcal{D}(f) \\ x \neq 0}} \frac{|f(x)|}{||x||} = 0 \Leftrightarrow |f(x)| = 0 \ \ \forall x \in \mathcal{D}(f),\ x \neq 0 \Leftrightarrow$$
$$\Leftrightarrow f(x) = 0 \ \ \forall x \in \mathcal{D}(f),\ x \neq 0 \Leftrightarrow f = 0.$$

3. $||\lambda f|| = |\lambda| \, ||f||$ for every $f \in X'$ and every scalar λ.

$$||\lambda f|| = \sup_{\substack{x \in \mathcal{D}(f) \\ ||x||=1}} |\lambda f(x)| = |\lambda| \sup_{\substack{x \in \mathcal{D}(f) \\ ||x||=1}} |f(x)| = |\lambda| \, ||f||.$$

4. $||f+g|| \le ||f|| + ||g||$ for every $f, g \in X'$.

$$\begin{aligned}||f+g|| &= \sup_{\substack{x\in\mathcal{D}(f)\\ ||x||=1}} |(f+g)(x)| = \sup_{\substack{x\in\mathcal{D}(f)\\ ||x||=1}} |f(x)+g(x)| \le \\ &\le [\text{using the triangle inequality}] \le \sup_{\substack{x\in\mathcal{D}(f)\\ ||x||=1}} (|f(x)|+|g(x)|) \le \\ &\le \sup_{\substack{x\in\mathcal{D}(f)\\ ||x||=1}} |f(x)| + \sup_{\substack{x\in\mathcal{D}(f)\\ ||x||=1}} |g(x)| = ||f|| + ||g||.\end{aligned}$$

□

Theorem 1.18 ([8, Theorem 2.10-4])**.** *The dual space X' of a normed space X is a Banach space.*

Definition 1.19. An *inner product space* is a vector space X with an inner product on X. A *Hilbert space* is a complete inner product space.

Here, an *inner product* on X is a mapping from $X \times X$ into the scalar field K of X such that for all vectors $x, y, z \in X$ and scalars α we have

$$\begin{aligned}\langle x+y, z\rangle &= \langle x, z\rangle + \langle y, z\rangle \\ \langle \alpha x, y\rangle &= \alpha\langle x, y\rangle \\ \langle x, y\rangle &= \overline{\langle y, x\rangle} \\ \langle x, x\rangle &\ge 0 \\ \langle x, x\rangle &= 0 \Leftrightarrow x = 0.\end{aligned}$$

Definition 1.20. A subspace M of a space X is said to be *dense* if for every $x \in X$ either $x \in M$ or x is a limit point of M.

Definition 1.21. A space X is called *separable* if it contains a countable dense subset.

Definition 1.22. A *partial ordering on a set M* is a binary relation which is written $\leqq$ and satisfies

(i) Reflexivity: $a \leqq a$ for every $a \in M$;

(ii) Antisymmetry: for $a, b \in M$ if $a \leqq b$ and $b \leqq a$, then $a = b$;

(iii) Transitivity: if $a \leqq b$ and $b \leqq c$, then $a \leqq c$.

Definition 1.23. Two elements a and b are called *comparable* if they satisfy $a \leqq b$ or $b \leqq a$.

Definition 1.24. A *totally ordered set* or *chain* is a partially ordered set such that every two elements of the set are comparable.

Zorn's Lemma 1.25 (M-A.Zorn, 1935)**.** *If every chain (that is, every totally ordered subset) in a partially ordered set X has an upper bound, then X has a maximal element.*

Definition 1.26. A subset M of a metric space X is said to be

(a) *rare* (or *nowhere dense*) in X if its closure $\overline{M}$ has no interior points,

(b) *meager* (or *of the first category*) in X if M is the union of countably many sets each of which is rare in X,

(c) *nonmeager* (or *of the second category*) in X if M is not meager in X.

Baire's Category Theorem 1.27 (R-L.Baire, 1899)**.** *If a metric space $X \neq \varnothing$ is complete, it is nonmeager in itself. Hence if $X \neq \varnothing$ is complete and*

$$X = \bigcup_{k=1}^{\infty} A_k, \quad (A_k \text{ closed})$$

then at least one A_k contains a nonempty open subset.

Proof. see [8, Theorem 4.7-2]. □

Chapter 2

Bases

When operating with elements in a vector space, the most easiest way is by using bases. This makes it possible to express functions and vectors in vector spaces by means of their coefficients.

2.1 Hamel Basis

We will start talking about the simplest basis, the Hamel basis. As we will notice during this chapter, Hamel basis is slightly different from the other bases.

Definition 2.1. Let V be a vector space (not necessarily finite dimensional). A family of vectors $(e_i)_{i\in I}$ is a *Hamel basis* for V if

(a) the set of all finite linear combinations (finite linear span) of $(e_i)_{i\in I}$ is V,

(b) every finite subset of $(e_i)_{i\in I}$ is linearly independent.

We do not require the index set I of a Hamel basis to be countable.

In finite dimensional linear algebra, a Hamel basis is usually just called a "basis". However, in Banach spaces the term "basis" is usually reserved for Schauder basis (Section 2.2).

Example 2.2. Let c_{00} be the space of all real sequences which have only finitely many non-zero terms. Then $(e_i; i \in \mathbb{N})$, where the sequence e_i is given by $e_i^{(k)} = \delta_{ki}$, that is, e_i is the sequence whose ith term is 1 and all other terms are zero, is a Hamel basis of this space.

We can state the first theorem, which will help us to understand the importance of Hamel bases.

Theorem 2.3 ([8, Theorem 4.1-7]). *Every vector space $X \neq \{0\}$ has a Hamel basis.*

Proof. Let $\mathcal{M}$ be the set of all finite linearly independent subsets of X. Since $X \neq \{0\}$, we know that $\mathcal{M}$ has at least one element, namely x such that $x \neq 0$ and hence $\mathcal{M} \neq \{0\}$.

Easily, we can see that set inclusion is a partial ordering on $\mathcal{M}$:

(i) $x \subset x$ for every $x \in \mathcal{M}$;

(ii) for $x, y \in \mathcal{M}$ if $x \subset y$ and $y \subset x$ then $x = y$;

(iii) if $x \subset y$ and $y \subset z$, then $x \subset z$, with $x, y, z \in \mathcal{M}$.

Hence, there exists at least one chain in $\mathcal{M}$. Every chain $\mathcal{C} \subset \mathcal{M}$ has an upper bound, namely, the union of all subsets of X which are elements of $\mathcal{C}$. By Zorn's lemma 1.25, $\mathcal{M}$ has a maximal element B. Now, we need to show that B is a Hamel basis for X.
Let $Y = \text{span}(B)$. Since $B \subset X$ and X is a vector space, Y is a subspace of X. Assume that $Y \neq X$ and take $\{z\} \in X \setminus Y$, so $\{z\} \notin \text{span}(B)$. Then, z and B are linearly independent. Thus $z \cup B$ is a linearly independent set contained in X, but we defined B as the maximal element of $\mathcal{M}$ so we come up with a contradiction. So, $Y = X$ and B is a Hamel basis. □

In the definition of a Hamel basis we mentioned that they do not have to be countable. Furthermore, in the proposition above we showed that for every space there is always a Hamel basis.
It is easy to show that for any finite dimensional vector space the Hamel bases will be countable, in fact finite, therefore, our next step is to study in which case a Hamel basis is not countable. We present the proposition bellow.

Proposition 2.4 ([10, Proposition 8.4.3]). If X is an infinite dimensional Banach space, then every Hamel basis for X is uncountable.

Proof. Suppose that $(e_i)_{i=1}^{\infty}$ is a countable Hamel basis for X and put $X_k = \text{span}(e_1, \ldots, e_k)$, $k = 1, 2, \ldots$. Each X_k is a finite dimensional vector space, so it is closed, and hence a Banach space by itself. Since $(e_i)_{i=1}^{\infty}$ is supposed to be a Hamel basis for X, $X = \bigcup_{k=1}^{\infty} X_k$.
Using contradiction, we will show that every X_k is nowhere dense.

Suppose there exists $k > 0$ such that X_k has interior points, and take such an interior point $x \in X_k$. Let $r > 0$ such that $B(x,r) \subset X_k$ and take $y \in X \setminus X_k$. Let us construct a point, namely z, as $z = x + \frac{r}{2}\frac{y}{||y||}$.
Clearly, $z \in B(x,r)$, which means that $x + \frac{r}{2}\frac{y}{||y||} \in X_k$. Since X_k is a vector space, $\frac{r}{2}\frac{y}{||y||} \in X_k$ and thus $y \in X_k$, which is a contradiction.
Hence, X_k cannot contain interior points, and are thus nowhere dense in X. This fact contradicts Baire's theorem 1.27. □

Hamel bases are not very used in infinite dimensional Banach spaces. Even so, there are some interesting properties about the cardinality of Hamel bases in those spaces. As an example, there is the following statement.

The set E^f of all linear functions $E \to \mathbb{R}$ on an infinite dimensional Banach space E has cardinality $2^{|E|}$, where $|E|$ is the cardinality of a Hamel basis of E.

This theorem and other related theorems are discussed in the article [5].

From now we will consider bases in Banach spaces, which are defined to be sequences such that every element can be written uniquely as an infinite linear combination of the basis elements.

2.2 Schauder basis

Recall that an infinite series is defined to be the limit of its sequence of partial sums.

Definition 2.5. A sequence $(e_i)_{i=1}^{\infty}$ in a Banach space X is called a *Schauder basis (also called Countable basis)* of X if for every $x \in X$ there is a unique sequence of scalars $(a_i)_{i=1}^{\infty}$ so that $x = \sum_{i=1}^{\infty} a_i e_i$. A sequence $(e_i)_{i=1}^{\infty}$ which is a Schauder basis of its closed linear span is called a *basic sequence.*

Notice that the sequence $(e_i)_{i=1}^{\infty}$ is linearly independent.

One can get confused with the notions of Hamel and Schauder bases; however, there are some important differences between them. Looking carefully at the definitions one can notice that a Hamel basis uses only finite dimensional linear combinations, while a Schauder basis uses infinite ones.

Example 2.6. The space ℓ^p has a Schauder basis, namely $(e_i)_{i=1}^{\infty}$, where $e_i = (\delta_{ij})$ with the norm defined in Example 1.8 . Indeed, let $x = (a_1, a_2, \dots) \in \ell^p$. Then,

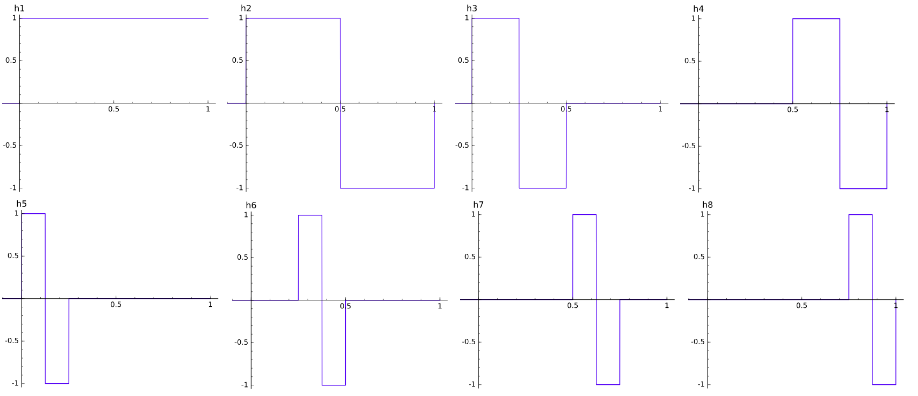
$$\left\| x - \sum_{i=1}^{N} a_i e_i \right\|^p = \left\| (a_1, a_2, \dots) - \sum_{i=1}^{N} a_i e_i \right\|^p = \\ = \sum_{i=N+1}^{\infty} |a_i|^p \to 0 \text{ as } N \to \infty, \text{ since } x \in \ell^p.$$

The next example, the Haar system, is essential to the study of wavelets and signal processing. During this chapter we will get through different properties of this system related with bases.

Example 2.7. The *Haar system* is a Schauder basis in the space $L^p[0,1]$, for $1 \le p < \infty$. The Haar system $(h_i)_{i=1}^{\infty}$ of functions on $[0,1]$ is defined as follows: $h_1(t) = 1$. If $i = 2^n + k$, where $1 \le k \le 2^n$ are integers (note the existence and uniqueness of such expression), then

$$h_i(t) = \begin{cases} 1 & \text{if } \frac{2k-2}{2^{n+1}} \le t < \frac{2k-1}{2^{n+1}}, \\ -1 & \text{if } \frac{2k-1}{2^{n+1}} \le t < \frac{2k}{2^{n+1}}, \\ 0 & \text{otherwise.} \end{cases}$$

For a better understanding, the graphs of the first 8 Haar functions are shown bellow:



As one can see, Haar functions are rescaled square-shaped wavelets defined in $[0, 1]$. Hence it is not difficult to imagine that the collection of all these functions could be a wavelet basis.

Once we know the meaning of a Schauder basis we can start looking more deeply into some important properties.

Proposition 2.8 ([10, Theorem 8.4.4])**.** If X has a Schauder basis, then X is separable.

Proof. Let X be a Banach space and $(e_i)_{i=1}^{\infty}$ be a Schauder basis of X. Without loss of generality, suppose $||e_i|| = 1$ for all $i \geq 1$. Now we consider the set

$$\mathcal{Q} = \left\{ \sum_{i=1}^{n} q_i e_i \ : \ n \in \mathbb{N}, q_i \in \mathbb{Q} \right\},$$

which is clearly countable.
Let $x = \sum_{i=1}^{\infty} a_i e_i \in X$ and $\varepsilon > 0$. Then, by definition there exists $n \geq 0$ such that $||x - \sum_{i=1}^{n} a_i e_i|| \leq \varepsilon$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, for each a_i we can find $q_i \in \mathbb{Q}$ such that $|q_i - a_i| \leq \frac{\varepsilon}{n}$. Hence, if we take $y = \sum_{i=1}^{n} q_i e_i \in \mathcal{Q}$ we obtain:

$$\begin{aligned} ||x - y|| &= \left|\left| x - \sum_{i=1}^{n} q_i e_i + \sum_{i=1}^{n} a_i e_i - \sum_{i=1}^{n} a_i e_i \right|\right| \leq \\ &\leq \left|\left| x - \sum_{i=1}^{n} a_i e_i \right|\right| + \left|\left| \sum_{i=1}^{n} (a_i e_i - q_i e_i) \right|\right| \leq \\ &\leq \left|\left| x - \sum_{i=1}^{n} a_i e_i \right|\right| + \sum_{i=1}^{n} |a_i - q_i| \cdot |e_i| \leq \varepsilon + \sum_{i=1}^{n} \frac{\varepsilon}{n} = 2\varepsilon. \end{aligned}$$

Thus, finite linear combinations with rational coefficients of the vectors in the basis are dense in X, so X is separable. $\square$

Since ℓ^∞ is not separable, the proposition implies that it does not have a Schauder basis.
In 1973, the Swedish mathematician Per Enflo gave an example[1] of a separable Banach space without a Schauder basis, thus refuting the conjecture by Stefan Banach from 1930 stating that every separable Banach space has a Schauder basis.

Now, we introduce the concept of canonical projections, which will help us manipulating Schauder bases.

Definition 2.9. If $(e_i)_{i=1}^\infty$ is a Schauder basis of a normed space X, then the *canonical projections* $P_n : X \to X$ are defined for $n \in \mathbb{N}$ by

$$P_n\left(\sum_{i=1}^{\infty} a_i e_i\right) = \sum_{i=1}^{n} a_i e_i.$$

Each P_n is a linear projection from X onto the linear subspace spanned by $(e_i : i = 1, 2, \ldots, n)$.

Proposition 2.10 ([3, Lemma 4.7])**.** Let (e_i) be a Schauder basis of a Banach space X. The canonical projections P_n satisfy:

(i) $\dim(P_n(X)) = n$,

(ii) $P_nP_m = P_mP_n = P_{\min(m,n)}$,

(iii) $P_n(x) \to x$ in X for every $x \in X$.

Proof. Let $(e_i)_{i=1}^\infty$ be a Schauder basis of a Banach space X and let $n, m \in \mathbb{N}$. Let also $x = \sum_{i=1}^\infty a_i e_i$.

(i) By definition, $P_n(x) = \sum_{i=1}^n a_i e_i$.
Thus, by definition 1.2 we arrive to the conclusion that all $y \in P_n(X)$ are made by n linearly independent elements. Hence, $\dim(P_n(X)) = n$.

(ii)

$$P_n(P_m(x)) = P_n\left(\sum_{i=1}^{m} a_i e_i\right) = \begin{cases} \sum_{i=1}^m a_i e_i & \text{if } n \geq m \\ \sum_{i=1}^n a_i e_i & \text{if } m \geq n. \end{cases} =$$

$$= \sum_{i=1}^{\min(n,m)} a_i e_i = P_{\min(n,m)}(x).$$

[1] The example can be found in [2]

With the same procedure one can see that $P_m(P_n(x)) = P_{\min(m,n)}(x)$. Clearly $P_{\min(n,m)}(x) = P_{\min(m,n)}(x)$, hence $P_n(P_m(x)) = P_m(P_n(x))$.

(iii) For every $x \in X$,

$$\lim_{n\to\infty} P_n(x) = \lim_{n\to\infty} \sum_{i=1}^{n} a_i e_i = \sum_{i=1}^{\infty} a_i e_i = x.$$

□

Proposition 2.11 ([9, Proposition 1.a.2]). Let X be a Banach space with a Schauder basis $(e_i)_{i=1}^{\infty}$. Then the projections $P_n : X \to X$ are bounded linear operators and $\sup_n ||P_n|| < \infty$.

Proof. The projections are linear operators by their definition. To see boundedness, we will use the following lemma:

Lemma 2.12. Let $T : X \to Y$ be a linear operator, where X, Y are normed spaces. Then, T is continuous if and only if T is bounded.[2]

In our case, P_n are linear projections from the Banach space X to itself. Hence, we show that the projections are continuous:
Let $(x_k) \in X$ be a sequence defined as $x_k = \sum_{i=1}^{\infty} a_{k,i} e_i$, where $(e_i)_{i=i}^{\infty}$ is a Schauder basis with $||e_i|| = 1$ for every i. Assume that $\lim_{k\to\infty} x_k = x \in X$. Suppose $\lim_{k\to\infty} P_n(x_k) = y \in X$. Then, since the e_i's are linearly independent,

$$\begin{aligned} y = \lim_{k\to\infty} P_n(x_k) = \lim_{k\to\infty} \sum_{i=1}^{n} a_{k,i} e_i = \\ = \sum_{i=1}^{n} \lim_{k\to\infty} a_{k,i} e_i = P_n(\lim_{k\to\infty} x_k) = P_n(x). \end{aligned}$$

Thus, the projections are bounded linear operators.
For the second part, take $x = \sum_{i=1}^{\infty} a_i e_i$ and define

$$||x||_0 := \sup_{N\geq 1} \left|\left| \sum_{i=1}^{N} a_i e_i \right|\right| = \sup_{N\geq 1} ||P_N(x)||.$$

[2] A complete proof of the Lemma can be found in [8, Theorem 2.7-9]

It is not difficult to prove that $||\cdot||_0$ is a norm. Define the normed space Y, which is X with the norm $||\cdot||_0$.
Notice that

$$||x|| = \left|\left|\lim_{N\to\infty}\sum_{i=1}^{N} a_i e_i\right|\right| \leq \sup_{N\geq 1}\left|\left|\sum_{i=1}^{N} a_i e_i\right|\right| = ||x||_0.$$

We show that $||\cdot||$ and $||\cdot||_0$ are equivalent norms.
Let $\iota : Y \to X$ be the formal inclusion map, which is bijective. Suppose that Y is a Banach space. Then, the Open Mapping Theorem 1.15 implies that ι has a continuous inverse, and hence, $||\cdot||_0$ is equivalent to $||\cdot||$.

So, we just have to prove that Y is complete.[3]
Given $x = \sum_{i=1}^{\infty} a_i e_i$ and $m \geq 1$, by the definition of $||\cdot||_0$,

$$\begin{aligned} |a_m| = ||a_m e_m||\,||e_m||^{-1} = ||e_m||^{-1}\left|\left|\sum_{i=1}^{m} a_i e_i - \sum_{i=1}^{m-1} a_i e_i\right|\right| \leq \\ \leq ||e_m||^{-1}\left(\left|\left|\sum_{i=1}^{m} a_i e_i\right|\right| + \left|\left|\sum_{i=1}^{m-1} a_i e_i\right|\right|\right) \leq 2||e_m||^{-1}||x||_0. \end{aligned} \tag{2.1}$$

Let $(x_k)_{k=1}^{\infty}$ be a Cauchy sequence in Y such that

$$x_k = \sum_{i=1}^{\infty} a_{k,i} e_i \quad (k \geq 1).$$

Then, for every $\varepsilon > 0$ there exists N_ε such that for $r, s \geq N_\varepsilon$,

$$\sup_{N\geq 1}\left|\left|\sum_{i=1}^{N}(a_{r,i} - a_{s,i})e_i\right|\right| = ||x_r - x_s||_0 < \varepsilon.$$

In (2.1) we saw that for every $m \geq 1$ the sequence of scalars $(a_{k,m})_{k=1}^{\infty}$ is bounded, and hence is a Cauchy sequence. Let b_m be the limit of each such Cauchy sequence.
We want to show that $y = \sum_{i=1}^{\infty} b_i e_i$ converges in Y.
Let $n \geq N_\varepsilon$. Then for each $N \geq 1$ we have

$$\left|\left|\sum_{i=1}^{N}(b_i - a_{n,i})e_i\right|\right| = \left|\left|\sum_{i=1}^{N}\lim_{r\to\infty}(a_{r,i} - a_{n,i})e_i\right|\right| = \lim_{r\to\infty}\left|\left|\sum_{i=1}^{N}(a_{r,i} - a_{n,i})e_i\right|\right| < \varepsilon. \tag{2.2}$$

[3] This part of the proof was extracted from [1].

We now have to see that the limit $y = \sum_{i=1}^{\infty} b_i e_i$ exists. Take $n > 0$ and let $y_n = P_n(y)$. Then,

$$||y_n - y||_0 = \sup_{N>n} \left|\left| \sum_{i=n+1}^{N} b_i e_i \right|\right| \leq \sup_{N>n} \left|\left| \sum_{i=n+1}^{N} (b_i - a_{s,i})e_i \right|\right| + \sup_{N>n} \left|\left| \sum_{i=n+1}^{N} a_{s,i} e_i \right|\right| \tag{2.3}$$

The first term in (2.3) is estimated using (2.2) as

$$\sup_{N>n} \left|\left| \sum_{i=n+1}^{N} (b_i - a_{s,i})e_i \right|\right| \leq \sup_{N>n} \left(\left|\left| \sum_{i=1}^{N} (b_i - a_{s,i})e_i \right|\right| + \left|\left| \sum_{i=1}^{n} (b_i - a_{s,i})e_i \right|\right| \right) < 2\varepsilon.$$

Similarly, we approximate the second term of (2.3) using the fact that $x_s = \sum_{i=1}^{\infty} a_{s,i} e_i$.

$$\sup_{N>n} \left|\left| \sum_{i=n+1}^{N} a_{s,i} e_i \right|\right| \leq \sup_{N>n} \left(\left|\left| \sum_{i=1}^{N} a_{s,i} e_i - x_s \right|\right| + \left|\left| \sum_{i=1}^{n} a_{s,i} e_i - x_s \right|\right| \right) < \varepsilon.$$

Hence, $||y_n - y||_0 < 3\varepsilon$ for large n.
Since $|| \cdot ||_0 \geq || \cdot ||$, this implies that the sum converges in X. Hence, y exists, as X is complete and $y_n \to y$ in Y.

Finally, we should see that $x_s \to y$ in Y. From (2.2) we see

$$\begin{aligned} ||x_s - y||_0 &= \left|\left| \sum_{i=1}^{\infty} x_{s,i} e_i - \sum_{i=1}^{\infty} b_i e_i \right|\right|_0 = \\ &= \sup_{N \geq 1} \left|\left| \sum_{i=1}^{N} (b_i - a_{s,i})e_i \right|\right| < \varepsilon \quad \text{if } s \geq N_\varepsilon. \end{aligned}$$

Thus Y is also complete, which concludes the proof. □

Proposition 2.13 ([9, Proposition 1.a.3])**.** Let $(e_i)_{i=1}^{\infty}$ be a sequence of vectors in X. Then $(e_i)_{i=1}^{\infty}$ is a Schauder basis if and only if the following three conditions hold.

(i) $e_i \neq 0$ for all i.

(ii) There is a constant K so that, for every choice of scalars $(a_i)_{i=1}^{\infty}$ and integers $n < m$, we have

$$\left|\left| \sum_{i=1}^{n} a_i e_i \right|\right| \leq K \left|\left| \sum_{i=1}^{m} a_i e_i \right|\right|. \tag{2.4}$$

(iii) The closed linear span of $(e_i)_{i=1}^{\infty}$ is all of X.

Proof. Suppose that $(e_i)_{i=1}^{\infty}$ is a Schauder basis in X. Since the e_i's are linearly independent, condition (i) holds. Condition (iii) means that the span of $(e_i)_{i=1}^{\infty}$ is dense in X, which we have seen in the proof of Proposition 2.8.
From Proposition 2.11 we know that the projections are bounded linear operators and $\sup_n ||P_n|| < \infty$.
Let $x = \sum_{i=1}^{\infty} a_i e_i \in X$. Fix $m \in \mathbb{N}$ and define $y = \sum_{i=1}^{\infty} b_i e_i \in X$ such that $b_i = a_i$ for $1 \leq i \leq m$ and $b_i = 0$ otherwise. Then, for $n < m$,

$$\begin{aligned} ||P_n(x)|| = ||P_n(y)|| \leq \sup_n ||P_n(y)|| \leq [\text{by Proposition 2.11}] \leq K\,||y|| = \\ = K\,||P_m(y)|| = K\,||P_m(x)||\,, \end{aligned}$$

which gives us (ii).

Conversely, if (i) and (ii) hold then suppose $x = \sum_{i=1}^{\infty} a_i e_i = 0$. Fix n in (2.4) and let $m \to \infty$. Then,

$$||P_n(x)|| \leq K \left|\left| \lim_{m\to\infty} P_m(x) \right|\right| = 0.$$

Hence, $||P_n(x)|| = 0$ for any $n > 0$. Then,

$$||P_1(x)|| = ||a_1 e_1|| = |a_1|\;||e_1|| = 0$$

so $|a_1| = 0$, i.e. $a_1 = 0$.
Suppose that there exists $n > 1$ such that $a_i = 0$ for $1 \leq i < n$. Then,

$$K \left|\left| \sum_{i=1}^{n} a_i e_i \right|\right| = K||a_n e_n|| = K|a_n|\;||e_n|| = 0.$$

Thus, $a_n = 0$ for all $n > 0$. This proves the uniqueness of the expansion in terms of $(e_i)_{i=1}^{\infty}$.

Now we just need to prove the existence of the expansion for every $x \in X$.
Let $Y = \text{span}\{e_1, e_2, \dots\}$. Suppose (iii) holds. Take $x \in X$. Then, there exists a sequence $(x_k)_{k=1}^{\infty} \in Y$ such that $\lim_k x_k = x$ with $x_k = \sum_{i=1}^{n_k} a_{k,i} e_i$. Then, by (ii),

$$0 \leq ||a_{k,1} e_1|| \leq K \left|\left| \sum_{i=1}^{n_k} a_{k,i} e_i \right|\right| = K||x_k||.$$

Since $(x_k)_{k=1}^{\infty}$ is a bounded sequence and $||a_{k,1}e_1|| = |a_{k,1}|\ ||e_1||$, the sequence $(a_{k,1})_{k=1}^{\infty}$ is bounded.
Hence, we can choose a subsequence $\left(a_{k,1}^{(1)}\right)_{k=1}^{\infty}$ of $(a_{k,1})_{k=1}^{\infty}$ so that it converges to some number $a_1 \in \mathbb{R}$ and which satisfies for all $k \geq 1$,

$$\left|a_{k,1}^{(1)} - a_1\right| < 2^{-k-1}.$$

Now, we consider the corresponding subsequence $\left(x_k^{(1)}\right)_{k=1}^{\infty}$ of $(x_k)_{k=1}^{\infty}$, say

$$x_k^{(1)} = \sum_{i=1}^{n_k^{(1)}} a_{k,i}^{(1)} e_i \quad \text{for every } k > 0.$$

Similarly,

$$0 \leq \left|\left|a_{k,2}^{(1)}e_2\right|\right| \leq \left|\left|a_{k,1}^{(1)}e_1 + a_{k,2}^{(1)}e_2\right|\right| + \left|\left|a_{k,1}^{(1)}e_1\right|\right| \leq 2K||x_k||.$$

So $\left(a_{k,2}^{(1)}\right)_{k=1}^{\infty}$ is bounded and we can pick a subsequence $\left(a_{k,2}^{(2)}\right)_{k=1}^{\infty}$ which converges to $a_2 \in \mathbb{R}$ and so that for all $k \geq 1$

$$\left|a_{k,2}^{(2)} - a_2\right| < 2^{-k-2}.$$

Notice that since $\left(a_{k,1}^{(2)}\right)_{k=1}^{\infty}$ is a subsequence of $\left(a_{k,1}^{(1)}\right)_{k=1}^{\infty}$ we also have that

$$\left|a_{k,1}^{(2)} - a_1\right| < 2^{-k-1}.$$

We extract another subsequence $\left(x_k^{(2)}\right)_{k=1}^{\infty}$, say

$$x_k^{(2)} = \sum_{i=1}^{n_k^{(2)}} a_{k,i}^{(2)} e_i \quad \text{for every } k > 0.$$

We do the same procedure for each $j = 3, 4, \ldots$:
By (ii),

$$0 \leq \left|\left|\sum_{i=1}^{j} a_{k,i}^{(j-1)} e_i\right|\right| \leq K||x_n||$$

and

$$0 \leq \left|\left|\sum_{i=1}^{j-1} a_{k,i}^{(j-1)} e_i\right|\right| \leq K||x_n||.$$

So,

$$0 \leq \left|\left|a_{k,j}^{(j-1)} e_j\right|\right| \leq \left|\left|\sum_{i=1}^{j} a_{k,i}^{(j-1)} e_i\right|\right| + \left|\left|\sum_{i=1}^{j-1} a_{k,i}^{(j-1)} e_i\right|\right| \leq 2K||x_n||.$$

Since $\left|\left|a_{k,j}^{(j-1)} e_j\right|\right| = \left|a_{k,j}^{(j-1)}\right| \; ||e_j||$, the sequence $\left(a_{k,j}^{(j-1)}\right)_{k=1}^{\infty}$ is bounded, so there exists a convergent subsequence $\left(a_{k,j}^{(j)}\right)_{k=1}^{\infty}$ with limit $a_j \in \mathbb{R}$ such that

$$\left|a_{k,j}^{(j)} - a_j\right| < 2^{-k-j} \quad \forall k \geq 1,$$

and we consider the subsequence $x_k^{(j)} = \sum_{i=1}^{n_k^{(j)}} a_{k,i}^{(j)} e_i, \quad k = 1, 2, \dots$.
Again, since $\left(a_{k,i}^{(j)}\right)_{k=1}^{\infty}$ is a subsequence of $\left(a_{k,i}^{(i)}\right)_{k=1}^{\infty}$ for all $i < j$, then

$$\left|a_{k,i}^{(j)} - a_i\right| < 2^{-k-i} \quad \forall k \geq 1.$$

Using Cantor's Diagonal Argument we extract the subsequence

$$\overline{x}_k = x_k^{(k)}, \quad k = 1, 2, \dots$$

which means that $\overline{x}_k$ is made of the red sequences bellow:

For each $1 \leq i \leq n_k$, the sequences $x_k^{(j)}$ have the following coefficients at e_i:

$$\begin{array}{cccccccccc}
x_k^{(1)} & \color{red}{a_{1,i}^{(1)}} & a_{2,i}^{(1)} & a_{3,i}^{(1)} & a_{4,i}^{(1)} & a_{5,i}^{(1)} & a_{6,i}^{(1)} & a_{7,i}^{(1)} & \cdots \\
x_k^{(2)} & a_{1,i}^{(2)} & \color{red}{a_{2,i}^{(2)}} & a_{3,i}^{(2)} & a_{4,i}^{(2)} & a_{5,i}^{(2)} & a_{6,i}^{(2)} & a_{7,i}^{(2)} & \cdots \\
x_k^{(3)} & a_{1,i}^{(3)} & a_{2,i}^{(3)} & \color{red}{a_{3,i}^{(3)}} & a_{4,i}^{(3)} & a_{5,i}^{(3)} & a_{6,i}^{(3)} & a_{7,i}^{(3)} & \cdots \\
x_k^{(4)} & a_{1,i}^{(4)} & a_{2,i}^{(4)} & a_{3,i}^{(4)} & \color{red}{a_{4,i}^{(4)}} & a_{5,i}^{(4)} & a_{6,i}^{(4)} & a_{7,i}^{(4)} & \cdots \\
x_k^{(5)} & a_{1,i}^{(5)} & a_{2,i}^{(5)} & a_{3,i}^{(5)} & a_{4,i}^{(5)} & \color{red}{a_{5,i}^{(5)}} & a_{6,i}^{(5)} & a_{7,i}^{(5)} & \cdots \\
x_k^{(6)} & a_{1,i}^{(6)} & a_{2,i}^{(6)} & a_{3,i}^{(6)} & a_{4,i}^{(6)} & a_{5,i}^{(6)} & \color{red}{a_{6,i}^{(6)}} & a_{7,i}^{(6)} & \cdots \\
x_k^{(7)} & a_{1,i}^{(7)} & a_{2,i}^{(7)} & a_{3,i}^{(7)} & a_{4,i}^{(7)} & a_{5,i}^{(7)} & a_{6,i}^{(7)} & \color{red}{a_{7,i}^{(7)}} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

So, for every $k \geq 1$ we have

$$\overline{x}_k = a_{k,1}^{(k)} e_1 + a_{k,2}^{(k)} e_2 + a_{k,3}^{(k)} e_3 + a_{k,4}^{(k)} e_4 + a_{k,5}^{(k)} e_5 + \cdots$$

$$\left\| \sum_{i=1}^{n_k^{(k)}} a_i e_i - \overline{x}_k \right\| = \left\| \sum_{i=1}^{n_k^{(k)}} \left(a_i - a_{k,i}^{(k)} \right) e_i \right\| \leq \text{[Triangle inequality]} \leq$$
$$\leq \sum_{i=1}^{n_k^{(k)}} \left\| \left(a_i - a_{k,i}^{(k)} \right) e_i \right\| = \sum_{i=1}^{n_k^{(k)}} \left| a_i - a_{k,i}^{(k)} \right| \, ||e_i|| \leq$$
$$\leq \sum_{i=1}^{n_k^{(k)}} 2^{-k-i} \leq 2^{-k}.$$

Then, taking limits we obtain

$$\lim_{k\to\infty} \left\| \sum_{i=1}^{n_k^{(k)}} a_i e_i - \overline{x}_k \right\| \leq \lim_{k\to\infty} 2^{-k} = 0.$$

Previously we said that $\lim_{k\to\infty} x_k = x$, i.e. $||\overline{x}_k - x|| \to 0$. Hence, by the uniqueness of the limit,

$$\lim_{k\to\infty} \left\| \sum_{i=1}^{n_k^{(k)}} a_i e_i - x \right\| \leq \lim_{k\to\infty} \left\| \sum_{i=1}^{n_k^{(k)}} a_i e_i - \overline{x}_k \right\| + \lim_{k\to\infty} ||\overline{x}_k - x|| = 0.$$

So, we obtain

$$x = \sum_{i=1}^{\infty} a_i e_i.$$

□

Theorem 2.14 ([9, Theorem 1.a.5])**.** *Every infinite dimensional Banach space contains a basic sequence.*

Proof. The proof of this theorem can be found in [9]. □

Example 2.15. If $X = c_0$ or $X = \ell_p$ for $p \in [1, \infty)$, then the sequence $(e_i)_{i=1}^{\infty}$ of the standard unit vectors is a Schauder basis of X.

When we talked about Hamel basis, we showed that for infinite dimensional Banach spaces, Hamel bases were uncountable. Since Schauder basis are always countable, it is clear that it is not possible to have a Hamel-Schauder basis (i.e. a basis which is Hamel and Schauder) of an infinite dimensional Banach space. Even so, we can still study the case when the Banach space is finite dimensional.

Proposition 2.16. Any Hamel basis of a finite dimensional Banach space X is a Schauder basis of X.

Proof. Let X be a finite dimensional Banach space with $\dim X = n < \infty$. For a Hamel basis B of X it holds that every finite subset of B is linearly independent, hence, B must have n elements (we discarded the case of B having less than n elements since it would not span all of X). Thus, B is finite.
Let $(e_i)_{i=1}^n$ be a Hamel basis of X. Since every element in X can be written as a linear combination of a subset of $(e_i)_{i=1}^n$, for every $x \in X$ there exists $J \subset I := \{1, 2, \dots, n\}$ such that

$$x = \sum_{i \in J} a_i e_i.$$

Finally, for $i \in I \setminus J$ and $i > n$ we define $a_i = 0$. Then,

$$x = \sum_{i=1}^{\infty} a_i e_i.$$

The only thing that remains to prove is the uniqueness of $(a_i)_{i=1}^n$, but this is trivial since the e_i's are linearly independent. □

2.3 Orthonormal basis

Definition 2.17. A *total set* (or fundamental set) in a normed space X is a subset $M \subset X$ whose finite linear span is dense in X. An orthonormal basis is an orthonormal sequence in an inner product space X which is total in X.

Note that Orthonormal bases can be either countable or uncountable.

Example 2.18. The sequence $(e_i)_{i=1}^{\infty}$ is an orthonormal basis for ℓ^2.

Example 2.19. The orthonormal system

$$\phi_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}, \ \ n = 0, \pm 1, \pm 2, \dots,$$

is an orthonormal basis in the space $L^2([-\pi, \pi])$.

Proposition 2.20. Any orthonormal basis of a separable Hilbert space H is a Schauder basis of H.

Proof. Let $(e_i)_{i \in I}$ be an Orthonormal basis of the separable Hilbert space H. Since H is separable, by Definition 1.21 the space H contains a countable dense subset and, since all bases in Hilbert spaces have the same cardinality our Orthonormal basis will be countable. So, we can write $(e_i)_{i=1}^{\infty}$.
Put $Y = \text{span}\{e_1, e_2, \dots\}$.
Since Y is dense in H, we know that $\overline{Y} = H$. By Bessel's inequality, for all $x \in H: \ \sum_{i=1}^{\infty}(x, e_i)e_i$ converges to some $y_0 \in H$.
Take $x \in H$ and let $y_0 := \sum_{i=1}^{\infty}(x, e_i)e_i$.

Then, for all $k \in \mathbb{N}$

$$\begin{aligned}(x - y_0, e_k) &= \left(x - \sum_{i=1}^{\infty}(x, e_i)e_i, e_k\right) = \\ &= (x, e_k) - \left(\sum_{i=1}^{\infty}(x, e_i)e_i, e_k\right) = \\ &= (x, e_k) - \sum_{i=1}^{\infty}(x, e_i)(e_i, e_k) = \\ &= (x, e_k) - (x, e_k) = 0.\end{aligned}$$

So, $x - y_0 \in \overline{Y}^{\perp}$, but we already have that $x - y_0 \in \overline{Y}$. Hence, $x - y_0 = 0$, i.e. $x = \sum_{i=1}^{\infty}(x, e_i)e_i$.

So, if we denote $a_i = (x, e_i)$ we obtain

$$x = \sum_{i=1}^{\infty} a_i e_i.$$

Hence, $(e_i)_{i=1}^{\infty}$ is a Schauder basis of H. □

Example 2.21. The Haar system is an orthonormal system in $L_2[0, 1]$.

Theorem 2.22 ([10, Theorem 9.5.11.])**.** *A Hilbert space H has a countable orthonormal basis if and only if H is separable.*

Proof. If $\dim(H) < \infty$, there is nothing to prove, so suppose that $\dim(H) = \infty$.

Let $(e_i)_{i=1}^{\infty}$ be a countable orthonormal basis of the Hilbert space H. Since $(e_i)_{i=1}^{\infty}$ is countable and dense in H we apply Proposition 2.20. Then, $(e_i)_{i=1}^{\infty}$ is a Schauder basis and hence, by Proposition 2.8 H is separable.

For the other implication suppose that a sequence $(e_i)_{i=1}^{\infty}$ is dense in H. If we remove the linearly dependent elements in the sequence we can assume that $(e_i)_{i=1}^{\infty}$ is linearly independent with span dense in H. Using the Gram-Schmidt process for orthonormalizing a linearly independent sequence in an inner product space, we can also assume that $(e_i)_{i=1}^{\infty}$ is orthonormal.
Let $x \in H$ and let $\varepsilon > 0$ be arbitrary.
Then, we can find a vector $y = \sum_{i=1}^{n_\varepsilon} a_i e_i$ such that $||x - y|| < \varepsilon$. It now follows from the properties of the orthogonal projection that if $N \geq n_\varepsilon$, then

$$\left|\left| x - \sum_{i=1}^{N} (x, e_i) e_i \right|\right| \leq \left|\left| x - \sum_{i=1}^{n_\varepsilon} (x, e_i) e_i \right|\right| \leq ||x - y|| < \varepsilon.$$

Since ε was arbitrary, this shows that $x = \sum_{i=1}^{\infty} (x, e_i) e_i$, so $(e_i)_{i=1}^{\infty}$ is an orthonormal basis. □

If we are in a non complete inner product space it is clear that an orthonormal basis cannot be a Schauder basis, since Schauder bases are just defined in Banach spaces. Otherwise, in Theorem 2.3 we saw that for every vector space there is always a Hamel basis. Hence, are orthonormal basis and Hamel basis related somehow in inner product spaces which are not Hilbert spaces? We come up with the following theorem.

Theorem 2.23. *An Orthonormal basis of a Hilbert space X is a Hamel basis if and only if X is finite dimensional.*

Proof. Let $(e_i)_{i\in I}$ be an Orthonormal-Hamel basis of a Hilbert space, that is, an orthonormal basis which is also a Hamel basis. In [7] page 93 it says that in infinite dimensional Hilbert spaces *"an orthonormal basis is never large enough to be a vector-space basis"*. So, X must be finite dimensional.

Now, let X be an inner product space with $\dim X = n < \infty$ (i.e. a Hilbert space), and let $(e_i)_{i=1}^n$ be an Orthonormal basis of X (since X is finite dimensional it is clear that the orthonormal basis is countable, even finite). Since $(e_i)_{i=1}^n$ is linearly independent and its linear span is all X, it will also be a Hamel basis. □

Chapter 3

Results

The idea of the project was to compare bases and study implications between them. The cases that we have to study are the ones where there can be more than one different type of bases. Since Schauder bases are defined in Banach spaces and orthonormal bases are defined in inner product spaces, we just have to study the cases in the following figure:

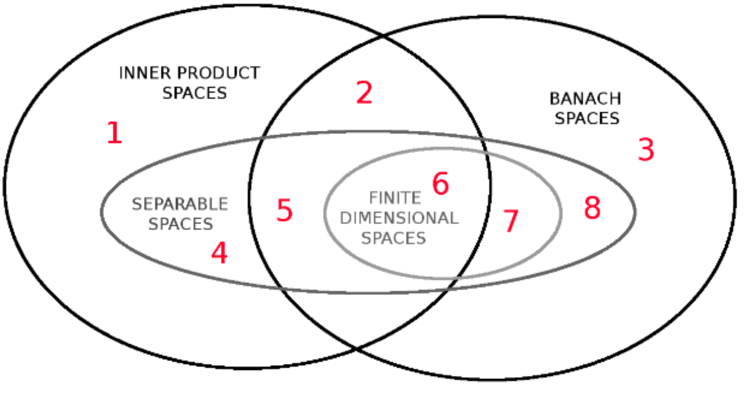


Figure 3.1: Spaces and cases to be studied

Let us explain each of these cases.

3.1 Non complete inner product spaces

By definition, in non complete inner product spaces (also called pre Hilbert spaces) we cannot have a Schauder basis.
We cannot say anything about the relations between Hamel bases and orthonormal bases in this case.

We could think about extending Theorem 2.23 to inner product spaces.
This would mean that in this case (**1** and **4** from the figure above) there cannot exist a basis which is Hamel and orthonormal. This hypothesis is not true as one can see in the example below.

Example 3.1. The space c_{00} with the inner product $||\cdot||_2$ is an inner space. It is not complete since, for example, the Cauchy sequence $(x_i)_{i=1}^{\infty} \in c_{00}$ defined as $x_k = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{k}, 0, \dots)$ does not converge in c_{00}.
In Example 2.2 we said that the sequence $(e_i)_{i=1}^{\infty}$ of the standard unit vectors is a Hamel basis on c_{00}, and it is quite easy to show that this sequence is also an orthonormal basis on the same space. Thus, we have an orthonormal-Hamel basis of an infinite dimensional space.

3.2 Banach spaces without inner product

Now, we will study the case when the space is Banach but it does not have an inner product.
The definition of orthonormal basis requires the space to have an inner product. Hence, the spaces of this group can have Hamel bases and Schauder bases.

3.2.1 Infinite dimensional spaces

When we studied Hamel basis we showed the Proposition 2.4, which says:

If X is an infinite dimensional Banach space, then every Hamel basis for X is uncountable.

The definition of Schauder basis indicates that it is always countable.
Therefore, in this section (**3** and **8**) we cannot have any relation between our bases.

3.2.2 Finite dimensional spaces

In contrast to the infinite dimensional case, in this case we can apply the Proposition 2.16 and hence, we know that in **7** any Hamel basis will be a Schauder basis.

3.3 Hilbert spaces

This part is the most interesting one since all three bases are defined there.

3.3.1 Non separable spaces

Since we are looking at non separable Hilbert spaces it is clear that the spaces will be infinite dimensional. Hence, as we have seen in Section 3.2.1 Hamel basis and Schauder basis cannot be related.
From Theorem 2.22 we can conclude that all Orthonormal bases in **2** are uncountable. Hence, There is no such relation between orthonormal basis and Schauder basis.
Finally, we should study the relation between Hamel basis and orthonormal basis. Theorem 2.23 shows that an orthonormal basis of a Hilbert space X is a Hamel basis if and only if X is finite dimensional. Thus, there is no such relation between these two bases.

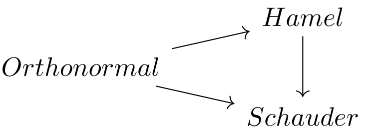
3.3.2 Infinite dimensional and separable spaces

As we mentioned before, in an infinite dimensional Banach space Hamel bases and Schauder bases cannot be related. Proposition 2.20 gives us that any orthonormal basis will be a Schauder basis.
As a corollary, in infinite dimensional separable Hilbert spaces it does not exists any relation between orthonormal bases and Hamel bases.

Example 3.2. During Chapter 2 we introduced the Haar system. Haar system is an orthonormal-Schauder basis with respect to the space $L^2[0, 1]$.

3.3.3 Finite dimensional spaces

Finally, our most fascinating part. As we mentioned before, in finite dimensional Banach spaces every Hamel basis is also a Schauder basis. Furthermore, we saw that in separable Hilbert spaces every orthonormal basis is also a Schauder basis. Lastly, from Theorem 2.23 any orthonormal basis of a finite dimensional Hilbert space is a Hamel basis. Hence, we have



So, every orthonormal basis of a finite dimensional Hilbert space will be Hamel and Schauder.
Thus, we can find an orthonormal-Hamel-Schauder basis for some space.

Example 3.3. The standard basis $(e_i)_{i=1}^n$ of the space $\mathbb{R}^n$ defined as $e_1 = (1, 0, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, $\ldots$, $e_n = (0, \ldots, 0, 1)$ is an Orthonormal-Hamel-Schauder basis of $\mathbb{R}^n$.

Chapter 4

Conclusions and Further Research

In this project we studied three bases and we compared them in all possible cases. We talked about interesting results such as orthonormal-Hamel-Schauder bases of finite dimensional Hilbert spaces and the non-possible orthonormal-Schauder basis of a non-separable Hilbert space.

This project has a lot of possibilities to be extended. Since we want to create a network of relations between bases, the continuation would be to add more types of bases to the network until it is complete.
We also did not find any general theorem for bases in non complete inner product spaces, so one could try to find it.
Next step on the study of bases would be to introduce the next basis, *unconditional basis.*

4.1 Unconditional basis

Before defining our new basis we present an absolutely necessary term.

Proposition 4.1 ([9, Proposition 1.c.1])**.** Let $(x_n)_{n=1}^{\infty}$ be a sequence of vectors in a Banach space X. Then the following conditions are equivalent.

(i) The series $\sum_{n=1}^{\infty} x_{\pi(n)}$ converges for every permutation π of the integers.

(ii) The series $\sum_{i=1}^{\infty} x_{n_i}$ converges for every choice of $n_1 < n_2 < \dots$.

(iii) The series $\sum_{i=1}^{\infty} \theta_n x_n$ converges for every choice of signs θ_n (i.e. $\theta_n = \pm 1$).

(iv) For every $\varepsilon > 0$ there exists an integer n so that $||\sum_{i\in\sigma} x_i|| < \varepsilon$ for every finite set of integers σ which satisfies $\min\{i \in \sigma\} > n$.

A series $\sum_{i=1}^{\infty} x_n$ which satisfies one, and thus all of the above conditions, is said to be *unconditionally convergent.*

Definition 4.2. A basis $(x_n)_{n=1}^{\infty}$ of a Banach space X is said to be *unconditional* if for every $x \in X$, its expansion in terms of the basis $\sum_{n=1}^{\infty} a_n x_n$ converges unconditionally.

To clarify more the meaning of an unconditional basis we state the following proposition.

Proposition 4.3 ([9, Proposition 1.c.6])**.** A basic sequence $(x_n)_{n=1}^{\infty}$ is unconditional if and only if any of the following conditions holds.

(i) For every permutation π of the integers the sequence $(x_{\pi(n)})_{n=1}^{\infty}$ is a basic sequence.

(ii) For every subset θ of the integers the convergence of $\sum_{n=1}^{\infty} a_n x_n$ implies the convergence of $\sum_{n\in\theta} a_n x_n$.

(iii) The convergence of $\sum_{n=1}^{\infty} a_n x_n$ implies the convergence of $\sum_{n=1}^{\infty} b_n x_n$ whenever $|b_n| \leq |a_n|$, for all n.

Now that we know what is an unconditional basis, we can study examples that we have seen during the thesis, such as:

Example 4.4. In the Example 2.15 we said that the standard basis for c_0 is a Schauder basis. It is not difficult to see that it is also an unconditional basis of c_0.

With more examples of other infinite dimensional Banach spaces we could try to evaluate which additional properties an infinite dimensional Banach space must have to possess an unconditional-Schauder basis.

Example 4.5. The Haar system defined in Example 2.7 is an unconditional-orthonormal-Schauder basis of $L^p[0,1]$, for $1 < p < \infty$.

Again, we could think about more examples to finally assess which spaces can have an unconditional-orthonormal-Schauder basis.

Furthermore, the next proposition shows us two examples where there does not exist any unconditional basis.

Proposition 4.6. [9] L_1 and $C(0,1)$ fail to have an unconditional basis.

The proof of the proposition above can be found in [9]. Looking carefully to this proof one could come up with other statements which would help us with a better understanding of unconditional bases.

We have seen some examples about when does and does not exist an unconditional basis. Hence, next step would be to find relations between this basis and the other three bases that we have already studied.

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