Department of Mathematics

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Fredrik Berntsson, Matts Karlsson, Vladimir Kozlov and Sergei A. Nazarov

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Department of Mathematics
Linköping University
S-581 83 Linköping

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Fredrik Berntsson [*] Matts Karlsson[*] Vladimir Kozlov[*]
Sergei A. Nazarov[†]

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Abstract

One dimensional models for fluid flow in tubes are frequently used to model complex systems, such as the arterial tree where a large number of vessels are linked together at bifurcations. At the junctions transmission conditions are needed. One popular option is the classic Kirchhoff conditions which means conservation of mass at the bifurcation and prescribes a continuous pressure at the joint.

In reality the boundary layer phenomena predicts fast local changes to both velocity and pressure inside the bifurcation. Thus it is not appropriate for a one dimensional model to assume a continuous pressure. In this work we present a modification to the classic Kirchhoff conditions, with a symmetric pressure drop matrix, that is more suitable for one dimensional flow models. An asymptotic analysis, that has been carried out previously shows that the new transmission conditions has en exponentially small error.

The modified transmission conditions take the geometry of the bifurcation into account and can treat two outlets differently. The conditions can also be written in a form that is suitable for implementation in a finite difference solver. Also, by appropriate choice of the pressure drop matrix we show that the new transmission conditions can produce head loss coefficients similar to experimentally obtained ones.

1 Introduction

Mathematical models of fluid flow in tubes have many important applications. Computer simulations are used for understanding the behavior of fluids in complex systems consisting of many different types of components.

[*]Linköping University, SE-58183 Linköping, Sweden. Emails: fredrik.berntsson@liu.se, matts.karlsson@liu.se, and vladimir.kozlov@liu.se

[†]Institute of Problems of Mechanical Engineering RAS, St Petersburg State University, Russia. Email: srgnazarov@yahoo.co.uk

In the most general case a computer simulations are based on describing the fluid using the Navier-Stokes equations. Here the fluid is kept confined in a system consisting of tubes with rigid walls, and, e.g., bifurcations or fluid reservoirs. This means that naturally certain quantities such as the pressure $p(\vec{x}, t)$ and the velocity $\vec{v}(\vec{x}, t)$, $\vec{x} \in \Omega \subset \mathbb{R}^3$ are continuous, or even differentiable, functions.

Solving the full Navier-Stokes equations for realistically complex systems is extremely time consuming. This creates a demand for simpler models that, maybe, has a lower accuracy but can be solved much easier. The simplest types of models are one dimensional ones, where the flow in a pipe is described using a fixed flow profile, e.g. the Posieuille flow, with a spatial coordinate z, directed along the pipe, which parametrized by, e.g., the average pressure $\bar{p}(z, t)$, for the cross section of the pipe, and the average velocity $\bar{v}(z, t)$ in the direction of the flow. Using one dimensional models a complex system would be modelled as, essentially, line segments and a bifurcation would be modelled as a point where three line segments meet. Thus, within this model, the bifurcation is effectively reduced to a point where suitable transmission conditions are supplied.

The simplest transmission condition is the classic Kirchhoff conditions [1] which prescribes a continuous pressure, and also a zero net flux, at the bifurcation. The classic transmission conditions do not take any information regarding the geometry of the bifurcation into account. However, for instance, it is well known that the flow at the bifurcation depends strongly on the angles involved [12].

The bifurcation of thin blood vessels was considered in [2, 3], where a dimension reduction of the three dimensional problem was performed. As a result a one dimensional model of a bifurcating vessel, with new transmission conditions at the bifurcation point, were derived. The new transmission conditions can be considered as a modification of the classic Kirchhoff conditions, see also [4] for a similar analysis. The advantages of the new transmission conditions, as observed in [2, 3], are as follows: First, the discrepancy in the approximation of the solutions of the three dimensional problem by solutions of the one dimensional problem are of order $\mathcal{O}(\mathrm{e}^{-\delta/h})$, where $\delta > 0$ is a number dependent on the cross-section, and h is the relative thickness of the vessel, whereas for the classic Kirchhoff conditions the discrepancy is $\mathcal{O}(h^3)$ for the velocities and $\mathcal{O}(h)$ for the pressure field. The difference is essential if a large system of bifurcating vessels, like in the bifurcating arterial tree, is considered. Second, the modified transmission conditions do take geometric information, such as angles, into account [3]. Finally, the new transmission conditions are of a type that can easily be implemented directly in, e.g., a finite difference solver.

It was shown in [2] that the modified Kirchhoff conditions can be reduced to the classical ones by changing the lengths of the bifurcating vessels in an appropriate way, i.e. to introduce appropriate one dimensional images of

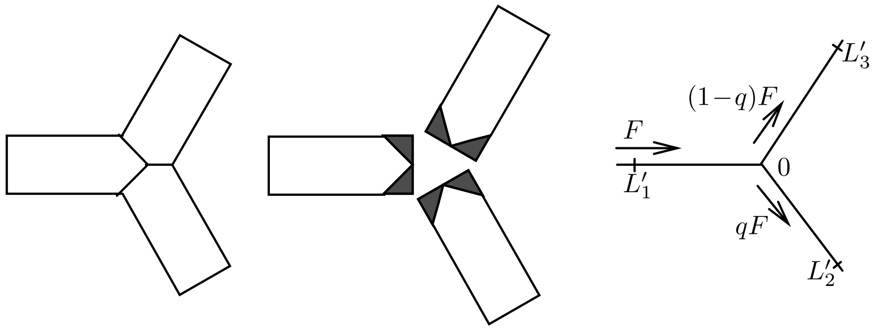


Figure 1: The bifurcation consists of three pipe segments. For each segment a local coordinate $z_\alpha \in [0, L_\alpha]$, for $\alpha = 1, 2, 3$, is introduced. The total flux F is divided between the two outlets. By breaking the joint in three parts we obtain pointed cylinders. Since the Poiseuille ansatz is derived for straight cylinders the $1D$ model, with Kirchhoff conditions, is accompanied by an increase in fluid volume. Thus for the one dimensional model modified lengths L'_α are used.

the thin channels. This is important since it gives a way to recover defects in a system of bifurcating vessels by comparing the measured lengths and the lengths which can be obtained by solving an inverse problem based on our one-dimensional model, with the classic transmission conditions and unknown lengths. The difference in these lengths must indicate the presence of defects [2, 3].

There is a bulk of engineering experience, see [5, 11], regarding the influence of a bifurcation on the flow inside a system. Generally the effect of a bifurcation can be described in terms of a head loss coefficient that describes the energy loss due to the flow profiles reconfiguring inside the bifurcation. In this paper we will investigate different types of transmission coefficients that can be used to describe a dividing, or combining, flow at a bifurcation.

2 Preliminaries of Fluid Flow

In our work we only consider a steady state flow in pipes with rigid walls, i.e. we have the Posieuille flow, which occur in straight cylinders governed by the Navier-Stokes equations with no-slip boundary conditions,

$$v_z(r,\varphi,z) = \frac{1}{4\mu}(r^2 - R^2)\partial_z\bar{p}(z), \quad v_\varphi(r,\varphi,z) = 0, \quad \text{and} \quad v_r(r,z) = 0, \quad (1)$$

where $\bar{p}(z)$ denotes the average pressure, μ is the dynamic viscosity of the fluid, and R is the radius of the pipe. By integrating the flow profile we obtain the flux through the cross-section,

$$F(z) = 2\pi \int_{r=0}^{R} v_z(r,z) r dr = -\frac{\pi R^4}{8\mu} \partial_z \bar{p}(z) = A\bar{v}, \tag{2}$$

where $\bar{v}$ is the average flow velocity, in the z-direction, and $A = \pi R^2$ is the cross section area.

Conservation of energy for a fluid is expressed in terms of the Bernoulli equation. The energy, per unit weight, of a fluid a location z, is called the *head* [5, 10] and is given by

$$h(z) = e(z) + \frac{\bar{p}(z)}{\rho g} + \frac{\bar{v}(z)^2}{2g} \tag{3}$$

where $e(z)$ is the elevation, ρ is the density of the fluid, and g is the acceleration due to gravity. The dimension of $h(z)$ is meters. We remark that differences in elevation are not considered further in the paper.

For a component, e.g. a pipe segment or a bend, the head loss coefficient, see e.g. [5], is defined as

$$K_{12} = \frac{h_1 - h_2}{\bar{v}_1^2/2g} \tag{4}$$

where h_1 and h_2 is the head measured directly before and after the component.

For a straight pipe segment of length L, with a constant cross section area, the loss coefficient can be calculated by the observation that $\bar{v}$ is constant which allows us to evaluate the pressure drop using (1). The result is

$$\bar{p}(L) - \bar{p}(0) = \int_0^L \partial_z \bar{p} dz = -\frac{8\mu}{\pi R^4} LF, \text{ and } K^{(li)} = \frac{16\pi\mu}{\rho}\frac{L}{F}. \tag{5}$$

This is a linear head loss coefficient due to a constant pressure gradient.

3 Transmission conditions at a Bifurcation

A bifurcation consists of three pipe segments that are connected at a joint. The bifurcation is illustrated in Figure 1. Note that a local spatial coordinate z_α, $\alpha = 1, 2, 3$, is introduced for each of the pipe segments.

The principle of conservation of mass means that the net flux into the bifurcation is zero. Assuming that a flux F at the inlet, and that a fixed proportion q goes into each of the two outlets, then

$$F = F_1 = F_2 + F_3 = qF + (1-q)F \text{ or } A_1\bar{v}_1 = A_2\bar{v}_2 + A_3\bar{v}_3 = qA_1\bar{v}_1 + (1-q)A_1\bar{v}_1, \tag{6}$$

where A_α and $\bar{v}_\alpha$, are the cross section areas and mean velocities in each of the pipe segments. This means that the mean velocities $\bar{v}_i$ in each segment are uniquely determined by F, q, and the respective cross section areas A_α.

Remark 3.1. If $F_\alpha \geq 0$ we have a situation with one inlet and two outlets, and $0 \leq q \leq 1$. It is also possible to consider a situation where, e.g., F_1 , $F_3 > 0$, but $F_2 < 0$. In that case we would effectively have two inlets and one outlet. This would correspond to $q > 1$.

3.1 The Classic Kirchhoff Conditions

In addition to conservation of mass, e.g. (6), the classic Kirchhoff transmission assumes that the pressure is continuous at the bifurcation, e.g.

$$\bar{p}_1(0) = \bar{p}_2(0) = \bar{p}_3(0), \tag{7}$$

which can be interpreted as the point $z_\alpha = 0$ substituting for the entire junction zone.

The above assumption allows us to calculate the loss coefficient K_{12}, see (4). We note that, within the corresponding one dimensional model,

$$\frac{\Delta \bar{p}}{\rho} = \frac{\bar{p}(L_1)}{\rho} - \frac{\bar{p}(L_2)}{\rho} = \frac{8\pi\mu F}{\rho}(\frac{L_1}{A_1^2} + q\frac{L_2}{A_2^2}), \tag{8}$$

and since $\bar{v}_2 = F_2/A_2 = qF/A_2$ we obtain

$$K_{12} = K_{12}^{(li)} + K_{12}^{(a)}, \tag{9}$$

where

$$K_{12}^{(li)} = \frac{16\mu\pi}{\rho F}(L_1 + qL_2(\frac{A_1}{A_2})^2) \text{ and } K_{12}^{(a)} = 1 - q^2(\frac{A_1}{A_2})^2. \tag{10}$$

Thus we have split the loss coefficient in two distinct parts. First there is the linear pressure loss $K_{12}^{(li)}$ due to the distance $L_1 + L_2$ between the measurement points. Second, there is a head loss $K_{12}^{(a)}$ resulting from a change of cross section area at the bifurcation. No other effects from the geometry of the bifurcation are included.

Remark 3.2. Note that the factor $K_{12}^{(a)}$ appear specifically due to the bifurcation and only takes the cross-section areas of the respective tubes into account. If we consider the case $A_1 = A_2$ then $K_{12}^{(a)} = 1 - q^2$, as illustrated in Figure 2. Note that for the case $q = 1$, i.e. no flux in the other outlet, then $K_{12}^{(a)} = 0$. This means that the bifurcation has no effect at all and could be replaced by a straight tube. This is obviously unphysical.

Remark 3.3. Similarly we can calculate K_{13} by the assumption $F_3 = (1 - q)F$. Also, recall that the flow F_α in each pipe segments may be positive or negative.

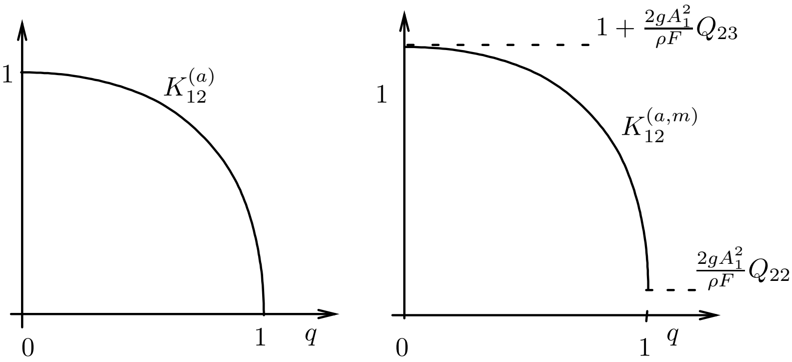


Figure 2: The loss coefficient $K_{12}^{(a)}$, for the case $A_1 = A_2$, as a function of q (left). This is the part that appears specifically due to the bifurcation when the classic Kirchhoff transmission conditions are used. We also display $K_{12}^{(a)} + K_{12}^{(m)}$, for the case $A_1 = A_2$, as a function of q (right). This is the loss coefficient for the bifurcation when the modified transmission conditions are used.

3.2 The Modified Kirchhoff Conditions

In our previous work, see [2, 6], we have carried out an asymptotic analysis of the Navier-Stokes flow in a junction of long, but finite, tubes. In addition to the usual dimension-reduction procedure in straight tube segments, implying the use of the Poiseuille ansatz and leading to the classical Reynolds equation, one must deal with the phenomenon of the boundary layer describing the local field perturbation in the junction zone. The analysis shows that the effects of the boundary layer can be taken into account by introducing a pressure drop matrix Q described below.

In order to explain the appearance of additional pressure drops, and their inclusion into a one-dimensional model, we conduct a mental experiment: break the joint into three parts as is seen in Figure 1. As a result, we obtain three cylinders with pointed ends. The Poiseuille ansatz is derived for a cylinder with straight ends, and therefore a one-dimensional model with the Kirchhoff transmission conditions is accompanied by an increase in the volume of the fluid in the junction zone. This does not affect the balance of fluxes, that are stable integral characteristics of inlets and outlets. However, the pressure can change in ways not inherent in the classical model. In the refined model these changes are described by the symmetric pressure drop matrix Q, which is of size 2×2 for a bifurcation, that reflects the geometry of the junction zone. As mentioned previously, its introduction into the transmission conditions at the bifurcation point is sufficient to significantly improve the quality of the approximation of three-dimensional fields and we have exponentially small errors.

In the modified Kirchhoff conditions the pressure is no longer continuous at the bifurcation point. Rather we have

$$p_2(0)=p_1(0)+Q_{22}qF+Q_{23}(1-q)F \text{ and } p_3(0)=p_1(0)+Q_{33}(1-q)F+Q_{32}qF, \quad (11)$$

where

$$Q=\begin{pmatrix} Q_{22} & Q_{23} \\ Q_{23} & Q_{33} \end{pmatrix}, \quad (12)$$

is the symmetric pressure drop matrix. The modified transmission conditions allows us to calculate a pressure drop

$$\Delta\bar{p}=\bar{p}(L_1)-\bar{p}(L_2)=8\pi\mu F(\frac{L_1}{A_1^2}+q\frac{L_2}{A_2^2})+Q_{22}qF+Q_{23}(1-q)F \quad (13)$$

As previously we can calculate a loss coefficient

$$K_{12}=K_{12}^{(li)}+K_{12}^{(a)}+K_{12}^{(m)}, \quad K_{12}^{(m)}=\frac{2A_1^2g}{\rho F}(Q_{22}q+Q_{23}(1-q)), \quad (14)$$

where $K_{12}^{(li)}$ and $K_{12}^{(a)}$ are the same as for the classic Kirchhoff conditions, see (9), and $K_{12}^{(m)}$ is the loss specifically due to geometry of the junction zone. The presence of an obstacle in the outlet tube, just after the bifurcation, would mean an increased value for the coefficient Q_{22}. In the extreme case when the outlet tube is entirely closed the coefficient Q_{22} would be infinitely large.

Remark 3.4. In order to illustrate the effect of the modified transmission conditions we again consider the case $A_1 = A_2$. In Figure 2 we display $K_{12}^{(a)}+K_{12}^{(m)}$, for $0 \le q \le 1$. The modification $K_{12}^{(m)}$ represents the loss specifically due to the shape of the bifurcation. In contrast to Remark 3.2 the bifurcation now has en effect on the flow even in the case $q=1$.

Remark 3.5. In the transmission condition (11) the number q is in practice not known before the flow problem is solved. However, $qF = A_2\bar{v}_2$ and $(1-q)F = A_3\bar{v}_3$ so the condition can be rewritten as

$$\bar{p}_2(0)=\bar{p}_1(0)+Q_{22}A_2\bar{v}_2+Q_{23}A_3\bar{v}_3, \quad (15)$$

which can be implemented in, e.g., a finite difference solver. The above version of the transmission condition can also be used for the case of a time dependent flow.

Remark 3.6. Finding the pressure drop matrix for particular configuration is a difficult problem. In principle we would need to solve the $3D$ Navier-Stokes equations in the union of three semi-infinite tubes and analyze the behavior of the pressure components at infinity. An alternative approach would be to determine the Q matrix by solving an inverse problem where the model is compared with experimentally obtained loss coefficients.

3.3 Corrected one dimensional image of the Bifurcation

Let us rewrite (11) in the form

$$\bar{p}_1(0)=Q_{23}F=\bar{p}_2(0)+(Q_{22}-Q_{23})qF=\bar{p}_3(0)+(Q_{33}-Q_{32})(1-q)F, \quad (16)$$

and recall that $\bar{p}_\alpha$ are linear functions in the coordinate z_α, i.e.

$$\bar{p}_1(z)=-\frac{z}{B}F+\bar{p}_1(0),\ \bar{p}_2(z)=\frac{z}{B}qF+\bar{p}_2(0),\ \bar{p}_3(z)=\frac{z}{B}(1-q)F+\bar{p}_3(0), \quad (17)$$

where $B=\pi R^4/8\mu$. Thus we can rewrite (16) as

$$\bar{p}_1(-BQ_{23})=\bar{p}_2(B(Q_{22}-Q_{23}))=\bar{p}_3(B(Q_{33}-Q_{32})). \quad (18)$$

The arguments ζ_α of the pressure functions $\bar{p}_\alpha$ imply new origins for the local coordinates z_α, for the inlets and outlets, as seen in Figure 1. The new one dimensional problem on the intervals (ζ_α, L_α), with the Classic Kirchhoff transmission conditions gives a new loss coefficient $\hat{K}_{12}^{(a)}$, which equals $K_{12}^{(a)}+K_{12}^{(m)}$, see (14), obtained for our modified transmission conditions. Thus the introduction of the pressure drop matrix is equivalent to adjusting the lengths of the line segments used in the one dimensional model.

We also mention the papers [4, 7, 8, 9] where low-order terms in the asymptotic expansion of the solution to the Navier-Stokes problem in the junction of thin pipes were discussed. However, these papers do not attain the same order of accuracy in their one dimensional models.

4 Results and Discussion

In order to illustrate our results we use a number of test cases and compute the loss coefficients. The head loss coefficient is expressed in terms of the Reynolds number for the flow in the inlet pipe, e.g.

$$\mathbf{Re}=\frac{\bar{v}_1 D_1}{\nu}, \quad (19)$$

where $\nu=\mu/\rho$ is the kinematic viscosity, $\bar{v}_1$ is the mean inlet velocity and D_1 is the diameter of the inlet pipe. This is convenient since the same velocity $\bar{v}_1$ is used to compute the kinetic energy $\bar{v}_1^2/2$ needed for making the head loss coefficient non-dimensional. In all cases the physical parameters roughly correspond to an oil with a low mean flow velocity. More precisely we use $\nu=1.3\cdot 10^{-5}\,m^2/s$ and $\rho=960\,kg/m^3$.

Example 4.1. In the first test we consider a bifurcation where the diameter of both the inlet and the two outlets is $D_\alpha=10\,mm$. The mean inlet velocity $\bar{v}_1$ varies in the range 0.01–1.00 m/s. This gives a Reynolds number in the range 7–770 and a laminar flow. In order to calculate the head loss

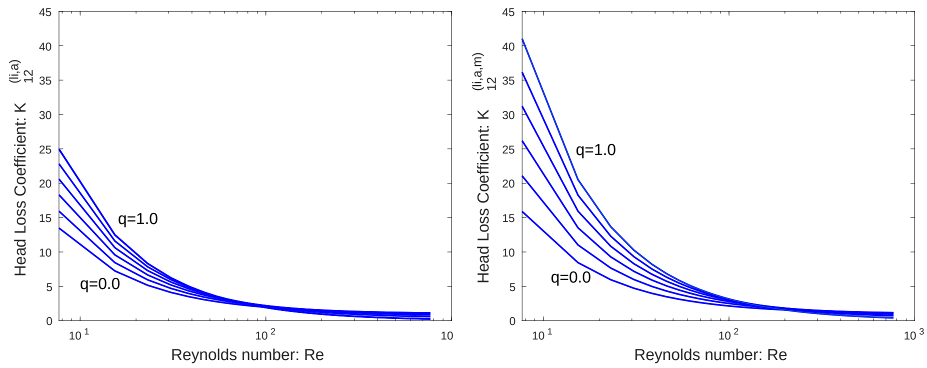


Figure 3: We show the head loss coefficient K_{12} calculated by using the classic Kirchhoff conditions (left) and also the coefficient obtained using our modified Kirchhoff conditions (right). In both cases the proportion of the flow that goes into the respective outlets is $q = 0.0, 0.2, \ldots, 0.8, 1.0$.

coefficient K_{12}, see (4), we set $L_\alpha = 15\,mm$. This means that we get a relatively large contribution from the linear pressure loss $K_{12}^{(li)}$. In Figure 3 we show the resulting head loss coefficient for a range of q values. For the modified Kirchhoff conditions, see (14), we used a pressure drop matrix Q as given by

$$Q = \begin{pmatrix} Q_{22} & Q_{23} \\ Q_{23} & Q_{33} \end{pmatrix} = 10^5 \cdot \begin{pmatrix} 1.0 & 0.15 \\ 0.15 & 1.0 \end{pmatrix}. \tag{20}$$

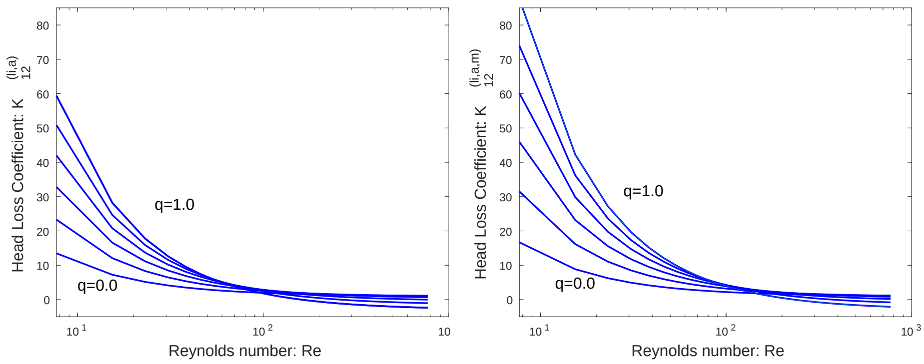
This case roughly corresponds to the results obtained from the T-junction experiment reported in [10].

Example 4.2. In the second test we consider a bifurcation where the diameter of the inlet pipe is $D_1 = 10\,mm$ and $D_2 = D_3 = 10/\sqrt{2}\,mm$ so that the total cross section area remains constant. The pressure drop matrix Q is given by

$$Q = \begin{pmatrix} Q_{22} & Q_{23} \\ Q_{23} & Q_{33} \end{pmatrix} = 10^5 \cdot \begin{pmatrix} 1.75 & 0.20 \\ 0.20 & 0.80 \end{pmatrix}. \tag{21}$$

The remaining parameters are the same as previously. In Figure 4 we show the resulting head loss coefficient for a range of q values.

The addition of a pressure drop matrix Q to the Kirchhoff transmission conditions gives a possibility to model differences in the geometry of the bifurcation by treating the two outlets differently.



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