Hopf Invariants in Real and Rational Homotopy Theory

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Abstract

In this thesis we use the theory of algebraic operads to define a complete invariant of real and rational homotopy classes of maps of topological spaces and manifolds. More precisely let f,g : M -> N be two smooth maps between manifolds M and N. To construct the invariant, we define a homotopy Lie structure on the space of linear maps between the homology of M and the homotopy groups of N, and a map mc from the set of based maps from M to N, to the set of Maurer-Cartan elements in the convolution algebra between the homology and homotopy. Then we show that the maps f and g are real (rational) homotopic if and only if mc(f) is gauge equivalent to mc(g), in this homotopy Lie convolution algebra. In the last part we show that in the real case, the map mc can be computed by integrating certain differential forms over certain subspaces of M. We also give a method to determine in certain cases, if the Maurer-Cartan elements mc(f) and mc(g) are gauge equivalent or not.

Keywords: *Rational homotopy theory, Real homotopy theory, operads, Hopf invariants.*

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Abstract

In this thesis we use the theory of algebraic operads to define a complete invariant of real and rational homotopy classes of maps of topological spaces and manifolds. More precisely let $f, g : M \to N$ be two smooth maps between manifolds M and N. To construct the invariant, we define an L_∞-structure on the space of linear maps $Hom_{\mathbb{R}}(H_*(M;\mathbb{R}), \pi_*(N) \otimes \mathbb{R})$ and a map

$$mc : Map_*(M, N) \to MC(Hom_{\mathbb{R}}(H_*(M;\mathbb{R}), \pi_*(N) \otimes \mathbb{R})),$$

from the set of based maps from M to N, to the set of Maurer-Cartan elements in $Hom_{\mathbb{R}}(H_*(M;\mathbb{R}), \pi_*(N) \otimes \mathbb{R})$. Then we show that the maps f and g are real (rational) homotopic if and only if $mc(f)$ is gauge equivalent to $mc(g)$, in this L_∞-algebra.

In the last part we show that in the real case, the map mc can be computed by integrating certain differential forms over certain subspaces of M. We also give a method to determine in certain cases, if the Maurer-Cartan elements $mc(f)$ and $mc(g)$ are gauge equivalent or not.

Abstract in Swedish

I denna avhandling används teorin för algebraiska operader för att definiera fullständiga invarianter för reella och rationella homotopiklasser av avbildningar av topologiska rum eller mångfalder. Mer specifikt, låt $f, g : M \to N$ vara två släta avbildningar mellan mångfalder M och N. För att konstruera invarianten definieras en L_∞-struktur på rummet av linjära avbildningar $Hom_{\mathbb{R}}(H_*(M;\mathbb{R}), \pi_*(N) \otimes \mathbb{R})$ och en avbildning

$$mc : Map_*(M, N) \to MC(Hom_{\mathbb{R}}(H_*(M;\mathbb{R}), \pi_*(N) \otimes \mathbb{R})),$$

från mängeden av baserade avbildningar från M till N till mängden av Maurer-Cartan-element i $Hom_{\mathbb{R}}(H_*(M;\mathbb{R}), \pi_*(N) \otimes \mathbb{R})$. Vi visar att avbildningarna f och g är reellt (rationellt) homotopa om och endast om $mc(f)$ är gaugeekvivalent med $mc(g)$ i denna L_∞-algebra.

I den sista delen visar vi, i det reella fallet, att avbildningen mc kan beräknas genom att integrera specifika differentialformer över specifika delmångfalder av M. Vi framför även en metod för att i specifika fall kunna bestämma om $mc(f)$ och $mc(g)$ är gaugeekvivalenta eller inte.

List of Papers

The following papers, referred to in the text by their Roman numerals, are included in this thesis.

PAPER I: **Algebraic Hopf Invariants**
Felix Wierstra, *submitted*

PAPER II: **Homotopy Morphisms Between Convolution Homotopy Lie Algebras**
Daniel Robert-Nicoud, Felix Wierstra

PAPER III: **Hopf Invariants and Differential Forms**
Felix Wierstra

Contents

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Introduction

"I hate when I'm on a flight and I wake up with a water bottle next to me like oh great now I gotta be responsible for this water bottle"
-Kanye West

1 The central question

The goal of this thesis is to use the theory of algebraic operads to solve questions in homotopy and algebraic topology. In particular we are mainly interested in the question:

Question 1.1. Given two maps $f, g : X \to Y$ between topological spaces X and Y, are f and g homotopic?

This is one of the most elementary questions in topology and is also extremely hard to answer in full generality. In this thesis we will solve a special case of this question by combining the theory of L_∞-algebras and algebraic operads with the theory of integration to answer Question 1.1 in the real and rational case. To do this we will first recall what real homotopic means.

Definition 1.2. Two smooth maps $f, g : M \to N$ between simply-connected smooth manifolds M and N are called real homotopic if the induced maps $\Omega_\bullet(f), \Omega_\bullet(g) : \Omega_\bullet(N) \to \Omega_\bullet(M)$ between the de Rham complexes of M and N are homotopic as maps of commutative differential graded algebras.

The central question that motivated this thesis then becomes:

Question 1.3. Given two smooth maps $f, g : M \to N$ from a compact simply-connected smooth manifold M to a simply-connected smooth manifold N of finite $\mathbb{R}$-type, are the maps f and g real homotopic?

Our goal is to define a sequence of invariants of the maps f and g such that f and g are real homotopic if and only if these invariants agree. Our approach to defining these invariants is to combine the latest developments in operad theory with the theory of differential forms and integration to obtain computable invariants of real homotopy classes of maps between manifolds.

Convention 1.4. Since we want to use the de Rham complex we will assume that all manifolds are smooth and that all maps between manifolds are smooth as well. Further we will assume that all manifolds are orientable. We will also assume that all our algebras, coalgebras, operads and cooperads are differential graded.

Remark 1.5. In this introduction we will only focus on manifolds, in Section 10 of Paper III we will give a brief explanation how we can apply the ideas of this thesis to simplicial sets as well.

2 Invariants of homotopy classes of maps using integration

The theory of differential forms and integration has a long history in algebraic topology and has often been very effective in answering many questions, for a more detailed description of applications of differential forms in algebraic topology we refer to the book [3]. In this section we will describe some examples of invariants of homotopy classes of maps which use differential forms and integration. These examples are the degree of a map, the classical Hopf invariant and the Hopf invariants in rational homotopy theory defined by Sinha and Walter (see [20]). The work in this thesis generalizes the ideas of these examples to give a complete invariant of real homotopy classes of maps between two simply-connected manifolds M and N, when M is compact.

2.1. The degree of a map The first example is the degree of a map. This is the simplest example where differential forms are used to define invariants of homotopy classes of maps.

Definition 2.1. Let $f : M \to N$ be a smooth map between a compact manifold M and a manifold N, assume that $dim(M) = dim(N)$. Denote by ω_N a representative of the top cohomology class of N and by M the fundamental class of M. The degree of the map f is defined as the following integral

$$\int_M f^* \omega_N.$$

This invariant is not particularly powerful, since we need to require that the manifolds M and N have the same dimension it is only defined in special cases. The degree of a map will also detect at most as much as the induced map in cohomology will detect, which is usually not enough to distinguish homotopy classes of maps. It is however strong enough to distinguish self maps of the n-sphere S^n.

Theorem 2.2. Let $f, g : S^n \to S^n$ be two smooth maps. The maps f and g are homotopic if and only if they have the same degree, i.e.

$$\int_{S^n} f^* \omega_{S^n} = \int_{S^n} g^* \omega_{S^n}.$$

For a proof see Lemma 17.10.2 of [3].

2.2. The classical Hopf invariant As we have seen in the previous section the degree of a map has the problem that it can not see anything above the top dimension of the target N. To solve this we will introduce the bar construction, this is a new chain complex associated to the de Rham complex $\Omega_\bullet(N)$, which has the property that it has more cohomology and therefore detects more. Using the bar construction we can define a version of the classical Hopf invariant, which is an invariant of maps $f : S^{4n-1} \to S^{2n}$. More details about the bar constructions can be found in Chapter 2 of [16]. The rest of this section is based on Section 1 of [20].

Definition 2.3. Let A be an associative algebra. The bar construction BA is defined as the cofree coassociative coalgebra generated by sA, the suspension of A. This space has as basis elements of the form $sa_1 \mid ... \mid sa_n$. We will call n the weight of an element $sa_1 \mid ... \mid sa_n$. The coproduct

$$\Delta : BA \to BA \otimes BA$$

is given by

$$\Delta(sa_1 \mid ... \mid sa_n) = \sum_{i=1}^{n} sa_1 \mid ... \mid sa_i \otimes sa_{i+1} \mid ... \mid sa_n.$$

The differential d_{BA} consists of two parts $d_{BA} = d_1 + d_2$. The first part is given by the extension of d_A by the Leibniz rule, i.e.

$$d_1(sa_1 \mid ... \mid sa_n) = \sum_{i=1}^{n} (-1)^{i-1|a_1|+...|a_{i-1}|} sa_1 \mid ... \mid s(d_A a_i) \mid ... \mid a_n.$$

The second part d_2 is coming from the product of A and is defined by

$$d_2(sa_1 \mid ... \mid sa_n) = \sum_{i=1}^{n-1} (-1)^{i-1+|a_1|+...+|a_i|} sa_1 \mid ... \mid s(a_i a_{i+1}) \mid ... \mid sa_n,$$

where $a_i a_{i+1}$ is the multiplication in A of a_i and a_{i+1}.

Remark 2.4. In this section we include a suspension in our definition of the bar construction, in the papers we not do this. Since we need more refined versions of the bar construction it is more natural to put the suspensions in the operads instead.

Proposition 2.5. The bar construction is a functor

$$B : As - alg \to Coas - coalg$$

from associative algebras to coassociative coalgebras.

Definition 2.6. Let $\pi : BA \to A$ denote the degree -1 linear map defined by $\pi(sa_1) = a_1$ and $\pi(sa_1 \mid \ldots \mid sa_n) = 0$ for $n \geq 2$.

Using the bar construction we can now define the classical Hopf invariant. The classical Hopf invariant is an invariant of maps $f : S^{4n-1} \to S^{2n}$ and is defined as follows.

Let $\omega \in \Omega_{2n}(S^{2n})$ be a cocycle representative for the top dimensional cohomology class. Since $\Omega_k(S^{2n})$ is zero for $k > 2n$, the wedge product $\omega \wedge \omega$ is also zero. It follows that the element $s\omega \mid s\omega \in B\Omega_\bullet(S^{2n})$ is a cocycle in the bar construction.

Fix a map $f : S^{4n-1} \to S^{2n}$. Since the bar construction is a functor we can pull $s\omega \mid s\omega$ back along f and get a cocycle $sf^*\omega \mid sf^*\omega \in B\Omega_\bullet(S^{4n-1})$. Since $H^{2n}(S^{4n-1}) = 0$, the element $f^*\omega$ is null homologous in $\Omega_\bullet(S^{4n-1})$. It is therefore possible to find an element $d^{-1}f^*\omega$ which bounds the cycle $f^*\omega$. A straightforward calculation then shows that the element $sd^{-1}f^*\omega \mid sf^*\omega$ is a coboundary, cobounding $sf^*\omega \mid sf^*\omega$ and $s(d^{-1}f^*\omega \wedge f^*\omega)$.

The next step is to apply the map π to the cocycle $s(d^{-1}f^*\omega \wedge f^*\omega)$. By doing this we obtain a $(4n-1)$-form $d^{-1}f^*\omega \wedge f^*\omega$ on S^{4n-1}, which we can integrate. The Hopf invariant is now defined as follows.

Definition 2.7. Let $f : S^{4n-1} \to S^{2n}$ be a smooth map. The Hopf invariant $H(f)$ of the map f is defined as the integral

$$H(f) = \int_{S^{4n-1}} d^{-1}f^*\omega \wedge f^*\omega.$$

Proposition 2.8 (Proposition 17.22, [3])**.** The definition of the Hopf invariant is independent of choice of ω and is an invariant of the homotopy class of f, i.e. homotopic maps have the same Hopf invariant.

The Hopf invariant is unfortunately not a complete invariant of maps $f : S^{4n-1} \to S^{2n}$, that is there are maps with the same Hopf invariant that are not homotopic. It is however possible to show that it is a complete invariant of real or rational homotopy classes of maps. We will state this result in Theorem 2.13 in the next section.

2.3. The Sinha-Walter Hopf invariants The classical Hopf invariant can be generalized to construct a complete invariant of real homotopy classes of maps. This was done in [20] by replacing the associative bar construction by a Lie coalgebraic version of the bar construction.

Definition 2.9 ([20], Definition 2.1)**.** Let A be a commutative algebra. The Lie coalgebraic bar construction is defined as the cofree Lie coalgebra co-generated by A with a certain differential coming from the differential of A and the multiplication of A. We will denote the Lie coalgebraic bar construction by $B_{Lie}A$.

Remark 2.10. In [21] and [20] Sinha and Walter also develop some computational techniques for getting an explicit basis for the free Lie coalgebra and for obtaining information about the Lie coalgebraic bar construction. In this introduction we omit this. Specially since we replace these techniques in Papers I and III by the Homotopy Transfer Theorem.

We will now define a similar construction as in the case of the classical Hopf invariant. Let $f : S^n \to Y_{\mathbb{Q}}$ be a continuous map from S^n to a rational space $Y_{\mathbb{Q}}$, denote by $A^*_{PL}(Y_{\mathbb{Q}})$ the polynomial de Rham forms on $Y_{\mathbb{Q}}$ (see Chapter 10 of [9]). This is a generalization of the smooth de Rham form due to Sullivan, which is defined for more general spaces than manifolds.

Definition 2.11. Let X be a simply-connected space and $f : S^n \to X$ be a map. Let $\omega \in B_{Lie}A^*_{PL}(X)$ be a cocycle of weight n, we call ω weight reducible if ω is cohomology equivalent to a cocylce of weight 1. If ω is weight reducible we will denote by $\tau(\omega)$ a choice of weight 1 cocycle cohomologous to ω.

Definition 2.12. Let $Y_{\mathbb{Q}}$ be a simply-connected rational space of finite $\mathbb{Q}$-type, $f : S^n \to Y_{\mathbb{Q}}$ a continuous map and let $\omega \in B_{Lie}A_{PL}(Y_{\mathbb{Q}})$ be a cocycle. The Hopf pairing

$$\eta : \pi_*(Y_{\mathbb{Q}}) \otimes H^*(B_{Lie}A_{PL}(Y_{\mathbb{Q}})) \to \mathbb{Q},$$

is defined by the following evaluation

$$\eta(f,\omega) = \int_{S^n} \tau(f^*(\omega)).$$

Here $\tau(f^*\omega)$ is a cocycle of weight 1 cohomology equivalent to $f^*\omega$.

It can be shown that this pairing is independent of choices, this is done in [20].

Theorem 2.13 ([20], Theorem 2.10)**.** The pairing η from Definition 2.12 is perfect, i.e. by taking the adjoint of the pairing we get an isomorphism

$$\eta^{\dagger} : H^*(B_{Lie}A_{PL}(Y_{\mathbb{Q}})) \to Hom_{\mathbb{Z}}(\pi_*(Y_{\mathbb{Q}}), \mathbb{Q}).$$

This theorem has as a corollary that it can distinguish homotopy classes of maps between S^n and $Y_{\mathbb{Q}}$.

Corollary 2.14. Two maps $f, g : S^n \to Y_{\mathbb{Q}}$ are homotopic if and only if

$$\int_{S^n} \tau(f^*\omega) = \int_{S^n} \tau(g^*\omega)$$

for all $\omega \in H^*(B_{Lie}(A^*_{PL}(Y_{\mathbb{Q}})))$.

When the space Y is not rational we get the following statement.

Corollary 2.15. Let Y be a simply-connected space of finite $\mathbb{Q}$-type and let $f, g : S^n \to Y$ be two maps, the maps f and g are rationally homotopic if and only if

$$\int_{S^n} \tau(f^*\omega) = \int_{S^n} \tau(g^*\omega),$$

for all $\omega \in H^*(B_{Lie}(A^*_{PL}(Y)))$.

3 Algebraic operads

The other essential ingredient of this thesis is the notion of an operad. Intuitively an operad is a type of algebraic structure that encodes certain types of algebras. Some examples of such types algebras are an associative product on a vector space, a Lie bracket on a vector space, or the multiplication of loops in the loop space of a topological space.

Operads originated in algebraic topology in the study of loop spaces and the word operad was coined by May in [19] to define the Recognition Principle. This theorem states that a topological space is weakly equivalent to an n-fold loop space if and only if it is an algebra over the little n-disks operad. But since then operads have found many applications in other fields. For examples, see the book [18].

Because of the great success of operads and the many applications to other fields, the field of algebraic operads became a topic of research in its own right and today there are numerous papers about algebraic operads. The main reference about algebraic operads is [16] and unless stated otherwise we will use the definitions and notation of [16] in this introduction.

To give an answer to Question 1.3 we will first use the theory of algebraic operads to define invariants between maps of algebras. Because of Definition 1.2 determining whether two maps $f, g : M \to N$ are real homotopic is equivalent to determining whether the induced maps on the de Rahm complexes are homotopic as maps of commutative differential graded algebras. To do this we will first need some preliminaries.

Convention 3.1. From now on we will work in the category of chain complexes over a field $\mathbb{K}$ of characteristic 0. In most situations $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{Q}$.

3.1. The bar and cobar construction relative to a twisting morphism Two of the most important ingredients in all our constructions are the bar and the cobar construction and in the previous sections we have already seen two examples of the associative and the Lie bar constructions. These constructions have many good properties, but what will be most important for this thesis is that they help to us construct fibrant and cofibrant objects in the model categories of algebras and coalgebras over an operad or cooperad. The main reference for this section is Chapter 11 of [16].

Definition 3.2 ([16], Section 11.2.2)**.** Let $\tau : \mathcal{C} \to \mathcal{P}$ be an operadic twisting morphism and A be a $\mathcal{P}$-algebra. Then we define $B_\tau A$ the bar construction relative to τ as $(\mathcal{C}(A), d)$, the free $\mathcal{C}$-coalgebra cogenerated by A with a certain differential coming from the differential and multiplication of A.

Dually we also have the cobar construction.

Definition 3.3 ([16], Section 11.2.8)**.** Let $\tau : \mathcal{C} \to \mathcal{P}$ be an operadic twisting morphism and C a $\mathcal{C}$-coalgebra. The cobar construction $\Omega_\tau C$ is defined as $(\mathcal{P}(C), d)$ the free $\mathcal{P}$ algebra generated by C with a certain differential coming from the differential and coproduct of C.

For more details about the bar and cobar differentials see Chapter 11 of [16].

3.2. The homotopy theory of algebras over an operad To determine whether two algebra maps are homotopic it is important that we first define what it means for algebra maps to be homotopic. To do this we will recall Hinich's model category structure on the category of $\mathcal{P}$-algebras, for some operad $\mathcal{P}$. Then we explain what it means for two algebra maps to be homotopic.

Theorem 3.4 ([14] Theorem 4.1)**.** Let $\mathcal{P}$ be an operad. The category of algebras over $\mathcal{P}$ admits a model structure in which the weak equivalences are given by the quasi-isomorphisms, the fibrations by surjective maps and the cofibrations are the maps which have the left lifting property with respect to acyclic fibrations.

A cylinder object in the category of $\mathcal{P}$-algebras is defined as follows.

Definition 3.5. Let A be a $\mathcal{P}$-algebra and R a commutative algebra, then we define the extension of scalars of A by R as the algebra with underlying vector space $R \otimes A$ and with a $\mathcal{P}$-algebra structure given by $\mu_n(r_1 \otimes a_1, ..., r_n \otimes a_n) = r_1...r_n \otimes \mu_n(a_1, ..., a_n)$ for $r_i \otimes a_i \in R \otimes A$ and $\mu_n \in \mathcal{P}(n)$.

We can now define what it means for algebra maps to be homotopic.

Definition 3.6. Let $\Omega_1 = \Lambda(t, dt)$ be the commutative algebra generated by an element t of degree 0 and an element dt of degree 1, the differential is given by $d(t) = dt$. Two algebra maps $f : A \to B$ between $\mathcal{P}$ algebras A and B are called homotopic if there exists a map $H : \Omega_1 \otimes A \to B$ such that H restricted to $t = 0$ and $dt = 0$ is f and H restricted to $t = 1$ and $dt = 0$ is equal to g.

For this introduction we will also need a description of the fibrant and some of the cofibrant objects in this category.

Proposition 3.7 ([14])**.** Let $\mathcal{P}$ be an operad and $B_{op}\mathcal{P}$ be the operadic bar construction on $\mathcal{P}$. Let $\pi : B_{op}\mathcal{P} \to \mathcal{P}$ be the canonical twisting morphism. In the model category of $\mathcal{P}$-algebras every object is fibrant and a class of cofibrant objects is given by by algebras of the form $\Omega_\pi C$, where Ω_π is the bar construction from 3.1 and C is a $B_{op}\mathcal{P}$-coalgebra.

Similar to algebras we also have a model structure on the category of coalgebras over a cooperad $\mathcal{C}$. This was done in [22] and [8].

Theorem 3.8 ([8], Theorem 3.11)**.** Let $\tau : \mathcal{C} \to \mathcal{P}$ be an operadic twisting morphism. The category of $\mathcal{C}$-coalgebras has a model structure in which the weak equivalences are given by maps $f : C \to D$, such that $\Omega_\tau f : \Omega_\tau C \to \Omega_\tau D$ is a quasi isomorphism. The cofibrations are the injective maps and the fibrations are the maps with the right lifting property with respect to acyclic cofibrations.

The fibrant and cofibrant objects are described in the following proposition.

Proposition 3.9 ([22], Theorem 2.1)**.** In the model category of $\mathcal{C}$-coalgebras relative to the Koszul twisting morphism $\iota : \mathcal{C} \to \Omega_{op}\mathcal{C}$ every object is cofibrant. A coalgebra C is fibrant in this model category if it is isomorphic to a quasi-free coalgebra. In particular the bar construction $B_\iota A$ on a $\Omega_{op}\mathcal{C}$-algebra A is fibrant. Where the twisting morphism $\iota : \mathcal{C} \to \Omega_{op}\mathcal{C}$, is the canonical twisting morphism from the cooperad $\mathcal{C}$ to its operadic bar construction $\Omega_{op}\mathcal{C}$.

3.3. L_∞-algebras The next essential ingredient we need for this thesis is the notion of an L_∞-algebra. L_∞-algebras are Lie algebras up to homotopy and since their discovery they have been appearing everywhere in mathematics. They played for example an important role in Kontsevich's deformation quantization result (see [15]), rational homotopy theory (see [11]) and in many more areas of mathematics. For the basics of the theory of L_∞-algebras see [11].

Definition 3.10. An L_∞-algebra L is an algebra over the L_∞-operad, where the L_∞-operad is given by $\Omega_{op}\mathcal{COCOM}$, the cobar construction on the cocommutative cooperad. This is equivalent to a differential graded vector space L together with a sequence of skew symmetric maps $l_n : L^{\otimes n} \to L$ of degree -1 for $n \geq 1$ satisfying certain identities.

Remark 3.11. In this thesis we will use a different grading convention for our L_∞-algebras than usual. In Paper I and Paper II we will give an explanation why we think our grading conventions are more natural.

In an L_∞-algebra there is a certain set of special elements called Maurer-Cartan elements.

Definition 3.12. Let L be an L_∞-algebra, a Maurer-Cartan element in L is an element τ that satisfies the Maurer-Cartan equation

$$\sum_{n\geq 1} \frac{1}{n!} l_n(\tau, ..., \tau) = 0.$$

The set of Maurer-Cartan elements in L will be denoted by $MC_0(L)$.

Remark 3.13. For the Maurer-Cartan equation to converge it is necessary to put some restrictions on the L_∞-algebras, like nilpotence, in this introduction we will ignore these restrictions and assume that the Maurer-Cartan equation always converges.

In [13] and [11], Hinich and Getzler define a functor from L_∞-algebras to simplicial sets.

Definition 3.14. Let L be a nilpotent L_∞-algebra. Then we have a functor $MC_\bullet : L_\infty - \mathrm{alg} \to sSet$ from L_∞-algebras to simplicial sets, defined by sending an L_∞-algebra to the simplicial set whose n-simplices are the Maurer-Cartan elements of the L_∞-algebra $\Omega_n \otimes L$, where Ω_n is the commutative algebra of polynomial de Rham forms on the simplex and the tensor product is taken as in Definition 3.5. The face and degeneracy maps are the face and degeneracy maps induced by the face and degeneracy maps of $\Omega_\bullet$.

Definition 3.15. An L_∞-algebra L is called a rational model for a simplicial set $X_\bullet$, if there exists a zig-zag of rational homotopy equivalences

$$MC_\bullet(L) \leftarrow ... \rightarrow X_\bullet.$$

The simplicial set $MC_\bullet(L)$ gives us a notion of homotopy or gauge equivalence between different Maurer-Cartan elements.

Definition 3.16. Two Maurer-Cartan elements in an L_∞-algebra L are called gauge equivalent if they are connected by a 1-simplex in $MC_\bullet(L)$.

Remark 3.17. In the literature this equivalence relation is both called homotopy equivalence and gauge equivalence. Since homotopy is already overused in this thesis we will call it gauge equivalence.

Definition 3.18. The moduli space of Maurer-Cartan elements in an L_∞-algebra L is defined as $MC_0(L)$ modulo the gauge equivalence relation. The moduli space of Maurer-Cartan elements will be denoted by $\mathcal{MC}(L)$.

3.4. Convolution algebras In Section 3.2 we saw that the bar and cobar constructions are fibrant-cofibrant objects in the categories of $\mathcal{C}$-coalgebras and $\mathcal{P}$-algebras. So to study the homotopy theory of $\mathcal{C}$-coalgebras and $\mathcal{P}$-algebras better we need a better understanding about maps into the bar construction and maps out of the cobar construction.

Let $\Omega_\tau C$ be the cobar construction on a $\mathcal{C}$-coalgebra and let A be a $\mathcal{P}$-algebra. Since the underlying $\mathcal{P}$-algebra of the cobar construction is free, every map $f:\Omega_\tau C \to A$ is completely determined by the image of the generators of $\Omega_\tau C$, i.e. the map f is determined by a linear map $\tilde{f}: C \to A$. Since the map $f:\Omega_\tau C \to A$ still needs to commute with differentials, not every linear map $\tilde{f}: C \to A$ extends to a map $f:\Omega_\tau C \to A$. The map $\tilde{f}$ therefore needs to satisfy a certain equation, called the Maurer-Cartan equation. One of the results of Paper I is that we show that this Maurer-Cartan equation comes from an L_∞-structure on $Hom_{\mathbb{R}}(C,A)$. Before we state the theorem we will first discuss the dual case as well.

Let $B_\tau A$ be the bar construction on a $\mathcal{P}$-algebra A and C be a $\mathcal{C}$-coalgebra. Let $f: C \to B_\tau A$ be a morphism of $\mathcal{C}$-coalgebras. Since $B_\tau A$ is cofree this is again completely determined by a linear map $\tilde{f}: C \to A$. Again not every linear map $\tilde{f}: C \to A$ determines a map $f: C \to B_\tau A$, it needs to satisfy a Maurer-Cartan equation.

Theorem 3.19 (Paper I, Theorem 7.1)**.** Let $\tau : \mathcal{C} \to \mathcal{P}$ be an operadic twisting morphism, C a $\mathcal{C}$-coalgebra and A a $\mathcal{P}$-algebra. Then there exists an L_∞-structure on $Hom_{\mathbb{R}}(C, A)$ which is natural with respect to strict morphisms of $\mathcal{C}$-coalgebras and $\mathcal{P}$-algebras, such that

1. The Maurer-Cartan elements in the L_∞-algebra $Hom_{\mathbb{R}}(C, L)$ are in bijection with the algebra maps $\Omega_\tau A \to C$ and in bijection with the coalgebra maps $C \to B_\tau A$.

2. Two $\mathcal{P}$-algebra maps $f, g : \Omega_\tau C \to A$ are homotopic if and only if the corresponding Maurer-Cartan elements $\tilde{f}, \tilde{g} : C \to A$ are gauge equivalent in $Hom_{\mathbb{R}}(C, A)$.

Remark 3.20. This L_∞-structure is defined for every field of characteristic 0 and not just $\mathbb{R}$.

One of the main results of Paper II is that the L_∞-algebra structure from Theorem 3.19 is not only natural with respect to strict morphisms, but also natural with respect to ∞-morphisms in one of the variables.

Remark 3.21. An ∞-morphism of $\mathcal{P}$-algebras is a morphism of $\mathcal{P}$-algebras up to homotopy, see Paper II for a detailed definition of ∞-morphism.

Theorem 3.22 (Paper II, Theorem 5.1)**.** The bifunctor

$$Hom_{\mathbb{R}}(-,-) : (\mathcal{C} - coalg)^{op} \times \mathcal{P} - alg \to L_\infty - alg$$

can be extended to bifunctors

$$Hom_{\mathbb{R}}(-,-) : (\infty - \mathcal{C} - coalg)^{op} \times \mathcal{P} - alg \to \infty - L_\infty - alg,$$

$$Hom_{\mathbb{R}}(-,-) : (\mathcal{C} - coalg)^{op} \times \infty - \mathcal{P} - alg \to \infty - L_\infty - alg.$$

Where $\infty - \mathcal{P} - alg$ denotes the category of $\mathcal{P}$-algebras with ∞-morphisms and $\infty - \mathcal{C} - coalg$ is the category of $\mathcal{C}$-coalgebras with ∞-morphisms.

Using this theorem we can prove the following theorem which, under some assumptions on C and L, turns $Hom_{\mathbb{R}}(C, L)$ into a rational model for a mapping space. This theorem is a generalization of Theorem 1.4 of [1]. We improve Berglund's Theorem by not only allowing C to be strictly cocommutative , but also cocommutative up to homotopy. The rational homotopy theory of mapping spaces has also been studied in [6], [6], [17], [10], [4], [2], [12] and [5].

Theorem 3.23 (Paper II, Corollary 8.19)**.** Let X be a simply-connected simplicial set of finite $\mathbb{Q}$-type and C a simply-connected C_∞-coalgebra model of finite type for X. Let Y be another simply-connected simplicial set of finite $\mathbb{Q}$-type and L a simply-connected locally finite L_∞-model for Y. Then there exists an L_∞-structure on $Hom_{\mathbb{Q}}(C,L)$ such that $Hom_{\mathbb{Q}}(C,L)$ becomes a rational model for the based mapping space $Map_*(X,Y)$, i.e. we have a homotopy equivalence

$$MC_\bullet(Hom_{\mathbb{Q}}(C,L)) \simeq Map_*(X,Y_{\mathbb{Q}}),$$

where $Y_{\mathbb{Q}}$ is the rationalization of Y.

Remark 3.24. We call an algebra A simply-connected if A is concentrated in degrees 2 and higher, similarly we will call a coalgebra C simply connected if C is concentrated in degree 2 and higher. For the definition of locally finite see Definition 8.2 of Paper II.

As a corollary of Theorem 3.23 we get an alternative proof of Theorem 3.2 of [7].

Corollary 3.25 (Paper I, Corollary 11.1 and [7] Theorem 3.2)**.** Let X and Y be simply connected simplicial sets of finite $\mathbb{Q}$-type. There exists an L_∞-structure on $Hom_{\mathbb{Q}}(H_*(X;\mathbb{Q}),\pi_*(Y)\otimes\mathbb{Q})$ such that $Hom_{\mathbb{Q}}(C,L)$ becomes a rational model for the based mapping space $Map_*(X,Y)$.

3.5. Algebraic Hopf invariants In this section we will use all the theory from the previous sections to describe the invariants of maps between algebras and maps between coalgebras.

We start by defining invariants of maps between coalgebras. We will do this in several steps. Let C and D be $\mathcal{C}$-coalgebras and let $\tau:\mathcal{C}\to\mathcal{P}$ be a Koszul operadic twisting morphism. We would like to use Theorem 3.19 to identify the space of maps between C and D with the set of Maurer-Cartan elements in a certain L_∞-algebra.

Unfortunately we can not do this directly since D is not necessarily given by the bar construction on something. Therefore we first apply the cobar construction and study instead maps between $\Omega_\tau C$ and $\Omega_\tau D$. In Paper I Lemma 9.1, it is shown that two maps $f,g:C\to D$ are homotopic if and only if $\Omega_\tau f$ is homotopic to $\Omega_\tau g$. So according to Theorem 3.19 the maps $\Omega_\tau f$ and $\Omega_\tau g$ are homotopic if and only if the corresponding Maurer-Cartan elements $\widetilde{\Omega_\tau f}$ and $\widetilde{\Omega_\tau g}$ are gauge equivalent in $Hom_{\mathbb{R}}(C,\Omega_\tau D)$.

The main problem with this approach is that the coalgebra models C and D are often quite large and not well suited for calculations. To solve this problem we construct strict morphisms of $\mathcal{P}$-algebras $i:\Omega_\tau H_*(C)\to\Omega_\tau C$ and $p:\Omega_\tau D\to H_*(\Omega_\tau D)$ in Section 9 of Paper I.

Remark 3.26. There are a few technical details that are omitted here, which are solved in Paper I.

Using the maps i and p we can define the algebraic Hopf invariant map $mc\colon Hom_{\mathcal{C}-coalg}(C,D) \to Hom_{\mathbb{R}}(H_*(C), H_*(\Omega_\tau D))$.

Definition 3.27 (Paper I, Definition 9.1)**.** The algebraic Hopf invariant maps are defined by

$$mc\colon Hom_{\mathcal{C}-coalg}(C,D) \to Hom_{\mathbb{R}}(H_*(C), H_*(\Omega_\tau D))$$

$$mc(f) = p \circ \Omega_\iota f \circ i.$$

The map

$$mc_\infty\colon Hom_{\mathcal{C}-coalg}(C,D) \to \mathcal{MC}(Hom_{\mathbb{R}}(H_*(C), H_*(\Omega_\tau D))$$

is defined by sending f to the gauge equivalence class of $mc(f) = p \circ \Omega_\iota f \circ i$ in the moduli space of Maurer-Cartan elements.

Remark 3.28. The map $mc(f)$ is not an invariant of the homotopy class of a map $f : C \to D$. It does however, have the property that it reduces the question whether two maps are homotopic to a question about gauge equivalence in a finite dimensional L_∞-algebra. Which is in general much easier to solve.

The following theorem is one of the main theorems of Paper I. This theorem states that the algebraic Hopf invariants are a complete invariant of the rational homotopy classes of maps

Theorem 3.29 (Paper I, Theorem 10.1)**.** The maps $f,g : C \to D$ between $\mathcal{C}$ coalgebras C and D are homotopic if and only if $mc_\infty(f) = mc_\infty(g)$.

4 The Hopf invariants

In this section we will describe how we can use the algebraic Hopf invariants from Section 3.5 to construct invariants of maps between manifolds. We will first explain how we can use integration to compute the map mc and then we will explain how to get invariants of homotopy classes of maps using mc.

4.1. From manifolds to algebra: How to compute the map mc To get invariants of maps between manifolds we can first apply the functor $\Omega_\bullet$ and then apply the algebraic Hopf invariant map to $Hom_{CDGA}(\Omega_\bullet(N), \Omega_\bullet(M))$. One of the problems with this approach is that the complexes $\Omega_\bullet(M)$ and

$\Omega_\bullet(N)$ are extremely large, so it is in practice nearly impossible to compute the algebraic Hopf invariant maps directly. In this section we will state one of the main theorems of Paper III in which we compute the map mc by solving a finite number of integrals.

Theorem 4.1 (Paper III, Theorem 6.13)**.** Let $\{\varphi_{i,j}\}$ be a basis for $Hom_{\mathbb{R}}(H_*(M),\pi_*(N))$ and let $f: M \to N$ be a smooth map. The Maurer-Cartan element $mc(f)$ can be expressed in this basis as $mc(f) = \sum_{i,j} \lambda_{i,j}\varphi_{i,j}$, with $\lambda_{i,j} \in \mathbb{R}$. The coefficients $\lambda_{i,j}$ can be computed as an integral of a certain differential form over a certain subspace of M.

The exact statement of the theorem can be found in Paper III as Theorem 6.13. The integrals of Theorem 4.1 are given by very explicit formulas and can often be computed, for a few examples of such computations see Section 8 of Paper III. The purpose of this theorem is that it reduces the question whether two maps $f, g: M \to N$ are real homotopic to a question whether two Maurer-Cartan elements in a finite dimensional L_∞-algebra are gauge equivalent or not. This last question can still be hard but is a lot easier to solve than the original question. In the next section we will also state some results on how to avoid checking whether the elements $mc(f)$ and $mc(g)$ are gauge equivalent.

4.2. Invariants of maps between manifolds Let $f: M \to N$ be a map between simply-connected smooth manifolds, such that M is compact. So far we have only explained how to compute the map $mc(f)$ and we did not explain how to compute $mc_\infty(f)$. So far we have therefore not yet defined any invariants of the map f. In Paper III we develop a technique to get a better understanding of the moduli space of Maurer-Cartan elements. We do this by defining an algebraic analog of a CW-complex to deal with this question in many favorable cases.

The idea is that when the manifold M has a CW-decomposition we can use the CW-decomposition to define a filtration on the homology of M. This filtration will induce a tower of fibrations on the Maurer-Cartan simplicial set $MC_\bullet(Hom_{\mathbb{R}}(H_*(M),\pi_*(N)))$ and will give us the possibility to compare Maurer-Cartan elements stage by stage in this tower.

In this section we will give two examples of the results we can obtain this way.

Theorem 4.2 (Paper III, Theorem 7.24)**.** Let $f: M \to N$ be a smooth map between simply-connected manifolds M and N. The map f is real homotopic to the constant map if and only if all the coefficients $\lambda^f_{i,j}$ are equal to zero.

Another example of an application of our theory is Corollary 7.21 of Paper III.

Theorem 4.3. Let $H_*(S^n \times S^m)$ be the homology of $S^n \times S^m$. This vector space has as basis α, β, γ, such that $|\alpha| = n$, $|\beta| = m$ $|\gamma| = n+m$. Let $f, g : S^n \times S^m \to Y$ be two maps. The maps f and g are homotopic if and only if

- The coefficients $\lambda^f_{\alpha,i} = \lambda^g_{\alpha,i}$ for all $i \in \pi_n(Y)$.
- And $\lambda^f_{\beta,j} = \lambda^g_{\beta,j}$ for all $j \in \pi_m(Y)$.
- And $\lambda^f_{\gamma,k} = \lambda^g_{\gamma,k}$ for a certain subset of $\pi_4(Y)$.

In Paper III we will give a precise definition of what a certain subset means. An example of such a subset is given in the following example.

Example 4.4. Let $f, g : S^2 \times S^2 \to S^2 \times S^2 \setminus \{*\}$ be two maps. The maps f and g are homotopic if and only if

- If $\lambda^f_{\alpha,i} = \lambda^g_{\alpha,i}$ and $\lambda^f_{\beta,i} = \lambda^g_{\beta,i}$ for all $i \in \pi_2(S^2 \times S^2 \setminus *)$ and at least one of the coefficients $\lambda^f_{\alpha,i}$ or $\lambda^f_{\beta,i}$ is non zero.
- Or if $\lambda^f_{\alpha,i} = \lambda^g_{\alpha,i} = \lambda^f_{\beta,i} = \lambda^g_{\beta,i} = 0$ for all $i \in \pi_2(S^2 \times S^2 \setminus *)$ and $\lambda^f_{\gamma,j} = \lambda^g_{\gamma,j}$ for all $j \in \pi_4(S^2 \times S^2 \setminus *)$.

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