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A Nonlinear Optimization Approach to $\mathcal{H}_2$-Optimal Modeling and Control

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A Nonlinear Optimization Approach to $\mathcal{H}_2$-Optimal Modeling and Control

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To Maria, Wilmer and Elsa!

Abstract

Mathematical models of physical systems are pervasive in engineering. These models can be used to analyze properties of the system, to simulate the system, or synthesize controllers. However, many of these models are too complex or too large for standard analysis and synthesis methods to be applicable. Hence, there is a need to reduce the complexity of models. In this thesis, techniques for reducing complexity of large linear time-invariant (LTI) state-space models and linear parameter-varying (LPV) models are presented. Additionally, a method for synthesizing controllers is also presented.

The methods in this thesis all revolve around a system theoretical measure called the $\mathcal{H}_2$-norm, and the minimization of this norm using nonlinear optimization. Since the optimization problems rapidly grow large, significant effort is spent on understanding and exploiting the inherent structures available in the problems to reduce the computational complexity when performing the optimization.

The first part of the thesis addresses the classical model-reduction problem of LTI state-space models. Various $\mathcal{H}_2$ problems are formulated and solved using the proposed structure-exploiting nonlinear optimization technique. The standard problem formulation is extended to incorporate also frequency-weighted problems and norms defined on finite frequency intervals, both for continuous and discrete-time models. Additionally, a regularization-based method to account for uncertainty in data is explored. Several examples reveal that the method is highly competitive with alternative approaches.

Techniques for finding LPV models from data, and reducing the complexity of LPV models are presented. The basic ideas introduced in the first part of the thesis are extended to the LPV case, once again covering a range of different setups. LPV models are commonly used for analysis and synthesis of controllers, but the efficiency of these methods depends highly on a particular algebraic structure in the LPV models. A method to account for and derive models suitable for controller synthesis is proposed. Many of the methods are thoroughly tested on a realistic modeling problem arising in the design and flight clearance of an Airbus aircraft model.

Finally, output-feedback $\mathcal{H}_2$ controller synthesis for LPV models is addressed by generalizing the ideas and methods used for modeling. One of the ideas here is to skip the LPV modeling phase before creating the controller, and instead synthesize the controller directly from the data, which classically would have been used to generate a model to be used in the controller synthesis problem. The method specializes to standard output-feedback $\mathcal{H}_2$ controller synthesis in the LTI case, and favorable comparisons with alternative state-of-the-art implementations are presented.

Populärvetenskaplig sammanfattning

Inom många naturvetenskapliga och tekniska områden används matematiska modeller för att beskriva olika system, till exempel för att beskriva hur ett flygplan kommer att röra sig givet att piloten ställer ut ett visst roderutslag. Dessa matematiska modeller kan exempelvis användas för att spara resurser genom att testa olika prototyper med simuleringar utan att behöva ha den fysiska prototypen. Dessa modeller kan skapas genom fysikaliska principer eller genom att en modell har byggts upp med hjälp av insamlad data.

Dagens moderna och komplexa system kan leda till väldigt stora och komplicerade matematiska modeller och dessa kan ibland vara för stora för att simulera eller analysera. Då behöver man kunna *reducera* komplexiteten på dessa modeller för att det skall vara möjligt att använda dem. Kravet på den reducerade modellen är att den skall kunna beskriva den stora komplexa modellen tillräckligt väl för det ändamål som krävs.

Det finns många olika slags matematiska modeller av olika grader av komplexitet. Den enklaste typen av modeller är *linjära modeller* och för dessa modeller är det möjligt att analysera egenskaper och dra viktiga slutsatser om systemet. Linjära modeller har dock nackdelen att de är begränsade i hur mycket de kan beskriva. Om vi igen tar ett flygplan som exempel, kan man säga att en linjär modell kan beskriva vad som händer med flygplanet om det håller sig på en specifik höjd med en specifik fart. Dock klarar inte den linjära modellen av att beskriva vad som händer om flygplanet avviker från dessa specifika värden på fart och höjd för mycket. En annan typ av modeller är *linjärt parametervarierande modeller*. Dessa modeller beror på en eller flera parametrar som kan beskriva vissa tillstånd. Flygplanet som vi förut beskrev med en linjär modell för en specifik fart och höjd, skulle nu istället kunna beskrivas med en parametervarierande modell. Denna parametervarierande modell kan, till exempel, vara beroende av dessa parametrar, höjd och fart, och kan då även beskriva vad som händer när flygplanet stiger till en ny höjd och ändrar farten.

I denna avhandling utvecklar vi metoder för att kunna reducera stora komplexa, linjära och linjära parametervarierande modeller till mindre, mer överkomliga modeller. Kravet är att dessa modeller fortfarande ska kunna beskriva det ursprungliga systemet väl så att de kan användas, till exempel, för att analysera systemet.

Med de metoder som har utvecklats för att reducera stora komplexa modeller till mindre modeller som utgångspunkt har även metoder för att kunna konstruera regulatorer för att styra dessa stora komplexa modeller utvecklats.

Acknowledgments

First of all, I would like to thank my supervisor Dr. Johan Löfberg and my co-supervisor Professor Lennart Ljung for all your patience and support. Especially Johan, for his vast (this time I got it right) knowledge in optimization and always having an open door and taking time to answer my questions.

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For financial support, I would like to thank the European Commission under contract No. AST5-CT-2006-030768-COFCLUO.

Finally, I would like to thank the person who has meant the most. Thank you Maria! Thank you for all the support and encouragement and thank you for bringing me two of the most important persons in my life; Wilmer and Elsa.

Linköping, August 2013
Daniel Petersson

Contents

Notation

Symbols, Operators and Functions

Notation	Meaning
$\mathbb{N}$	the set of natural numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\mathcal{O}$	Ordo
$\in$	belongs to
$[a, b]$	the closed interval from a to b
$\triangleq$	equal by definition
i	$\sqrt{-1}$
$\overline{a}$	the complex conjugate of a
$\operatorname{Re} a$	the real part of a
$\operatorname{Im} a$	the complex part of a
$\dot{x}(t)$	the time derivative of the function $x(t)$
$\mathbf{e}_i$	the unit vector with a one in the i:th element
$\overline{\mathbf{a}}$	the element-wise complex conjugate of the vector $\mathbf{a}$
$\mathbf{A}$	matrices are denoted by bold, upright capitalized letters
$\mathbb{I}$	the identity matrix
$\mathbf{0}$	a matrix with only zeros
$[\mathbf{A}]_{ij}$	element (i, j) of the matrix $\mathbf{A}$
$\mathbf{A}^\mathsf{T}$	the transpose of $\mathbf{A}$
$\mathbf{A}^*$	the complex conjugate transpose of $\mathbf{A}$
$\mathbf{A}^{-1}$	the inverse of $\mathbf{A}$
$\mathbf{A} \succ (\succeq) \mathbf{0}$	$\mathbf{A}$ is a positive (semi-)definite matrix
$\mathbf{A} \prec (\preceq) \mathbf{0}$	$\mathbf{A}$ is a negative (semi-)definite matrix
$\operatorname{tr} \mathbf{A}$	the trace of the matrix $\mathbf{A}$
$\operatorname{rank} \mathbf{A}$	the rank of the matrix $\mathbf{A}$

Symbols, Operators and Functions

Notation	Meaning
$\frac{\partial \mathbf{A}}{\partial a}$	denotes the element wise differentiation of the matrix $\mathbf{A}$ with respect to the scalar variable a
$\Vert \cdot \Vert_2$	for vectors the two norm and for matrices the induced two norm
$\Vert \cdot \Vert_F$	the Frobenius norm
$\Vert \cdot \Vert_{\mathcal{H}_2}$	the $\mathcal{H}_2$-norm for dynamical systems
$\Vert \cdot \Vert_{\mathcal{H}_2,\omega}$	the frequency-limited $\mathcal{H}_2$-norm for dynamical systems, defined in Chapter 3
$\Vert \cdot \Vert_{\mathcal{H}_\infty}$	the $\mathcal{H}_\infty$-norm for dynamical systems
$\mathcal{N}\left(\mu, \sigma^2\right)$	the Gaussian distribution with mean μ and variance σ^2
$\mathrm{E}\left(X\right)$	the expected value of the random variable X
$\mathrm{Cov}\left(X\right)$	the covariance matrix of the random variable X

Abbreviations

Abbreviation	Meaning
LTI	Linear time invariant
LPV	Linear parameter varying
LTV	Linear time varying
LFT	Linear fractional transformation
LFR	Linear fractional representation
SISO	Single input single output
MISO	Multiple input single output
SIMO	Single input multiple output
MIMO	Multiple input multiple output
OE	Output error
QP	Quadratic programming
SDP	Semidefinite programming
NLP	Nonlinear programming
LMI	Linear matrix inequality
BMI	Bilinear matrix inequality
BFGS	Broyden-Fletcher-Goldfarb-Shanno
COFCLUO	Clearance of flight control laws using optimization
LS	Least squares
LASSO	Least absolute shrinkage and selection operator
SVD	Singular value decomposition

1
Introduction

Mathematical models of physical systems are pervasive in engineering. These models can be used to analyze properties of the systems, to simulate the systems, or synthesize controllers. However, many of these models are too complex or too large for standard analysis and synthesis methods to be applicable. Hence, there is a need to be able to reduce the complexity of models. The main goal of this thesis is to develop methods for reducing the complexity of different systems by minimizing the $\mathcal{H}_2$-norm between the large complex system and the reduced system.

Many of the early methods for controller synthesis and model reduction relies on linear algebra and solutions to Lyapunov and Riccati equations. Later, when solvers for more general and advanced optimization methods were developed, it was possible to formulate many of the problems in control theory as, for example, semidefinite programs to be solved using interior-point solvers. However, many of these programs included, not only linear matrix inequalities, LMIs, but also bilinear matrix inequalities, BMIs, which make the problems non-convex. This and the fact that semidefinite programs generally do not scale well with the number of variables sometimes make these problems time consuming and difficult to solve. In this thesis, we take a step back, and instead try to keep the original structure of the problem and formulate a general nonlinear optimization problem using linear algebra and Lyapunov equations, and use a general quasi-Newton solver to solve the problem. The problems formulated in this thesis are still non-convex, but since the original structure of the problem is kept and a more direct approach is used, it is possible to, for example, impose certain structural constraints on the system matrices and still be able to use the methods for medium-scale systems.

1.1 Outline of the Thesis

Most of the results in this thesis concern the minimization of the $\mathcal{H}_2$-norm of various linear time-invariant (LTI) systems with different structures and how to utilize the different characteristics of the different problems. Most of the results are based on standard concepts in matrix theory, linear systems theory and optimization. A brief overview of the necessary concepts in matrix theory, linear systems theory and optimization are presented in Chapter 2.

In Chapter 3, the concepts of frequency-limited Gramians are presented, Additionally, complete derivations for both the discrete-time case and continuous-time case are presented. These are then used to form a frequency-limited $\mathcal{H}_2$-norm, which is later used in some of the proposed algorithms.

In Chapter 4, a short overview of the model-reduction problem is presented before a number of model-reduction algorithms are presented. These algorithms all try to utilize the different structures of the equations to be able to solve the problems efficiently using quasi-Newton methods.

In Chapter 5, a number of methods for generating linear parameter-varying models, using the model-reduction methods in Chapter 4 as a foundation, are presented.

In Chapter 6, methods for designing $\mathcal{H}_2$ controllers, both for linear time-invariant systems and linear parameter-varying systems, are presented. These methods are based on the same procedure as the methods in Chapter 4 and Chapter 5.

Chapter 7 presents two larger examples that highlight some properties and applications for the model reduction and linear parameter-varying algorithms. One example shows a flight clearance application of an Airbus aircraft model and the other example highlights the connections between $\mathcal{H}_2$ model reduction and system identification.

Finally in Chapter 8 some concluding remarks about the results and suggestions about future research directions are presented.

1.2 Contributions

The first main contributions in the thesis are the model-reduction methods presented in Chapter 4 and especially the frequency-limited model reduction in Section 4.4.3 and the unified and complete derivation of the frequency-limited Gramians and frequency-limited $\mathcal{H}_2$-norm in Chapter 3, which are based on the publication

> Daniel Petersson and Johan Löfberg. Model reduction using a frequency-limited $\mathcal{H}_2$-cost. *arXiv preprint arXiv:1212.1603*, December 2012a. URL `http://arxiv.org/abs/1212.1603`,

which has been submitted to *Systems and Control Letters*.

The second main contributions in the thesis are the linear parameter-varying generating methods in Chapter 5. To be able to reduce the complexity of a linear parameter-varying model, the idea of model reduction is used to have methods that are invariant to state transformations. These results are based on the publication

> Daniel Petersson and Johan Löfberg. Optimization based LPV-approximation of multi-model systems. In *Proceedings of the European Control Conference*, pages 3172–3177, Budapest, Hungary, 2009,

which was extended with

> Daniel Petersson and Johan Löfberg. Robust generation of LPV state-space models using a regularized $\mathcal{H}_2$-cost. In *Proceedings of the IEEE International Symposium on Computer-Aided Control System Design*, pages 1170–1175, Yokohama, Japan, 2010,

to be able to handle uncertainties in the data. These publications with some extensions have also been published in

> Daniel Petersson. *Nonlinear optimization approaches to $\mathcal{H}_2$-norm based LPV modelling and control*. Licentiate thesis no. 1453, Department of Electrical Engineering, Linköping University, 2010,

and

> Daniel Petersson and Johan Löfberg. *Optimization Based Clearance of Flight Control Laws - A Civil Aircraft Application*, chapter Identification of LPV State-Space Models Using $\mathcal{H}_2$ -Minimisation, pages 111–128. Springer, 2012b,

and have been submitted as

> Daniel Petersson and Johan Löfberg. Optimization-based modeling of LPV systems using an $\mathcal{H}_2$ objective. *Submitted to International Journal of Control*, December 2012c.

Additionally, an extension of the linear parameter-varying generating methods is presented, where it is possible to control the rank of the coefficient matrices in the resulting linear parameter-varying model.

The third main contributions are the $\mathcal{H}_2$ controller-synthesis methods in Chapter 6, which use similar ideas as the other contributions to synthesize $\mathcal{H}_2$ controllers instead. This chapter is partly based on the publication

> Daniel Petersson and Johan Löfberg. LPV $\mathcal{H}_2$-controller synthesis using nonlinear programming. In *Proceedings of the 18th IFAC World Congress*, pages 6692–6696, Milan, Italy, 2011.

2
Preliminaries

This chapter begins by presenting some theory and concepts for system theory. Some basic optimization background with focus on the concept of quasi-Newton methods will then be presented. The chapter will finish with some matrix theory that will be used in the thesis, where, for example, the concepts of matrix functions are presented.

2.1 System Theory

This section reviews some of the standard system theoretical concepts and explains some system norms that will be used in the thesis.

2.1.1 Basic Theory and Notation

In engineering, mathematical models are often described, in continuous time, by ordinary differential equations. An important subclass of these models is the class of systems of linear ordinary differential equations with constant coefficients. The models in this class, which are called *linear time-invariant models*, LTI models, can mathematically be described, for a continuous-time model, as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \tag{2.1a}$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t), \tag{2.1b}$$

and for a discrete-time model with sample time T_S as

$$\mathbf{x}(t + T_S) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \tag{2.2a}$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t), \tag{2.2b}$$

where $\mathbf{x}(t) \in \mathbb{R}^{n_x}$ is a vector containing the states of the system, $\mathbf{u}(t) \in \mathbb{R}^{n_u}$ is a vector containing the input to the system and $\mathbf{y}(t) \in \mathbb{R}^{n_y}$ is a vector containing the output of the system. The matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ are constant matrices of suitable dimensions, where $\mathbf{A}$ describes the dynamics of the system, $\mathbf{B}$ describes how the input enters the system and $\mathbf{C}$ and $\mathbf{D}$ describes what is being measured from the system. The system in (2.1) is expressed in state-space form, the corresponding transfer-function form, for the system from $\mathbf{u}(t)$ to $\mathbf{y}(t)$, is

$$Y(s) = G(s)U(s),$$

where $U(s)$ and $Y(s)$ are the Laplace transforms of $\mathbf{u}(t)$ and $\mathbf{y}(t)$ and

$$G(s) = \mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \triangleq \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right].$$

Here, the notation $\left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]$ is introduced as the transfer function of the system given a particular realization, $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$.

In discrete time, difference equations are used to describe the dynamics of the system, (2.2), and consequently use the z-transform instead of the Laplace transform to express the transfer function, i.e., given the discrete-time system in (2.2) the transfer function becomes $G(z) = \mathbf{C}(z\mathbb{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$.

The vector $\mathbf{x}$, describing the states, can be transformed into a new basis, $\hat{\mathbf{x}}$, using an invertible matrix, $\mathbf{T}$, i.e., $\hat{\mathbf{x}} \triangleq \mathbf{T}\mathbf{x}$. This yields the realization

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}\hat{\mathbf{x}}(t) + \mathbf{T}\mathbf{B}\mathbf{u}(t), \tag{2.3a}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{T}^{-1}\hat{\mathbf{x}}(t) + \mathbf{D}\mathbf{u}(t). \tag{2.3b}$$

The transfer function for this system is

$$\hat{G}(s) \triangleq \mathbf{C}\mathbf{T}^{-1}(s\mathbb{I} - \mathbf{T}\mathbf{A}\mathbf{T}^{-1})^{-1}\mathbf{T}\mathbf{B} + \mathbf{D} = \mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} = G(s), \tag{2.4}$$

thus, there exists infinitely many realizations of a system.

2.1.2 Gramians

Two important entities when it comes to system theory and determining system properties are the *controllability Gramian*, $\mathbf{P}$ and the *observability Gramian*, $\mathbf{Q}$. The equations for these differ in continuous and discrete time and the rest of the section is split up into two subsections, one for continuous time and one for discrete time.

Continuous-Time Systems

Definition 2.1. The controllability and observability Gramians, in the continuous-time domain, of the system (2.1) are defined as

$$\mathbf{P} \triangleq \int_0^\infty \mathrm{e}^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^\mathsf{T}\mathrm{e}^{\mathbf{A}^\mathsf{T}\tau}\mathrm{d}\tau, \tag{2.5a}$$

$$\mathbf{Q} \triangleq \int_0^\infty \mathrm{e}^{\mathbf{A}^\mathsf{T}\tau}\mathbf{C}^\mathsf{T}\mathbf{C}\mathrm{e}^{\mathbf{A}\tau}\mathrm{d}\tau. \tag{2.5b}$$

The Gramians in (2.5) can also be written as the stationary solutions to the differential equations

$$\dot{\mathbf{P}} = \mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^\mathsf{T} + \mathbf{B}\mathbf{B}^\mathsf{T}, \tag{2.6a}$$

$$\dot{\mathbf{Q}} = \mathbf{A}^\mathsf{T}\mathbf{Q} + \mathbf{Q}\mathbf{A} + \mathbf{C}^\mathsf{T}\mathbf{C}, \tag{2.6b}$$

i.e., having $\dot{\mathbf{P}} = \dot{\mathbf{Q}} = \mathbf{0}$, thus becoming solutions to the algebraic equations, called Lyapunov equations,

$$\mathbf{0} = \mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{B}\mathbf{B}^\mathsf{T}, \tag{2.7a}$$

$$\mathbf{0} = \mathbf{A}^\mathsf{T}\mathbf{Q} + \mathbf{Q}\mathbf{A} + \mathbf{C}^\mathsf{T}\mathbf{C}. \tag{2.7b}$$

By using Parseval's identity on (2.5), the Gramians can be expressed in the frequency domain.

Definition 2.2. The controllability and observability Gramians, in frequency domain, for the system (2.1) are defined as

$$\mathbf{P} \triangleq \frac{1}{2\pi}\int_{-\infty}^\infty \mathbf{H}(i\nu)\,\mathbf{B}\mathbf{B}^\mathsf{T}\mathbf{H}^*(i\nu)\,\mathrm{d}\nu, \tag{2.8a}$$

$$\mathbf{Q} \triangleq \frac{1}{2\pi}\int_{-\infty}^\infty \mathbf{H}^*(i\nu)\,\mathbf{C}^\mathsf{T}\mathbf{C}\mathbf{H}(i\nu)\,\mathrm{d}\nu, \tag{2.8b}$$

where $\mathbf{H}(i\omega) \triangleq (\mathbb{I}i\omega - \mathbf{A})^{-1}$ and $\mathbf{H}^*$ denote the conjugate transpose of $\mathbf{H}$.

One important observation to make, both for the Gramians in continuous time and discrete time (see Section 2.1.2), is that the Gramians are dependent on which state basis that is used. If a state transformation is performed, $\hat{\mathbf{x}} = \mathbf{T}\mathbf{x}$, $\mathbf{T}$ is invertible, the Gramians change

$$\mathbf{P}_\mathbf{T} = \mathbf{T}^{-1}\mathbf{P}\mathbf{T}^{-\mathsf{T}}, \tag{2.9a}$$

$$\mathbf{Q}_\mathbf{T} = \mathbf{T}^\mathsf{T}\mathbf{Q}\mathbf{T}. \tag{2.9b}$$

Hence, the eigenvalues of the Gramians change if a state transformation is performed. However, the eigenvalues of the product of the Gramians, $\lambda(\mathbf{PQ})$, are invariant to state transformations, since

$$\lambda_i(\mathbf{P_T Q_T}) = \lambda_i\left(\mathbf{T}^{-1}\mathbf{P}\mathbf{T}^{-\mathsf{T}}\mathbf{T}^{\mathsf{T}}\mathbf{QT}\right) = \lambda_i\left(\mathbf{T}^{-1}\mathbf{PQT}\right) = \lambda_i(\mathbf{PQ}) \triangleq \sigma_i^2, \tag{2.10}$$

where σ_i is called a *Hankel singular value* of the system.

The Gramians, both in continuous time and discrete time, can be interpreted physically (see, e.g., Skogestad and Postlethwaite [2007] or Antoulas [2005]). Given a state $\mathbf{x}$, the smallest amount of energy needed to steer a system from 0 to $\mathbf{x}$ is given by

$$\mathbf{x}^{\mathsf{T}}\mathbf{P}^{-1}\mathbf{x}, \tag{2.11}$$

and the observability Gramian describes the energy obtained by observing the output of a system with initial condition $\mathbf{x}$ and given no other input and is described by

$$\mathbf{x}^{\mathsf{T}}\mathbf{Q}\mathbf{x}. \tag{2.12}$$

This goes for both continuous- and discrete-time systems.

Discrete-Time Systems

Definition 2.3. The controllability and observability Gramians, in discrete time, of the system (2.2) are defined as

$$\mathbf{P} \triangleq \sum_{k=0}^{\infty} \mathbf{A}^k\mathbf{B}\mathbf{B}^{\mathsf{T}}\left(\mathbf{A}^k\right)^{\mathsf{T}}, \tag{2.13a}$$

$$\mathbf{Q} \triangleq \sum_{k=0}^{\infty} \left(\mathbf{A}^k\right)^{\mathsf{T}}\mathbf{C}^{\mathsf{T}}\mathbf{C}\mathbf{A}^k. \tag{2.13b}$$

These Gramians also satisfy the discrete Lyapunov equations

$$\mathbf{0} = \mathbf{APA}^{\mathsf{T}} - \mathbf{P} + \mathbf{BB}^{\mathsf{T}}, \tag{2.14a}$$

$$\mathbf{0} = \mathbf{A}^{\mathsf{T}}\mathbf{QA} - \mathbf{Q} + \mathbf{C}^{\mathsf{T}}\mathbf{C}. \tag{2.14b}$$

The definition of the discrete-time Gramians in frequency domain becomes

Definition 2.4. The controllability and observability Gramians, in frequency domain, for the system (2.2) are defined as

$$\mathbf{P} \triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbf{H}(\nu)\mathbf{B}\mathbf{B}^{\mathsf{T}}\mathbf{H}^*(\nu)\,\mathrm{d}\nu, \tag{2.15a}$$

$$\mathbf{Q} \triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbf{H}^*(i\nu)\mathbf{C}^{\mathsf{T}}\mathbf{C}\mathbf{H}(i\nu)\,\mathrm{d}\nu, \tag{2.15b}$$

where $\mathbf{H}\left(e^{i\omega}\right) = \left(\mathbb{I}e^{i\omega} - \mathbf{A}\right)^{-1}$ and $\mathbf{H}^*$ denote the conjugate transpose of $\mathbf{H}$.

2.1.3 System Norms

System norms are important tools when it comes to comparing and analyzing systems. In this thesis, mainly the $\mathcal{H}_2$-norm will be used. In this section, the two most commonly used norms in system theory, namely the $\mathcal{H}_2$-norm and the $\mathcal{H}_\infty$-norm are presented and defined.

Given a system $G = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]$ such that

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{w}(t), \tag{2.16a}$$

$$\mathbf{z}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{w}(t), \tag{2.16b}$$

where $\mathbf{x}$ is the state, $\mathbf{w}$ is a disturbance and $\mathbf{z}$ is the output of interest. Suppose a system that guarantees a certain performance is wanted, e.g., $\mathbf{w}$ does not influence $\mathbf{z}$ too much. The system norms are functions that quantify this into something computationally tractable, with different interpretations. System norms can be interpreted as norms that answer the question: "given information about the allowed input, how large can the output be?".

To be able to do this, two signal norms that will be used to interpret the system norms are defined.

Definition 2.5 ($\mathcal{L}_2$, 2-norm in time). The $\mathcal{L}_2$-norm for square integrable signals is defined by

$$\|\mathbf{e}(t)\|_{\mathcal{L}_2} \triangleq \sqrt{\int_0^\infty \|\mathbf{e}(\tau)\|_2^2 \, d\tau}. \tag{2.17}$$

$\|\mathbf{e}(t)\|_{\mathcal{L}_2}$ is also referred to as the energy of the signal $\mathbf{e}(t)$.

Definition 2.6 ($\mathcal{L}_\infty$, ∞-norm in time). The $\mathcal{L}_\infty$-norm for magnitude-bounded signals is defined as

$$\|\mathbf{e}(t)\|_{\mathcal{L}_\infty} \triangleq \sup_{\tau\geq 0} \|\mathbf{e}(\tau)\|_2 . \tag{2.18}$$

For a scalar signal $\mathbf{e}(t)$, $\|\mathbf{e}(t)\|_{\mathcal{L}_\infty}$ is simply the peak of the signal.

These signal norms are used to define some system norms in the next section.

Continuous-Time $\mathcal{H}_2$-Norm

For a SISO system G, which has the realization (2.16) with $\mathbf{A}$ Hurwitz and $\mathbf{D} = \mathbf{0}$, the $\mathcal{H}_2$-norm can be defined as

$$\|G\|_{\mathcal{H}_2} \triangleq \sup_{\|\mathbf{w}(t)\|_{\mathcal{L}_2} \leq 1} \|\mathbf{z}(t)\|_{\mathcal{L}_\infty}. \tag{2.19}$$

For some physical interpretations of the $\mathcal{H}_2$-norm, see for example Skogestad and Postlethwaite [2007], Skelton et al. [1998] or Zhou et al. [1996]. However, the definition that will be used mostly in this thesis is

Definition 2.7 ($\mathcal{H}_2$-norm). For an asymptotically stable ($\mathbf{A}$ Hurwitz) and strictly proper ($\mathbf{D} = \mathbf{0}$) continuous-time system, G, the $\mathcal{H}_2$-norm is defined as

$$\|G\|_{\mathcal{H}_2} \triangleq \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{tr} G^*(i\nu)G(i\nu)\mathrm{d}\nu}. \tag{2.20}$$

One important thing to note about the $\mathcal{H}_2$-norm is that it is, in contrast to the $\mathcal{H}_\infty$-norm (see Section 2.1.3), *not* an induced norm and does *not*, in general, satisfy the multiplicative property, $\|GF\|_{\mathcal{H}_2} \leq \|G\|_{\mathcal{H}_2} \|F\|_{\mathcal{H}_2}$, with G and F being two LTI systems. This property, if true, makes it possible to analyze individual systems in series to conclude facts about the interconnected system.

The forms in (2.19) and (2.20) are not suitable for actual evaluation of the $\mathcal{H}_2$-norm. However, the $\mathcal{H}_2$-norm can be expressed in a more computationally friendly form. The $\mathcal{H}_2$-norm in (2.20) can be rewritten, given a system G with a realization as in (2.16), using the Gramians in (2.5), to

$$\begin{aligned}
\|G\|_{\mathcal{H}_2}^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{tr} G^*(i\nu)G(i\nu)\mathrm{d}\nu = \frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{tr} G(i\nu)G^*(i\nu)\mathrm{d}\nu \\
&= \frac{1}{2\pi} \operatorname{tr} \int_{-\infty}^{\infty} \mathbf{B}^\mathsf{T}\mathbf{H}^*\big(i\nu\big)\mathbf{C}^\mathsf{T}\mathbf{C}\mathbf{H}\big(i\nu\big)\mathbf{B}\mathrm{d}\nu = \operatorname{tr} \mathbf{B}^\mathsf{T}\mathbf{Q}\mathbf{B} && (2.21\text{a}) \\
&= \frac{1}{2\pi} \operatorname{tr} \int_{-\infty}^{\infty} \mathbf{C}\mathbf{H}\big(i\nu\big)\mathbf{B}\mathbf{B}^\mathsf{T}\mathbf{H}^*\big(i\nu\big)\mathbf{C}^\mathsf{T}\mathrm{d}\nu = \operatorname{tr} \mathbf{C}\mathbf{P}\mathbf{C}^\mathsf{T}, && (2.21\text{b})
\end{aligned}$$

where $\mathbf{P}$ and $\mathbf{Q}$ satisfy

$$\mathbf{0} = \mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^\mathsf{T} + \mathbf{B}\mathbf{B}^\mathsf{T}, \tag{2.22a}$$

$$\mathbf{0} = \mathbf{A}^\mathsf{T}\mathbf{Q} + \mathbf{Q}\mathbf{A} + \mathbf{C}^\mathsf{T}\mathbf{C}. \tag{2.22b}$$

Discrete-Time $\mathcal{H}_2$-Norm

All the material for the continuous-time case is readily extended to the discrete-time case.

Definition 2.8 ($\mathcal{H}_2$-norm). For an asymptotically stable ($\mathbf{A}$ Schur) discrete-time system, G, the $\mathcal{H}_2$-norm is defined as

$$\|G\|_{\mathcal{H}_2} \triangleq \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{tr} G^*(\mathrm{e}^{i\nu})G(\mathrm{e}^{i\nu})\mathrm{d}\nu}. \tag{2.23}$$

An important observation here is that the system does not have to be strictly proper for the $\mathcal{H}_2$-norm to be defined. As in the continuous-time case, the above definition is not in a computationally friendly form, and (2.23) can be reformulated using the definitions of the discrete-time Gramians, (2.13), which yields

$$\begin{aligned}\|G\|_{\mathcal{H}_2}^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{tr} G^*(\mathrm{e}^{i\nu})G(\mathrm{e}^{i\nu})\mathrm{d}\nu = \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{tr} G(\mathrm{e}^{i\nu})G^*(\mathrm{e}^{i\nu})\mathrm{d}\nu \\ &= \operatorname{tr}\left(\mathbf{B}^\mathsf{T}\mathbf{Q}\mathbf{B} + \mathbf{D}^\mathsf{T}\mathbf{D}\right) &(2.24a)\\ &= \operatorname{tr}\left(\mathbf{C}\mathbf{P}\mathbf{C}^\mathsf{T} + \mathbf{D}\mathbf{D}^\mathsf{T}\right), &(2.24b)\end{aligned}$$

where $\mathbf{P}$ and $\mathbf{Q}$ satisfy

$$\mathbf{0} = \mathbf{A}\mathbf{P}\mathbf{A}^\mathsf{T} - \mathbf{P} + \mathbf{B}\mathbf{B}^\mathsf{T}, \tag{2.25a}$$

$$\mathbf{0} = \mathbf{A}^\mathsf{T}\mathbf{Q}\mathbf{A} - \mathbf{Q} + \mathbf{C}^\mathsf{T}\mathbf{C}. \tag{2.25b}$$

Continuous-Time $\mathcal{H}_\infty$-Norm

Although our proposed methods revolve around the $\mathcal{H}_2$-measure, the $\mathcal{H}_\infty$-measure will be used in various comparisons. Hence, the definition of it will be presented in this section. As with the $\mathcal{H}_2$-norm, the $\mathcal{H}_\infty$-norm can be defined using the signal norms presented in Section 2.1.3. Given an asymptotically stable ($\mathbf{A}$ Hurwitz) continuous-time system, G, the $\mathcal{H}_\infty$-norm is

$$\|G\|_{\mathcal{H}_\infty} \triangleq \max_{\mathbf{w}(t)\neq 0} \frac{\|\mathbf{z}(t)\|_{\mathcal{L}_2}}{\|\mathbf{w}(t)\|_{\mathcal{L}_2}} = \max_{\|\mathbf{w}(t)\|_{\mathcal{L}_2}=1} \|\mathbf{z}(t)\|_{\mathcal{L}_2}. \tag{2.26}$$

Looking at (2.26), it can be observed that, the $\mathcal{H}_\infty$-norm is indeed an induced norm, and hence satisfies the multiplicative property $\|GF\|_{\mathcal{H}_\infty} \leq \|G\|_{\mathcal{H}_\infty}\|F\|_{\mathcal{H}_\infty}$. This is one reason for the popularity of this norm.

The definition for the $\mathcal{H}_\infty$-norm in the frequency domain is

Definition 2.9 ($\mathcal{H}_\infty$-norm). For an asymptotically stable ($\mathbf{A}$ Hurwitz) continuous-time system, G, the $\mathcal{H}_\infty$-norm is, in the frequency domain, defined as

$$\|G\|_{\mathcal{H}_\infty} \triangleq \max_{\omega\in\mathbb{R}} \bar{\sigma}\left(G(i\omega)\right). \tag{2.27}$$

Observe that for the $\mathcal{H}_\infty$-norm, the system does not have to be strictly proper.

The $\mathcal{H}_\infty$-norm is however not as straightforward to compute as the $\mathcal{H}_2$-norm. One way to compute the $\mathcal{H}_\infty$-norm is to compute the smallest value γ such that the Hamiltonian matrix $\mathbf{W}$ has no eigenvalues on the imaginary axis, where

$$\mathbf{W} \triangleq \begin{pmatrix} \mathbf{A} + \mathbf{B}\mathbf{R}^{-1}\mathbf{D}^\mathsf{T}\mathbf{C} & \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\mathsf{T} \\ -\mathbf{C}^\mathsf{T}\left(\mathbb{I} + \mathbf{D}\mathbf{R}^{-1}\mathbf{D}^\mathsf{T}\right)\mathbf{C} & -\left(\mathbf{A} + \mathbf{B}\mathbf{R}^{-1}\mathbf{D}^\mathsf{T}\mathbf{C}\right)^\mathsf{T} \end{pmatrix} \tag{2.28}$$

and $\mathbf{R} \triangleq \gamma^2 - \mathbf{D}^\mathsf{T}\mathbf{D}$.

Discrete-Time $\mathcal{H}_\infty$-Norm

The material for the continuous-time case is readily extended to the discrete-time case. The definition for the $\mathcal{H}_\infty$-norm in discrete time becomes

Definition 2.10 ($\mathcal{H}_\infty$-norm). For an asymptotically stable ($\mathbf{A}$ Schur) discrete-time system, G, the $\mathcal{H}_\infty$-norm is, in the frequency domain, defined as

$$\|G\|_{\mathcal{H}_\infty} \triangleq \max_{\omega\in[-\pi,\pi]} \bar{\sigma}\left(G(\mathrm{e}^{i\omega})\right). \tag{2.29}$$

2.1.4 Output-Feedback Controller

An *output-feedback controller*, K, of order n_K can be described as a linear system

$$\dot{\mathbf{x}}_K(t) = \mathbf{K}_A\mathbf{x}_K(t) + \mathbf{K}_B\mathbf{y}(t) \tag{2.30a}$$
$$\mathbf{u}(t) = \mathbf{K}_C\mathbf{x}_K(t) + \mathbf{K}_D\mathbf{y}(t) \tag{2.30b}$$

where $\mathbf{x}_K \in \mathbb{R}^{n_K}$ is the state vector of the controller, $\mathbf{y} \in \mathbb{R}^{n_y}$ the measurement signal and $\mathbf{u} \in \mathbb{R}^{n_u}$ the control signal. A commonly used model for analyzing systems and measure performance, which will be used in this thesis, is

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \mathbf{z} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{C}_1 & \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{C}_2 & \mathbf{D}_{21} & \mathbf{D}_{22} \end{pmatrix}\begin{pmatrix} \mathbf{x} \\ \mathbf{w} \\ \mathbf{u} \end{pmatrix}, \tag{2.31}$$

where $\mathbf{x} \in \mathbb{R}^{n_x}$ is the state vector, $\mathbf{w} \in \mathbb{R}^{n_w}$ the disturbance signal, $\mathbf{u} \in \mathbb{R}^{n_u}$ the control signal, $\mathbf{z} \in \mathbb{R}^{n_z}$ the performance measure and $\mathbf{y} \in \mathbb{R}^{n_y}$ the measurement signal. Here, the matrix $\mathbf{D}_{22}$ is assumed, without loss of generality, to be zero, see Zhou et al. [1996]. Combine equations (2.31) and (2.30) to arrive at a state-space representation of the closed-loop system from $\mathbf{w}$ to $\mathbf{z}$, see Figure 2.1,

$$T_{w,z} = \left[\begin{array}{c|c} \begin{pmatrix} \mathbf{A} + \mathbf{B}_2\mathbf{K}_D\mathbf{C}_2 & \mathbf{B}_2\mathbf{K}_C \\ \mathbf{K}_B\mathbf{C}_2 & \mathbf{K}_A \end{pmatrix} & \begin{pmatrix} \mathbf{B}_1 + \mathbf{B}_2\mathbf{K}_D\mathbf{D}_{21} \\ \mathbf{K}_B\mathbf{D}_{21} \end{pmatrix} \\ \hline \begin{pmatrix} \mathbf{C}_1 + \mathbf{D}_{12}\mathbf{K}_D\mathbf{C}_2 & \mathbf{D}_{12}\mathbf{K}_C \end{pmatrix} & \begin{pmatrix} \mathbf{D}_{11} + \mathbf{D}_{12}\mathbf{K}_D\mathbf{D}_{21} \end{pmatrix} \end{array}\right]. \tag{2.32}$$

The two types of controllers that will be mentioned in this thesis are $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controllers. These controllers are designed to minimize the $\mathcal{H}_2$ or $\mathcal{H}_\infty$-norm of

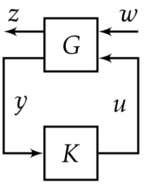

Figure 2.1: *Feedback*

the closed-loop system, $T_{w,z}$. The problem of finding an $\mathcal{H}_2$ or $\mathcal{H}_\infty$ controller can be divided into three cases. The simple case, both in the case of $\mathcal{H}_\infty$ and $\mathcal{H}_2$ controllers, is to find a full order controller, $n_K = n_x$, see e.g., Skogestad and Postlethwaite [2007] or Zhou et al. [1996]. The two more difficult cases are to find a *reduced-order controller*, $0 < n_K < n_x$, or a *static output-feedback controller*, $n_K = 0$. However, the problem of computing a reduced-order controller can be reformulated as a static controller problem, this is shown in El Ghaoui et al. [1997] and restated here for clarification.

To see that the problem of finding a reduced-order controller can be reformulated as a static output-feedback controller, first create the augmented system, G_{aug}.

$$G_{aug} = \left[\begin{array}{c|c} \mathbf{A}_{aug} & \begin{pmatrix}\mathbf{B}_{1,aug} & \mathbf{B}_{2,aug}\end{pmatrix} \\ \hline \begin{pmatrix}\mathbf{C}_{1,aug} \\ \mathbf{C}_{2,aug}\end{pmatrix} & \begin{pmatrix}\mathbf{D}_{11,aug} & \mathbf{D}_{12,aug} \\ \mathbf{D}_{21,aug} & \mathbf{D}_{22,aug}\end{pmatrix}\end{array}\right], \text{ where}$$

$$\mathbf{A}_{aug} = \begin{pmatrix}\mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{pmatrix}, \mathbf{B}_{1,aug} = \begin{pmatrix}\mathbf{B}_1 \\ \mathbf{0}\end{pmatrix}, \mathbf{B}_{2,aug} = \begin{pmatrix}\mathbf{0} & \mathbf{B}_2 \\ \mathbb{I} & \mathbf{0}\end{pmatrix},$$

$$\mathbf{C}_{1,aug} = \begin{pmatrix}\mathbf{C}_1 & \mathbf{0}\end{pmatrix}, \mathbf{D}_{11,aug} = \mathbf{D}_{11}, \mathbf{D}_{12,aug} = \begin{pmatrix}\mathbf{0} & \mathbf{D}_{12}\end{pmatrix},$$

$$\mathbf{C}_{2,aug} = \begin{pmatrix}\mathbf{0} & \mathbb{I} \\ \mathbf{C}_2 & \mathbf{0}\end{pmatrix}, \mathbf{D}_{21,aug} = \begin{pmatrix}\mathbf{0} \\ \mathbf{D}_{21}\end{pmatrix}, \mathbf{D}_{22,aug} = \mathbf{0},$$

with the new state space vector augmented with $\mathbf{x}_K \in \mathbb{R}^{n_K}$, $\mathbf{x}_{aug} = \begin{pmatrix}\mathbf{x} \\ \mathbf{x}_K\end{pmatrix}$, the new control signal augmented with $\mathbf{u}_K \in \mathbb{R}^{n_K}$, $\mathbf{u}_{aug} = \begin{pmatrix}\mathbf{u}_K \\ \mathbf{u}\end{pmatrix}$ and the new measurement signal augmented with $\mathbf{y}_K \in \mathbb{R}^{n_K}$, $\mathbf{y}_{aug} = \begin{pmatrix}\mathbf{y}_K \\ \mathbf{y}\end{pmatrix}$. The $\mathbf{0}$'s are matrices of compatible sizes with all elements zero and $\mathbb{I}$ are identity matrices of compatible sizes.

Now use the static controller, $\mathbf{u}_{aug} = \mathbf{K}_{aug}\mathbf{y}_{aug}$, on G_{aug}, where $\mathbf{K}_{aug}$ has the structure

$$\mathbf{K}_{aug} = \begin{pmatrix}\mathbf{K}_A & \mathbf{K}_B \\ \mathbf{K}_C & \mathbf{K}_D\end{pmatrix},$$

where $\mathbf{K}_A, \mathbf{K}_B, \mathbf{K}_C$ and $\mathbf{K}_D$ are the matrices from the controller in (2.30). Computing the closed-loop equations for this feedback system will lead to obtaining the same equations as in (2.32). This shows that any method for computing a static output-feedback controller can also be used to compute a reduced-order controller.

2.1.5 LPV Systems

A natural generalization of LTI systems is *linear time-varying systems*, LTV systems, where the state-space matrices can be dependent on time. The drawback is that LTV systems are very hard to analyze and work with. This raises the need of an intermediate step to represent systems, and this is where *linear parameter-varying systems*, LPV systems, comes in. LPV systems depend on scheduling parameters, $\mathbf{p}$, that varies with time, *but are measurable*. A general LPV system can be written, in state-space representation, in continuous time, (see Tóth [2008]), as

$$G(\mathbf{p}) : \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{p})\mathbf{x}(t) + \mathbf{B}(\mathbf{p})\mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{C}(\mathbf{p})\mathbf{x}(t) + \mathbf{D}(\mathbf{p})\mathbf{u}(t), \end{cases} \tag{2.33}$$

where $\mathbf{p}$ is the vector of *scheduling parameters*. Note that there is no restriction on how the LPV system depends on the scheduling parameters, hence it can be nonlinear and also depend on the time derivative of $\mathbf{p}$. LPV systems have the property that if the scheduling parameters in the LPV system are kept constant, the system becomes a regular LTI system.

As with ordinary LTI systems, the state-space representation for an LPV system is not unique and it is possible, by applying a state transformation, to change the basis of the states. As with the system matrices, when generalizing to LPV systems from LTI systems, the state transformations can depend on the scheduling parameters, i.e.,

$$\mathbf{x} = \mathbf{T}(\mathbf{p})\hat{\mathbf{x}}, \tag{2.34}$$

where $\mathbf{T}(\mathbf{p})$ is a nonsingular continuously differentiable matrix for all t. Applying this similarity transformation to the system in (2.33) yields

$$\hat{G}(\mathbf{p}) = \left[\begin{array}{c|c} \mathbf{T}^{-1}(\mathbf{p})\mathbf{A}(\mathbf{p})\mathbf{T}(\mathbf{p}) + \mathbf{T}^{-1}(\mathbf{p})\dot{\mathbf{T}}(\mathbf{p}) & \mathbf{T}^{-1}(\mathbf{p})\mathbf{B}(\mathbf{p}) \\ \hline \mathbf{C}(\mathbf{p})\mathbf{T}(\mathbf{p}) & \mathbf{D}(\mathbf{p}) \end{array}\right]. \tag{2.35}$$

Note that there is a term in the new $\mathbf{A}$-matrix that depends on the time derivative of the state transformation.

A general discrete-time state-space LPV system can be written as, see Kulcsar and Tóth [2011],

$$G(\mathbb{P}_k) = \left[\begin{array}{c|c} \mathbf{A}(\mathbb{P}_k) & \mathbf{B}(\mathbb{P}_k) \\ \hline \mathbf{C}(\mathbb{P}_k) & \mathbf{D}(\mathbb{P}_k) \end{array}\right], \tag{2.36}$$

where $\mathbb{P}_k = \left\{\mathbf{p}_{k+j}\right\}_{j=-\infty}^{\infty}$. By applying a similarity transformation (which can depend on the parameters), i.e.,

$$\mathbf{x}_k = \mathbf{T}(\mathbf{p}_k)\hat{\mathbf{x}}_k, \tag{2.37}$$

where $\mathbf{T}(\mathbf{p}_k)$ is a nonsingular and bounded matrix for all k, an LPV system with the same behavior but with another state-space representation is constructed,

$$\hat{G}(\mathbb{P}_k) = \left[\begin{array}{c|c} \mathbf{T}(\mathbf{p}_{k+1})\mathbf{A}(\mathbb{P}_k)\mathbf{T}(\mathbf{p}_k) & \mathbf{T}(\mathbf{p}_{k+1})\mathbf{B}(\mathbb{P}_k) \\ \hline \mathbf{C}(\mathbb{P}_k)\mathbf{T}(\mathbf{p}_k) & \mathbf{D}(\mathbb{P}_k) \end{array}\right]. \tag{2.38}$$

Looking at how the state transformations work for the LPV system above, one realizes that in one state base the state-space matrices can depend on only the current value of the parameter and in another it can also depend the derivative (in discrete time, the parameter values at other time steps than the current). Similar behavior can be seen when going from an LPV system described in state-space form to an input-output model structure of the LPV system. For example, study an example from Tóth et al. [2012], where a second order state-space representation of an LPV system is used,

$$\begin{aligned} \mathbf{x}_{k+1} &= \begin{pmatrix} 0 & a_2(p_k) \\ 1 & a_1(p_k) \end{pmatrix} \mathbf{x}_k + \begin{pmatrix} b_2(p_k) \\ b_1(p_k) \end{pmatrix} u_k, \\ y_k &= \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x}_k. \end{aligned}$$

This system only depends on the current parameter value, i.e., p_k. However, the equivalent input-output form becomes

$$y_k = a_1(p_{k-1})y_{k-1} + a_2(p_{k-2})y_{k-2} + b_1(p_{k-1})u_{k-1} + b_2(p_{k-2})u_{k-2},$$

which is clearly not only dependent of only the current parameter value. Hence, it is important to note, when working with LPV systems, if one is working with state-space or input-output forms, since these can give rise to different dependencies of the parameters.

2.2 Optimization

This section starts by giving a brief presentation of optimization and some methods that can be used to solve optimization problems. The presentation will closely follow relevant sections in Nocedal and Wright [2006].

Most optimization problems can mathematically be written as

$$\begin{aligned} &\underset{\mathbf{x}}{\text{minimize}}\ f(\mathbf{x}) \\ &\quad \text{subject to } g_{I,i}(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m_I \\ &\qquad\qquad\quad\ g_{E,i}(\mathbf{x}) = 0, \quad i = 1, \ldots, m_E \end{aligned}$$

where $f(\mathbf{x})$ is the *cost function*, $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, and $g_{I,i}(\mathbf{x}), g_{E,i}(\mathbf{x})$ are the *constraint functions*. A vector $\mathbf{x}^\star$ is called optimal if it produces the smallest value of the cost function of all the $\mathbf{x}$ that satisfy the constraints. In this thesis, the problems will mostly be unconstrained, i.e., problems without any $g_{I,i}(\mathbf{x})$ or $g_{E,i}(\mathbf{x})$. The value attained at the solution, $\mathbf{x}^\star$, to the optimization problem, $f(\mathbf{x}^\star)$, is called a *minimum*. This can either be a local or global minimum and the point where this value is attained, $\mathbf{x}^\star$ is called a *minimizer* (local or global). One way to be able to classify when a minimum is attained is to use first order necessary conditions.

Optimization problems can be divided into two classes, *convex optimization problems* and *non-convex optimization problems*. The problems of interest in this thesis will be non-convex. To explain what a non-convex problem is, a convex problem is presented first.

First, define a convex set. A convex set, $\mathcal{N}$, is a set, such that any point, $\mathbf{z}$, on a line between any two points, $\mathbf{x}, \mathbf{y}$, in the set, this point, $\mathbf{z}$, should also lie in the set, i.e.,

$$\theta\mathbf{x} + (1-\theta)\mathbf{y} = \mathbf{z} \in \mathcal{N}, \quad \forall\theta \in [0,1], \quad \mathbf{x}, \mathbf{y} \in \mathcal{N}. \tag{2.39}$$

A convex function is defined in the same manner. A function is convex if it satisfies

$$f(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \leq \theta f(x) + (1-\theta)f(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in \mathcal{N}$ and $\theta \in [0,1]$, where $\mathcal{N}$ is a convex set.

A convex optimization problem is an optimization problem where both the cost function and the feasible set, the set of $\mathbf{x}$'s defined by the constraints, are convex. Convex optimization problems have the feature that *a local minimizer is always a global minimizer*. This means that when a minimum is found in a convex optimization problem it is the global minimum. This guarantee does not exist in general for non-convex optimization problems. The problem of finding the global minimizer for a general non-convex optimization problem is difficult and often only local minimizers are sought. For further reading see e.g., Nocedal and Wright [2006].

2.2.1 Local Methods

One approach to solve non-convex optimization problems is to use *local methods*, methods that seek for a local minimizer, i.e., a point that in a neighborhood of feasible points has the smallest value of the cost function. A class of local methods which is widely used today in solving nonlinear non-convex problems is the class of *quasi-Newton line-search methods*. These methods typically require that the cost function is twice continuously differentiable, at least for the convergence theory to hold. However, in practice, these methods have been shown to work well on certain non-smooth problems as well, see for example Lewis and Overton [2012].

The line search strategy is to find a direction $\mathbf{p}_k$, and a step α_k, such that

$$\mathbf{f}_k \triangleq f(\mathbf{x}_k) > f(\mathbf{x}_k + \alpha_k\mathbf{p}_k). \tag{2.40}$$

There exist many suggestions of how to find the direction $\mathbf{p}_k$ and the step length α_k. One suggestion, and maybe the most obvious, is to take the *steepest descent* direction, which is $\mathbf{p}_k = -\frac{\nabla\mathbf{f}_k}{\|\nabla\mathbf{f}_k\|}$ and choose α_k as

$$\alpha_k \triangleq \arg\min_{\alpha} f(\mathbf{x}_k - \alpha\mathbf{p}_k).$$

A benefit with the choice $\mathbf{p}_k = -\frac{\nabla \mathbf{f}_k}{\|\nabla \mathbf{f}_k\|}$, is that only information about the gradient is needed and no second-order information, i.e., information about the Hessian. The problem of choosing the steepest descent direction is that the convergence can be extremely slow.

By exploiting second-order information about the cost function a better search direction can be produced. Assume a model function

$$m_k(\mathbf{p}) \triangleq \mathbf{f}_k + \mathbf{p}^\mathsf{T} \nabla \mathbf{f}_k + \mathbf{p}^\mathsf{T} \nabla^2 \mathbf{f}_k \mathbf{p},$$

that approximates the function $\mathbf{f}$ well in a neighborhood of $\mathbf{x}_k$, then define $\mathbf{p}_k$ to be the solution to

$$\underset{\mathbf{p}}{\text{minimize}}\ m_k(\mathbf{p}),$$

i.e., $\mathbf{p}_k = -(\nabla^2 \mathbf{f}_k)^{-1} \nabla \mathbf{f}_k$ and α_k is chosen according some conditions, for more detail see, for example, Nocedal and Wright [2006]. A method with this choice of direction is called a *Newton method*. There are however two major drawbacks with this method, the Hessian has to be computed which can be very time consuming, and the Hessian has to be positive definite.

Quasi-Newton Methods

Quasi-Newton methods are methods that resemble Newton methods but in some way tries to approximate the Hessian in a computationally efficient manner. As in the Newton method, start with a quadratic model function

$$m_k(\mathbf{p}) \triangleq \mathbf{f}_k + \nabla \mathbf{f}_k^\mathsf{T} \mathbf{p} + \frac{1}{2} \mathbf{p}^\mathsf{T} \mathbf{B}_k \mathbf{p},$$

where $\mathbf{B}_k$ is a symmetric positive definite matrix. Instead of computing a new $\mathbf{B}_k$ for every iteration only an update of $\mathbf{B}_k$ is wanted to obtain $\mathbf{B}_{k+1}$. As for the Newton method, the minimizer to the model function is $\mathbf{p}_k = -\mathbf{B}_k^{-1} \nabla \mathbf{f}_k$, which is then used to calculate $\mathbf{x}_{k+1}$ as

$$\mathbf{x}_{k+1} \triangleq \mathbf{x}_k + \alpha_k \mathbf{p}_k.$$

As in the Newton method, α_k is chosen according to some conditions which will not be further discussed here, see e.g., Nocedal and Wright [2006] for further reading.

One way of updating $\mathbf{B}_k$ is to let $\mathbf{B}_{k+1}$ be the solution to the optimization problem

$$\underset{\mathbf{B}}{\text{minimize}}\ \|\mathbf{B} - \mathbf{B}_k\|_{\mathbf{G}_k^{-1}} \tag{2.41a}$$

$$\text{subject to } \mathbf{B} = \mathbf{B}^\mathsf{T}, \quad \mathbf{B}\mathbf{s}_k = \mathbf{y}_k \tag{2.41b}$$

where $\mathbf{s}_k \triangleq \alpha_k \mathbf{p}_k$ and $\mathbf{y}_k \triangleq \nabla \mathbf{f}_{k+1} - \nabla \mathbf{f}_k$. The norm that is used in the optimization problem is the weighted Frobenius norm,

$$\|\mathbf{B}\|_{\mathbf{G}_k^{-1}} \triangleq \left\|\mathbf{G}_k^{-\frac{1}{2}}\mathbf{B}\mathbf{G}_k^{-\frac{1}{2}}\right\|_F, \quad \mathbf{G}_k \triangleq \int_0^1 \nabla f(\mathbf{x}_k + \tau\alpha_k\mathbf{p}_k)d\tau.$$

The structure of the optimization problem (2.41) can be explained like this. The constraint that $\mathbf{B}$, which is an approximation of the Hessian, should be symmetric is obvious for a function that is a twice differentiable function. The second constraint, the *secant equation*, ensures that $\mathbf{B}$ generates a consistent expression for a first-order approximation of the Hessian using the gradient. To determine $\mathbf{B}_{k+1}$ uniquely, the $\mathbf{B}$, in some sense, closest to $\mathbf{B}_k$ is chosen. Additionally, the minimization problem is made scale-invariant and dimensionless, which explains the minimization and the choice of norm and weights.

The optimization problem (2.41) has a closed form solution,

$$\mathbf{B}_{k+1} = (\mathbb{I} - \rho_k\mathbf{y}_k\mathbf{s}_k^\mathsf{T})\mathbf{B}_k(\mathbb{I} - \rho_k\mathbf{s}_k\mathbf{y}_k^\mathsf{T}) + \rho_k\mathbf{y}_k\mathbf{y}_k^\mathsf{T}, \quad \rho_k \triangleq \frac{1}{\mathbf{y}_k^\mathsf{T}\mathbf{s}_k}.$$

This update of $\mathbf{B}_k$ is called the DFP (which stands for Davidon-Fletcher-Powell) updating formula. To compute the direction $\mathbf{p}_k = -\mathbf{B}_k^{-1}\nabla\mathbf{f}_k$, the inverse of $\mathbf{B}_k$ is needed. Since $\mathbf{B}_{k+1}$ is a rank two update of $\mathbf{B}_k$, the inverse of $\mathbf{B}_{k+1} \triangleq \mathbf{H}_{k+1}^{-1}$ can be expressed in closed form as

$$\mathbf{H}_{k+1} = \mathbf{H}_k - \frac{\mathbf{H}_k\mathbf{y}_k\mathbf{y}_k^\mathsf{T}\mathbf{H}_k}{\mathbf{y}_k^\mathsf{T}\mathbf{H}_k\mathbf{y}_k} + \frac{\mathbf{s}_k\mathbf{s}_k^\mathsf{T}}{\mathbf{y}_k^\mathsf{T}\mathbf{s}_k}.$$

An even better updating formula is the BFGS (which stands for Broyden-Fletcher-Goldfarb-Shanno) updating formula where a similar optimization problem as before, but for $\mathbf{H}_{k+1}$ instead, is solved. $\mathbf{H}_{k+1}$ is the solution to the optimization problem

$$\begin{aligned} &\underset{\mathbf{H}}{\text{minimize}} \ \|\mathbf{H} - \mathbf{H}_k\|_{\mathbf{G}_k} \\ &\text{subject to } \mathbf{H} = \mathbf{H}^\mathsf{T}, \quad \mathbf{H}\mathbf{y}_k = \mathbf{s}_k \end{aligned}$$

which has the solution

$$\mathbf{H}_{k+1} \triangleq (\mathbb{I} - \rho_k\mathbf{s}_k\mathbf{y}_k^\mathsf{T})\mathbf{H}_k(\mathbb{I} - \rho_k\mathbf{y}_k\mathbf{s}_k^\mathsf{T}) + \rho_k\mathbf{s}_k\mathbf{s}_k^\mathsf{T}.$$

The benefit with quasi-Newton methods is that every iteration in the optimization scheme now can be performed with complexity $\mathcal{O}(n^2)$, not including function and gradient evaluations.

The BFGS-scheme will be used extensively in the strategies proposed in this thesis.

2.3 Matrix Theory

This section will briefly present, for the sake of easy reference in the later chapters, some basic matrix-theory concepts and definitions. The presented theory can also be found in Higham [2008], Skelton et al. [1998] and Lancaster and Tismenetsky [1985].

2.3.1 Properties for Dynamical Systems

In this thesis, linear dynamical systems plays an important role, especially asymptotically stable linear systems. Two useful matrix definitions for discrete and continuous-time linear systems are,

Definition 2.11. Let λ_i be the eigenvalues to the square matrix $\mathbf{A}$. If $\operatorname{Re} \lambda_i < 0, \forall i$, then $\mathbf{A}$ is called Hurwitz.

Definition 2.12. Let λ_i be the eigenvalues to the square matrix $\mathbf{A}$. If $|\lambda_i| < 1, \forall i$, then $\mathbf{A}$ is called Schur.

For a continuous-time (discrete-time) linear system it holds that, if the $\mathbf{A}$-matrix is Hurwitz (Schur), then the system is asymptotically stable.

As was explained in Section 2.1.2, the Gramians for linear systems are an important part in this thesis. To compute these Gramians a number of Lyapunov equations (both continuous and discrete), as in (2.7) and (2.14), have to be solved. An important question to ask is; when do these equations have a unique solution?

Theorem 2.1 (Corollary 3.3.3 in Skelton et al. [1998]). *A matrix $\mathbf{X}$ solving a Lyapunov equation*

$$\mathbf{0} = \mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{A}^{\mathsf{T}} + \mathbf{Y}, \quad \mathbf{Y} \succeq \mathbf{0} \tag{2.42}$$

is unique if and only if there are no two eigenvalues of $\mathbf{A}$ that are symmetrically located about the imaginary axis.

Proof: The left eigenvalues $\mathbf{v}_i$ of $\mathbf{A}$ satisfy $\mathbf{v}_i^*\mathbf{A} = \lambda_i \mathbf{v}_i^*$. Multiplying (2.42) from left and right by $\mathbf{v}_i^*$ and $\mathbf{v}_j$, respectively, to obtain

$$\mathbf{0} = \mathbf{v}_i^*\mathbf{A}\mathbf{X}\mathbf{v}_j + \mathbf{v}_i^*\mathbf{X}\mathbf{A}^{\mathsf{T}}\mathbf{v}_j + \mathbf{v}_i^*\mathbf{Y}\mathbf{v}_j = \mathbf{v}_i^*\mathbf{X}\mathbf{v}_j\left(\lambda_i + \overline{\lambda}_j\right) + \mathbf{v}_i^*\mathbf{Y}\mathbf{v}_j. \tag{2.43}$$

This yields unique values for the elements of the transformed $\hat{\mathbf{X}}$:

$$\hat{\mathbf{X}}_{ij} \triangleq \left[\mathbf{V}^{-1}\mathbf{X}\mathbf{V}^{-*}\right]_{ij} = \mathbf{v}_i^*\mathbf{X}\mathbf{v}_j = -\frac{\mathbf{v}_i^*\mathbf{Y}\mathbf{v}_j}{\lambda_i + \overline{\lambda}_j}, \forall i,j, \; \mathbf{V}^{-*} = [\mathbf{v}_1 \; \cdots \; \mathbf{v}_n] \tag{2.44}$$

if and only if $\lambda_i + \overline{\lambda}_j \neq 0$ for all i and j. □

Theorem 2.2 (Corollary 3.4.1 in Skelton et al. [1998]). *A matrix* $\mathbf{X}$ *solving the discrete Lyapunov equation*

$$\mathbf{0} = \mathbf{A}^\mathsf{T}\mathbf{X}\mathbf{A} - \mathbf{X} + \mathbf{Y}, \quad \mathbf{Y} \succeq \mathbf{0} \tag{2.45}$$

is unique if and only if $\lambda_i(\mathbf{A}) \neq \left(\lambda_j(\mathbf{A})\right)^{-1}$ *for all* i *and* j.

Proof: Multiply (2.45) from the left and right with the matrix of left eigenvectors of $\mathbf{A}$ (where $\lambda_i \mathbf{v}_i^* = \mathbf{v}_i^*\mathbf{A}$, $\mathbf{V}^{-*} = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_n]$, $\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \mathbf{\Lambda} = \mathrm{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n)$), as follows,

$$\begin{aligned}
\mathbf{V}^{-1}\mathbf{X}\mathbf{V}^{-*} &= \mathbf{V}^{-1}\left(\mathbf{A}\mathbf{X}\mathbf{A}^\mathsf{T} + \mathbf{Y}\right)\mathbf{V}^{-*} \\
&= \mathbf{V}^{-1}\mathbf{A}\mathbf{V}\mathbf{V}^{-1}\mathbf{X}\mathbf{V}^{-*}\mathbf{V}^{*}\mathbf{A}^\mathsf{T}\mathbf{V}^{-*} + \mathbf{V}^{-1}\mathbf{Y}\mathbf{V}^{-*} \\
&= \mathbf{\Lambda}\mathbf{V}^{-1}\mathbf{X}\mathbf{V}^{-*}\mathbf{\Lambda} + \mathbf{V}^{-1}\mathbf{Y}\mathbf{V}^{-*}.
\end{aligned}$$

This yields unique values for the elements of the transformed $\hat{\mathbf{X}}$,

$$\hat{\mathbf{X}}_{ij} \triangleq \left[\mathbf{V}^{-1}\mathbf{X}\mathbf{V}^{-*}\right] = \mathbf{v}_i^*\mathbf{X}\mathbf{v}_j = \left(1 - \lambda_i\overline{\lambda}_j\right)^{-1}\mathbf{v}_i^*\mathbf{Y}\mathbf{v}_j, \tag{2.46}$$

if and only if $\lambda_i\overline{\lambda}_j \neq 1$, for all i and j. □

The two theorems above tells us that, given an asymptotically stable system ($\mathbf{A}$ Hurwitz for continuous time and $\mathbf{A}$ Schur for discrete time), then the solutions to the Lyapunov equations for the Gramians are unique.

2.3.2 Matrix Functions

This section will give some definitions of matrix functions and present some theory that will be useful in the later chapters of the thesis.

As stated in Higham [2008], there exist many ways of defining matrix functions, $f(\mathbf{A})$. Presented here, is the definition via Jordan canonical form, which exists for all matrices, see for example Lancaster and Tismenetsky [1985].

Definition 2.13 (Definition 1.1 in Higham [2008]). The function f is said to be defined on the spectrum of $\mathbf{A} \in \mathbb{C}^{n\times n}$ if the values

$$f^{(j)}(\lambda_i), \quad j = 0, 1, \ldots, n_i - 1, \quad i = 1, 2, \ldots, s \tag{2.47}$$

exist. These are called the values of the function f on the spectrum of $\mathbf{A}$. n_i are the sizes of the individual Jordan blocks in $\mathbf{A}$ and s is the number of individual eigenvalues.

Now, if f is defined on the spectrum of the matrix, then it is possible to define $f(\mathbf{A})$.

Definition 2.14 (Definition 1.2 in Higham [2008]). Let f be defined on the spectrum of $\mathbf{A} \in \mathbb{C}^{n\times n}$ and let $\mathbf{J}_k$ denote a Jordan block in $\mathbf{A}$ with $\mathbf{A} \triangleq \mathbf{Z}\mathbf{J}\mathbf{Z}^{-1} =$

$\mathbf{Z}\,\mathrm{diag}\,(\mathbf{J}_k)\,\mathbf{Z}^{-1}$ and λ_k denote an eigenvalue of $\mathbf{A}$. Then

$$f(\mathbf{A}) \triangleq \mathbf{Z}f(\mathbf{J})\mathbf{Z}^{-1} = \mathbf{Z}\,\mathrm{diag}\,(f(\mathbf{J}_k))\,\mathbf{Z}^{-1}, \tag{2.48}$$

where

$$f(\mathbf{J}_k) \triangleq \begin{pmatrix} f(\lambda_k) & f'(\lambda_k) & \dots & \frac{f^{(n_k-1)}(\lambda_k)}{(n_k-1)!} \\ & f(\lambda_k) & \ddots & \vdots \\ & & \ddots & f'(\lambda_k) \\ & & & f(\lambda_k) \end{pmatrix}. \tag{2.49}$$

For example, given the function $f(x) = \sin x$, and we want to compute $f(\mathbf{A})$. Then the definition above can be used to compute $f(\mathbf{A})$, given a diagonalizable matrix $\mathbf{A} = \mathbf{Z}\mathbf{D}\mathbf{Z}^{-1} = \mathbf{Z}\,\mathrm{diag}(\lambda_i)\mathbf{Z}^{-1}$, as

$$\sin \mathbf{A} = \mathbf{Z}\,(\sin \mathbf{D})\,\mathbf{Z}^{-1} = \mathbf{Z}\,\mathrm{diag}(\sin \lambda_i)\mathbf{Z}^{-1}. \tag{2.50}$$

A number of properties for general matrix functions, to be able to use them more efficiently, can be derived.

Theorem 2.3 (Theorem 1.18 in Higham [2008]). *Let f be analytic on an open subset $\Omega \subseteq \mathbb{C}$ such that each connected component of Ω is closed under conjugation. Consider the corresponding matrix function f on its natural domain in $\mathbb{C}^{n\times n}$, the set $\mathcal{D} = \{\mathbf{A} \in \mathbb{C}^{n\times n} : \Lambda(\mathbf{A}) \subseteq \Omega\}$. Then the following are equivalent:*

(a) $f(\mathbf{A}^*) = f^*(\mathbf{A})$ *for all* $\mathbf{A} \in \mathcal{D}$.

(b) $f(\overline{\mathbf{A}}) = \overline{f(\mathbf{A})}$ *for all* $\mathbf{A} \in \mathcal{D}$.

(c) $f(\mathbb{R}^{n\times n} \cap \mathcal{D}) \subseteq \mathbb{R}^{n\times n}$.

(d) $f(\mathbb{R} \cap \Omega) \subseteq \mathbb{R}$.

Theorem 2.4 (Theorem 1.19 in Higham [2008]). *Let $\mathcal{D}$ be an open subset of $\mathbb{R}$ or $\mathbb{C}$ and let f be $n-1$ times continuously differentiable on $\mathcal{D}$. Then $f(\mathbf{A})$ is a continuous matrix function on the set of matrices $\mathbf{A} \in \mathbb{C}^{n\times n}$ with spectrum in $\mathcal{D}$.*

Theorem 2.5 (Theorem 1.20 in Higham [2008]). *Let f satisfy the conditions in Theorem 2.4. Then $f(\mathbf{A}) = \mathbf{0}$ for all $\mathbf{A} \in \mathbb{C}^{n\times n}$ with spectrum in $\mathcal{D}$ if and only if $f(\mathbf{A}) = \mathbf{0}$ for all diagonalizable $\mathbf{A} \in \mathbb{C}^{n\times n}$ with spectrum in $\mathcal{D}$.*

Theorem 2.5 (together with Theorem 2.4) can be interpreted as, if a function satisfies some mild continuity conditions (see Theorem 2.4), then to check the validity of a matrix identity it is sufficient to only check it for diagonalizable matrices.

One matrix function that will be used extensively in this thesis is the matrix logarithm, defined below.

Definition 2.15. Assume $\mathbf{A} \in \mathbb{C}^{n \times n}$ and that $\mathbf{A}$ does not have any eigenvalues on $\mathbb{R}^-$. Let $\mathbf{A}$ satisfy the equation $\mathbf{A} = e^{\mathbf{B}}$ for a matrix $\mathbf{B} \in \mathbb{C}^{n \times n}$, then it holds that $\mathbf{B} = \ln \mathbf{A}$, where ln denotes the principal logarithm.

This means, for a diagonalizable matrix $\mathbf{A} = \mathbf{Z}\mathbf{D}\mathbf{Z}^{-1} = \mathbf{Z}\,\mathrm{diag}(\lambda_i)\mathbf{Z}^{-1}$, the complex logarithm of the matrix $\mathbf{A}$ can be written as

$$\ln \mathbf{A} = \mathbf{Z}\,\mathrm{diag}\left(\ln|\lambda_i| + i \arg \lambda_i\right)\mathbf{Z}^{-1}. \tag{2.51}$$

Since computing the matrix logarithm can be computationally heavy, it can be beneficial, when having a sum of logarithm evaluations, to combine them, when possible, to one matrix logarithm computation, e.g., $\ln \mathbf{A} + \ln \mathbf{B} = \ln \mathbf{AB}$. The next two theorems will guide us to when this is possible.

Theorem 2.6 (Theorem 11.2 in Higham [2008]). *For $\mathbf{A} \in \mathbb{C}^{n \times n}$ with no eigenvalues on $\mathbb{R}^-$ and $\alpha \in [-1, 1]$ it holds that $\ln \mathbf{A}^{\alpha} = \alpha \ln \mathbf{A}$. In particular, $\ln \mathbf{A}^{-1} = -\ln \mathbf{A}$ and $\ln \mathbf{A}^{1/2} = \frac{1}{2} \ln \mathbf{A}$.*

Theorem 2.7 (Theorem 11.3 in Higham [2008]). *Suppose $\mathbf{B}, \mathbf{C} \in \mathbb{C}^{n \times n}$ both have no eigenvalues on $\mathbb{R}^-$ and that $\mathbf{BC} = \mathbf{CB}$. If for every eigenvalue λ_j of $\mathbf{B}$ and the corresponding eigenvalue μ_j of $\mathbf{C}$,*

$$\left|\arg \lambda_j + \arg \mu_j\right| < \pi, \tag{2.52}$$

then $\ln \mathbf{BC} = \ln \mathbf{B} + \ln \mathbf{C}$.

The methods that will be derived in this thesis will be gradient-based optimization algorithms. Hence, it will be required to compute the Fréchet derivative of the matrix logarithm. The Fréchet derivative can be seen as generalization of the ordinary derivative for matrix functions.

Theorem 2.8 (See Chapter 11 in Higham [2008]). *Let $L(\mathbf{A}, \mathbf{E})$ denote the Fréchet derivative of the matrix logarithm, defined in Definition 2.15, at $\mathbf{A} \in \mathbb{C}^{n \times n}$ in the direction $\mathbf{E} \in \mathbb{C}^{n \times n}$. Then it holds that*

$$L(\mathbf{A}, \mathbf{E}) = \int_0^1 \left(t(\mathbf{A} - \mathbb{I}) + \mathbb{I}\right)^{-1} \mathbf{E} \left(t(\mathbf{A} - \mathbb{I}) + \mathbb{I}\right)^{-1} \mathrm{d}t. \tag{2.53}$$

As written in (2.51) and (2.53), these equations are not suitable for computational evaluation. Thankfully, there exists computationally efficient and stable algorithms to compute these entities, e.g., the Schur-Parlett algorithm (see, e.g., Higham [2008]) can be used to compute $\ln(\mathbf{A})$, and all other functions that are analytic, and an algorithm for computing the Fréchet derivative of the matrix logarithm is described in Al-Mohy et al. [2012].

3

Frequency-Limited $\mathcal{H}_2$-Norm

In this chapter, a new $\mathcal{H}_2$-measure that, instead of taking the whole frequency interval into account, only focuses on pre-specified intervals is presented. The chapter starts by defining some new Gramians that are based on the ordinary Gramians in Section 2.1.2, but are limited to a limited frequency interval. These new Gramians are then used to define a new $\mathcal{H}_2$-measure that computes the $\mathcal{H}_2$-norm for a limited frequency interval.

3.1 Frequency-Limited Gramians

This section presents the framework that the new measure, that is presented in Section 3.2, is based on, the frequency-limited Gramians. These Gramians were introduced in Gawronski and Juang [1990] (continuous time) and Horta et al. [1993] (discrete time). The section starts by defining the frequency-limited Gramians and continues by deriving some properties of the Gramians. Ways to efficiently compute the Gramians are also presented. The results for the continuous-time case, which are also presented in Gawronski and Juang [1990] and Gawronski [2004], are presented, both for the sake of completeness, and to give a more thorough derivation. Theorem 3.1 and Theorem 3.2, describing the frequency-limited Gramians, are results that already exist in Gawronski [2004]. However, in this section, the results are presented using the given notation and in more detail. The reformulations of $\mathbf{S}_\omega$ and $\mathbf{S}_\Omega$ presented in Theorem 3.3 and Corollary 3.1 have not been published elsewhere.

The results for the discrete-time case contain a new derivation which differs from Horta et al. [1993], both in approach and result.

3.1.1 Continuous Time

In this section, it is assumed that the system that is used, G, is asymptotically stable, with a realization

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \tag{3.1a}$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t). \tag{3.1b}$$

G being asymptotically stable is equivalent to having $\mathbf{A}$ Hurwitz. For this system we have that the standard controllability and observability Gramians are

$$\mathbf{P} \triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{H}\big(i\nu\big)\mathbf{B}\mathbf{B}^{\mathsf{T}}\mathbf{H}^{*}\big(i\nu\big)\mathrm{d}\nu, \tag{3.2a}$$
$$\mathbf{Q} \triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{H}^{*}\big(i\nu\big)\mathbf{C}^{\mathsf{T}}\mathbf{C}\mathbf{H}\big(i\nu\big)\mathrm{d}\nu, \tag{3.2b}$$

where $\mathbf{H}\big(i\nu\big) \triangleq (\mathbb{I}i\nu - \mathbf{A})^{-1}$. The controllability and observability Gramians also satisfy the Lyapunov equations

$$\mathbf{0} = \mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^{\mathsf{T}} + \mathbf{B}\mathbf{B}^{\mathsf{T}}, \tag{3.3a}$$
$$\mathbf{0} = \mathbf{A}^{\mathsf{T}}\mathbf{Q} + \mathbf{Q}\mathbf{A} + \mathbf{C}^{\mathsf{T}}\mathbf{C}. \tag{3.3b}$$

Narrowing the frequency band in (3.2), from $(-\infty, \infty)$ to $(-\omega, \omega)$, where $\omega < \infty$, leads to the definition of the frequency-limited Gramians, see Gawronski and Juang [1990].

Definition 3.1. The frequency-limited controllability and observability Gramians for the system (3.1), are defined as

$$\mathbf{P}_{\omega} \triangleq \frac{1}{2\pi}\int_{-\omega}^{\omega} \mathbf{H}\big(i\nu\big)\mathbf{B}\mathbf{B}^{\mathsf{T}}\mathbf{H}^{*}\big(i\nu\big)\mathrm{d}\nu, \tag{3.4a}$$
$$\mathbf{Q}_{\omega} \triangleq \frac{1}{2\pi}\int_{-\omega}^{\omega} \mathbf{H}^{*}\big(i\nu\big)\mathbf{C}^{\mathsf{T}}\mathbf{C}\mathbf{H}\big(i\nu\big)\mathrm{d}\nu, \tag{3.4b}$$

with $\omega < \infty$.

As with the ordinary Gramians, the frequency-limited Gramians can also be written as solutions to two Lyapunov equations.

Theorem 3.1. *Given a system* $G = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]$*, where* $\mathbf{A}$ *is Hurwitz, it holds that*

$$\mathbf{P}_{\omega} \triangleq \mathbf{S}_{\omega}\mathbf{P} + \mathbf{P}\mathbf{S}_{\omega}^{\mathsf{T}}, \tag{3.5}$$

where $\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^{\mathsf{T}} + \mathbf{B}\mathbf{B}^{\mathsf{T}} = \mathbf{0}$ *and* $\mathbf{S}_{\omega} = \frac{1}{2\pi}\int_{-\omega}^{\omega}\mathbf{H}\big(i\nu\big)\mathrm{d}\nu$*. Furthermore,* $\mathbf{P}_{\omega}$ *can also* *be computed as a solution to*

$$\mathbf{A}\mathbf{P}_\omega + \mathbf{P}_\omega\mathbf{A}^\top + \mathbf{S}_\omega\mathbf{B}\mathbf{B}^\top + \mathbf{B}\mathbf{B}^\top\mathbf{S}_\omega^\top = \mathbf{0}. \tag{3.6}$$

Lemma 3.1. *For the ordinary controllability and observability Gramians,* $\mathbf{P}$ *and* $\mathbf{Q}$*, in (3.3), it holds that*

$$\mathbf{H}\big(i\nu\big)\mathbf{B}\mathbf{B}^\top\mathbf{H}^*\big(i\nu\big) = \mathbf{P}\mathbf{H}^*\big(i\nu\big) + \mathbf{H}\big(i\nu\big)\mathbf{P}, \tag{3.7a}$$

$$\mathbf{H}^*\big(i\nu\big)\mathbf{C}^\top\mathbf{C}\mathbf{H}\big(i\nu\big) = \mathbf{Q}\mathbf{H}\big(i\nu\big) + \mathbf{H}^*\big(i\nu\big)\mathbf{Q}. \tag{3.7b}$$

Proof: Using the definition of $\mathbf{H}\big(i\nu\big)$ and starting with a variant of the right hand side of (3.7a), it holds that

$$\mathbf{H}^{-1}\big(i\nu\big)\mathbf{P} + \mathbf{P}\mathbf{H}^{-*}\big(i\nu\big) = (i\nu\mathbb{I} - \mathbf{A})\,\mathbf{P} + \mathbf{P}\left(-i\nu\mathbb{I} - \mathbf{A}^\top\right) = -\left(\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^\top\right) = \mathbf{B}\mathbf{B}^\top, \tag{3.8}$$

which can be written as (3.7a) by multiplying with $\mathbf{H}\big(i\nu\big)$ and $\mathbf{H}^*\big(i\nu\big)$ from left and right, respectively. Similarly, it holds that

$$\mathbf{H}^{-*}\big(i\nu\big)\mathbf{Q} + \mathbf{Q}\mathbf{H}^{-1}\big(i\nu\big) = \left(-i\nu\mathbb{I} - \mathbf{A}^\top\right)\mathbf{Q} + \mathbf{Q}\,(i\nu\mathbb{I} - \mathbf{A}) = -\left(\mathbf{A}^\top\mathbf{Q} + \mathbf{Q}\mathbf{A}\right) = \mathbf{C}^\top\mathbf{C} \tag{3.9}$$

which can be written as (3.7b) by multiplying with $\mathbf{H}^*\big(i\nu\big)$ and $\mathbf{H}\big(i\nu\big)$ from left and right, respectively. □

Proof of Theorem 3.1: Using the definition of $\mathbf{P}_\omega$ in (3.4a) and Lemma 3.1, $\mathbf{P}_\omega$ can be written as

$$\begin{aligned}\mathbf{P}_\omega &= \frac{1}{2\pi}\int_{-\omega}^{\omega}\mathbf{H}\big(i\nu\big)\mathbf{B}\mathbf{B}^\top\mathbf{H}^*\big(i\nu\big)\mathrm{d}\nu = \mathbf{P}\frac{1}{2\pi}\int_{-\omega}^{\omega}\mathbf{H}^*\big(i\nu\big)\mathrm{d}\nu + \frac{1}{2\pi}\int_{-\omega}^{\omega}\mathbf{H}\big(i\nu\big)\mathrm{d}\nu\mathbf{P}\\ &= \mathbf{P}\mathbf{S}_\omega^* + \mathbf{S}_\omega\mathbf{P}.\end{aligned}$$

Hence, it holds that $\mathbf{P}_\omega = \mathbf{P}\mathbf{S}_\omega^* + \mathbf{S}_\omega\mathbf{P}$, with $\mathbf{S}_\omega = \frac{1}{2\pi}\int_{-\omega}^{\omega}\mathbf{H}\big(i\nu\big)\mathrm{d}\nu$.

Before showing that (3.6) holds, observe that

$$\begin{aligned}\mathbf{A}\mathbf{S}_\omega &= \mathbf{A}\left(\frac{1}{2\pi}\int_{-\omega}^{\omega}\mathbf{H}\big(i\nu\big)\mathrm{d}\nu\right) = \mathbf{A}\left(\frac{1}{2\pi}\int_{-\omega}^{\omega}(i\nu\mathbb{I} - \mathbf{A})^{-1}\,\mathrm{d}\nu\right)\\ &= \left(\frac{1}{2\pi}\int_{-\omega}^{\omega}(i\nu\mathbb{I} - \mathbf{A})^{-1}\,\mathrm{d}\nu\right)\mathbf{A} = \mathbf{S}_\omega\mathbf{A},\end{aligned}$$

i.e., the matrices $\mathbf{A}$ and $\mathbf{S}_\omega$ commute. Using the newly shown result $\mathbf{P}_\omega = \mathbf{P}\mathbf{S}_\omega^* + \mathbf{S}_\omega\mathbf{P}$ together with the fact that $\mathbf{A}$ and $\mathbf{S}_\omega$ commute, $\mathbf{A}\mathbf{P}_\omega + \mathbf{P}_\omega\mathbf{A}^\top$ can be written as

$$\begin{aligned}\mathbf{A}\mathbf{P}_\omega + \mathbf{P}_\omega\mathbf{A}^\top &= \mathbf{A}\left(\mathbf{S}_\omega\mathbf{P} + \mathbf{P}\mathbf{S}_\omega^*\right) + \left(\mathbf{S}_\omega\mathbf{P} + \mathbf{P}\mathbf{S}_\omega^*\right)\mathbf{A}^\top \\ &= \mathbf{S}_\omega\left(\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^\top\right) + \left(\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^\top\right)\mathbf{S}_\omega^* = -\mathbf{S}_\omega\mathbf{B}\mathbf{B}^\top - \mathbf{B}\mathbf{B}^\top\mathbf{S}_\omega^*.\end{aligned}$$

Hence, (3.6) holds. □

The same can be stated for the observability Gramian

Theorem 3.2. *Given a system* $G = \left[\begin{array}{c|c}\mathbf{A} & \mathbf{B}\\ \hline \mathbf{C} & \mathbf{D}\end{array}\right]$, *where* $\mathbf{A}$ *is Hurwitz, it holds that*

$$\mathbf{Q}_\omega \triangleq \mathbf{S}_\omega^\top\mathbf{Q} + \mathbf{Q}\mathbf{S}_\omega, \tag{3.10}$$

where $\mathbf{A}^\top\mathbf{Q} + \mathbf{Q}\mathbf{A} + \mathbf{B}^\top\mathbf{B} = \mathbf{0}$ *and* $\mathbf{S}_\omega = \frac{1}{2\pi}\int_{-\omega}^{\omega}\mathbf{H}\big(i\nu\big)\mathrm{d}\nu$. *Furthermore,* $\mathbf{Q}_\omega$ *can also be computed as a solution to*

$$\mathbf{A}^\top\mathbf{Q}_\omega + \mathbf{Q}_\omega\mathbf{A} + \mathbf{S}_\omega^\top\mathbf{C}^\top\mathbf{C} + \mathbf{C}^\top\mathbf{C}\mathbf{S}_\omega = \mathbf{0}, \tag{3.11}$$

Proof: The proof is analogous with the proof in the previous theorem, with the controllability Gramian. □

To be able to compute the limited-frequency Gramians $\mathbf{P}_\omega$ and $\mathbf{Q}_\omega$ we need to have a more computationally tractable expression for the matrix $\mathbf{S}_\omega$.

Theorem 3.3. *The matrix* $\mathbf{S}_\omega = \frac{1}{2\pi}\int_{-\omega}^{\omega}\mathbf{H}\big(i\nu\big)\mathrm{d}\nu$ *can be written as*

$$\mathbf{S}_\omega = \mathrm{Re}\left[\frac{i}{\pi}\ln\left(-\mathbf{A} - i\omega\mathbb{I}\right)\right]. \tag{3.12}$$

Proof: We have that

$$\mathbf{S}_\omega \triangleq \frac{1}{2\pi}\int_{-\omega}^{\omega}\mathbf{H}\big(i\nu\big)\mathrm{d}\nu = \frac{1}{2\pi}\int_{-\omega}^{\omega}\left(i\nu\mathbb{I} - \mathbf{A}\right)^{-1}\mathrm{d}\nu \triangleq f(\mathbf{A}). \tag{3.13}$$

With $f(x) = \frac{1}{2\pi}\int_{-\omega}^{\omega}\left(i\nu\mathbb{I} - x\right)^{-1}\mathrm{d}\nu$, Theorem 2.5 states that it is sufficient to calculate the function on the spectrum of $\mathbf{A}$. Let λ be an eigenvalue of $\mathbf{A}$ and since $\mathbf{A}$ is Hurwitz, it holds that $\mathrm{Re}\,\lambda < 0$. Hence

$$\frac{1}{2\pi}\int_{-\omega}^{\omega}\frac{1}{i\nu - \lambda}\mathrm{d}\nu = \frac{1}{2\pi}\left[-i\ln\left(i\nu - \lambda\right)\right]_{-\omega}^{\omega} = \frac{1}{2\pi}\left(i\ln\left(-i\omega - \lambda\right) - i\ln\left(i\omega - \lambda\right)\right), \tag{3.14}$$

where $\ln\lambda$ denotes the principal branch of the complex logarithm, namely $\ln\lambda = \ln|\lambda| + i\arg\lambda$, $-\pi < \arg\lambda \leq \pi$. Going back to the matrix form entails

$$\mathbf{S}_\omega = \frac{1}{2\pi}\int_{-\omega}^{\omega}\mathbf{H}\big(i\nu\big)\mathrm{d}\nu = \frac{1}{2\pi}\left[i\ln\left(-i\omega - \mathbf{A}\right) - i\ln\left(i\omega - \mathbf{A}\right)\right]. \tag{3.15}$$

Since the principal branch of the logarithm is used, Theorem 2.3 is applicable, which for this case means that given a matrix $\mathbf{C} \in \mathbb{C}^{n\times n}$ it holds that $\ln \overline{\mathbf{C}} = \overline{\ln \mathbf{C}}$. $\mathbf{S}_\omega$ becomes

$$\mathbf{S}_\omega = \frac{i}{2\pi}\ln(-\mathbf{A} - i\omega\mathbb{I}) + \overline{\frac{i}{2\pi}\ln(-\mathbf{A} - i\omega\mathbb{I})} = \mathrm{Re}\left[\frac{i}{\pi}\ln(-\mathbf{A} - i\omega\mathbb{I})\right].$$

□

Remark 3.1. An interesting property to investigate is what happens when ω tends to infinity. First note that if $x \in \mathbb{C} \setminus \mathbb{R}^-$, then $\mathrm{Re}\,[i \ln x] = -\arg x$. Now, let λ be an eigenvalue to $\mathbf{A}$ with $\mathrm{Re}\,\lambda < 0$ since $\mathbf{A}$ is Hurwitz, then $\mathrm{Re}\left[\frac{i}{\pi}\ln(-\mathbf{A} - i\omega\mathbb{I})\right]$ will approach $\frac{1}{2}$ when ω approaches infinity. Hence, $\mathbf{S}_\omega$ will approach $\frac{\mathbb{I}}{2}$ and the Lyapunov equations (3.6) and (3.11) will approach the Lyapunov equations for the regular Gramians (3.3) when ω approaches infinity.

Until now, only a single frequency band $(-\omega, \omega)$ around 0 has been considered. It is also possible to have arbitrary segments in the frequency domain, e.g., $\mathbf{Q}_\Omega$, $\Omega = [-\omega_4, -\omega_3] \cup [-\omega_2, -\omega_1] \cup [\omega_1, \omega_2] \cup [\omega_3, \omega_4]$, $0 < \omega_1 < \omega_2 < \omega_3 < \omega_4$.

Corollary 3.1. *For a union of disjunct frequency intervals*

$$\Omega = \bigcup_{k=1}^{N} [-\omega_{2k}, -\omega_{2k-1}] \cup [\omega_{2k-1}, \omega_{2k}], \text{ with } 0 \leq \omega_1 < \omega_2 < \cdots < \omega_{2N} < \infty,$$

it holds that

$$\mathbf{P}_\Omega = \frac{1}{2\pi}\int_\Omega \mathbf{H}\big(i\nu\big)\mathbf{B}\mathbf{B}^\mathsf{T}\mathbf{H}^*\big(i\nu\big)\mathrm{d}\nu, \tag{3.16a}$$

$$\mathbf{Q}_\Omega = \frac{1}{2\pi}\int_\Omega \mathbf{H}^*\big(i\nu\big)\mathbf{C}^\mathsf{T}\mathbf{C}\mathbf{H}\big(i\nu\big)\mathrm{d}\nu, \tag{3.16b}$$

satisfy the Lyapunov equations

$$\mathbf{0} = \mathbf{A}\mathbf{P}_\Omega + \mathbf{P}_\Omega\mathbf{A}^\mathsf{T} + \mathbf{S}_\Omega\mathbf{B}\mathbf{B}^\mathsf{T} + \mathbf{B}\mathbf{B}^\mathsf{T}\mathbf{S}_\Omega^\mathsf{T}, \tag{3.17a}$$

$$\mathbf{0} = \mathbf{A}^\mathsf{T}\mathbf{Q}_\omega + \mathbf{Q}_\omega\mathbf{A} + \mathbf{S}_\Omega^\mathsf{T}\mathbf{C}^\mathsf{T}\mathbf{C} + \mathbf{C}^\mathsf{T}\mathbf{C}\mathbf{S}_\Omega, \tag{3.17b}$$

where

$$\mathbf{S}_\Omega = \mathrm{Re}\left\{\frac{i}{\pi}\ln\left[\prod_{k=1}^{N}(-\mathbf{A} - i\omega_{2k}\mathbb{I})(-\mathbf{A} - i\omega_{2k-1}\mathbb{I})^{-1}\right]\right\}. \tag{3.18}$$

Proof: The corollary is proven for the observability Gramian, the proof for the controllability Gramian follows the same procedure. Splitting the integral in (3.16b) into two different sums with limits of the integral centered around 0, yields

$$\mathbf{Q}_\Omega \triangleq \frac{1}{2\pi}\int_\Omega \mathbf{H}\big(i\nu\big)\mathbf{B}\mathbf{B}^\mathsf{T}\mathbf{H}^*\big(i\nu\big)\mathrm{d}\nu = \sum_{k=1}^{N}\frac{1}{2\pi}\int_{-\omega_{2k}}^{\omega_{2k}} \mathbf{H}\big(i\nu\big)\mathbf{B}\mathbf{B}^\mathsf{T}\mathbf{H}^*\big(i\nu\big)\mathrm{d}\nu$$
$$- \frac{1}{2\pi}\int_{-\omega_{2k-1}}^{\omega_{2k-1}} \mathbf{H}\big(i\nu\big)\mathbf{B}\mathbf{B}^\mathsf{T}\mathbf{H}^*\big(i\nu\big)\mathrm{d}\nu = \sum_{k=1}^{N}\mathbf{Q}_{\omega_{2k}} - \mathbf{Q}_{\omega_{2k-1}}. \quad (3.19)$$

Define $\mathbf{L}_{\omega_i} \triangleq \mathbf{A}^\mathsf{T}\mathbf{Q}_{\omega_i} + \mathbf{Q}_{\omega_i}\mathbf{A} + \mathbf{S}_{\omega_i}^\mathsf{T}\mathbf{C}^\mathsf{T}\mathbf{C} + \mathbf{C}^\mathsf{T}\mathbf{C}\mathbf{S}_{\omega_i} = \mathbf{0}$. Using the fact that $\mathbf{L}_{\omega_i} = \mathbf{0}$ entails

$$\mathbf{0} = \sum_{k=1}^{N}\mathbf{L}_{\omega_{2k}} - \mathbf{L}_{\omega_{2k-1}} = \mathbf{A}^\mathsf{T}\left(\sum_{k=1}^{N}\mathbf{Q}_{\omega_{2k}} - \mathbf{Q}_{\omega_{2k-1}}\right) + \left(\sum_{k=1}^{N}\mathbf{Q}_{\omega_{2k}} - \mathbf{Q}_{\omega_{2k-1}}\right)\mathbf{A}$$
$$+ \left(\sum_{k=1}^{N}\mathbf{S}_{\omega_{2k}} - \mathbf{S}_{\omega_{2k-1}}\right)^\mathsf{T}\mathbf{C}^\mathsf{T}\mathbf{C} + \mathbf{C}^\mathsf{T}\mathbf{C}\left(\sum_{k=1}^{N}\mathbf{S}_{\omega_{2k}} - \mathbf{S}_{\omega_{2k-1}}\right)$$
$$= \mathbf{A}^\mathsf{T}\mathbf{Q}_\Omega + \mathbf{Q}_\Omega\mathbf{A} + \mathbf{S}_\Omega^\mathsf{T}\mathbf{C}^\mathsf{T}\mathbf{C} + \mathbf{C}^\mathsf{T}\mathbf{C}\mathbf{S}_\Omega. \quad (3.20)$$

Hence, it is proven that (3.17b) holds. If $\mathbf{S}_\Omega$ can be computed, then only one Lyapunov equation has to be solved to obtain $\mathbf{Q}_\Omega$. $\mathbf{S}_\Omega$ is for the moment a sum of $2N$ matrix logarithms, which, using Theorem 2.6, can be rewritten as

$$\mathbf{S}_\Omega = \sum_{k=1}^{N}\mathbf{S}_{\omega_{2k}} - \mathbf{S}_{\omega_{2k-1}} = \mathrm{Re}\left\{\frac{i}{\pi}\sum_{k=1}^{N}\left[\ln\left(-\mathbf{A} - i\omega_{2k}\mathbb{I}\right) - \ln\left(-\mathbf{A} - i\omega_{2k-1}\mathbb{I}\right)\right]\right\}$$
$$= \mathrm{Re}\left\{\frac{i}{\pi}\sum_{k=1}^{N}\Big[\ln\left(-\mathbf{A} - i\omega_{2k}\mathbb{I}\right) + \ln\left(-\mathbf{A} - i\omega_{2k-1}\mathbb{I}\right)^{-1}\Big]\right\} \quad (3.21)$$

Now, we want to show that this sum can be combined into one matrix logarithm evaluation. Theorem 2.5 states that it is sufficient to calculate the function on the spectrum of $\mathbf{A}$ to show this. Let λ be an eigenvalue to $\mathbf{A}$ and since $\mathbf{A}$ is Hurwitz, it holds that $\mathrm{Re}\,\lambda < 0$. Define $x_i = -\lambda - i\omega_i$, with $\omega_i > 0$ then it holds that $-\pi/2 < \arg x_i < \arg x_j < \pi/2$ for $i > j$. Note that $\arg x_i^{-1} = -\arg x_i$. Start with $\sum_{k=1}^{N}\Big[\ln x_{2k} + \ln x_{2k-1}^{-1}\Big]$ and reorder the terms

$$\sum_{k=1}^{N}\Big[\ln x_{2k} + \ln x_{2k-1}^{-1}\Big] = \ln x_{2N} + \ln x_1^{-1} + \sum_{k=1}^{N-1}\ln x_{2k} + \ln x_{2k+1}^{-1}. \quad (3.22)$$

Analyzing the argument of the first two terms, x_{2N} and x_1, gives

$$-\pi < \arg x_{2N} + \arg x_1^{-1} < 0, \tag{3.23}$$

hence, using Theorem 2.7,

$$\ln x_{2N} + \ln x_1^{-1} = \ln x_{2N} x_1^{-1}, \; -\pi < \arg x_{2N} x_1^{-1} < 0. \tag{3.24}$$

Analyzing the argument for the last sum in (3.22), yields that for all k, it holds that $0 < \arg x_{2k} + \arg x_{2k+1}^{-1} < \pi$. Hence, $\ln x_{2k} + \ln x_{2k+1}^{-1} = \ln x_{2k} x_{2k+1}^{-1}$. Now, since $0 < \omega_1 < \omega_2 < \cdots < \omega_N$ and all x_i are in the open right half plane, it holds that

$$0 < \sum_{k=1}^{N-1} \arg x_{2k} x_{2k+1}^{-1} < \pi. \tag{3.25}$$

Hence, using Theorem 2.7,

$$\sum_{k=1}^{N-1} \ln x_{2k} + \ln x_{2k+1}^{-1} = \ln \prod_{k=1}^{N-1} x_{2k} x_{2k+1}^{-1}, \; 0 < \arg \prod_{k=1}^{N-1} x_{2k} x_{2k+1}^{-1} < \pi. \tag{3.26}$$

Returning to (3.22),

$$\begin{aligned}\sum_{k=1}^{N} \left[\ln x_{2k} + \ln x_{2k-1}^{-1}\right] &= \ln x_{2N} + \ln x_1^{-1} + \sum_{k=1}^{N-1} \ln x_{2k} + \ln x_{2k+1}^{-1} \\ &= \ln x_{2N} x_1^{-1} + \ln \prod_{k=1}^{N-1} x_{2k} x_{2k+1}^{-1} = \ln \prod_{k=1}^{N} x_{2k} x_{2k-1}^{-1} \end{aligned} \tag{3.27}$$

since $-\pi < \arg x_{2N} x_1^{-1} + \arg \prod_{k=1}^{N-1} x_{2k} x_{2k+1} < \pi$. This holds for all eigenvalues of $\mathbf{A}$, and therefore it also holds that

$$\begin{aligned}\mathbf{S}_\Omega = \sum_{k=1}^{N} \mathbf{S}_{\omega_{2k}} - \mathbf{S}_{\omega_{2k-1}} &= \mathrm{Re}\left\{\frac{i}{\pi} \sum_{k=1}^{N} \left[\ln(-\mathbf{A} - i\omega_{2k}\mathbb{I}) - \ln(-\mathbf{A} - i\omega_{2k-1}\mathbb{I})\right]\right\} \\ &= \mathrm{Re}\left\{\frac{i}{\pi} \ln\left[\prod_{k=1}^{N} (-\mathbf{A} - i\omega_{2k}\mathbb{I})(-\mathbf{A} - i\omega_{2k-1}\mathbb{I})^{-1}\right]\right\}. \end{aligned} \tag{3.28}$$

□

Theorem 3.1 tells us that, by using addition of two or more frequency-limited Gramians corresponding to different frequency intervals, it is possible to construct a frequency-limited Gramian for a combined frequency interval, e.g., you can construct the frequency-limited controllability Gramian, $\mathbf{P}_\Omega$, for the interval $\omega \in \Omega = \Omega_1 \cup \Omega_2$, with $\Omega_1 = [-\omega_2, -\omega_1] \cup [\omega_1, \omega_2]$ and $\Omega_2 = [-\omega_4, -\omega_3] \cup [\omega_3, \omega_4]$

as

$$\mathbf{A}\mathbf{P}_\Omega + \mathbf{P}_\Omega\mathbf{A}^\top + \mathbf{S}_\Omega\mathbf{B}\mathbf{B}^\top + \mathbf{B}\mathbf{B}^\top\mathbf{S}_\Omega^\top = \mathbf{0}, \tag{3.29}$$

with $\mathbf{S}_\Omega$ computed as in Corollary 3.1.

Remark 3.2. It is also possible to use, with abuse of notation, $\omega = \infty$ as the end frequency, in that case the ordinary controllability Gramian, $\mathbf{P}$ can be used in combination with the frequency-limited Gramians.

3.1.2 Discrete Time

The equations for the discrete-time frequency-limited Gramians are similar to the ones in the continuous-time case. However, since the derivation in Horta et al. [1993] is not as straightforward and yields an erroneous result, we will present our derivation in this section.

Given an asymptotically stable system $G = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]$. G being asymptotically stable means having $\mathbf{A}$ Schur. For this system the frequency-limited controllability and observability Gramians can be defined.

Definition 3.2. The frequency-limited controllability and observability Gramians for the system $G = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]$, are defined as

$$\mathbf{P}_\omega \triangleq \frac{1}{2\pi} \int_{-\omega}^{\omega} \mathbf{H}\left(e^{i\nu}\right)\mathbf{B}\mathbf{B}^\top\mathbf{H}^*\left(e^{i\nu}\right)d\nu, \tag{3.30}$$

$$\mathbf{Q}_\omega \triangleq \frac{1}{2\pi} \int_{-\omega}^{\omega} \mathbf{H}^*\left(e^{i\nu}\right)\mathbf{C}^\top\mathbf{C}\mathbf{H}\left(e^{i\nu}\right)d\nu, \tag{3.31}$$

with $\omega < \pi$ and $\mathbf{H}\left(e^{i\omega}\right) = \left(\mathbb{I}e^{i\omega} - \mathbf{A}\right)^{-1}$.

Inspired by the continuous-time case, the frequency-limited Gramians in discrete-time can be written as solutions to two discrete-time Lyapunov equations.

Theorem 3.4. *Given a discrete-time system $G = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]$, where $\mathbf{A}$ is Schur, it holds that*

$$\mathbf{P}_\omega \triangleq \mathbf{S}_\omega\mathbf{P} + \mathbf{P}\mathbf{S}_\omega^\top, \tag{3.32}$$

where $\mathbf{A}\mathbf{P}\mathbf{A}^\top - \mathbf{P} + \mathbf{B}\mathbf{B}^\top = \mathbf{0}$ *and* $\mathbf{S}_\omega = \frac{1}{4\pi}\int_{-\omega}^{\omega}\left(\mathbb{I} - e^{-i\nu}\mathbf{A}\right)^{-1}\left(\mathbb{I} + \mathbf{A}e^{-i\nu}\right)d\nu$. *Furthermore,* $\mathbf{P}_\omega$ *can be computed as a solution to*

$$\mathbf{A}\mathbf{P}_\omega\mathbf{A}^\top - \mathbf{P}_\omega + \mathbf{S}_\omega\mathbf{B}\mathbf{B}^\top + \mathbf{B}\mathbf{B}^\top\mathbf{S}_\omega^\top = \mathbf{0}. \tag{3.33}$$

To prove Theorem 3.4, a lemma is first presented.

Lemma 3.2. *For the ordinary Gramians* $\mathbf{P}$ *and* $\mathbf{Q}$*, in* (2.14)*, it holds that*

$$\mathbf{H}\left(e^{i\omega}\right)\mathbf{B}\mathbf{B}^{\mathsf{T}}\mathbf{H}^{*}\left(e^{i\omega}\right) = e^{-i\omega}\mathbf{P}\mathbf{H}^{*}\left(e^{i\omega}\right) + e^{i\omega}\mathbf{H}\left(e^{i\omega}\right)\mathbf{P} - \mathbf{P}, \tag{3.34a}$$

$$\mathbf{H}^{*}\left(e^{i\omega}\right)\mathbf{C}^{\mathsf{T}}\mathbf{C}\mathbf{H}\left(e^{i\omega}\right) = e^{i\omega}\mathbf{Q}\mathbf{H}\left(e^{i\omega}\right) + e^{-i\omega}\mathbf{H}^{*}\left(e^{i\omega}\right)\mathbf{Q} - \mathbf{Q}. \tag{3.34b}$$

Proof: Using the definition of $\mathbf{H}\left(e^{i\omega}\right) = \left(e^{i\omega}\mathbb{I} - \mathbf{A}\right)^{-1}$. Straightforward calculations yields

$$\begin{aligned} &e^{-i\omega}\mathbf{P}\mathbf{H}^{*}\left(e^{i\omega}\right) + e^{i\omega}\mathbf{H}\left(e^{i\omega}\right)\mathbf{P} - \mathbf{P} \\ &\quad = e^{-i\omega}\mathbf{H}^{-1}\left(e^{i\omega}\right)\mathbf{P} + e^{i\omega}\mathbf{P}\mathbf{H}^{-*}\left(e^{i\omega}\right) - \mathbf{H}^{-1}\left(e^{i\omega}\right)\mathbf{P}\mathbf{H}^{-*}\left(e^{i\omega}\right) \\ &\quad = e^{-i\omega}\left(e^{i\omega}\mathbb{I} - \mathbf{A}\right)\mathbf{P} + e^{i\omega}\mathbf{P}\left(e^{i\omega}\mathbb{I} - \mathbf{A}\right)^{*} - \left(e^{i\omega}\mathbb{I} - \mathbf{A}\right)\mathbf{P}\left(e^{i\omega}\mathbb{I} - \mathbf{A}\right)^{*} \\ &\quad = -\left(\mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{T}} - \mathbf{P}\right) = \mathbf{B}\mathbf{B}^{\mathsf{T}}, \end{aligned} \tag{3.35}$$

which can be written as (3.34a) by multiplying with $\mathbf{H}\left(e^{i\omega}\right)$ and $\mathbf{H}^{*}\left(e^{i\omega}\right)$ from left and right, respectively. Similarly, it holds that

$$\begin{aligned} &e^{i\omega}\mathbf{Q}\mathbf{H}\left(e^{i\omega}\right) + e^{-i\omega}\mathbf{H}^{*}\left(e^{i\omega}\right)\mathbf{Q} - \mathbf{Q} \\ &\quad = e^{i\omega}\mathbf{H}^{-*}\left(e^{i\omega}\right)\mathbf{Q} + e^{-i\omega}\mathbf{Q}\mathbf{H}^{-1}\left(e^{i\omega}\right) - \mathbf{H}^{-*}\left(e^{i\omega}\right)\mathbf{Q}\mathbf{H}^{-1}\left(e^{i\omega}\right) \\ &\quad = e^{i\omega}\left(e^{i\omega}\mathbb{I} - \mathbf{A}\right)^{*}\mathbf{Q} + e^{-i\omega}\mathbf{Q}\left(e^{i\omega}\mathbb{I} - \mathbf{A}\right) - \left(e^{i\omega}\mathbb{I} - \mathbf{A}\right)^{*}\mathbf{Q}\left(e^{i\omega}\mathbb{I} - \mathbf{A}\right) \\ &\quad = -\left(\mathbf{A}^{\mathsf{T}}\mathbf{Q}\mathbf{A} - \mathbf{Q}\right) = \mathbf{C}^{\mathsf{T}}\mathbf{C} \end{aligned} \tag{3.36}$$

which can be written as (3.34b) by multiplying with $\mathbf{H}^{*}\left(e^{i\omega}\right)$ and $\mathbf{H}\left(e^{i\omega}\right)$ from left and right, respectively. □

Proof of Theorem 3.4: Using the definition of $\mathbf{P}_{\omega}$ in (3.30) and Lemma 3.2, $\mathbf{P}_{\omega}$ can be written as

$$\mathbf{P}_\omega = \frac{1}{2\pi}\int_{-\omega}^{\omega} \mathbf{H}\left(e^{i\nu}\right)\mathbf{B}\mathbf{B}^\top\mathbf{H}^*\left(e^{i\nu}\right)d\nu = \frac{1}{2\pi}\int_{-\omega}^{\omega}\left(e^{-i\nu}\mathbf{P}\mathbf{H}^*\left(e^{i\nu}\right) + e^{i\nu}\mathbf{H}\left(e^{i\nu}\right)\mathbf{P} - \mathbf{P}\right)d\nu$$
$$= \frac{1}{2\pi}\int_{-\omega}^{\omega}\left(e^{i\nu}\mathbf{H}\left(e^{i\nu}\right) - \frac{\mathbb{I}}{2}\right)d\nu\mathbf{P} + \mathbf{P}\frac{1}{2\pi}\int_{-\omega}^{\omega}\left(e^{i\nu}\mathbf{H}\left(e^{i\nu}\right) - \frac{\mathbb{I}}{2}\right)^* d\nu$$
$$= \frac{1}{4\pi}\int_{-\omega}^{\omega}\left[\left(\mathbb{I} - e^{-i\nu}\mathbf{A}\right)^{-1}\left(\mathbb{I} + \mathbf{A}e^{-i\nu}\right)\right]d\nu\mathbf{P} + \mathbf{P}\frac{1}{4\pi}\int_{-\omega}^{\omega}\left[\left(\mathbb{I} - e^{-i\nu}\mathbf{A}\right)^{-1}\left(\mathbb{I} + \mathbf{A}e^{-i\nu}\right)\right]^* d\nu$$
$$= \mathbf{S}_\omega\mathbf{P} + \mathbf{P}\mathbf{S}_\omega^*. \quad (3.37)$$

Hence, it holds that $\mathbf{P}_\omega = \mathbf{S}_\omega\mathbf{P} + \mathbf{P}\mathbf{S}_\omega^*$, with

$$\mathbf{S}_\omega = \frac{1}{4\pi}\int_{-\omega}^{\omega}\left[\left(\mathbb{I} - e^{-i\nu}\mathbf{A}\right)^{-1}\left(\mathbb{I} + \mathbf{A}e^{-i\nu}\right)\right]d\nu.$$

Before showing that (3.33) holds, observe that

$$\mathbf{A}\mathbf{S}_\omega = \mathbf{A}\left(\frac{1}{4\pi}\int_{-\omega}^{\omega}\left[\left(\mathbb{I} - e^{-i\nu}\mathbf{A}\right)^{-1}\left(\mathbb{I} + \mathbf{A}e^{-i\nu}\right)\right]d\nu\right)$$
$$= \left(\frac{1}{4\pi}\int_{-\omega}^{\omega}\left[\left(\mathbb{I} - e^{-i\nu}\mathbf{A}\right)^{-1}\left(\mathbb{I} + \mathbf{A}e^{-i\nu}\right)\right]d\nu\right)\mathbf{A} = \mathbf{S}_\omega\mathbf{A}, \quad (3.38)$$

i.e., the matrices $\mathbf{A}$ and $\mathbf{S}_\omega$ commute. Using that $\mathbf{P}_\omega = \mathbf{S}_\omega\mathbf{P} + \mathbf{P}\mathbf{S}_\omega^*$ and the fact that $\mathbf{A}$ and $\mathbf{S}_\omega$ commute, $\mathbf{A}\mathbf{P}_\omega\mathbf{A}^\top - \mathbf{P}_\omega$ can be written as

$$\mathbf{A}\mathbf{P}_\omega\mathbf{A}^\top - \mathbf{P}_\omega = \mathbf{A}\left(\mathbf{S}_\omega\mathbf{P} + \mathbf{P}\mathbf{S}_\omega^*\right)\mathbf{A}^\top - \left(\mathbf{S}_\omega\mathbf{P} + \mathbf{P}\mathbf{S}_\omega^*\right)$$
$$\mathbf{S}_\omega\left(\mathbf{A}\mathbf{P}\mathbf{A}^\top - \mathbf{P}\right) + \left(\mathbf{A}\mathbf{P}\mathbf{A}^\top - \mathbf{P}\right)\mathbf{S}_\omega^* = -\left(\mathbf{S}_\omega\mathbf{B}\mathbf{B}^\top + \mathbf{B}\mathbf{B}^\top\mathbf{S}_\omega^*\right). \quad (3.39)$$

Hence, (3.33) holds. □

The same can be shown for the observability Gramian.

Theorem 3.5. *Given a discrete-time system* $G = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]$, *where* $\mathbf{A}$ *is Schur, it holds that*

$$\mathbf{Q}_\omega \triangleq \mathbf{S}_\omega^\top\mathbf{Q} + \mathbf{Q}\mathbf{S}_\omega, \quad (3.40)$$

where $\mathbf{A}^\top\mathbf{Q}\mathbf{A} - \mathbf{Q} + \mathbf{C}^\top\mathbf{C} = \mathbf{0}$ *and* $\mathbf{S}_\omega = \frac{1}{4\pi}\int_{-\omega}^{\omega}\left(\mathbb{I} - e^{-i\nu}\mathbf{A}\right)^{-1}\left(\mathbb{I} + \mathbf{A}e^{-i\nu}\right)d\nu$. *Fur-*

thermore, $\mathbf{Q}_\omega$ can be computed as a solution to

$$\mathbf{A}^\mathsf{T}\mathbf{Q}_\omega\mathbf{A} - \mathbf{Q}_\omega + \mathbf{S}_\omega^\mathsf{T}\mathbf{C}^\mathsf{T}\mathbf{B} + \mathbf{C}^\mathsf{T}\mathbf{C}\mathbf{S}_\omega = \mathbf{0}. \tag{3.41}$$

Proof: The proof is analogous to the one for the controllability Gramian. □

Theorem 3.6. *The matrix $\mathbf{S}_\omega = \frac{1}{4\pi}\int_{-\omega}^{\omega}\left(\mathbb{I} - \mathrm{e}^{-i\nu}\mathbf{A}\right)^{-1}\left(\mathbb{I} + \mathbf{A}\mathrm{e}^{-i\nu}\right)\mathrm{d}\nu$ can be written as*

$$\mathbf{S}_\omega = \frac{1}{2\pi}\operatorname{Re}\left[\omega\mathbb{I} - 2i\ln\left(\mathbb{I} - \mathbf{A}\mathrm{e}^{-i\omega}\right)\right]. \tag{3.42}$$

Proof: We have that

$$\mathbf{S}_\omega = \frac{1}{4\pi}\int_{-\omega}^{\omega}\left(\mathbb{I} - \mathrm{e}^{-i\nu}\mathbf{A}\right)^{-1}\left(\mathbb{I} + \mathbf{A}\mathrm{e}^{-i\nu}\right)\mathrm{d}\nu \triangleq f(\mathbf{A}). \tag{3.43}$$

With

$$f(x) = \frac{1}{4\pi}\int_{-\omega}^{\omega}\left(\mathbb{I} - \mathrm{e}^{-i\nu}x\right)^{-1}\left(\mathbb{I} + x\mathrm{e}^{-i\nu}\right)\mathrm{d}\nu,$$

Theorem 2.5 states that it is sufficient to calculate the function on the spectrum of $\mathbf{A}$. Let λ be an eigenvalue to $\mathbf{A}$ and since $\mathbf{A}$ is Schur, it holds that $|\lambda| < 1$. Hence

$$\begin{aligned}
&\int_{-\omega}^{\omega}\frac{1 + \lambda\mathrm{e}^{-i\nu}}{1 - \lambda\mathrm{e}^{-i\nu}}\mathrm{d}\nu = i\left[\ln\mathrm{e}^{-i\nu} - 2i\ln\left(1 - \lambda\mathrm{e}^{-i\nu}\right)\right]_{-\omega}^{\omega} \\
&= \left[\nu - 2i\ln\left(1 - \lambda\mathrm{e}^{-i\nu}\right)\right]_{-\omega}^{\omega} = 2\omega - 2\left[i\ln\left(1 - \lambda\mathrm{e}^{-i\omega}\right) - i\ln\left(1 - \lambda\mathrm{e}^{i\omega}\right)\right]
\end{aligned} \tag{3.44}$$

where $\ln z$ denotes the principal branch of the complex logarithm, namely $\ln z = \ln|z| + i\arg z$, $-\pi < \arg z \leq \pi$. Going back to the matrix equation entails

$$\begin{aligned}
\mathbf{S}(\omega) &= \frac{1}{4\pi}\int_{-\omega}^{\omega}\left(\mathbb{I} - \mathrm{e}^{-i\nu}\mathbf{A}\right)^{-1}\left(\mathbb{I} + \mathbf{A}\mathrm{e}^{-i\nu}\right)\mathrm{d}\nu \\
&= \frac{1}{2\pi}\left\{\omega\mathbb{I} - i\left[\ln\left(\mathbb{I} - \mathbf{A}\mathrm{e}^{-i\omega}\right) - \ln\left(\mathbb{I} - \mathbf{A}\mathrm{e}^{i\omega}\right)\right]\right\}.
\end{aligned} \tag{3.45}$$

Since the principal branch of the logarithm is used, Theorem 2.3 is applicable. For this case it means that given a matrix $\mathbf{C} \in \mathbb{C}^{n\times n}$ it holds that $\ln\overline{\mathbf{C}} = \overline{\ln\mathbf{C}}$. $\mathbf{S}_\omega$ becomes

$$\begin{aligned}
\mathbf{S}(\omega) &= \frac{1}{2\pi}\left\{\omega\mathbb{I} - i\left[\ln\left(\mathbb{I} - \mathbf{A}\mathrm{e}^{-i\omega}\right) - \ln\left(\mathbb{I} - \mathbf{A}\mathrm{e}^{i\omega}\right)\right]\right\} \\
&= \frac{1}{2\pi}\left\{\omega\mathbb{I} - \left[i\ln\left(\mathbb{I} - \mathbf{A}\mathrm{e}^{-i\omega}\right) + \overline{i\ln\left(\mathbb{I} - \mathbf{A}\mathrm{e}^{-i\omega}\right)}\right]\right\} \\
&= \frac{1}{2\pi}\operatorname{Re}\left[\omega\mathbb{I} - 2i\ln\left(\mathbb{I} - \mathbf{A}\mathrm{e}^{-i\omega}\right)\right].
\end{aligned}$$

$\square$

Remark 3.3. If $\omega = \pi$, then $\mathbf{S}_\omega = \frac{\mathbb{I}}{2} - \frac{1}{\pi}\operatorname{Re}\left[i \ln(\mathbb{I} + \mathbf{A})\right]$, and since the logarithm of a real matrix is a real matrix, it follows that $\mathbf{S}_\omega = \frac{\mathbb{I}}{2}$. Thus, the frequency-limited Gramians coincides with the regular Gramians when $\omega = \pi$.

3.2 Frequency-Limited $\mathcal{H}_2$-Norm

In this section, we will introduce a new frequency-limited $\mathcal{H}_2$-norm that uses the frequency-limited Gramians defined in the previous section. This new measure can for example be used to compare different models on limited frequency intervals, instead of the whole frequency domain.

3.2.1 Continuous Time

As presented in Section 2.1.3, the $\mathcal{H}_2$-norm of a continuous-time system $G = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]$, which is asymptotically stable ($\mathbf{A}$ is Hurwitz) and strictly proper ($\mathbf{D} = \mathbf{0}$), can be described by

$$\|G\|_{\mathcal{H}_2}^2 = \frac{1}{2\pi} \operatorname{tr} \int_{-\infty}^{\infty} G(i\nu)G^*(i\nu)\mathrm{d}\nu \tag{3.46a}$$

$$= \frac{1}{2\pi} \operatorname{tr} \int_{-\infty}^{\infty} \mathbf{C}\mathbf{H}\big(i\nu\big)\mathbf{B}\mathbf{B}^\mathsf{T}\mathbf{H}^*\big(i\nu\big)\mathbf{C}^\mathsf{T}\mathrm{d}\nu = \operatorname{tr} \mathbf{C}\mathbf{P}\mathbf{C}^\mathsf{T} \tag{3.46b}$$

$$= \frac{1}{2\pi} \operatorname{tr} \int_{-\infty}^{\infty} \mathbf{B}^\mathsf{T}\mathbf{H}^*\big(i\nu\big)\mathbf{C}^\mathsf{T}\mathbf{C}\mathbf{H}\big(i\nu\big)\mathbf{B}\mathrm{d}\nu = \operatorname{tr} \mathbf{B}^\mathsf{T}\mathbf{Q}\mathbf{B}. \tag{3.46c}$$

In this section, a new frequency-limited $\mathcal{H}_2$-like norm, that uses the frequency-limited Gramians presented in the previous section, is defined and is denoted as $\|G\|_{\mathcal{H}_2,\omega}$.

Definition 3.3. For an asymptotically stable system G and $0 < \omega < \infty$, define

$$\|G\|_{\mathcal{H}_2,\omega}^2 \triangleq \frac{1}{2\pi} \operatorname{tr} \int_{-\omega}^{\omega} G(i\nu)G^*(i\nu)\mathrm{d}\nu. \tag{3.47}$$

To be able to use the limited-frequency $\mathcal{H}_2$-norm in practice, it has to be expressed in a more computationally friendly way.

Theorem 3.7. *For an asymptotically stable system* $G = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]$ *and* $0 < \omega < \infty$*, the limited-frequency* $\mathcal{H}_2$*-norm can be computed as*

$$\|G\|^2_{\mathcal{H}_2,\omega} = \operatorname{tr} \mathbf{C}\mathbf{P}_\omega\mathbf{C}^\mathsf{T} + 2\operatorname{tr}\left[\left(\mathbf{C}\mathbf{S}_\omega\mathbf{B} + \mathbf{D}\frac{\omega}{2\pi}\right)\mathbf{D}^\mathsf{T}\right], \tag{3.48}$$

or

$$\|G\|^2_{\mathcal{H}_2,\omega} = \operatorname{tr} \mathbf{B}^\mathsf{T}\mathbf{Q}_\omega\mathbf{B} + 2\operatorname{tr}\left[\left(\mathbf{C}\mathbf{S}_\omega\mathbf{B} + \mathbf{D}\frac{\omega}{2\pi}\right)\mathbf{D}^\mathsf{T}\right], \tag{3.49}$$

where

$$\mathbf{0} = \mathbf{A}\mathbf{P}_\omega + \mathbf{P}_\omega\mathbf{A}^\mathsf{T} + \mathbf{S}_\omega\mathbf{B}\mathbf{B}^\mathsf{T} + \mathbf{B}\mathbf{B}^\mathsf{T}\mathbf{S}^\mathsf{T}_\omega, \tag{3.50a}$$

$$\mathbf{0} = \mathbf{A}^\mathsf{T}\mathbf{Q}_\omega + \mathbf{Q}_\omega\mathbf{A} + \mathbf{S}^\mathsf{T}_\omega\mathbf{C}^\mathsf{T}\mathbf{C} + \mathbf{C}^\mathsf{T}\mathbf{C}\mathbf{S}_\omega, \tag{3.50b}$$

$$\mathbf{S}_\omega = \operatorname{Re}\left[\frac{i}{\pi}\ln\left(-\mathbf{A} - i\omega\mathbb{I}\right)\right]. \tag{3.50c}$$

Proof: Using Theorem 3.1 we can rewrite equation (3.47),

$$\begin{aligned}
\|G\|^2_{\mathcal{H}_2,\omega} &= \frac{1}{2\pi}\operatorname{tr}\int_{-\omega}^{\omega} G(i\nu)G^*(i\nu)\mathrm{d}\nu \\
&= \frac{1}{2\pi}\operatorname{tr}\int_{-\omega}^{\omega}\left[\mathbf{C}\mathbf{H}\big(i\nu\big)\mathbf{B} + \mathbf{D}\right]\left[\mathbf{B}^\mathsf{T}\mathbf{H}^*\big(i\nu\big)\mathbf{C}^\mathsf{T} + \mathbf{D}^\mathsf{T}\right]\mathrm{d}\nu \\
&= \operatorname{tr}\mathbf{C}\frac{1}{2\pi}\int_{-\omega}^{\omega}\mathbf{H}\big(i\nu\big)\mathbf{B}\mathbf{B}^\mathsf{T}\mathbf{H}^*\big(i\nu\big)\mathrm{d}\nu\,\mathbf{C}^\mathsf{T} + \operatorname{tr}\frac{1}{2\pi}\int_{-\omega}^{\omega}\mathbf{D}\mathbf{D}^\mathsf{T}\mathrm{d}\nu \\
&\quad + \operatorname{tr}\left(\mathbf{C}\frac{1}{2\pi}\int_{-\omega}^{\omega}\mathbf{H}\big(i\nu\big)\mathrm{d}\nu\,\mathbf{B}\mathbf{D}^\mathsf{T} + \mathbf{D}\mathbf{B}^\mathsf{T}\frac{1}{2\pi}\int_{-\omega}^{\omega}\mathbf{H}^*\big(i\nu\big)\mathrm{d}\nu\,\mathbf{C}^\mathsf{T}\right) \\
&= \operatorname{tr}\mathbf{C}\mathbf{P}_\omega\mathbf{C}^\mathsf{T} + 2\operatorname{tr}\left[\left(\mathbf{C}\mathbf{S}_\omega\mathbf{B} + \mathbf{D}\frac{\omega}{2\pi}\right)\mathbf{D}^\mathsf{T}\right].
\end{aligned}$$

The same procedure can be used, using Theorem 3.2 and the fact that $\|G\|^2_{\mathcal{H}_2,\omega}$ also can be written as $\|G\|^2_{\mathcal{H}_2,\omega} = \frac{1}{2\pi}\operatorname{tr}\int_{-\omega}^{\omega} G^*(i\nu)G(i\nu)\mathrm{d}\nu$, to show equation (3.49). Theorem 3.3 shows how $\mathbf{S}_\omega$ can be computed. □

Using Corollary 3.1 it is possible, also for the limited-frequency $\mathcal{H}_2$-norm, to compute the $\mathcal{H}_2$-norm on arbitrary segments in the frequency domain, $\|G\|^2_{\mathcal{H}_2,\Omega}$, $\Omega = [-\omega_4, -\omega_3] \cup [-\omega_2, -\omega_1] \cup [\omega_1, \omega_2] \cup [\omega_3, \omega_4]$, $0 < \omega_1 < \omega_2 < \omega_3 < \omega_4$.

One important thing to note that differs between the limited-frequency $\mathcal{H}_2$-norm and the ordinary $\mathcal{H}_2$-norm, is that, if we do not include an infinite interval in Ω, i.e., include $\omega = \infty$ as the end frequency, then the system does not have to be strictly proper. This means that it is possible, in this case, to have $\mathbf{D} \neq \mathbf{0}$.

3.2.2 Discrete Time

In this section, the new frequency-limited $\mathcal{H}_2$-like norm for discrete-time systems, that uses the frequency-limited Gramians presented in Section 3.1.2, is defined.

Definition 3.4. For an asymptotically stable discrete-time system G and $0 < \omega < \pi$, define

$$\|G\|^2_{\mathcal{H}_2,\omega} \triangleq \frac{1}{2\pi} \operatorname{tr} \int_{-\omega}^{\omega} G(i\nu)G^*(i\nu)\mathrm{d}\nu. \tag{3.51}$$

Analogous to the continuous-time case, (3.51) can be expressed in a more computationally friendly way.

Theorem 3.8. *For an asymptotically stable discrete-time system* $G = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]$ *and* $0 < \omega < \pi$*, the limited-frequency* $\mathcal{H}_2$*-norm can be computed as*

$$\|G\|^2_{\mathcal{H}_2,\omega} = \operatorname{tr} \mathbf{C}\mathbf{P}_\omega\mathbf{C}^\mathsf{T} + 2\operatorname{tr}\left[\left(\mathbf{C}\mathbf{R}_\omega\mathbf{B} + \mathbf{D}\frac{\omega}{2\pi}\right)\mathbf{D}^\mathsf{T}\right], \tag{3.52}$$

or

$$\|G\|^2_{\mathcal{H}_2,\omega} = \operatorname{tr} \mathbf{B}^\mathsf{T}\mathbf{Q}_\omega\mathbf{B} + 2\operatorname{tr}\left[\left(\mathbf{C}\mathbf{R}_\omega\mathbf{B} + \mathbf{D}\frac{\omega}{2\pi}\right)\mathbf{D}^\mathsf{T}\right], \tag{3.53}$$

where

$$\begin{aligned}
\mathbf{0} &= \mathbf{A}\mathbf{P}_\omega\mathbf{A}^\mathsf{T} - \mathbf{P}_\omega + \mathbf{S}_\omega\mathbf{B}\mathbf{B}^\mathsf{T} + \mathbf{B}\mathbf{B}^\mathsf{T}\mathbf{S}_\omega^\mathsf{T}, &\text{(3.54a)}\\
\mathbf{0} &= \mathbf{A}^\mathsf{T}\mathbf{Q}_\omega\mathbf{A} - \mathbf{Q}_\omega + \mathbf{S}_\omega^\mathsf{T}\mathbf{C}^\mathsf{T}\mathbf{C} + \mathbf{C}^\mathsf{T}\mathbf{C}\mathbf{S}_\omega, &\text{(3.54b)}\\
\mathbf{S}_\omega &= \frac{1}{2\pi}\operatorname{Re}\left[\omega\mathbb{I} - 2i\ln\left(\mathbb{I} - \mathbf{A}e^{-i\omega}\right)\right], &\text{(3.54c)}\\
\mathbf{R}_\omega &= -\frac{1}{\pi}\mathbf{A}^{-1}\operatorname{Re}\left[i\ln\left(\mathbb{I} - \mathbf{A}e^{-i\omega}\right)\right]. &\text{(3.54d)}
\end{aligned}$$

Proof: By using Theorem 3.4 and Theorem 3.5, $\|G\|^2_{\mathcal{H}_2,\omega}$ can easily be rewritten to

$$\begin{aligned}
\|G\|^2_{\mathcal{H}_2,\omega} &= \operatorname{tr} \mathbf{C}\mathbf{P}_\omega\mathbf{C}^\mathsf{T} + 2\operatorname{tr}\left[\left(\mathbf{C}\mathbf{R}_\omega\mathbf{B} + \mathbf{D}\frac{\omega}{2\pi}\right)\mathbf{D}^\mathsf{T}\right], &\text{(3.55a)}\\
\|G\|^2_{\mathcal{H}_2,\omega} &= \operatorname{tr} \mathbf{B}^\mathsf{T}\mathbf{Q}_\omega\mathbf{B} + 2\operatorname{tr}\left[\left(\mathbf{C}\mathbf{R}_\omega\mathbf{B} + \mathbf{D}\frac{\omega}{2\pi}\right)\mathbf{D}^\mathsf{T}\right], &\text{(3.55b)}
\end{aligned}$$

where

$$\mathbf{R}_\omega = \frac{1}{2\pi}\int_{-\omega}^{\omega} \mathbf{H}\left(e^{i\nu}\right)\mathrm{d}\nu = \frac{1}{2\pi}\int_{-\omega}^{\omega}\left(e^{i\nu}\mathbb{I} - \mathbf{A}\right)^{-1}\mathrm{d}\nu. \tag{3.56}$$

This integral can be computed and simplified similarly to what is shown in the proof for Theorem 3.4, which leads to

$$\mathbf{R}_\omega = -\frac{1}{\pi}\mathbf{A}^{-1}\,\mathrm{Re}\left[i\ln\left(\mathbb{I}-\mathbf{A}e^{-i\omega}\right)\right]. \tag{3.57}$$

□

3.3 Concluding Remarks

In this chapter, the frequency-limited Gramians and their derivations have been presented. Computationally more efficient expressions than those presented in the original papers (Gawronski and Juang [1990] and Horta et al. [1993]), were derived. A detailed derivation of the discrete-time frequency-limited Gramians was presented, using the same notation and framework as in the continuous-time case and correcting errors in the available literature. Additionally, the frequency-limited $\mathcal{H}_2$-norm that uses these Gramians, both for continuous and discrete time, were presented. This frequency-limited $\mathcal{H}_2$-norm will be used for frequency-limited model reduction in Chapter 4.

4

Model Reduction

This chapter starts by introducing the model-reduction problem in Section 4.1. In Section 4.2, one of the most commonly used methods, balanced truncation (including frequency weighted and frequency limited), will be presented. Then in Section 4.3 some existing methods that use an $\mathcal{H}_2$-measure for model reduction are presented. Then the proposed methods for ordinary, robust, frequency-weighted and frequency-limited model reduction will be presented in Section 4.4. The material in this chapter is based on an extended version of the results in Petersson and Löfberg [2012a].

4.1 Introduction

Direct numerical simulation of dynamical systems has been a successful strategy for studying complex physical phenomena. However, deriving sufficiently detailed mathematical models, e.g., for designing controllers or analyzing performance, can be extremely difficult and can result in large and unnecessarily complicated models. This is the case particularly for systems pertaining to circuit simulations or dynamical systems coming from discretized partial differential equations. These large-scale models can make it difficult to analyze the system, due to memory-limitations, time-limitations, ill-conditioning or computationally expensive analysis methods. Hence, there is a need for smaller models that can describe large complex systems well. One way of creating these low-order models is through model reduction.

Given an LTI model,

$$G : \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t), \end{cases}$$

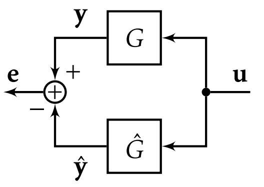

Figure 4.1: *Model reduction*

where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$ and $\mathbf{D} \in \mathbb{R}^{p\times m}$. For this model, the model-reduction problem is to find a reduced-order model

$$\hat{G} : \begin{cases} \dot{\hat{\mathbf{x}}}(t) = \hat{\mathbf{A}}\hat{\mathbf{x}}(t) + \hat{\mathbf{B}}\mathbf{u}(t), \\ \hat{\mathbf{y}}(t) = \hat{\mathbf{C}}\hat{\mathbf{x}}(t) + \hat{\mathbf{D}}\mathbf{u}(t), \end{cases}$$

with $\hat{\mathbf{A}} \in \mathbb{R}^{\hat{n}\times \hat{n}}$, $\hat{\mathbf{B}} \in \mathbb{R}^{\hat{n}\times m}$, $\hat{\mathbf{C}} \in \mathbb{R}^{p\times \hat{n}}$ and $\hat{\mathbf{D}} \in \mathbb{R}^{p\times m}$ and $\hat{n} < n$, where this reduced-order model, $\hat{G}$, describes the original model, G, well in some metric. One way to quantify the discrepancy between G and $\hat{G}$, is through the difference in their respective outputs. Particularly, given a certain input, $\mathbf{u}(t)$, the difference in the output, $\mathbf{e}(t) = \mathbf{y}(t) - \hat{\mathbf{y}}(t)$, should be small in some norm, see Figure 4.1.

This can be written as an optimization problem

$$\underset{\hat{G}}{\text{minimize}} \; \left\|G - \hat{G}\right\|, \text{ s.t. } \deg \hat{G} = \hat{n},$$

where $\deg \hat{G}$ denotes the size of the system, i.e., the number of states in the system, and $\mathcal{H}_\infty$ or $\mathcal{H}_2$ are two examples of norms that could be used. There are a number of methods that address this problem, for example using balanced truncation (see Section 4.2), e.g., Enns [1984], Moore [1981], Glover [1984], or using optimization, e.g., Flagg et al. [2010], Beattie and Gugercin [2007], Beattie and Gugercin [2009], Antoulas [2005], Poussot-Vassal [2011], Helmersson [1994] and the material in Section 4.4.

In many applications one is mainly interested in a low-order model that describes the system well only in a certain frequency interval. This leads us to investigate frequency-weighted model reduction. For the frequency-weighted model reduction, weighting filters are utilized, and in order to also facilitate MIMO-systems an input-filter (W_i) and an output-filter (W_o) are needed. Example of such methods are, for example, Enns [1984], Diab et al. [2000], Halevi [1992], Sreeram and Sahlan [2009], Zhou [1995]. Writing the frequency-weighted model-reduction problem as an optimization problem, results in

$$\underset{\hat{G}}{\text{minimize}} \; \left\|W_o(G - \hat{G})W_i\right\|, \text{ s.t. } \deg \hat{G} = \hat{n}.$$

In the frequency-weighted case, the weights have to be given by the user and are in practice often difficult to choose. However, in many applications it is the case that a system should be approximated over a limited frequency interval, while the other frequencies are not important at all. In this case one would like to use an ideal band-pass filter, but approximating an ideal band-pass filter requires a large number of states in the weighting filters, and can lead to other problems. To address this issue there are methods, that could be classified as a special class of frequency-weighted model-reduction methods, that will be called frequency-limited model reduction. This class of methods uses approaches that behave as though ideal band-pass filters have been used, e.g., Gawronski and Juang [1990], Huang et al. [2001], Horta et al. [1993], Sahlan et al. [2012] and Poussot-Vassal and Vuillemin [2012], and we will introduce a new method using this strategy in Section 4.4.3.

4.2 Balanced Truncation

One of the most commonly used model-reduction schemes is called balanced truncation, introduced in Moore [1981]. The physical interpretation of the balanced truncation is very simple, remove the states that induce a small amount of energy in the output and at the same time require a large amount of energy to excite. By understanding how the observability and controllability Gramians connect to these energies, see Section 2.1.2, one realizes that the system has to be expressed in a basis where the observability and controllability Gramians are equal and diagonal. Recall that the elements on the diagonal in the Gramians are the Hankel singular values of the system, see Section 2.1.2. This basis describes the states that can be classified as both difficult to control and observe, these states that can be removed. These are the states that correspond to the small Hankel singular values. When a system is expressed in such a basis the system is called *balanced*. Given a system with the observability Gramian $\mathbf{Q}$ and controllability Gramian $\mathbf{P}$, where $\mathbf{P}$ have the Cholesky factor $\mathbf{U}$, $\mathbf{P} = \mathbf{U}\mathbf{U}^*$, and $\mathbf{U}^*\mathbf{Q}\mathbf{U} = \mathbf{K}\Sigma^2\mathbf{K}^*$, it can be shown that the transformation needed to balance the system can be written as

$$\mathbf{T} = \Sigma^{1/2}\mathbf{K}^*\mathbf{U}^{-1} \text{ and } \mathbf{T}^{-1} = \mathbf{U}\mathbf{K}\Sigma^{-1/2}, \tag{4.1}$$

and the new states are given by $\tilde{\mathbf{x}} = \mathbf{T}\mathbf{x}$, see for example Antoulas [2005].

Theorem 4.1 (Balanced reduction, Theorem 7.9 in Antoulas [2005]). *Given a balanced system* $G = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]$, *which is asymptotically stable, with the Gramians equal to* Σ *and given the partitioning*

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix} \in \mathbb{R}^{n\times n},\ \mathbf{B} = \begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{pmatrix} \in \mathbb{R}^{n\times m},\ \mathbf{C} = \begin{pmatrix} \mathbf{C}_1 & \mathbf{C}_2 \end{pmatrix} \in \mathbb{R}^{p\times n},\ \Sigma = \begin{pmatrix} \Sigma_1 & \mathbf{0} \\ \mathbf{0} & \Sigma_2 \end{pmatrix}. \tag{4.2}$$

Then $\hat{G} = \left[\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{B}_1 \\ \hline \mathbf{C}_1 & \mathbf{D} \end{array}\right]$, $\mathbf{A}_{11} \in \mathbb{R}^{\hat{n}\times\hat{n}}$ *is a reduced-order system of order* $\hat{n} < n$, *which is both stable and balanced. Additionally, it holds that*

$$\left\|G - \hat{G}\right\|_{\mathcal{H}_\infty} \leq 2 \sum_{i=\hat{n}+1}^{n} \sigma_i, \tag{4.3}$$

where σ_i are the Hankel singular values of the system in descending order of magnitude.

Proof: See Theorem 7.9 in Antoulas [2005] □

There are several variations of the balanced-truncation method, which allow us to perform model reduction in a more computationally robust and efficient manner, e.g., Safonov and Chiang [1989], Safonov et al. [1990], Glover [1984]. Two properties that most of the balanced-truncation methods have in common (which make them very popular) are the preservation of stability and the *a priori* computable error bounds. Important to note is that a system resulting from a balanced truncation scheme is not a minimizer to a specific system norm optimization (for example $\mathcal{H}_2$ and $\mathcal{H}_\infty$).

As mentioned in Section 4.1, one important class of balanced-truncation methods are the frequency-weighted balanced-truncation methods and they are described in the following way. Let $G = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]$, be an asymptotically stable system to be reduced. Also assume that an input weighting, $W_i(s)$, and an output weighting, $W_o(s)$, are given. Define the weighted controllability and observability Gramians as

$$\mathbf{P}_i = \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\omega\mathbb{I} - \mathbf{A})^{-1} \mathbf{B} W_i(i\omega) W_i^*(i\omega) \mathbf{B}^* (i\omega\mathbb{I} - \mathbf{A})^{-*} \, \mathrm{d}\omega, \tag{4.4a}$$

$$\mathbf{Q}_o = \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\omega\mathbb{I} - \mathbf{A})^{-*} \mathbf{C}^* W_o^*(i\omega) W_o(i\omega) \mathbf{C} (i\omega\mathbb{I} - \mathbf{A})^{-1} \, \mathrm{d}\omega. \tag{4.4b}$$

and compute the state transformation that simultaneously diagonalizes $\mathbf{P}_i$ and $\mathbf{Q}_o$. Frequency-weighted balanced-truncation methods then utilize this transformation that diagonalizes $\mathbf{P}_i$ and $\mathbf{Q}_o$, to do a balanced truncation. This approach to frequency-weighted balanced truncation was first introduced in Enns [1984]. If either $W_i = \mathbb{I}$ or $W_o = \mathbb{I}$ this method guarantee stability of the reduced model. However, if both input and output weightings are used at the same time nothing can be guaranteed. Modifications of this method that guarantee stability when both input and output weights are used are discussed in, e.g., Lin and Chiu [1992], Varga and Anderson [2001].

Another important class of model-reduction methods, which was mentioned in Section 4.1, is frequency-limited balanced truncation. This was introduced by Gawronski and Juang [1990] for continuous-time systems and Horta et al. [1993] for discrete-time systems. In these articles they use frequency-limited Gramians (see Section 3.1) and simultaneously diagonalize these, to obtain a basis in which the truncation is done. The method in Gawronski and Juang [1990] can be seen as a special case of the method in Enns [1984] by choosing the weighting filters to be ideal bandpass filters (see Gugercin and Antoulas [2004]). However, the method in Gawronski and Juang [1990] cannot guarantee stability. A modification to this method that guarantee stability, has been presented in Gugercin and Antoulas [2004].

4.3 Overview of Model-Reduction Methods using the $\mathcal{H}_2$-Norm

The problem of finding a reduced-order model that, in $\mathcal{H}_2$ sense, resembles the original model well has been a goal in many investigations. Especially since the work of Meier and Luenberger [1967], and especially Wilson [1970], in which they derive first-order optimality conditions for minimization of the $\mathcal{H}_2$-norm, see also, for example, Lepschy et al. [1991], Beattie and Gugercin [2007], Fulcheri and Olivi [1998], Yan and Lam [1999] and references therein. One reason for this could be the fact that the $\mathcal{H}_2$ criterion provides a meaningful characterization of the error, both in deterministic and stochastic contexts. For example, given two discrete-time asymptotically-stable SISO systems G and $\hat{G}$, with the outputs $y(t)$ and $\hat{y}(t)$ respectively, and a white-noise input $u(t)$ (i.e., the input spectrum is $\Phi_u(\omega) = 1$), then it holds that

$$\begin{aligned}\underset{\hat{G}}{\text{minimize}}\ \mathrm{E}\left[(y-\hat{y})^2\right] &= \underset{\hat{G}}{\text{minimize}} \int_{-\pi}^{\pi} \left|G(\mathrm{e}^{i\omega}) - \hat{G}(\mathrm{e}^{i\omega})\right|^2 \Phi_u(\omega)\mathrm{d}\omega \\ &= \underset{\hat{G}}{\text{minimize}} \int_{-\pi}^{\pi} \left|G(\mathrm{e}^{i\omega}) - \hat{G}(\mathrm{e}^{i\omega})\right|^2 \mathrm{d}\omega = \underset{\hat{G}}{\text{minimize}}\ \left\|G-\hat{G}\right\|_{\mathcal{H}_2}^2. \end{aligned} \tag{4.5}$$

Finding global minimizers for the $\mathcal{H}_2$ approximation problem is very difficult, it is in fact a nonlinear non-convex optimization problem (see Example 4.1). The existing methods for $\mathcal{H}_2$ approximation have the more modest goal of finding local minimizers and can crudely be categorized into two categories; methods using tangential interpolation techniques or methods using gradient-flow techniques.

Example 4.1: Non-Convexity

To show that the cost function

$$V = \left\|\hat{G} - G_{\text{true}}\right\|_{\mathcal{H}_2}^2$$

is non-convex, we start with the system

$$G_{\text{true}} = \left[\begin{array}{c|c} -1 & 1 \\ \hline 1 & 0 \end{array}\right].$$

A system

$$\hat{G} = \left[\begin{array}{c|c} a & b \\ \hline c & 0 \end{array}\right],$$

that approximates the system G_{true}, is sought, where a, b and c are the decision variables. Consider an initial guess in an optimization formulation to be the system

$$G_0 = \left[\begin{array}{c|c} -8 & -4 \\ \hline -2 & 0 \end{array}\right].$$

Now, given the system G_0, pick a descent direction for the cost function $V(t)$, for example $(\delta a, \delta b, \delta c)^{\mathsf{T}} = (7, 5, 5)^{\mathsf{T}}$, such that

$$\hat{G}(t) = \left[\begin{array}{c|c} -8 + 7t & -4 + 5t \\ \hline -2 + 5t & 0 \end{array}\right], t \in [0, 1],$$

then the value of the cost function, $V(t) = \left\|\hat{G}(t) - G_{\text{true}}\right\|^2_{\mathcal{H}_2}$, along the descent direction is non-convex, see Figure 4.2.

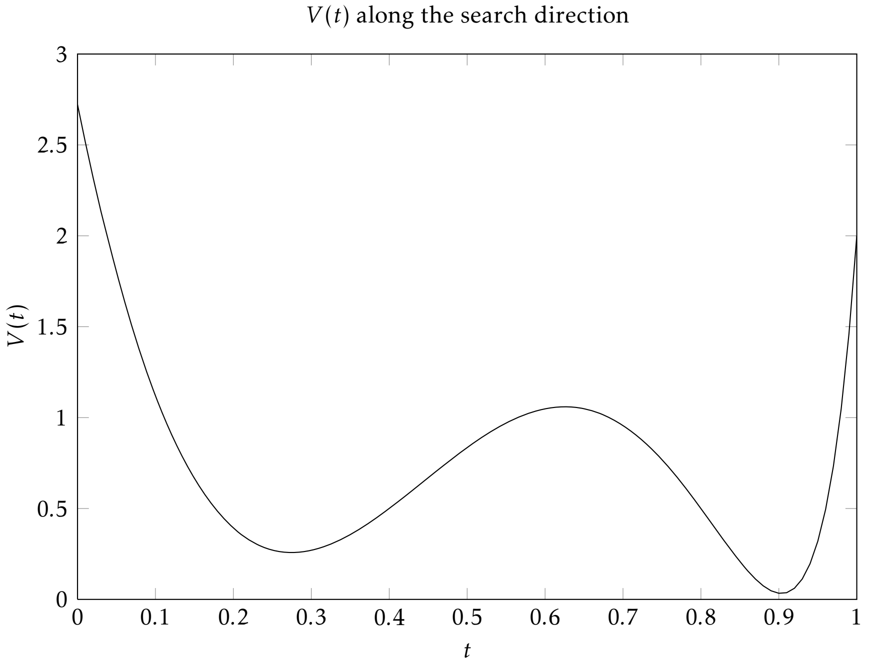

Figure 4.2: *The value of the cost function along the search direction described in Example 4.1. The function clearly demonstrates the presence of local minimas along the search direction.*

The gradient-flow algorithms use the gradients of $\left\|G - \hat{G}\right\|_{\mathcal{H}_2}$ with respect to the state-space matrices, derived in Wilson [1974] and let these evolve in time to find a local approximation of the given system, see for example Yan and Lam [1999], Fulcheri and Olivi [1998] and Huang et al. [2001]. The different algorithms in this class use different techniques to assure that the reduced model is stable, to speed up the process and to guarantee convergence.

The interpolation-based $\mathcal{H}_2$ model-reduction techniques tries to find a model whose transfer function interpolates the transfer function of the full-order system (and its derivative) at selected interpolation points. These methods often use computationally effective Krylov-based algorithms which makes these techniques suitable for large-scale problems. Examples of these algorithms are Xu and Zeng [2011], Beattie and Gugercin [2007] and Poussot-Vassal [2011].

4.4 Model Reduction using an $\mathcal{H}_2$-Measure

In this section, the proposed methods for model reduction are presented. We consider the following description for the model-reduction problem. Given a system G, search for the system $\hat{G}$ such that

$$\hat{G} = \arg\min_{\hat{G}} \left\| W_o \left(G - \hat{G} \right) W_i \right\|^2_{\mathcal{H}_2,\omega}. \tag{4.6}$$

It is assumed that the systems G and $\hat{G}$ have the state-space realizations

$$G = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array} \right], \quad \hat{G} = \left[\begin{array}{c|c} \hat{\mathbf{A}} & \hat{\mathbf{B}} \\ \hline \hat{\mathbf{C}} & \hat{\mathbf{D}} \end{array} \right], \tag{4.7}$$

where

$$\begin{aligned} &\mathbf{A} \in \mathbb{R}^{n \times n}, \quad \mathbf{B} \in \mathbb{R}^{n \times m}, \quad \mathbf{C} \in \mathbb{R}^{p \times n}, \quad \mathbf{D} \in \mathbb{R}^{p \times m}, \\ &\hat{\mathbf{A}} \in \mathbb{R}^{\hat{n} \times \hat{n}}, \quad \hat{\mathbf{B}} \in \mathbb{R}^{\hat{n} \times m}, \quad \hat{\mathbf{C}} \in \mathbb{R}^{p \times \hat{n}}, \quad \hat{\mathbf{D}} \in \mathbb{R}^{p \times m}. \end{aligned} \tag{4.8}$$

Since the $\mathcal{H}_2$-norm is used, it is also assumed that the system that is to be reduced, G, is asymptotically stable. Since, otherwise, the $\mathcal{H}_2$-norm is not defined.

The idea with the proposed methods is to try an approach that tries to tackle the model-reduction problem head on. In Helmersson [1994] the model reduction problem (in $\mathcal{H}_\infty$-norm) is rewritten as an SDP problem with BMIs, which, even for small models, leads to large optimization problems that are hard to solve. In Anić et al. [2013] they rewrite the model-reduction problem to an interpolation problem which makes it hard to incorporate structure in the system matrices. The proposed technique to solve the model-reduction problem is instead to use a nonlinear optimization approach and simply use a quasi-Newton algorithm. Using this technique, the problem is not rewritten in any other format, which makes it possible to both use and incorporate structure in the system matrices. Additionally, by taking caution when differentiating the different cost functions, and using the structure, the computational complexity can be kept low (in general an overhead cost of $\mathcal{O}(n^3)$ and $\mathcal{O}(n^2\hat{n} + n\hat{n}^2)$ per iteration).

4.4.1 Standard Model Reduction

The method presented in this section was proposed already in Wilson [1970] for continuous time, however as a special case. The derivation in this section will include weighting filters and also the discrete-time case. In this thesis, a different derivation will be used, compared to Wilson [1970], with focus on being computationally efficient and also laying a foundation for the methods to come in the following sections.

The objective is to minimize the error between the given model, G, and the sought reduced-order model, $\hat{G}$, in the $\mathcal{H}_2$-norm with weighting filters, W_i and W_o, i.e.,

$$\hat{G} = \arg\min_{\hat{G}} \ \|E\|^2_{\mathcal{H}_2}, \ E = W_o \left(G - \hat{G}\right) W_i, \tag{4.9}$$

where it is assumed that W_i and W_o are given by the user and have the realizations

$$W_i = \left[\begin{array}{c|c} \mathbf{A}_i & \mathbf{B}_i \\ \hline \mathbf{C}_i & \mathbf{D}_i \end{array}\right], \quad W_o = \left[\begin{array}{c|c} \mathbf{A}_o & \mathbf{B}_o \\ \hline \mathbf{C}_o & \mathbf{D}_o \end{array}\right], \tag{4.10}$$

where

$$\begin{array}{llll} \mathbf{A}_i \in \mathbb{R}^{n_i \times n_i}, & \mathbf{B}_i \in \mathbb{R}^{n_i \times m}, & \mathbf{C}_i \in \mathbb{R}^{p \times n_i}, & \mathbf{D}_i \in \mathbb{R}^{p \times m}, \\ \mathbf{A}_o \in \mathbb{R}^{n_o \times n_o}, & \mathbf{B}_o \in \mathbb{R}^{n_o \times m}, & \mathbf{C}_o \in \mathbb{R}^{p \times n_o}, & \mathbf{D}_o \in \mathbb{R}^{p \times m}. \end{array} \tag{4.11}$$

Using the realizations of G, $\hat{G}$, W_i and W_o, E can be realized as

$$E = \left[\begin{array}{c|c} \mathbf{A}_E & \mathbf{B}_E \\ \hline \mathbf{C}_E & \mathbf{D}_E \end{array}\right] = \left[\begin{array}{c|c} \begin{pmatrix} \mathbf{A} & \mathbf{0} & \mathbf{B}\mathbf{C}_i & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{A}} & \hat{\mathbf{B}}\mathbf{C}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_i & \mathbf{0} \\ \mathbf{B}_o\mathbf{C} & -\mathbf{B}_o\hat{\mathbf{C}} & \mathbf{0} & \mathbf{A}_o \end{pmatrix} & \begin{pmatrix} \mathbf{B}\mathbf{D}_i \\ \hat{\mathbf{B}}\mathbf{D}_i \\ \mathbf{B}_i \\ \mathbf{0} \end{pmatrix} \\ \hline \begin{pmatrix} \mathbf{D}_o\mathbf{C} & -\mathbf{D}_o\hat{\mathbf{C}} & \mathbf{0} & \mathbf{C}_o \end{pmatrix} & \mathbf{D}_o\left(\mathbf{D} - \hat{\mathbf{D}}\right)\mathbf{D}_i \end{array}\right]. \tag{4.12}$$

To be able to use the structure in the realization of E, a partitioning of the Gramians, $\mathbf{P}_E$ and $\mathbf{Q}_E$, is introduced

$$\mathbf{P}_E = \begin{pmatrix} \mathbf{P} & \mathbf{P}_{12} & \mathbf{P}_{13} & \mathbf{P}_{14} \\ \mathbf{P}_{12}^\mathsf{T} & \hat{\mathbf{P}} & \mathbf{P}_{23} & \mathbf{P}_{24} \\ \mathbf{P}_{13}^\mathsf{T} & \mathbf{P}_{23}^\mathsf{T} & \mathbf{P}_i & \mathbf{P}_{34} \\ \mathbf{P}_{14}^\mathsf{T} & \mathbf{P}_{24}^\mathsf{T} & \mathbf{P}_{34}^\mathsf{T} & \mathbf{P}_o \end{pmatrix}, \quad \mathbf{Q}_E = \begin{pmatrix} \mathbf{Q} & \mathbf{Q}_{12} & \mathbf{Q}_{13} & \mathbf{Q}_{14} \\ \mathbf{Q}_{12}^\mathsf{T} & \hat{\mathbf{Q}} & \mathbf{Q}_{23} & \mathbf{Q}_{24} \\ \mathbf{Q}_{13}^\mathsf{T} & \mathbf{Q}_{23}^\mathsf{T} & \mathbf{Q}_i & \mathbf{Q}_{34} \\ \mathbf{Q}_{14}^\mathsf{T} & \mathbf{Q}_{24}^\mathsf{T} & \mathbf{Q}_{34}^\mathsf{T} & \mathbf{Q}_o \end{pmatrix}. \tag{4.13}$$

Since there will be some differences between the continuous and the discrete-time cases, both cases will be presented. However, due to many similarities between the two, the continuous-time case will be presented in more detail than the discrete-time case.

Continuous Time

In the continuous-time case, it is assumed that the system is strictly proper, otherwise the $\mathcal{H}_2$-norm will be unbounded, i.e., $\mathbf{D}_o\left(\mathbf{D} - \hat{\mathbf{D}}\right)\mathbf{D}_i = \mathbf{0}$. Assuming this, the cost function in (4.9) can be written as, see Section 2.1.3,

$$\|E\|^2_{\mathcal{H}_2} = \operatorname{tr} \mathbf{B}_E^\mathsf{T}\mathbf{Q}_E\mathbf{B}_E \tag{4.14a}$$

$$= \operatorname{tr} \mathbf{C}_E\mathbf{P}_E\mathbf{C}_E^\mathsf{T}, \tag{4.14b}$$

which are two equivalent ways of computing the cost function, where $\mathbf{P}_E$ and $\mathbf{Q}_E$ are the controllability and observability Gramians respectively, for the error system E, satisfying the equations

$$\mathbf{A}_E\mathbf{P}_E + \mathbf{P}_E\mathbf{A}_E^{\mathsf{T}} + \mathbf{B}_E\mathbf{B}_E^{\mathsf{T}} = \mathbf{0}, \tag{4.15a}$$

$$\mathbf{A}_E^{\mathsf{T}}\mathbf{Q}_E + \mathbf{Q}_E\mathbf{A}_E + \mathbf{C}_E^{\mathsf{T}}\mathbf{C}_E = \mathbf{0}. \tag{4.15b}$$

Using (4.14) and (4.15) it is possible to state the general necessary conditions for optimality, in which the gradients of the problem readily can be extracted to be used in a quasi-Newton algorithm. In order to be as general as possible, we first neglect the structure in (4.12).

Theorem 4.2 (Necessary conditions for optimality). *Assume that $G, \hat{G}, W_i$ and W_o are asymptotically stable and that E is strictly proper, for the $\mathcal{H}_2$-norm to be defined, i.e., $\mathbf{A}, \hat{\mathbf{A}}, \mathbf{A}_i$ and $\mathbf{A}_o$ are Hurwitz and $\mathbf{D}_o\left(\mathbf{D} - \hat{\mathbf{D}}\right)\mathbf{D}_i = \mathbf{0}$. In order for the matrices $\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}$ to be optimal for the problem (4.9), it is necessary that they satisfy the equations in (4.15) and that*

$$\frac{\partial \|E\|_{\mathcal{H}_2}^2}{\partial \hat{\mathbf{A}}} = 2\hat{\mathbf{E}}^{\mathsf{T}}\mathbf{Q}_E\mathbf{P}_E\hat{\mathbf{E}} = \mathbf{0}, \tag{4.16a}$$

$$\frac{\partial \|E\|_{\mathcal{H}_2}^2}{\partial \hat{\mathbf{B}}} = 2\hat{\mathbf{E}}^{\mathsf{T}}\left(\mathbf{Q}_E\mathbf{P}_E\mathbf{E}_i\mathbf{C}_i^{\mathsf{T}} + \mathbf{Q}_E\mathbf{B}_E\mathbf{D}_i\right) = \mathbf{0}, \tag{4.16b}$$

$$\frac{\partial \|E\|_{\mathcal{H}_2}^2}{\partial \hat{\mathbf{C}}} = -2\left(\mathbf{B}_o^{\mathsf{T}}\mathbf{E}_o^{\mathsf{T}}\mathbf{Q}_E\mathbf{P}_E + \mathbf{D}_o^{\mathsf{T}}\mathbf{C}_E\mathbf{P}_E\right)\hat{\mathbf{E}} = \mathbf{0}, \tag{4.16c}$$

where

$$\hat{\mathbf{E}} = \begin{pmatrix} \mathbf{0}_{n\times\hat{n}} \\ \mathbb{I}_{\hat{n}\times\hat{n}} \\ \mathbf{0}_{n_i\times\hat{n}} \\ \mathbf{0}_{n_o\times\hat{n}} \end{pmatrix}, \quad \mathbf{E}_i = \begin{pmatrix} \mathbf{0}_{n\times\hat{n}} \\ \mathbf{0}_{\hat{n}\times\hat{n}} \\ \mathbb{I}_{n_i\times\hat{n}} \\ \mathbf{0}_{n_o\times\hat{n}} \end{pmatrix}, \quad \mathbf{E}_o = \begin{pmatrix} \mathbf{0}_{n\times\hat{n}} \\ \mathbf{0}_{\hat{n}\times\hat{n}} \\ \mathbf{0}_{n_i\times\hat{n}} \\ \mathbb{I}_{n_o\times\hat{n}} \end{pmatrix}. \tag{4.17}$$

Before proving the theorem above, two lemmas are needed to simplify the proof.

Lemma 4.1. *If* $\mathbf{M}$ *and* $\mathbf{N}$ *satisfy the Sylvester equations*

$$\mathbf{AM} + \mathbf{MB} + \mathbf{C} = \mathbf{0}, \quad \mathbf{NA} + \mathbf{BN} + \mathbf{D} = \mathbf{0},$$

then $\operatorname{tr}\mathbf{CN} = \operatorname{tr}\mathbf{DM}$.

Proof of Lemma 4.1: Multiplying the first Sylvester equation from the left with $\mathbf{N}$ and the second from the right with $\mathbf{M}$, entails

$$\mathbf{NAM} + \mathbf{NMB} + \mathbf{NC} = \mathbf{0}, \quad \mathbf{NAM} + \mathbf{BNM} + \mathbf{DM} = \mathbf{0}.$$

Now taking the trace of both equations yields

$$-\operatorname{tr}(\mathbf{NAM} + \mathbf{NMB}) = \operatorname{tr}\mathbf{CN}, \quad -\operatorname{tr}(\mathbf{NAM} + \mathbf{NMB}) = \operatorname{tr}\mathbf{DM}.$$

Hence, it holds that $\operatorname{tr}\mathbf{CN} = \operatorname{tr}\mathbf{DM}$. □

Lemma 4.2. *If* $\mathbf{A} \in \mathbb{R}^{n\times p}$, $\mathbf{B} \in \mathbb{R}^{m\times n}$ *and* $\mathbf{C} \in \mathbb{R}^{p\times m}$ *and* $a_{ij} = [\mathbf{A}]_{ij}$*, then it holds that*

$$\operatorname{tr}\left(\mathbf{B}\frac{\partial \mathbf{A}}{\partial a_{ij}}\mathbf{C}\right) = \left[\mathbf{B}^\top\mathbf{C}^\top\right]_{ij}\ \forall i, j \quad \textit{or equivalently} \quad \frac{\partial}{\partial \mathbf{A}}(\operatorname{tr}\mathbf{BAC}) = \mathbf{B}^\top\mathbf{C}^\top.$$

Proof of Lemma 4.2: First note that $\frac{\partial \mathbf{A}}{\partial a_{ij}} = \mathbf{e}_i\mathbf{e}_j^\top$, which is a matrix with a one in element (i, j) and zeros elsewhere. Now, it holds that

$$\operatorname{tr}\left(\mathbf{B}\frac{\partial \mathbf{A}}{\partial a_{ij}}\mathbf{C}\right) = \operatorname{tr}\left(\mathbf{B}\mathbf{e}_i\mathbf{e}_j^\top\mathbf{C}\right) = \operatorname{tr}\left(\mathbf{e}_j^\top\mathbf{C}\mathbf{B}\mathbf{e}_i\right) = \mathbf{e}_j^\top\mathbf{C}\mathbf{B}\mathbf{e}_i = [\mathbf{CB}]_{ji} = \left[\mathbf{B}^\top\mathbf{C}^\top\right]_{ij}.$$

□

Now, continuing with the proof for Theorem 4.2.

Proof of Theorem 4.2: If $\mathbf{A}, \hat{\mathbf{A}}, \mathbf{A}_i$ and $\mathbf{A}_o$ are Hurwitz, then all the equations in (4.15) are uniquely solvable. The solutions to the equations in (4.15) are needed to compute the cost function and its gradient. Now, the gradient of the cost function with respect to $\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}$ have to be computed. Let a_{ij}, b_{ij} and c_{ij} denote element (i, j) in $\hat{\mathbf{A}}, \hat{\mathbf{B}}$ and $\hat{\mathbf{C}}$ respectively, now differentiating (4.14) with respect to a_{ij}, b_{ij} and c_{ij} entails

$$\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial a_{ij}} = \operatorname{tr}\frac{\partial \mathbf{Q}_E}{\partial a_{ij}}\mathbf{B}_E\mathbf{B}_E^\top, \tag{4.18a}$$

$$\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial b_{ij}} = \operatorname{tr}\left(2\frac{\partial \mathbf{B}_E^\top}{\partial b_{ij}}\mathbf{Q}_E\mathbf{B}_E + \frac{\partial \mathbf{Q}_E}{\partial b_{ij}}\mathbf{B}_E\mathbf{B}_E^\top\right), \tag{4.18b}$$

$$\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial c_{ij}} = \operatorname{tr}\left(2\frac{\partial \mathbf{C}_E^\top}{\partial c_{ij}}\mathbf{C}_E\mathbf{P}_E + \frac{\partial \mathbf{P}_E}{\partial c_{ij}}\mathbf{C}_E^\top\mathbf{C}_E\right). \tag{4.18c}$$

Differentiate (4.15) with respect to a_{ij}, b_{ij} and c_{ij},

$$\mathbf{A}_E^\top\frac{\partial \mathbf{Q}_E}{\partial a_{ij}} + \frac{\partial \mathbf{Q}_E}{\partial a_{ij}}\mathbf{A}_E + \frac{\partial \mathbf{A}_E^\top}{\partial a_{ij}}\mathbf{Q}_E + \mathbf{Q}_E\frac{\partial \mathbf{A}_E}{\partial a_{ij}}, \tag{4.19a}$$

$$\mathbf{A}_E^\top\frac{\partial \mathbf{Q}_E}{\partial b_{ij}} + \frac{\partial \mathbf{Q}_E}{\partial b_{ij}}\mathbf{A}_E + \frac{\partial \mathbf{A}_E^\top}{\partial b_{ij}}\mathbf{Q}_E + \mathbf{Q}_E\frac{\partial \mathbf{A}_E}{\partial b_{ij}}, \tag{4.19b}$$

$$\mathbf{A}_E\frac{\partial \mathbf{P}_E}{\partial c_{ij}} + \frac{\partial \mathbf{P}_E}{\partial c_{ij}}\mathbf{A}_E^\top + \frac{\partial \mathbf{A}_E}{\partial c_{ij}}\mathbf{P}_E + \mathbf{P}_E\frac{\partial \mathbf{A}_E^\top}{\partial c_{ij}}. \tag{4.19c}$$

Using Lemma 4.1 with (4.18) and (4.19) yields

$$\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial a_{ij}} = 2\,\mathrm{tr}\,\frac{\partial \mathbf{A}_E^\top}{\partial a_{ij}}\mathbf{Q}_E\mathbf{P}_E, \tag{4.20a}$$

$$\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial b_{ij}} = 2\,\mathrm{tr}\left(\frac{\partial \mathbf{A}_E^\top}{\partial b_{ij}}\mathbf{Q}_E\mathbf{P}_E + \frac{\partial \mathbf{B}_E^\top}{\partial b_{ij}}\mathbf{Q}_E\mathbf{B}_E\right), \tag{4.20b}$$

$$\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial c_{ij}} = 2\,\mathrm{tr}\left(\frac{\partial \mathbf{A}_E}{\partial c_{ij}}\mathbf{Q}_E\mathbf{P}_E + \frac{\partial \mathbf{C}_E^\top}{\partial c_{ij}}\mathbf{C}_E\mathbf{P}_E\right). \tag{4.20c}$$

Using the structure in the realization of E, (4.12), and Lemma 4.2, entails

$$\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \hat{\mathbf{A}}} = 2\hat{\mathbf{E}}^\top\mathbf{Q}_E\mathbf{P}_E\hat{\mathbf{E}} = \mathbf{0}, \tag{4.21a}$$

$$\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \hat{\mathbf{B}}} = 2\hat{\mathbf{E}}^\top\left(\mathbf{Q}_E\mathbf{P}_E\mathbf{E}_i\mathbf{C}_i^\top + \mathbf{Q}_E\mathbf{B}_E\mathbf{D}_i\right) = \mathbf{0}, \tag{4.21b}$$

$$\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \hat{\mathbf{C}}} = -2\left(\mathbf{B}_o^\top\mathbf{E}_o^\top\mathbf{Q}_E\mathbf{P}_E + \mathbf{D}_o^\top\mathbf{C}_E\mathbf{P}_E\right)\hat{\mathbf{E}} = \mathbf{0}, \tag{4.21c}$$

where

$$\hat{\mathbf{E}} = \begin{pmatrix}\mathbf{0}_{n\times\hat{n}}\\ \mathbb{I}_{\hat{n}\times\hat{n}}\\ \mathbf{0}_{n_i\times\hat{n}}\\ \mathbf{0}_{n_o\times\hat{n}}\end{pmatrix},\ \mathbf{E}_i = \begin{pmatrix}\mathbf{0}_{n\times n_i}\\ \mathbf{0}_{\hat{n}\times n_i}\\ \mathbb{I}_{n_i\times n_i}\\ \mathbf{0}_{n_o\times n_i}\end{pmatrix},\ \mathbf{E}_o = \begin{pmatrix}\mathbf{0}_{n\times n_o}\\ \mathbf{0}_{\hat{n}\times n_o}\\ \mathbf{0}_{n_i\times n_o}\\ \mathbb{I}_{n_o\times n_o}\end{pmatrix}. \tag{4.22}$$

□

At a first glance, it can seem restrictive to have a technique that operates on system matrices, since one is given a model in a specific realization. Does this influence the realization of the resulting model or in other ways restrict the sought model? As can be seen in Theorem 4.3 below, this is not the case since the optimization problem becomes invariant to the realization of the given model to be reduced.

Theorem 4.3. *The cost function in the optimization problem (4.6) and its gradient, given in Theorem 4.2, are invariant under state transformations of the systems G, W_i and W_o.*

Proof: Given the realizations of G, W_i and W_o in (4.7) and (4.10). The realizations of the transformed systems, given the transformations matrices $\mathbf{T}$,$\mathbf{T}_i$ and $\mathbf{T}_o$, become

$$G = \left[\begin{array}{c|c}\bar{\mathbf{A}} & \bar{\mathbf{B}}\\ \hline \bar{\mathbf{C}} & \bar{\mathbf{D}}\end{array}\right] = \left[\begin{array}{c|c}\mathbf{T}^{-1}\mathbf{A}\mathbf{T} & \mathbf{T}^{-1}\mathbf{B}\\ \hline \mathbf{C}\mathbf{T} & \mathbf{D}\end{array}\right],$$

$$W_i = \left[\begin{array}{c|c} \bar{\mathbf{A}}_i & \bar{\mathbf{B}}_i \\ \hline \bar{\mathbf{C}}_i & \bar{\mathbf{D}}_i \end{array}\right] = \left[\begin{array}{c|c} \mathbf{T}_i^{-1}\mathbf{A}_i\mathbf{T}_i & \mathbf{T}_i^{-1}\mathbf{B}_i \\ \hline \mathbf{C}_i\mathbf{T}_i & \mathbf{D}_i \end{array}\right],$$
$$W_o = \left[\begin{array}{c|c} \bar{\mathbf{A}}_o & \bar{\mathbf{B}}_o \\ \hline \bar{\mathbf{C}}_o & \bar{\mathbf{D}}_o \end{array}\right] = \left[\begin{array}{c|c} \mathbf{T}_o^{-1}\mathbf{A}_o\mathbf{T}_o & \mathbf{T}_o^{-1}\mathbf{B}_o \\ \hline \mathbf{C}_o\mathbf{T}_o & \mathbf{D}_o \end{array}\right].$$

This can be written as

$$E = \left[\begin{array}{c|c} \bar{\mathbf{A}}_E & \bar{\mathbf{B}}_E \\ \hline \bar{\mathbf{C}}_E & \bar{\mathbf{D}}_E \end{array}\right] = \left[\begin{array}{c|c} \mathbf{T}_E^{-1}\mathbf{A}_E\mathbf{T}_E & \mathbf{T}_E^{-1}\mathbf{B}_E \\ \hline \mathbf{C}_E\mathbf{T}_E & \mathbf{D}_E \end{array}\right], \ \mathbf{T}_E = \begin{pmatrix} \mathbf{T} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbb{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{T}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{T}_o \end{pmatrix}. \tag{4.23}$$

The matrices $\mathbf{P}_E$ and $\mathbf{Q}_E$ will be transformed as

$$\mathbf{P}_E = \mathbf{T}_E\bar{\mathbf{P}}_E\mathbf{T}_E^{\mathsf{T}}, \ \mathbf{Q}_E = \mathbf{T}_E^{-\mathsf{T}}\bar{\mathbf{Q}}_E\mathbf{T}_E^{-1}. \tag{4.24}$$

Now it is easy to see that the cost function (4.14) is invariant under the transformation $\mathbf{T}_E$, since

$$\|E\|_{\mathcal{H}_2}^2 = \operatorname{tr} \mathbf{B}_E^{\mathsf{T}}\mathbf{Q}_E\mathbf{B}_E = \operatorname{tr} \bar{\mathbf{B}}_E^{\mathsf{T}}\mathbf{T}_E^{\mathsf{T}}\mathbf{T}_E^{-\mathsf{T}}\bar{\mathbf{Q}}_E\mathbf{T}_E^{-1}\mathbf{T}_E\bar{\mathbf{B}}_E = \operatorname{tr} \bar{\mathbf{B}}_E^{\mathsf{T}}\bar{\mathbf{Q}}_E\bar{\mathbf{B}}_E. \tag{4.25}$$

Before continuing with the gradient, the matrix products $\hat{\mathbf{E}}^{\mathsf{T}}\mathbf{T}_E^{-\mathsf{T}}$, $\mathbf{T}_E^{\mathsf{T}}\mathbf{E}_i$ and $\mathbf{E}_o^{\mathsf{T}}\mathbf{T}_E^{-\mathsf{T}}$ are evaluated,

$$\hat{\mathbf{E}}^{\mathsf{T}}\mathbf{T}_E^{-\mathsf{T}} = \hat{\mathbf{E}}^{\mathsf{T}}, \ \mathbf{T}_E^{\mathsf{T}}\mathbf{E}_i = \mathbf{E}_i\mathbf{T}_i^{\mathsf{T}}, \ \mathbf{E}_o^{\mathsf{T}}\mathbf{T}_E^{-\mathsf{T}} = \mathbf{T}_o^{-\mathsf{T}}\mathbf{E}_o^{\mathsf{T}}. \tag{4.26}$$

Using (4.26) when computing the gradient entails,

$$\frac{\partial \|E\|_{\mathcal{H}_2}^2}{\partial \hat{\mathbf{A}}} = 2\hat{\mathbf{E}}^{\mathsf{T}}\mathbf{Q}_E\mathbf{P}_E\hat{\mathbf{E}} = 2\hat{\mathbf{E}}^{\mathsf{T}}\mathbf{T}_E^{-\mathsf{T}}\bar{\mathbf{Q}}_E\mathbf{T}_E^{-1}\mathbf{T}_E\bar{\mathbf{P}}_E\mathbf{T}_E^{\mathsf{T}}\hat{\mathbf{E}} = 2\hat{\mathbf{E}}^{\mathsf{T}}\bar{\mathbf{Q}}_E\bar{\mathbf{P}}_E\hat{\mathbf{E}},$$

$$\begin{aligned} \frac{\partial \|E\|_{\mathcal{H}_2}^2}{\partial \hat{\mathbf{B}}} &= 2\hat{\mathbf{E}}^{\mathsf{T}}\left(\mathbf{Q}_E\mathbf{P}_E\mathbf{E}_i\mathbf{C}_i^{\mathsf{T}} + \mathbf{Q}_E\mathbf{B}_E\mathbf{D}_i\right) \\ &= 2\hat{\mathbf{E}}^{\mathsf{T}}\mathbf{T}_E^{-\mathsf{T}}\left(\mathbf{Q}_E\mathbf{T}_E^{-1}\mathbf{T}_E\bar{\mathbf{P}}_E\mathbf{T}_E^{\mathsf{T}}\mathbf{E}_i\mathbf{T}_i^{-\mathsf{T}}\bar{\mathbf{C}}_i^{\mathsf{T}} + \mathbf{Q}_E\mathbf{T}_E^{-1}\mathbf{T}_E\mathbf{B}_E\bar{\mathbf{D}}_i\right) \\ &= 2\hat{\mathbf{E}}^{\mathsf{T}}\left(\bar{\mathbf{Q}}_E\bar{\mathbf{P}}_E\mathbf{E}_i\bar{\mathbf{C}}_i^{\mathsf{T}} + \bar{\mathbf{Q}}_E\bar{\mathbf{B}}_E\bar{\mathbf{D}}_i\right), \end{aligned}$$

$$
\begin{aligned}
\frac{\partial \|E\|_{\mathcal{H}_2}^2}{\partial \hat{\mathbf{C}}} &= -2\left(\mathbf{B}_o^\mathsf{T}\mathbf{E}_o^\mathsf{T}\mathbf{Q}_E\mathbf{P}_E + \mathbf{D}_o^\mathsf{T}\mathbf{C}_E\mathbf{P}_E\right)\hat{\mathbf{E}} \\
&= -2\left(\bar{\mathbf{B}}_o^\mathsf{T}\mathbf{T}_o^\mathsf{T}\mathbf{E}_o^\mathsf{T}\mathbf{T}_E^{-\mathsf{T}}\bar{\mathbf{Q}}_E\mathbf{T}_E^{-1}\mathbf{T}_E\bar{\mathbf{P}}_E + \bar{\mathbf{D}}_o^\mathsf{T}\bar{\mathbf{C}}_E\mathbf{T}_E^{-1}\mathbf{T}_E\bar{\mathbf{P}}_E\right)\mathbf{T}_E^\mathsf{T}\hat{\mathbf{E}} \\
&= -2\left(\bar{\mathbf{B}}_o^\mathsf{T}\mathbf{E}_o^\mathsf{T}\bar{\mathbf{Q}}_E\bar{\mathbf{P}}_E + \bar{\mathbf{D}}_o^\mathsf{T}\bar{\mathbf{C}}_E\bar{\mathbf{P}}_E\right)\hat{\mathbf{E}}.
\end{aligned}
$$

□

Looking at the special case when not having any weighting filters, i.e., $W_i = \mathbb{I}$ and $W_o = \mathbb{I}$, $n_i = n_o = 0$, yields the cost function

$$\|E\|_{\mathcal{H}_2}^2 = \operatorname{tr}\left(\mathbf{B}^\mathsf{T}\mathbf{Q}\mathbf{B} + 2\mathbf{B}^\mathsf{T}\mathbf{Q}_{12}\hat{\mathbf{B}} + \hat{\mathbf{B}}^\mathsf{T}\hat{\mathbf{Q}}\hat{\mathbf{B}}\right), \tag{4.27a}$$

$$\|E\|_{\mathcal{H}_2}^2 = \operatorname{tr}\left(\mathbf{C}\mathbf{P}\mathbf{C}^\mathsf{T} - 2\mathbf{C}\mathbf{P}_{12}\hat{\mathbf{C}}^\mathsf{T} + \hat{\mathbf{C}}\hat{\mathbf{P}}\hat{\mathbf{C}}^\mathsf{T}\right), \tag{4.27b}$$

and the first-order conditions for the gradient simplify to

$$\frac{\partial \|E\|_{\mathcal{H}_2}^2}{\partial \hat{\mathbf{A}}} = 2\left(\hat{\mathbf{Q}}\hat{\mathbf{P}} + \mathbf{Q}_{12}^\mathsf{T}\mathbf{P}_{12}\right) = \mathbf{0}, \tag{4.28a}$$

$$\frac{\partial \|E\|_{\mathcal{H}_2}^2}{\partial \hat{\mathbf{B}}} = 2\left(\hat{\mathbf{Q}}\hat{\mathbf{B}} + \mathbf{Q}_{12}^\mathsf{T}\mathbf{B}\right) = \mathbf{0}, \tag{4.28b}$$

$$\frac{\partial \|E\|_{\mathcal{H}_2}^2}{\partial \hat{\mathbf{C}}} = 2\left(\hat{\mathbf{C}}\hat{\mathbf{P}} - \mathbf{C}\mathbf{P}_{12}\right) = \mathbf{0}. \tag{4.28c}$$

$\mathbf{P}, \mathbf{Q}, \hat{\mathbf{P}}, \hat{\mathbf{Q}}, \mathbf{P}_{12}$ and $\mathbf{Q}_{12}$ satisfy the equations

$$\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^\mathsf{T} + \mathbf{B}\mathbf{B}^\mathsf{T} = \mathbf{0}, \tag{4.29a}$$

$$\mathbf{A}\mathbf{P}_{12} + \mathbf{P}_{12}\hat{\mathbf{A}}^\mathsf{T} + \mathbf{B}\hat{\mathbf{B}}^\mathsf{T} = \mathbf{0}, \tag{4.29b}$$

$$\hat{\mathbf{A}}\hat{\mathbf{P}} + \hat{\mathbf{P}}\hat{\mathbf{A}}^\mathsf{T} + \hat{\mathbf{B}}\hat{\mathbf{B}}^\mathsf{T} = \mathbf{0}, \tag{4.29c}$$

$$\mathbf{A}^\mathsf{T}\mathbf{Q} + \mathbf{Q}\mathbf{A} + \mathbf{C}^\mathsf{T}\mathbf{C} = \mathbf{0}, \tag{4.29d}$$

$$\mathbf{A}^\mathsf{T}\mathbf{Q}_{12} + \mathbf{Q}_{12}\hat{\mathbf{A}} - \mathbf{C}^\mathsf{T}\hat{\mathbf{C}} = \mathbf{0}, \tag{4.29e}$$

$$\hat{\mathbf{A}}^\mathsf{T}\hat{\mathbf{Q}} + \hat{\mathbf{Q}}\hat{\mathbf{A}} + \hat{\mathbf{C}}^\mathsf{T}\hat{\mathbf{C}} = \mathbf{0}. \tag{4.29f}$$

Note that $\mathbf{P}$ and $\mathbf{Q}$ satisfy the Lyapunov equations for the controllability and observability Gramians for the given system, G, and $\hat{\mathbf{P}}$ and $\hat{\mathbf{Q}}$ satisfy the Lyapunov equations for the controllability and observability Gramians for the sought system, $\hat{G}$.

For this special case it is also quite straightforward to derive the Hessian for the cost function. Using differentiated (with respect to a_{ij}, b_{ij}, c_{ij}) versions of the equations in (4.29) and using Lemma 4.1 and Lemma 4.2, yields

$$\frac{\partial^2 V}{\partial a_{ij}\partial a_{kl}} = 2\left(\left[\hat{\mathbf{Q}}\frac{\partial \hat{\mathbf{P}}}{\partial a_{ij}}\right]_{kl} + \left[\hat{\mathbf{Q}}\frac{\partial \hat{\mathbf{P}}}{\partial a_{kl}}\right]_{ij} + \left[\mathbf{Q}_{12}^{\top}\frac{\partial \mathbf{P}_{12}}{\partial a_{ij}}\right]_{kl} + \left[\mathbf{Q}_{12}^{\top}\frac{\partial \mathbf{P}_{12}}{\partial a_{kl}}\right]_{ij}\right), \tag{4.30a}$$

$$\frac{\partial^2 V}{\partial b_{ij}\partial b_{kl}} = \begin{cases} 2\left[\hat{\mathbf{Q}}\right]_{ik}, & l = j \\ 0, & l \neq j \end{cases}, \tag{4.30b}$$

$$\frac{\partial^2 V}{\partial c_{ij}\partial c_{kl}} = \begin{cases} 2\left[\hat{\mathbf{P}}\right]_{lj}, & i = k \\ 0, & i \neq k \end{cases}, \tag{4.30c}$$

$$\frac{\partial^2 V}{\partial a_{ij}\partial b_{kl}} = 2\left[\hat{\mathbf{Q}}\frac{\partial \hat{\mathbf{P}}}{\partial b_{kl}}\right]_{ij} + 2\left[\mathbf{Q}_{12}^{\top}\frac{\partial \mathbf{P}_{12}}{\partial b_{ij}}\right]_{kl}, \tag{4.30d}$$

$$\frac{\partial^2 V}{\partial c_{ij}\partial a_{kl}} = 2\left[\hat{\mathbf{C}}\frac{\partial \hat{\mathbf{P}}}{\partial a_{kl}}\right]_{ij} - 2\left[\mathbf{C}\frac{\partial \mathbf{P}_{12}}{\partial a_{kl}}\right]_{ij}, \tag{4.30e}$$

$$\frac{\partial^2 V}{\partial c_{ij}\partial b_{kl}} = 2\left[\hat{\mathbf{C}}\frac{\partial \hat{\mathbf{P}}}{\partial b_{kl}}\right]_{ij} - 2\left[\mathbf{C}\frac{\partial \mathbf{P}_{12}}{\partial b_{kl}}\right]_{ij}. \tag{4.30f}$$

The explicit equations for the cost function, the gradient and the Lyapunov equations for the case when having both input and output filters are included in Appendix 4.B.1.

Discrete Time

In the discrete-time case the cost function in (4.6) can be rewritten as, see Section 2.1.3,

$$\|E\|_{\mathcal{H}_2}^2 = \operatorname{tr} \mathbf{B}_E^{\top}\mathbf{Q}_E\mathbf{B}_E + \mathbf{D}_E^{\top}\mathbf{D}_E \tag{4.31a}$$

$$= \operatorname{tr} \mathbf{C}_E\mathbf{P}_E\mathbf{C}_E^{\top} + \mathbf{D}_E\mathbf{D}_E^{\top}, \tag{4.31b}$$

which are two equivalent ways of computing the cost function. The matrices $\mathbf{P}_E$ and $\mathbf{Q}_E$ are the controllability and observability Gramians respectively, for the error system E, and in this case they satisfy the discrete Lyapunov equations

$$\mathbf{A}_E\mathbf{P}_E\mathbf{A}_E^{\top} - \mathbf{P}_E + \mathbf{B}_E\mathbf{B}_E^{\top} = \mathbf{0}, \tag{4.32a}$$

$$\mathbf{A}_E^{\top}\mathbf{Q}_E\mathbf{A}_E - \mathbf{Q}_E + \mathbf{C}_E^{\top}\mathbf{C}_E = \mathbf{0}. \tag{4.32b}$$

Note that in the discrete-time case, the system E does not any longer have to be strictly proper, however it still has to be asymptotically stable for the $\mathcal{H}_2$-norm to be defined.

Theorem 4.4 (Necessary conditions for optimality). *Assume that G, $\hat{G}$, W_i and W_o are asymptotically stable, for the $\mathcal{H}_2$-norm to be defined, i.e., $\mathbf{A}, \hat{\mathbf{A}}, \mathbf{A}_i$ and $\mathbf{A}_o$ are Schur. In order for the matrices $\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$ to be optimal for the problem*

(4.9), it is necessary that they satisfy the equations in (4.32) and that

$$\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \hat{\mathbf{A}}} = 2\hat{\mathbf{E}}^\mathsf{T}\mathbf{Q}_E\mathbf{A}_E\mathbf{P}_E\hat{\mathbf{E}} = \mathbf{0}, \tag{4.33a}$$

$$\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \hat{\mathbf{B}}} = 2\hat{\mathbf{E}}^\mathsf{T}\left(\mathbf{Q}_E\mathbf{A}_E\mathbf{P}_E\mathbf{E}_i\mathbf{C}_i^\mathsf{T} + \mathbf{Q}_E\mathbf{B}_E\mathbf{D}_i\right) = \mathbf{0}, \tag{4.33b}$$

$$\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \hat{\mathbf{C}}} = -2\left(\mathbf{B}_o^\mathsf{T}\mathbf{E}_o^\mathsf{T}\mathbf{Q}_E\mathbf{A}_E\mathbf{P}_E + \mathbf{D}_o^\mathsf{T}\mathbf{C}_E\mathbf{P}_E\right)\hat{\mathbf{E}} = \mathbf{0}, \tag{4.33c}$$

$$\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \hat{\mathbf{D}}} = 2\mathbf{D}_o^\mathsf{T}\mathbf{D}_o\left(\hat{\mathbf{D}} - \mathbf{D}\right)\mathbf{D}_i\mathbf{D}_i^\mathsf{T} = \mathbf{0}, \tag{4.33d}$$

where

$$\hat{\mathbf{E}} = \begin{pmatrix} \mathbf{0}_{n\times\hat{n}} \\ \mathbb{I}_{\hat{n}\times\hat{n}} \\ \mathbf{0}_{n_i\times\hat{n}} \\ \mathbf{0}_{n_o\times\hat{n}} \end{pmatrix}, \quad \mathbf{E}_i = \begin{pmatrix} \mathbf{0}_{n\times\hat{n}} \\ \mathbf{0}_{\hat{n}\times\hat{n}} \\ \mathbb{I}_{n_i\times\hat{n}} \\ \mathbf{0}_{n_o\times\hat{n}} \end{pmatrix}, \quad \mathbf{E}_o = \begin{pmatrix} \mathbf{0}_{n\times\hat{n}} \\ \mathbf{0}_{\hat{n}\times\hat{n}} \\ \mathbf{0}_{n_i\times\hat{n}} \\ \mathbb{I}_{n_o\times\hat{n}} \end{pmatrix}. \tag{4.34}$$

Proof: The proof is analogous to the proof of Theorem 4.2 for the continuous time case. □

Theorem 4.5. *The cost function of the optimization problem (4.6) and its gradient, given in Theorem 4.4, are invariant under state transformations of the systems G, W_i and W_o.*

Proof: The proof is analogous with the proof for Theorem 4.3. □

Now looking at the special case when not having any weighting filters, i.e., $W_i = \mathbb{I}$ and $W_o = \mathbb{I}$, $n_i = n_o = 0$, yields the cost function

$$\|E\|^2_{\mathcal{H}_2} = \operatorname{tr}\left(\mathbf{B}^\mathsf{T}\mathbf{Q}\mathbf{B} + 2\hat{\mathbf{B}}^\mathsf{T}\mathbf{Q}_{12}^\mathsf{T}\mathbf{B} + \hat{\mathbf{B}}^\mathsf{T}\hat{\mathbf{Q}}\hat{\mathbf{B}} + \mathbf{D}^\mathsf{T}\mathbf{D} - 2\hat{\mathbf{D}}^\mathsf{T}\mathbf{D} + \hat{\mathbf{D}}^\mathsf{T}\hat{\mathbf{D}}\right), \tag{4.35a}$$

$$\|E\|^2_{\mathcal{H}_2} = \operatorname{tr}\left(\mathbf{C}\mathbf{P}\mathbf{C}^\mathsf{T} - 2\hat{\mathbf{C}}\mathbf{P}_{12}^\mathsf{T}\mathbf{C}^\mathsf{T} + \hat{\mathbf{C}}\hat{\mathbf{P}}\hat{\mathbf{C}}^\mathsf{T} + \mathbf{D}\mathbf{D}^\mathsf{T} - 2\mathbf{D}\hat{\mathbf{D}}^\mathsf{T} + \hat{\mathbf{D}}\hat{\mathbf{D}}^\mathsf{T}\right), \tag{4.35b}$$

and the first-order conditions for the gradient simplify to

$$\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \hat{\mathbf{A}}} = 2\left(\hat{\mathbf{Q}}\hat{\mathbf{A}}\hat{\mathbf{P}} + \mathbf{Q}_{12}^\mathsf{T}\mathbf{A}\mathbf{P}_{12}\right) = \mathbf{0}, \tag{4.36a}$$

$$\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \hat{\mathbf{B}}} = 2\left(\hat{\mathbf{Q}}\hat{\mathbf{B}} + \mathbf{Q}_{12}^\mathsf{T}\mathbf{B}\right) = \mathbf{0}, \tag{4.36b}$$

$$\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \hat{\mathbf{C}}} = 2\left(\hat{\mathbf{C}}\hat{\mathbf{P}} - \mathbf{C}\mathbf{P}_{12}\right) = \mathbf{0}, \tag{4.36c}$$

$$\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \hat{\mathbf{D}}} = 2\left(\hat{\mathbf{D}} - \mathbf{D}\right) = \mathbf{0}, \tag{4.36d}$$

where $\mathbf{P}, \mathbf{Q}, \hat{\mathbf{P}}, \hat{\mathbf{Q}}, \mathbf{P}_{12}$ and $\mathbf{Q}_{12}$ satisfy the equations

$$\mathbf{A}\mathbf{P}\mathbf{A}^{\mathsf{T}} - \mathbf{P} + \mathbf{B}\mathbf{B}^{\mathsf{T}} = \mathbf{0}, \tag{4.37a}$$
$$\mathbf{A}\mathbf{P}_{12}\hat{\mathbf{A}}^{\mathsf{T}} - \mathbf{P}_{12} + \mathbf{B}\hat{\mathbf{B}}^{\mathsf{T}} = \mathbf{0}, \tag{4.37b}$$
$$\hat{\mathbf{A}}\hat{\mathbf{P}}\hat{\mathbf{A}}^{\mathsf{T}} - \hat{\mathbf{P}} + \hat{\mathbf{B}}\hat{\mathbf{B}}^{\mathsf{T}} = \mathbf{0}, \tag{4.37c}$$
$$\mathbf{A}^{\mathsf{T}}\mathbf{Q}\mathbf{A} - \mathbf{Q} + \mathbf{C}^{\mathsf{T}}\mathbf{C} = \mathbf{0}, \tag{4.37d}$$
$$\hat{\mathbf{A}}^{\mathsf{T}}\mathbf{Q}_{12}^{\mathsf{T}}\mathbf{A} - \mathbf{Q}_{12}^{\mathsf{T}} - \hat{\mathbf{C}}^{\mathsf{T}}\mathbf{C} = \mathbf{0}, \tag{4.37e}$$
$$\hat{\mathbf{A}}^{\mathsf{T}}\hat{\mathbf{Q}}\hat{\mathbf{A}} - \hat{\mathbf{Q}} + \hat{\mathbf{C}}^{\mathsf{T}}\hat{\mathbf{C}} = \mathbf{0}. \tag{4.37f}$$

Note that $\mathbf{P}$ and $\mathbf{Q}$ satisfy the Lyapunov equations for the controllability and observability Gramians for the given system, G, and $\hat{\mathbf{P}}$ and $\hat{\mathbf{Q}}$ satisfy the Lyapunov equations for the controllability and observability Gramians for the sought system, $\hat{G}$. For this special case, in discrete time, it is also quite straightforward to derive the Hessian for the cost function. Using differentiated (with respect to a_{ij}, b_{ij}, c_{ij}) versions of the equations in (4.37) and using Lemma 4.1 and Lemma 4.2, entails

$$\begin{aligned}\frac{\partial^2 V}{\partial a_{ij}\partial a_{kl}} =& 2\left[\mathbf{Q}_{12}^{\mathsf{T}}\mathbf{A}\frac{\partial \mathbf{P}_{12}}{\partial a_{ij}}\right]_{kl} + 2\left[\mathbf{Q}_{12}^{\mathsf{T}}\mathbf{A}\frac{\partial \mathbf{P}_{12}}{\partial a_{kl}}\right]_{ij} + 2\left[\hat{\mathbf{Q}}\right]_{ik}\left[\hat{\mathbf{P}}\right]_{lj} \\ &+ 2\left[\hat{\mathbf{Q}}\hat{\mathbf{A}}\frac{\partial \hat{\mathbf{P}}}{\partial a_{ij}}\right]_{kl} + 2\left[\hat{\mathbf{Q}}\hat{\mathbf{A}}\frac{\partial \hat{\mathbf{P}}}{\partial a_{kl}}\right]_{ij}.\end{aligned} \tag{4.38a}$$

$$\frac{\partial^2 V}{\partial b_{ij}\partial b_{kl}} = \begin{cases} 2\left[\hat{\mathbf{Q}}\right]_{ik}, & l = j \\ 0, & l \neq j \end{cases}, \tag{4.38b}$$

$$\frac{\partial^2 V}{\partial c_{ij}\partial c_{kl}} = \begin{cases} 2\left[\hat{\mathbf{P}}\right]_{lj}, & i = k \\ 0, & i \neq k \end{cases}, \tag{4.38c}$$

$$\frac{\partial^2 V}{\partial d_{ij}\partial d_{kl}} = \begin{cases} 2, & i = k,\ j = l \\ 0, & \text{otherwise} \end{cases}, \tag{4.38d}$$

$$\frac{\partial^2 V}{\partial a_{ij}\partial b_{kl}} = 2\left[\hat{\mathbf{Q}}\hat{\mathbf{A}}\frac{\partial \hat{\mathbf{P}}}{\partial b_{kl}}\right]_{ij} + 2\left[\mathbf{Q}_{12}^{\mathsf{T}}\mathbf{A}\frac{\partial \mathbf{P}_{12}}{\partial b_{kl}}\right]_{ij}, \tag{4.38e}$$

$$\frac{\partial^2 V}{\partial c_{ij}\partial a_{kl}} = 2\left[\hat{\mathbf{C}}\frac{\partial \hat{\mathbf{P}}}{\partial a_{kl}}\right]_{ij} - 2\left[\mathbf{C}\frac{\partial \mathbf{P}_{12}}{\partial a_{kl}}\right]_{ij}, \tag{4.38f}$$

$$\frac{\partial^2 V}{\partial c_{ij}\partial b_{kl}} = 2\left[\hat{\mathbf{C}}\frac{\partial \hat{\mathbf{P}}}{\partial b_{kl}}\right]_{ij} - 2\left[\mathbf{C}\frac{\partial \mathbf{P}_{12}}{\partial b_{kl}}\right]_{ij}, \tag{4.38g}$$

$$\frac{\partial^2 V}{\partial a_{ij}\partial d_{kl}} = \frac{\partial^2 V}{\partial b_{ij}\partial d_{kl}} = \frac{\partial^2 V}{\partial c_{ij}\partial d_{kl}} = 0. \tag{4.38h}$$

The explicit equations for the cost function, the gradient and the Lyapunov equations for the case when having both input and output filters are included in Appendix 4.B.2.

4.4.2 Robust Model Reduction

In the previous section, it has been tacitly assumed that the given data, (i.e., the state-space matrices) are exact. In a more realistic setting, the presence of errors (e.g., modeling, truncation or round-off) in these data can be assumed. The question is how to cope with these errors and take them into account. This can for example be done using *robust optimization*. However, this is a very difficult problem, see, e.g., Bertsimas et al. [2011] or Ben-Tal and Nemirovski [2002]. In this section, a different view of robust optimization is investigated, that is to use *regularization* as a proxy for robust optimization, which can be seen as a worst-case optimization approach.

Before presenting the equations for the regularized model-reduction problem, the idea is first presented by using a more general description to get an intuition for the idea. The idea is then exemplified using a least-squares (LS) problem and a quadratic programming (QP) problem.

Regularization can be used to make ill-posed problems well posed or to make a solution less sensitive when having small amount of data. Commonly used regularization methods are for example ℓ_1- and ℓ_2-regularization, for least-squares problems referred to, in the ℓ_1-case as LASSO and in the ℓ_2-case Tikhonov regularization or ridge regression, see e.g., Hastie et al. [2001]. In these regularizations an extra term, $V_{\text{rob}}(\mathbf{x})$, is added to the cost function, $V_{\text{original}}(\mathbf{x})$, to penalize the ℓ_1- or ℓ_2-norm of the sought variables, i.e.,

$$V_{\text{reg}}(\mathbf{x}) = V_{\text{original}}(\mathbf{x}) + \lambda V_{\text{rob}}(\mathbf{x}). \tag{4.39}$$

The regularization parameter, here denoted λ, is seen as a design parameter and is in most cases hard to tune (see for example Bauer and Lukas [2011]).

In many applications, there is no *a priori* knowledge about the variables, e.g., that they should be small (typically achieved by ℓ_2-regularization) or that the solution should be sparse (typically achieved using ℓ_1-regularization). Instead, one would like to make the solution less sensitive to uncertainties. As mentioned above, in this section, regularization will be used as a proxy for robust optimization. The idea is to penalize the first-order derivative (with respect to *data*) of the cost function to make it less sensitive to uncertainties in data. This can be interpreted as doing a first-order approximation of the general robust optimization problem

$$\underset{\mathbf{x}}{\text{minimize}} \max_{\|\Lambda\|_2 \le \lambda} V(\mathbf{x}, \hat{\mathbf{y}}), \quad \hat{\mathbf{y}} \triangleq \mathbf{y} + \Lambda, \tag{4.40}$$

where $\hat{\mathbf{y}} \in \mathbb{R}^m$ is the given data, $\mathbf{y} \in \mathbb{R}^m$ is the unperturbed data, $\Lambda \in \mathbb{R}^m$ represents the uncertainty in the data and $\mathbf{x} \in \mathbb{R}^n$ is the sought variable. To see how a regularization can be an approximation of the robust optimization problem, a Taylor expansion of the cost function with respect to the data is made. Assuming that the cost function is differentiable in the data variables, the cost function can be expressed as

$$\begin{aligned} V(\mathbf{x},\hat{\mathbf{y}}) &= V(\mathbf{x},\mathbf{y}) + (\hat{\mathbf{y}}-\mathbf{y})^{\mathsf{T}}\nabla_{\hat{\mathbf{y}}} V(\mathbf{x},\mathbf{y}) + \mathcal{O}(\|\hat{\mathbf{y}}-\mathbf{y}\|_2^2) \\ &= V(\mathbf{x},\mathbf{y}) + \Lambda^{\mathsf{T}} f_{\hat{\mathbf{y}}}(\mathbf{x},\mathbf{y}) + \mathcal{O}(\|\Lambda\|^2). \end{aligned} \tag{4.41}$$

Limiting the uncertainty to be bounded, i.e., $\|\Lambda\|_2 \leq \lambda$, and computing the maximum of (4.41), yields

$$\begin{aligned} \max_{\|\Lambda\|_2\leq\lambda} V(\mathbf{x},\hat{\mathbf{y}}) &= \max_{\|\Lambda\|_2\leq\lambda} V(\mathbf{x},\mathbf{y}) + \Lambda^{\mathsf{T}}\nabla_{\hat{\mathbf{y}}} V(\mathbf{x},\mathbf{y}) + \mathcal{O}(\|\Lambda\|_2^2) \\ &= V(\mathbf{x},\mathbf{y}) + \lambda \left\|\nabla_{\hat{\mathbf{y}}} V(\mathbf{x},\mathbf{y})\right\|_2 + \mathcal{O}(\lambda^2). \end{aligned} \tag{4.42}$$

To make this more clear, some examples are presented for an LS problem and a QP problem.

Example 4.2: Robust LS and QP

Let us start with one of the most common problems, an LS problem. Assume that the *data* $\mathbf{A}$ and $\mathbf{b}$ are given and a solution $\mathbf{x}$, fulfilling

$$\mathbf{x} \triangleq \arg\min_{\mathbf{x}} V(\mathbf{x},\mathbf{A},\mathbf{b}) = \arg\min_{\mathbf{x}} (\mathbf{A}\mathbf{x}-\mathbf{b})^{\mathsf{T}}(\mathbf{A}\mathbf{x}-\mathbf{b}), \tag{4.43}$$

is sought. To see how, for example, the $\mathbf{A}$-matrix influence the cost function, the cost function is differentiated with respect to $\mathbf{A}$, i.e.

$$\frac{\partial V(\mathbf{x},\mathbf{A},\mathbf{b})}{\partial a_{ij}} = 2\,\mathrm{tr}\left(\mathbf{e}_j\mathbf{e}_i^{\mathsf{T}}\left[\mathbf{A}\mathbf{x}-\mathbf{b}\right]\mathbf{x}^{\mathsf{T}}\right), \tag{4.44}$$

where a_{ij} is the (i,j) element in $\mathbf{A}$. This yields

$$\frac{\partial V(\mathbf{x},\mathbf{A},\mathbf{b})}{\mathbf{A}} = 2\,(\mathbf{A}\mathbf{x}-\mathbf{b})\,\mathbf{x}^{\mathsf{T}}. \tag{4.45}$$

Hence,

$$\left\|\frac{\partial V(\mathbf{x},\mathbf{A},\mathbf{b})}{\mathbf{A}}\right\|_2 = 2\,\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2\,\|\mathbf{x}\|_2\,. \tag{4.46}$$

An interesting fact about the term in (4.46) is that it can be rewritten as

$$2\,\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2\,\|\mathbf{x}\|_2 = 2\frac{\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2}{\|\mathbf{x}\|_2}\,\|\mathbf{x}\|_2^2 = \mu(\mathbf{x})\,\|\mathbf{x}\|_2^2\,,$$

where $\mu(\mathbf{x})$ resembles Miller's choice of regularization parameter (see El Ghaoui and Lebret [1997] or Miller [1970]). In Miller [1970] the regularization parameter $\mu(\mathbf{x})$ is determined iteratively.

It is also possible to differentiate with respect to $\mathbf{b}$ in the LS problem. This term, together with the terms coming from differentiating with respect to $\mathbf{H}$ and $\mathbf{f}$ in a QP problem

$$V(\mathbf{x};\mathbf{H},\mathbf{f}) = \mathbf{x}^{\mathsf{T}}\mathbf{H}\mathbf{x} + \mathbf{f}^{\mathsf{T}}\mathbf{x}, \tag{4.47}$$

are collected in Table 4.1.

Table 4.1: The different regularization terms for the different variables in the special cases, LS *problem and* QP *problem*

Problem	Variable	Uncertainty in	Reg. term
LS	$\mathbf{A}$	$\lVert\Lambda\rVert_F \leq \lambda$	$\lambda \lVert\mathbf{x}\rVert_2 \lVert\mathbf{Ax}-\mathbf{b}\rVert_2 = \lambda \frac{\lVert\mathbf{Ax}-\mathbf{b}\rVert_2}{\lVert\mathbf{x}\rVert_2} \lVert\mathbf{x}\rVert_2^2$
LS	$\mathbf{b}$	$\lVert\Lambda\rVert_2 \leq \lambda$	$\lambda \lVert\mathbf{Ax}\rVert_2$
QP	$\mathbf{H}$ (not sym.)	$\lVert\Lambda\rVert_F \leq \lambda$	$\frac{1}{2}\lambda \lVert\mathbf{x}\rVert_2^2$
QP	$\mathbf{H}$ (sym.)	$\lVert\Lambda\rVert_F \leq \lambda$	$\lambda \lVert\mathbf{x}\rVert_2^2 \sqrt{1 - \frac{3}{4}\frac{\mathrm{tr}(\mathbf{xx}^\mathsf{T} \odot \mathbf{xx}^\mathsf{T})}{\lVert\mathbf{x}\rVert_2^4}}$
QP	$\mathbf{f}$	$\lVert\Lambda\rVert_2 \leq \lambda$	$\lambda \lVert\mathbf{x}\rVert_2$

Now, the regularization strategy explained above will be used as an extension to the special case of the model-reduction method in Section 4.4.1, having no weighting filters. To reduce the influence of errors in data, the unregularized cost function (4.27) is regularized by adding three new terms. These are the Frobenius norms of the derivatives of the cost function with respect to *the given data*, $\mathbf{A}$, $\mathbf{B}$,$\mathbf{C}$ and $\mathbf{D}$, i.e., the solution obtained is inclined to be less sensitive to uncertainties in *the data*.

The optimization problem with these new terms becomes

$$\min_{\hat{\mathbf{A}},\hat{\mathbf{B}},\hat{\mathbf{C}},\hat{\mathbf{D}}} \lVert E\rVert_{\mathcal{H}_2}^2 + V_{\mathrm{rob}}, \quad E = G - \hat{G}, \tag{4.48}$$

where

$$V_{\mathrm{rob}} = \epsilon_{\mathbf{A}} \left\lVert \frac{\partial \lVert E\rVert_{\mathcal{H}_2}^2}{\partial \mathbf{A}} \right\rVert_F + \epsilon_{\mathbf{B}} \left\lVert \frac{\partial \lVert E\rVert_{\mathcal{H}_2}^2}{\partial \mathbf{B}} \right\rVert_F + \epsilon_{\mathbf{C}} \left\lVert \frac{\partial \lVert E\rVert_{\mathcal{H}_2}^2}{\partial \mathbf{C}} \right\rVert_F + \epsilon_{\mathbf{D}} \left\lVert \frac{\partial \lVert E\rVert_{\mathcal{H}_2}^2}{\partial \mathbf{D}} \right\rVert_F. \tag{4.49}$$

Note that here the term V_{rob} includes the regularization parameters, $\epsilon_{\mathbf{A}}$, $\epsilon_{\mathbf{B}}$, $\epsilon_{\mathbf{C}}$ and $\epsilon_{\mathbf{D}}$.

V_{rob} becomes different in the continuous-time case and the discrete-time case. By exploiting the symmetry in (4.27), (4.28) and (4.29) with respect to $(\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}})$ and $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ we obtain, in continuous time that V_{rob} is

$$V_{\mathrm{rob}} = 2\left(\epsilon_{\mathbf{A}} \left\lVert \mathbf{QP} + \mathbf{Q}_{12}\mathbf{P}_{12}^\mathsf{T} \right\rVert_F + \epsilon_{\mathbf{B}} \left\lVert \mathbf{QB} + \mathbf{Q}_{12}\hat{\mathbf{B}} \right\rVert_F + \epsilon_{\mathbf{C}} \left\lVert \mathbf{CP} - \hat{\mathbf{C}}\mathbf{P}_{12}^\mathsf{T} \right\rVert_F \right), \tag{4.50}$$

and in the discrete-time cases it becomes

$$\begin{aligned} V_{\mathrm{rob}} = 2\Big(&\epsilon_{\mathbf{A}} \left\lVert \mathbf{QAP} + \mathbf{Q}_{12}\hat{\mathbf{A}}\mathbf{P}_{12}^\mathsf{T} \right\rVert_F + \epsilon_{\mathbf{B}} \left\lVert \mathbf{QB} + \mathbf{Q}_{12}\hat{\mathbf{B}} \right\rVert_F \\ &+ \epsilon_{\mathbf{C}} \left\lVert \mathbf{CP} - \hat{\mathbf{C}}\mathbf{P}_{12}^\mathsf{T} \right\rVert_F + \epsilon_{\mathbf{D}} \left\lVert \mathbf{D} - \hat{\mathbf{D}} \right\rVert_F \Big). \end{aligned} \tag{4.51}$$

By differentiating the cost function (4.48) it is possible to state the necessary conditions for optimality, both for the continuous-time case and the discrete-time case.

Theorem 4.6 (Necessary conditions for optimality in continuous time). *Assume that G and $\hat{G}$ are asymptotically stable and that E is strictly proper, for the $\mathcal{H}_2$-norm to be defined. In order for the matrices $\hat{\mathbf{A}}, \hat{\mathbf{B}}$ and $\hat{\mathbf{C}}$ to be optimal for (4.48), in continuous time, it is necessary that they satisfy the equations in* (4.29) *and the equations*

$$\hat{\mathbf{A}}^\mathsf{T}\mathbf{W}_1 + \mathbf{W}_1\mathbf{A} + \mathbf{Q}_{12}^\mathsf{T}\left(\mathbf{Q}\mathbf{P} + \mathbf{Q}_{12}\mathbf{P}_{12}^\mathsf{T}\right) = \mathbf{0}, \tag{4.52a}$$

$$\mathbf{A}\mathbf{W}_2 + \mathbf{W}_2\hat{\mathbf{A}}^\mathsf{T} + \left(\mathbf{Q}\mathbf{P} + \mathbf{Q}_{12}\mathbf{P}_{12}^\mathsf{T}\right)\mathbf{P}_{12} = \mathbf{0}, \tag{4.52b}$$

$$\mathbf{A}\mathbf{W}_3 + \mathbf{W}_3\hat{\mathbf{A}}^\mathsf{T} + \left(\mathbf{Q}\mathbf{B} + \mathbf{Q}_{12}\hat{\mathbf{B}}\right)\hat{\mathbf{B}}^\mathsf{T} = \mathbf{0}, \tag{4.52c}$$

$$\hat{\mathbf{A}}^\mathsf{T}\mathbf{W}_4 + \mathbf{W}_4\mathbf{A} + \hat{\mathbf{C}}^\mathsf{T}\left(\hat{\mathbf{C}}\mathbf{P}_{12}^\mathsf{T} - \mathbf{C}\mathbf{P}\right) = \mathbf{0}, \tag{4.52d}$$

and that

$$\frac{\partial \|E\|_{\mathcal{H}_2}^2}{\partial \hat{\mathbf{A}}} + \frac{\partial V_{rob}}{\partial \hat{\mathbf{A}}} = \mathbf{0}, \tag{4.53a}$$

$$\frac{\partial \|E\|_{\mathcal{H}_2}^2}{\partial \hat{\mathbf{B}}} + \frac{\partial V_{rob}}{\partial \hat{\mathbf{B}}} = \mathbf{0}, \tag{4.53b}$$

$$\frac{\partial \|E\|_{\mathcal{H}_2}^2}{\partial \hat{\mathbf{C}}} + \frac{\partial V_{rob}}{\partial \hat{\mathbf{C}}} = \mathbf{0}. \tag{4.53c}$$

With

$$\begin{aligned}
\frac{\partial V_{rob}}{\partial \hat{\mathbf{A}}} =& 4\epsilon_\mathbf{A}\frac{\mathbf{W}_1\mathbf{P}_{12} + \mathbf{Q}_{12}^\mathsf{T}\mathbf{W}_2}{\left\|\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial\mathbf{A}}\right\|_F} + 4\epsilon_\mathbf{B}\frac{\mathbf{Q}_{12}^\mathsf{T}\mathbf{W}_3}{\left\|\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial\mathbf{B}}\right\|_F} + 4\epsilon_\mathbf{C}\frac{\mathbf{W}_4\mathbf{P}_{12}}{\left\|\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial\mathbf{C}}\right\|_F}, \\
\frac{\partial V_{rob}}{\partial \hat{\mathbf{B}}} =& 4\epsilon_\mathbf{A}\frac{\mathbf{W}_1\mathbf{B}}{\left\|\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial\mathbf{A}}\right\|_F} + 4\epsilon_\mathbf{B}\frac{\mathbf{Q}_{12}^\mathsf{T}\left(\mathbf{Q}\mathbf{B} + \mathbf{Q}_{12}\hat{\mathbf{B}}\right)}{\left\|\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial\mathbf{B}}\right\|_F} + 4\epsilon_\mathbf{C}\frac{\mathbf{W}_4\mathbf{B}}{\left\|\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial\mathbf{C}}\right\|_F}, \\
\frac{\partial V_{rob}}{\partial \hat{\mathbf{C}}} =& -4\epsilon_\mathbf{A}\frac{\mathbf{C}\mathbf{W}_2}{\left\|\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial\mathbf{A}}\right\|_F} - 4\epsilon_\mathbf{B}\frac{\mathbf{C}\mathbf{W}_3}{\left\|\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial\mathbf{B}}\right\|_F} - 4\epsilon_\mathbf{C}\frac{\left(\mathbf{C}\mathbf{P} - \hat{\mathbf{C}}\mathbf{P}_{12}^\mathsf{T}\right)\mathbf{P}_{12}}{\left\|\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial\mathbf{C}}\right\|_F},
\end{aligned}$$

and $\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial\hat{\mathbf{A}}}$, $\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial\hat{\mathbf{B}}}$ *and* $\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial\hat{\mathbf{C}}}$ *as in* (4.28).

Proof: If G and $\hat{G}$ are asymptotically stable, the equations in (4.29) and (4.52) are uniquely solvable. The solutions to the equations in (4.29) and (4.52) are needed to compute the cost function and its gradient. Now the gradient of the cost function with respect to $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}$ has to be computed. The first part of the gradient $\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial\hat{\mathbf{A}}}$, $\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial\hat{\mathbf{B}}}$ and $\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial\hat{\mathbf{C}}}$ has been computed in Theorem 4.2 and can be found in (4.28). Only the equations for the gradient of the V_{rob}-part is left to be calculated, since this part enters as an additive term in the cost function. The calculations of this part of the gradient are moved to Appendix 4.A. □

An analogous result can be stated in discrete time.

Theorem 4.7 (Necessary conditions for optimality in discrete time). *Assume that G and $\hat{G}$ are asymptotically stable, for the $\mathcal{H}_2$-norm to be defined. In order for the matrices $\hat{\mathbf{A}}, \hat{\mathbf{B}}$, $\hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$ to be optimal for (4.48), in discrete time, it is necessary that they satisfy the equations in* (4.37) *and the equations*

$$\hat{\mathbf{A}}^{\mathsf{T}}\mathbf{W}_1\mathbf{A} - \mathbf{W}_1 + \hat{\mathbf{A}}^{\mathsf{T}}\mathbf{Q}_{12}^{\mathsf{T}}\left(\mathbf{QAP} + \mathbf{Q}_{12}\hat{\mathbf{A}}\mathbf{P}_{12}^{\mathsf{T}}\right) = \mathbf{0}, \tag{4.55a}$$

$$\mathbf{A}\mathbf{W}_2\hat{\mathbf{A}}^{\mathsf{T}} - \mathbf{W}_2 + \left(\mathbf{QAP} + \mathbf{Q}_{12}\hat{\mathbf{A}}\mathbf{P}_{12}^{\mathsf{T}}\right)\mathbf{P}_{12}\hat{\mathbf{A}}^{\mathsf{T}} = \mathbf{0}, \tag{4.55b}$$

$$\mathbf{A}\mathbf{W}_3\hat{\mathbf{A}}^{\mathsf{T}} - \mathbf{W}_3 + \left(\mathbf{QB} + \mathbf{Q}_{12}\hat{\mathbf{B}}\right)\hat{\mathbf{B}}^{\mathsf{T}} = \mathbf{0}, \tag{4.55c}$$

$$\hat{\mathbf{A}}^{\mathsf{T}}\mathbf{W}_4\mathbf{A} - \mathbf{W}_4 + \hat{\mathbf{C}}^{\mathsf{T}}\left(\hat{\mathbf{C}}\mathbf{P}_{12}^{\mathsf{T}} - \mathbf{CP}\right) = \mathbf{0}, \tag{4.55d}$$

$$\mathbf{Q}_{12}^{\mathsf{T}}\left(\mathbf{Q}_{12}\hat{\mathbf{A}}\mathbf{P}_{12}^{\mathsf{T}} + \mathbf{QAP}\right)\mathbf{P}_{12} = \mathbf{W}_5, \tag{4.55e}$$

and that

$$\frac{\partial \|E\|_{\mathcal{H}_2}^2}{\partial \hat{\mathbf{A}}} + \frac{\partial V_{rob}}{\partial \hat{\mathbf{A}}} = \mathbf{0}, \tag{4.56a}$$

$$\frac{\partial \|E\|_{\mathcal{H}_2}^2}{\partial \hat{\mathbf{B}}} + \frac{\partial V_{rob}}{\partial \hat{\mathbf{B}}} = \mathbf{0}, \tag{4.56b}$$

$$\frac{\partial \|E\|_{\mathcal{H}_2}^2}{\partial \hat{\mathbf{C}}} + \frac{\partial V_{rob}}{\partial \hat{\mathbf{C}}} = \mathbf{0}. \tag{4.56c}$$

With

$$\begin{aligned}
\frac{\partial V_{rob}}{\partial \hat{\mathbf{A}}} =& 4\epsilon_{\mathbf{A}} \frac{\mathbf{W}_5 + \mathbf{W}_1\mathbf{A}\mathbf{P}_{12} + \mathbf{Q}_{12}^{\mathsf{T}}\mathbf{A}\mathbf{W}_2}{\left\|\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial \mathbf{A}}\right\|_F} + 4\epsilon_{\mathbf{B}} \frac{\mathbf{Q}_{12}^{\mathsf{T}}\mathbf{A}\mathbf{W}_3}{\left\|\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial \mathbf{B}}\right\|_F} + 4\epsilon_{\mathbf{C}} \frac{\mathbf{W}_4\mathbf{A}\mathbf{P}_{12}}{\left\|\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial \mathbf{C}}\right\|_F}, \\
\frac{\partial V_{rob}}{\partial \hat{\mathbf{B}}} =& 4\epsilon_{\mathbf{A}} \frac{\mathbf{W}_1\mathbf{B}}{\left\|\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial \mathbf{A}}\right\|_F} + 4\epsilon_{\mathbf{B}} \frac{\mathbf{Q}_{12}^{\mathsf{T}}\left(\mathbf{QB} + \mathbf{Q}_{12}\hat{\mathbf{B}}\right)}{\left\|\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial \mathbf{B}}\right\|_F} + 4\epsilon_{\mathbf{C}} \frac{\mathbf{W}_4\mathbf{B}}{\left\|\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial \mathbf{C}}\right\|_F}, \\
\frac{\partial V_{rob}}{\partial \hat{\mathbf{C}}} =& -4\epsilon_{\mathbf{A}} \frac{\mathbf{C}\mathbf{W}_2}{\left\|\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial \mathbf{A}}\right\|_F} - 4\epsilon_{\mathbf{B}} \frac{\mathbf{C}\mathbf{W}_3}{\left\|\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial \mathbf{B}}\right\|_F} - 4\epsilon_{\mathbf{C}} \frac{\left(\mathbf{CP} - \hat{\mathbf{C}}\mathbf{P}_{12}^{\mathsf{T}}\right)\mathbf{P}_{12}}{\left\|\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial \mathbf{C}}\right\|_F}, \\
\frac{\partial V_{rob}}{\partial \hat{\mathbf{D}}} =& 4\epsilon_{\mathbf{D}} \frac{\hat{\mathbf{D}} - \mathbf{D}}{\left\|\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial \mathbf{D}}\right\|_F},
\end{aligned}$$

and $\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial \hat{\mathbf{A}}}$, $\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial \hat{\mathbf{B}}}$ *and* $\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial \hat{\mathbf{C}}}$ *as in* (4.36).

Proof: The proof is analogous with the one for Theorem 4.6. □

4.4.3 Frequency-Limited Model Reduction

The method proposed in this section is a new method that was introduced in Petersson and Löfberg [2012a]. The method relies heavily on the theory in Chapter 3. The variants of this method for continuous and discrete time are similar and, therefore, the continuous-time case will be presented in full detail and we will not provide as much detail for the discrete-time case.

The method proposed in this section is a model-reduction method that given a model G, finds a reduced order model $\hat{G}$, which is a good approximation of G on a chosen frequency interval, e.g., $[0, \omega]$. The objective is to minimize the discrepancy between the given model and the sought reduced-order model in a frequency-limited $\mathcal{H}_2$-norm, using the frequency-limited Gramians. Correspondingly, the optimization problem for this purpose is as follows

$$\hat{G} = \arg\min_{\hat{G}} \; \|E\|^2_{\mathcal{H}_2,\omega}, \; E = G - \hat{G}, \tag{4.58}$$

where $\|E\|^2_{\mathcal{H}_2,\omega}$ is defined in Chapter 3.

Given the realization in (4.7), the error system can be realized, in state-space form, as

$$E : \left[\begin{array}{c|c} \mathbf{A}_E & \mathbf{B}_E \\ \hline \mathbf{C}_E & \mathbf{D}_E \end{array}\right] = \left[\begin{array}{c|c} \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{A}} \end{pmatrix} & \begin{pmatrix} \mathbf{B} \\ \hat{\mathbf{B}} \end{pmatrix} \\ \hline \begin{pmatrix} \mathbf{C} & -\hat{\mathbf{C}} \end{pmatrix} & \mathbf{D} - \hat{\mathbf{D}} \end{array}\right]. \tag{4.59}$$

Continuous Time

In the continuous-time case, the cost function of the optimization problem in (4.58) can be rewritten as, see Section 3.2.1

$$\begin{aligned} \|E\|^2_{\mathcal{H}_2,\omega} &= \operatorname{tr} \mathbf{C}_E \mathbf{P}_{E,\omega} \mathbf{C}_E^{\mathsf{T}} + 2 \operatorname{tr} \left[\left(\mathbf{C}_E \mathbf{S}_{E,\omega} \mathbf{B}_E + \mathbf{D}_E \frac{\omega}{2\pi} \right) \mathbf{D}_E^{\mathsf{T}} \right] & \text{(4.60a)} \\ &= \operatorname{tr} \mathbf{B}_E^{\mathsf{T}} \mathbf{Q}_{E,\omega} \mathbf{B}_E + 2 \operatorname{tr} \left[\left(\mathbf{C}_E \mathbf{S}_{E,\omega} \mathbf{B}_E + \mathbf{D}_E \frac{\omega}{2\pi} \right) \mathbf{D}_E^{\mathsf{T}} \right]. & \text{(4.60b)} \end{aligned}$$

where

$$\begin{aligned} \mathbf{A}_E \mathbf{P}_{E,\omega} + \mathbf{P}_{E,\omega} \mathbf{A}_E^{\mathsf{T}} + \mathbf{S}_{E,\omega} \mathbf{B}_E \mathbf{B}_E^{\mathsf{T}} + \mathbf{B}_E \mathbf{B}_E^{\mathsf{T}} \mathbf{S}_{E,\omega}^* &= \mathbf{0}, & \text{(4.61a)} \\ \mathbf{A}_E^{\mathsf{T}} \mathbf{Q}_{E,\omega} + \mathbf{Q}_{E,\omega} \mathbf{A}_E + \mathbf{S}_{E,\omega}^* \mathbf{C}_E^{\mathsf{T}} \mathbf{C}_E + \mathbf{C}_E^{\mathsf{T}} \mathbf{C}_E \mathbf{S}_{E,\omega} &= \mathbf{0}, & \text{(4.61b)} \end{aligned}$$

with

$$\mathbf{S}_{E,\omega} = \operatorname{Re} \left[\frac{i}{2\pi} \ln \left(-\mathbf{A}_E - i\omega \mathbb{I} \right) \right]. \tag{4.62}$$

Now, the cost function (4.60) can be rewritten using the inherent structure in the problem. This is done by using the realization given in (4.59) and by partitioning the Gramians $\mathbf{P}_{E,\omega}$ and $\mathbf{Q}_{E,\omega}$ as

$$\mathbf{P}_{E,\omega} = \begin{pmatrix} \mathbf{P}_\omega & \mathbf{P}_{12,\omega} \\ \mathbf{P}_{12,\omega}^\mathsf{T} & \hat{\mathbf{P}}_\omega \end{pmatrix}, \quad \mathbf{Q}_{E,\omega} = \begin{pmatrix} \mathbf{Q}_\omega & \mathbf{Q}_{12,\omega} \\ \mathbf{Q}_{12,\omega}^\mathsf{T} & \hat{\mathbf{Q}}_\omega \end{pmatrix}, \tag{4.63}$$

and $\mathbf{S}_{E,\omega}$ as

$$\mathbf{S}_{E,\omega} = \begin{pmatrix} \mathbf{S}_\omega & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{S}}_\omega \end{pmatrix}. \tag{4.64}$$

$\mathbf{P}_\omega, \mathbf{Q}_\omega, \hat{\mathbf{P}}_\omega, \hat{\mathbf{Q}}_\omega, \mathbf{P}_{12,\omega}$ and $\mathbf{Q}_{12,\omega}$ satisfy, by (4.61), the Sylvester and Lyapunov equations

$$\mathbf{A}\mathbf{P}_\omega + \mathbf{P}_\omega\mathbf{A}^\mathsf{T} + \mathbf{S}_\omega\mathbf{B}\mathbf{B}^\mathsf{T} + \mathbf{B}\mathbf{B}^\mathsf{T}\mathbf{S}_\omega^* = \mathbf{0}, \tag{4.65a}$$

$$\mathbf{A}\mathbf{P}_{12,\omega} + \mathbf{P}_{12,\omega}\hat{\mathbf{A}}^\mathsf{T} + \mathbf{S}_\omega\mathbf{B}\hat{\mathbf{B}}^\mathsf{T} + \mathbf{B}\hat{\mathbf{B}}^\mathsf{T}\hat{\mathbf{S}}_\omega^* = \mathbf{0}, \tag{4.65b}$$

$$\hat{\mathbf{A}}\hat{\mathbf{P}}_\omega + \hat{\mathbf{P}}_\omega\hat{\mathbf{A}}^\mathsf{T} + \hat{\mathbf{S}}_\omega\hat{\mathbf{B}}\hat{\mathbf{B}}^\mathsf{T} + \hat{\mathbf{B}}\hat{\mathbf{B}}^\mathsf{T}\hat{\mathbf{S}}_\omega^* = \mathbf{0}, \tag{4.65c}$$

$$\mathbf{A}^\mathsf{T}\mathbf{Q}_\omega + \mathbf{Q}_\omega\mathbf{A} + \mathbf{S}_\omega^*\mathbf{C}^\mathsf{T}\mathbf{C} + \mathbf{C}^\mathsf{T}\mathbf{C}\mathbf{S}_\omega = \mathbf{0}, \tag{4.65d}$$

$$\mathbf{A}^\mathsf{T}\mathbf{Q}_{12,\omega} + \mathbf{Q}_{12,\omega}\hat{\mathbf{A}} - \mathbf{S}_\omega^*\mathbf{C}^\mathsf{T}\hat{\mathbf{C}} - \mathbf{C}^\mathsf{T}\hat{\mathbf{C}}\hat{\mathbf{S}}_\omega = \mathbf{0}, \tag{4.65e}$$

$$\hat{\mathbf{A}}^\mathsf{T}\hat{\mathbf{Q}}_\omega + \hat{\mathbf{Q}}_\omega\hat{\mathbf{A}} + \hat{\mathbf{S}}_\omega^*\hat{\mathbf{C}}^\mathsf{T}\hat{\mathbf{C}} + \hat{\mathbf{C}}^\mathsf{T}\hat{\mathbf{C}}\hat{\mathbf{S}}_\omega = \mathbf{0}, \tag{4.65f}$$

with

$$\mathbf{S}_\omega = \mathrm{Re}\left[\frac{i}{2\pi}\ln\left(-\mathbf{A} - i\omega\mathbb{I}\right)\right], \quad \hat{\mathbf{S}}_\omega = \mathrm{Re}\left[\frac{i}{2\pi}\ln\left(-\hat{\mathbf{A}} - i\omega\mathbb{I}\right)\right]. \tag{4.66}$$

Note that $\mathbf{P}_\omega$ and $\mathbf{Q}_\omega$ satisfy the Lyapunov equations for the frequency-limited controllability and observability Gramians for the given model, and $\hat{\mathbf{P}}_\omega$ and $\hat{\mathbf{Q}}_\omega$ satisfy the Lyapunov equations for the frequency-limited controllability and observability Gramians for the sought model, see Section 3.1.1.

With the partitioning of $\mathbf{P}_{E,\omega}$ and $\mathbf{Q}_{E,\omega}$, it is possible to rewrite (4.60) in two alternative forms

$$\begin{aligned} \|E\|_{\mathcal{H}_2,\omega}^2 &= \mathrm{tr}\left(\mathbf{B}^\mathsf{T}\mathbf{Q}_\omega\mathbf{B} + 2\mathbf{B}^\mathsf{T}\mathbf{Q}_{12,\omega}\hat{\mathbf{B}} + \hat{\mathbf{B}}^\mathsf{T}\hat{\mathbf{Q}}_\omega\hat{\mathbf{B}}\right) \\ &\quad + 2\,\mathrm{tr}\left[\mathbf{C}\mathbf{S}_\omega\mathbf{B} + \mathbf{D}\frac{\omega}{2\pi} - \left(\hat{\mathbf{C}}\hat{\mathbf{S}}_\omega\hat{\mathbf{B}} + \hat{\mathbf{D}}\frac{\omega}{2\pi}\right)\right]\left(\mathbf{D}^\mathsf{T} - \hat{\mathbf{D}}^\mathsf{T}\right), \end{aligned} \tag{4.67a}$$

$$\begin{aligned} \|E\|_{\mathcal{H}_2,\omega}^2 &= \mathrm{tr}\left(\mathbf{C}\mathbf{P}_\omega\mathbf{C}^\mathsf{T} - 2\mathbf{C}\mathbf{P}_{12,\omega}\hat{\mathbf{C}}^\mathsf{T} + \hat{\mathbf{C}}\hat{\mathbf{P}}_\omega\hat{\mathbf{C}}^\mathsf{T}\right) \\ &\quad + 2\,\mathrm{tr}\left[\mathbf{C}\mathbf{S}_\omega\mathbf{B} + \mathbf{D}\frac{\omega}{2\pi} - \left(\hat{\mathbf{C}}\hat{\mathbf{S}}_\omega\hat{\mathbf{B}} + \hat{\mathbf{D}}\frac{\omega}{2\pi}\right)\right]\left(\mathbf{D}^\mathsf{T} - \hat{\mathbf{D}}^\mathsf{T}\right). \end{aligned} \tag{4.67b}$$

Of course, as in Chapter 3, it is possible to have arbitrary segments in the frequency domain, e.g., $\|E\|_{\mathcal{H}_2,\Omega}^2$, $\Omega = [-\omega_4, -\omega_3] \cup [-\omega_2, -\omega_1] \cup [\omega_1, \omega_2] \cup [\omega_3, \omega_4]$, $0 < \omega_1 < \omega_2 < \omega_3 < \omega_4$. Important to note, is that if Ω does not contain an infinite interval, then neither the given system to be reduced, G, nor the reduced system, $\hat{G}$, have to be strictly proper.

An appealing feature of the proposed optimization problem (4.58), is that the corresponding cost function, (4.67), is differentiable in the system matrices, $\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}$

and $\hat{\mathbf{D}}$. In addition, the closed-form expressions obtained when differentiating the cost function is expressed in the given data ($\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$), the optimization variables ($\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$) and solutions to the equations in (4.65). This makes it possible to formulate necessary conditions for optimality for the optimization problem (4.58).

Theorem 4.8 (Necessary conditions for optimality). *Assume that G and $\hat{G}$ are asymptotically stable, for the frequency-limited $\mathcal{H}_2$-norm to be defined, i.e., $\mathbf{A}$ and $\hat{\mathbf{A}}$ are Hurwitz. In order for the matrices $\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$ to be optimal for the problem (4.58), it is necessary that they satisfy the equations in (4.65) and the equations in (4.29) and that*

$$\frac{\partial \|E\|^2_{\mathcal{H}_2,\omega}}{\partial \hat{\mathbf{A}}} = 2\left(\mathbf{Q}^{\mathsf{T}}_{12,\omega}\mathbf{P}_{12} + \hat{\mathbf{Q}}_{\omega}\hat{\mathbf{P}}\right) - 2\mathbf{W} = \mathbf{0}, \tag{4.68a}$$

$$\frac{\partial \|E\|^2_{\mathcal{H}_2,\omega}}{\partial \hat{\mathbf{B}}} = 2\left(\hat{\mathbf{Q}}_{\omega}\hat{\mathbf{B}} + \mathbf{Q}^{\mathsf{T}}_{12,\omega}\mathbf{B} - \hat{\mathbf{S}}^{\mathsf{T}}_{\omega}\hat{\mathbf{C}}^{\mathsf{T}}\left[\mathbf{D} - \hat{\mathbf{D}}\right]\right) = \mathbf{0}, \tag{4.68b}$$

$$\frac{\partial \|E\|^2_{\mathcal{H}_2,\omega}}{\partial \hat{\mathbf{C}}} = 2\left(\hat{\mathbf{C}}\hat{\mathbf{P}}_{\omega} - \mathbf{C}\mathbf{P}_{12,\omega} - \left[\mathbf{D} - \hat{\mathbf{D}}\right]\hat{\mathbf{B}}^{\mathsf{T}}\hat{\mathbf{S}}^{\mathsf{T}}_{\omega}\right) = \mathbf{0}, \tag{4.68c}$$

$$\frac{\partial \|E\|^2_{\mathcal{H}_2,\omega}}{\partial \hat{\mathbf{D}}} = -2\left(\mathbf{C}\mathbf{S}_{\omega}\mathbf{B} + \mathbf{D}\frac{\omega}{\pi} - \hat{\mathbf{C}}\hat{\mathbf{S}}_{\omega}\hat{\mathbf{B}} - \hat{\mathbf{D}}\frac{\omega}{\pi}\right) = \mathbf{0}, \tag{4.68d}$$

where

$$\mathbf{W} = \mathrm{Re}\left(\frac{i}{\pi}L\left(-\hat{\mathbf{A}} - i\omega\mathbb{I}, \mathbf{V}\right)\right)^{\mathsf{T}}, \tag{4.68e}$$

$$\mathbf{V} = \hat{\mathbf{C}}^{\mathsf{T}}\hat{\mathbf{C}}\hat{\mathbf{P}} - \hat{\mathbf{C}}^{\mathsf{T}}\mathbf{C}\mathbf{P}_{12} - \hat{\mathbf{C}}^{\mathsf{T}}\left(\mathbf{D} - \hat{\mathbf{D}}\right)\hat{\mathbf{B}}^{\mathsf{T}} \tag{4.68f}$$

with the function $L(\,\cdot\,,\,\cdot\,)$ being the Frechét derivative of the matrix logarithm, see Higham [2008].

Proof: If $\mathbf{A}$ and $\hat{\mathbf{A}}$ are Hurwitz, then the equations in (4.65) are uniquely solvable, see Theorem 2.1. These are needed to compute the cost function and its gradient. Now, the gradient of the cost function with respect to $\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$ have to be calculated. However, this is done in Appendix 4.C, since the calculations are quite long. □

As in Section 4.4.1 the optimization problem in this section also becomes invariant to the realization of the given model to be reduced, as can be seen in the following theorem.

Theorem 4.9. *The cost function in the optimization problem* (4.58) *and its gradient, given in Theorem 4.8, are invariant under state transformations of the system G.*

Proof: Given the realization of G in (4.7) and a transformations matrix $\mathbf{T}$, the realization of the transformed system becomes

$$G = \left[\begin{array}{c|c} \bar{\mathbf{A}} & \bar{\mathbf{B}} \\ \hline \bar{\mathbf{C}} & \bar{\mathbf{D}} \end{array}\right] = \left[\begin{array}{c|c} \mathbf{T}^{-1}\mathbf{A}\mathbf{T} & \mathbf{T}^{-1}\mathbf{B} \\ \hline \mathbf{C}\mathbf{T} & \mathbf{D} \end{array}\right].$$

Realizing that $\mathbf{S}_\omega = \mathbf{T}^{-1}\bar{\mathbf{S}}_\omega\mathbf{T}$, since

$$\begin{aligned} \mathbf{S}_\omega = \mathrm{Re}\left[\frac{i}{2\pi}\ln(-\mathbf{A} - i\omega\mathbb{I})\right] &= \mathrm{Re}\left[\frac{i}{2\pi}\ln\left(\mathbf{T}^{-1}\left[-\bar{\mathbf{A}} - i\omega\mathbb{I}\right]\mathbf{T}\right)\right] \\ &= \mathbf{T}^{-1}\,\mathrm{Re}\left[\frac{i}{2\pi}\ln\left(\left[-\bar{\mathbf{A}} - i\omega\mathbb{I}\right]\right)\right]\mathbf{T} = \mathbf{T}^{-1}\bar{\mathbf{S}}_\omega\mathbf{T}, \end{aligned}$$

the proof is analogous to the proof in Theorem 4.3. □

Discrete Time

In the discrete-time case, the cost function in (4.58) can be written as, see Section 3.2.2,

$$\begin{aligned} \|G\|^2_{\mathcal{H}_2,\omega} &= \mathrm{tr}\,\mathbf{C}\mathbf{P}_\omega\mathbf{C}^\mathsf{T} + \mathrm{tr}\,\hat{\mathbf{C}}\hat{\mathbf{P}}_\omega\hat{\mathbf{C}}^\mathsf{T} - 2\,\mathrm{tr}\,\mathbf{C}\mathbf{P}_{12,\omega}\hat{\mathbf{C}}^\mathsf{T} \\ &\quad + 2\,\mathrm{tr}\left(\mathbf{C}\mathbf{R}_\omega\mathbf{B} + \frac{\omega}{2}\mathbf{D} - \hat{\mathbf{C}}\hat{\mathbf{R}}_\omega\hat{\mathbf{B}} - \frac{\omega}{2}\hat{\mathbf{D}}\right)\left(\mathbf{D} - \hat{\mathbf{D}}\right)^\mathsf{T} && (4.69a) \\ &= \mathrm{tr}\,\mathbf{B}^\mathsf{T}\mathbf{Q}_\omega\mathbf{B} + 2\,\mathrm{tr}\,\mathbf{B}^\mathsf{T}\mathbf{Q}_{12,\omega}\hat{\mathbf{B}} + \mathrm{tr}\,\hat{\mathbf{B}}^\mathsf{T}\hat{\mathbf{Q}}_\omega\hat{\mathbf{B}} \\ &\quad + 2\,\mathrm{tr}\left(\mathbf{D} - \hat{\mathbf{D}}\right)^\mathsf{T}\left(\mathbf{C}\mathbf{R}_\omega\mathbf{B} + \frac{\omega}{2}\mathbf{D} - \hat{\mathbf{C}}\hat{\mathbf{R}}_\omega\hat{\mathbf{B}} - \frac{\omega}{2}\hat{\mathbf{D}}\right), && (4.69b) \end{aligned}$$

where

$$\mathbf{A}\mathbf{P}_\omega\mathbf{A}^\mathsf{T} - \mathbf{P}_\omega + \mathbf{S}_\omega\mathbf{B}\mathbf{B}^\mathsf{T} + \mathbf{B}\mathbf{B}^\mathsf{T}\mathbf{S}_\omega^\mathsf{T} = \mathbf{0}, \tag{4.70a}$$

$$\mathbf{A}^\mathsf{T}\mathbf{Q}_\omega\mathbf{A} - \mathbf{Q}_\omega + \mathbf{S}_\omega^\mathsf{T}\mathbf{C}^\mathsf{T}\mathbf{C} + \mathbf{C}^\mathsf{T}\mathbf{C}\mathbf{S}_\omega = \mathbf{0}, \tag{4.70b}$$

$$\mathbf{A}\mathbf{P}_{12,\omega}\mathbf{A}^\mathsf{T} - \mathbf{P}_{12,\omega} + \mathbf{S}_\omega\mathbf{B}\mathbf{B}^\mathsf{T} + \mathbf{B}\mathbf{B}^\mathsf{T}\mathbf{S}_\omega^\mathsf{T} = \mathbf{0}, \tag{4.70c}$$

$$\mathbf{A}^\mathsf{T}\mathbf{Q}_{12,\omega}\mathbf{A} - \mathbf{Q}_{12,\omega} + \mathbf{S}_\omega^\mathsf{T}\mathbf{C}^\mathsf{T}\mathbf{C} + \mathbf{C}^\mathsf{T}\mathbf{C}\mathbf{S}_\omega = \mathbf{0}, \tag{4.70d}$$

with

$$\mathbf{S}_\omega = \frac{1}{2\pi}\mathrm{Re}\left(\omega\mathbb{I} - 2i\ln\left(\mathbb{I} - \mathbf{A}e^{-i\omega}\right)\right), \hat{\mathbf{S}}_\omega = \frac{1}{2\pi}\mathrm{Re}\left(\omega\mathbb{I} - 2i\ln\left(\mathbb{I} - \hat{\mathbf{A}}e^{-i\omega}\right)\right), \tag{4.70e}$$

$$\mathbf{R}_\omega = -\frac{1}{\pi}\mathbf{A}^{-1}\,\mathrm{Re}\left(i\ln\left(\mathbb{I} - \mathbf{A}e^{-i\omega}\right)\right), \hat{\mathbf{R}}_\omega = -\frac{1}{\pi}\hat{\mathbf{A}}^{-1}\,\mathrm{Re}\left(i\ln\left(\mathbb{I} - \hat{\mathbf{A}}e^{-i\omega}\right)\right). \tag{4.70f}$$

For the discrete-time case it is also possible to calculate a closed form expression for the gradient of the cost function, and again this makes it possible to formulate necessary conditions for optimality.

Theorem 4.10 (Necessary conditions for optimality). *Assume that G and $\hat{G}$ are asymptotically stable, for the frequency-limited $\mathcal{H}_2$-norm to be defined, i.e., $\mathbf{A}$ and $\hat{\mathbf{A}}$ are Schur. In order for the matrices $\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$ to be optimal for the problem in (4.58), it is necessary that they satisfy the equations in (4.70) and the* *equations in (4.37) and that*

$$\begin{aligned}
\frac{\partial \|E\|^2_{\mathcal{H}_2,\omega}}{\partial \hat{\mathbf{A}}} =& 2\left(\mathbf{Q}^{\mathsf{T}}_{12,\omega}\mathbf{A}\mathbf{P}_{12} + \hat{\mathbf{Q}}_{\omega}\hat{\mathbf{A}}\hat{\mathbf{P}}\right) + \mathbf{W} \\
&+ 2\hat{\mathbf{A}}^{-\mathsf{T}}\hat{\mathbf{C}}^{\mathsf{T}}\left(\mathbf{D} - \hat{\mathbf{D}}\right)\hat{\mathbf{B}}^{\mathsf{T}}\hat{\mathbf{R}}^{\mathsf{T}} = \mathbf{0}, \qquad (4.71a)
\end{aligned}$$

$$\frac{\partial \|E\|^2_{\mathcal{H}_2,\omega}}{\partial \hat{\mathbf{B}}} = 2\left(\hat{\mathbf{Q}}_{\omega}\hat{\mathbf{B}} + \mathbf{Q}^{\mathsf{T}}_{12,\omega}\mathbf{B} - \hat{\mathbf{R}}^{\mathsf{T}}_{\omega}\hat{\mathbf{C}}^{\mathsf{T}}\left[\mathbf{D} - \hat{\mathbf{D}}\right]\right) = \mathbf{0}, \qquad (4.71b)$$

$$\frac{\partial \|E\|^2_{\mathcal{H}_2,\omega}}{\partial \hat{\mathbf{C}}} = 2\left(\hat{\mathbf{C}}\hat{\mathbf{P}}_{\omega} - \mathbf{C}\mathbf{P}_{12,\omega} - \left[\mathbf{D} - \hat{\mathbf{D}}\right]\hat{\mathbf{B}}^{\mathsf{T}}\hat{\mathbf{R}}^{\mathsf{T}}_{\omega}\right) = \mathbf{0}, \qquad (4.71c)$$

$$\frac{\partial \|E\|^2_{\mathcal{H}_2,\omega}}{\partial \hat{\mathbf{D}}} = -2\left(\mathbf{C}\mathbf{R}_{\omega}\mathbf{B} + \mathbf{D}\omega - \hat{\mathbf{C}}\hat{\mathbf{R}}_{\omega}\hat{\mathbf{B}} - \hat{\mathbf{D}}\omega\right) = \mathbf{0}, \qquad (4.71d)$$

where

$$\mathbf{W} = \mathrm{Re}\left(\frac{i}{\pi}\mathrm{e}^{-i\pi\omega}L\left(\mathbb{I} - \hat{\mathbf{A}}\mathrm{e}^{-i\omega}, \mathbf{V}\right)\right)^{\mathsf{T}}, \qquad (4.72a)$$

$$\mathbf{V} = \hat{\mathbf{P}}\hat{\mathbf{C}}^{\mathsf{T}}\hat{\mathbf{C}} - \mathbf{P}^{\mathsf{T}}_{12}\mathbf{C}^{\mathsf{T}}\hat{\mathbf{C}} - \hat{\mathbf{B}}\left(\mathbf{D} - \hat{\mathbf{D}}\right)^{\mathsf{T}}\hat{\mathbf{C}}\hat{\mathbf{A}}^{-1} \qquad (4.72b)$$

with the function $L(\,\cdot\,,\,\cdot\,)$ being the Frechét derivative of the matrix logarithm, see Higham [2008].

Proof: The proof is analogous to the proof for Theorem 4.8 for continuous time. □

Theorem 4.11. *The cost function to the optimization problem (4.6) and its gradient, given in Theorem 4.10, are invariant under state transformations of the system G.*

Proof: Realizing that $\mathbf{S}_{\omega} = \mathbf{T}^{-1}\bar{\mathbf{S}}_{\omega}\mathbf{T}$ and $\mathbf{R}_{\omega} = \mathbf{T}^{-1}\bar{\mathbf{R}}_{\omega}\mathbf{T}$, makes the proof analogous to the proof in Theorem 4.3. □

4.5 Computational Aspects of the Optimization Problems

In this section, suggestions for how to initialize the optimization and how the optimization can be performed efficiently, by using the inherent structure to speed up the computations, will be presented.

For all the methods that have been presented in Section 4.4, a cost function has been given and necessary conditions for optimality. The gradients for all the methods are readily extracted from the necessary conditions for optimality for the methods. With this information it is straightforward to, for example, use any quasi-Newton solver, see Section 2.2.1, to solve the optimization problem in (4.6). For two special cases, the Hessians were also calculated, which can be used to

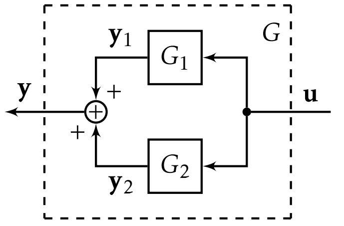

Figure 4.3: *Models in parallel*

initialize the Hessian in the quasi-Newton solver. Computing the Hessian in all iterations would be to computationally expensive.

4.5.1 Structure in Variables

In some cases, the system matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ have a certain structure, that is desired to preserve while computing $\hat{G}$. In other words, it is desirable to have a similar structure in the system matrices for $\hat{G}$, i.e., $\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$. For example, assume that G has the structure as given in Figure 4.3, with two systems in parallel where we want to use model reduction on the system G, but also keep the internal parallel structure. In this case a block diagonal $\hat{\mathbf{A}}$-matrix is desired.

Looking at all the cost functions in Section 4.4, there is nothing holding us back from introducing structure in the system matrices, e.g., block diagonal $\hat{\mathbf{A}}$, when formulating our optimization problem. The question is if the derived gradients are still usable when having structure in the system matrices, and the answer is, yes. This is because all the steps in deriving the gradients have been done element wise for all the system matrices. If, for example, a diagonal $\hat{\mathbf{A}}$ is desirable, only the diagonal elements in the gradient for $\hat{\mathbf{A}}$ are relevant and are hence used. In general, for this purpose, the so called structure variables $\mathbf{S}_{\hat{\mathbf{A}}}, \mathbf{S}_{\hat{\mathbf{B}}}, \mathbf{S}_{\hat{\mathbf{C}}}$ and $\mathbf{S}_{\hat{\mathbf{D}}}$, are introduced, which holds the structure of the system matrices, i.e., element (i, j) in $\mathbf{S}_{\hat{\mathbf{A}}}$ is 1 if element (i, j) is a variable in the sought system matrix and 0 otherwise. The gradients now become

$$\frac{\partial \|E\|^2_{\mathcal{H}_2,\omega}}{\partial \hat{\mathbf{A}}} \odot \mathbf{S}_{\hat{\mathbf{A}}}, \quad \frac{\partial \|E\|^2_{\mathcal{H}_2,\omega}}{\partial \hat{\mathbf{B}}} \odot \mathbf{S}_{\hat{\mathbf{B}}},$$
$$\frac{\partial \|E\|^2_{\mathcal{H}_2,\omega}}{\partial \hat{\mathbf{C}}} \odot \mathbf{S}_{\hat{\mathbf{C}}}, \quad \frac{\partial \|E\|^2_{\mathcal{H}_2,\omega}}{\partial \hat{\mathbf{D}}} \odot \mathbf{S}_{\hat{\mathbf{D}}},$$

where $\odot$ denotes the Hadamard (element wise) product of two matrices.

Furthermore, with $\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$ initialized with structure according to $\mathbf{S}_{\hat{\mathbf{A}}}, \mathbf{S}_{\hat{\mathbf{B}}}, \mathbf{S}_{\hat{\mathbf{C}}}$ and $\mathbf{S}_{\hat{\mathbf{D}}}$, the structure will remain when moving along a quasi-Newton step.

4.5.2 Initialization

The optimization problem in (4.6), is both nonlinear and non-convex, see, for instance, Example 4.1. This makes the initialization an important part of the problem. For the methods proposed in this chapter, the model used for initialization has to be asymptotically stable. Since there exists numerous methods for model reduction, which are easily computed and produces asymptotically stable reduced models, e.g., balanced truncation, see Section 4.2, any of them can be used to create a model for initialization. In the special cases, in Section 4.4.1, where there are no input or output filters, even more can be done for the initialization. Looking at the cost functions, (4.27) and (4.35), one sees that the cost functions becomes quadratic in $\hat{\mathbf{B}}$ (or $\hat{\mathbf{C}}$) if $\hat{\mathbf{A}}$ and $\hat{\mathbf{C}}$ (or $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$) are fixed, and since $\hat{\mathbf{Q}}$ (and $\hat{\mathbf{P}}$) is positive semidefinite, the quadratic program is solvable. Hence, first a basic initialization is used to obtain a model with the correct number of states, e.g., using balanced truncation. This model is then used in the quadratic program described above to obtain a better initialization for $\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}$.

4.5.3 Structure in Equations

In this section, the inherent structure in the equations will be used to speed up the computations. First, remember that the problem is a model *reduction* problem, and in most cases $\hat{n} << n$. The analysis in this section will be based on the continuous-time case, but the same results are also valid for the discrete-time case. Consider the cost function for the general case, when using input and output filters, (4.98). The terms $\mathbf{D}_i^\top \mathbf{B}^\top \mathbf{Q} \mathbf{B} \mathbf{D}_i$ and $\mathbf{D}_o \mathbf{C} \mathbf{P} \mathbf{C}^\top \mathbf{D}_o^\top$ do not depend on any of the optimization variables and are the only terms that include the matrices $\mathbf{P}$ and $\mathbf{Q}$ (see (4.96), (4.97) and (4.98)). Hence, $\mathbf{P}$ and $\mathbf{Q}$ does not have to be computed. The same applies for the terms $\mathbf{B}^\top \mathbf{Q}_\omega \mathbf{B}$ and $\mathbf{C} \mathbf{P}_\omega \mathbf{C}^\top$ and the matrices $\mathbf{P}_\omega$ and $\mathbf{Q}_\omega$ in (4.65).

In all the presented methods, for every iteration in the solver, both the cost function and its gradient have to be computed. To do this a number of Lyapunov and Sylvester equations have to be solved. This is where most of the computational time is spent. Therefore, before starting to analyze what is done in every iteration, a brief explanation on how to solve a general Sylvester equation is presented. A general Sylvester equation can be written as

$$\mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{B} + \mathbf{C} = \mathbf{0}, \ \mathbf{A} \in \mathbb{R}^{n\times n}, \ \mathbf{B} \in \mathbb{R}^{\hat{n}\times\hat{n}}, \ \mathbf{C} \in \mathbb{R}^{n\times\hat{n}}. \tag{4.73}$$

The first main step when solving a Sylvester equation is to Schur factorize (see e.g., Golub and Van Loan [1996] or Bartels and Stewart [1972]) $\mathbf{A}$ and $\mathbf{B}$, which can be done in $\mathcal{O}(n^3)$ operations for $\mathbf{A}$ and $\mathcal{O}(\hat{n}^3)$ operations for $\mathbf{B}$. Now the equation

$$\mathbf{A}_S \mathbf{X}_S + \mathbf{X}_S \mathbf{B}_S = \mathbf{C}_S \tag{4.74}$$

has to be solved, where $\mathbf{A}_S = \mathbf{U}^\top \mathbf{A} \mathbf{U}$ and $\mathbf{B}_S = \mathbf{V}^\top \mathbf{B} \mathbf{V}$ are block upper triangular, computed using the Schur factorization and $\mathbf{C}_S = \mathbf{U}^\top \mathbf{C} \mathbf{V}$ and $\mathbf{X}_S = \mathbf{U}^\top \mathbf{X} \mathbf{V}$. It is not hard to verify that the new system of linear equations, (4.74), can be solved in $\mathcal{O}(n^2\hat{n} + n\hat{n}^2)$ complexity, and the solution to (4.73) is computed as, $\mathbf{X} = \mathbf{U}\mathbf{X}_S\mathbf{V}^\top$ which also costs $\mathcal{O}(n^2\hat{n} + n\hat{n}^2)$. It can be concluded that when solving several Sylvester equations with the same factors $\mathbf{A}$ and $\mathbf{B}$ but different $\mathbf{C}$:s, speed can be gained in the computations if $\mathbf{A}$ and $\mathbf{B}$ are Schur factorized before solving the equations. It can also be concluded that it is computationally much more efficient to use the structure in the realizations (4.12) and (4.59) and split up the large Lyapunov equations for $\mathbf{P}_E$ and $\mathbf{Q}_E$ in a number of smaller Lyapunov/Sylvester equations, as described in (4.96) and (4.97), which can be solved much more efficiently.

For the methods in Section 4.4.1 and Section 4.4.3, which are invariant under state transformations, the given system G (and the input and/or output filter if they are present) can be transformed to a basis such that the $\mathbf{A}$-matrices are upper triangular (Schur factorize the $\mathbf{A}$-matrices). In other words, given a Schur factorization of $\mathbf{A}$, such that $\mathbf{A} = \mathbf{U}\bar{\mathbf{A}}\mathbf{U}^\mathsf{T}$, where $\bar{\mathbf{A}}$ is block upper triangular and $\mathbf{U}$ is orthogonal, we can transform the system as follows,

$$G = \left[\begin{array}{c|c} \mathbf{U}^\mathsf{T}\mathbf{A}\mathbf{U} & \mathbf{U}^\mathsf{T}\mathbf{B} \\ \hline \mathbf{C}\mathbf{U} & \mathbf{D} \end{array}\right] = \left[\begin{array}{c|c} \bar{\mathbf{A}} & \bar{\mathbf{B}} \\ \hline \bar{\mathbf{C}} & \bar{\mathbf{D}} \end{array}\right], \tag{4.75}$$

and use this realization during the iterations. Additionally, looking at the Lyapunov/Sylvester equations needed to be solved (equations (4.96) and (4.97) or equations (4.65) or (4.52)), one observes that they all have the same underlying structure, i.e., their factors in the equations are $\mathbf{A}$, $\hat{\mathbf{A}}$, $\mathbf{A}_i$, and $\mathbf{A}_o$. Assuming that $\mathbf{A}$ (and $\mathbf{A}_i$ and $\mathbf{A}_o$) is given in real Schur form, then for every iteration only the matrix $\hat{\mathbf{A}}$ has to be Schur factorized, which is small compared to $\mathbf{A}$, to be able to solve all Lyapunov/Sylvester equations at a maximum cost of $\mathcal{O}(n^2\hat{n} + n\hat{n}^2)$.

4.6 Examples

In this section, some examples that show the applicability of the proposed methods will be presented. Where it is possible, comparisons with other relevant methods will be made. To be able to measure how well different methods perform, the relative error for the particular norm in use will be utilized, i.e.,

$$\frac{\left\|G - \hat{G}\right\|_{\mathcal{H}}}{\|G\|_{\mathcal{H}}}. \tag{4.76}$$

To shorten the names and make the figures more readable our proposed methods will be denoted as

- H$_2$NL– the ordinary model-reduction method without weights, described in Section 4.4.1
- WH$_2$NL– the ordinary model-reduction method with weights, described in Section 4.4.1
- FLH$_2$NL– the frequency-limited model-reduction method, described in Section 4.4.3
- RH$_2$NL– the robust model-reduction method, described in Section 4.4.2

The methods that will be used for comparison, in the different examples, are

- BT– ordinary balanced truncation, the implementation used is the function `schurmr` in Robust Control Toolbox in MATLAB
- WBT– weighted balanced truncation, an implementation of the method in Enns [1984]
- FLBT– frequency-limited balanced truncation, an implementation of the method in Gawronski and Juang [1990]
- MFLBT– modified frequency-limited balanced truncation, an implementation of the method in Gugercin and Antoulas [2004]
- ITIA– iterative tangential interpolation algorithm, the implementation in the MORE-toolbox is used (see Poussot-Vassal and Vuillemin [2012])
- ISTIA– iterative SVD-tangential interpolation algorithm (see Poussot-Vassal and Vuillemin [2012]), the implementation in the MORE-toolbox is used
- FLISTIA– frequency-limited iterative tangential interpolation algorithm(see Vuillemin et al. [2013]), the implementation in the MORE-toolbox is used

We start with an example to illustrate that the balanced truncation method can be used for initialization of the proposed methods.

Example 4.3: $\mathcal{H}_2$ Model Reduction

In this example 10000 random asymptotically stable and strictly proper SISO systems with 20 states using the function `rss` in Control System Toolbox in MATLAB are generated. On each of these systems, the number of states are reduced to 10 with $\mathrm{H_2NL}$ and BT. When reducing the order of a system with $\mathrm{H_2NL}$, the reduced model from BT is used as the initial point. In this case $\mathrm{H_2NL}$ works as a refinement step on top of BT.

In Figure 4.4, two histograms are plotted. They show the histograms of the entities $\frac{\|G-\hat{G}_{\mathrm{BT}}\|_{\mathcal{H}_2}}{\|G-\hat{G}_{\mathrm{H_2NL}}\|_{\mathcal{H}_2}}$ and $\frac{\|G-\hat{G}_{\mathrm{BT}}\|_{\mathcal{H}_\infty}}{\|G-\hat{G}_{\mathrm{H_2NL}}\|_{\mathcal{H}_\infty}}$ respectively. In other words, they show how much the systems reduced using BT have been improved, in $\mathcal{H}_2$-norm and $\mathcal{H}_\infty$-norm, using $\mathrm{H_2NL}$. $\mathrm{H_2NL}$ works well as a model-reduction method and can in most cases decrease the model reduction error 1-6 times, measured in the $\mathcal{H}_2$-norm. The average improvement in $\mathcal{H}_2$-norm is 4.15. Observe that also the $\mathcal{H}_\infty$-norm can be improved when using $\mathrm{H_2NL}$, this is because of the fact that BT is not a solution to a minimum norm, $\mathcal{H}_2$ or $\mathcal{H}_\infty$, problem. In average a run with $\mathrm{H_2NL}$ takes 1.82 seconds and with BT it takes 0.07 seconds.

We continue with two more examples based on a medium-scale model of a clamped beam. For the first example we use ordinary model reduction without weights and for the second one the frequency-limited model-reduction method is utilized.

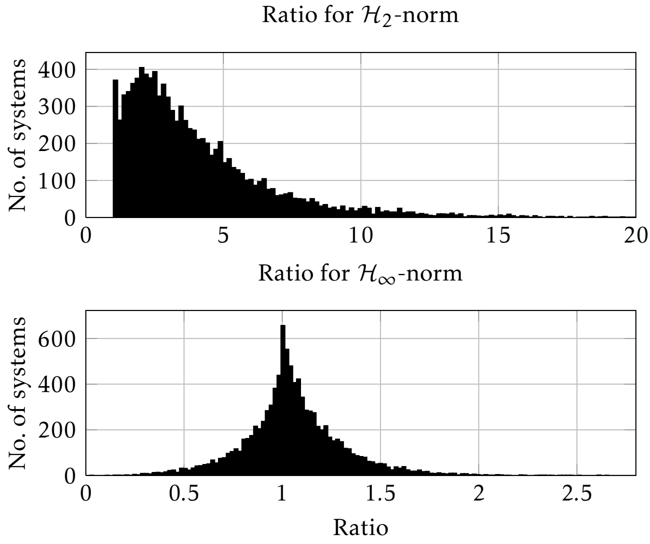

Figure 4.4: *The figure illustrates, in two histograms, how much a system reduced using BT has been improved using H_2NL. The x-axis is the quotient between the $\mathcal{H}_\infty$-norm and $\mathcal{H}_2$-norm of the error system from using BT and the error system from using H_2NL, i.e.,* $\frac{\|G-\hat{G}_{BT}\|_{\mathcal{H}}}{\|G-\hat{G}_{H_2NL}\|_{\mathcal{H}}}$.

Example 4.4: Clamped Beam Model, varying order

In this example a model of a clamped beam, a SISO model with 348 states which can be found in Leibfritz and Lipinski [2003], is used. The model will be reduced to different orders, $n_r \in [4, 30]$, with H_2NL. The reduced models using H_2NL will be compared with models reduced using ISTIA, ITIA and BT. In the left plot of Figure 4.5, it can be observed that for small n_r, H_2NL, ITIA and ISTIA are better than BT, for the $\mathcal{H}_2$-norm, and for larger n_r the error approaches zero for all methods. It can also be observed, in the right plot of Figure 4.5, that, even though we are minimizing the $\mathcal{H}_2$-norm, the $\mathcal{H}_\infty$-norm remains small for all the methods.

Example 4.5: Clamped Beam Model, limited frequency interval

In this example, the model of the clamped beam from the previous example is reused. This time, instead of trying different orders, the focus will be on finding reduced models for different frequency intervals, $[0, \omega]$, $\omega \in [2, 40]$ and fix the reduced-order model to have 12 states, $n_r = 12$. The proposed method FLH_2NL will be used and it will be compared with the frequency-limited methods FLISTIA, FLBT and MFLBT. Additionally, the methods WH_2NL and WBT will be used, both with a tenth order Butterworth low-pass filter, with the cut-off frequency equal to ω. Looking at the left plot of Figure 4.6, it can be observed that for small ω, all the $\mathcal{H}_2$ optimal methods do very well. However, for $\omega > 7$, H_2NL gives better result than all the other methods. As in the previous example, the relative $\mathcal{H}_\infty$-

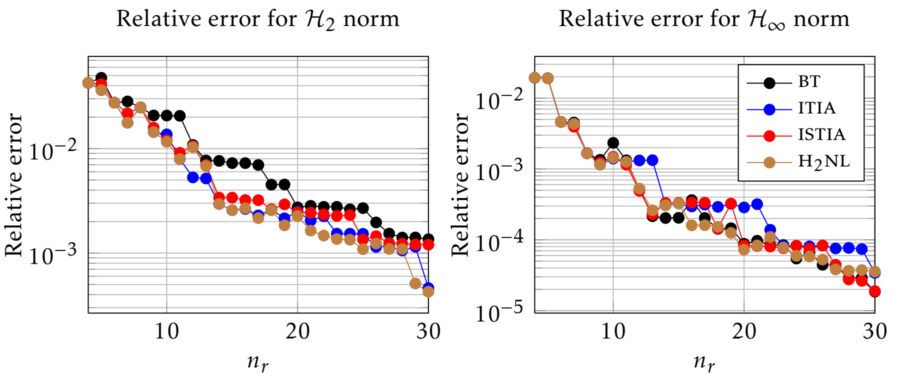

Figure 4.5: *Reduction of a clamped beam model to different orders using $\mathrm{H_2NL}$, ITIA, ISTIA and BT. To the left, the relative $\mathcal{H}_2$ error and to the right the relative $\mathcal{H}_\infty$ error.*

norm remains low, for almost all ω, the $\mathcal{H}_2$ optimal methods have better relative $\mathcal{H}_\infty$-error than the methods using balanced truncation.

Now, two smaller examples are presented to show how models coming from frequency-limited methods can look in the frequency region of interest and outside this region. We start with a small toy example.

Example 4.6: Small toy example

This example considers a small model with four states. The model is composed of two second-order models in series, one with a resonance frequency at $\omega = 1$ and the other at $\omega = 3$. The frequency range is limited to $\omega \in [0, 2]$ to try to only capture the first model. The model used is

$$G = G_1 G_2 = \frac{1}{s^2 + 0.2s + 1} \frac{9}{s^2 + 0.003s + 9}. \tag{4.77}$$

The methods $\mathrm{FLH_2NL}$, FLISTIA, FLBT and MFLBT are compared. These methods are also compared with the methods $\mathrm{WH_2NL}$ and WBT using a tenth order low-pass Butterworth filter with a cut-off frequency of 2, see Figure 4.7. The results from the different methods can be seen in Figure 4.8, Figure 4.9 and Table 4.2. As can be seen in the result, $\mathrm{FLH_2NL}$, $\mathrm{WH_2NL}$, FLISTIA and FLBT are successful in finding a good model for the relevant frequencies, especially $\mathrm{FLH_2NL}$, which is almost six times better, in $\mathcal{H}_2$-norm, than the second best model, $\mathrm{WH_2NL}$, see Table 4.2. MFLBT captures the wrong resonance mode (from our perspective) and fails completely in the lower frequency region, and WBT misses to capture the gain at both the resonance frequency and at the cut-off frequency. Interesting to note is how the methods, that does a good job, sacrifices the model fit at higher frequencies for the lower.

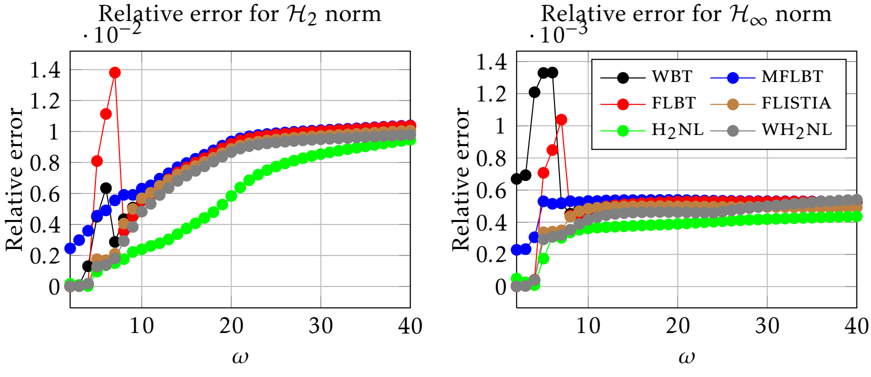

Figure 4.6: *Reduction of a clamped beam model to 12 states with focus on the frequency interval* $[0, \omega]$, $\omega \in [2, 40]$ *using* FLH2NL, WH2NL, FLISTIA, FLBT *and* MFLBT. *The filter used for the weighted methods is a tenth order Butterworth low-pass filter with cut-off frequency* ω. *To the left, the relative* $\mathcal{H}_2$ *error and to the right the relative* $\mathcal{H}_\infty$ *error.*

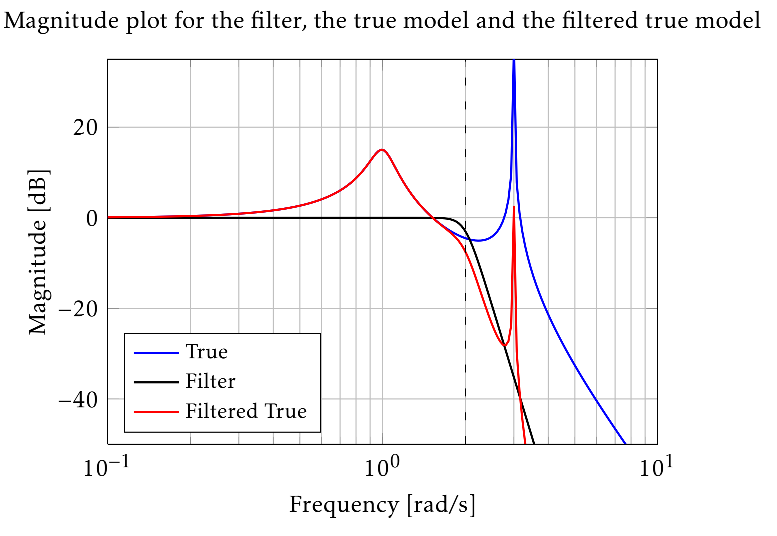

Figure 4.7: *The true and filtered model and the low-pass filter for Example 4.6. The dashed vertical line denotes* $\omega = 2$.

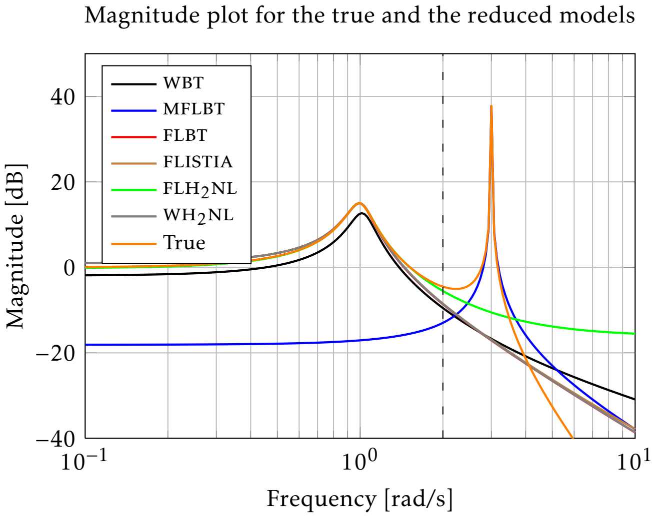

Figure 4.8: *The true and reduced-order models for Example 4.6. The dashed vertical line denotes $\omega = 2$. FLH$_2$NL, WH$_2$NL, FLISTIA and FLBT are successful in finding a good model for the relevant frequencies while MFLBT and WBT fails.*

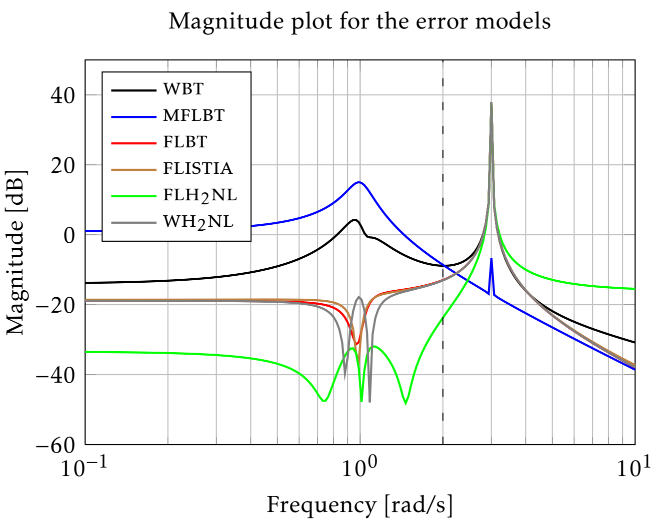

Figure 4.9: *The error models for the different methods for Example 4.6. The dashed vertical line denotes $\omega = 2$. FLH$_2$NL, WH$_2$NL, FLISTIA and FLBT are successful in finding a good model for the relevant frequencies while MFLBT and WBT fails.*

Table 4.2: *Numerical results for Example 4.6*

	$\frac{\|\|G-\hat{G}\|\|_{\mathcal{H}_2,\omega}}{\|\|G\|\|_{\mathcal{H}_2,\omega}}$	$\frac{\|\|G-\hat{G}\|\|_{\mathcal{H}_\infty,\omega}}{\|\|G\|\|_{\mathcal{H}_\infty,\omega}}$	Re $\lambda_{\max}$
WBT	3.01e-01	2.91e-01	-1.00e-01
MFLBT	1.00e+00	1.00e+00	-1.51e-03
FLBT	6.31e-02	4.00e-02	-9.93e-02
FLISTIA	6.38e-02	3.96e-02	-9.99e-02
FLH_2NL	1.02e-02	1.15e-02	-1.01e-01
WH_2NL	5.97e-02	3.95e-02	-1.00e-01

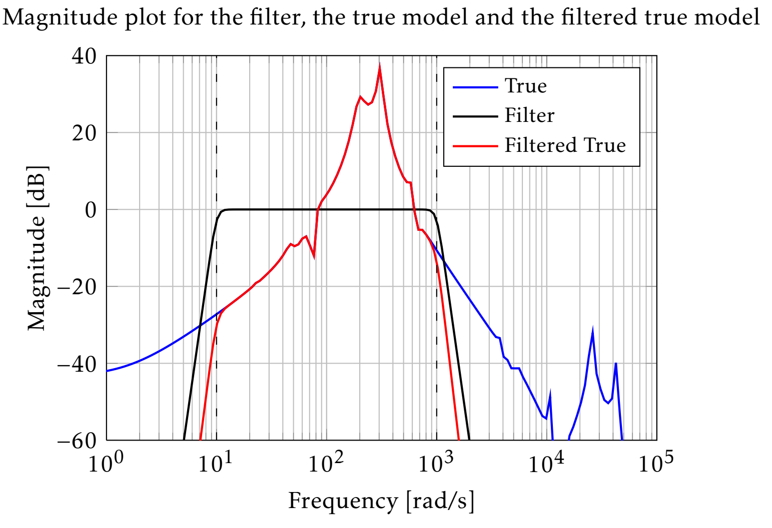

Figure 4.10: *The true and filtered model and the band-pass filter for Example 4.7. The dashed vertical lines denote $\omega = 10$ and $\omega = 10000$.*

Example 4.7: CD player

This example uses a slightly larger model, a model of a compact-disc player with 120 states and two inputs and two outputs, see Leibfritz and Lipinski [2003]. In this example, to illustrate the result in the same way as in the previous example, only one SISO part of the transfer function is chosen, namely the transfer function from the second input to the first output of the model. Here, focus will be on a banded frequency interval, $\omega \in [10, 1000]$ where the main peak gain is, see Figure 4.10. The methods that will be compared are the frequency-limited methods FLBT, MFLBT, FLISTIA and FLH_2NL and the weighted methods WBT and WH_2NL with a tenth order Butterworth band-pass filter with cut-off frequencies equal to $\omega = 10$ and $\omega = 1000$. Looking at the results in Figure 4.11, Figure 4.12 and Table 4.3 all the methods, except FLISTIA, does a good job, and again FLH_2NL finds the best model.

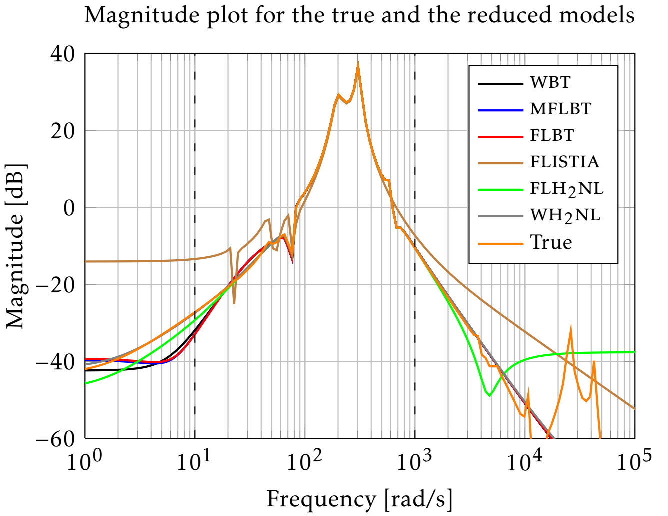

Figure 4.11: *The true and reduced order models for Example 4.7. The dashed vertical lines denote $\omega = 10$ and $\omega = 10000$.* FLH$_2$NL*,* WH$_2$NL*,* MFLBT*,* WBT *and* FLBT *are successful in finding a good model for the relevant frequencies. However, in this example the method* FLISTIA *fails.*

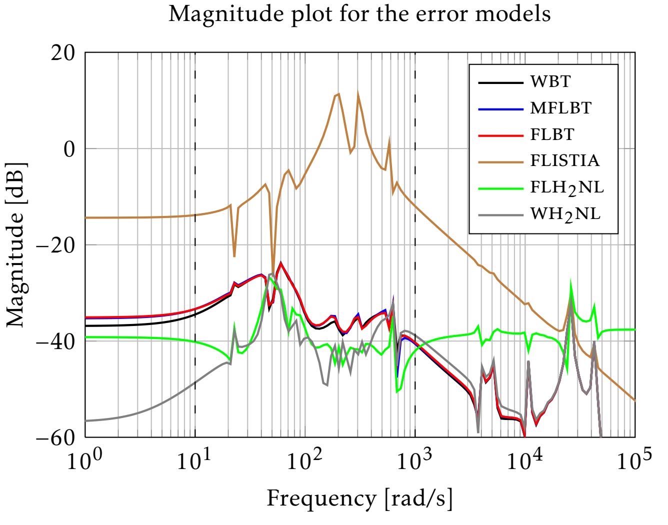

Figure 4.12: *The error models for the different methods for Example 4.7. The dashed vertical lines denote $\omega = 10$ and $\omega = 10000$.* FLH$_2$NL*,* WH$_2$NL*,* MFLBT*,* WBT *and* FLBT *are successful in finding a good model for the relevant frequencies. However, in this example the method* FLISTIA *fails.*

Table 4.3: Numerical results for Example 4.7

	$\frac{\|G-\hat{G}\|_{\mathcal{H}_2,\omega}}{\|G\|_{\mathcal{H}_2,\omega}}$	$\frac{\|G-\hat{G}\|_{\mathcal{H}_\infty,\omega}}{\|G\|_{\mathcal{H}_\infty,\omega}}$	Re $\lambda_{\max}$
WBT	1.24e-03	9.50e-04	-5.55e+00
MFLBT	1.25e-03	9.43e-04	-5.54e+00
FLBT	1.24e-03	9.41e-04	-5.54e+00
FLISTIA	8.23e-02	5.64e-02	-2.26e-01
FLH$_2$NL	6.95e-04	6.83e-04	-5.63e+00
WH$_2$NL	8.94e-04	7.76e-04	-5.80e+00

Example 4.8: CD player with perturbed poles

In this example, the model of the CD player from the last example is used again. However, in this case the system matrices of the model are perturbed such that

$$\begin{aligned}\mathbf{A}_{\text{pert}} &= \mathbf{A} + \mathbf{E}_{\mathbf{A}} \odot \mathbf{A},\\ \mathbf{B}_{\text{pert}} &= \mathbf{B} + \mathbf{E}_{\mathbf{B}} \odot \mathbf{B},\\ \mathbf{C}_{\text{pert}} &= \mathbf{C} + \mathbf{E}_{\mathbf{C}} \odot \mathbf{C},\end{aligned}$$

where the elements in $\mathbf{E}_{\mathbf{A}}$, $\mathbf{E}_{\mathbf{B}}$ and $\mathbf{E}_{\mathbf{C}}$ are independent random variables with the distribution $\mathcal{N}\left(0, 0.05^2\right)$. The perturbed model will be reduced to a fifteenth order model using H$_2$NL. This will be compared with reducing the model with RH$_2$NL with different values of the regularization parameter. This procedure is repeated 250 times with different realizations of the random variables and the average is computed. The result from the optimization can be seen in Figure 4.13. In this figure, the average relative error between the true, unperturbed, model and the reduced models, as a function of the regularization parameter, for RH$_2$NL and H$_2$NL are plotted.

In Figure 4.13 one observes that for the tested values of the regularization parameters it is possible, in this case, to find a better model. Even for the $\mathcal{H}_\infty$-norm, it is possible to find a model that performs better than the unregularized method.

Some more examples using model reduction methods will be performed in Chapter 7, where two larger examples are presented which need more background.

4.7 Conclusions

In this chapter, three model-reduction methods (in both continuous and discrete time) based on minimizing the $\mathcal{H}_2$-norm using optimization have been presented. For these methods, both cost functions and gradients have been derived, which makes it possible to efficiently use of-the-shelves quasi-Newton solvers. For a few cases the Hessians have been derived, which also can be utilized in the quasi-Newton solver. The derivation of the methods enables us to impose structural

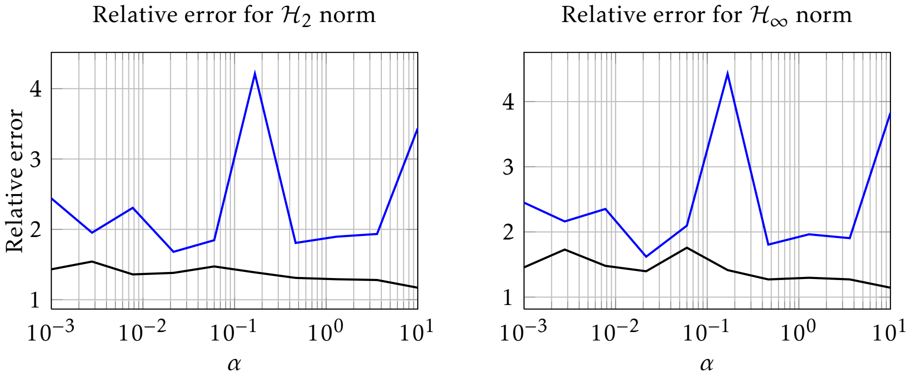

Figure 4.13: *The black line is the average relative error, over different perturbations, for RH$_2$NL using different values of the regularization parameter and the blue line is the average relative error, over different perturbations, for H$_2$NL ($\mathcal{H}_2$-norm in the left plot and $\mathcal{H}_\infty$-norm in the right).*

constraints, e.g., block diagonal $\hat{\mathbf{A}}$-matrix, in the system matrices. Additionally, a number of examples showing the applicability of the methods, both for small and medium-scale problems have been presented, for which the methods have performed well.

One of the drawbacks with the methods is the non-convexity of the problem. One way to possibly reduce the influence of the non-convexity is to have a better initialization, which is a subject of further research. However, for the examples presented in this chapter the proposed initialization procedure seems to work.

Appendix

4.A Gradient of V_{rob}

In this appendix, the derivation of the gradient, with respect to $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}$, for V_{rob} in (4.49) in Section 4.4.2 will be presented, where

$$V_{\text{rob}} = \epsilon_{\mathbf{A}} \left\| \frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{A}} \right\|_F + \epsilon_{\mathbf{B}} \left\| \frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{B}} \right\|_F + \epsilon_{\mathbf{C}} \left\| \frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{C}} \right\|_F + \epsilon_{\mathbf{D}} \left\| \frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{D}} \right\|_F. \quad (4.78)$$

To differentiate V_{rob}, we first need the definition of the Frobenius norm. Given a function $f(x)$, the Frobenius norm is defined as

$$\|f(x)\|_F \triangleq \sqrt{\operatorname{tr} f(x)^{\mathsf{T}} f(x)}. \quad (4.79)$$

Differentiating $\|f(x)\|_F$ with respect to x, yields

$$\frac{\partial \|f(x)\|_F}{\partial x} = \frac{\frac{\partial}{\partial x} \operatorname{tr} f(x)^{\mathsf{T}} f(x)}{2\|f(x)\|_F}. \quad (4.80)$$

This means that the still unknown part when calculating $\frac{\partial \|f(x)\|_F}{\partial x}$ given $f(x)$, is the numerator part. Given the structure of V_{rob} in (4.78) this means that to obtain an expression for V_{rob}, we need to calculate, for example, terms like

$$\frac{\partial}{\partial \hat{\mathbf{A}}} \operatorname{tr}\left(\left[\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{A}} \right]^{\mathsf{T}} \left[\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{A}} \right] \right).$$

In this appendix, elements in the matrices $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}$ will be denoted with a_{ij}, b_{ij} and c_{ij} respectively.

To simplify the equations later on, four new Sylvester equations are defined,

$$\hat{\mathbf{A}}^\top \mathbf{W}_1 + \mathbf{W}_1 \mathbf{A} + \mathbf{Q}_{12}^\top \left(\mathbf{Q}\mathbf{P} + \mathbf{Q}_{12}\mathbf{P}_{12}^\top \right) = \mathbf{0}, \tag{4.81a}$$

$$\mathbf{A}\mathbf{W}_2 + \mathbf{W}_2 \hat{\mathbf{A}}^\top + \left(\mathbf{Q}\mathbf{P} + \mathbf{Q}_{12}\mathbf{P}_{12}^\top \right) \mathbf{P}_{12} = \mathbf{0}, \tag{4.81b}$$

$$\mathbf{A}\mathbf{W}_3 + \mathbf{W}_3 \hat{\mathbf{A}}^\top + \left(\mathbf{Q}\mathbf{B} + \mathbf{Q}_{12}\hat{\mathbf{B}} \right) \hat{\mathbf{B}}^\top = \mathbf{0}, \tag{4.81c}$$

$$\hat{\mathbf{A}}^\top \mathbf{W}_4 + \mathbf{W}_4 \mathbf{A} + \hat{\mathbf{C}}^\top \left(\hat{\mathbf{C}}\mathbf{P}_{12}^\top - \mathbf{C}\mathbf{P} \right) = \mathbf{0}, \tag{4.81d}$$

whose origin will become clear soon. Differentiated versions of the equations in (4.29) will also be needed

$$\mathbf{A}\frac{\partial \mathbf{P}_{12}}{\partial a_{ij}} + \frac{\partial \mathbf{P}_{12}}{\partial a_{ij}}\hat{\mathbf{A}}^\top + \mathbf{P}_{12}\frac{\partial \hat{\mathbf{A}}^\top}{\partial a_{ij}} = \mathbf{0}, \tag{4.82a}$$

$$\hat{\mathbf{A}}^\top \frac{\partial \mathbf{Q}_{12}^\top}{\partial a_{ij}} + \frac{\partial \mathbf{Q}_{12}^\top}{\partial a_{ij}}\mathbf{A} + \frac{\partial \hat{\mathbf{A}}^\top}{\partial a_{ij}}\mathbf{Q}_{12}^\top = \mathbf{0}, \tag{4.82b}$$

$$\hat{\mathbf{A}}^\top \frac{\partial \hat{\mathbf{Q}}}{\partial a_{ij}} + \frac{\partial \hat{\mathbf{Q}}}{\partial a_{ij}}\hat{\mathbf{A}} + \frac{\partial \hat{\mathbf{A}}^\top}{\partial a_{ij}}\hat{\mathbf{Q}} + \hat{\mathbf{Q}}\frac{\partial \hat{\mathbf{A}}}{\partial a_{ij}} = \mathbf{0}, \tag{4.82c}$$

$$\mathbf{A}\frac{\partial \mathbf{P}_{12}}{\partial b_{ij}} + \frac{\partial \mathbf{P}_{12}}{\partial b_{ij}}\hat{\mathbf{A}}^\top + \mathbf{B}\frac{\partial \hat{\mathbf{B}}^\top}{\partial b_{ij}} = \mathbf{0}, \tag{4.82d}$$

$$\hat{\mathbf{A}}^\top \frac{\partial \mathbf{Q}_{12}^\top}{\partial c_{ij}} + \frac{\partial \mathbf{Q}_{12}^\top}{\partial c_{ij}}\mathbf{A} - \frac{\partial \hat{\mathbf{C}}^\top}{\partial c_{ij}}\mathbf{C} = \mathbf{0}. \tag{4.82e}$$

We start with the terms containing $\frac{\partial \|E\|_{\mathcal{H}_2}^2}{\partial \mathbf{A}}$,

$$\begin{aligned}\operatorname{tr}\left(\left[\frac{\partial \|E\|_{\mathcal{H}_2}^2}{\partial \mathbf{A}} \right]^\top \left[\frac{\partial \|E\|_{\mathcal{H}_2}^2}{\partial \mathbf{A}} \right] \right) &= \operatorname{tr}\left(4 \left[\mathbf{Q}\mathbf{P} + \mathbf{Q}_{12}\mathbf{P}_{12}^\top \right]^\top \left[\mathbf{Q}\mathbf{P} + \mathbf{Q}_{12}\mathbf{P}_{12}^\top \right] \right) \\ &= 4 \operatorname{tr}\left(\mathbf{P}\mathbf{Q}\mathbf{Q}\mathbf{P} + 2\mathbf{P}_{12}\mathbf{Q}_{12}^\top \mathbf{Q}\mathbf{P} + \mathbf{P}_{12}\mathbf{Q}_{12}^\top \mathbf{Q}_{12}\mathbf{P}_{12}^\top \right). \end{aligned} \tag{4.83}$$

Differentiating with respect to $\hat{\mathbf{A}}$:

$$\begin{aligned}&\frac{\partial}{\partial a_{ij}} \operatorname{tr}\left(\left[\frac{\partial \|E\|_{\mathcal{H}_2}^2}{\partial \mathbf{A}} \right]^\top \left[\frac{\partial \|E\|_{\mathcal{H}_2}^2}{\partial \mathbf{A}} \right] \right) = \\ &\quad = 8 \operatorname{tr}\left(\frac{\partial \mathbf{P}_{12}}{\partial a_{ij}} \mathbf{Q}_{12}^\top \left[\mathbf{Q}\mathbf{P} + \mathbf{Q}_{12}\mathbf{P}_{12}^\top \right] + \frac{\partial \mathbf{Q}_{12}^\top}{\partial a_{ij}} \left[\mathbf{Q}\mathbf{P} + \mathbf{Q}_{12}\mathbf{P}_{12}^\top \right] \mathbf{P}_{12} \right). \end{aligned} \tag{4.84}$$

Differentiating with respect to $\hat{\mathbf{B}}$:

$$\frac{\partial}{\partial b_{ij}} \operatorname{tr}\left(\left[\frac{\partial \|E\|_{\mathcal{H}_2}^2}{\partial \mathbf{A}} \right]^\top \left[\frac{\partial \|E\|_{\mathcal{H}_2}^2}{\partial \mathbf{A}} \right] \right) = 8 \operatorname{tr}\left(\frac{\partial \mathbf{P}_{12}}{\partial b_{ij}} \mathbf{Q}_{12}^\top \left[\mathbf{Q}\mathbf{P} + \mathbf{Q}_{12}\mathbf{P}_{12}^\top \right] \right). \tag{4.85}$$

Differentiating with respect to $\hat{\mathbf{C}}$:

$$\frac{\partial}{\partial c_{ij}} \operatorname{tr}\left(\left[\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{A}}\right]^\top \left[\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{A}}\right]\right) = 8 \operatorname{tr}\left(\frac{\partial \mathbf{Q}_{12}^\top}{\partial x_{ij}} \left[\mathbf{QP} + \mathbf{Q}_{12}\mathbf{P}_{12}^\top\right] \mathbf{P}_{12}\right). \tag{4.86}$$

Now, continue with the terms containing $\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{B}}$,

$$\begin{aligned}\operatorname{tr}\left(\left[\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{B}}\right]^\top \left[\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{B}}\right]\right) &= \operatorname{tr}\left(4\left[\mathbf{QB} + \mathbf{Q}_{12}\hat{\mathbf{B}}\right]^\top \left[\mathbf{QB} + \mathbf{Q}_{12}\hat{\mathbf{B}}\right]\right) \\ &= \operatorname{tr}\left(\mathbf{B}^\top\mathbf{QQB} + 2\hat{\mathbf{B}}^\top\mathbf{Q}_{12}^\top\mathbf{QB} + \hat{\mathbf{B}}^\top\mathbf{Q}_{12}^\top\mathbf{Q}_{12}\hat{\mathbf{B}}\right). \end{aligned} \tag{4.87}$$

Differentiating with respect to $\hat{\mathbf{A}}$:

$$\frac{\partial}{\partial a_{ij}} \operatorname{tr}\left(\left[\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{B}}\right]^\top \left[\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{B}}\right]\right) = 8 \operatorname{tr}\left(\frac{\partial \mathbf{Q}_{12}^\top}{\partial a_{ij}} \left[\mathbf{QB} + \mathbf{Q}_{12}\hat{\mathbf{B}}\right] \hat{\mathbf{B}}^\top\right). \tag{4.88}$$

Differentiating with respect to $\hat{\mathbf{B}}$:

$$\frac{\partial}{\partial b_{ij}} \operatorname{tr}\left(\left[\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{B}}\right]^\top \left[\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{B}}\right]\right) = 8 \operatorname{tr}\left(\frac{\partial \hat{\mathbf{B}}^\top}{\partial b_{ij}} \mathbf{Q}_{12}^\top \left[\mathbf{QB} + \mathbf{Q}_{12}\hat{\mathbf{B}}\right]\right). \tag{4.89}$$

Differentiating with respect to $\hat{\mathbf{C}}$:

$$\frac{\partial}{\partial c_{ij}} \operatorname{tr}\left(\left[\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{B}}\right]^\top \left[\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{B}}\right]\right) = 8 \operatorname{tr}\left(\frac{\partial \mathbf{Q}_{12}^\top}{\partial c_{ij}} \left[\mathbf{QB} + \mathbf{Q}_{12}\hat{\mathbf{B}}\right] \hat{\mathbf{B}}^\top\right) \tag{4.90}$$

Continuing with the terms containing $\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{C}}$,

$$\begin{aligned}\operatorname{tr}\left(\left[\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{C}}\right]^\top \left[\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{C}}\right]\right) &= \operatorname{tr}\left(4\left[\mathbf{CP} - \hat{\mathbf{C}}\mathbf{P}_{12}^\top\right]^\top \left[\mathbf{CP} - \hat{\mathbf{C}}\mathbf{P}_{12}^\top\right]\right) \\ &= 4 \operatorname{tr}\left(\mathbf{PC}^\top\mathbf{CP} - 2\mathbf{P}_{12}\hat{\mathbf{C}}^\top\mathbf{CP} + \mathbf{P}_{12}\hat{\mathbf{C}}^\top\hat{\mathbf{C}}\mathbf{P}_{12}^\top\right) \end{aligned} \tag{4.91}$$

Differentiating with respect to $\hat{\mathbf{A}}$:

$$\frac{\partial}{\partial a_{ij}} \operatorname{tr}\left(\left[\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{C}}\right]^\top \left[\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{C}}\right]\right) = 8 \operatorname{tr}\left(\frac{\partial \mathbf{P}_{12}}{\partial a_{ij}} \hat{\mathbf{C}}^\top \left[\hat{\mathbf{C}}\mathbf{P}_{12}^\top - \mathbf{CP}\right]\right) \tag{4.92}$$

Differentiating with respect to $\hat{\mathbf{B}}$:

$$\frac{\partial}{\partial b_{ij}}\operatorname{tr}\left(\left[\frac{\partial\,\|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{C}}\right]^\mathsf{T}\left[\frac{\partial\,\|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{C}}\right]\right) = 8\operatorname{tr}\left(\frac{\partial \mathbf{P}_{12}}{\partial b_{ij}}\hat{\mathbf{C}}^\mathsf{T}\left[\hat{\mathbf{C}}\mathbf{P}_{12}^\mathsf{T} - \mathbf{C}\mathbf{P}\right]\right) \tag{4.93}$$

Differentiating with respect to $\hat{\mathbf{C}}$:

$$\frac{\partial}{\partial c_{ij}}\operatorname{tr}\left(\left[\frac{\partial\,\|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{C}}\right]^\mathsf{T}\left[\frac{\partial\,\|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{C}}\right]\right) = -8\operatorname{tr}\left(\frac{\partial \hat{\mathbf{C}}^\mathsf{T}}{\partial c_{ij}}\left[\mathbf{C}\mathbf{P} - \hat{\mathbf{C}}\mathbf{P}_{12}^\mathsf{T}\right]\mathbf{P}_{12}\right) \tag{4.94}$$

Here is where the equations for $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_3$ and $\mathbf{W}_4$ from (4.81) comes in. Using Lemma 4.1 with the equations in (4.81) and (4.82) together with the equations above entails

$$\frac{\partial V_{\text{rob}}}{\partial \hat{\mathbf{A}}} = 4\epsilon_{\mathbf{A}}\frac{\mathbf{W}_1\mathbf{P}_{12} + \mathbf{Q}_{12}^\mathsf{T}\mathbf{W}_2}{\left\|\frac{\partial\|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{A}}\right\|_F} + 4\epsilon_{\mathbf{B}}\frac{\mathbf{Q}_{12}^\mathsf{T}\mathbf{W}_3}{\left\|\frac{\partial\|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{B}}\right\|_F} + 4\epsilon_{\mathbf{C}}\frac{\mathbf{W}_4\mathbf{P}_{12}}{\left\|\frac{\partial\|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{C}}\right\|_F}, \tag{4.95a}$$

$$\frac{\partial V_{\text{rob}}}{\partial \hat{\mathbf{B}}} = 4\epsilon_{\mathbf{A}}\frac{\mathbf{W}_1\mathbf{B}}{\left\|\frac{\partial\|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{A}}\right\|_F} + 4\epsilon_{\mathbf{B}}\frac{\mathbf{Q}_{12}^\mathsf{T}\left(\mathbf{Q}\mathbf{B} + \mathbf{Q}_{12}\hat{\mathbf{B}}\right)}{\left\|\frac{\partial\|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{B}}\right\|_F} + 4\epsilon_{\mathbf{C}}\frac{\mathbf{W}_4\mathbf{B}}{\left\|\frac{\partial\|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{C}}\right\|_F}, \tag{4.95b}$$

$$\frac{\partial V_{\text{rob}}}{\partial \hat{\mathbf{C}}} = -4\epsilon_{\mathbf{A}}\frac{\mathbf{C}\mathbf{W}_2}{\left\|\frac{\partial\|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{A}}\right\|_F} - 4\epsilon_{\mathbf{B}}\frac{\mathbf{C}\mathbf{W}_3}{\left\|\frac{\partial\|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{B}}\right\|_F} - 4\epsilon_{\mathbf{C}}\frac{\left(\mathbf{C}\mathbf{P} - \hat{\mathbf{C}}\mathbf{P}_{12}^\mathsf{T}\right)\mathbf{P}_{12}}{\left\|\frac{\partial\|E\|^2_{\mathcal{H}_2}}{\partial \mathbf{C}}\right\|_F}. \tag{4.95c}$$

4.B Equations for Frequency-Weighted Model Reduction

In this appendix, the equations that comes from partitioning $\mathbf{P}_E$ and $\mathbf{Q}_E$ as in (4.13) and using the realization (4.12) of E, will be presented, both for continuous and discrete time.

4.B.1 Continuous Time

Splitting the equations in (4.15) using the partitioning in (4.13) yields the equations

$$\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^\top + \mathbf{B}\mathbf{C}_i\mathbf{P}_{13}^\top + \mathbf{P}_{13}\mathbf{C}_i^\top\mathbf{B}^\top + \mathbf{B}\mathbf{D}_i\mathbf{D}_i^\top\mathbf{B}^\top = \mathbf{0}, \tag{4.96a}$$

$$\hat{\mathbf{A}}\hat{\mathbf{P}} + \hat{\mathbf{P}}\hat{\mathbf{A}}^\top + \hat{\mathbf{B}}\mathbf{C}_i\mathbf{P}_{23}^\top + \mathbf{P}_{23}\mathbf{C}_i^\top\hat{\mathbf{B}}^\top + \hat{\mathbf{B}}\mathbf{D}_i\mathbf{D}_i^\top\hat{\mathbf{B}}^\top = \mathbf{0}, \tag{4.96b}$$

$$\mathbf{A}_i\mathbf{P}_i + \mathbf{P}_i\mathbf{A}_i^\top + \mathbf{B}_i\mathbf{B}_i^\top = \mathbf{0}, \tag{4.96c}$$

$$\mathbf{A}_o\mathbf{P}_o + \mathbf{P}_o\mathbf{A}_o^\top + \mathbf{B}_o\mathbf{C}\mathbf{P}_{14} + \mathbf{P}_{14}^\top\mathbf{C}^\top\mathbf{B}_o^\top - \mathbf{B}_o\hat{\mathbf{C}}\mathbf{P}_{24} - \mathbf{P}_{24}^\top\hat{\mathbf{C}}^\top\mathbf{B}_o^\top = \mathbf{0}, \tag{4.96d}$$

$$\mathbf{A}\mathbf{P}_{12} + \mathbf{P}_{12}\hat{\mathbf{A}}^\top + \mathbf{B}\mathbf{C}_i\mathbf{P}_{23}^\top + \mathbf{P}_{13}\mathbf{C}_i^\top\hat{\mathbf{B}}^\top + \mathbf{B}\mathbf{D}_i\mathbf{D}_i^\top\hat{\mathbf{B}}^\top = \mathbf{0}, \tag{4.96e}$$

$$\mathbf{A}\mathbf{P}_{13} + \mathbf{P}_{13}\mathbf{A}_i^\top + \mathbf{B}\mathbf{C}_i\mathbf{P}_i + \mathbf{B}\mathbf{D}_i\mathbf{B}_i^\top = \mathbf{0}, \tag{4.96f}$$

$$\mathbf{A}\mathbf{P}_{14} + \mathbf{P}_{14}\mathbf{A}_o^\top + \mathbf{B}\mathbf{C}_i\mathbf{P}_{34} + \mathbf{P}\mathbf{C}^\top\mathbf{B}_o^\top - \mathbf{P}_{12}\hat{\mathbf{C}}^\top\mathbf{B}_o^\top = \mathbf{0}, \tag{4.96g}$$

$$\hat{\mathbf{A}}\mathbf{P}_{23} + \mathbf{P}_{23}\mathbf{A}_i^\top + \hat{\mathbf{B}}\mathbf{C}_i\mathbf{P}_i + \hat{\mathbf{B}}\mathbf{D}_i\mathbf{B}_i^\top = \mathbf{0}, \tag{4.96h}$$

$$\hat{\mathbf{A}}\mathbf{P}_{24} + \mathbf{P}_{24}\mathbf{A}_o^\top + \hat{\mathbf{B}}\mathbf{C}_i\mathbf{P}_{34} + \mathbf{P}_{12}^\top\mathbf{C}^\top\mathbf{B}_o^\top - \hat{\mathbf{P}}\hat{\mathbf{C}}^\top\mathbf{B}_o^\top = \mathbf{0}, \tag{4.96i}$$

$$\mathbf{A}_i\mathbf{P}_{34} + \mathbf{P}_{34}\mathbf{A}_o^\top + \mathbf{P}_{13}^\top\mathbf{C}^\top\mathbf{B}_o^\top - \mathbf{P}_{23}^\top\hat{\mathbf{C}}^\top\mathbf{B}_o^\top = \mathbf{0}, \tag{4.96j}$$

and

$$\mathbf{Q}\mathbf{A} + \mathbf{A}^\top\mathbf{Q} + \mathbf{Q}_{14}\mathbf{B}_o\mathbf{C} + \mathbf{C}^\top\mathbf{B}_o^\top\mathbf{Q}_{14}^\top + \mathbf{C}^\top\mathbf{D}_o^\top\mathbf{D}_o\mathbf{C} = \mathbf{0}, \tag{4.97a}$$

$$\hat{\mathbf{Q}}\hat{\mathbf{A}} + \hat{\mathbf{A}}^\top\hat{\mathbf{Q}} - \mathbf{Q}_{24}\mathbf{B}_o\hat{\mathbf{C}} - \hat{\mathbf{C}}^\top\mathbf{B}_o^\top\mathbf{Q}_{24}^\top + \hat{\mathbf{C}}^\top\mathbf{D}_o^\top\mathbf{D}_o\hat{\mathbf{C}} = \mathbf{0}, \tag{4.97b}$$

$$\mathbf{Q}_i\mathbf{A}_i + \mathbf{A}_i^\top\mathbf{Q}_i + \mathbf{Q}_{13}^\top\mathbf{B}\mathbf{C}_i + \mathbf{C}_i^\top\mathbf{B}^\top\mathbf{Q}_{13} + \mathbf{Q}_{23}^\top\hat{\mathbf{B}}\mathbf{C}_i + \mathbf{C}_i^\top\hat{\mathbf{B}}^\top\mathbf{Q}_{23} = \mathbf{0}, \tag{4.97c}$$

$$\mathbf{Q}_o\mathbf{A}_o + \mathbf{A}_o^\top\mathbf{Q}_o + \mathbf{C}_o^\top\mathbf{C}_o = \mathbf{0}, \tag{4.97d}$$

$$\mathbf{Q}_{12}\hat{\mathbf{A}} + \mathbf{A}^\top\mathbf{Q}_{12} - \mathbf{Q}_{14}\mathbf{B}_o\hat{\mathbf{C}} + \mathbf{C}^\top\mathbf{B}_o^\top\mathbf{Q}_{24}^\top - \mathbf{C}^\top\mathbf{D}_o^\top\mathbf{D}_o\hat{\mathbf{C}} = \mathbf{0}, \tag{4.97e}$$

$$\mathbf{Q}_{13}\mathbf{A}_i + \mathbf{A}^\top\mathbf{Q}_{13} + \mathbf{Q}\mathbf{B}\mathbf{C}_i + \mathbf{Q}_{12}\hat{\mathbf{B}}\mathbf{C}_i + \mathbf{C}^\top\mathbf{B}_o^\top\mathbf{Q}_{34}^\top = \mathbf{0}, \tag{4.97f}$$

$$\mathbf{Q}_{14}\mathbf{A}_o + \mathbf{A}^\top\mathbf{Q}_{14} + \mathbf{C}^\top\mathbf{B}_o^\top\mathbf{Q}_o + \mathbf{C}^\top\mathbf{D}_o^\top\mathbf{C}_o = \mathbf{0}, \tag{4.97g}$$

$$\mathbf{Q}_{23}\mathbf{A}_i + \hat{\mathbf{A}}^\top\mathbf{Q}_{23} + \mathbf{Q}_{12}^\top\mathbf{B}\mathbf{C}_i + \hat{\mathbf{Q}}\hat{\mathbf{B}}\mathbf{C}_i - \hat{\mathbf{C}}^\top\mathbf{B}_o^\top\mathbf{Q}_{34}^\top = \mathbf{0}, \tag{4.97h}$$

$$\mathbf{Q}_{24}\mathbf{A}_o + \hat{\mathbf{A}}^\top\mathbf{Q}_{24} - \hat{\mathbf{C}}^\top\mathbf{B}_o^\top\mathbf{Q}_o - \hat{\mathbf{C}}^\top\mathbf{D}_o^\top\mathbf{C}_o = \mathbf{0}, \tag{4.97i}$$

$$\mathbf{Q}_{34}\mathbf{A}_o + \mathbf{A}_i^\top\mathbf{Q}_{34} + \mathbf{C}_i^\top\mathbf{B}^\top\mathbf{Q}_{14} + \mathbf{C}_i^\top\hat{\mathbf{B}}^\top\mathbf{Q}_{24} = \mathbf{0}. \tag{4.97j}$$

Splitting the cost function, (4.14), using the realization of E, (4.12) and the partitioning of $\mathbf{P}_E$ and $\mathbf{Q}_E$, yields

$$\begin{aligned}\|E\|_{\mathcal{H}_2}^2 = \operatorname{tr}\Big(&\mathbf{D}_i^\top\mathbf{B}^\top\mathbf{Q}\mathbf{B}\mathbf{D}_i + 2\mathbf{D}_i^\top\hat{\mathbf{B}}^\top\mathbf{Q}_{12}^\top\mathbf{B}\mathbf{D}_i + \mathbf{D}_i^\top\hat{\mathbf{B}}^\top\hat{\mathbf{Q}}\hat{\mathbf{B}}\mathbf{D}_i \\ &+ \mathbf{B}_i^\top\mathbf{Q}_i\mathbf{B}_i + 2\mathbf{B}_i^\top\mathbf{Q}_{13}^\top\mathbf{B}\mathbf{D}_i + 2\mathbf{B}_i^\top\mathbf{Q}_{23}^\top\hat{\mathbf{B}}\mathbf{D}_i\Big),\end{aligned} \tag{4.98a}$$

$$\begin{aligned}\|E\|_{\mathcal{H}_2}^2 = \operatorname{tr}\Big(&\mathbf{D}_o\mathbf{C}\mathbf{P}\mathbf{C}^\top\mathbf{D}_o^\top - 2\mathbf{D}_o\hat{\mathbf{C}}\mathbf{P}_{12}^\top\mathbf{C}^\top\mathbf{D}_o^\top + \mathbf{D}_o\hat{\mathbf{C}}\hat{\mathbf{P}}\hat{\mathbf{C}}^\top\mathbf{D}_o^\top \\ &+ \mathbf{C}_o\mathbf{P}_o\mathbf{C}_o^\top + 2\mathbf{D}_o\mathbf{C}\mathbf{P}_{14}\mathbf{C}_o^\top - 2\mathbf{D}_o\hat{\mathbf{C}}\mathbf{P}_{24}\mathbf{C}_o^\top\Big).\end{aligned} \tag{4.98b}$$

The gradient becomes

$$\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \hat{\mathbf{A}}} = 2\left(\hat{\mathbf{Q}}\hat{\mathbf{P}} + \mathbf{Q}_{12}^\top \mathbf{P}_{12} + \mathbf{Q}_{23}\mathbf{P}_{23}^\top + \mathbf{Q}_{24}\mathbf{P}_{24}^\top\right), \tag{4.99a}$$

$$\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \hat{\mathbf{B}}} = 2\left(\mathbf{Q}_{12}^\top \mathbf{P}_{13} + \hat{\mathbf{Q}}\mathbf{P}_{23} + \mathbf{Q}_{23}\mathbf{P}_i + \mathbf{Q}_{24}\mathbf{P}_{34}^\top\right)\mathbf{C}_i^\top + 2\left(\hat{\mathbf{Q}}\hat{\mathbf{B}}\mathbf{D}_i + \mathbf{Q}_{12}^\top \mathbf{B}\mathbf{D}_i + \mathbf{Q}_{23}\mathbf{B}_i\right)\mathbf{D}_i^\top, \tag{4.99b}$$

$$\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \hat{\mathbf{C}}} = -2\mathbf{B}_o^\top\left(\mathbf{Q}_{14}^\top \mathbf{P}_{12} + \mathbf{Q}_{24}^\top \hat{\mathbf{P}} + \mathbf{Q}_{34}^\top \mathbf{P}_{23}^\top + \mathbf{Q}_o\mathbf{P}_{24}^\top\right) + 2\mathbf{D}_o^\top\left(\mathbf{D}_o\hat{\mathbf{C}}\hat{\mathbf{P}} - \mathbf{D}_o\mathbf{C}\mathbf{P}_{12} - \mathbf{C}_o\mathbf{P}_{24}^\top\right). \tag{4.99c}$$

4.B.2 Discrete Time

Splitting the equations in (4.32) using the partitioning in (4.13) yields the equations

$$\mathbf{A}\mathbf{P}\mathbf{A}^\top - \mathbf{P} + \mathbf{B}\mathbf{C}_i\mathbf{P}_{13}^\top\mathbf{A}^\top + \mathbf{A}\mathbf{P}_{13}\mathbf{C}_i^\top\mathbf{B}^\top + \mathbf{B}\mathbf{C}_i\mathbf{P}_i\mathbf{C}_i^\top\mathbf{B}^\top + \mathbf{B}\mathbf{D}_i\mathbf{D}_i^\top\mathbf{B}^\top = \mathbf{0}, \tag{4.100a}$$

$$\hat{\mathbf{A}}\hat{\mathbf{P}}\hat{\mathbf{A}}^\top - \hat{\mathbf{P}} + \hat{\mathbf{B}}\mathbf{C}_i\mathbf{P}_{23}^\top\hat{\mathbf{A}}^\top + \hat{\mathbf{A}}\mathbf{P}_{23}\mathbf{C}_i^\top\hat{\mathbf{B}}^\top + \hat{\mathbf{B}}\mathbf{C}_i\mathbf{P}_i\mathbf{C}_i^\top\hat{\mathbf{B}}^\top + \hat{\mathbf{B}}\mathbf{D}_i\mathbf{D}_i^\top\hat{\mathbf{B}}^\top = \mathbf{0}, \tag{4.100b}$$

$$\mathbf{A}_i\mathbf{P}_i\mathbf{A}_i^\top - \mathbf{P}_i + \mathbf{B}_i\mathbf{B}_i^\top = \mathbf{0}, \tag{4.100c}$$

$$\mathbf{A}_o\mathbf{P}_o\mathbf{A}_o^\top - \mathbf{P}_o + \mathbf{B}_o\mathbf{C}\mathbf{P}_{14}\mathbf{A}_o^\top + \mathbf{A}_o\mathbf{P}_{14}^\top\mathbf{C}^\top\mathbf{B}_o^\top - \mathbf{B}_o\hat{\mathbf{C}}\mathbf{P}_{24}\mathbf{A}_o^\top - \mathbf{A}_o\mathbf{P}_{24}^\top\hat{\mathbf{C}}^\top\mathbf{B}_o^\top - \mathbf{B}_o\hat{\mathbf{C}}\mathbf{P}_{12}\mathbf{C}^\top\mathbf{B}_o^\top - \mathbf{B}_o\mathbf{C}\mathbf{P}_{12}^\top\hat{\mathbf{C}}^\top\mathbf{B}_o + \mathbf{B}_o\mathbf{C}\mathbf{P}\mathbf{C}^\top\mathbf{B}_o^\top + \mathbf{B}_o\hat{\mathbf{C}}\hat{\mathbf{P}}\hat{\mathbf{C}}^\top\mathbf{B}_o^\top = \mathbf{0}, \tag{4.100d}$$

$$\mathbf{A}\mathbf{P}_{12}\hat{\mathbf{A}}^\top - \mathbf{P}_{12} + \mathbf{B}\mathbf{C}_i\mathbf{P}_{23}^\top\hat{\mathbf{A}}^\top + \mathbf{A}\mathbf{P}_{13}\mathbf{C}_i^\top\hat{\mathbf{B}}^\top + \mathbf{B}\mathbf{C}_i\mathbf{P}_i\mathbf{C}_i^\top\hat{\mathbf{B}}^\top + \mathbf{B}\mathbf{D}_i\mathbf{D}_i^\top\hat{\mathbf{B}}^\top = \mathbf{0}, \tag{4.100e}$$

$$\mathbf{A}\mathbf{P}_{13}\mathbf{A}_i^\top - \mathbf{P}_{13} + \mathbf{B}\mathbf{C}_i\mathbf{P}_i\mathbf{A}_i^\top + \mathbf{B}\mathbf{D}_i\mathbf{B}_i^\top = \mathbf{0}, \tag{4.100f}$$

$$\mathbf{A}\mathbf{P}_{14}\mathbf{A}_o^\top - \mathbf{P}_{14} + \mathbf{B}\mathbf{C}_i\mathbf{P}_{34}\mathbf{A}_o^\top + \mathbf{A}\mathbf{P}\mathbf{C}^\top\mathbf{B}_o^\top - \mathbf{A}\mathbf{P}_{12}\hat{\mathbf{C}}^\top\mathbf{B}_o^\top + \mathbf{B}\mathbf{C}_i\mathbf{P}_{13}^\top\mathbf{C}^\top\mathbf{B}_o^\top - \mathbf{B}\mathbf{C}_i\mathbf{P}_{23}^\top\hat{\mathbf{C}}^\top\mathbf{B}_o^\top = \mathbf{0}, \tag{4.100g}$$

$$\hat{\mathbf{A}}\mathbf{P}_{23}\mathbf{A}_i^\top - \mathbf{P}_{23} + \hat{\mathbf{B}}\mathbf{C}_i\mathbf{P}_i\mathbf{A}_i^\top + \hat{\mathbf{B}}\mathbf{D}_i\mathbf{B}_i^\top = \mathbf{0}, \tag{4.100h}$$

$$\hat{\mathbf{A}}\mathbf{P}_{24}\mathbf{A}_o^\top - \mathbf{P}_{24} + \hat{\mathbf{B}}\mathbf{C}_i\mathbf{P}_{34}\mathbf{A}_o^\top + \hat{\mathbf{A}}\mathbf{P}_{12}^\top\mathbf{C}^\top\mathbf{B}_o^\top - \hat{\mathbf{A}}\hat{\mathbf{P}}\hat{\mathbf{C}}^\top\mathbf{B}_o^\top + \hat{\mathbf{B}}\mathbf{C}_i\mathbf{P}_{13}^\top\mathbf{C}^\top\mathbf{B}_o^\top - \hat{\mathbf{B}}\mathbf{C}_i\mathbf{P}_{23}^\top\hat{\mathbf{C}}^\top\mathbf{B}_o^\top = \mathbf{0}, \tag{4.100i}$$

$$\mathbf{A}_i\mathbf{P}_{34}\mathbf{A}_o^\top - \mathbf{P}_{34} + \mathbf{A}_i\mathbf{P}_{13}^\top\mathbf{C}^\top\mathbf{B}_o^\top - \mathbf{A}_i\mathbf{P}_{23}^\top\hat{\mathbf{C}}^\top\mathbf{B}_o^\top = \mathbf{0}, \tag{4.100j}$$

and

$$\mathbf{A}^\mathsf{T}\mathbf{Q}\mathbf{A} - \mathbf{Q} + \mathbf{A}^\mathsf{T}\mathbf{Q}_{14}\mathbf{B}_o\mathbf{C} + \mathbf{C}^\mathsf{T}\mathbf{B}_o^\mathsf{T}\mathbf{Q}_{14}^\mathsf{T}\mathbf{A} + \mathbf{C}^\mathsf{T}\mathbf{B}_o^\mathsf{T}\mathbf{Q}_o\mathbf{B}_o\mathbf{C} + \mathbf{C}^\mathsf{T}\mathbf{D}_o^\mathsf{T}\mathbf{D}_o\mathbf{C} = \mathbf{0}, \tag{4.101a}$$

$$\hat{\mathbf{A}}^\mathsf{T}\hat{\mathbf{Q}}\hat{\mathbf{A}} - \hat{\mathbf{Q}} - \hat{\mathbf{A}}^\mathsf{T}\mathbf{Q}_{24}\mathbf{B}_o\hat{\mathbf{C}} - \hat{\mathbf{C}}^\mathsf{T}\mathbf{B}_o^\mathsf{T}\mathbf{Q}_{24}^\mathsf{T}\hat{\mathbf{A}} + \hat{\mathbf{C}}^\mathsf{T}\mathbf{B}_o^\mathsf{T}\mathbf{Q}_o\mathbf{B}_o\hat{\mathbf{C}} + \hat{\mathbf{C}}^\mathsf{T}\mathbf{D}_o^\mathsf{T}\mathbf{D}_o\hat{\mathbf{C}} = \mathbf{0}, \tag{4.101b}$$

$$\begin{aligned}\mathbf{A}_i^\mathsf{T}\mathbf{Q}_i\mathbf{A}_i - \mathbf{Q}_i + \mathbf{A}_i^\mathsf{T}\mathbf{Q}_{13}^\mathsf{T}\mathbf{B}\mathbf{C}_i + \mathbf{C}_i^\mathsf{T}\mathbf{B}^\mathsf{T}\mathbf{Q}_{13}\mathbf{A}_i + \mathbf{A}_i^\mathsf{T}\mathbf{Q}_{23}^\mathsf{T}\hat{\mathbf{B}}\mathbf{C}_i + \mathbf{C}_i^\mathsf{T}\hat{\mathbf{B}}^\mathsf{T}\mathbf{Q}_{23}\mathbf{A}_i \\ +\mathbf{C}_i^\mathsf{T}\hat{\mathbf{B}}^\mathsf{T}\mathbf{Q}_{12}^\mathsf{T}\mathbf{B}\mathbf{C}_i + \mathbf{C}_i^\mathsf{T}\mathbf{B}^\mathsf{T}\mathbf{Q}_{12}\hat{\mathbf{B}}\mathbf{C}_i + \mathbf{C}_i^\mathsf{T}\mathbf{B}^\mathsf{T}\mathbf{Q}\mathbf{B}\mathbf{C}_i + \mathbf{C}_i^\mathsf{T}\hat{\mathbf{B}}^\mathsf{T}\hat{\mathbf{Q}}\hat{\mathbf{B}}\mathbf{C}_i = \mathbf{0},\end{aligned} \tag{4.101c}$$

$$\mathbf{A}_o^\mathsf{T}\mathbf{Q}_o\mathbf{A}_o - \mathbf{Q}_o + \mathbf{C}_o^\mathsf{T}\mathbf{C}_o = \mathbf{0}, \tag{4.101d}$$

$$\begin{aligned}\mathbf{A}^\mathsf{T}\mathbf{Q}_{12}\hat{\mathbf{A}} - \mathbf{Q}_{12} - \mathbf{A}^\mathsf{T}\mathbf{Q}_{14}\mathbf{B}_o\hat{\mathbf{C}} + \mathbf{C}^\mathsf{T}\mathbf{B}_o^\mathsf{T}\mathbf{Q}_{24}^\mathsf{T}\hat{\mathbf{A}} \\ -\mathbf{C}^\mathsf{T}\mathbf{B}_o^\mathsf{T}\mathbf{Q}_o\mathbf{B}_o\hat{\mathbf{C}} - \mathbf{C}^\mathsf{T}\mathbf{D}_o^\mathsf{T}\mathbf{D}_o\hat{\mathbf{C}} = \mathbf{0},\end{aligned} \tag{4.101e}$$

$$\begin{aligned}\mathbf{A}^\mathsf{T}\mathbf{Q}_{13}\mathbf{A}_i - \mathbf{Q}_{13} + \mathbf{A}^\mathsf{T}\mathbf{Q}\mathbf{B}\mathbf{C}_i + \mathbf{A}^\mathsf{T}\mathbf{Q}_{12}\hat{\mathbf{B}}\mathbf{C}_i + \mathbf{C}^\mathsf{T}\mathbf{B}_o^\mathsf{T}\mathbf{Q}_{34}^\mathsf{T}\mathbf{A}_i \\ +\mathbf{C}^\mathsf{T}\mathbf{B}_o^\mathsf{T}\mathbf{Q}_{14}^\mathsf{T}\mathbf{B}\mathbf{C}_i + \mathbf{C}^\mathsf{T}\mathbf{B}_o^\mathsf{T}\mathbf{Q}_{24}^\mathsf{T}\hat{\mathbf{B}}\mathbf{C}_i = \mathbf{0},\end{aligned} \tag{4.101f}$$

$$\mathbf{A}^\mathsf{T}\mathbf{Q}_{14}\mathbf{A}_o - \mathbf{Q}_{14} + \mathbf{C}^\mathsf{T}\mathbf{B}_o^\mathsf{T}\mathbf{Q}_o\mathbf{A}_o + \mathbf{C}^\mathsf{T}\mathbf{D}_o^\mathsf{T}\mathbf{C}_o = \mathbf{0}, \tag{4.101g}$$

$$\begin{aligned}\hat{\mathbf{A}}^\mathsf{T}\mathbf{Q}_{23}\mathbf{A}_i - \mathbf{Q}_{23} + \hat{\mathbf{A}}^\mathsf{T}\mathbf{Q}_{12}^\mathsf{T}\mathbf{B}\mathbf{C}_i + \hat{\mathbf{A}}^\mathsf{T}\hat{\mathbf{Q}}\hat{\mathbf{B}}\mathbf{C}_i - \hat{\mathbf{C}}^\mathsf{T}\mathbf{B}_o^\mathsf{T}\mathbf{Q}_{34}^\mathsf{T}\mathbf{A}_i \\ -\hat{\mathbf{C}}^\mathsf{T}\mathbf{B}_o^\mathsf{T}\mathbf{Q}_{14}^\mathsf{T}\mathbf{B}\mathbf{C}_i - \hat{\mathbf{C}}\mathbf{B}_o^\mathsf{T}\mathbf{Q}_{24}\hat{\mathbf{B}}\mathbf{C}_i = \mathbf{0},\end{aligned} \tag{4.101h}$$

$$\hat{\mathbf{A}}^\mathsf{T}\mathbf{Q}_{24}\mathbf{A}_o - \mathbf{Q}_{24} - \hat{\mathbf{C}}^\mathsf{T}\mathbf{B}_o^\mathsf{T}\mathbf{Q}_o\mathbf{A}_o - \hat{\mathbf{C}}^\mathsf{T}\mathbf{D}_o^\mathsf{T}\mathbf{C}_o = \mathbf{0}, \tag{4.101i}$$

$$\mathbf{A}_i^\mathsf{T}\mathbf{Q}_{34}\mathbf{A}_o - \mathbf{Q}_{34} + \mathbf{C}_i^\mathsf{T}\mathbf{B}^\mathsf{T}\mathbf{Q}_{14}\mathbf{A}_o + \mathbf{C}_i^\mathsf{T}\hat{\mathbf{B}}^\mathsf{T}\mathbf{Q}_{24}\mathbf{A}_o = \mathbf{0}. \tag{4.101j}$$

Using the partitioning in (4.13) again, yields the cost function

$$\begin{aligned}\|E\|_{\mathcal{H}_2}^2 = \operatorname{tr}\Big(&\mathbf{D}_i^\mathsf{T}\mathbf{B}^\mathsf{T}\mathbf{Q}\mathbf{B}\mathbf{D}_i + 2\mathbf{D}_i^\mathsf{T}\hat{\mathbf{B}}^\mathsf{T}\mathbf{Q}_{12}^\mathsf{T}\mathbf{B}\mathbf{D}_i + \mathbf{D}_i^\mathsf{T}\hat{\mathbf{B}}^\mathsf{T}\hat{\mathbf{Q}}\hat{\mathbf{B}}\mathbf{D}_i + \mathbf{B}_i^\mathsf{T}\mathbf{Q}_i\mathbf{B}_i \\ &+ 2\mathbf{B}_i^\mathsf{T}\mathbf{Q}_{13}^\mathsf{T}\mathbf{B}\mathbf{D}_i + 2\mathbf{B}_i^\mathsf{T}\mathbf{Q}_{23}^\mathsf{T}\hat{\mathbf{B}}\mathbf{D}_i + \mathbf{D}_i^\mathsf{T}\left[\mathbf{D}^\mathsf{T} - \hat{\mathbf{D}}^\mathsf{T}\right]\mathbf{D}_o^\mathsf{T}\mathbf{D}_o\left[\mathbf{D} - \hat{\mathbf{D}}\right]\mathbf{D}_i\Big),\end{aligned} \tag{4.102a}$$

$$\begin{aligned}\|E\|_{\mathcal{H}_2}^2 = \operatorname{tr}\Big(&\mathbf{D}_o\mathbf{C}\mathbf{P}\mathbf{C}^\mathsf{T}\mathbf{D}_o^\mathsf{T} - 2\mathbf{D}_o\hat{\mathbf{C}}\mathbf{P}_{12}^\mathsf{T}\mathbf{C}^\mathsf{T}\mathbf{D}_o^\mathsf{T} + \mathbf{D}_o\hat{\mathbf{C}}\hat{\mathbf{P}}\hat{\mathbf{C}}^\mathsf{T}\mathbf{D}_o^\mathsf{T} + \mathbf{C}_o\mathbf{P}_o\mathbf{C}_o^\mathsf{T} \\ &+ 2\mathbf{D}_o\mathbf{C}\mathbf{Q}_{14}\mathbf{C}_o^\mathsf{T} - 2\mathbf{D}_o\hat{\mathbf{C}}\mathbf{Q}_{24}\mathbf{C}_o^\mathsf{T} + \mathbf{D}_o\left[\mathbf{D} - \hat{\mathbf{D}}\right]\mathbf{D}_i\mathbf{D}_i^\mathsf{T}\left[\mathbf{D}^\mathsf{T} - \hat{\mathbf{D}}^\mathsf{T}\right]\mathbf{D}_o^\mathsf{T}\Big).\end{aligned} \tag{4.102b}$$

The gradient becomes

$$\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \hat{\mathbf{A}}} = 2\Big(\hat{\mathbf{Q}}\hat{\mathbf{A}}\hat{\mathbf{P}} + \mathbf{Q}_{12}^{\mathsf{T}}\mathbf{A}\mathbf{P}_{12} + \mathbf{Q}_{23}\mathbf{A}_i\mathbf{P}_{23}^{\mathsf{T}} + \mathbf{Q}_{24}\mathbf{A}_o\mathbf{P}_{24}^{\mathsf{T}} + \mathbf{Q}_{24}\mathbf{B}_o\Big[\mathbf{C}\mathbf{P}_{12} - \hat{\mathbf{C}}\hat{\mathbf{P}}\Big]\Big), \tag{4.103a}$$

$$\begin{aligned}\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \hat{\mathbf{B}}} = 2\Big(&\mathbf{Q}_{12}^{\mathsf{T}}\mathbf{A}\mathbf{P}_{13} + \hat{\mathbf{Q}}\hat{\mathbf{A}}\mathbf{P}_{23} + \mathbf{Q}_{23}\mathbf{A}_i\mathbf{P}_i + \mathbf{Q}_{24}\mathbf{A}_o\mathbf{Q}_{34}^{\mathsf{T}} \\ &+ \mathbf{Q}_{24}\mathbf{B}_o\Big[\mathbf{C}\mathbf{P}_{13} - \hat{\mathbf{C}}\mathbf{P}_{23}\Big] + \Big[\mathbf{Q}_{12}^{\mathsf{T}}\mathbf{B} + \hat{\mathbf{Q}}\hat{\mathbf{B}}\Big]\mathbf{C}_i\mathbf{P}_i\Big)\mathbf{C}_i^{\mathsf{T}} \\ &+ 2\Big(\hat{\mathbf{Q}}\hat{\mathbf{B}}\mathbf{D}_i + \mathbf{Q}_{12}^{\mathsf{T}}\mathbf{B}\mathbf{D}_i + \mathbf{Q}_{23}\mathbf{B}_i\Big)\mathbf{D}_i^{\mathsf{T}},\end{aligned} \tag{4.103b}$$

$$\begin{aligned}\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \hat{\mathbf{C}}} = -2\mathbf{B}_o^{\mathsf{T}}\Big(&\mathbf{Q}_{14}^{\mathsf{T}}\mathbf{A}\mathbf{P}_{12} + \mathbf{Q}_{24}^{\mathsf{T}}\hat{\mathbf{A}}\hat{\mathbf{P}} + \mathbf{Q}_{34}^{\mathsf{T}}\mathbf{A}_i\mathbf{P}_{23}^{\mathsf{T}} + \mathbf{Q}_o\mathbf{A}_o\mathbf{P}_{24}^{\mathsf{T}} \\ &+ \mathbf{Q}_o\mathbf{B}_o\Big[\mathbf{C}\mathbf{P}_{12} - \hat{\mathbf{C}}\hat{\mathbf{P}}\Big] + \Big[\mathbf{Q}_{14}^{\mathsf{T}}\mathbf{B} + \mathbf{Q}_{24}^{\mathsf{T}}\hat{\mathbf{B}}\Big]\mathbf{C}_i\mathbf{P}_{23}^{\mathsf{T}}\Big) \\ &+ 2\mathbf{D}_o^{\mathsf{T}}\Big(\mathbf{D}_o\hat{\mathbf{C}}\hat{\mathbf{P}} - \mathbf{D}_o\mathbf{C}\mathbf{P}_{12} - \mathbf{C}_o\mathbf{P}_{24}^{\mathsf{T}}\Big),\end{aligned} \tag{4.103c}$$

$$\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \hat{\mathbf{D}}} = 2\mathbf{D}_o^{\mathsf{T}}\mathbf{D}_o\Big(\hat{\mathbf{D}} - \mathbf{D}\Big)\mathbf{D}_i\mathbf{D}_i^{\mathsf{T}}. \tag{4.103d}$$

4.C Gradient of the Frequency-Limited Case

In this section, the derivation of the gradient of the cost function (4.67) will be presented. We start by differentiating the cost function (4.67) with respect to $\hat{\mathbf{B}}, \hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$. First, note that neither $\mathbf{Q}_\omega, \mathbf{Q}_{12,\omega}$ nor $\hat{\mathbf{Q}}_\omega$ in equation (4.67a) depend on $\hat{\mathbf{B}}$. This means that (4.67a) is quadratic in $\hat{\mathbf{B}}$. Analogous observations can be made with equation (4.67b) and the variable $\hat{\mathbf{C}}$ and similarly with $\hat{\mathbf{D}}$. Hence, the derivative of the cost function with respect $\hat{\mathbf{B}}, \hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$ becomes

$$\frac{\partial \|E\|^2_{\mathcal{H}_2,\omega}}{\partial \hat{\mathbf{B}}} = 2\Big(\hat{\mathbf{Q}}_\omega\hat{\mathbf{B}} + \mathbf{Q}_{12,\omega}^{\mathsf{T}}\mathbf{B}\hat{\mathbf{S}}_\omega^{\mathsf{T}}\hat{\mathbf{C}}^{\mathsf{T}}\Big[\mathbf{D} - \hat{\mathbf{D}}\Big]\Big), \tag{4.104a}$$

$$\frac{\partial \|E\|^2_{\mathcal{H}_2,\omega}}{\partial \hat{\mathbf{C}}} = 2\Big(\hat{\mathbf{C}}\hat{\mathbf{P}}_\omega - \mathbf{C}\mathbf{P}_{12,\omega} - \Big[\mathbf{D} - \hat{\mathbf{D}}\Big]\hat{\mathbf{B}}^{\mathsf{T}}\hat{\mathbf{S}}_\omega^{\mathsf{T}}\Big), \tag{4.104b}$$

$$\frac{\partial \|E\|^2_{\mathcal{H}_2,\omega}}{\partial \hat{\mathbf{D}}} = -2\Big(\mathbf{C}\mathbf{S}_\omega\mathbf{B} + \mathbf{D}\omega - \hat{\mathbf{C}}\hat{\mathbf{S}}_\omega\hat{\mathbf{B}} - \hat{\mathbf{D}}\omega + \Big[\mathbf{D} - \hat{\mathbf{D}}\Big]\omega\Big). \tag{4.104c}$$

When differentiating with respect to $\hat{\mathbf{A}}$ it is important to remember that $\hat{\mathbf{Q}}_\omega$ and $\mathbf{Q}_{12,\omega}$ depend on $\hat{\mathbf{A}}$.

$$\frac{\partial \|E\|^2_{\mathcal{H}_2,\omega}}{\partial a_{ij}} = \operatorname{tr}\left(2\hat{\mathbf{B}}\mathbf{B}^\top \frac{\partial \mathbf{Q}_{12,\omega}}{\partial a_{ij}} + \hat{\mathbf{B}}\hat{\mathbf{B}}^\top \frac{\partial \hat{\mathbf{Q}}_\omega}{\partial a_{ij}} - 2\hat{\mathbf{C}}\frac{\partial \hat{\mathbf{S}}_\omega}{\partial a_{ij}}\hat{\mathbf{B}}\left(\mathbf{D}^\top - \hat{\mathbf{D}}^\top\right)\right), \tag{4.105}$$

where $\frac{\partial \mathbf{Q}_{12,\omega}}{\partial a_{ij}}$ and $\frac{\partial \hat{\mathbf{Q}}_\omega}{\partial a_{ij}}$ depend on $\hat{\mathbf{A}}$ via the differentiated versions of the equations in (4.65),

$$\hat{\mathbf{A}}^\top \frac{\partial \mathbf{Q}^\top_{12,\omega}}{\partial a_{ij}} + \frac{\partial \mathbf{Q}^\top_{12,\omega}}{\partial a_{ij}}\mathbf{A} + \frac{\partial \hat{\mathbf{A}}^\top}{\partial a_{ij}}\mathbf{Q}^\top_{12,\omega} - \frac{\partial \hat{\mathbf{S}}^\top_\omega}{\partial a_{ij}}\hat{\mathbf{C}}^\top\mathbf{C} = \mathbf{0}, \tag{4.106a}$$

$$\hat{\mathbf{A}}^\top \frac{\partial \hat{\mathbf{Q}}_\omega}{\partial a_{ij}} + \frac{\partial \hat{\mathbf{Q}}_\omega}{\partial a_{ij}}\hat{\mathbf{A}} + \frac{\partial \hat{\mathbf{A}}^\top}{\partial a_{ij}}\hat{\mathbf{Q}}_\omega + \hat{\mathbf{Q}}_\omega\frac{\partial \hat{\mathbf{A}}}{\partial a_{ij}} + \frac{\partial \hat{\mathbf{S}}^\top_\omega}{\partial a_{ij}}\hat{\mathbf{C}}^\top\hat{\mathbf{C}} + \hat{\mathbf{C}}^\top\hat{\mathbf{C}}\frac{\partial \hat{\mathbf{S}}_\omega}{\partial a_{ij}} = \mathbf{0}. \tag{4.106b}$$

Using Lemma 4.1 on (4.105) with the equations in (4.29) and (4.106) yields

$$\begin{aligned}\frac{\partial \|E\|^2_{\mathcal{H}_2,\omega}}{\partial a_{ij}} = 2\operatorname{tr}\left(\frac{\partial \hat{\mathbf{A}}^\top}{\partial a_{ij}}\left[\mathbf{Y}^\top_\omega\mathbf{X} + \hat{\mathbf{Q}}_\omega\hat{\mathbf{P}}\right] + \frac{\partial \hat{\mathbf{S}}^\top_\omega}{\partial a_{ij}}\left[\hat{\mathbf{C}}^\top\hat{\mathbf{C}}\hat{\mathbf{P}} - \hat{\mathbf{C}}^\top\mathbf{C}\mathbf{X}\right]\right)\\ - 2\operatorname{tr}\left(\frac{\partial \hat{\mathbf{S}}_\omega}{\partial a_{ij}}\left[\hat{\mathbf{B}}\left(\mathbf{D}^\top - \hat{\mathbf{D}}^\top\right)\hat{\mathbf{C}}\right]\right).\end{aligned} \tag{4.107}$$

What remains is to rewrite the two last terms in (4.107), which includes $\frac{\partial \hat{\mathbf{S}}_\omega}{\partial a_{ij}}$ and $\frac{\partial \hat{\mathbf{S}}^\top_\omega}{\partial a_{ij}}$. Recall the definition of $\hat{\mathbf{S}}_\omega$,

$$\hat{\mathbf{S}}_\omega \triangleq \operatorname{Re}\left(\frac{i}{\pi}\ln\left(-\hat{\mathbf{A}} - i\omega\mathbb{I}\right)\right) \tag{4.108}$$

and differentiate with respect to an element in $\hat{\mathbf{A}}$, i.e., a_{ij}. This yields

$$\begin{aligned}\frac{\partial \hat{\mathbf{S}}_\omega}{\partial a_{ij}} &= \operatorname{Re}\left(\frac{i}{2\pi}L\left(-\hat{\mathbf{A}} - i\omega\mathbb{I}, \frac{\partial}{\partial a_{ij}}\left(-\hat{\mathbf{A}} - i\omega\mathbb{I}\right)\right)\right)\\ &= \operatorname{Re}\left(\frac{i}{2\pi}L\left(-\hat{\mathbf{A}} - i\omega\mathbb{I}, -\frac{\partial \hat{\mathbf{A}}}{\partial a_{ij}}\right)\right),\end{aligned} \tag{4.109}$$

where $L(\mathbf{A}, \mathbf{E})$ is the Frechét derivative of the matrix logarithm with

$$L(\mathbf{A}, \mathbf{E}) = \int_0^1 \left(t(\mathbf{A} - \mathbb{I}) + \mathbb{I}\right)^{-1}\mathbf{E}\left(t(\mathbf{A} - \mathbb{I}) + \mathbb{I}\right)^{-1} \mathrm{d}t, \tag{4.110}$$

see Higham [2008].

The function $L(\mathbf{A}, \mathbf{E})$ can be efficiently evaluated using the algorithm in Higham [2008] or Al-Mohy et al. [2012]. Substituting (4.109) into (4.107) and using (4.110) with the fact that the tr-operator and the integral can be interchanged, yields

$$\begin{aligned}\frac{\partial \|E\|^2_{\mathcal{H}_2,\omega}}{\partial a_{ij}} &= 2\,\mathrm{tr}\left(\frac{\partial \hat{\mathbf{A}}^\top}{\partial a_{ij}}\left[\mathbf{Q}^\top_{12,\omega}\mathbf{P}_{12} + \hat{\mathbf{Q}}_\omega\hat{\mathbf{P}}\right] + \frac{\partial \hat{\mathbf{S}}^\top_\omega}{\partial a_{ij}}\left[\hat{\mathbf{C}}^\top\hat{\mathbf{C}}\hat{\mathbf{P}} - \hat{\mathbf{C}}^\top\mathbf{C}\mathbf{P}_{12}\right]\right) \\ &\qquad - 2\,\mathrm{tr}\left(\frac{\partial \hat{\mathbf{S}}_\omega}{\partial a_{ij}}\left[\hat{\mathbf{B}}\left\{\mathbf{D}^\top - \hat{\mathbf{D}}^\top\right\}\hat{\mathbf{C}}\right]\right) \\ &= 2\,\mathrm{tr}\left(\frac{\partial \hat{\mathbf{A}}^\top}{\partial a_{ij}}\left[\mathbf{Q}^\top_{12,\omega}\mathbf{P}_{12} + \hat{\mathbf{Q}}_\omega\hat{\mathbf{P}}\right]\right) - 2\,\mathrm{tr}\left(\frac{\partial \hat{\mathbf{A}}^\top}{\partial a_{ij}}\,\mathrm{Re}\left[\frac{i}{\pi}L\left(-\hat{\mathbf{A}} - i\omega\mathbb{I}, \mathbf{V}\right)\right]^\top\right) \\ &= \mathrm{tr}\left(\frac{\partial \hat{\mathbf{A}}^\top}{\partial a_{ij}}\left[2\left\{\mathbf{Q}^\top_{12,\omega}\mathbf{P}_{12} + \hat{\mathbf{Q}}_\omega\hat{\mathbf{P}}\right\} - 2\mathbf{W}\right]\right), \end{aligned} \tag{4.111}$$

where

$$\mathbf{W} = \mathrm{Re}\left(\frac{i}{\pi}L\left(-\hat{\mathbf{A}} - i\omega\mathbb{I}, \mathbf{V}\right)\right)^\top, \tag{4.112}$$

$$\mathbf{V} = \hat{\mathbf{C}}^\top\hat{\mathbf{C}}\hat{\mathbf{P}} - \hat{\mathbf{C}}^\top\mathbf{C}\mathbf{P}_{12} - \hat{\mathbf{C}}^\top\left(\mathbf{D} - \hat{\mathbf{D}}\right)\hat{\mathbf{B}}^\top. \tag{4.113}$$

5

LPV Modeling

In this chapter, local methods to approximate LPV models are developed. The methods use an approach that tries to preserve the input-output relations from the given models in the resulting LPV model. This is done by minimizing the sum of the $\mathcal{H}_2$-norms of the difference between the given models and a parametrized LPV model. When developing the methods, large effort is made on making the method computationally efficient. The material in this chapter is largely based on Petersson and Löfberg [2012c].

5.1 Introduction

In the last decades, intensive research has been carried out on linear parameter-varying models (LPV models), see e.g., Rugh and Shamma [2000], Leith and Leithead [2000], Tóth [2008], Lovera et al. [2011] or Mohammadpour and Scherer [2012]. An important reason for this interest is that it is a powerful tool for modeling and analysis of nonlinear systems, such as aircrafts (see Marcos and Balas [2004]) or wafer stages (see Wassink et al. [2005]). Some advanced robustness analysis methods, such as IQC-analysis and μ-analysis, see e.g., Megretski and Rantzer [1997], Zhou et al. [1996], require a conversion of the LPV model into a linear fractional representation (LFR), see e.g., Zhou et al. [1996]. For this to be possible it is necessary that the parametric matrices $\mathbf{A}(\mathbf{p})$, $\mathbf{B}(\mathbf{p})$, $\mathbf{C}(\mathbf{p})$ and $\mathbf{D}(\mathbf{p})$ of the LPV model are rational in $\mathbf{p}$. This requirement is often violated in LPV models generated directly from a non-fractional model description, either due to presence of non-fractional parametric expressions or tabulated data in the model. In both cases, rational approximations must be used to obtain a suitable model. This motivates a method that both can approximate a nonlinear model with an LPV model and approximate a complex LPV model with a less complex one.

As described in Section 2.1.5, LPV-models can be described by linear differential equations whose coefficients depend on scheduling parameters,

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{p})\mathbf{x}(t) + \mathbf{B}(\mathbf{p})\mathbf{u}(t), \tag{5.1}$$

$$\mathbf{y}(t) = \mathbf{C}(\mathbf{p})\mathbf{x}(t) + \mathbf{D}(\mathbf{p})\mathbf{u}(t), \tag{5.2}$$

where $\mathbf{x}(t)$ is the state, $\mathbf{u}(t)$ and $\mathbf{y}(t)$ are the input and output signals and $\mathbf{p}(t)$ is the vector of scheduling parameters. For example, in flight control applications, the components of $\mathbf{p}(t)$ are typically mass, position of centre of gravity and various aerodynamic coefficients, but can also include state dependent parameters such as altitude and velocity, specifying current flight conditions.

Generation of LPV models can simplistically be divided into two main families of methods, global methods (see e.g., Nemani et al. [1995], Lee and Poolla [1999], Bamieh and Giarre [2002], Felici et al. [2007], Tóth [2008]) and local methods (see e.g., Steinbuch et al. [2003], Wassink et al. [2005], Lovera and Mercere [2007], De Caigny et al. [2011], Pfifer and Hecker [2008], De Caigny et al. [2012]). A survey of existing methods can be found in Tóth [2008]. The global methods will only be mentioned briefly, since the main focus will be on local methods.

5.2 Global Methods

In the class of global methods, a global identification experiment is performed by exciting the system while the scheduling parameters change the dynamics of the system. An advantage with this approach, of generating LPV models, is that it is also possible to capture the rate of change of the parameters and how they can vary between different operating points. However, one drawback is that it is sometimes, for example in some flight applications, not possible to perform such an experiment.

5.3 Local Methods

In the class of local methods, a set of LTI models, $\mathcal{M} = \left\{ G_i = \left[\begin{array}{c|c} \mathbf{A}_i & \mathbf{B}_i \\ \hline \mathbf{C}_i & \mathbf{D}_i \end{array} \right], \mathbf{p}_i \right\}_{i=1}^{N}$, are interpolated, or in some other way combined, to generate an LPV model. These local models, G_i, can, for example, have been identified using a set of input-output measurements where the parameters have been kept constant, for which there exists several methods, see e.g., Ljung [1999], or by linearizing a nonlinear model in different operating points.

In this family of methods it is assumed that the system can operate at different fixed operating points, where the scheduling parameters are "frozen". There are of course systems where this is not possible and where this family of methods is inapplicable, requiring the use of global methods. Another drawback with this family of methods is that it does not take time variations of the scheduling parameters into account, thus limiting local methods to systems where the scheduling parameters vary slowly in time, which is a commonly used assumption in gain scheduling, see Shamma and Athans [1992]. To see this more clearly, write the LPV system as

$$G(\mathbf{p}, \dot{\mathbf{p}}, \dots) = \left[\begin{array}{c|c} \mathbf{A}_S(\mathbf{p}) & \mathbf{B}_S(\mathbf{p}) \\ \hline \mathbf{C}_S(\mathbf{p}) & \mathbf{D}_S(\mathbf{p}) \end{array}\right] + \left[\begin{array}{c|c} \mathbf{A}_D(\mathbf{p}, \dot{\mathbf{p}}, \dots) & \mathbf{B}_D(\mathbf{p}, \dot{\mathbf{p}}, \dots) \\ \hline \mathbf{C}_D(\mathbf{p}, \dot{\mathbf{p}}, \dots) & \mathbf{D}_D(\mathbf{p}, \dot{\mathbf{p}}, \dots) \end{array}\right]$$
$$= G_S(\mathbf{p}) + G_D(\mathbf{p}, \dot{\mathbf{p}}, \dots), \quad (5.3)$$

where $G_S(\mathbf{p})$ only depends on the current parameter value and does not include any dynamic dependence of the parameters, and $G_D(\mathbf{p}, \dot{\mathbf{p}}, \dots)$ includes all the dynamic dependence of the parameters. G_D has the property that $G_D(\mathbf{p}, \mathbf{0}, \mathbf{0}, \dots) = 0$. If the parameters are kept constant and the models, G_i, are generated

$$G(\mathbf{p}_i, 0, 0, , \dots) = G_S(\mathbf{p}_i) + G_D(\mathbf{p}_i, 0, 0, \dots) = G_S(\mathbf{p}_i),$$

one observes that the information in G_D is lost. This is one reason why one has to be careful when doing model interpolation. A paper that explains the pitfalls of interpolation is Tóth et al. [2007].

A common drawback of many of the local methods is that they need the local models to be given in the same state-space basis, see e.g., Pfifer and Hecker [2008]. However, the LTI models given in $\mathcal{M}$ are related to the true LPV system as

$$G_i = \left[\begin{array}{c|c} \mathbf{A}_i & \mathbf{B}_i \\ \hline \mathbf{C}_i & \mathbf{D}_i \end{array}\right] = \left[\begin{array}{c|c} \mathbf{T}_i^{-1}\mathbf{A}_S(\mathbf{p})\mathbf{T}_i & \mathbf{T}_i^{-1}\mathbf{B}_S(\mathbf{p}) \\ \hline \mathbf{C}_S(\mathbf{p})\mathbf{T}_i & \mathbf{D}_S(\mathbf{p}) \end{array}\right],$$

for some invertible matrices $\mathbf{T}_i$, which are unknown. Hence, one cannot, in general, assume that the given LTI models are described in the same state-space basis. Some methods have remedies to this, by trying to find invertible matrices $\hat{\mathbf{T}}_i$ to be able to transform the LTI models, G_i, to a common basis that encourage interpolation, usually some canonical form, see e.g., Steinbuch et al. [2003]. However, these LTI models in canonical forms may suffer from bad numerics. In De Caigny et al. [2012] they solve this problem by fixing one of the given models as a reference model and transforming the other models to state-space bases that are consistent with the reference model.

5.4 LPV Modeling using an $\mathcal{H}_2$-Measure

The methods that will be described in this section are based on the model-reduction techniques introduced in Section 4.4 and are in the family of local methods.

The goal with the methods proposed in this section is to try to preserve the input-output relations of the given LTI models in $\mathcal{M}$, instead of doing direct interpolation of system matrices. Let $G(\mathbf{p})$ denote the true LPV system, then ideally the goal would be to find an LPV model, $\hat{G}(\mathbf{p})$, that is optimal with respect to some global discrepancy measure on the model error, for instance the following integral

$$\min_{\hat{\mathbf{A}}(\mathbf{p}),\hat{\mathbf{B}}(\mathbf{p}),\hat{\mathbf{C}}(\mathbf{p}),\hat{\mathbf{D}}(\mathbf{p})} \int \left\| G(\mathbf{p}) - \hat{G}(\mathbf{p}) \right\|^2_{\mathcal{H}_2,\omega} \mathrm{d}\mathbf{p}, \tag{5.4}$$

where

$$\hat{G}(\mathbf{p}) : \left\{ \begin{array}{rcl} \dot{\mathbf{x}}(t) & = & \hat{\mathbf{A}}(\mathbf{p})\mathbf{x}(t) + \hat{\mathbf{B}}(\mathbf{p})\mathbf{u}(t) \\ \mathbf{y}(t) & = & \hat{\mathbf{C}}(\mathbf{p})\mathbf{x}(t) + \hat{\mathbf{D}}(\mathbf{p})\mathbf{u}(t) \end{array} \right. . \tag{5.5}$$

This is not always practical or even tractable. In many applications, e.g., flight applications, one often only have a simulation model available or a model that is used for computational fluid-dynamic calculations and not an analytical nonlinear model and it is only possible to extract linearized models for discrete values of the scheduling parameters, $\mathbf{p}_i$, i.e., we are given the model set $\mathcal{M} = \{G_i, \mathbf{p}_i\}_{i=1}^N$. Having this in mind (5.4) is changed into a discretized, in the parameters, version,

$$\min_{\hat{\mathbf{A}}(\mathbf{p}),\hat{\mathbf{B}}(\mathbf{p}),\hat{\mathbf{C}}(\mathbf{p}),\hat{\mathbf{D}}(\mathbf{p})} \sum_{i=1}^{N} \left\| G_i - \hat{G}(\mathbf{p}_i) \right\|^2_{\mathcal{H}_2,\omega}. \tag{5.6}$$

The two most widely used norms in system theory are the $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms, both capturing the input-output relation of the system. As indicated in (5.4) and (5.6), the norm that will be used here is the $\mathcal{H}_2$-norm (or the frequency-limited $\mathcal{H}_2$-norm). The main reason for this choice is, as in Chapter 4, that the cost function, again, becomes differentiable with respect to the optimization variables, with readily computed gradients.

5.4.1 General Properties

Since the LPV methods in this section will be based on the methods in Section 4.4, they also inherit the property that they are invariant under state transformations of the given LTI systems. This was useful in the model-reduction scheme since it does not matter in which state basis the given system is described. For the LPV methods, this fact can be utilized again. As explained in Section 5.3, what we are searching for in the local methods is the $G_S(\mathbf{p})$-part of the LPV model, which is related to the model set $\mathcal{M}$ as

$$\mathcal{M} = \{G_i, \mathbf{p}_i\}_{i=1}^N, \; G_i = \left[\begin{array}{c|c} \mathbf{A}_i & \mathbf{B}_i \\ \hline \mathbf{C}_i & \mathbf{D}_i \end{array}\right] = \left[\begin{array}{c|c} \mathbf{T}_i^{-1}\mathbf{A}_S(\mathbf{p})\mathbf{T}_i & \mathbf{T}_i^{-1}\mathbf{B}_S(\mathbf{p}) \\ \hline \mathbf{C}_S(\mathbf{p})\mathbf{T}_i & \mathbf{D}_S(\mathbf{p}) \end{array}\right],$$

where $\mathbf{T}_i$ are some unknown invertible matrices, which, generally, are not related to each other. Since the methods are invariant under state transformations we do not seek to find these $\mathbf{T}_i$, only $G_S(\mathbf{p})$, which is an advantage compared to most other local methods.

One thing that has been left out so far, is how the system matrices $\hat{\mathbf{A}}(\mathbf{p})$,$\hat{\mathbf{B}}(\mathbf{p})$,$\hat{\mathbf{C}}(\mathbf{p})$ and $\hat{\mathbf{D}}(\mathbf{p})$ are parametrized. These matrices are taken to be linear combinations of some basis functions $w_k(\mathbf{p})$, e.g., in the polynomial case, monomials. The system matrices in the LPV model, $\hat{G}(\mathbf{p})$, will then depend on $\mathbf{p}$ as

$$\hat{\mathbf{A}}(\mathbf{p}) = \sum_k w_k(\mathbf{p})\hat{\mathbf{A}}^{(k)}, \tag{5.7a}$$

$$\hat{\mathbf{B}}(\mathbf{p}) = \sum_k w_k(\mathbf{p})\hat{\mathbf{B}}^{(k)}, \tag{5.7b}$$

$$\hat{\mathbf{C}}(\mathbf{p}) = \sum_k w_k(\mathbf{p})\hat{\mathbf{C}}^{(k)}, \tag{5.7c}$$

$$\hat{\mathbf{D}}(\mathbf{p}) = \sum_k w_k(\mathbf{p})\hat{\mathbf{D}}^{(k)}, \tag{5.7d}$$

where the functions $w_k(\mathbf{p})$ are design choices that can be hard to choose to not make the model class to restrictive. However, it is not as restrictive as one might think. To see this, start by looking at how an LPV model changes when doing a state transformation, which can depend on the parameters. Given the state transformation

$$\bar{\mathbf{x}} = \bar{\mathbf{T}}(\mathbf{p})\mathbf{x}, \tag{5.8}$$

where $\bar{\mathbf{T}}(\mathbf{p})$, in the continuous-time case is a nonsingular continuously differentiable matrix for all valid parameter values, and in the discrete-time case is a matrix rational function of $\mathbf{p}$ and invertible for all $\mathbf{p}_k$. For the continuous-time case, given an LPV model as in (5.3), entails

$$\begin{aligned}\bar{G}(\mathbf{p}, \dot{\mathbf{p}}, \dots) &= \left[\begin{array}{c|c} \bar{\mathbf{T}}(\mathbf{p})\mathbf{A}_S(\mathbf{p})\bar{\mathbf{T}}^{-1}(\mathbf{p}) & \bar{\mathbf{T}}(\mathbf{p})\mathbf{B}_S(\mathbf{p}) \\ \hline \mathbf{C}_S(\mathbf{p})\bar{\mathbf{T}}^{-1}(\mathbf{p}) & \mathbf{D}_S(\mathbf{p}) \end{array}\right] \\ &+ \left[\begin{array}{c|c} \bar{\mathbf{T}}(\mathbf{p})\mathbf{A}_D(\mathbf{p}, \dot{\mathbf{p}}, \dots)\bar{\mathbf{T}}^{-1}(\mathbf{p}) + \dot{\bar{\mathbf{T}}}(\mathbf{p})\bar{\mathbf{T}}^{-1}(\mathbf{p}) & \bar{\mathbf{T}}(\mathbf{p})\mathbf{B}_D(\mathbf{p}, \dot{\mathbf{p}}, \dots) \\ \hline \mathbf{C}_D(\mathbf{p}, \dot{\mathbf{p}}, \dots)\bar{\mathbf{T}}^{-1}(\mathbf{p}) & \mathbf{D}_D(\mathbf{p}, \dot{\mathbf{p}}, \dots) \end{array}\right] \\ &= \bar{G}_S(\mathbf{p}) + \bar{G}_D(\mathbf{p}, \dot{\mathbf{p}}, \dots). \end{aligned} \tag{5.9}$$

Important to note here is that the part $G_S(\mathbf{p})$ is transformed using only a static dependence in the parameters and, hence, it will, after the transformation, still only depend statically on the parameters. This fact can be used to realize that the choices of $w_k(\mathbf{p})$ in (5.7) are not as restrictive as one can think. Let us illustrate this with an example.

Example 5.1: Effect of State Transformations

Assume samples from the continuous-time LPV model, $G(p)$ are given. $G(p)$ do not have any dynamic dependence of the parameters, i.e., $G(p) = G_S(p) = \left[\begin{array}{c|c} \mathbf{A}(p) & \mathbf{B}(p) \\ \hline \mathbf{C}(p) & \mathbf{D}(p) \end{array}\right]$, where

$$\mathbf{A}(p) = \begin{pmatrix} 0.4p^2 + 3p - 3.6 & -\frac{0.4(p^3-24p-40)}{p} & \frac{0.2(27p^3+55p^2+37p-160)}{p} \\ 0.4p^2 + 3.6p - 3.2 & -\frac{0.2(2p^3+3p^2-46p-10)}{p} & \frac{0.2(27p^3+23p^2-96p-20)}{p} \\ 1.6p - 1.6 & -\frac{0.2(8p^2-33p-5)}{p} & \frac{0.2(23p^2-68p-10)}{p} \end{pmatrix},$$

$$
\begin{aligned}
\mathbf{B}(p) &= \begin{pmatrix} 8+7p+p^2 \\ 6+2p+p^2 \\ 3 \end{pmatrix}, \\
\mathbf{C}(p) &= \begin{pmatrix} 0.2+0.2p & -\frac{0.2(-9p+p^2-10)}{p} & -\frac{0.8(-p+4p^2-5)}{p} \end{pmatrix}, \\
\mathbf{D}(p) &= 0.
\end{aligned}
$$

This LPV model does not have any dynamic dependence in the parameter and to be certain to be in the correct model class we can use the basis functions, $w_k(p) = \{p^{-1}, 1, p, p^2\}$. However, a different realization of this model is given by

$$
\begin{aligned}
\mathbf{A}_T(p) &= \bar{\mathbf{T}}(p)\mathbf{A}(p)\bar{\mathbf{T}}^{-1}(p) = \begin{pmatrix} -2+p & 3+p & 5+2p \\ 2+2p & -4+3p & 1+5p \\ -8+8p & 1+5p & -2+3p \end{pmatrix}, \\
\mathbf{B}_T(p) &= \bar{\mathbf{T}}(p)\mathbf{B}(p) = \begin{pmatrix} 1+p \\ 2+p \\ 3 \end{pmatrix}, \\
\mathbf{C}_T(p) &= \mathbf{C}(p)\bar{\mathbf{T}}^{-1}(p) = \begin{pmatrix} 1+p & 2+2p & 3+3p \end{pmatrix}, \\
\mathbf{D}_T(p) &= \mathbf{D}(p) = 0, \\
\bar{\mathbf{T}}(p) &= \begin{pmatrix} 5 & p & 1 \\ 0 & p & 2 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

Obviously, in this realization, the model is only affine in p. This means that it is also possible to find the correct model using only the basis functions $\{1, p\}$.

In the example above, it can be observed that the choice of $w_k(\mathbf{p})$ can sometimes be a little forgiving, since we have methods that are invariant to the state basis that the given models are represented in.

5.4.2 The Optimization Problem

The general optimization problem that will be studied can be written as

$$
\begin{aligned}
&\underset{\hat{\mathbf{A}}^{(k)},\hat{\mathbf{B}}^{(k)},\hat{\mathbf{C}}^{(k)},\hat{\mathbf{D}}^{(k)}}{\text{minimize}} \sum_{i=1}^{N} \left\| W_{o,i}\left(G_i - \hat{G}(\mathbf{p}_i)\right) W_{i,i} \right\|_{\mathcal{H}_2,\omega}^2 \\
&\qquad = \underset{\hat{\mathbf{A}}^{(k)},\hat{\mathbf{B}}^{(k)},\hat{\mathbf{C}}^{(k)},\hat{\mathbf{D}}^{(k)}}{\text{minimize}} \sum_{i=1}^{N} \left\| W_{o,i} E_i W_{i,i} \right\|_{\mathcal{H}_2,\omega}^2, \; E_i = G_i - \hat{G}(\mathbf{p}_i).
\end{aligned} \tag{5.10}
$$

To study the problem in (5.10), start by looking at the case when there is only one model and see what can be concluded. This problem becomes, almost, identical to the problems in Section 4.4. The only difference is that the system matrices are parametrized as in (5.7). However, the new optimization variables $\hat{\mathbf{A}}^{(k)}$, $\hat{\mathbf{B}}^{(k)}$, $\hat{\mathbf{C}}^{(k)}$ and $\hat{\mathbf{D}}^{(k)}$ enter linearly in $\hat{\mathbf{A}}(\mathbf{p})$, $\hat{\mathbf{B}}(\mathbf{p})$, $\hat{\mathbf{C}}(\mathbf{p})$ and $\hat{\mathbf{D}}(\mathbf{p})$ which makes it easy to express the gradient in the new variables instead, for example,

$$\frac{\partial \left\| W_{o,i} \left(G_i - \hat{G}(\mathbf{p}_i) \right) W_{i,i} \right\|_{\mathcal{H}_2}^2}{\partial \hat{\mathbf{A}}^{(k)}} = 2 w_k(\mathbf{p}_i) \hat{\mathbf{E}}^\mathsf{T} \mathbf{Q}_{E,i} \mathbf{P}_{E,i} \hat{\mathbf{E}}. \tag{5.11}$$

Now returning to the original problem, (5.10), when having a number of LTI models given, instead of just one. This is also a simple extension of the problems in Section 4.4, since this is a sum of the $\mathcal{H}_2$-norm over a number of LTI models, which yields the structure

$$\begin{aligned} \frac{\partial \sum_i \left\| W_{o,i} \left(G_i - \hat{G}(\mathbf{p}_i) \right) W_{i,i} \right\|_{\mathcal{H}_2}^2}{\partial \hat{\mathbf{A}}^{(k)}} &= \sum_{i=1}^{N} \frac{\partial \left\| W_{o,i} \left(G_i - \hat{G}(\mathbf{p}_i) \right) W_{i,i} \right\|_{\mathcal{H}_2}^2}{\partial \hat{\mathbf{A}}^{(k)}} \\ &= 2 \sum_{i=1}^{N} w_k(\mathbf{p}_i) \hat{\mathbf{E}}^\mathsf{T} \mathbf{Q}_{E,i} \mathbf{P}_{E,i} \hat{\mathbf{E}}. \end{aligned} \tag{5.12}$$

When converting the model-reduction methods in Section 4.4 into LPV methods, they will not only inherit the properties, but also the prerequisites of the methods, that is, when extending all the methods, it is required that the given LTI models in $\mathcal{M}$ are all asymptotically stable, and additionally for the continuous-time case and the methods in Section 4.4.1 and Section 4.4.2, the LTI models require the error system to be strictly proper, i.e., $\mathbf{D}_i = \hat{\mathbf{D}}(\mathbf{p}_i)$. For these methods, the problem of finding $\hat{\mathbf{D}}(\mathbf{p})$ can be seen as a separate problem.

Before stating the necessary conditions for optimality for the proposed LPV methods (derived from the model-reduction methods), some notation has to be established. The given systems, G_i in the set $\mathcal{M}$ are assumed to have the realizations

$$G_i = \left[\begin{array}{c|c} \mathbf{A}_i & \mathbf{B}_i \\ \hline \mathbf{C}_i & \mathbf{D}_i \end{array} \right], \tag{5.13}$$

and correspond to the parameter values $\mathbf{p}_i$. The notation and partitioning will be the same as in Section 4.4, with the exception that all variables will have a subscript i corresponding to the parameter value considered, $\mathbf{p}_i$. Only the necessary conditions for the continuous-time cases are stated, since most of the details are covered in Section 4.4 and the discrete-time cases are analogous with the continuous-time case. From the necessary conditions for optimality, the expressions for the gradients can be readily extracted to be used in, for example, a quasi-Newton algorithm.

The necessary conditions for optimality for the LPV version of the method in Section 4.4.1 can be stated as follows.

Theorem 5.1 (Necessary conditions for optimality). *Assume that G_i, $\hat{G}_i$, $W_{i,i}$ and $W_{o,i}$ are asymptotically stable and that E_i is strictly proper, for the $\mathcal{H}_2$-norm* *to be defined, i.e.,* $\mathbf{A}_i, \hat{\mathbf{A}}_i, \mathbf{A}_{i,i}$ *and* $\mathbf{A}_{o,i}$ *are Hurwitz and* $\mathbf{D}_{o,i}\left(\mathbf{D}_i - \hat{\mathbf{D}}_i\right)\mathbf{D}_{i,i} = \mathbf{0}$ *for all* i*. In order for the matrices* $\hat{\mathbf{A}}^{(k)}, \hat{\mathbf{B}}^{(k)}, \hat{\mathbf{C}}^{(k)}$ *to be optimal for the problem*

$$\underset{\hat{\mathbf{A}}^{(k)},\hat{\mathbf{B}}^{(k)},\hat{\mathbf{C}}^{(k)}}{\text{minimize}} \sum_i \|E_i\|^2_{\mathcal{H}_2}, \quad E_i = W_{o,i}\left(G_i - \hat{G}_i\right)W_{i,i}, \tag{5.14}$$

it is necessary that they satisfy the equations

$$\mathbf{A}_{E,i}\mathbf{P}_{E,i} + \mathbf{P}_{E,i}\mathbf{A}_{E,i}^\mathsf{T} + \mathbf{B}_{E,i}\mathbf{B}_{E,i}^\mathsf{T} = \mathbf{0}, \tag{5.15a}$$

$$\mathbf{A}_{E,i}^\mathsf{T}\mathbf{Q}_{E,i} + \mathbf{Q}_{E,i}\mathbf{A}_{E,i} + \mathbf{C}_{E,i}^\mathsf{T}\mathbf{C}_{E,i} = \mathbf{0}, \tag{5.15b}$$

for all i*:s, and that*

$$\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \hat{\mathbf{A}}^{(k)}} = 2\sum_i w_k(\mathbf{p}_i)\hat{\mathbf{E}}^\mathsf{T}\mathbf{Q}_{E,i}\mathbf{P}_{E,i}\hat{\mathbf{E}} = \mathbf{0}, \tag{5.16a}$$

$$\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \hat{\mathbf{B}}^{(k)}} = 2\sum_i w_k(\mathbf{p}_i)\hat{\mathbf{E}}^\mathsf{T}\left(\mathbf{Q}_{E,i}\mathbf{P}_{E,i}\mathbf{E}_i\mathbf{C}_i^\mathsf{T} + \mathbf{Q}_{E,i}\mathbf{B}_{E,i}\mathbf{D}_i\right) = \mathbf{0}, \tag{5.16b}$$

$$\frac{\partial \|E\|^2_{\mathcal{H}_2}}{\partial \hat{\mathbf{C}}^{(k)}} = -2\sum_i w_k(\mathbf{p}_i)\left(\mathbf{B}_o^\mathsf{T}\mathbf{E}_o^\mathsf{T}\mathbf{Q}_{E,i}\mathbf{P}_{E,i} + \mathbf{D}_o^\mathsf{T}\mathbf{C}_{E,i}\mathbf{P}_{E,i}\right)\hat{\mathbf{E}} = \mathbf{0}, \tag{5.16c}$$

where

$$\hat{\mathbf{E}} = \begin{pmatrix} \mathbf{0}_{n\times\hat{n}} \\ \mathbb{I}_{\hat{n}\times\hat{n}} \\ \mathbf{0}_{n_i\times\hat{n}} \\ \mathbf{0}_{n_o\times\hat{n}} \end{pmatrix}, \quad \mathbf{E}_i = \begin{pmatrix} \mathbf{0}_{n\times\hat{n}} \\ \mathbf{0}_{\hat{n}\times\hat{n}} \\ \mathbb{I}_{n_i\times\hat{n}} \\ \mathbf{0}_{n_o\times\hat{n}} \end{pmatrix}, \quad \mathbf{E}_o = \begin{pmatrix} \mathbf{0}_{n\times\hat{n}} \\ \mathbf{0}_{\hat{n}\times\hat{n}} \\ \mathbf{0}_{n_i\times\hat{n}} \\ \mathbb{I}_{n_o\times\hat{n}} \end{pmatrix}. \tag{5.17}$$

Proof: The proof is analogous with Theorem 4.2. □

The necessary conditions for optimality for the LPV version of the method presented in Section 4.4.2, the robust extension, can be stated as

Theorem 5.2 (Necessary conditions for optimality). *Assume that* $G_i, \hat{G}_i, W_{i,i}$ *and* $W_{o,i}$ *are asymptotically stable and that* E_i *is strictly proper, for the* $\mathcal{H}_2$*-norm to be defined, i.e.,* $\mathbf{A}_i, \hat{\mathbf{A}}_i, \mathbf{A}_{i,i}$ *and* $\mathbf{A}_{o,i}$ *are Hurwitz and* $\mathbf{D}_{o,i}\left(\mathbf{D}_i - \hat{\mathbf{D}}_i\right)\mathbf{D}_{i,i} = \mathbf{0}$ *for all* i*. In order for the matrices* $\hat{\mathbf{A}}^{(k)}, \hat{\mathbf{B}}^{(k)}, \hat{\mathbf{C}}^{(k)}$ *to be optimal for the problem*

$$\min_{\hat{\mathbf{A}}^{(k)},\hat{\mathbf{B}}^{(k)},\hat{\mathbf{C}}^{(k)}} \sum_i \|E_i\|^2_{\mathcal{H}_2} + V_{rob},$$

$$\begin{aligned} V_{rob} = \sum_i 2\epsilon_\mathbf{A}\left\|\mathbf{Q}_i\mathbf{P}_i + \mathbf{Q}_{12,i}\mathbf{P}_{12,i}^\mathsf{T}\right\|_F \\ + 2\epsilon_\mathbf{B}\left\|\mathbf{Q}_i\mathbf{B}_i + \mathbf{Q}_{12,i}\hat{\mathbf{B}}_i\right\|_F + 2\epsilon_\mathbf{C}\left\|\mathbf{C}_i\mathbf{P}_i - \hat{\mathbf{C}}_i\mathbf{P}_{12,i}^\mathsf{T}\right\|_F, \end{aligned} \tag{5.18}$$

it is necessary that they satisfy the equations in (5.15) *(for* $W_{i,i} = W_{o,i} = \mathbb{I}$*) and* *the equations*

$$\hat{\mathbf{A}}_i^\mathsf{T}\mathbf{W}_{1,i} + \mathbf{W}_{1,i}\mathbf{A}_i + \mathbf{Q}_{12,i}^\mathsf{T}\left(\mathbf{Q}_i\mathbf{P}_i + \mathbf{Q}_{12,i}\mathbf{P}_{12,i}^\mathsf{T}\right) = \mathbf{0}, \tag{5.19a}$$

$$\mathbf{A}_i\mathbf{W}_{2,i} + \mathbf{W}_{2,i}\hat{\mathbf{A}}_i^\mathsf{T} + \left(\mathbf{Q}_i\mathbf{P}_i + \mathbf{Q}_{12,i}\mathbf{P}_{12,i}^\mathsf{T}\right)\mathbf{P}_{12,i} = \mathbf{0}, \tag{5.19b}$$

$$\mathbf{A}_i\mathbf{W}_{3,i} + \mathbf{W}_{3,i}\hat{\mathbf{A}}_i^\mathsf{T} + \left(\mathbf{Q}_i\mathbf{B}_i + \mathbf{Q}_{12,i}\hat{\mathbf{B}}_i\right)\hat{\mathbf{B}}_i^\mathsf{T} = \mathbf{0}, \tag{5.19c}$$

$$\hat{\mathbf{A}}_i^\mathsf{T}\mathbf{W}_{4,i} + \mathbf{W}_{4,i}\mathbf{A}_i + \hat{\mathbf{C}}_i^\mathsf{T}\left(\hat{\mathbf{C}}_i\mathbf{P}_{12,i}^\mathsf{T} - \mathbf{C}_i\mathbf{P}_i\right) = \mathbf{0}. \tag{5.19d}$$

for all i and that

$$\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial\hat{\mathbf{A}}^{(k)}} + \frac{\partial V_{rob}}{\partial\hat{\mathbf{A}}^{(k)}} = \mathbf{0}, \tag{5.20a}$$

$$\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial\hat{\mathbf{B}}^{(k)}} + \frac{\partial V_{rob}}{\partial\hat{\mathbf{B}}^{(k)}} = \mathbf{0}, \tag{5.20b}$$

$$\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial\hat{\mathbf{C}}^{(k)}} + \frac{\partial V_{rob}}{\partial\hat{\mathbf{C}}^{(k)}} = \mathbf{0}. \tag{5.20c}$$

With

$$\begin{aligned}\frac{\partial V_{rob}}{\partial\hat{\mathbf{A}}^{(k)}} = 4\sum_i w_k(\mathbf{p}_i)\Bigg(&\epsilon_\mathbf{A}\frac{\mathbf{W}_{1,i}\mathbf{P}_{12,i} + \mathbf{Q}_{12,i}^\mathsf{T}\mathbf{W}_{2,i}}{\left\|\mathbf{Q}_i\mathbf{P}_i + \mathbf{Q}_{12,i}\mathbf{P}_{12,i}^\mathsf{T}\right\|_F}\\ &+ \epsilon_\mathbf{B}\frac{\mathbf{Q}_{12,i}^\mathsf{T}\mathbf{W}_{3,i}}{\left\|\mathbf{Q}_i\mathbf{B}_i + \mathbf{Q}_{12,i}\hat{\mathbf{B}}_i\right\|_F} + \epsilon_\mathbf{C}\frac{\mathbf{W}_{4,i}\mathbf{P}_{12,i}}{\left\|\mathbf{C}_i\mathbf{P}_i - \hat{\mathbf{C}}_i\mathbf{P}_{12,i}^\mathsf{T}\right\|_F}\Bigg),\end{aligned}$$

$$\begin{aligned}\frac{\partial V_{rob}}{\partial\hat{\mathbf{B}}^{(k)}} = 4\sum_i w_i(\mathbf{p}_i)\Bigg(&\epsilon_\mathbf{A}\frac{\mathbf{W}_{1,i}\mathbf{B}_i}{\left\|\mathbf{Q}_i\mathbf{P}_i + \mathbf{Q}_{12,i}\mathbf{P}_{12,i}^\mathsf{T}\right\|_F}\\ &+ \epsilon_\mathbf{B}\frac{\mathbf{Q}_{12,i}^\mathsf{T}\left(\mathbf{Q}_i\mathbf{B}_i + \mathbf{Q}_{12,i}\hat{\mathbf{B}}_i\right)}{\left\|\mathbf{Q}_i\mathbf{B}_i + \mathbf{Q}_{12,i}\hat{\mathbf{B}}_i\right\|_F} + \epsilon_\mathbf{C}\frac{\mathbf{W}_{4,i}\mathbf{B}_i}{\left\|\mathbf{C}_i\mathbf{P}_i - \hat{\mathbf{C}}_i\mathbf{P}_{12,i}^\mathsf{T}\right\|_F}\Bigg),\end{aligned}$$

$$\begin{aligned}\frac{\partial V_{rob}}{\partial\hat{\mathbf{C}}^{(k)}} = -4\sum_i w_k(\mathbf{p}_i)\Bigg(&\epsilon_\mathbf{A}\frac{\mathbf{C}_i\mathbf{W}_{2,i}}{\left\|\mathbf{Q}_i\mathbf{P}_i + \mathbf{Q}_{12,i}\mathbf{P}_{12,i}^\mathsf{T}\right\|_F}\\ &+ \epsilon_\mathbf{B}\frac{\mathbf{C}_i\mathbf{W}_{3,i}}{\left\|\mathbf{Q}_i\mathbf{B}_i + \mathbf{Q}_{12,i}\hat{\mathbf{B}}_i\right\|_F} + \epsilon_\mathbf{C}\frac{\left(\mathbf{C}_i\mathbf{P}_i - \hat{\mathbf{C}}_i\mathbf{P}_{12,i}^\mathsf{T}\right)\mathbf{P}_{12,i}}{\left\|\mathbf{C}_i\mathbf{P}_i - \hat{\mathbf{C}}_i\mathbf{P}_{12,i}^\mathsf{T}\right\|_F}\Bigg).\end{aligned}$$

and $\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial\hat{\mathbf{A}}^{(k)}}$, $\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial\hat{\mathbf{B}}^{(k)}}$ *and* $\frac{\partial\|E\|_{\mathcal{H}_2}^2}{\partial\hat{\mathbf{C}}^{(k)}}$ *as in* (5.16).

Proof: The proof is analogous with the proof for Theorem 4.6. □

For the LPV version of the frequency-limited method, described in Section 4.4.3, the necessary conditions for optimality can be stated as

Theorem 5.3. *Assume that all G_i and $\hat{G}_i$ are asymptotically stable, for the limited $\mathcal{H}_2$-norm to be defined, i.e., all $\mathbf{A}_i$ and $\hat{\mathbf{A}}_i$ are Hurwitz for all i. In order for the matrices $\hat{\mathbf{A}}^{(k)}$, $\hat{\mathbf{B}}^{(k)}$, $\hat{\mathbf{C}}^{(k)}$ and $\hat{\mathbf{D}}^{(k)}$ to be optimal for the problem*

$$\underset{\hat{\mathbf{A}}^{(k)},\hat{\mathbf{B}}^{(k)},\hat{\mathbf{C}}^{(k)},\hat{\mathbf{D}}^{(k)}}{\text{minimize}} \sum_i \|E_i\|^2_{\mathcal{H}_2,\omega}, \; E_i = G_i - \hat{G}_i, \tag{5.22}$$

where $\|E_i\|^2_{\mathcal{H}_2,\omega}$ is defined in Chapter 3, it is necessary that they satisfy the equations in (4.65) and the equations in (4.29) for all i and that

$$\frac{\partial \sum_i \|E_i\|^2_{\mathcal{H}_2,\omega}}{\partial \hat{\mathbf{A}}^{(k)}} = 2\sum_i w_k(\mathbf{p}_i)\left(\left[\mathbf{Q}^{\mathsf{T}}_{12,\omega,i}\mathbf{P}_{12,i} + \hat{\mathbf{Q}}_{\omega,i}\hat{\mathbf{P}}_i\right] - \mathbf{W}_i\right) = \mathbf{0}, \tag{5.23a}$$

$$\frac{\partial \sum_i \|E_i\|^2_{\mathcal{H}_2,\omega}}{\partial \hat{\mathbf{B}}^{(k)}} = 2\sum_i w_k(\mathbf{p}_i)\left(\hat{\mathbf{Q}}_{\omega,i}\hat{\mathbf{B}}_i + \mathbf{Q}^{\mathsf{T}}_{12,\omega,i}\mathbf{B}_i - \hat{\mathbf{S}}^{\mathsf{T}}_{\omega,i}\hat{\mathbf{C}}^{\mathsf{T}}_i\left[\mathbf{D}_i - \hat{\mathbf{D}}_i\right]\right) = \mathbf{0}, \tag{5.23b}$$

$$\frac{\partial \sum_i \|E_i\|^2_{\mathcal{H}_2,\omega}}{\partial \hat{\mathbf{C}}^{(k)}} = 2\sum_i w_k(\mathbf{p}_i)\left(\hat{\mathbf{C}}_i\hat{\mathbf{P}}_{\omega,i} - \mathbf{C}_i\mathbf{P}_{12,\omega,i} - \left[\mathbf{D}_i - \hat{\mathbf{D}}_i\right]\hat{\mathbf{B}}^{\mathsf{T}}_i\hat{\mathbf{S}}^{\mathsf{T}}_{\omega,i}\right) = \mathbf{0}, \tag{5.23c}$$

$$\frac{\partial \sum_i \|E_i\|^2_{\mathcal{H}_2,\omega}}{\partial \hat{\mathbf{D}}^{(k)}} = -2\sum_i w_k(\mathbf{p}_i)\left(\mathbf{C}_i\mathbf{S}_{\omega,i}\mathbf{B}_i + \mathbf{D}_i\frac{\omega}{\pi} - \hat{\mathbf{C}}_i\hat{\mathbf{S}}_{\omega,i}\hat{\mathbf{B}}_i - \hat{\mathbf{D}}_i\frac{\omega}{\pi}\right) = \mathbf{0}, \tag{5.23d}$$

where

$$\mathbf{W}_i = \mathrm{Re}\left(\frac{i}{\pi}L\left(-\hat{\mathbf{A}}_i - i\omega\mathbb{I}, \mathbf{V}_i\right)\right)^{\mathsf{T}}, \tag{5.23e}$$

$$\mathbf{V}_i = \hat{\mathbf{C}}^{\mathsf{T}}_i\hat{\mathbf{C}}_i\hat{\mathbf{P}}_i - \hat{\mathbf{C}}^{\mathsf{T}}_i\mathbf{C}_i\mathbf{P}_{12,i} - \hat{\mathbf{C}}^{\mathsf{T}}_i\left(\mathbf{D}_i - \hat{\mathbf{D}}_i\right)\hat{\mathbf{B}}^{\mathsf{T}}_i. \tag{5.23f}$$

With the function $L(\cdot,\cdot)$ being the Frechét derivative of the matrix logarithm, see Higham [2008].

Proof: The proof is analogous with the proof for Theorem 4.8. □

Low Rank Coefficient Matrices

For some applications it can be preferable to be able to control the rank of some of the matrices $\hat{\mathbf{A}}^{(k)}$, $\hat{\mathbf{B}}^{(k)}$, $\hat{\mathbf{C}}^{(k)}$ and $\hat{\mathbf{D}}^{(k)}$. See, for instance, the example in Section 7.1, where this is important.

One way of controlling the rank of the coefficient matrices, is to parametrize them as

$$\hat{\mathbf{A}}^{(k)} = \mathbf{V}^{(k)}_A\mathbf{W}^{(k)\mathsf{T}}_A, \tag{5.24a}$$

$$\hat{\mathbf{B}}^{(k)} = \mathbf{V}^{(k)}_B\mathbf{W}^{(k)\mathsf{T}}_B, \tag{5.24b}$$

$$\hat{\mathbf{C}}^{(k)} = \mathbf{V}^{(k)}_C\mathbf{W}^{(k)\mathsf{T}}_C. \tag{5.24c}$$

If, for example, it is assumed that the resulting LPV model should have n_r states,

$\hat{\mathbf{A}}^{(k)} \in \mathbb{R}^{n_r \times n_r}, \forall k$, and the rank of the matrix $\hat{\mathbf{A}}^{(k)}$ should be $n_k < n_r$, then $\mathbf{V}_A^{(k)} \in \mathbb{R}^{n_r \times n_k}$ and $\mathbf{W}_A^{(k)} \in \mathbb{R}^{n_r \times n_k}$ is chosen. This type of parametrization have, with success, been used in, for example, Burer and Monteiro [2003] for semidefinite programs.

If this new parametrization is introduced, the only change in Theorem 5.1, Theorem 5.2 and Theorem 5.3, will be a small change in the gradients. For example, the gradient for $\hat{\mathbf{A}}^{(k)}$ in Theorem 5.1 was computed as

$$\frac{\partial \|E\|_{\mathcal{H}_2}^2}{\partial \hat{\mathbf{A}}^{(k)}} = 2 \sum_i w_k(\mathbf{p}_i) \hat{\mathbf{E}}^\mathsf{T} \mathbf{Q}_{E,i} \mathbf{P}_{E,i} \hat{\mathbf{E}}.$$

The new equations for the gradient, given the parametrization in (5.24), would be

$$\frac{\partial \|E\|_{\mathcal{H}_2}^2}{\partial \mathbf{V}_A^{(k)}} = 2 \sum_i w_k(\mathbf{p}_i) \hat{\mathbf{E}}^\mathsf{T} \mathbf{Q}_{E,i} \mathbf{P}_{E,i} \hat{\mathbf{E}} \mathbf{W}_A^{(k)}, \tag{5.25a}$$

$$\frac{\partial \|E\|_{\mathcal{H}_2}^2}{\partial \mathbf{W}_A^{(k)}} = 2 \sum_i w_k(\mathbf{p}_i) \left(\hat{\mathbf{E}}^\mathsf{T} \mathbf{Q}_{E,i} \mathbf{P}_{E,i} \hat{\mathbf{E}}\right)^\mathsf{T} \mathbf{V}_A^{(k)}. \tag{5.25b}$$

The equations for $\mathbf{V}_B^{(k)}$, $\mathbf{W}_B^{(k)\mathsf{T}}$, $\mathbf{V}_B^{(k)}$ and $\mathbf{W}_C^{(k)\mathsf{T}}$ follow analogously.

5.5 Computational Aspects of the Optimization Problems

In this section, as in Section 4.5, an initialization will be suggested and again how to use both the structure in the variables and the equations to speed up the computations is shown.

As with the methods in Section 4.5, both the cost functions and gradients are given for the LPV methods and it is straightforward to use, for example, any quasi-Newton solver to solve the optimization problem.

5.5.1 Structure in Variables and Equations

What was explained in Section 4.5.1, about structure in the sought system matrices, is applicable, with the same motivation, for the LPV methods. Hence, it is easy to impose structure in the system matrices, e.g., block-diagonal $\mathbf{A}$-matrix.

In Section 4.5.3, it was explained how to use the inherent structure of the equations in the problem to, more efficiently, compute the Lyapunov/Sylvester equations that is needed to compute the cost function and the gradient. For the model-reduction case it was possible to reduce the complexity for every iteration to $\mathcal{O}\left(n^2\hat{n} + n\hat{n}^2\right)$. For the LPV case, the same structure can be utilized for every LTI model in $\mathcal{M}$ and iteration. This means that the complexity per iteration will be $\mathcal{O}\left(N\left[n^2\hat{n} + n\hat{n}^2\right]\right)$, where N is the number of LTI models in $\mathcal{M}$.

5.5.2 Initialization

A subject that needs more attention though, is the initialization. It is assumed, for the initializations described here, that one basis function for the sought system matrices is $w_k(\mathbf{p}) = 1$, i.e., there is a constant term in the parametrization (5.7). A simple initialization is to use one of the given models in $\mathcal{M}$ and set the constant matrix coefficient terms to this model.

As with the model reduction problem, a bit more can be done in then case when there are no input or output filters. The cost functions in this case becomes

$$\sum_i \|E_i\|^2_{\mathcal{H}_2} = \sum_i \operatorname{tr}\left(\mathbf{B}_i^\mathsf{T}\mathbf{Q}_i\mathbf{B}_i + 2\mathbf{B}_i^\mathsf{T}\mathbf{Q}_{12,i}\hat{\mathbf{B}}_i + \hat{\mathbf{B}}_i^\mathsf{T}\hat{\mathbf{Q}}_i\hat{\mathbf{B}}_i\right), \tag{5.26a}$$

$$\sum_i \|E_i\|^2_{\mathcal{H}_2} = \sum_i \operatorname{tr}\left(\mathbf{C}_i\mathbf{P}_i\mathbf{C}_i^\mathsf{T} - 2\mathbf{C}_i\mathbf{P}_{12,i}\hat{\mathbf{C}}_i^\mathsf{T} + \hat{\mathbf{C}}_i\hat{\mathbf{P}}_i\hat{\mathbf{C}}_i^\mathsf{T}\right). \tag{5.26b}$$

These functions are quadratic in the $\hat{\mathbf{B}}^{(k)}$ (or $\hat{\mathbf{C}}^{(k)}$) matrices if the $\hat{\mathbf{A}}^{(k)}$ and $\hat{\mathbf{C}}^{(k)}$ (or $\hat{\mathbf{A}}^{(k)}$ and $\hat{\mathbf{C}}^{(k)}$) matrices are fixed. First, to have a system to start from, any of the given LTI models in $\mathcal{M}$ is chosen, denote this system as $\tilde{G} = \left[\begin{array}{c|c} \tilde{\mathbf{A}} & \tilde{\mathbf{B}} \\ \hline \tilde{\mathbf{C}} & \tilde{\mathbf{D}} \end{array}\right]$. Now set $\hat{\mathbf{A}}(\mathbf{p}) = \tilde{\mathbf{A}}$, i.e., choose $\hat{\mathbf{A}}$ to be a constant matrix that does not depend on $\mathbf{p}$, and do the same thing for $\hat{\mathbf{C}}$. The problem of finding $\hat{\mathbf{B}}^{(k)}$ is now a quadratic problem which can be solved as explained below.

$\hat{\mathbf{B}}_i$ can be written as

$$\begin{aligned}\hat{\mathbf{B}}_i = &\begin{pmatrix}\mathbb{I}w_1(\mathbf{p}_i) & \mathbb{I}w_2(\mathbf{p}_i) & \mathbb{I}w_3(\mathbf{p}_i) & \dots & \mathbb{I}w_{N_w}(\mathbf{p}_i)\end{pmatrix} \\ &\times \begin{pmatrix}\left(\hat{\mathbf{B}}^{(1)}\right)^\mathsf{T} & \left(\hat{\mathbf{B}}^{(2)}\right)^\mathsf{T} & \left(\hat{\mathbf{B}}^{(3)}\right)^\mathsf{T} & \dots & \left(\hat{\mathbf{B}}^{(N_w)}\right)^\mathsf{T}\end{pmatrix}^\mathsf{T} = \bar{\mathbf{p}}_i\bar{\mathbf{B}},\end{aligned} \tag{5.27}$$

where $w_k(\mathbf{p}_i)$ and $\hat{\mathbf{B}}$ are defined in (5.7) and $\mathbb{I}$ is the identity matrix of compatible size.

Now rewrite the cost function (5.26a) as

$$\begin{aligned} V &= \sum_i \operatorname{tr}\left(\mathbf{B}_i^\mathsf{T}\mathbf{Q}_i\mathbf{B}_i + 2\mathbf{B}_i^\mathsf{T}\mathbf{Q}_{12,i}\hat{\mathbf{B}}_i + \hat{\mathbf{B}}_i^\mathsf{T}\hat{\mathbf{Q}}_i\hat{\mathbf{B}}_i\right) \\ &= \sum_i \operatorname{tr}\left(\mathbf{B}_i^\mathsf{T}\mathbf{Q}_i\mathbf{B}_i + 2\mathbf{B}_i^\mathsf{T}\mathbf{Q}_{12,i}\bar{\mathbf{p}}_i\bar{\mathbf{B}} + \bar{\mathbf{B}}^\mathsf{T}\bar{\mathbf{p}}_i^\mathsf{T}\hat{\mathbf{Q}}_i\bar{\mathbf{p}}_i\bar{\mathbf{B}}\right) \\ &= \operatorname{tr}\left(\sum_i \mathbf{B}^\mathsf{T}\mathbf{Q}_i\mathbf{B}_i + \left[2\sum_i \mathbf{B}_i^\mathsf{T}\mathbf{Q}_{12,i}\bar{\mathbf{p}}_i\right]\bar{\mathbf{B}} + \frac{1}{2}\bar{\mathbf{B}}^\mathsf{T}\left[2\sum_i \bar{\mathbf{p}}_i^\mathsf{T}\hat{\mathbf{Q}}_i\bar{\mathbf{p}}_i\right]\bar{\mathbf{B}}\right) \\ &= \operatorname{tr}\left(\mathbf{b}_1 + \mathbf{b}_2\bar{\mathbf{B}} + \frac{1}{2}\bar{\mathbf{B}}^\mathsf{T}\mathbf{b}_3\bar{\mathbf{B}}\right). \end{aligned} \tag{5.28}$$

The solution to the problem $\min_{\bar{\mathbf{B}}} V$, which always exists since $\mathbf{b}_3$ is positive semidefinite, is the solution to the linear system of equations

$$\mathbf{b}_3\bar{\mathbf{B}} = -\mathbf{b}_2^\mathsf{T}. \tag{5.29}$$

Analogous calculations can be used to compute a starting point for $\hat{\mathbf{C}}^{(k)}$; define $\bar{\mathbf{C}} = \left(\hat{\mathbf{C}}^{(1)}, \hat{\mathbf{C}}^{(2)}, \ldots, \hat{\mathbf{C}}^{(N_w)}\right)$, use the same $\hat{\mathbf{A}}$ as for finding $\hat{\mathbf{B}}^{(k)}$ but use the $\hat{\mathbf{B}}$ that was found solving the quadratic problem described above. Now, the equations

$$V = \operatorname{tr}\left(\mathbf{c}_1 + \mathbf{c}_2\bar{\mathbf{C}}^\mathsf{T} + \frac{1}{2}\bar{\mathbf{C}}\mathbf{c}_3\bar{\mathbf{C}}^\mathsf{T}\right) \tag{5.30}$$

are obtained, where $\mathbf{c}_1 = \sum_i \mathbf{C}_i\mathbf{P}_i\mathbf{C}_i^\mathsf{T}, \mathbf{c}_2 = -2\sum_i \mathbf{C}_i\mathbf{P}_{12,i}\bar{\mathbf{p}}_i$ and $\mathbf{c}_3 = 2\sum_i \bar{\mathbf{p}}_i^\mathsf{T}\hat{\mathbf{P}}_i\bar{\mathbf{p}}_i$. The solution to the quadratic problem in this case, which also always exists since $\mathbf{c}_3$ is positive semidefinite, is the solution to the system of linear equations

$$\bar{\mathbf{C}}\mathbf{c}_3 = -\mathbf{c}_2. \tag{5.31}$$

These are suggestions for finding initial values for $\hat{\mathbf{A}}$,$\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}$.

When using the fact that the rank of the $\hat{\mathbf{A}}^{(k)}$, $\hat{\mathbf{B}}^{(k)}$ and $\hat{\mathbf{C}}^{(k)}$ matrices can be controlled, the initialization strategy above has to be used with caution. In the above strategy, using the parametrization

$$\hat{\mathbf{A}}(\mathbf{p}) = \hat{\mathbf{A}}^{(1)} + \sum_k w_k(\mathbf{p})\mathbf{V}_A^{(k)}\mathbf{W}_A^{(k)\mathsf{T}}, \tag{5.32}$$

$\hat{\mathbf{V}}^{(k)}$ and $\hat{\mathbf{W}}^{(k)}$ are initialized as matrices with all zeros. Looking at (5.25), it can be realized that doing this will cause the gradient for $\hat{\mathbf{V}}_A^{(k)}$ and $\hat{\mathbf{W}}_A^{(k)}$ to stay zero for all iterations. This can be solved by, e.g., initializing one of $\hat{\mathbf{V}}^{(k)}$ and $\hat{\mathbf{W}}^{(k)}$ to zero and the other one to a matrix with random values, or more generally to two orthogonal matrices. This will avoid the problem described above.

5.6 Examples

In this section, an illustrative example to shed light on some properties of the proposed methods will be presented. A larger more extensive example using the methods in this chapter will be presented in Chapter 7, since it requires more background material.

When solving the example, the function `fminunc` in MATLAB is used as the quasi-Newton solver framework. To generate a starting point for the solver, which is an extremely important problem in need of significant amounts of research, the initialization procedure explained in Section 5.5.2 is used.

As a comparison, a method that will be called SMILE is used. The method is described in detail in De Caigny et al. [2012]. This method uses interpolation of the system matrices, by first changing all the given LTI models to a common basis and then do a standard interpolation of the elements in the system matrices.

Example 5.2: Small LPV Approximation Example

To show the potential of the LPV approximation and illustrate the importance of addressing system properties, a small example is studied.

The system in this example is defined by a connection of two second-order systems, i.e., a system with four states, with parameter dependent damping,

$$G = G_1 G_2, \quad \text{where } G_1 = \frac{1}{s^2 + 2\zeta_1 s + 1}, \; G_2 = \frac{9}{s^2 + 6\zeta_2 s + 9}, \tag{5.33a}$$

$$\zeta_1 = 0.1 + 0.9p, \; \zeta_2 = 0.1 + 0.9(1-p), \; p \in [0,1]. \tag{5.33b}$$

The system was sampled by selecting 10 equidistant points in $p \in [0,1]$, i.e., 10 linear models with four states each are given as data to the method.

The data is given in a state basis where all the LTI models are balanced. The elements in the system matrices happen to depend nonlinearly on the parameter p, see the gray dashed lines in Figure 5.1. The interesting and obvious property of this example is that there exists state bases (for example, observable canonical form) for which the model has affine dependence on p; in fact only two elements of the system matrix $\mathbf{A}$ are affine in p while all other matrix elements in $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are constants, see the black solid lines in Figure 5.1.

The method H_2NL will be used with an affine parametrization with respect to the parameters, and we investigate if it is possible to find a representation of the true system with this structure, given the data where the individual elements in the system matrices depends nonlinearly on the parameter. Additionally, the method H_2NL, where we control the rank in $\hat{\mathbf{A}}^{(1)}$, where $\hat{\mathbf{A}}(p) = \hat{\mathbf{A}}^{(0)} + p\hat{\mathbf{A}}^{(1)}$, will also be used. We will choose the rank to be two, since there exists a state-space basis where only two elements of the system matrix $\mathbf{A}$ are affine in p and all other elements are constant, see the black lines in Figure 5.1.

From the results in Table 5.1 it can be observed that when H_2NL is used, both with rank 4 and rank 2, a high accuracy low order (indeed affine) LPV model of the system can be found.

Using SMILE with an affine parametrization, a much worse model is obtained. Achieving comparable results using the SMILE strategy requires polynomials of order two. To further illustrate the accuracy, 100 validation points are generated from (5.33) and the relative $\mathcal{H}_2$-norm for the error model in these points is shown in Figure 5.2.

In this example, the importance of addressing the behavior of the system instead of interpolating the system matrices can be seen. First of all, it is hard to find base transformations such that all the given LTI models are represented in the same basis (called a coherent state basis in De Caigny et al. [2012]), and second you cannot control how the system depends on the parameters in this basis, as is illustrated with the SMILE method using different orders in the polynomial of the parameter.

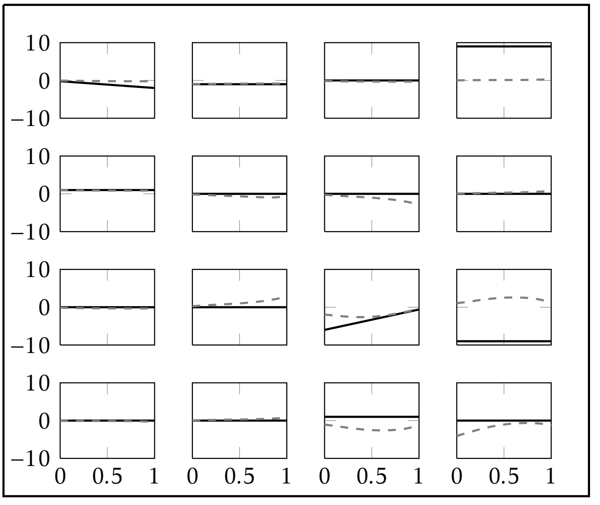

Figure 5.1: *The elements in the* $\mathbf{A}$*-matrices as function of* p *for the four state* LPV *system* (5.33) *for two different state bases. The gray dashed lines represents the elements in the* $\mathbf{A}$*-matrix when any* LTI *model extracted is given in a balanced form. For this state basis, the elements depend nonlinearly on* p*, This is also the basis for which the* LTI *models that are given as data are extracted from. The black lines represents the elements in the* $\mathbf{A}$*-matrix for another state basis when only two elements depend affine on* p *and the rest are constant. This state basis is shown here to show that there exist another, input-output equivalent, system which has a simple structure.*

5.7 Conclusions

In this chapter, new local methods for computing an LPV model, given a set of LTI models are proposed. These methods use a nonlinear optimization approach that is based on the model-reduction techniques in Chapter 4. The proposed methods try to preserve the input-output behavior of the given systems by minimizing the $\mathcal{H}_2$-norm of the error systems. The cost functions and their gradients are derived to be computationally efficient. This enables us to have a measure of first order optimality and to efficiently use standard quasi-Newton solvers to solve the problem. The method has been shown to work both conceptually, on small examples, and on real-world examples, as we will see in Chapter 7.

There are two main advantages with the proposed methods, compared to existing local methods. The first one is that it is possible to impose structure in the elements in the system matrices. The other one is that the method tries to capture the input-output behavior of the given systems. However, this comes at the price of computational burden, which makes the method slower than many existing local methods. The fact that the methods consider the input-output behavior, using the $\mathcal{H}_2$-norm, implies that the method is invariant to which state-space bases

__Table 5.1:__ Numerical results from Example 5.2

Method	$\sum_i \|E_i\|_{\mathcal{H}_2}$	Degree
H_2NL, rank 2	$1.44 \cdot 10^{-4}$	1
H_2NL, rank 4	$2.54 \cdot 10^{-5}$	1
SMILE	$2.54 \cdot 10^{-13}$	2
SMILE	6.70	1

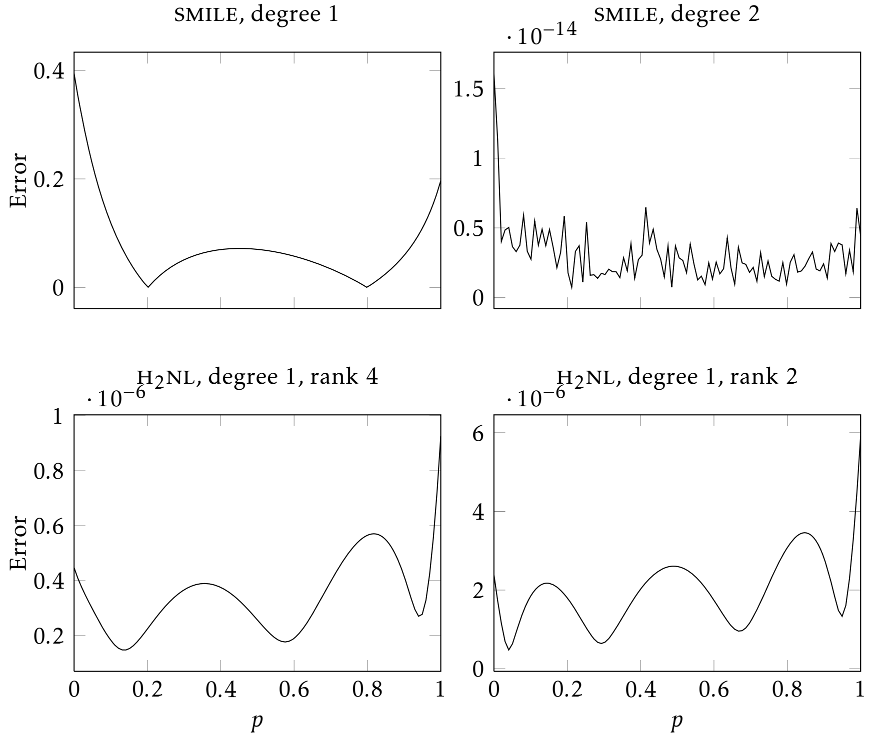

Figure 5.2: *The figure illustrates the relative $\mathcal{H}_2$-norm of the error system in 100 validation points for the different methods. Note the different scales and that it takes a polynomial of order two using the SMILE approach to obtain a satisfactory result, as with the proposed method using an affine function.*

the given local LTI models are represented in and even how many states the given models have. It also implies that it is possible to find an LPV model with low dependence on the parameters, despite apparently complex dependence of the parameter.

6

Controller Synthesis

Let us start by quoting a sentence from Syrmos et al. [1997]: *"The static output feedback problem is one of the basic problems in feedback design, which, in the multivariable case, is still analytically unsolved."* In Blondel and Tsitsiklis [1997] they show that the static output-feedback stabilization problem is indeed NP-hard if one constrains the coefficients of the controller to lie in prespecified intervals. They also conjecture that already the unconstrained problem is NP-hard.

This chapter does not include a revolutionary solution to this problem, instead it proposes a computational method for finding locally optimal solutions to the mentioned problem and as will be shown, the method works for medium-scale systems and for controllers that have structural constraints. A method for synthesizing controllers for LPV systems, based on the first method, is also presented. The methods use, as the methods in the previous chapters, a general nonlinear optimization approach.

6.1 Overview

The problem of finding an unstructured state-feedback $\mathcal{H}_2$ or $\mathcal{H}_\infty$ controller is well known to be a problem that, under certain assumptions, see, e.g., Zhou et al. [1996], easily can be solved. However, the problem of finding a static output-feedback $\mathcal{H}_2$ (or $\mathcal{H}_\infty$) controller is generally a non-convex problem and not solved as easily. The problem of finding an $\mathcal{H}_2$ controller is closely related to the problem of finding an optimal controller with a quadratic performance criterion. This problem was introduced in Kalman [1960] and has been studied since then. The problem has been attacked in different ways, both using direct general-purpose minimization, see, e.g., Rautert and Sachs [1997], and using semidefinite programs (SDP) see, e.g., Stingl [2006]. These methods can handle problems of moderate sizes but can experience problems already for small-scale systems. SDP has been a hot topic during the last years, but the problem with the SDP approach is that it scales badly with the dimension of the problem. When formulating this particular optimization problem, of finding a reduced-order controller, it involves bilinear matrix inequalities (BMIs) that makes the problem even more difficult to solve, see Mesbahi et al. [1995]. Another approach that very recently has been published is the more direct approach in Lin et al. [2009] (and Fardad et al. [2009]) that formulates the problem as a general nonlinear optimization problem and uses a dedicated quasi-Newton algorithm to solve the problem. The first method presented in this chapter resembles closely the method presented in Lin et al. [2009], but has been independently derived with an, in our opinion, more straightforward derivation. The main focus in Lin et al. [2009] is on the ability to create structured controllers, e.g., interconnected systems subject to architectural constraints on the distributed controller. In this chapter the main goal is to find a method that is applicable to medium-scale systems and is expandable to a framework for creating robust $\mathcal{H}_2$ controllers or controllers for LPV system, e.g., controllers for systems with parametric uncertainties. The first method is then extended to handle controller synthesis for LPV systems, much as how the methods in Chapter 5 are extensions of the methods in Chapter 4.

The methods proposed in this chapter, for controller synthesis, both for LTI and LPV systems, will of course have at least two drawbacks. The first one is that the methods need a stabilizing controller to be able to start the optimization and finding a stabilizing controller is most likely an NP-hard problem. The second one is that, given a stabilizing controller, the problem of finding a static output-feedback $\mathcal{H}_2$ controller is a non-convex problem, therefore the proposed methods can not guarantee to find a globally optimal controller but only a locally optimal one.

6.2 Static Output-Feedback $\mathcal{H}_2$-Controllers

In this section, a method for synthesizing static output-feedback $\mathcal{H}_2$ controllers for LTI systems will be presented, and as explained in Section 2.1.4, this method can also be used to synthesize reduced-order controllers. The proposed method will, as the methods presented in Chapter 4 and Chapter 5, be based on minimizing the $\mathcal{H}_2$-norm.

The goal with the optimization problem in this section is to formulate an optimization problem for synthesizing a static output-feedback controller. When formulating this optimization problem, great care need to be taken when deriving the expression for the cost function and its gradient to make sure that the expressions can be evaluated efficiently. The method presented in this section is designed to work on medium-scale systems, which will be shown later, and it also works with structural constraints in the controller.

As described in Section 2.1.4, the model that will be used to measure the performance of a system is

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \mathbf{z} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{C}_1 & \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{C}_2 & \mathbf{D}_{21} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \\ \mathbf{u} \end{pmatrix}, \tag{6.1}$$

where $\mathbf{x} \in \mathbb{R}^{n_x}$ is the state vector, $\mathbf{w} \in \mathbb{R}^{n_w}$ the disturbance signal, $\mathbf{u} \in \mathbb{R}^{n_u}$ the control signal, $\mathbf{z} \in \mathbb{R}^{n_z}$ the performance measure and $\mathbf{y} \in \mathbb{R}^{n_y}$ the measurement signal.

Closing the loop with a static output-feedback controller, $\mathbf{u} = \mathbf{K}\mathbf{y}$, where $\mathbf{K}$ is a matrix describing the controller, yields the closed-loop system

$$T_{w,z} = \left[\begin{array}{c|c} \mathbf{A}_T & \mathbf{B}_T \\ \hline \mathbf{C}_T & \mathbf{D}_T \end{array}\right] = \left[\begin{array}{c|c} \mathbf{A} + \mathbf{B}_2\mathbf{K}\mathbf{C}_2 & \mathbf{B}_1 + \mathbf{B}_2\mathbf{K}\mathbf{D}_{21} \\ \hline \mathbf{C}_1 + \mathbf{D}_{12}\mathbf{K}\mathbf{C}_2 & \mathbf{D}_{11} + \mathbf{D}_{12}\mathbf{K}\mathbf{D}_{21} \end{array}\right]. \tag{6.2}$$

Now, let us formulate the optimization problem of minimizing the $\mathcal{H}_2$-norm of the closed-loop system from $\mathbf{w}$ to $\mathbf{z}$, $T_{w,z}$, in (6.2), i.e.,

$$\min_{\mathbf{K}} \left\| T_{w,z} \right\|_{\mathcal{H}_2}^2. \tag{6.3}$$

Since the equations will differ in continuous and discrete time but the general ideas are the same, both versions will be presented but with less detail in the discrete-time case.

6.2.1 Continuous Time

For the $\mathcal{H}_2$-norm to be defined, the system $T_{w,z}$ has to be asymptotically stable and strictly proper, i.e., $\mathbf{A} + \mathbf{B}_2\mathbf{K}\mathbf{C}_2$ has to be Hurwitz and $\mathbf{D}_{11} + \mathbf{D}_{12}\mathbf{K}\mathbf{D}_{21} = \mathbf{0}$. Note that already the problem of finding a $\mathbf{K}$ that stabilizes the system is, as explained in the beginning of this chapter, most likely an NP-hard problem. Because of this, for the rest of the chapter, if nothing else is mentioned, it will be assumed that $\mathbf{K}$ stabilizes the system.

To compute the cost function for the optimization problem (6.3), the cost function have to be expressed in a more suitable form for evaluation. Using (2.21), the cost function for the optimization problem (6.3) can be expressed as

$$\left\| T_{w,z} \right\|_{\mathcal{H}_2}^2 = \operatorname{tr} \mathbf{B}_T^\mathsf{T} \mathbf{Q}_T \mathbf{B}_T \tag{6.4a}$$

$$= \operatorname{tr} \mathbf{C}_T \mathbf{P}_T \mathbf{C}_T^\mathsf{T}, \tag{6.4b}$$

where $\mathbf{Q}_T$ and $\mathbf{P}_T$ satisfy the Lyapunov equations

$$\mathbf{A}_T\mathbf{P}_T + \mathbf{P}_T\mathbf{A}_T^\mathsf{T} + \mathbf{B}_T\mathbf{B}_T^\mathsf{T} = \mathbf{0}, \tag{6.5a}$$

$$\mathbf{A}_T^\mathsf{T}\mathbf{Q}_T + \mathbf{Q}_T\mathbf{A}_T + \mathbf{C}_T^\mathsf{T}\mathbf{C}_T = \mathbf{0}. \tag{6.5b}$$

Now, with the equations in (6.4) and (6.5) it is possible to state necessary conditions for optimality for (6.3). In the theorem below, which states the necessary conditions for optimality, the gradient of the cost function for the optimization problem (6.3) can be readily extracted to be used in, for example, a quasi-Newton algorithm.

Theorem 6.1 (Necessary conditions for optimality). *Given a system G as in* (6.1) *and a static output-feedback controller, described by the matrix* $\mathbf{K}$*, such that* $\mathbf{u} = \mathbf{K}\mathbf{y}$*. The system G and the controller are given such that the closed-loop system, $T_{w,z}$ in* (6.2)*, is asymptotically stable and strictly proper, i.e.,* $\mathbf{A}_T$ *is Hurwitz and* $\mathbf{D}_T = \mathbf{0}$*. In order for the matrix* $\mathbf{K}$ *to be optimal for the problem (6.3), it is necessary that* $\mathbf{K}$ *satisfies the equations in (6.5) and that*

$$\frac{\partial \left\|T_{w,z}\right\|_{\mathcal{H}_2}^2}{\partial \mathbf{K}} = 2\left(\mathbf{B}_2^\mathsf{T}\mathbf{Q}_T\mathbf{P}_T\mathbf{C}_2^\mathsf{T} + \mathbf{B}_2^\mathsf{T}\mathbf{Q}_T\mathbf{B}_T\mathbf{D}_{21}^\mathsf{T} + \mathbf{D}_{21}^\mathsf{T}\mathbf{C}_T\mathbf{P}_T\mathbf{C}_2^\mathsf{T}\right) = \mathbf{0}. \tag{6.6}$$

Proof: If $\mathbf{A}_T$ is Hurwitz, then the equations in (6.5) are uniquely solvable. These are needed to compute the cost function and its gradient. Now the gradient of the cost function with respect to $\mathbf{K}$ has to be computed. Let k_{ij} denote element (i, j) in $\mathbf{K}$. First differentiate (6.5b) with respect to k_{ij}, which will be needed later on, which entails

$$\mathbf{A}_T^\mathsf{T}\frac{\partial \mathbf{Q}_T}{\partial k_{ij}} + \frac{\partial \mathbf{Q}_T}{\partial k_{ij}}\mathbf{A}_T + \frac{\partial \mathbf{A}_T^\mathsf{T}}{\partial k_{ij}}\mathbf{Q}_T + \mathbf{Q}_T\frac{\partial \mathbf{A}_T}{\partial k_{ij}} + \frac{\partial \mathbf{C}_T^\mathsf{T}}{\partial k_{ij}}\mathbf{C}_T + \mathbf{C}_T^\mathsf{T}\frac{\partial \mathbf{C}_T}{\partial k_{ij}} = \mathbf{0}. \tag{6.7}$$

Now differentiate the cost function (6.4a) with respect to k_{ij},

$$\frac{\partial \left\|T_{w,z}\right\|_{\mathcal{H}_2}^2}{\partial k_{ij}} = 2\,\mathrm{tr}\,\frac{\partial \mathbf{B}_T^\mathsf{T}}{\partial k_{ij}}\mathbf{Q}_T\mathbf{B}_T + \mathrm{tr}\,\frac{\partial \mathbf{Q}_T}{\partial k_{ij}}\mathbf{B}_T\mathbf{B}_T^\mathsf{T}. \tag{6.8}$$

Using Lemma 4.1 on the equation above together with equations (6.5a) and (6.7) entails

$$\frac{\partial \left\|T_{w,z}\right\|_{\mathcal{H}_2}^2}{\partial k_{ij}} = 2\,\mathrm{tr}\left(\frac{\partial \mathbf{A}_T^\mathsf{T}}{\partial k_{ij}}\mathbf{Q}_T\mathbf{P}_T + \frac{\partial \mathbf{B}_T^\mathsf{T}}{\partial k_{ij}}\mathbf{Q}_T\mathbf{B}_T + \frac{\partial \mathbf{C}_T^\mathsf{T}}{\partial k_{ij}}\mathbf{C}_T\mathbf{P}_T\right). \tag{6.9}$$

Using the structure of the variables $\mathbf{A}_T$, $\mathbf{B}_T$ and $\mathbf{C}_T$ in (6.2) and Lemma 4.2 yields

$$\frac{\partial \left\|T_{w,z}\right\|_{\mathcal{H}_2}^2}{\partial \mathbf{K}} = 2\left(\mathbf{B}_2^\mathsf{T}\mathbf{Q}_T\mathbf{P}_T\mathbf{C}_2^\mathsf{T} + \mathbf{B}_2^\mathsf{T}\mathbf{Q}_T\mathbf{B}_T\mathbf{D}_{21}^\mathsf{T} + \mathbf{D}_{12}^\mathsf{T}\mathbf{C}_T\mathbf{P}_T\mathbf{C}_2^\mathsf{T}\right). \tag{6.10}$$

□

For the optimization problem (6.3) it is also quite straightforward to derive the Hessian, in the same manner as deriving the gradient, ending up in

$$
\begin{aligned}
\frac{\partial^2 \left\|T_{w,z}\right\|_{\mathcal{H}_2}^2}{\partial k_{ij}\partial k_{kl}} &= 2\,\mathrm{tr}\left(\frac{\partial \mathbf{K}^\mathsf{T}}{\partial k_{ij}}\left[\mathbf{B}_2^\mathsf{T}\frac{\partial \mathbf{Q}_T}{\partial k_{kl}}\mathbf{B}_T\mathbf{D}_{21}^\mathsf{T} + \mathbf{D}_{21}^\mathsf{T}\mathbf{C}_T\frac{\partial \mathbf{P}_T}{\partial k_{kl}}\mathbf{C}_2^\mathsf{T} + \mathbf{B}_2^\mathsf{T}\frac{\partial \mathbf{Q}_T}{\partial k_{kl}}\mathbf{P}_T\mathbf{C}_2^\mathsf{T}\right.\right.\\
&\quad \left.\left. +\mathbf{B}_2^\mathsf{T}\mathbf{Q}_T\frac{\partial \mathbf{P}_T}{\partial k_{kl}}\mathbf{C}_2^\mathsf{T} + \mathbf{B}_2^\mathsf{T}\mathbf{Q}_T\mathbf{B}_2\frac{\partial \mathbf{K}}{\partial k_{kl}}\mathbf{D}_{21}\mathbf{D}_{21}^\mathsf{T} + \mathbf{D}_{21}^\mathsf{T}\mathbf{D}_{21}\frac{\partial \mathbf{K}}{\partial k_{kl}}\mathbf{C}_2\mathbf{P}_T\mathbf{C}_2^\mathsf{T}\right]\right)\\
&= 2\left[\mathbf{D}_{12}^\mathsf{T}\mathbf{C}_T\frac{\partial \mathbf{P}_T}{\partial k_{kl}}\mathbf{C}_2^\mathsf{T}\right]_{ij} + 2\left[\mathbf{D}_{12}^\mathsf{T}\mathbf{C}_T\frac{\partial \mathbf{P}_T}{\partial k_{ij}}\mathbf{C}_2^\mathsf{T}\right]_{kl} + 2\left[\mathbf{B}_2^\mathsf{T}\mathbf{Q}_T\frac{\partial \mathbf{P}_T}{\partial k_{kl}}\mathbf{C}_2^\mathsf{T}\right]_{ij}\\
&\quad + 2\left[\mathbf{B}_2^\mathsf{T}\mathbf{Q}_T\frac{\partial \mathbf{P}_T}{\partial k_{ij}}\mathbf{C}_2^\mathsf{T}\right]_{kl} + 2\left[\mathbf{B}_2^\mathsf{T}\mathbf{Q}_T\mathbf{B}_2\right]_{ik}\left[\mathbf{D}_{21}\mathbf{D}_{21}^\mathsf{T}\right]_{lj} + 2\left[\mathbf{D}_{12}^\mathsf{T}\mathbf{D}_{12}\right]_{ik}\left[\mathbf{C}_2\mathbf{P}_T\mathbf{C}_2^\mathsf{T}\right]_{lj}.
\end{aligned} \tag{6.11}
$$

6.2.2 Discrete Time

In discrete time, for the $\mathcal{H}_2$-norm to be defined, the system $T_{w,z}$ must be asymptotically stable, i.e., $\mathbf{A} + \mathbf{B}_2\mathbf{K}\mathbf{C}_2$ has to be Schur. To compute the cost function in (6.3) for discrete-time systems the equations

$$\left\|T_{w,z}\right\|_{\mathcal{H}_2}^2 = \mathrm{tr}\,\mathbf{B}_T^\mathsf{T}\mathbf{Q}_T\mathbf{B}_T + \mathrm{tr}\,\mathbf{D}_T\mathbf{D}_T^\mathsf{T} \tag{6.12a}$$

$$= \mathrm{tr}\,\mathbf{C}_T\mathbf{P}_T\mathbf{C}_T^\mathsf{T} + \mathrm{tr}\,\mathbf{D}_T^\mathsf{T}\mathbf{D}_T, \tag{6.12b}$$

can be used, where $\mathbf{Q}_T$ and $\mathbf{P}_T$ satisfy the discrete-time Lyapunov equations

$$\mathbf{A}_T\mathbf{P}_T\mathbf{A}_T^\mathsf{T} - \mathbf{P}_T + \mathbf{B}_T\mathbf{B}_T^\mathsf{T} = \mathbf{0}, \tag{6.13a}$$

$$\mathbf{A}_T^\mathsf{T}\mathbf{Q}_T\mathbf{A}_T + \mathbf{Q}_T + \mathbf{C}_T^\mathsf{T}\mathbf{C}_T = \mathbf{0}. \tag{6.13b}$$

Theorem 6.2 (Necessary conditions for optimality). *Given a system G as in* (6.1) *and a static output-feedback controller, described by the matrix $\mathbf{K}$, such that $\mathbf{u} = \mathbf{K}\mathbf{y}$. The system G and the controller are given such that the closed-loop system, $T_{w,z}$ in* (6.2)*, is asymptotically stable, i.e., $\mathbf{A}_T$ is Schur. In order for the matrix $\mathbf{K}$ to be optimal for the problem (6.3), it is necessary that it satisfies the equations in (6.13) and that*

$$\frac{\partial \left\|T_{w,z}\right\|_{\mathcal{H}_2}^2}{\partial \mathbf{K}} = 2\left(\mathbf{B}_2^\mathsf{T}\mathbf{Q}_T\mathbf{A}_T\mathbf{P}_T\mathbf{C}_2^\mathsf{T} + \mathbf{B}_2^\mathsf{T}\mathbf{Q}_T\mathbf{B}_T\mathbf{D}_{21}^\mathsf{T} + \mathbf{D}_{21}^\mathsf{T}\mathbf{C}_T\mathbf{P}\mathbf{C}_2^\mathsf{T} + \mathbf{D}_{12}^\mathsf{T}\mathbf{D}_T\mathbf{D}_{21}^\mathsf{T}\right) = \mathbf{0}. \tag{6.14}$$

Proof: The proof is analogous to the proof for Theorem 6.1 □

As in the continuous-time case it also here possible to compute the Hessian, which becomes

$$
\begin{aligned}
\frac{\partial^2 \left\|T_{w,z}\right\|_{\mathcal{H}_2}^2}{\partial k_{ij}\partial k_{kl}} &= 2\left[\mathbf{D}_{12}^{\mathsf{T}}\mathbf{C}_T\frac{\partial \mathbf{P}_T}{\partial k_{kl}}\mathbf{C}_2^{\mathsf{T}}\right]_{ij} + 2\left[\mathbf{D}_{12}^{\mathsf{T}}\mathbf{C}_T\frac{\partial \mathbf{P}_T}{\partial k_{ij}}\mathbf{C}_2^{\mathsf{T}}\right]_{kl} \\
&+ 2\left[\mathbf{B}_2^{\mathsf{T}}\mathbf{Q}_T\mathbf{A}_T\frac{\partial \mathbf{P}_T}{\partial k_{kl}}\mathbf{C}_2^{\mathsf{T}}\right]_{ij} + 2\left[\mathbf{B}_2^{\mathsf{T}}\mathbf{Q}_T\mathbf{A}_T\frac{\partial \mathbf{P}_T}{\partial k_{ij}}\mathbf{C}_2^{\mathsf{T}}\right]_{kl} + 2\left[\mathbf{B}_2^{\mathsf{T}}\mathbf{Q}_T\mathbf{B}_2\right]_{ik}\left[\mathbf{C}_2\mathbf{P}_T\mathbf{C}_2^{\mathsf{T}}\right]_{lj} \\
&+2\left[\mathbf{B}_2^{\mathsf{T}}\mathbf{Q}_T\mathbf{B}_2\right]_{ik}\left[\mathbf{D}_{21}\mathbf{D}_{21}^{\mathsf{T}}\right]_{lj}+2\left[\mathbf{D}_{12}^{\mathsf{T}}\mathbf{D}_{12}\right]_{ik}\left[\mathbf{C}_2\mathbf{P}_T\mathbf{C}_2^{\mathsf{T}}\right]_{lj}+2\left[\mathbf{D}_{12}^{\mathsf{T}}\mathbf{D}_{12}\right]_{ik}\left[\mathbf{D}_{21}\mathbf{D}_{21}^{\mathsf{T}}\right]_{lj}.
\end{aligned} \tag{6.15}
$$

6.3 Static Output-Feedback $\mathcal{H}_2$ LPV Controllers

The controller synthesis method for LPV systems presented in this section will be an extension of the method presented in the previous section, much as how the methods in Chapter 5 are extensions of the methods in Chapter 4. The goal with the optimization problem in this section, is to synthesize a static output-feedback linear parameter-varying $\mathcal{H}_2$ controller. The idea is to be able to directly synthesize a controller using data, instead of first identifying an LPV model and then from that model synthesize a controller. As talked about in Chapter 5, given an LPV model,

$$
G(\mathbf{p}) : \begin{pmatrix} \dot{\mathbf{x}} \\ \mathbf{z} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{A}(\mathbf{p}) & \mathbf{B}_1(\mathbf{p}) & \mathbf{B}_2(\mathbf{p}) \\ \mathbf{C}_1(\mathbf{p}) & \mathbf{D}_{11}(\mathbf{p}) & \mathbf{D}_{12}(\mathbf{p}) \\ \mathbf{C}_2(\mathbf{p}) & \mathbf{D}_{21}(\mathbf{p}) & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \\ \mathbf{u} \end{pmatrix} \tag{6.16}
$$

what ideally is wanted, is to minimize the integral

$$
\min_{\mathbf{K}(\mathbf{p})} \int \left\|T_{w,z}(\mathbf{p})\right\|_{\mathcal{H}_2}^2 \,\mathrm{d}\mathbf{p}, \tag{6.17}
$$

where $T_{w,z}(\mathbf{p})$ is the closed-loop system when closing the loop for the LPV model $G(\mathbf{p})$ with the LPV controller $K(\mathbf{p})$. However, in this section, it is assumed that a set, $\mathcal{M}$, of N LTI models, G_i, for different fixed parameter values, $\mathbf{p}_i$, just as in Chapter 5, is given. This will of course lead to the fact that it is not possible to control the dynamic behavior coming from when the parameters are not fixed, as discussed in Chapter 5, since this information is not present in the data given. However, this is a common problem when working with gain-scheduling and it is assumed in this thesis that the parameters move slowly such that the dynamics from the parameters do not influence the system much, a commonly used assumption, see Shamma and Athans [1992]. The optimization problem now becomes

$$
\underset{\mathbf{K}(\mathbf{p})}{\text{minimize}} \sum_{i=1}^{N} \left\|T_{w,z}(\mathbf{p}_i)\right\|_{\mathcal{H}_2}^2, \tag{6.18}
$$

which for a fixed i becomes equivalent to the problem in Section 6.2.

The parametrization of the controller

$$K(\mathbf{p}) : \mathbf{u}(t) = \mathbf{K}(\mathbf{p})\mathbf{y}(t) \tag{6.19}$$

with respect to the parameters is taken as

$$\mathbf{K}(\mathbf{p}) = \sum_k w_k(\mathbf{p})\mathbf{K}^{(k)}. \tag{6.20}$$

As when identifying LPV models, the functions $w_k(\mathbf{p})$ are design choices that can be hard to choose. However, given such a parametrization and a set of LTI models, as in (6.16), where the given LTI models are denoted as $G(\mathbf{p}_i) = G_i$, the controller as $K(\mathbf{p}_i) = K_i$ and the closed loop system as $T_{w,z}(\mathbf{p}_i) = T_{w,z,i}$, the optimization problem can be written as

$$\underset{\mathbf{K}^{(k)}}{\text{minimize}} \sum_{i=1}^{N} \left\| T_{w,z,i} \right\|_{\mathcal{H}_2}^2 \tag{6.21}$$

where $\mathbf{Q}_{T,i}$ and $\mathbf{P}_{T,i}$ satisfy the Lyapunov equations

$$\mathbf{A}_{T,i}\mathbf{P}_{T,i} + \mathbf{P}_{T,i}\mathbf{A}_{T,i}^{\mathsf{T}} + \mathbf{B}_{T,i}\mathbf{B}_{T,i}^{\mathsf{T}} = \mathbf{0}, \tag{6.22a}$$

$$\mathbf{A}_{T,i}^{\mathsf{T}}\mathbf{Q}_{T,i} + \mathbf{Q}_{T,i}\mathbf{A}_{T,i} + \mathbf{C}_{T,i}^{\mathsf{T}}\mathbf{C}_{T,i} = \mathbf{0}, \tag{6.22b}$$

for the continuous-time case and their discrete-time counterpart in the discrete-time case.

Now we formulate the necessary conditions for optimality for this method in continuous time, the conditions for the discrete-time case are analogous.

Theorem 6.3 (Necessary conditions for optimality). *Assume that $\mathbf{K}_i$ stabilizes the system G_i and that all closed-loop systems, $T_{w,z,i}$ are strictly proper, i.e., $\mathbf{A}_{T,i}$ is Hurwitz and $\mathbf{D}_{T,i} = \mathbf{0}$ for all i. In order for the matrices $\mathbf{K}^{(k)}$ to be optimal for the problem (6.21), it is necessary that $\mathbf{K}(p)$ satisfies the equations in (6.22) for all i, and that*

$$\begin{aligned}\frac{\partial \sum_i \left\| T_{w,z,i} \right\|_{\mathcal{H}_2}^2}{\partial \mathbf{K}} = 2\sum_{i=1}^{N} w_k(\mathbf{p}_i)\Big(&\mathbf{B}_{2,i}^{\mathsf{T}}\mathbf{Q}_{T,i}\mathbf{P}_{T,i}\mathbf{C}_{2,i}^{\mathsf{T}} + \mathbf{B}_{2,i}^{\mathsf{T}}\mathbf{Q}_{T,i}\mathbf{B}_{T,i}\mathbf{D}_{21,i}^{\mathsf{T}} \\ &+ \mathbf{D}_{12,i}^{\mathsf{T}}\mathbf{C}_{T,i}\mathbf{P}_{T,i}\mathbf{C}_{2,i}^{\mathsf{T}}\Big) = \mathbf{0}.\end{aligned} \tag{6.23}$$

Proof: The proof is analogous with the proof for Theorem 6.1. □

6.4 Computational Aspects

In this section, a suggestion of how the methods in this chapter can be initialized and how to speed up the computations will be presented.

As with the methods in the previous chapters, both cost functions and their gradients have been calculated and can easily be used in, e.g., a quasi-Newton algorithm to solve the optimization problem. For the methods described in this chapter also the Hessians have been calculated, which can be utilized in a quasi-Newton algorithm to initialize the Hessian approximation in, e.g., BFGS. We do not want to use the Hessian information in every iteration since this would be too heavy, computationally.

The derivations for the gradients and the Hessians in Section 6.2, have been done element wise, as with the methods in the previous chapters. This means that it is possible, also for these methods, to introduce structure in the controller, e.g. a diagonal controller.

For the methods in this chapter it is, however, not as straightforward to utilize the structure in the Lyapunov equations (6.22) (or (6.5)) since, in the realization (6.2), there is no obvious structure that can be exploited. What can be used, is that if both the cost function and the gradient have to be computed, both $\mathbf{Q}_T$ and $\mathbf{P}_T$ must be computed, and $\mathbf{Q}_T$ and $\mathbf{P}_T$ can be solved efficiently together by using the fact that $\mathbf{A}_T$ is the factor in both of the Lyapunov equations, see for example Benner et al. [1998].

The optimization problems (6.3) and (6.21) are both non-convex and nonlinear, which makes the initialization an important problem. Additionally, it is required that the initializing controller is stabilizing, which is probably an NP-hard problem, see Blondel and Tsitsiklis [1997]. If the given system (or systems if given a set of models) is asymptotically stable, then the initialization used is a controller with all zeros. However, if given an unstable system for which an $\mathcal{H}_2$ controller should be computed we take use of other, existing, methods/algorithms to try and stabilize the system and then start our method with this stabilizing controller. The algorithm used to find a stabilizing controller is HIFOO (see Gumussoy et al. [2009]).

6.5 Examples

In this section, we will try to show the applicability of the methods presented in this chapter using some examples. We begin with an example where the method presented in Section 6.2 is used on some systems in the COMPl$_e$ib benchmark collection (see Leibfritz and Lipinski [2003]).

Example 6.1: COMPl$_e$ib-Systems

In this example our goal is to compare the method presented in Section 6.2, which will be called H$_2$NLCTRL, with the SDP-method described in Stingl [2006], called STINGL, and the method described in Arzelier et al. [2011], called HIFOO.

The systems used in this example comes from COMPl$_e$ib (see Leibfritz and Lipinski [2003]), and are systems ranging from 2 to 1100 states. In all systems we have $\mathbf{D}_{11} = \mathbf{0}$ and $\mathbf{D}_{12} = \mathbf{0}$ or $\mathbf{D}_{21} = \mathbf{0}$, to make sure that $\mathbf{D}_{11} + \mathbf{D}_{12}\mathbf{K}\mathbf{D}_{21} = \mathbf{0}$, so that the $\mathcal{H}_2$-norm of the closed-loop system is defined.

To initialize H$_2$NLCTRL and HIFOO, a heuristic approach is used. First it is checked if the system is open-loop stable and if that is the case, then the optimization is initialized with $\mathbf{K} = \mathbf{0}$. If this does not hold then the optimization package HIFOO, see Gumussoy et al. [2009], is called to minimize the real part of the largest eigenvalue of the matrix $\mathbf{A} + \mathbf{B}_2\mathbf{K}\mathbf{C}_2$. Only cases where a stabilizing controller is found are reported in the tables.

In Table 6.1, Table 6.2, Table 6.3, Table 6.4 and Table 6.5, results from the numerical benchmark are presented. The name of the COMPl$_e$ib-system is displayed in the first column. In the second column, the relevant sizes, i.e., number of states, number of output and number of inputs are display and are denoted n_x, n_y and n_u respectively. In columns three, four and five the $\mathcal{H}_2$-norm for the resulting closed-loop systems are displayed and in columns six, seven and eight how long time it took for the methods H$_2$NLCTRL, HIFOO and STINGL to find the controller are displayed. For the first 31 systems, which also occur in the test performed in Stingl [2006], the results are compared to the results reported in Stingl [2006]. For the reminder of systems a "–" in the fourth column denotes that we do not have any other results from Stingl [2006] to compare with. In Stingl [2006] they could not find a controller for these systems, mostly because of numerical problems and rapid growth of the SDPs, since they optimize over both $\mathbf{K}$ and the Lyapunov matrices $\mathbf{P}$ or $\mathbf{Q}$.

When the results using HIFOO and H$_2$NLCTRL are compared with the results from STINGL, HIFOO and H$_2$NLCTRL find for almost all systems the same value for $\left\|T_{w,z}\right\|_{\mathcal{H}_2}$, however, generally, much faster. When comparing H$_2$NLCTRL and HIFOO they perform very similar for most of the systems regarding the value $\left\|T_{w,z}\right\|_{\mathcal{H}_2}$. However, for a large number of the systems, H$_2$NLCTRL is able to find the controller faster than HIFOO and for a few systems HIFOO is not able to compute a controller due to out of memory, denoted with "–" in the fifth column, where H$_2$NLCTRL can.

The results in the tables below also show the benefit with the new method, apart from being able to handle structure in the controller, it can handle medium-scale systems. The amount of optimization variables does not grow with the amount of states in the systems, as in the SDP case used by Stingl [2006], but only depends on the size of the controller.

Table 6.1: *Numerical results for Example 6.1. The systems are taken from COMPl$_e$ib (see Leibfritz and Lipinski [2003]) with their name displayed first followed by the sizes of the system; number of states, number of outputs and number of inputs (n_x, n_y and n_u). The result, $\|T_{w,z}\|_{\mathcal{H}_2}$, for the method proposed in Section 6.2 are compared with HIFOO (see Gumussoy et al. [2009]), and when possible, to results from Stingl [2006].*

Model	$n_x/n_y/n_u$	$\|T_{w,z}\|_{\mathcal{H}_2}$ H_2NLCTRL	$\|T_{w,z}\|_{\mathcal{H}_2}$ STINGL	$\|T_{w,z}\|_{\mathcal{H}_2}$ HIFOO	Time [s] H_2NLCTRL	Time [s] HIFOO	Time [s] STINGL
AC1	5/3/3	2.37e-03	1.01e-03	1.28e-11	5.78e-01	1.53e+00	7.90e-01
AC11	5/4/2	3.96e+00	3.94e+00	3.94e+00	1.87e-01	7.27e-01	3.56e+00
AC15	4/3/2	1.26e+01	1.26e+01	1.26e+01	2.10e-01	5.83e-01	5.30e-01
AC16	4/4/2	1.23e+01	1.23e+01	1.23e+01	2.55e-01	6.25e-01	2.66e+00
AC17	4/2/1	4.11e+00	4.11e+00	4.11e+00	9.75e-02	1.10e-01	3.50e-01
AC2	5/3/3	5.03e-02	5.04e-02	5.03e-02	6.85e-01	5.10e+00	1.60e+00
AC3	5/4/2	4.57e+00	4.57e+00	4.57e+00	3.10e-01	9.58e-01	7.20e-01
AC6	7/4/2	3.80e+00	3.80e+00	3.80e+00	1.27e-01	3.58e-01	3.22e+00
AGS	12/2/2	7.00e+00	7.00e+00	7.00e+00	9.93e-02	4.80e-01	1.67e+02
BDT1	11/3/3	9.84e-03	1.75e-02	1.07e-02	3.36e-01	1.19e-01	3.56e+00
CSE1	20/10/2	8.81e-03	1.21e-02	8.97e-03	1.05e+00	1.65e-01	5.10e+01
DIS1	8/4/4	2.66e+00	2.66e+00	2.66e+00	4.27e-01	1.20e+00	6.45e+00
DIS2	3/2/2	1.42e+00	1.42e+00	1.42e+00	1.20e-01	2.27e-01	2.30e-01
DIS3	6/4/4	1.84e+00	1.84e+00	1.84e+00	2.20e-01	2.45e-01	2.47e+00
DIS4	6/6/4	1.69e+00	1.69e+00	1.69e+00	2.14e-01	2.38e-01	3.73e+00
HE1	4/1/2	9.54e-02	9.55e-02	9.54e-02	8.84e-02	1.28e-01	2.20e-01
HE2	4/2/2	3.43e+00	3.34e+00	3.43e+00	1.78e-01	5.53e-01	7.60e-01
HE4	8/6/4	2.13e+01	2.08e+01	2.08e+01	7.07e-01	1.06e+00	2.17e+01
IH	21/10/11	1.80e-13	8.26e-04	5.96e-07	2.93e+00	1.08e+01	3.76e+02
MFP	4/2/3	9.16e+00	9.16e+00	9.16e+00	1.43e-01	2.14e-01	5.90e-01
NN11	16/5/3	7.85e-02	1.20e+02	1.20e-01	2.77e-01	3.44e-01	2.85e+02

Table 6.2: *Numerical results for Example 6.1. The systems are taken from COMPl$_e$ib (see Leibfritz and Lipinski [2003]) with their name displayed first followed by the sizes of the system; number of states, number of outputs and number of inputs (n_x, n_y and n_u). The result, $\|T_{w,z}\|_{\mathcal{H}_2}$, for the method proposed in Section 6.2 are compared with HIFOO (see Gumussoy et al. [2009]), and when possible, to results from Stingl [2006].*

Model	$n_x/n_y/n_u$	$\|T_{w,z}\|_{\mathcal{H}_2}$ H_2NLCTRL	$\|T_{w,z}\|_{\mathcal{H}_2}$ STINGL	$\|T_{w,z}\|_{\mathcal{H}_2}$ HIFOO	Time [s] H_2NLCTRL	Time [s] HIFOO	Time [s] STINGL
NN15	3/2/2	4.85e-02	4.90e-02	4.86e-02	8.41e+00	1.94e+00	3.50e-01
NN16	8/4/4	2.91e-01	3.07e-01	2.91e-01	1.20e+00	2.43e+02	4.66e+01
NN2	2/1/1	1.57e+00	1.56e+00	1.57e+00	8.12e-02	1.42e-01	2.80e-01
NN4	4/3/2	1.88e+00	1.88e+00	1.88e+00	1.77e-01	4.85e-01	3.80e-01
NN8	3/2/2	2.28e+00	2.28e+00	2.28e+00	1.07e-01	1.48e-01	3.40e-01
PSM	7/3/2	1.50e+00	1.50e+00	1.50e+00	1.17e-01	1.89e-01	2.57e+00
REA1	4/3/2	1.82e+00	1.82e+00	1.82e+00	1.18e-01	2.11e-01	9.50e-01
REA2	4/2/2	1.86e+00	1.86e+00	1.86e+00	9.51e-02	1.85e-01	5.50e-01
REA3	12/3/1	1.21e+01	1.21e+01	1.21e+01	1.91e-01	4.12e-01	2.63e+01
TG1	10/2/2	2.23e+01	2.23e+01	2.23e+01	2.14e-01	6.62e-01	1.07e+02
AC10	55/2/2	2.78e+01	–	2.78e+01	3.72e-01	2.50e+00	–
AC13	28/4/3	1.32e+02	–	1.32e+02	4.13e-01	1.12e+00	–
AC18	10/2/2	1.97e+01	–	1.97e+01	4.21e-01	2.58e+00	–
AC5	82/4/4	7.95e-01	–	7.95e-01	1.73e+00	2.86e+00	–
BDT2	60/30/2	8.87e-03	–	9.11e-03	1.66e+01	2.91e-01	–
CSE2	40/2/2	2.39e+03	–	1.07e+03	4.06e-01	8.70e-01	–
DLR2	40/2/2	1.13e+03	–	1.13e+03	6.98e-01	1.38e+00	–
DLR3	5/3/1	3.03e+07	–	1.69e+04	6.22e-02	1.21e+00	–
FS	130/2/1	5.82e-02	–	5.82e-02	7.25e-01	1.29e+00	–
HF1	5/3/2	7.12e+04	–	7.12e+04	1.72e-01	4.98e-01	–
HF2D10	5/3/2	8.51e+04	–	8.51e+04	2.34e-01	8.43e-01	–

Table 6.3: *Numerical results for Example 6.1. The systems are taken from COMPl$_e$ib (see Leibfritz and Lipinski [2003]) with their name displayed first followed by the sizes of the system; number of states, number of outputs and number of inputs (n_x, n_y and n_u). The result, $\left\|T_{w,z}\right\|_{\mathcal{H}_2}$, for the method proposed in Section 6.2 are compared with HIFOO (see Gumussoy et al. [2009]), and when possible, to results from Stingl [2006].*

Model	$n_x/n_y/n_u$	$\left\|T_{w,z}\right\|_{\mathcal{H}_2}$ H$_2$NLCTRL	$\left\|T_{w,z}\right\|_{\mathcal{H}_2}$ STINGL	$\left\|T_{w,z}\right\|_{\mathcal{H}_2}$ HIFOO	Time [s] H$_2$NLCTRL	Time [s] HIFOO	Time [s] STINGL
HF2D11	5/4/2	6.29e+05	–	6.29e+05	5.87e-01	1.87e+00	–
HF2D12	5/4/2	7.96e+04	–	7.96e+04	2.67e-01	6.68e-01	–
HF2D13	5/4/2	3.74e+05	–	3.74e+05	2.80e-01	1.51e+00	–
HF2D14	5/4/2	2.97e+05	–	2.97e+05	2.45e-01	1.20e+00	–
HF2D15	5/4/2	2.85e+05	–	2.85e+05	4.00e-01	2.09e+00	–
HF2D16	5/4/2	3.76e+05	–	3.76e+05	2.03e-01	1.80e+00	–
HF2D17	5/2/2	2.78e+01	–	2.78e+01	1.10e-01	2.67e+00	–
HF2D18	316/3/2	1.31e+04	–	–	3.02e+01	–	–
HF2D1_M316	541/3/2	1.07e+04	–	–	2.44e+02	–	–
HF2D1_M541	316/3/2	1.34e+04	–	–	1.83e+01	–	–
HF2D2_M316	541/3/2	1.35e+04	–	–	1.20e+02	–	–
HF2D2_M541	289/4/2	2.17e+04	–	–	1.80e+01	–	–
HF2D5_M289	529/4/2	2.08e+04	–	–	1.97e+02	–	–
HF2D5_M529	289/4/2	1.93e+04	–	–	2.22e+01	–	–
HF2D6_M289	529/4/2	2.55e+04	–	–	2.39e+02	–	–
HF2D6_M529	256/2/2	4.67e+00	–	–	2.64e+01	–	–
HF2D9_M256	484/2/2	4.64e+00	–	–	7.13e+01	–	–
HF2D9_M484	256/2/2	7.65e+00	–	7.65e+00	1.37e+01	4.80e+01	–
HF2D_CD1_M256	484/2/2	6.11e+00	–	–	1.96e+02	–	–
HF2D_CD1_M484	256/2/2	1.04e+01	–	1.04e+01	1.52e+01	5.63e+01	–
HF2D_CD2_M256	484/2/2	4.72e+00	–	–	1.51e+02	–	–

Table 6.4: *Numerical results for Example 6.1. The systems are taken from COMPl$_e$ib (see Leibfritz and Lipinski [2003]) with their name displayed first followed by the sizes of the system; number of states, number of outputs and number of inputs (n_x, n_y and n_u). The result, $\left\|T_{w,z}\right\|_{\mathcal{H}_2}$, for the method proposed in Section 6.2 are compared with HIFOO (see Gumussoy et al. [2009]), and when possible, to results from Stingl [2006].*

Model	$n_x/n_y/n_u$	$\left\|T_{w,z}\right\|_{\mathcal{H}_2}$ H$_2$NLCTRL	$\left\|T_{w,z}\right\|_{\mathcal{H}_2}$ STINGL	$\left\|T_{w,z}\right\|_{\mathcal{H}_2}$ HIFOO	Time [s] H$_2$NLCTRL	Time [s] HIFOO	Time [s] STINGL
HF2D_CD2_M484	324/2/2	8.66e+00	–	–	1.44e+01	–	–
HF2D_CD3_M324	576/2/2	5.00e+00	–	–	2.36e+02	–	–
HF2D_CD3_M576	7/2/2	4.56e+01	–	4.56e+01	2.30e-01	3.86e-01	–
HF2D_CD4	7/2/2	3.14e+01	–	3.14e+01	1.69e-01	6.53e-01	–
HF2D_CD5	7/2/2	3.49e+01	–	3.49e+01	1.67e-01	4.53e-01	–
HF2D_CD6	361/4/2	4.94e+05	–	–	1.77e+01	–	–
HF2D_IS1_M361	529/4/2	5.10e+04	–	–	2.56e+02	–	–
HF2D_IS1_M529	361/4/2	2.80e+04	–	–	3.98e+01	–	–
HF2D_IS2_M361	529/4/2	2.10e+04	–	–	2.33e+02	–	–
HF2D_IS2_M529	256/2/2	1.16e+01	–	1.16e+01	1.13e+01	8.18e+01	–
HF2D_IS3_M256	484/2/2	1.11e+01	–	–	9.25e+01	–	–
HF2D_IS3_M484	256/2/2	7.69e+00	–	7.69e+00	2.48e+01	2.36e+02	–
HF2D_IS4_M256	484/2/2	1.70e+01	–	–	1.96e+02	–	–
HF2D_IS4_M484	5/4/2	1.24e+06	–	1.24e+06	6.77e-01	3.39e+00	–
HF2D_IS5	5/4/2	2.49e+05	–	2.49e+05	3.31e-01	1.25e+00	–
HF2D_IS6	5/2/2	6.13e+01	–	6.13e+01	1.02e-01	4.06e-01	–
HF2D_IS7	5/2/2	9.94e+01	–	9.94e+01	1.14e-01	4.22e-01	–
HF2D_IS8	30/5/3	6.25e+01	–	6.31e+01	1.35e+00	3.63e+00	–
JE1	21/3/3	9.61e+02	–	9.61e+02	4.40e-01	1.67e+00	–
JE2	3/2/1	4.18e+01	–	4.18e+01	8.90e-02	1.73e-01	–
NN1	6/2/2	1.89e+01	–	1.89e+01	8.98e-01	6.02e+00	–

Table 6.5: *Numerical results for Example 6.1. The systems are taken from COMPl$_e$ib (see Leibfritz and Lipinski [2003]) with their name displayed first followed by the sizes of the system; number of states, number of outputs and number of inputs (n_x, n_y and n_u). The result, $\|T_{w,z}\|_{\mathcal{H}_2}$, for the method proposed in Section 6.2 are compared with HIFOO (see Gumussoy et al. [2009]), and when possible, to results from Stingl [2006].*

Model	$n_x/n_y/n_u$	$\|T_{w,z}\|_{\mathcal{H}_2}$ H$_2$NLCTRL	$\|T_{w,z}\|_{\mathcal{H}_2}$ STINGL	$\|T_{w,z}\|_{\mathcal{H}_2}$ HIFOO	Time [s] H$_2$NLCTRL	Time [s] HIFOO	Time [s] STINGL
NN10	3/1/2	9.46e+00	–	9.46e+00	1.05e-01	1.45e-01	–
NN12	1006/1/1	3.68e-01	–	3.68e-01	4.57e+02	1.39e+02	–
NN17	7/2/1	1.41e+02	–	1.41e+02	1.07e-01	1.33e-01	–
NN18	9/4/1	1.35e+03	–	1.31e+03	5.13e-01	1.12e+00	–
NN3	9/4/1	1.33e+02	–	1.33e+02	2.01e-01	7.08e-01	–
NN5	5/3/1	1.10e+07	–	3.80e-03	6.67e-02	5.99e-01	–
NN6	6/4/2	2.80e-01	–	2.75e-01	2.87e-01	8.41e-01	–
NN7	9/2/2	1.63e+02	–	1.63e+02	3.19e-01	1.13e+00	–
PAS	7/5/3	6.91e-03	–	1.81e-03	3.60e-01	8.80e-01	–
ROC10	5/3/2	3.41e-01	–	3.41e-01	2.79e-01	7.98e-01	–
ROC3	256/2/2	1.96e+09	–	–	2.51e+00	–	–
ROC4	8/2/2	2.59e-14	–	1.33e-12	9.98e-02	2.28e+01	–
ROC5	10/4/3	7.36e+00	–	7.36e+00	4.20e-01	1.73e+00	–
ROC7	10/4/3	7.70e+00	–	7.70e+00	3.96e-01	1.07e+00	–
TL	10/4/3	8.15e+00	–	8.15e+00	4.21e-01	1.41e+00	–

Table 6.6: Numerical values for the coefficients in the LPV controllers in Example 6.2 and the time to compute them.

Model	$\mathbf{K}^{(0)}$	$\mathbf{K}^{(1)}$	$\mathbf{K}^{(2)}$	Time [s]
Constant	0.1638	–	–	0.09 s
Linear	0.259	-0.305	–	0.12 s
Quadratic	0.2936	-0.935	0.6903	0.11 s

A small example of an LPV controller synthesis problem is now presented to show the potential of the method proposed in Section 6.3.

Example 6.2

The system in this example is the same as in Example 5.2

$$G = G_1 G_2, \quad \text{where} \tag{6.24a}$$

$$G_1 = \frac{1}{s^2 + 2\zeta_1 s + 1},\ G_2 = \frac{9}{s^2 + 6\zeta_2 s + 9}, \tag{6.24b}$$

$$\zeta_1 = 0.1 + 0.9p,\ \zeta_2 = 0.1 + 0.9(1-p),\ p \in [0,1]. \tag{6.24c}$$

From these equations we obtain $\mathbf{A}(p), \mathbf{B}_2(p), \mathbf{C}_2(p)$ and $\mathbf{D}_{22}(p)$, using the notation in (2.31), that represents the dynamical system. Then we create the matrices

$$\mathbf{B}_1(p) = \mathbb{I}_{4\times 4}, \quad \mathbf{C}_1(p) = \mathbb{I}_{4\times 4}$$
$$\mathbf{D}_{11}(p) = \mathbf{0}_{4\times 4}, \quad \mathbf{D}_{12}(p) = \begin{pmatrix} \mathbf{0}_{3\times 1} \\ 1 \end{pmatrix}, \quad \mathbf{D}_{21}(p) = \mathbf{0}_{1\times 4}$$

to have a fully defined performance measure of the system. From this system we extract five systems representing five equidistant points in $p \in [0,1]$, i.e., we are given five LTI models, extracted from the LPV system (6.24), with four states each.

The LPV system is expressed in a balanced state basis. In this state basis the LPV system depend nonlinearly on the parameter p, see Figure 6.1. Hence, judging from the given data, one could easily suspect that a complex LPV controller would be required. However, in this example, using the proposed method from Section 6.3, we will try to find three static output-feedback LPV controllers of different complexity, one that is constant and independent of the parameter p, one that is linear in p and one that is quadratic in p. For example, the quadratic LPV controller has the structure,

$$u(t) = \mathbf{K}(p)y(t), \quad \mathbf{K}(p) = \mathbf{K}^{(0)} + \mathbf{K}^{(1)}p + \mathbf{K}^{(2)}p^2. \tag{6.25}$$

The specific values for the resulting LPV controllers and times for computing them can be found in Table 6.6

To validate the controllers, 100 validation points were generated from (6.24), for $p \in [0,1]$. For each of these 100 models, an optimal static output-feedback

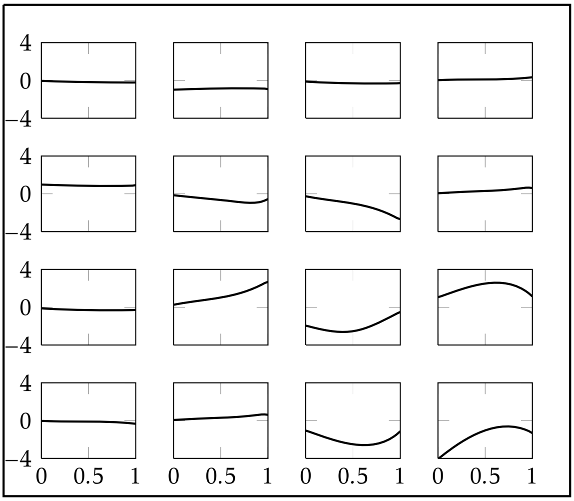

Figure 6.1: *The elements in the* $\mathbf{A}$*-matrices as function of* p *for the four state* LPV *system* (6.24) *in the given state basis.*

controller was created (the associated optimization problem is a scalar problem, hence trivially solved using, e.g., gridding). In Figure 6.2 the ratio between the $\mathcal{H}_2$-performance for the different LPV controllers and the $\mathcal{H}_2$-performance with the optimal static output-feedback controller in the different validation points is shown, i.e., the closer the curve is to the value one the closer the LPV controller is to the optimal controller. In Figure 6.2, we see that method is able to find LPV controllers that, depending on the complexity of the LPV controller, is close to the optimal reference controller in the validation points.

In Figure 6.3, the reference controller and the resulting LPV controllers (constant, linear and quadratic in p) are plotted. Looking at both Figure 6.2 and Figure 6.3 one can see that with an LPV controller that is quadratic in p we find a controller that is very similar to the globally optimal one.

6.6 Conclusions

In this chapter, two methods for synthesizing $\mathcal{H}_2$ controllers have been presented, one for LTI systems and one for LPV systems. The methods use a direct nonlinear optimization approach to solve the problem which makes it possible to control the structure of the controller to create, e.g., a diagonal or bidiagonal controller. For these methods, both cost functions, gradients and hessians have been derived, which makes it possible to effectively use of-the-shelf quasi-Newton solvers and makes it possible to solve problems of medium-scale size. One of the drawbacks with the methods is the non-convexity of the problems and the possible fact that finding a stabilizing controller is an NP-hard problem. However, this is a problem

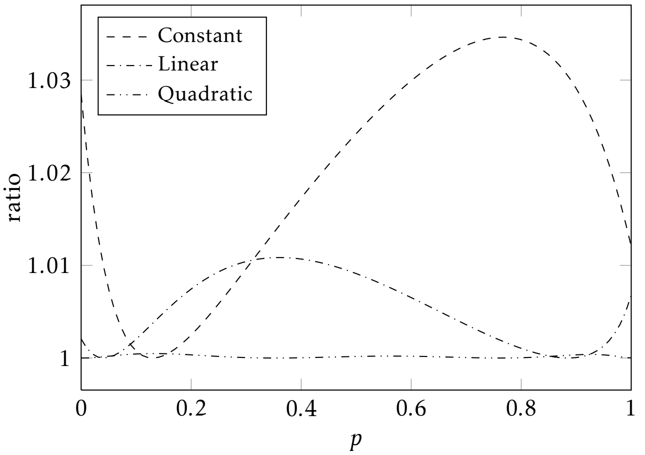

Figure 6.2: *The ratio between the $\mathcal{H}_2$-performance with the different LPV controllers and the $\mathcal{H}_2$-performance with the optimal static output-feedback controller in the different validation points. The closer the curve is to the value one the closer the LPV controller is to the optimal controller.*

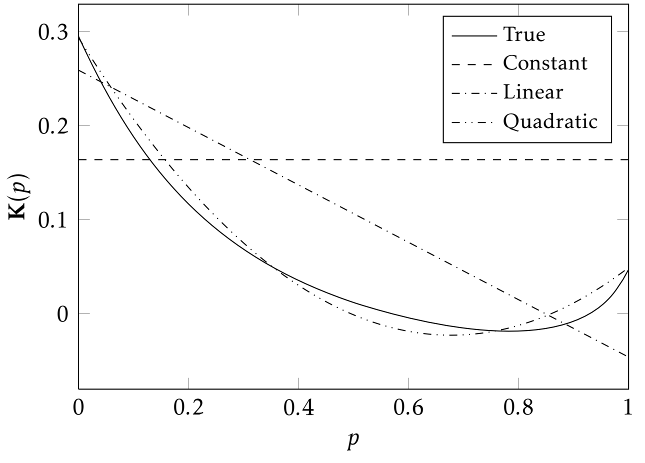

Figure 6.3: *The reference controller (solid line) and the resulting LPV controllers (linear in p, dashed line, quadratic in p, dash-dotted line and cubic in p, dotted line) plotted as functions of the parameter, p.*

that the methods have in common with other methods too, and is one of the problems that need more attention in the future. One possible direct extension that has not been tested is to use the idea of controlling the rank of the system matrices, as in Section 5.4.2. By using a method that can control the rank, one could, for example, enforce the controller to have integrators.

7

Examples of Applications

In this chapter, the methods from Chapter 4 and Chapter 5 are illustrated with two more elaborate examples. In the first example, both model-reduction methods from Chapter 4 and LPV generation methods from Chapter 5 are used on an Airbus aircraft model, to show the applicability of the methods on a real-world example. In the second example, we show how model-reduction methods can be used in system identification to obtain better estimates for certain model structures.

7.1 Aircraft Example

The models used in this section, are models of an Airbus aircraft that were developed and used in an EU project called COFCLUO (Clearance Of Flight Control Laws Using Optimization, see `http://cofcluo.isy.liu.se/` and Varga et al. [2012]). The main objective of the COFCLUO project was to develop methods that use optimization techniques to make clearance of flight control laws more efficient and reliable, see for example Garulli et al. [2013]. The clearance of flight control laws is an important part of the certification and qualification process for the airplane industry. The models used in the examples below are three LPV models that, with different complexity, describe an airplane in closed loop in the longitudinal direction. All models are SISO LPV models with 22 states and all depend polynomially on the parameters. The difference between the LPV models is that they depend on one (different configurations for the center tank), two (different configurations for the center tank and the outer tank) or three parameters (different configurations for the center tank, the outer tank and payload) respectively.

7.1.1 LPV Simplification

To be able to use certain analysis methods for evaluating performance criteria for flight clearance, the LPV models have to be represented as linear fractional representations, LFRs, (see, e.g., Zhou et al. [1996] or Hecker [2006]). To be able to use the analysis methods efficiently the LFRs have to be of low order. Generally, any LPV model with rational dependence in the parameters can be turned into an LFR. However, it is a difficult problem to guarantee that the resulting LFR is of minimal order. There exist some special cases for when this is possible, for example, when the LPV depends affinely on the parameters, see Hecker [2006]). Take an LPV model

$$G(\mathbf{p}) = \left[\begin{array}{c|c} \mathbf{A}(\mathbf{p}) & \mathbf{B}(\mathbf{p}) \\ \hline \mathbf{C}(\mathbf{p}) & \mathbf{D}(\mathbf{p}) \end{array}\right],$$

where the system matrices depend affinely on the parameters in $\mathbf{p}$, i.e., $\mathbf{A}(\mathbf{p}) = \mathbf{A}^{(0)} + \mathbf{A}^{(1)}p_1 + \mathbf{A}^{(2)}p_2 + \cdots + \mathbf{A}^{(N)}p_N$ and the same for $\mathbf{B}(\mathbf{p})$, $\mathbf{C}(\mathbf{p})$ and $\mathbf{D}(\mathbf{p})$. Now create the matrices $\mathbf{F}^{(0)}, \mathbf{F}^{(1)}, \mathbf{F}^{(2)}, \ldots, \mathbf{F}^{(N)}$ as

$$\mathbf{F}^{(0)} = \begin{pmatrix} \mathbf{A}^{(0)} & \mathbf{B}^{(0)} \\ \mathbf{C}^{(0)} & \mathbf{D}^{(0)} \end{pmatrix}, \quad \mathbf{F}^{(1)} = \begin{pmatrix} \mathbf{A}^{(1)} & \mathbf{B}^{(1)} \\ \mathbf{C}^{(1)} & \mathbf{D}^{(1)} \end{pmatrix}, \quad \mathbf{F}^{(2)} = \begin{pmatrix} \mathbf{A}^{(2)} & \mathbf{B}^{(2)} \\ \mathbf{C}^{(2)} & \mathbf{D}^{(2)} \end{pmatrix}, \quad \ldots.$$

The minimal order the LFR, generated from $G(\mathbf{p})$, can have is $\sum_{i=1}^{N} \operatorname{rank} \mathbf{F}^{(i)}$ and this LFR is easy to compute, see Hecker [2006].

In this example, the LPV generation methods described in Chapter 5 will be used to reduce the complexity, with respect to the parameters, of the original LPV models. The strategy that will be used is to sample a number of LTI models from the three given LPV models and choose an affine parametrization for the generated LPV models to be able to guarantee that a low order LFR can be computed from the generated LPV models.

The given LPV models are not strictly proper, which is a problem when using methods based on the $\mathcal{H}_2$-norm, since the $\mathcal{H}_2$-norm is infinite if $\mathbf{D} \neq \mathbf{0}$. To circumvent this problem, the $\mathbf{D}$ matrices are first ignored and an affine LPV model is computed using only the $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ matrices. To find the resulting $\mathbf{D}$ matrices a simple element-wise interpolation problem is solved. However, since the $\mathbf{D}$ matrices are unaffected by state transformations the complexity cannot as easy be reduced for the $\mathbf{D}$ matrices and a higher order polynomial might be necessary in the interpolation to obtain a sufficiently good approximation.

As mentioned above, an LFR of low order is preferred. The first step towards this was to use an affine parametrization. However, by using the rank controlling method described in Section 5.4.2, it is possible to control the rank of the coefficient matrices ($\mathbf{F}^{(1)}$, $\mathbf{F}^{(2)}$, ...) in the generated LPV. Hence, using the rank controlling method described in Section 5.4.2 the complexity of the resulting LFR can be lowered even more by constraining the appropriate matrices to have low rank.

In this example, we sample 10, 100 and 125 LTI models from the one, two and three parameter LPV models, respectively. The LTI models are sampled equidistantly in the parameter space. These LTI models are used as inputs to the proposed methods. Two LPV models will be generated for the data sets from the LPV models with one and two parameters, one with full rank in all the coefficient matrices and one with rank deficient $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ matrices.

A few different ranks for the $\mathbf{A}^{(1)}$, $\mathbf{A}^{(2)}$ and $\mathbf{A}^{(3)}$ matrices were tested and for the one parameter model set, rank two was chosen for the matrix $\mathbf{A}^{(1)}$ and for the two parameter model set, rank eleven was used for both $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$. For the three parameter model set, no sufficiently good model, for the ranks tested, was found and only the result using coefficient matrices with full rank will be presented.

The validity of the resulting LPV models are evaluated by sampling a new, different, set of LTI models from each of the given LPV models and compare these with the generated LPV models. The models are compared both using the relative $\mathcal{H}_2$-norm, ignoring the $\mathbf{D}$ matrices and the relative $\mathcal{H}_\infty$-norm, including the $\mathbf{D}$ matrices. The results from the LPV generation are displayed for the one parameter case in Figure 7.1. For the two parameter case, the full rank case is displayed in Figure 7.2 and for the low rank case in Figure 7.3. The result from the three parameter case is displayed in Figure 7.4.

In Figure 7.1 – 7.4, we can see that all the generated LPV models have a low relative $\mathcal{H}_2$-norm for all validation models. This suggests that we have found good approximations of the original LPV models. Not only is the relative $\mathcal{H}_2$-norm low, but also the relative $\mathcal{H}_\infty$-norm, which gives another certificate that the generated models approximates the given LPV models well. Looking at Table 7.1, we can also see that complexity of the resulting LFR have decreased in most cases and especially in the cases where we were able to find LPV models with rank deficient coefficient matrices. These facts suggest that the proposed LPV methods can be used to reduce the complexity of LPV models and their LFRs. Another interesting fact that can be seen in Figure 7.1 is that for the one parameter model, the resulting model using a rank deficient coefficient matrix finds a better model than the one with full rank. Two likely explanations are that it could be due to the non-convexity of the problem or that the full rank case is an over-parametrization and the low rank method works as a regularization to the problem.

7.1.2 Model Reduction

The three LPV models, described in the previous section, describe an aircraft, and more precisely a flexible aircraft. The original models were computed using finite element computations and were very large. These models were then reduced such that the dynamics above 15 rad/s in the models were truncated. Hence, the given LPV models are only valid up till 15 rad/s, which makes these models suitable for testing the frequency-limited model-reduction method, described in Section 4.4.3. As can be seen in Figure 7.5, which plots the magnitude curve of one of the LTI models, it would be beneficial to be able to ignore the dynamics after 15 rad/s when doing model reduction.

For this example we extract one LTI model from the one parameter LPV model

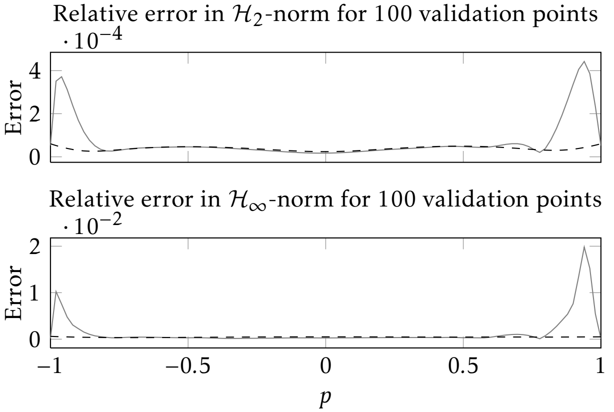

Figure 7.1: *Relative error in $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norm at 100 validation points, in the one parameter case. The gray line comes from the case when the coefficient matrix $\hat{\mathbf{A}}^{(1)}$ has full rank and the black dashed line from the case when $\hat{\mathbf{A}}^{(1)}$ has rank two. Interesting to note is that the low rank model performs better than the full rank one. This could, for example, be due to the non-convexity of the problem or over-parametrization.*

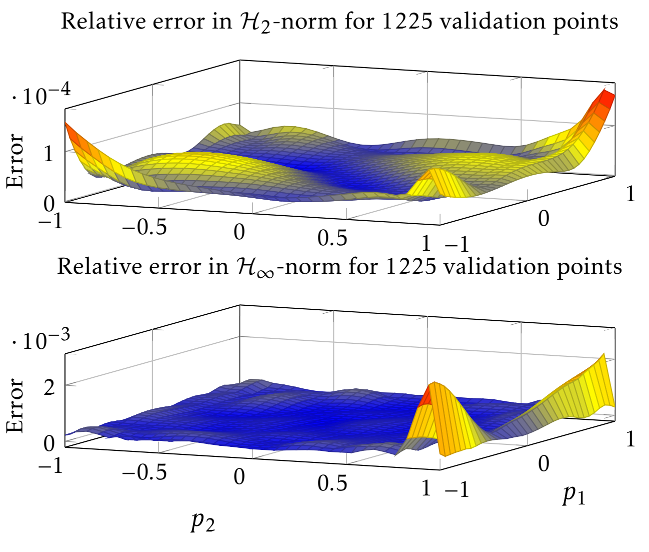

Figure 7.2: *Relative error in $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norm at 1225 validation points, in the two parameter case when the coefficient matrices $\hat{\mathbf{A}}^{(1)}$ and $\hat{\mathbf{A}}^{(2)}$ have full rank.*

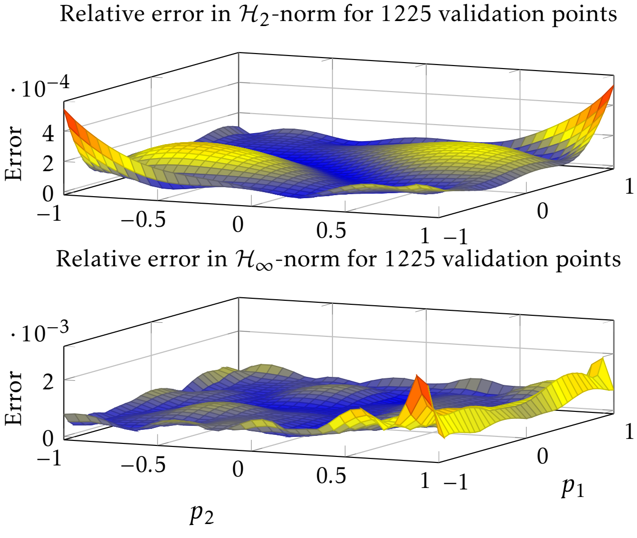

Figure 7.3: *Relative error in $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norm at 1225 validation points, in the two parameter case when the coefficient matrices $\hat{\mathbf{A}}^{(1)}$ and $\hat{\mathbf{A}}^{(2)}$ have rank 11.*

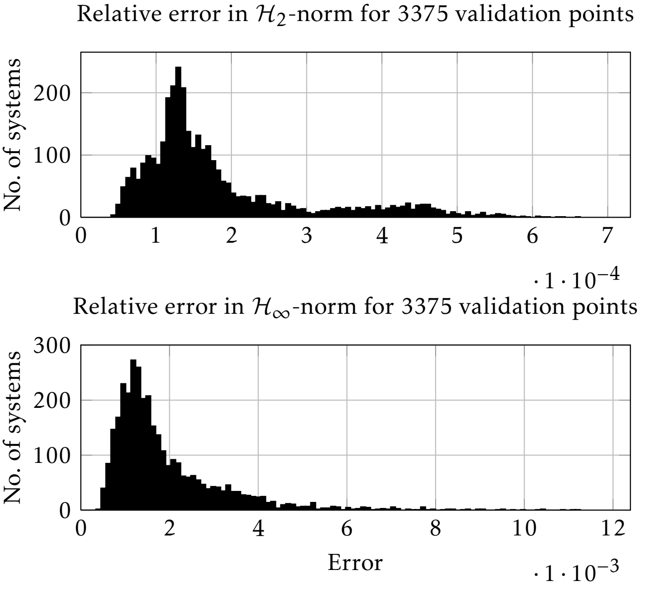

Figure 7.4: *A histogram over the relative error in $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norm at 3375 validation points, in the three parameter case when the coefficient matrices $\hat{\mathbf{A}}^{(1)}$,$\hat{\mathbf{A}}^{(2)}$ and $\hat{\mathbf{A}}^{(3)}$ have full rank.*

Table 7.1: *A table showing the amount of time it took to compute the different LPV models from Section 7.1.1 and the sizes of the corresponding LFRs. $\bar{n}_\Delta$ represents the size of the resulting LFR coming from the proposed methods and n_Δ represents the size of the resulting LFR from the original LPV model.*

LPV Model	$\bar{n}_\Delta$	n_Δ	Time
1 parameter, full rank	26	20	7m 56s
1 parameter, rank 2	6	20	8m 56s
2 parameters, full rank	52	62	1h 35m 31s
2 parameters, rank 11	32	62	50m 44s
3 parameters, full rank	94	98	1h 55m 53s

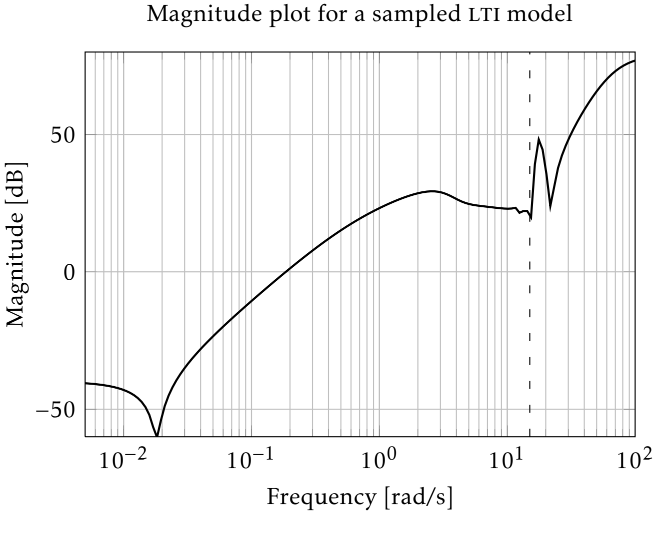

Figure 7.5: *A magnitude plot for a sampled LTI model from the one parameter LPV model. The dashed vertical line denotes $\omega = 15$ rad/s.*

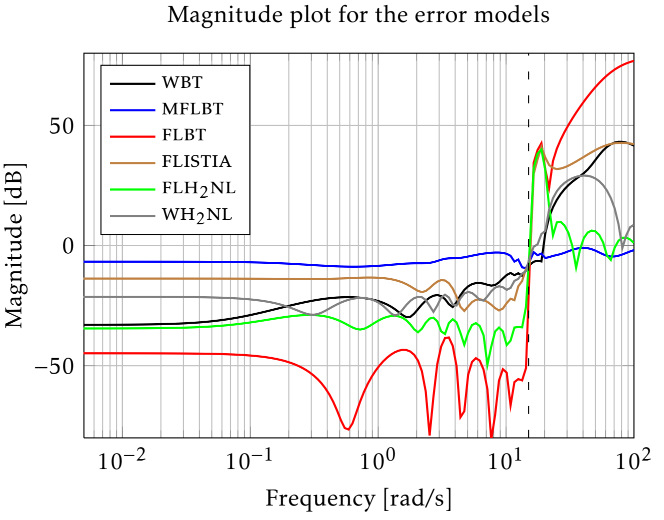

Figure 7.6: *The error models resulting from the different methods, from Section 7.1.2. The dashed vertical line denotes* $\omega = 15$ rad/s. *The red line (FLBT) seems to have found the best model. However, this model is unstable. The best model, in* $\mathcal{H}_2$*-norm, is then the green model, which is our proposed method from Section 4.4.3.*

at the nominal value $p = 0$. This model will be reduced using the methods described in Chapter 4 and will be compared with other model-reduction methods. The methods FLH$_2$NL (which is our proposed frequency-limited model-reduction method, see Section 4.4.3), FLISTIA, FLBT and MFLBT are compared. These methods are also compared with the methods WH$_2$NL (which is our proposed frequency-weighted model-reduction method, see Section 4.4.1) and WBT using a tenth order low-pass Butterworth filter with a cut-off frequency of 15 rad/s. The model is reduced from 22 states to 16 states.

The results from the different methods can be seen in Figure 7.6, showing the different error models, and Figure 7.7, showing the true and reduced models, and Table 7.2. In Figure 7.6 it seems that FLBT has found a good approximation. However, looking at Table 7.2 we see that the model from FLBT is unstable. All the other methods find models that are acceptable for the relevant frequency range and as in the examples in Section 4.6, FLH$_2$NL finds the model with the best $\mathcal{H}_2$ fit.

In this example we had a model that was only valid up till a certain frequency and looking at the result in Figure 7.6 and Figure 7.7 and Table 7.2, we see that the frequency-limited model-reduction methods sacrifices the model fit in the upper frequencies for the valid, lower, frequency regions. Hence, we see the importance of using methods that are able focus on the relevant region.

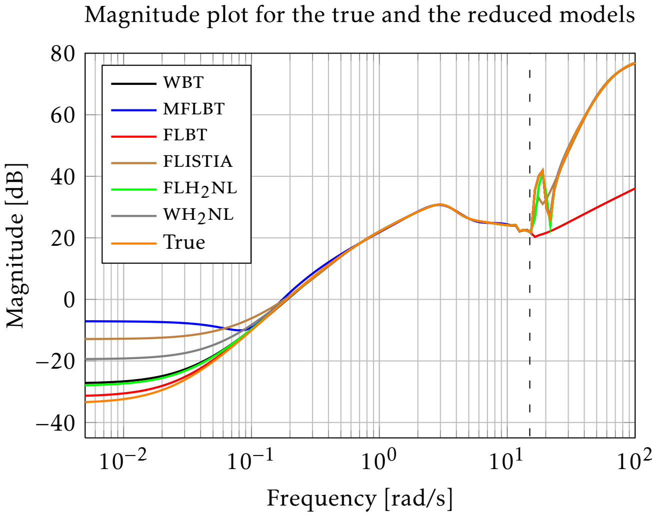

Figure 7.7: *The true and reduced-order models, for the different methods, from Section 7.1.2. The dashed vertical line denotes* $\omega = 15$ rad/s.

Table 7.2: *Numerical results for the example in Section 7.1.2.*

	$\frac{\|G-\hat{G}\|_{\mathcal{H}_2,\omega}}{\|G\|_{\mathcal{H}_2,\omega}}$	$\frac{\|G-\hat{G}\|_{\mathcal{H}_\infty,\omega}}{\|G\|_{\mathcal{H}_\infty,\omega}}$	Re $\lambda_{\max}$
WBT	9.90e-03	1.12e-02	-1.63e-01
MFLBT	2.90e-02	2.07e-02	-1.19e-01
FLBT	∞	3.87e-04	6.12e+00
ISTIA	7.79e-03	9.33e-03	-1.35e-01
FLH$_2$NL	1.68e-03	5.11e-03	-1.90e-01
WH$_2$NL	8.12e-03	1.37e-02	-1.82e-01

7.2 Model Reduction in System Identification

In this example we will show how model reduction can be used in system identification to obtain parameter estimates with a smaller covariance matrix than with direct system identification. The example that will be used is taken from Tjärnström [2003] where also the theoretical results are presented.

We will be work with a SISO discrete-time output-error (OE, see Ljung [1999]) model with $T_s = 1$. Let $y(t)$ denote the output of the system and $u(t)$ the input and N is the total number of measured data. The signal $y(t)$ is assumed to be generated from the true system, $G_0(q)$, as

$$y(t) = G_0(q)u(t) + e(t),$$

where q is the discrete-time shift operator and the additive noise, $e(t)$, is a zero-mean, white-noise sequence, independent of the input. The sought system is parametrized as an OE model and denoted $\hat{G}(q, \theta)$, where θ is a vector holding the parameters for the OE model. To identify a model using the input-output data the prediction-error method (PEM) (see Ljung [1999]) can be used. One cost function that is commonly used when doing system identification using PEM is

$$V_N(\theta) = \frac{1}{2N} \sum_{t=1}^{N} \epsilon^2(t, \theta),$$
$$\epsilon(t, \theta) = y(t) - \hat{G}(q, \theta)u(t),$$

and the estimate of θ given N data points, $\hat{\theta}_N$, is taken as

$$\hat{\theta}_N = \arg\min_{\theta} V_N(\theta).$$

Using the notation and definitions above we can state a connection between system identification, using PEM, and model reduction, using the $\mathcal{H}_2$-norm. Under weak conditions, it holds that

$$\hat{\theta}_N \to \theta^* = \arg\min_{\theta} \frac{1}{2}\bar{\mathrm{E}}\epsilon^2(t, \theta) \triangleq \bar{V}(\theta), \quad \text{as } N \to \infty,$$

where $\bar{E}f(t) \triangleq \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \mathrm{E}f(t)$ and using Parseval's formula and an OE model structure, we have that

$$\bar{V}(\theta) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \left|G(\mathrm{e}^{i\omega}) - \hat{G}(\mathrm{e}^{i\omega}, \theta)\right|^2 \Phi_u(\omega)\mathrm{d}\omega = \frac{1}{2}\left\|G - \hat{G}(\theta)\right\|^2_{\Phi_u, \mathcal{H}_2}.$$

Results in Tjärnström and Ljung [2002] and Tjärnström [2003] states, when estimating an OE model of low order (undermodeling), it is better to estimate the low-order model with model reduction of a high-order model compared to estimating the low-order model directly from data. This was exemplified already in Tjärnström [2003]. However, not by using a $\mathcal{H}_2$ model-reduction algorithm but by using a first-order approximation of the covariance expression for the parameters, see Tjärnström [2003]. First in this example we will use the method proposed in Section 4.4.1 to do the model reduction when having a white-noise input. Secondly, we will use an input signal with a frequency-limited spectrum that requires the use of the method proposed in Section 4.4.3.

In this example the true system is given by

$$y(t) = \frac{B(q)}{F(q)}u(t) + e(t),$$

where

$$B(q) = 2q^{-1} - q^{-2}$$
$$F(q) = 1 - 0.7q^{-1} + 0.52q^{-2} - 0.092q^{-3} - 0.1904q^{-4}.$$

The input, u, and noise, e, are jointly independent. The noise is a zero-mean white-noise process with variance 1.

First we will use a zero-mean white-noise process with variance 1 for the input. The system is simulated with this input with $N = 250$ to obtain a data set with input and output data. This data set is used first to directly estimate, using PEM, three low-order OE models with orders $\{n_b = 1, n_f = 1, n_k = 1\}$, $\{n_b = 2, n_f = 2, n_k = 1\}$ and $\{n_b = 3, n_f = 3, n_k = 1\}$ respectively. Now, using the same data set an OE model with order $\{n_b = 4, n_f = 4, n_k = 1\}$ is estimated using PEM and this estimated model of order $\{n_b = 4, n_f = 4, n_k = 1\}$ are reduced, using $\mathrm{H_2NL}$, to three OE models with orders $\{n_b = 1, n_f = 1, n_k = 1\}$, $\{n_b = 2, n_f = 2, n_k = 1\}$ and $\{n_b = 3, n_f = 3, n_k = 1\}$ respectively. This procedure is repeated 500 times and from the obtained estimates, Monte Carlo based estimates of the covariance matrices are computed. From each of the six covariance matrices, as in Tjärnström [2003], the eigenvalues are determined to represent the size of the covariance matrices. The results are presented in Table 7.3.

Table 7.3: *Numerical results for the example in Section 7.2 using a zero-mean white-noise process with variance 1 for the input. The cases marked "direct" means that the model comes from directly using* PEM *and "reduced" means that first a fourth order model is identified using* PEM *and then this model is then reduced using model reduction to the desired order.*

Model – Method	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
OE $(1, 1, 1)$ – direct	0.930	0.0859	–	–	–	–
OE $(1, 1, 1)$ – reduced	0.924	0.0671	–	–	–	–
OE $(2, 2, 1)$ – direct	1.87	0.916	0.0919	0.0440	–	–
OE $(2, 2, 1)$ – reduced	1.81	0.910	0.0871	0.0431	–	–
OE $(3, 3, 1)$ – direct	233	3.57	0.915	0.355	0.0413	0.0276
OE $(3, 3, 1)$ – reduced	179	2.11	0.952	0.305	0.0407	0.0265

In a second experiment we use an input with a limited spectrum. The input in this case is a zero-mean gaussian signal with a non-zero spectrum on the frequency interval $[0, \pi/2]$ and with variance 1. The same procedure as above is used to estimate six different OE models using the direct and reduced approach. The difference compared to the case above is that the proposed method from Section 4.4.3 is used instead. From each of the six covariance matrices, as in Tjärnström [2003], the eigenvalues are determined to represent the size of the covariance matrices. The results are presented in Table 7.4

This example repeats the results from Tjärnström [2003], that $\mathcal{H}_2$ model reduction can in some cases be used to find better estimates in system identification, by finding smaller covariance matrices, see Table 7.3 and Table 7.4. However, this time using an $\mathcal{H}_2$ model-reduction algorithm, both for the case of having white-noise input and an input with limited spectrum. This example is meant to highlight the connection between system identification and $\mathcal{H}_2$ model reduction, and illustrate yet another application of our results.

Table 7.4: *Numerical results for the example in Section 7.2 using a zero-mean gaussian process with a limited spectrum with variance 1 for the input. With "direct" means that the model comes from directly using* PEM *and with reduced means that first a fourth order model is identified using* PEM *and the this model is reduced, using model reduction to the correct order.*

Model – Method	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
OE $(1,1,1)$ – direct	43.2	0.411	–	–	–	–
OE $(1,1,1)$ – reduced	40.6	0.400	–	–	–	–
OE $(2,2,1)$ – direct	1290	80.9	10.1	0.214	–	–
OE $(2,2,1)$ – reduced	1210	65.9	8.18	0.246	–	–
OE $(3,3,1)$ – direct	3590	595	466	128	3.63	0.170
OE $(3,3,1)$ – reduced	1940	530	488	99.6	3.51	0.180

7.3 Conclusions

The two examples in this chapter have been chosen to highlight some properties and applications for the model reduction and LPV algorithms and to show their applicability on a real-world example. In the aircraft example in Section 7.1 we could see how the LPV generating algorithms could be used to lower the complexity of an existing LPV model and how the limited-frequency model-reduction algorithm can be used to capture relevant frequency regions when performing model reduction. In the system identification example in Section 7.2 we highlight the connection between system identification and $\mathcal{H}_2$ model reduction using an example that shows how the covariance matrix of the estimates can be made smaller using model reduction together with system identification.

8 Concluding Remarks

The previous chapters have introduced, and shown the applicability of, some new methods for reducing the complexity of LTI and LPV systems and for synthesizing $\mathcal{H}_2$ controllers. All methods are based on the same technique, which is minimizing the $\mathcal{H}_2$-norm of different systems, and utilizing the structure of the problems to make the methods more efficient. The methods have been developed such that an off-the-shelf quasi-Newton solver can be used to solve the problems using the equations derived in the thesis.

In Section 4.4.1 a method for model reduction, for which the basic idea is not new, was presented. However, we presented how to utilize the structure of the problem and also laid the foundation for the other methods that were presented.

In Section 4.4.2 a model-reduction method that tries to cope with errors in the given data was presented. The method uses the foundation laid in Section 4.4.1 together with a different view of robust optimization, namely using regularization as a proxy for robust optimization.

In Chapter 3 a more complete and uniform derivation, than in the existing literature, of frequency-limited Gramians were presented. In Section 4.4.3 a frequency-limited model-reduction method was presented. This method was based on the derivations in Chapter 3 together with the foundation laid in Section 4.4.1.

All the model-reduction methods in Chapter 4 were then extended into an LPV framework to be able to handle LPV systems and to be able to reduce the complexity both in the states and the parameters for the LPV systems. Many of the existing LPV generating methods have one drawback in common, which is that they are not invariant to the state basis the LTI models are given in. This drawback makes it hard for the existing models to be able to reduce the complexity of the LPV model. However, by using a model-reduction method as the foundation to the LPV generating methods in Chapter 5 this drawback is eliminated.

The model-reduction problem is closely related to the controller-synthesis problem and using the same techniques as in Chapter 4 and Chapter 5, $\mathcal{H}_2$ controller-synthesis methods were developed in Chapter 6. As discussed in Chapter 6, a possible extension of the methods for synthesizing controllers could be to use the idea of controlling the rank of the system matrices, as in Section 5.4.2. By using this idea, of controlling the rank, one could, for example, enforce the controller to have integrators.

The presented methods have been shown to work well on the presented examples, which are both small academic examples and relevant real-world examples, for example a model of an Airbus aircraft.

All the methods described in this thesis tries to solve non-convex optimization problems, which are difficult problems and only local solutions can be guaranteed. Hence, the initialization problem is a very important part of the methods presented in this thesis. We have presented some suggestions for initializing the methods and, in our examples, they have worked well. However, this is a part of the problem that is in need of further research and much can be gained by making even better initializations, e.g., faster and more reliable computations, since we can hopefully start even closer to an optimum.

Another problem that is in need of further research is the problem of finding a stabilizing controller, which is a problem that has not been discussed much in this thesis. The problem, of finding stabilizing controllers, is crucial to be able to use the methods in Chapter 6, and in this thesis only one simple suggestion that relies on existing methods is presented.

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