Real and Complex Operator Norms

Natalia Sabourova

Luleå University of Technology
Department of Mathematics

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by

Natalia Sabourova

Department of Mathematics
Luleå University of Technology
SE-97187 Luleå
Sweden

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Abstract

Any bounded linear operator between real (quasi-)Banach spaces $T : X \to Y$ has a natural bounded complex linear extension $T_C : X_C \to Y_C$ defined by the formula $T_C(x + iy) = Tx + iTy$ for $x, y \in X$, where $X_C$ and $Y_C$ are so called reasonable complexifications of $X$ and $Y$, respectively. We are interested in the exact relation between the norms of the operators $T_C$ and $T$. This relation can be expressed in terms of the constant $\gamma_{X,Y}$ appearing in the inequality

$$\|T_C\| \leq \gamma_{X,Y}\|T\|$$

considered for all bounded linear operators $T : X \to Y$ between (quasi-)Banach spaces. The work on the constant $\gamma_{L^p,L^q}$ for $0 < p, q \leq \infty$, or shortly $\gamma_{p,q}$, is traced back to M. Riesz, Thorin, Marcinkiewicz, Zygmund, Verbički, Krivine, Gasch, Maligranda, Defant and others. In this thesis we try to summarize the results of these authors. We also present some new estimates for $\gamma_{p,q}$ in the case when at least one of the spaces is quasi-Banach as well as in the case when the spaces are supplied with discrete measures. For example, we get that $\gamma_{p,q} \leq 2$ for all $0 < p, q \leq \infty$. Furthermore we obtain some new results concerning the relation between complex and real norms of the operators between spaces of functions of bounded $p$-variation and between mixed norm Lebesgue spaces. Looking for the criteria of the equality of real and complex norms of operators from a Banach lattice into the same Banach lattice we find a number of examples of two dimensional Orlicz spaces different from Lebesgue spaces and a simple operator between them with non-equal real and complex norms. We also consider in detail the Clarkson inequality which can be interpreted in terms of a certain operator norm inequality appearing as an example in many parts of the thesis. It turns out that complex norm of this operator can be easily obtained but to find the real one is not so trivial. With the help of the Clarkson inequality we construct an operator between Lebesgue spaces with non-atomic measures which has different real and complex norms. Finally, we consider both complex and real versions of the Riesz-Thorin interpolation theorem in the first quadrant and by using numbers $\gamma_{p,q}$ find, for example, that the real Riesz constant is bounded by 2 for all $0 < p, q \leq \infty$. 
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CHAPTER
ONE

Introduction

Any bounded linear operator between real (quasi-)Banach spaces \( T : X \to Y \) has a natural complex extension \( T_C : X_C \to Y_C \) defined by the formula

\[
T_C(x + iy) = Tx + iTy \quad \text{for} \quad x, y \in X.
\]

Now we just say that the complex (quasi-)Banach spaces \( X_C \) and \( Y_C \) are obtained from \( X \) and \( Y \) by some reasonable complexification procedure that we will explain in the next chapter. We call the norms of the operators \( T \) and \( T_C \) as real norm and complex norm, respectively. In this thesis we study the relation between complex and real norms of arbitrary bounded linear operator \( T \) between two real (quasi-)Banach spaces or; more precisely, we study the constant \( \gamma_{X,Y} \) in the inequality

\[
\|T_C\| \leq \gamma_{X,Y} \|T\|.
\]

The main reason for us to consider this relation is that it allows to interpret the results obtained for complex operator norms in the real settings as it is, for example, in the Riesz-Thorin interpolation theorem. Despite it is enough to have some good estimate of the constant \( \gamma_{X,Y} \) for doing such interpretation we are also interested in the techniques of finding the best constant \( \gamma_{X,Y} \) and put special emphasis on the case when \( \gamma_{X,Y} = 1 \), i.e. the case of equality of complex and real operator norms.

Probably the first interest to the question about the relation between complex and real operator norms appeared in the paper of Marcel Riesz \[50\] on convexity and bilinear forms from 1927 but published in 1926! In this paper Riesz proved his well-known convexity theorem for bilinear forms by "real-variable" techniques which can also be stated in the following way: the norm of the operator \( T : l^p_n \to l^q_k \) as a function of \((1/p, 1/q)\) is logarithmically convex in the "lower triangle" \( \{(\alpha, \beta) \in \mathbb{R}^2 : 0 \leq \beta \leq \alpha \leq 1\} \). Besides, Riesz raised a question on the validity of this theorem for complex Lebesgue spaces in the entire unit square, i.e. in \( \{(\alpha, \beta) \in \mathbb{R}^2 : 0 \leq \alpha, \beta \leq 1\} \). As a starting point for the following study of this question he provided an example of an operator between three-dimensional Lebesgue spaces with different real and complex norms. Moreover, he briefly presented arguments in support of the statement about the equality of real and complex norms of operators between finite-dimensional \( l^p \) spaces. The detailed proof of this statement can be found in the work of Taylor \[56\] (see also Crouzeix \[12\]). In 1939 Thorin \[57\], the student of Riesz, gave an affirmative answer to Riesz’s question publishing an elegant proof using an elementary convexity property of analytic functions, the so called "three lines lemma". In the thesis on convexity theorems \[58\] he pointed out that Riesz theorem
for real bilinear forms in the "lower triangle" can be obtained as a consequence of the proved by him complex case. Thorin also worked out an example of Riesz showing the impossibility of extending the real convexity theorem to a convex region beyond and containing the "lower triangle" even in the case of two-dimensional $L^p$ spaces. Thus, the importance of the relation between complex and real operator norms was recognized. Marcinkiewicz and Zygmund [44] used Gaussian variables to obtain vector-valued estimates of the operators between (quasi-)Banach $L^p - L^p$ spaces and some years later Zygmund included a simple proof of the equality of complex and real norms for the operators between $L^p[a, b]$ spaces in his book Trigonometric Series [65]. Verbičkiǐ and Sereda in [59] profitably employed the simple idea of Zygmund and established the equality of complex and real norms of the operators between Banach $L^p(\mu) - L^q(\nu)$ spaces with arbitrary $\sigma$-finite measures $\mu$ and $\nu$ and $p \leq q$. At the same time Verbičkiǐ in [60] began to consider constant $\gamma_{X,Y}$ for arbitrary Banach spaces $X$ and $Y$ paying special attention to the possible ways of complexification of real normed spaces. He improved the general estimate of the relation between complex and real norms for the operators between Lebesgue spaces, i.e. $\gamma_{p,q}$, and found the exact value of $\gamma_{X,Y}$ in the case of operators between spaces of trigonometric polynomials supplied with $L^p$ norms. Then the highly non-trivial works of Krivine [33]-[34] evolved. Using the advantage of tensor product technique Krivine found the exact value of $\gamma_{\infty,1}$, which is turned out to be the biggest among all $\gamma_{p,q}$ in the case $1 \leq p, q \leq \infty$. Utilizing further the tensor product technique Figiel, Iwaniec and Pełczyński [17] established the equality of complex and real norms for the vector-valued operators from $L^p$ into $L^q$ space with $1 \leq p \leq q$ as well as Defant extended the range of the values $p$ and $q$ for which the exact constant $\gamma_{p,q}$ was found. Some interest to this problem continues to appear in literature up to now. For example, Gasch, Maligranda [19] and Luna-Elizarrarás, Shapiro [36] continued the study of the relation between complex and real norms for vector-valued operators. Kirwan [31] and Muñoz, Sarantopoulos and Tonge [46] studied various procedures of complexification of real normed spaces and obtained the estimates for the complex norms of multilinear mappings and polynomials. Maligranda and Sabourova [43] closely considered real and complex versions of the classical Clarkson inequality in the content of the question about the relation between real and complex operator norms. They found an example of an operator between Lebesgue spaces with non-atomic measures with different real and complex norms. And finally we will mention the work of Holtz and Karow [23], where some of these questions appeared again.

It is not surprising that the most complete answer to the question about the relation between complex and real operator norms exists only in the case of $L^p$ spaces and for the spaces which are close to $L^p$ although there are still some open questions even in this case. So we describe the problem in the framework of $L^p$ spaces and then formulate it in the more general settings.

In this thesis we use standard notations. For $0 < p < \infty$ and arbitrary measure space $(\Omega, \Sigma, \mu)$ with a positive $\sigma$-finite measure $\mu$ we denote by $L^p_c(\Omega, \mu)$ (and even more shortly $L^p_c(\mu)$) the Banach (quasi-Banach if $0 < p < 1$) space of complex-valued $p$-integrable classes of functions on $\Omega$ with the norm:

$$\|f + ig\|_p = \left( \int_{\Omega} |f(x) + ig(x)|^p d\mu(x) \right)^{1/p}.$$
If $p = \infty$ we have a Banach space of complex-valued essentially bounded functions with the norm:

$$\|f + ig\|_{\infty} = \text{ess sup}\{|f(x) + ig(x)| : x \in \Omega\},$$

where

$$\text{ess sup}\{h(x) : x \in \Omega\} = \inf\{M > 0 : \mu\{x \in \Omega : h(x) > M\} = 0\}.$$  

When the underscript $C$ is omitted the functions are assumed to be real-valued on $\Omega$. Furthermore, if $X$ and $Y$ are arbitrary (quasi-)Banach spaces, then $\mathcal{L}(X,Y)$ denotes the collection of all bounded linear operators from $X$ into $Y$ endowed with the supremum (quasi-)norm and $\mathcal{L}(X)$ replaces $\mathcal{L}(X,X)$. In this thesis we work only with this kind of operators. For $1 < p < \infty$, let $p'$ denote its conjugate exponent in the sense that $1/p + 1/p' = 1$. In the case when $p = 1$ let define $p' = \infty$ and vice versa.

Any bounded linear operator $T \in \mathcal{L}(L^p(\mu), L^q(\nu))$ between real Lebesgue spaces has a natural complex linear extension $T_C \in \mathcal{L}(L^p_C(\mu), L^q_C(\nu))$ defined as follows

$$(1.1) \quad T_C(f + ig) = Tf + iTg,$$

where $f, g \in L^p(\mu)$ and the norm of the complexification operator $T_C$ is given by:

$$\|T_C\|_{p,q} = \sup_{\|f + ig\|_p \neq 0} \frac{\|T_C(f + ig)\|_q}{\|f + ig\|_p}.$$  

When no confusion can arise we simply write $\|T\|$ instead of $\|T\|_{p,q}$. For any bounded linear operator $T$ between normed spaces with so called reasonable complexifications we have a general estimate of the norms $\|T\| \leq \|T_C\| \leq 2\|T\|$ (see Chapter 2, Proposition 3). In particular, for any operator $T \in \mathcal{L}(L^p(\mu), L^q(\nu))$ with $1 \leq p, q \leq \infty$ we obtain

$$\|T_C(f + ig)\|_q = \|T f + iTg\|_q \leq \|T f\|_q + \|T g\|_q \leq \|T\|_{p,q} \|f\|_p + \|T\|_{p,q} \|g\|_p$$

$$\leq \|T\|_{p,q} (\|f^2 + |g^2|\|_2^{1/2})_p + \|T\|_{p,q} (\|f^2 + |g^2|\|_2^{1/2})_p$$

$$\leq 2 \|T\|_{p,q} (\|f^2 + |g^2|\|_2^{1/2})_p.$$

This relation establishes boundedness of the operator $T_C$. Hence, the natural complexification $T_C$ of any operator $T \in \mathcal{L}(L^p(\mu), L^q(\nu))$ defined by (1.1) is a complex bounded linear operator with the norm $\|T\|_{p,q} \leq \|T_C\|_{p,q} \leq 2\|T\|_{p,q}$ and it makes sense to consider the ratio $\|T_C\|_{p,q}/\|T\|_{p,q}$. Thus, we are interested in the numbers:

$$\gamma_{p,q} = \sup\{\inf \gamma \geq 1 : \|T_C\|_{p,q} \leq \gamma \|T\|_{p,q} \text{ for any } T \in \mathcal{L}(L^p(\mu), L^q(\nu))\} :$$

$$\text{for arbitrary positive } \sigma = \text{finite measures } \mu, \nu.$$  

In other words, for any $T \in \mathcal{L}(L^p(\mu), L^q(\nu))$ and $f, g \in L^p(\mu)$

$$\|T_C(f + ig)\|_q = \|T f + iTg\|_q = \|(|f|^2 + |g|^2)^{1/2}\|_q$$

$$\leq \|T_C\|_{p,q} \|f + ig\|_p \leq \gamma_{p,q} \|T\|_{p,q} (|f|^2 + |g|^2)^{1/2}\|_p,$$

means that $\gamma_{p,q}$ is the best constant in the inequality

$$(1.2) \quad \|(|f|^2 + |g|^2)^{1/2}\|_q \leq \gamma_{p,q} \|T\|_{p,q} (|f|^2 + |g|^2)^{1/2}\|_p$$

for all $f, g \in L^p(\mu)$ and all $T \in \mathcal{L}(L^p(\mu), L^q(\nu))$.

The above estimates show only that $1 \leq \gamma_{p,q} \leq 2$. We note however, that linearizing the expression under the norm sign on the left hand side of inequality
(1.2) in terms of a certain integral and using the possibility to exchange the order of taking the operator norm and integration, we could extract the norm of the real operator $T$ and following the same procedure we could get back the expression on the right-hand side. As a result of this procedure we could find a better estimate of $\gamma_{p,q}$ from above. Calculation of the exact value of $\gamma_{p,q}$ for all $0 < p, q \leq \infty$ seems to be a complicated problem. For this purpose we also should have a lower estimate of $\gamma_{p,q}$ for that special operators are needed.

We should mention here that the numbers $\gamma_{p,q}$ are just a special case of the constants appearing in so called vector-valued inequalities (cf. Marcinkiewicz-Zygmund [44], Gasch-Maligranda [19]): for $1 \leq p, q, r \leq \infty$ and $n = 2, 3, ...$

$$K^{(n)}_{p,q}(r) = \sup\{K_{L^p(\mu),L^q(\nu)}(r) : \text{for any positive } \sigma - \text{finite measures } \mu, \nu\},$$

where

$$K^{(r)}_{L^p(\mu),L^q(\nu)}(n) = \inf\{C \geq 1 : \|\sum_{k=1}^{n} |Tf_k|^r\|_q \leq C\|T\|\|\sum_{k=1}^{n} |f_k|^r\|_p, \text{ for all } f_1, f_2, ..., f_n \in L^p(\mu) \text{ and } T \in \mathcal{L}(L^p(\mu), L^q(\nu))\}.$$ 

In other words, constants $K^{(r)}_{p,q}(n)$ establish the biggest possible relation between the norm of a natural vector-valued operator extension $T_E : L^p(\mu, l^r_n) \rightarrow L^q(\nu, l^r_n)$ of the operator $T$ defined by $T_E(f_1, ..., f_n) = (Tf_1, ..., Tf_n)$ and the norm of the operator $T$. Thus, with the preceding notations $\gamma_{p,q} = K^{(2)}_{p,q}(2)$ and in this thesis we will focus on these constants connected to $L^p$ spaces.

The discussion around the complexification of Lebesgue spaces and operators between them can be completely adapted to arbitrary Banach lattices $X$ and $Y$ and as a consequence the same kind of inequality as (1.2) will define the constant $\gamma_{X,Y}$ (for explanation see Chapter 2). For arbitrary (quasi-)Banach spaces $X$ and $Y$ the constant $\gamma_{X,Y}$ is no longer necessary connected to inequality (1.2) as it is, for example, in the case of operators between spaces of functions of bounded $p$-variation. However, even in this case we will systematically use, when it is possible, the same tool, namely the linearization of the expression $(a^2 + b^2)^{1/2}$ in terms of a certain integral and the possibility to exchange the order of taking the operator norm and integration.

This thesis is organized as follows. In Chapter 2 we consider the question about "natural" complexifications of real (quasi-)normed spaces and define the notion of reasonable norm which will play a crucial role in getting that the relation between complex and real norms of operators between arbitrary Banach spaces is bounded by 2.

Chapter 3 is about the equality of real and complex operator norms. We consider the operators between Lebesgue spaces, mixed norm Lebesgue spaces, some other Banach lattices, spaces of functions of bounded $p$-variation. Looking for the criteria of the equality of real and complex norms of the operators between Banach lattices we found a connection of this question with the notions of $p$-convexity and $q$-concavity of these spaces that we present here. We also show some examples of simple operators from two-dimensional Orlicz space different from $L^p$ space into the same Orlicz space with non-equal real and complex norms.

In Chapter 4 we summarize the results obtained for the constant $\gamma_{X,Y}$, where $X$ and $Y$ are Lebesgue spaces, mixed norm Lebesgue spaces, spaces of functions of
bounded $p$-variation. The most interesting here is that we establish the estimate of the constant $\gamma_{X,Y}$ by 2 even for the quasi-Banach variants of these spaces.

Chapter 5 deals with real and complex versions of the so-called generalized Clarkson inequality which can be interpreted in terms of the operator norm inequality. This operator appears as an example in many parts of this thesis and sometimes we even need to know the exact values of its real and complex norms. We also present here some other applications of the Clarkson inequality, including a construction of an operator but now between Lebesgue spaces with non-atomic measures for which real and complex norms are different. We note, that this chapter was initially written as an article "Clarkson Inequality in the Real Case" which will appear in the journal Mathematische Nachrichten.

In Chapter 6 we summarize the basic properties of the constant $\gamma(l_p^n, l_q^m)$ which is just a particular case of the more general constant $\gamma_{p,q}$. The main reason why this chapter appears is that we wanted to know if the constant $\gamma_{p,q}$ is finite when $q$ tends to zero and tried to answer the question for the more simple constant $\gamma(l_p^n, l_q^n)$. However, the affirmative answer was obtained directly for $\gamma_{p,q}$. In spite of this fact we found it interesting to improve for some values of $p, q, n, m$ the estimates of $\gamma(l_p^n, l_q^n)$ obtained in Chapter 4 for $\gamma_{p,q}$.

Finally, Chapter 7 treats how with the knowledge of the constant $\gamma_{p,q}$ to get the real Riesz-Thorin interpolation theorem from its complex analogue. We consider this theorem in the whole first quadrant and what is new we show that even in the case when at least one of the spaces is quasi-Banach the so-called Riesz constant is not bigger than 2.
Complexification of Real Spaces and Operators

Complexification of a real normed space is taking place on two different levels. On the algebraic level we complexify the elements of the real linear space obtaining a complex linear space. Then, on the geometric level, we introduce on this complex linear space a new norm. To get a complex normed space which inherits some of the properties of the original space this new norm should be reasonable, the notion which we explain in this chapter. It turns also out that if complex normed spaces are supplied with such reasonable norms than the relation between complex and real norms of bounded linear operators between them does not exceed 2 and if the norms are not reasonable this estimate may not necessarily holds. We note, although we do not write it explicitly, that the complexification of real quasi-normed spaces follows the same procedure as we describe in this chapter for real normed spaces and the notion of reasonable quasi-norm is defined in exactly the same way as reasonable norm.

2.1. Complexification of real linear spaces and operators

Let $V$ be a real linear space. The natural complexification $V_C$ of $V$ can be constructed in the way similar to usual construction of the complex numbers from the reals. Thus, $V_C$ can be defined as the direct sum

$$V_C = V \oplus iV = \{ x + iy : x, y \in V \}$$

for which the complex linear structure given by the relations:

\[(i) \quad (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2), \quad x_1, x_2, y_1, y_2 \in V, \]
\[(ii) \quad (a + ib)(x + iy) = (ax - by) + i(bx + ay), \quad x, y \in V \text{ and } a, b \in \mathbb{R}. \]

Obviously, $V_C$ is a complex span of $V$. A briefly review of other possible alternative descriptions of complexification of a real linear space can be found in the paper of Muñoz-Sarantopoulos-Tonge [46].

Complexifying real linear spaces we extend linear bounded operators between them to the complex linear operators between complexification spaces. Namely, if $T : X \rightarrow Y$ is a linear operator between real linear spaces $X$ and $Y$, then its complex linear extension is naturally defined by

$$T_C(x + iy) = Tx + iTy,$$
where \(x, y \in X\). To check linearity of \(T_C\) we need to establish the additivity and homogeneity of \(T_C\). Let \(x, y, u, v \in X\) and \(a, b \in \mathbb{R}\), then
\[
T_C(x + iy + u + iv) = T(x + u) + iT(y + v)
\]
\[
= Tx + iTy + Tu + iTv = T_C(x + iy) + T_C(u + iv)
\]
\[
T_C((a + ib)(x + iy)) = T(ax - by + i(ay + bx))
\]
\[
= T(ax - by) + iT(ay + bx) = aTx - bTy + iaTy + ibTx
\]
\[
= (a + ib)(Tx + iTy) = (a + ib)T_C(x + iy).
\]

It is worth to remark that the equality \(T_C(x + i0) = Tx\) allows us to identify \(\mathcal{L}(X, Y)\) with a real subspace of \(\mathcal{L}(X_C, Y_C)\).

### 2.2. Complexification of real normed spaces

Complex extensions of real normed spaces require new norms. It is quite natural to look for the norms which replicate some basic properties of complex numbers.

**Definition 1.** Let \((X, \| \cdot \|_X)\) be a real Banach space. A norm \(\| \cdot \|_{X_C}\) on the complexification \(X_C\) is said to be reasonable if

(i) \(\|x\|_{X_C} = \|x\|_X\) for any \(x \in X\) (complexification norm)

(ii) \(\|e^{i\theta}z\|_{X_C} = \|z\|_{X_C}\) for \(z \in X_C\) and \(\theta \in \mathbb{R}\) (symmetry of the unit ball)

(iii) \(\|\overline{z}\|_{X_C} = \|z\|_{X_C}\) for any \(z \in X_C\) (conjugation-invariant norm)

Property (i) defines so called complexification norm and simply means that a reasonable norm \(\| \cdot \|_{X_C}\) extends the original real norm \(\| \cdot \|_X\). Property (ii) says that a reasonable norm possesses symmetry of the unit ball. Property (iii) describes a conjugation-invariant property. These properties imply, for example, that the constructed complexified space inherits separability and completeness of the original space. Moreover, due these properties all reasonable norms on the complexified space are equivalent and, what is the most important in our work, that the relation between complex and real norms of bounded linear operators between them does not exceed 2 (cf. Holtz-Karow [23], Kirwan [31]). The proofs of all these statements are based on the following easily obtained consequence of property (iii). Namely, property (iii) implies that for all \(z \in X_C\) it follows \(\|\text{Re } z\|_X, \|\text{Im } z\|_X \leq \|z\|_{X_C}\). Indeed, \(2\|\text{Re } z\|_X = \|\bar{z} + z\|_{X_C} \leq 2\|z\|_{X_C}\) and since \(\text{Im } z = \text{Re } (iz)\), then also \(\|\text{Im } z\|_X \leq \|z\|_{X_C}\). Conversely, if property (iii) does not holds we can construct a complexification of a real space for which \(\|\text{Re } z\|_X, \|\text{Im } z\|_X > \|z\|_{X_C}\). as it is shown in the following example.

**Example 1.** Let \(p \geq 1\) and \(0 < \alpha < (1/2)^p\). Consider a two-dimensional space \(X = \mathbb{C}^2\) equipped with the norm
\[
\|z\|_{X_C} = (\alpha|z_1 - i z_2|^p + (1 - \alpha)|z_1 + i z_2|^p)^{1/p}\quad \text{for all } z = (z_1, z_2) \in \mathbb{C}^2.
\]

It is easy to check that \(\| \cdot \|_{X_C}\) defines a complexification of usual norm on \(\ell_2^2\) and thus satisfies property (i) of Definition 1. Clearly, \(\| \cdot \|_{X_C}\) also satisfies property (ii). Now, if we take \(z = (1, i)\), then \(\|\text{Re } z\|_X = \|\text{Im } z\|_X = 1\) and \(\|z\|_{X_C} = 2\alpha^{1/p} < 1\) for \(0 < \alpha < (1/2)^p\). Hence, \(\|\text{Re } z\|_X, \|\text{Im } z\|_X < \|z\|_{X_C}\). Moreover, this example also shows that property (iii) is not a simple consequence of the first two. Taking the same \(z\) we get \(\|z\|_{X_C} = 2\alpha^{1/p}\) and \(\|\overline{z}\|_{X_C} = 2(1 - \alpha)^{1/p}\). Therefore \(\|z\|_{X_C} \neq \|\overline{z}\|_{X_C}\) since otherwise \(\alpha\) should be equal to 1/2 which is not the case in our settings.
Proposition 1 (cf. Muñoz-Sarantopoulos-Tonge [46]). Let $X_C$ be a reasonable complexification of a real Banach space $X$. For any $x, y \in X$ we have
\[
\sup_{0 \leq \theta \leq 2\pi} \|x \cos \theta - y \sin \theta\|_X \leq \|x + iy\|_{X_C}
\]
and
\[
\|x + iy\|_{X_C} \leq \inf_{0 \leq \theta \leq 2\pi} (\|x \cos \theta - y \sin \theta\|_X + \|x \sin \theta + y \cos \theta\|_X).
\]

Proof. Property (ii) of Definition 1 implies that for any $\theta \in \mathbb{R}$ we have
\[
\|e^{i\theta}(x + iy)\|_{X_C} = \|x + iy\|_{X_C}.
\]
Then, using property (iii) we conclude
\[
\|\Re e^{i\theta}(x + iy)\|_X = \|\Re(x \cos \theta - y \sin \theta + i(x \sin \theta + y \cos \theta))\|_X
= \|x \cos \theta - y \sin \theta\|_X \leq \|e^{i\theta}(x + iy)\|_{X_C} = \|x + iy\|_{X_C}.
\]
On the other hand,
\[
\|x + iy\|_{X_C} = \|e^{i\theta}(x + iy)\|_{X_C} = \|x \cos \theta - y \sin \theta + i(x \sin \theta + y \cos \theta)\|_{X_C}
\leq \|x \cos \theta - y \sin \theta\|_X + \|x \sin \theta + y \cos \theta\|_X.
\]
Thus, for all $x, y, \theta \in \mathbb{R}$ we have
\[
(2.1) \quad \|x \cos \theta - y \sin \theta\|_X \leq \|x + iy\|_{X_C} \leq \|x \cos \theta - y \sin \theta\|_X + \|x \sin \theta + y \cos \theta\|_X.
\]
Now, taking the supremum on both sides of the left inequality and infimum on both sides of the right inequality in (2.1) over $0 \leq \theta \leq 2\pi$ we obtain the statement of the proposition. \(\blacksquare\)

Proposition 1 also says that among all reasonable norms the smallest one is given by the formula
\[
\|x + iy\|_T = \sup_{0 \leq \theta \leq 2\pi} \|x \cos \theta - y \sin \theta\|_X.
\]
The notation $T$ stands for Taylor norm who first introduced this kind of reasonable norm in [45]. Another way to describe the Taylor norm is by
\[
\|x + iy\|_T = \sup_{0 \leq \theta \leq 2\pi} \|x \cos \theta - y \sin \theta\|_X
= \sup_{0 \leq \theta \leq 2\pi} \sup_{\|\varphi\|_{X^*} \leq 1} |\varphi(x) \cos \theta - \varphi(y) \sin \theta|
= \sup_{\|\varphi\|_{X^*} \leq 1} \sqrt{\varphi(x)^2 + \varphi(y)^2}.
\]
These two descriptions appeared after Taylor’s unsuccessful attempt to define the so called “natural” complexification norm in [55] by
\[
\|x + iy\|_{nat} = \sqrt{\|x\|^2 + \|y\|^2}.
\]
The problem with this norm is that, in general, it fails to have property (ii) of Definition 1 and hence it is not reasonable. It becomes reasonable only in the
settings of Hilbert spaces. Indeed, if the norm on $(X_{\mathbb{C}}, \|\cdot\|_{\text{nat}})$ defined by (2.2) is reasonable, then

\[
2 \|x + iy\|^2_{\text{nat}} = |1 + i|^2 \|x + iy\|^2_{\text{nat}} = \|(1 + i)(x + iy)\|^2_{\text{nat}} \\
= \|(x - y) + i(x + y)\|^2_{\text{nat}} \\
= \|x - y\|^2_X + \|x + y\|^2_X = 2(\|x\|^2_X + \|y\|^2_X).
\]

Hence, $(X, \|\cdot\|_{\mathbb{C}})$ satisfies the parallelogram law and hence is a Hilbert space. Conversely, assume that $(X, \|\cdot\|_{\mathbb{C}})$ is a Hilbert space, then property (ii) can be obtained from

\[
\|e^{i\theta}(x + iy)\|^2_{\text{nat}} = \|x \cos \theta - y \sin \theta\|^2_X + \|x \sin \theta + y \cos \theta\|^2_X \\
= (x \cos \theta - y \sin \theta, x \cos \theta - y \sin \theta) + (x \sin \theta + y \cos \theta, x \sin \theta + y \cos \theta) \\
= (x, x) \cos^2 \theta - 2(x, y) \cos \theta \sin \theta + (y, y) \sin^2 \theta \\
+ (x, y) \sin^2 \theta + 2(x, y) \cos \theta \sin \theta + (y, y) \cos^2 \theta \\
= \|x\|^2_X + \|y\|^2_X = \|x + iy\|^2_{\text{nat}}
\]

and properties (i) and (iii) hold obviously. Hence, the norm $\|\cdot\|_{\text{nat}}$ is reasonable in the settings of a Hilbert space.

There are also some interesting observations concerning the Taylor complexification. From Proposition 1 it easily follows that all other reasonable norms $\|\cdot\|$ on $X_{\mathbb{C}}$ are in the following relation to the Taylor norm

\[
\|x + iy\|_T \leq \|x + iy\|_{X_{\mathbb{C}}} \leq 2 \|x + iy\|_T.
\]

Moreover, Taylor complexification of real Banach spaces is also a way to extend linear bounded operators between real Banach spaces to the complex ones between their complexifications without increasing the norm.

**Proposition 2.** For any $T \in \mathcal{L}(X, Y)$, where $X, Y$ are real Banach spaces and $T_{\mathbb{C}} \in \mathcal{L}((X_{\mathbb{C}}, \|\cdot\|_{\mathbb{C}}), (Y_{\mathbb{C}}, \|\cdot\|_{\mathbb{C}}))$ we have $\|T_{\mathbb{C}}\| = \|T\|$. 

**Proof.** Obviously, $\|T_{\mathbb{C}}\| \geq \|T\|$. On the other hand,

\[
\|T_{\mathbb{C}}(x + iy)\|_{X_{\mathbb{C}}} = \sup_{0 \leq \theta \leq 2\pi} \|T(x) \cos \theta - T(y) \sin \theta\|_X \\
= \sup_{0 \leq \theta \leq 2\pi} \|T(x \cos \theta - y \sin \theta)\|_X \\
\leq \|T\| \sup_{0 \leq \theta \leq 2\pi} \|x \cos \theta - y \sin \theta\|_X \\
\leq \|T\| \|x + iy\|_{X_{\mathbb{C}}}
\]

and $\|T_{\mathbb{C}}\| \leq \|T\|$. Hence, $\|T_{\mathbb{C}}\| = \|T\|$. 

**Remark 1.** Thus, in the case of Hilbert spaces for which Taylor complexification is reasonable and natural we get the equality of complex and real norms of operators between them.

An alternative approach to construct a reasonable norm on the complexification space is to take into consideration all values of $\|e^{i\theta}z\|$ for $0 \leq \theta \leq 2\pi$. For example, Lindenstrauss and Tzafriri ([35, I, p. 81] used the following reasonable norm

\[
\|x + iy\|_{LT} = \sup_{0 \leq \theta \leq 2\pi} \left(\|x \cos \theta - y \sin \theta\|^2_X + \|x \cos \theta + y \sin \theta\|^2_X\right)^{1/2}.
\]
2.3. Complexification of real Banach lattices

This approach produces a plenty of other examples of reasonable norms which can be found in the papers of Kirwan [31] and Muñoz-Sarantopoulos-Tonge [46]. We want to mention here that not all complex Banach spaces can be produced as complexifications of real Banach spaces. The existence of such space was proved by Bourgain [8] using probabilistic arguments and the first explicit example was constructed by Kalton [25].

2.3. Complexification of real Banach lattices

Additional structure of real Banach lattices allows to introduce some special but natural procedure of complexification of these spaces (cf. Schaefer [52]).

Definition 2. A real Banach lattice is a Banach space over the real numbers endowed with a partial order \( \preceq \) such that any two elements \( x, y \in X \) have an infimum (denoted by \( x \wedge y := \inf \{ x, y \} \) and supremum (\( x \vee y := \sup \{ x, y \} \)), the positive cone \( E^+ := \{ x \in E : x \geq 0 \} \) is closed under addition and multiplication by nonnegative real numbers and the partial order is connected to the norm on \( X \) by the condition

\[
(2.3) \quad y \in X \text{ and } |x| \leq |y| \Rightarrow x \in X \text{ and } \|x\| \leq \|y\|,
\]

where \( |x| = x \vee (-x) \).

Now assume that we have complexified a real vector lattice \( X \) in a natural way, i.e. \( X_C = X + iX \) and consider for \( z = x + iy, x, y \in X \) the supremum:

\[
(2.4) \quad |z| = \sup_{0 \leq \theta \leq 2\pi} |x \cos \theta + y \sin \theta|.
\]

Obviously, this mapping satisfies the following properties of absolute value function

\[
|z| = 0 \text{ if and only if } z = 0
\]

\[
|\alpha z| = |\alpha||z| \text{ for all } \alpha \in \mathbb{C} \text{ and } z \in X_C
\]

\[
|z_1 + z_2| \leq |z_1| + |z_2| \text{ for all } z_1, z_2 \in X_C
\]

If \( X_C \) is a complex vector lattice and \( \|z\| \) is a lattice norm on \( X \), then the absolute value function (2.4) suggests the extension of the given lattice norm to \( X_C \) by

\[
\|z\|_{X_C} = \|z\|_X = \left\| (|x|^2 + |y|^2)^{1/2} \right\|_X \text{ for all } z = x + iy \in X_C.
\]

Clearly, with this notion of norm the complexified space will possess lattice property (2.3).

As an example of Banach lattice consider the space of real valued functions \( C(K) \) defined on a compact set \( K \). It was shown by Muñoz-Sarantopoulos-Tonge in [46] that the lattice complexification of this space coincides with the Taylor complexification. Thus, Proposition 2 implies the equality of the norms of any bounded linear operator between \( C(K) \) spaces and its complexification. Moreover, since \( L^\infty_C \) space is isometrically isomorphic to \( C_C(K) \) and since this isomorphism maps real functions into real functions, then the equality of real and complex norms holds even for operators between \( L^\infty \) spaces.
2.4. Complex norms of operators

Applying any complexification procedure to real Banach spaces we extend bounded linear operators between these spaces to the complex bounded linear operators between their complexifications whose complex norms will be clearly not less than the real ones. In some cases we may even expect that complex operator norms remain the same as real. The problem of finding complexifications preserving operator norms was studied, for example, in [46]. Here we show how property (iii) of reasonable complexification procedure affects the relation between complex and real operator norms. It turns out that if the relations \( \| z \|_X, \| \Im z \|_X \leq \| z \|_{X_C} \) hold for any \( z = x + iy \in X_C \), then we easily get the following rough estimate of \( \| T_C \|_{X_C,Y_C} / \| T \|_{X,Y} \).

**Proposition 3.** Let \((X,\|\cdot\|_X)\) and \((Y,\|\cdot\|_Y)\) be normed spaces with reasonable complexifications \((X_C,\|\cdot\|_{X_C})\) and \((Y_C,\|\cdot\|_{Y_C})\) respectively. Then for any \( T \in \mathcal{L}(X,Y) \) we have \( \| T_C \|_{X_C,Y_C} \leq 2 \| T \|_{X,Y} \), where \( T_C \) is a standard complexification of the operator \( T \), i.e. \( T_C = T + iTy \).

**Proof.** Recall that for any \( z \in X_C \) we have \( \| \Re z \|_X, \| \Im z \|_X \leq \| z \|_{X_C} \) by the consequence of the conjugation-invariant property defined in Definition 1. Then for \( z = x + iy \in X_C \)

\[
\frac{\| T(x + iy) \|}{\| x + iy \|} = \frac{\| Tx + iTy \|}{\| x + iy \|} \leq \frac{\| Tx \| + \| Ty \|}{\| x + iy \|} \leq \| T \| \frac{\| \Re z \| + \| \Im z \|}{\| z \|} \leq 2 \| T \|.
\]

Thus, \( \| T_C \|_{X_C,Y_C} \leq 2 \| T \|_{X,Y} \). In general, constant 2 cannot be improved. Supply, for example, \( Y_C \) with maximal reasonable norm \( (\| x + iy \|_T) \) and \( X_C \) with minimal reasonable norm \( (\| x + iy \|_T) \).

Now assume that the condition \( \| \Re z \|_X, \| \Im z \|_X \leq \| z \|_{X_C} \) does not hold and consider the following example of the complexification of a real Banach space showing that the relation between complex and real operator norms can be arbitrary large in this case. In particular, this relation is no longer necessarily bounded by constant 2 as we could intuitively expect.

**Example 2.** Let \( 1 \leq p < \infty \) and \( 0 < \alpha < 1/2 \). Consider space \( \mathbb{C}^2 \) as a natural complexification of \( \mathbb{R}^2 \) with the norm defined by:

\[
\| (z_1, z_2) \| = \| (x_1 - iz_2, 1 - \alpha)z_1 + iz_2 \|^{1/p}.
\]

As it was mentioned above this norm satisfies properties (i) and (ii) of Definition 1 but it is not conjugation-invariant. Clearly, for \((x, y) \in \mathbb{R}^2 \) we have \( \| (x, y) \| = \sqrt{x^2 + y^2} \). If we consider the operator \( T(x,y) = (x + y, x - y) : \mathbb{R}^2 \to \mathbb{R}^2 \), then real norm of this operator is \( \sqrt{2} \). On the other hand, taking \((z_1, z_2) = (1, i)\) we obtain \( \| (1, i) \| = 2\alpha^{1/p} \) and \( \| T_C(1, i) \| = \| (1 + i, 1 - i) \| = 2\sqrt{2}(1 - \alpha)^{1/p} \). Hence, for \( 0 < \alpha < 1/2 \)

\[
\frac{\| T_C \|}{\| T \|} \geq \frac{(1 - \alpha)^{1/p}}{\alpha^{1/p}} > 1.
\]

We note, if \( \alpha \) tends to zero, then \( (1 - \alpha)^{1/p}/\alpha^{1/p} \) tends to infinity. Thus, the relation between complex and real norms of this operator can be arbitrary large.
CHAPTER
THREE

Equality of Real and Complex Operator Norms

3.1. Operators between Lebesgue spaces

In 1958 A. E. Taylor [56], referring to the paper of M. Riesz on convexity and bilinear forms [50], reproved Riesz’s statement about the equality of the real and complex norms for any bounded linear operator between finite-dimensional spaces $l^p_n$ and $l^q_m$ whenever $p \leq q$ and pointed out the importance of this result. Taylor’s proof used only standard calculus. Marcinkiewicz and Zygmund [44] used Gaussian variables to obtain the equality of real and complex norms of vector-valued operators between (quasi-)Banach $L^p - L^p$ spaces. Later on A. Zygmund [64, p. 181] obtained the same result for the bounded linear operators from $L^p[a, b]$ into $L^p[a, b]$ space, relying his proof on a simple equality, showing that $L^p[0,1]$ contains an isometric copy of $l^2$. Using this nice idea of Zygmund, Verbickii and Sereda [59] in 1975 found a simple proof of the equality of real and complex norms for the operators between $L^p(\Omega_1, \mu)$ and $L^q(\Omega_2, \nu)$ spaces for $1 \leq p \leq q$ on any positive $\sigma$-finite measure spaces $(\Omega_1, \mu)$ and $(\Omega_2, \nu)$.

In this chapter we present an alternative proof of Verbickii-Sereda theorem (cf. Holtz-Karow [23]), as well as find out some other possibilities when the norm of a real operator becomes equal to the norm of its complexification. As it was shown in the previous chapter such equality holds for bounded linear operators between Hilbert spaces, between $C(K)$ spaces as well as between $L^\infty$ spaces. Here we obtain the same result for the positive operators between (quasi-)Banach lattices, for operators between some mixed (quasi-)norm Lebesgue spaces and between spaces of functions of bounded $p$-variation for some values of $p$. Finally, we give an example of an operator acting from an Orlicz space into the same Orlicz space for which the equality of complex and real norms does not hold.

To obtain the most part of these results we will mainly look for the possibility to estimate the complex operator norm by the real one. In these estimates the fact that complex numbers possess symmetry of the unit ball will play a crucial role. By symmetry of the unit ball we mean that $|a + ib| = |e^{-i2\pi t}|a + ib|$ holds for all $a, b, t \in \mathbb{R}$. Using Euler identity $e^{i2\pi t} = \cos 2\pi t + i\sin 2\pi t$ we get $|e^{-i2\pi t}|a + ib| = |a \cos 2\pi t + b \sin 2\pi t + i(a \sin 2\pi t - b \cos 2\pi t)|$. And since $|z| \geq |\text{Re} z|$ for any $z \in \mathbb{C}$ we have for all $a, b, t \in \mathbb{R}$

$$|a + ib| \geq |a \cos 2\pi t + b \sin 2\pi t|.$$
Taking, the $p$-power of the both sides of this inequality and integrating them in $t$ over $[0, 1]$ we get even equality which will be a very useful formula in our proofs. It is this formula was used by Zygmund when he showed the equality of complex and real norms for the operators between $L^p[a, b]$ spaces.

**Lemma 1.** For $0 < p \leq \infty$ and for all $a, b \in \mathbb{R}$ the following equality holds

\[
\left( \int_0^1 |a \cos 2\pi t + b \sin 2\pi t|^p \, dt \right)^{1/p} = d_p(a^2 + b^2)^{1/2},
\]

where $d_p = \left( \int_0^1 |\cos 2\pi t|^p \, dt \right)^{1/p} = \|\cos 2\pi t\|_{L^p[0,1]}$.

**Proof.** We can assume that $a^2 + b^2 > 0$, since otherwise we have equality. If we divide the both sides of (3.1) by $\sqrt{a^2 + b^2}$ we obtain by $2\pi$-periodicity of cosine that

\[
\left\| \frac{a}{\sqrt{a^2 + b^2}} \cos 2\pi t + \frac{b}{\sqrt{a^2 + b^2}} \sin 2\pi t \right\|_{L^p[0,1]} = \|\cos \theta \cos 2\pi t + \sin \theta \sin 2\pi t\|_{L^p[0,1]}
\]

\[
= \|\cos(2\pi t + \theta)\|_{L^p[0,1]} = \left( \int_0^1 |\cos(2\pi t + \theta)|^p \, dt \right)^{1/p}
\]

\[
= \left( \int_0^{2\pi + \theta} |\cos s|^p \frac{ds}{2\pi} \right)^{1/p} = \left( \int_0^{2\pi} |\cos s|^p \frac{ds}{2\pi} \right)^{1/p}
\]

\[
= \left( \int_0^1 |\cos 2\pi t|^p \, dt \right)^{1/p} = \|\cos 2\pi t\|_{L^p[0,1]}
\]

and the proof is complete.

**Remark 2.** Taking $p = \infty$ we get a useful particular case of this lemma, namely $(a^2 + b^2)^{1/2} = \sup \{ a \cos 2\pi t + b \sin 2\pi t : 0 \leq t \leq 1 \}$.

Lemma 1 enables us to give a simple explanation why the norm of bounded linear operator $T : L^p(\mu) \rightarrow L^p(\nu)$ will not be changed when we go to the complexified spaces. This proof was obtained by Zygmund [64, p. 181]. Thus, by Lemma 1 and the lattice structure of Lebesgue spaces we get

\[
d_p \|TC(f + ig)\|_p = d_p \left\| \left( |Tf|^2 + |Ty|^2 \right)^{1/2} \right\|_p = \left\| Tf \cos 2\pi t + Ty \sin 2\pi t \right\|_{L^p[0,1]}.\]

Applying twice Fubini theorem and using boundedness of the operator $T$ we obtain

\[
\left\| T(f \cos 2\pi t + g \sin 2\pi t) \right\|_{L^p[0,1]} \leq \left\| T \right\|_{p,p} \left\| f \cos 2\pi t + g \sin 2\pi t \right\|_{L^p[0,1]} = \left\| T \right\|_{p,p} \left\| f \cos 2\pi t + g \sin 2\pi t \right\|_{L^p[0,1]} = d_p \left\| T \right\|_{p,p} \|f + ig\|_p.
\]

Hence, $\|TC\|_{p,p} \leq \|T\|_{p,p}$ and, since the reverse inequality is obviously true, we have $\|TC\|_{p,p} = \|T\|_{p,p}$.

The constants $d_p$ will often appear in what it follows and that is why we emphasize from the beginning some of their simple properties.

**Lemma 2.** Function $d_p$ is increasing in $p$ for $0 < p \leq \infty$. 
Proof. Assume that $0 < p \leq q < \infty$, then by the H"{o}lder inequality with $q/p \geq 1$

$$d_p = \left( \int_0^1 |\cos 2\pi t|^p \, dt \right)^{\frac{1}{p}} \leq \left( \int_0^1 |\cos 2\pi t|^{p\frac{q}{p}} \, dt \right)^{\frac{p}{q}} \left( \int_0^1 1 \, dt \right)^{\frac{q}{p} - \frac{1}{q}}$$

$$= \left( \int_0^1 |\cos 2\pi t|^q \, dt \right)^{\frac{1}{q}} = d_q.$$ The case when $0 < p < q = \infty$ is obvious.

Remark 3. $d_1 = \frac{1}{\pi} < d_2 = \frac{1}{\sqrt{\pi}} < d_\infty = 1$. It is also possible to find the expression of constant $d_p$ in terms of gamma function $\Gamma$ using the following relation $\Gamma(2x) = \pi^{-1/2} 2^{2x-1} \Gamma(x)\Gamma(x + \frac{1}{2})$ and Euler function $B(x, y) = \int_0^1 u^{x-1}(1 - u)^{y-1} \, du = \Gamma(x)\Gamma(y)/\Gamma(x + y)$ (cf. Zygmund [65], Vol. II, p. 57):

$$d_p^p = \int_0^1 |\cos 2\pi t|^p \, dt = \int_0^1 |\sin 2\pi t|^p \, dt = 2^{p+1} \int_0^{1/2} (\sin \pi t \cos \pi t)^p \, dt$$

$$= \frac{2p}{\pi} \int_0^1 u^{\frac{p+1}{2}}(1 - u)^{\frac{p-1}{2}} \, du = \frac{2p}{\pi} B\left(\frac{p+1}{2}, \frac{p+1}{2}\right)$$

$$= \frac{2p}{\pi} \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma(p+1)} = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{p+1}{2}\right)}$$

To prove the equality of real and complex norms of the operators $T \in \mathcal{L}(L^p(\mu), L^q(\nu))$ for $p \leq q$ we will use the following generalization of Lemma 1.

Lemma 3. Let $f, g \in L^p(\Omega, \mu)$. If $0 < p \leq q \leq \infty$, then

$$\left( \int_0^1 \|f \cos 2\pi t + g \sin 2\pi t\|^q_p \, dt \right)^{\frac{1}{q}} \leq d_q \|f + ig\|_p.$$ The equality holds when $p = q$. If conversely, $0 < q \leq p \leq \infty$, then

$$\left( \int_0^1 \|f \cos 2\pi t + g \sin 2\pi t\|^q_p \, dt \right)^{\frac{1}{q}} \leq d_p \|f + ig\|_p.$$ Both these two inequalities can be written as one

$$(3.2) \quad \left( \int_0^1 \|f \cos 2\pi t + g \sin 2\pi t\|^q_p \, dt \right)^{\frac{1}{q}} \leq \max\{d_p, d_q\} \|f + ig\|_p.$$ Proof. To prove this lemma for $0 < p \leq q \leq \infty$ we use the integral Jensen inequality (see Rudin [51], p. 62) and Fubini theorem. Denoting $\cos 2\pi \theta_x = (f(x)/\sqrt{f^2(x) + g^2(x)}) = (f(x)/|f(x) + ig(x)|)$ we get

$$\left( \int_0^1 \|f \cos 2\pi t + g \sin 2\pi t\|^q_p \, dt \right)^{\frac{1}{q}}$$

$$= \left( \int_0^1 \left( \int_{\Omega} |f(x) \cos 2\pi t + g(x) \sin 2\pi t|^p d\mu(x) \right)^{q/p} \, dt \right)^{\frac{1}{q}}$$

$$= \left( \int_{\Omega} \left( \int_0^1 |\cos 2\pi(t - \theta_x)|^p |f(x) + ig(x)|^p d\mu(x) \right)^{q/p} \, dt \right)^{\frac{1}{q}}$$
by a simple modification of these inequalities. Applying boundedness of the operator \( \| \cdot \|_{p,q} \) holds then \( \| f \|_{p,q} \leq \| T \|_{p,q} \). Thus, we have proved that

\[
\| f \|_{p,q} \leq \| T \|_{p,q} \| f \|_{p,q}.
\]

The case when \( q = \infty \) (and \( p = \infty \)) can be obtained by a simple modification of these inequalities.

For \( p = q \) the equality holds in each step of this derivation. A simple modification of this proof together with Lemma 1 justifies the same statement for \( q = \infty \) (\( d_\infty = 1 \)).

When \( 0 < q \leq p \leq \infty \), then using the Hölder inequality and just proved inequality we obtain

\[
\left( \int_0^1 \| f \cos 2\pi t + g \sin 2\pi t \|_q^p \, dt \right)^{1/p} \leq \left( \int_0^1 \| f \cos 2\pi t + g \sin 2\pi t \|_p^p \, dt \right)^{1/p} = d_p \| f + ig \|_p.
\]

**Remark 4.** Equality (3.1) is an analogue of this lemma for real scalars. Note also that Lemma 3 can be extended to vector-valued functions (see Marcinkiewicz-Zygmund [44], Gasch-Maligranda [19]).

**Theorem 1.** If \( 0 < p \leq q \leq \infty \), then \( \| T \|_{p,q} = \| T \|_{p,q} \) for arbitrary \( T \in \mathcal{L}(L^p(\mu), L^q(\nu)) \), where \( \mu \) and \( \nu \) are positive \( \sigma \)-finite measures.

**Proof.** Lattice structure of \( L^\phi(\nu) \) and Lemma 3 imply

\[
d_q \| T_C(f + ig) \|_q^q = d_q \| |T f + i T g|\|_q^q = d_q \left( \left( \| T f \|^2 + \| T g \|^2 \right)^{1/2} \right)_q^q = \int_0^1 \| T f \cos 2\pi t + T g \sin 2\pi t \|_q^q \, dt.
\]

Applying boundedness of the operator \( T \) and Lemma 3 again we get

\[
d_q \| T_C(f + ig) \|_q^q \leq \left( \| T \|_{p,q} \right)^q \int_0^1 \| f \cos 2\pi t + g \sin 2\pi t \|_p^p \, dt \leq d_q \left( \| T \|_{p,q} \right)^q \| f + ig \|_p^p.
\]

Thus, we have proved that \( \| T \|_{p,q} \leq \| T \|_{p,q} \). Since the reverse inequality obviously holds then \( \| T \|_{p,q} = \| T \|_{p,q} \). The case when \( q = \infty \) (and \( p = \infty \)) can be obtained by a simple modification of these inequalities.
3.1. OPERATORS BETWEEN LEBESGUE SPACES

Remark 5. In the case of quasi-Banach Lebesgue spaces it can happen that the operators \( T : L^p(\mu) \to L^q(\nu) \) can be only trivial ones. For example, if \( T : L^p(\mu) \to L^q(\nu) \) is a bounded linear operator, where \( \mu \) is a non-atomic measure and \( 0 < p < \min\{1, q\} \), then \( T = 0 \) (see Brudnyi-Krugljak [9], p. 89, Maligranda [38], p. 150 and Day [13]). Of course, if \( \mu \) is a discrete measure Theorem 1 is not trivial.

Remark 6. There exist other proofs of Theorem 1. Some of them use tensor product technique and prove vector-valued version of Theorem 1 (Figiel-Iwaniec-Pelczyński [17], Reinov [49]), others explore random variables (Zygmund [64]). For discrete measures there are even proofs by standard differential calculus (Taylor [56], Crouzeix [12]).

Remark 7. As we will see later on the equality of complex and real norms of an operator \( T : L^p(\mu) \to L^q(\nu) \) for \( 1 \leq p \leq q \leq \infty \) follows from that for the operator \( T : L^p(\mu) \to L^q(\nu) \) due to monotonicity of the relation between these norms: this relation is increasing in \( p \) and decreasing in \( q \). But to prove this statement the more advanced techniques than those presented in Theorem 1 are required.

Recall that
\[ \gamma_{p,q} = \sup \{ \gamma \geq 1 : \| T_{\mathbb{C}} \|_{p,q} \leq \gamma \| T \|_{p,q} \text{ for any } T \in \mathcal{L}(L^p(\mu), L^q(\nu)) \} \]
and by Theorem 1 we have a straightforward

Corollary 1. If \( 0 < p \leq q \leq \infty \), then \( \gamma_{p,q} = 1 \).

The conclusion of Corollary 1 cannot in general be extended to other values of \( p \) and \( q \). Already Riesz found an example of operator which for each couple \((p, q)\) such that \( 1 \leq q < p \leq \infty \) has different real and complex norms. The same operator has different real and complex norms even for the case \( 0 < q < p \) that can be proved in exactly the same way as we pointed out below.

Example 3 (Riesz [50]). For \( \varepsilon > 0 \) let \( T^\varepsilon \in \mathcal{L}(l_1^p, l_1^q) \) be defined by
\[ T^\varepsilon(x_1, x_2, x_3) = (x_1 - \varepsilon(x_1 + x_2 + x_3), x_2 - \varepsilon(x_1 + x_2 + x_3), x_3 - \varepsilon(x_1 + x_2 + x_3)). \]
Then for all couples \((p, q)\) such as \( 0 < q < p \leq \infty \) there exists \( \varepsilon > 0 \) such that \( \| T^\varepsilon \| < \| T \| \). For the complete proof of this fact see Gasch-Maligranda [19].

Hence, if \( 0 < q < p \leq \infty \), then \( \gamma_{p,q} > 1 \), that together with Corollary 1 implies

Proposition 4. \( \gamma_{p,q} = 1 \) if and only if \( 0 < p \leq q \leq \infty \).

Another example of the operator with different real and complex norms \( T(x, y) = (x + y, x - y) : l_2^p \to l_2^q \) for \( 0 < q < 2 < p \leq \infty \) will be carefully investigated in Chapter 5.

There is a simple generalization of Theorem 1 to the bounded linear operators between mixed norm spaces \( L^p[L^q] \) and \( L^r[L^s] \).

Theorem 2. If \( 0 < p, q, r, s \leq \infty \) and \( \max\{p, q\} \leq \min\{r, s\} \), then \( \| T_{\mathbb{C}} \| = \| T \| \) for arbitrary \( T \in \mathcal{L}(L^p[L^q], L^r[L^s]) \).
Proof. The proof of this statement mainly relies on the integral Minkowski inequality. Let $\tau = \min\{r, s\}$, then
\[
d_{\tau}^{r} \| T_{\mathbb{C}}(f + ig) \|_{s, r}^{\tau} = d_{\tau}^{r} \| T f + iT g \|_{s, r}^{\tau} = \left\| \left\| T f \cos 2\pi t + T g \sin 2\pi t \right\|_{L^{r}[0, 1]}^{\tau} \right\|_{s}^{r}
\]
\[
= \left\| \left( \int_{\Omega_{2}} \left( \int_{0}^{1} |T f \cos 2\pi t + T g \sin 2\pi t|^{\tau} dt \right)^{s/\tau} d\mu_{1} \right)^{r/\tau} \right\|_{s}^{r}
\]
\[
\leq \left\| \left( \int_{0}^{1} \| T f \cos 2\pi t + T g \sin 2\pi t \|_{L^{r}[0, 1]}^{\tau} dt \right)^{s/\tau} \right\|_{s}^{r}
\]
\[
= \left( \int_{\Omega_{2}} \left( \int_{0}^{1} \| T f \cos 2\pi t + T g \sin 2\pi t \|_{s}^{\tau} dt \right)^{r/\tau} d\mu_{2} \right)^{s/\tau}
\]
\[
\leq \int_{0}^{1} \| T f \cos 2\pi t + T g \sin 2\pi t \|_{s, r}^{\tau} dt
\]
\[
\leq \| T \|\left( \int_{0}^{1} \| f \cos 2\pi t + g \sin 2\pi t \|_{q, p}^{\tau} dt \right)^{s/\tau}
\]
Recall, $\tau = \min\{r, s\} \geq \max\{p, q\}$. Therefore, again by the integral Minkowski inequality we obtain
\[
d_{\tau}^{r} \| T_{\mathbb{C}}(f + ig) \|_{s, r}^{\tau} \leq \| T \|\left( \int_{0}^{1} \| f \cos 2\pi t + g \sin 2\pi t \|_{q, p}^{\tau} dt \right)^{s/\tau}
\]
\[ \frac{\|T\|^r \left( \int_{\Omega_2} \left( \int_0^1 \left( \int_{\Omega_1} |f \cos 2\pi t + g \sin 2\pi t|^q \, d\mu_1 \right)^{\frac{\tau}{q}} \, dt \right)^{\frac{p/\tau}{p}} \, d\mu_2 \right)^{\tau/p}}{\|T\|^r \left( \int_{\Omega_2} \left( \int_0^1 \left( \int_{\Omega_1} |f \cos 2\pi t + g \sin 2\pi t|^q \, d\mu_1 \right)^{\frac{p/\tau}{p}} \, d\mu_2 \right)^{\tau/p}} \leq \|T\|^r \left( \int_{\Omega_2} \left( \int_0^1 \|f \cos 2\pi t + g \sin 2\pi t\|_{L^{\tau/q}[0,1]} \, d\mu_1 \right)^{\frac{p/\tau}{p}} \, d\mu_2 \right)^{\tau/p} \]

\[ = \|T\|^r \left( \int_{\Omega_2} \left( \int_0^1 \|f \cos 2\pi t + g \sin 2\pi t\|_{L^{\tau/q}[0,1]} \, d\mu_1 \right)^{\frac{p/\tau}{p}} \, d\mu_2 \right)^{\tau/p} \]

\[ = d_\tau \|T\|^r \|f + ig\|_{q,p}^r. \]

Thus, \( \|T_C\| \leq \|T\| \) and since the reverse inequality is obvious the proof is complete. ■

**Remark 8.** By induction the same statement can be proved for the operators \( T \in \mathcal{L}(L^{p_1}[\ldots[L^{p_n}]]_{\ldots}, L^{q_1}[\ldots[L^{q_m}]]_{\ldots}), \) where \( 0 < \max\{p_1, \ldots, p_n\} \leq \min\{q_1, \ldots, q_m\} < \infty. \)

3.2. Operators between spaces of bounded \( p \)-variation

Applying the same techniques as we used in the settings of Lebesgue spaces we can establish the equality of real and complex norms for operators between spaces of functions of bounded \( p \)-variation for some values of \( p \). These spaces for \( p = 2 \) were introduced by Wiener [62] when he investigated the convergence of Fourier series. The extension to \( 1 \leq p \leq \infty \) was done by L. C. Young and to all \( 0 < p \leq \infty \) is due Bergh and Petree [5].

**Definition 3.** Let \( 0 < p < \infty \). A function \( f \) defined on \([a, b]\) is said to be of bounded \( p \)-variation on \([a, b]\) if its total \( p \)-variation \( \text{Var}_p(f) \) on \([a, b]\) is finite, where

\[ \text{Var}_p(f) = \sup_{P_n} \left( \sum_{k=1}^n |f(t_k) - f(t_{k-1})|^p \right)^{1/p} \]

and the supremum is taken over all partitions \( P_n : a = t_0 < t_1 < \ldots < t_n = b \) of the interval \([a, b]\) with arbitrary \( n \in \mathbb{N} \). In the case when \( p = \infty \) we define

\[ \text{Var}_\infty(f) = \sup_{x \in [a, b]} |f(x)|. \]

All functions (real or complex-valued) of bounded \( p \)-variation on \([a, b]\) form a Banach space (quasi-Banach space for \( 0 < p < 1 \) denoted by \( BV_p[a, b] \) with the (quasi-)norm

\[ \|f\|_{BV_p} = (\|f(a)^p + \text{Var}_p^p(f)\|)^{1/p}, \quad \text{for } f \in BV_p[a, b], \text{ where } 0 < p < \infty \]

and

\[ \|f\|_{BV_\infty} = \sup_{x \in [a, b]} |f(x)|, \quad \text{for } f \in BV_\infty[a, b], \text{ where } p = \infty. \]
Clearly, the norm introduced in Definition 3 on the space $BV_p[a, b]$ is reasonable. Therefore, Proposition 3 proved for normed spaces with reasonable complexifications implies that $\|T\| \leq 2\|T\|$ for any $T \in \mathcal{L}(BV_p[a, b])$, where $p \geq 1$. However, as in the case of Lebesgue spaces this estimate can be improved for some values of $p$ and $q$. In this section we mainly consider the case when the complex norm of any operator $T \in \mathcal{L}(BV_p[a, b], BV_q[a, b])$ is the same as its real norm and show a general estimate of the relation $\|T\| / \|T\|$ valid even in quasi-Banach settings. Moreover, we obtain some cases of the equality of real and complex norms of bounded $p$-variation and another one is Lebesgue or Hilbert space.

Since we deal here with the quasi-Banach variant of the space $BV_p[a, b]$ for $0 < p < 1$ it is useful to note that this space is considerably small in the sense that it does not contain any continuous functions except the constants as it is shown in the following proposition (cf. Dudley-Norvaiša [16]).

**Proposition 5.** If $0 < p < 1$, then any continuous function in $BV_p[a, b]$ is a constant function.

**Proof.** Assume that there exists some continuous function $f \in BV_p[a, b]$ which is not a constant. That means, that there exists $x_1 \in [a, b]$ such that $f(a) \neq f(x_1)$. Without loss of generality, let assume $|f(a) - f(x_1)| = 1$ and put $a = x_0$. Now choose the smallest number $x_{1/2} \in (x_0, x_1)$ such that $|f(x_0) - f(x_{1/2})| = 1/2$ whereof we have $|f(x_1) - f(x_{1/2})| \geq 1/2$. On the next step we choose two numbers $x_{1/2} \in (x_0, x_{1/2})$ and $x_{3/2} \in (x_{1/2}, x_1)$ such that for $j = 1, \ldots, 2^k$ we have $|f(x_{j-1}) - f(x_j)| \geq 1/2^j$. We continue this procedure in a similar way and each time choose $2^k$ numbers, by one from each previously defined open segment, such that for $j = 1, \ldots, 2^k$ we have $|f(x_{j-1}) - f(x_j)| \geq 1/2^k$. Due to continuity of function $f$ this construction is possible. Thus, taking the partition $P$ as ordered rearrangement of $x_j$ we have

$$Var_p(f) \geq 2^k/2^{kp} = 2^{k(1-p)}$$

which tends to infinity as $k$ goes to infinity for $0 < p < 1$. We come to the contradiction that $f \notin BV_p[a, b]$ and therefore $f$ is a constant. \[\Box\]

**Definition 4.** A step function $f$ on the interval $[a, b]$ is a function such that $f(x) = r_i$ for $x_i < x < x_{i+1}$, where $\{x_i\}_{i=1}^{\infty}$ is a partition of $[a, b]$ and $r_i \in \mathbb{R}$ for $i \in \mathbb{N}$.

**Corollary 2.** If $0 < p < 1$, then any function in $BV_p[a, b]$ is a step function.

**Proof.** Take any function from $BV_p[a, b]$ for $0 < p < 1$. Any continuous part of $f$ is a constant by previous proposition. Assume that $f \in BV_p[a, b]$ is discontinuous at each point of some subinterval of $[a, b]$. Hence, $f$ has an uncountable number of jumps which blow up this function from the space $BV_p[a, b]$ (uncountable sum of positive terms is always infinite). These two observations imply that function $f$ is a step function with at most countable number of jumps. \[\Box\]

Equality of the real and complex norms of operators $T \in \mathcal{L}(BV_p[a, b], BV_q[a, b])$ for either $0 < p \leq 1$ and $p \leq q \leq \infty$ or $0 \leq p \leq q = \infty$ can be shown with the help of the following lemma. Note, that spaces $BV_p[a, b]$ for $q \geq 1$ are much richer than $BV_p[a, b]$ and for $0 < p \leq q < \infty$ we have $BV_p[a, b] \subseteq BV_q[a, b]$.
3.2. Operators between spaces of bounded p-variation

Lemma 4. If either $0 < p \leq 1$ and $p \leq q < \infty$ or $0 < p \leq q = \infty$, then

$$d_q \text{Var}_q(f + ig) \leq \left( \int_0^1 \text{Var}_p^q(f \cos 2\pi t + g \sin 2\pi t)dt \right)^{1/q} \leq d_q \text{Var}_p^q(f + ig)$$

for all $f, g \in BV_p^q[a, b]$. The equalities hold for either $0 < p = 1$ or $p = q = \infty$. Moreover, the left inequality holds even for all $0 < p \leq q \leq \infty$.

Proof. Consider the right inequality for $0 < p \leq 1$ and $p \leq q < \infty$. First we note that for $0 < p \leq 1$ a total p-variation of any function $h \in BV_p^q[a, b]$ ($h \in (BV_p^q)_C[a, b]$) can be represented as a limit of a sequence:

$$\text{Var}_p^q(h) = \lim_{n \to \infty} \left( \sum_{k=1}^n |h(x_k) - h(x_{k-1})|^p \right)^{q/p} \equiv \lim_{n \to \infty} S_n^{q/p},$$

where a partition $P_n : a = t_1 < t_2 < \ldots < t_n = b$ is a refinement of a partition $P_{n-1}$. Indeed, the sequence $\{S_n^{q/p}\}$ is increasing, that can be shown by Jensen inequality and therefore converges to its supremum, which is the $q$-power of $p$-variation of function $h$ by Definition 3.3.

Now, for $t \in [0, 1]$ we take

$$h(x, t) = f(x) \cos 2\pi t + g(x) \sin 2\pi t$$

and show that the sequence $\{S_n^{q/p}\}$ satisfies the conditions of Dini lemma. As it has just been shown the sequence $\{S_n^{q/p}\}$ is increasing. Each element of this sequence is now a continuous function of variable $t$ defined on $[0, 1]$ and the sequence converges to the function $\text{Var}_p^q(h)$ which is also continuous in $t$, that follows from that $\|h\|_p = |h(a)|^p + \text{Var}_p^q(h)$ is a non-homogeneous norm on $BV_p[a, b]$, which is continuous function. Then, Dini lemma asserts that the sequence $\{S_n^{q/p}\}$ is uniformly convergent and hence the order of integration and taking the limit of this sequence can be exchanged. Thus, from the discussion above and the Minkowski inequality we have

$$\int_0^1 \text{Var}_p^q(f \cos 2\pi t + g \sin 2\pi t)dt = \int_0^1 \lim_{n \to \infty} S_n^{q/p}dt = \lim_{n \to \infty} \int_0^1 S_n^{q/p}dt$$

$$= \lim_{n \to \infty} \int_0^1 \left( \sum_{k=1}^n |f(t_k) - f(t_{k-1})| \cos 2\pi t + (g(t_k) - g(t_{k-1})) \sin 2\pi t|^p \right)^{q/p} dt$$

$$\leq \lim_{n \to \infty} \left( \sum_{k=1}^n \|f(t_k) - f(t_{k-1})| \cos 2\pi t + (g(t_k) - g(t_{k-1})) \sin 2\pi t|^p \right)^{q/p} L_{q/p}[0, 1]$$. 

$$= \lim_{n \to \infty} \left( \int_0^1 \|f(t_k) - f(t_{k-1})| \cos 2\pi t + (g(t_k) - g(t_{k-1})) \sin 2\pi t|^p dt \right)^{q/p}$$

$$= \lim_{n \to \infty} d_q^p \left( \sum_{k=1}^n |f(t_k) - f(t_{k-1})| + i(g(t_k) - g(t_{k-1}))|^p \right)^{q/p}$$

$$= d_q^p \text{Var}_p^q(f + ig).$$
Obviously, when \(0 < p = q \leq 1\), then the equality holds in each step of this derivation.

If \(0 < p < q = \infty\), then \(\sup_{0 \leq t \leq 1} Var_p^p(f \cos 2\pi t + g \sin 2\pi t) =
\sup_{0 \leq t \leq 1} \sup_{n} \sum_{k=1}^{n} |(f_k - f_{k-1}) \cos 2\pi t + (g_k - g_{k-1}) \sin 2\pi t|^p
\leq \sup_{P_n} \sum_{k=1}^{n} \sup_{0 \leq t \leq 1} |(f_k - f_{k-1}) \cos 2\pi t + (g_k - g_{k-1}) \sin 2\pi t|^p
= \sup_{P_n} \sum_{k=1}^{n} |f_k - f_{k-1} + i(g_k - g_{k-1})|^p = Var_p^p(f + ig).

When \(p = \infty\), then we clearly get \(\sup_{0 \leq t \leq 1} Var_{\infty}(f \cos 2\pi t + g \sin 2\pi t) = Var_{\infty}(f + ig)\).

Considering the left inequality and taking into account that \(Var_q(f) \leq Var_p(f)\) for all \(f \in BV_p[a, b]\) and \(0 < p \leq q \leq \infty\) we get
\[
d_q Var_q(f + ig) = d_q \sup_{P_n} \left( \sum_{k=1}^{n} |f_k - f_{k-1} + i(g_k - g_{k-1})|^q \right)^{1/q}
\leq \left( \int_{0}^{1} \sup_{P_n} \left( \sum_{k=1}^{n} |(f_k - f_{k-1}) \cos 2\pi t + (g_k - g_{k-1}) \sin 2\pi t|^q \right) dt \right)^{1/q}
\leq \left( \int_{0}^{1} Var_q^q(f \cos 2\pi t + g \sin 2\pi t) dt \right)^{1/q}.
\]

**Corollary 3.** If either \(0 < p \leq 1\) and \(p \leq q < \infty\) or \(0 < p \leq q = \infty\), then
\(d_q \|f + ig\|_{BV_q} \leq \left( \int_{0}^{1} \|f \cos 2\pi t + g \sin 2\pi t\|_p^q dt \right)^{1/q} \leq d_q \|f + ig\|_{BV_p}\)
for all \(f, g \in BV_p[a, b]\). The equalities hold either for \(0 < p = q \leq 1\) or \(p = q = \infty\). Moreover, the left inequality holds even for all \(0 < p \leq q \leq \infty\).

**Proof.** The proof is obvious by Lemma 1, Lemma 4 and the Minkowski inequality. In fact,
\[
\left( \int_{0}^{1} \|f \cos 2\pi t + g \sin 2\pi t\|_p^q dt \right)^{1/q} = \left( \int_{0}^{1} \left( \|f(a) \cos 2\pi t + g(a) \sin 2\pi t\|_p^q + Var_p^p(f \cos 2\pi t + g \sin 2\pi t) \right)^{q/p} dt \right)^{1/q}
\leq \left( \left( \int_{0}^{1} \|f(a) \cos 2\pi t + g(a) \sin 2\pi t\|_p^q dt \right)^{1/p} + \right)_{L^{0/p}[0,1]}\]
\[ \left( \|f(a) \cos 2\pi t + g(a) \sin 2\pi t\|_{L^q[0,1]} + \|\varphi_p(f \cos 2\pi t + g \sin 2\pi t)\|_{L^q[0,1]} \right)^{1/p} \]

\[ \leq \left( d_q^q |f(a) + ig(a)|^q + \int_0^1 \varphi_p^q(f \cos 2\pi t + g \sin 2\pi t) dt \right)^{p/q} \]

\[ \leq (|f(a) + ig(a)|^q + \varphi_p^q(f + ig))^{1/q} = d_q \|f + ig\|_{BV_p}. \]

The case when \( 0 < p < q = \infty \) can be shown in the similar way. For \( p = \infty \) we obviously have equality \( \sup_{0 \leq t \leq 1} \|f \cos 2\pi t + g \sin 2\pi t\|_{BV_\infty} = \|f + ig\|_{BV_\infty}. \)

To obtain the left inequality we use Lemma 4 and thus if \( 0 < p \leq q \leq \infty \), then

\[ d_q \|f + ig\|_{BV_p} = d_q \left( |f(a) + ig(a)|^q + \varphi_p^q(f + ig) \right)^{1/q} \]

\[ \leq \left( \int_0^1 (|f(a) \cos 2\pi t + g(a) \sin 2\pi t|^q + \varphi_p^q(f \cos 2\pi t + g \sin 2\pi t)) dt \right)^{1/q} \]

\[ = \left( \int_0^1 \|f \cos 2\pi t + g \sin 2\pi t\|_{BV_q}^q dt \right)^{1/q} \leq \left( \int_0^1 \|f \cos 2\pi t + g \sin 2\pi t\|_{BV_p}^q dt \right)^{1/q}. \]

**Theorem 3.** If either \( 0 < p \leq 1 \) and \( p \leq q < \infty \) or \( 0 < p \leq q = \infty \), then for any \( T \in \mathcal{L}(BV_p[a, b], BV_q[a, b]) \) the equality \( \|T\| = \|T\| \) holds.

**Proof.** Let \( f, g \in BV_p[a, b] \). From Lemma 4, Corollary 3 and boundedness of the operator \( T \) it follows that for \( q < \infty \)

\[ d_q \|T(f + ig)\|_{BV_q} = d_q \|Tf + iTg\|_{BV_q} \]

\[ \leq \left( \int_0^1 \|T(f \cos 2\pi t + g \sin 2\pi t)\|_{BV_q}^q dt \right)^{1/q} \]

\[ \leq \|T\| \left( \int_0^1 \|f \cos 2\pi t + g \sin 2\pi t\|_{BV_p}^q dt \right)^{1/q} \]

\[ \leq d_q \|T\| \|f + ig\|_{BV_p}. \]

Therefore, \( \|T\| \leq \|T\| \) and since the reverse inequality is obviously true, the theorem is proved. For \( q = \infty \) the proof can be obtained by a simple modification of these inequalities.

**Remark 9.** Particular case of Theorem 3 is the case of Banach spaces. Thus, if \( q \geq 1 \), then for any \( T \in \mathcal{L}(BV_1[a, b], BV_q[a, b]) \) we have \( \|T\| = \|T\| \). Zygmund proved this statement for \( T \in \mathcal{L}(BV_1[a, b]) \) by using random variables (see [64]).

**Remark 10.** The estimate \( \|T\| / \|T\| \leq 1/d_q \) can now be easily obtained for all \( 0 < p, q \leq \infty \). Indeed, for all \( 0 < q \leq \infty \) by Corollary 3 and boundedness of the operator \( T \) we have

\[ d_q \|T(f + ig)\|_{BV_q} \leq \left( \int_0^1 \|T(f \cos 2\pi t + g \sin 2\pi t)\|_{BV_q}^q dt \right)^{1/q} \]

\[ \leq \|T\| \left( \int_0^1 \|f \cos 2\pi t + g \sin 2\pi t\|_{BV_p}^q dt \right)^{1/q}. \]
The latter expression is clearly less than \( \|T\| \sup_{0 \leq t \leq 1} \|f \cos 2\pi t + g \sin 2\pi t\|_{BV_p} \). Using again Corollary 3 we get

\[
d_q \|TC(f + ig)\|_{BV_q} \leq \|T\| \|f + ig\|_{BV_p}.
\]

If \( 0 < q < p \leq 1 \), then \( \|T_C\| / \|T\| \leq d_q / d_p \).

Due to the similarity of inequalities (3.2) and (3.5) we can establish the following proposition.

**Proposition 6.** For bounded linear operators \( T : BV_p[a, b] \to L^q(\mu) \), where \( 0 < p \leq 1 \) and \( p \leq q \leq \infty \), and \( T : H \to BV_q[a, b] \), where \( H \) is a Hilbert space we have equality \( \|T_C\| = \|T\| \).

**Proof.** The proofs of these statements are just replications of the proofs presented above and based on Lemma 3, Corollary 3 and the standard complexification of a Hilbert space.

**Problem 1.** Prove or disprove: if \( 1 < p < \infty \), then there exists a bounded linear operator \( T : BV_p[a, b] \to BV_p[a, b] \) whose complex and real norms are connected by a strong inequality \( \|T_C\| > \|T\| \).

### 3.3. Operators between Banach lattices

Besides the operators acting between Lebesgue spaces we are also interested in other examples of operators with equal real and complex norms. First, let consider positive operators acting between Banach lattices \( X \) and \( Y \) (see Chapter 2).

**Definition 5.** A linear operator \( T : X \to Y \) is said to be positive if for all \( f \in X \) with property \( f \geq 0 \) we have \( Tf \geq 0 \).

**Definition 6.** A linear operator \( T : X \to Y \) is said to be monotone if for all \( f, g \in X \) such that \( f \geq g \) it follows \( Tf \geq Tg \).

The following lemma establishes the equivalence of these two definitions.

**Lemma 5.** An operator \( T \) is positive if and only if \( T \) is monotone.

**Proof.** Assume that the operator \( T \) is positive. Consider functions \( f \) and \( g \) such that \( f \geq g \). Then \( f - g \geq 0 \) and by the linearity and positivity of the operator \( T \) we have \( Tf - Tg = T(f - g) \geq 0 \) or, equivalently, \( Tf \geq Tg \). The reverse implication is obvious, just take \( f \geq 0 = g \) and we get \( Tf \geq 0 \).

**Proposition 7.** The equality \( \|T_C\| = \|T\| \) holds for any positive operator \( T \in L(X, Y) \), where \( X, Y \) are real (quasi-)Banach lattices.

**Proof.** Lattice complexification of \( Y \) and Lemma 1 implies

\[
\|T_C(f + ig)\|_{Y_C} = \left\| \left( |Tf|^2 + |Tg|^2 \right)^{1/2} \right\|_Y = \left\| \sup_{0 \leq t \leq 1} (Tf \cos 2\pi t + Tg \sin 2\pi t) \right\|_Y.
\]
Hence, monotonicity of the operator $T$ and lattice complexification of $X$ imply
\begin{align*}
\left\| \sup_{0 \leq t \leq 1} T (f \cos 2\pi t + g \sin 2\pi t) \right\|_Y \\
\leq \left\| T \left( \sup_{0 \leq t \leq 1} (f \cos 2\pi t + g \sin 2\pi t) \right) \right\|_Y \\
\leq \|T\| \left\| \sup_{0 \leq t \leq 1} (f \cos 2\pi t + g \sin 2\pi t) \right\|_X \\
= \|T\| \|f + ig\|_X = \|T\| \|f + ig\|_{X^c}.
\end{align*}

Hence, $\|T_c\| \leq \|T\|$ and since the reverse inequality is obvious we have $\|T_c\| = \|T\|$.

Now let consider arbitrary operators between (quasi-)Banach lattices. Minkowski inequality and its reverse which were the main tools in the proof of Theorem 2 and which appeared implicitly in the proof of Theorem 1 can be generalized to these spaces. Such generalizations need the notions of $p$-concavity and $p$-convexity. Namely, for $0 < p, q \leq \infty$ we say that a (quasi-)normed lattice $(X, \|\cdot\|)$ is $p$-convex with a constant $M(p) < \infty$, respectively $q$-concave with a constant $M(q) < \infty$, if for all $x_1, \ldots, x_n \in X$
\begin{equation}
\left\| \left( \sum_{k=1}^n |x_k|^p \right)^{1/p} \right\| \leq M(p) \left\| \left( \sum_{k=1}^n \|x_k\|^p \right)^{1/p} \right\|. \tag{1}
\end{equation}

respectively,
\begin{equation}
\left\| \left( \sum_{k=1}^n \|x_k\|^q \right)^{1/q} \right\| \leq M(q) \left\| \left( \sum_{k=1}^n |x_k|^q \right)^{1/q} \right\|. \tag{2}
\end{equation}

If a (quasi-)normed lattice $X$ possesses the Fatou property, then the same estimates as above hold for the integrals instead of the sums (see Lindenstrauss-Tzafriri [35]).

**Theorem 4.** Let $X$ and $Y$ be real (quasi-)Banach lattices with the Fatou property. If $X$ is $p$-concave with constant 1 and $Y$ is $q$-convex with constant 1 for $0 < p \leq q \leq \infty$, then for arbitrary $T \in \mathcal{L}(X, Y)$ we have $\|T_c\| = \|T\|$.

**Proof.** Using Lemma 1, $q$-convexity of $Y$ and boundedness of the operator $T$ we can write
\begin{align*}
d_q \|Tf + iTg\|_Y &= \left\| \left( \int_0^1 |Tf \cos 2\pi t + Tg \sin 2\pi t|^q \right)^{1/q} \right\|_Y \\
&\leq \left( \int_0^1 \|Tf \cos 2\pi t + Tg \sin 2\pi t\|_Y^q \right)^{1/q} \\
&\leq \|T\| \left( \int_0^1 \|f \cos 2\pi t + g \sin 2\pi t\|^q_X \right)^{1/q}.
\end{align*}

Now we note that $p$-concavity of $X$ with constant one implies $q$-concavity of $X$ with the same constant for $p \leq q$ (see Lindenstrauss-Tzafriri [35], Vol. I, p. 49.
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Maligranda [40]) and we get
\[
\|T\| \left( \int_0^1 \|f \cos 2\pi t + g \sin 2\pi t\|_X^p dt \right)^{1/q} \leq \|T\| \left( \int_0^1 \|f \cos 2\pi t + g \sin 2\pi t\|_X^q dt \right)^{1/q} = d_q \|T\| \|f + ig\|_X.
\]

**Remark 11.** If in Theorem 4 \( X \) is \( p \)-concave with constant 1 and \( Y \) is \( q \)-convex with constant 1 for \( 0 < p \leq q \leq \infty \), then it is easy to show that \( \|Tc\| / \|T\| \leq d_p / d_q \) for arbitrary \( T \in \mathcal{L}(X,Y) \).

Recently, Hao-Kamińska-Tomczak-Jaegermann ([22]) have established a criteria of \( p \)-convexity and \( q \)-concavity with constant 1 for Orlicz function space with non-atomic measure. Let \( 0 < p < \infty \) and \( \varphi \) be an Orlicz function. Then the Orlicz space \( L^\varphi \) with non-atomic measure is \( p \)-convex (\( p \)-concave) with constant one if and only if \( \varphi(u^{1/p}) \) is convex (concave). In the case of Orlicz sequence space the additional condition for \( p \)-convexity (respectively \( p \)-concavity) of \( L^\varphi \) is required. Namely, \( \varphi \) should be such that \( \limsup_{u \to 0} \frac{\varphi(u^{1/p})}{\varphi((2u)^{1/p})} \geq 1 \) (respectively \( \liminf_{u \to 0} \frac{\varphi(u^{1/p})}{\varphi((2u)^{1/p})} \leq 1 \)), moreover it is enough to consider functions \( \varphi(u^{1/p}) \) on \((0,1)\). Thus, due to this result it is possible to construct Orlicz spaces different from Lebesgue spaces which satisfy the requirements of Proposition 4 and therefore bounded linear operators acting between these Orlicz spaces will have equal real and complex norms. By the same criteria an Orlicz space which is \( p \)-convex and \( p \)-concave with constants one for \( 0 < p < \infty \) simultaneously coincides with \( L^p(\mu) \).

**Problem 2.** Prove or disprove: Let \( X \) and \( Y \) be (quasi-)Banach lattices with the Fatou property. Then, \( \gamma_{X,Y} = 1 \) if and only if \( X \) is \( p \)-convex and \( Y \) is \( q \)-concave with constants one for some \( 0 < p \leq \infty \). We note, that this is actually true for Lebesgue spaces with non-atomic measures. Namely, by Proposition 4 we have \( \gamma_{p,q} = 1 \) if and only if \( 0 < p \leq q \). The latter condition is equivalent to either the function \( u^{p/q} \) is concave (\( u^{p/q} \) is convex) or \( u^{p/q} \) is convex (\( u^{p/q} \) is concave), that in turn by the criteria named above is equivalent to the condition that \( L^p \) is \( q \)-convex (\( p \)-concave) and \( L^q \) is \( q \)-convex (\( p \)-concave) with constants one. If we could get an affirmative answer on the preceding question, then by the same criteria we could conclude, that if \( X \) is an Orlicz space with the Fatou property, then \( \gamma_{X,X} = 1 \) if and only if \( X \) coincides with \( L^p(\mu) \).

### 3.4. Operators between Orlicz spaces

Up to now we have considered the case of the equality of real and complex norms of the operators from one (quasi-)Banach space into the same space. Recall, for example, operators \( T \in \mathcal{L}(L^p(\mu)) \). Here, we will present an operator between the same two-dimensional Orlicz space for which real and complex norms will differ. This Orlicz space will be constructed as an isometric copy of some symmetric space.

**Example 4.** Consider a two-dimensional symmetric space \( X = \mathbb{R}^2 \) equipped for \( 1 \leq p < 2 \) with the norm
\[
\|(x_1,x_2)\| = \frac{(|x_1|^p + |x_2|^p)^{1/p} + 2\frac{p}{1+p}\max\{|x_1|,|x_2|\}}{1 + 2\frac{p}{1+p}}, \quad x_1,x_2 \in \mathbb{R}
\]
and consider operator \( T : X \to X \) given by the formula \( T(x,y) = (x + y, x - y) \).
3.4. OPERATORS BETWEEN ORLICZ SPACES

First, using Theorem 11, we find the real norm of this operator:

\[
\|T(x_1, x_2)\| = \frac{\left( |x_1 + x_2|^p + |x_1 - x_2|^p \right)^{1/p} + 2\frac{\varphi}{\varphi(1)} \max\{ |x_1 + x_2|, |x_1 - x_2| \}}{1 + 2\frac{\varphi}{\varphi(1)}} \leq \frac{2 \max\{ |x_1|, |x_2| \} + 2\frac{\varphi}{\varphi(1)} \left( |x_1|^p + |x_2|^p \right)^{1/p}}{1 + 2\frac{\varphi}{\varphi(1)}} \leq 2^{1 - \frac{\varphi}{\varphi(1)}} \left( |x_1|^p + |x_2|^p \right)^{1/p} + 2\frac{\varphi}{\varphi(1)} \max\{ |x_1|, |x_2| \} = 2^{1 - \frac{\varphi}{\varphi(1)}} \|(x_1, x_2)\|.
\]

The equality holds for \((x_1, x_2) = (1, 1)\) and therefore \(\|T\| = 2^{1 - \frac{\varphi}{\varphi(1)}}\). On the other hand, the complex norm of this operator is strictly bigger than \(2^{1 - \frac{\varphi}{\varphi(1)}}\) since for \(1 < p < 2\) and \((z_1, z_2) = (1, i)\) we have

\[
\|T_C(z_1, z_2)\| / \|(z_1, z_2)\| = \sqrt{2} > 2^{1 - \frac{\varphi}{\varphi(1)}}.
\]

For \(p = 1\) we take \((z_1, z_2) = (1, 1 + i)\) and get

\[
\|T_C(z_1, z_2)\| / \|(z_1, z_2)\| = \left( (\sqrt{2} + 1)\sqrt{5} + 1 \right) / (3 + \sqrt{2}) > \sqrt{2}.
\]

Thus, \(\|T_C\| / \|T\| > 1\).

Remark 12. For \(p = 1\) this example was considered by Gasch-Maligranda [19]. We can even construct another example, taking \(2 < p \leq \infty\) and a symmetric space \(X = \mathbb{R}^2\) with the norm

\[
\|(x_1, x_2)\| = \frac{|x_1| + |x_2| + 2\frac{1}{p} \left( |x_1|^p + |x_2|^p \right)^{1/p}}{1 + 2\frac{1}{p}}, \quad x_1, x_2 \in \mathbb{R}
\]

and the same operator \(T\) as before. The real norm of this operator is equal to \(2^{1 + \frac{1}{p}}\).

Theorem 5. There exists a two-dimensional Orlicz space \(l_2^\varphi\) such that for the linear operator \(T : l_2^\varphi \to l_2^\varphi\) given by \(T(x, y) = (x + y, x - y)\) we have \(\|T_C\| > \|T\|\).

Proof. We use here the result of Grzašlewicz [20] saying that every compact symmetric convex subset of \(\mathbb{R}^2\) with non-empty interior is a unit ball of some Orlicz space \(l_2^\varphi\). For simplicity we consider two-dimensional space \(X\) from Example 4 with \(p = 1\). Clearly, the unit ball defined by the norm (3.6) is a compact symmetric convex non-empty subset of \(\mathbb{R}^2\) and therefore, due to Grzašlewicz result, it is a unit ball of some Orlicz space.

Let us take Orlicz space \(l_2^\varphi\), where

\[
\varphi(u) = \begin{cases} \frac{u}{\sqrt{u} - u}, & 0 \leq u \leq \sqrt{2}, \\ \frac{1}{\sqrt{u}}, & \frac{1}{\sqrt{2}} \leq u \end{cases}.
\]

It is simple to check that Luxemburg norm on \(l_2^\varphi\) which is given by

\[
\|(x_1, x_2)\|_\varphi = \inf\{k > 0 : \varphi\left(\frac{x_1}{k}\right) + \varphi\left(\frac{x_2}{k}\right) \leq 1\}
\]

coincides with the original norm (3.6) on \(X\). Indeed, without loss of generality let fix \(0 \leq x_1 \leq x_2\). Then, \(\|(x_1, x_2)\| = x_2 + \frac{1}{\sqrt{2}}\). To calculate \(\|(x_1, x_2)\|_\varphi = \inf\{k > 0 : \varphi\left(\frac{x_1}{k}\right) + \varphi\left(\frac{x_2}{k}\right) \leq 1\}\) we consider the following three cases obtained naturally from the representation of \(\varphi\), namely

\[
\|(x_1, x_2)\|_\varphi = \min\{\inf A, \inf B, \inf C\},
\]

where

\[
A = \left\{ k \geq 0 : \varphi\left(\frac{x_1}{k}\right) \leq 1, \frac{x_1}{k} \leq \sqrt{2} \right\},
\]

\[
B = \left\{ k \geq 0 : \varphi\left(\frac{x_2}{k}\right) \leq 1, \frac{x_2}{k} \leq \sqrt{2} \right\},
\]

\[
C = \left\{ k \geq 0 : \varphi\left(\frac{x_1}{k}\right) + \varphi\left(\frac{x_2}{k}\right) \leq 1 \right\}.
\]
where

\[ A = \{ k > 0 : k \leq \sqrt{2}x_1 \text{ and } \varphi\left(\frac{x_1}{k}\right) + \varphi\left(\frac{x_2}{k}\right) = \left(\frac{x_1 + x_2}{k}\right)(1 + \frac{1}{\sqrt{2}}) - \sqrt{2} \leq 1 \}. \]

\[ B = \{ k > 0 : \sqrt{2}x_1 \leq k \leq \sqrt{2}x_2 \text{ and } \varphi\left(\frac{x_1}{k}\right) + \varphi\left(\frac{x_2}{k}\right) = \frac{x_1}{\sqrt{2}k} + \frac{x_2}{k} \left(1 + \frac{\sqrt{2}}{2} - \frac{1}{\sqrt{2}} \right) \leq 1 \}. \]

\[ C = \{ k > 0 : k \geq \sqrt{2}x_2 \text{ and } \varphi\left(\frac{x_1}{k}\right) + \varphi\left(\frac{x_2}{k}\right) = \frac{x_1 + x_2}{k} \sqrt{2} \leq 1 \}. \]

The set \( A \) consists only of \( \sqrt{2}x_1 \) since \( \frac{x_1 + x_2}{k} \leq \frac{1 + \sqrt{2}}{1 + 1/\sqrt{2}} = \sqrt{2} \) and to satisfy \( \frac{x_1 + x_2}{k} \leq k \leq \sqrt{2}x_1 \leq \frac{x_1 + x_2}{\sqrt{2}2} \) we should have \( x_1 = x_2 \) and therefore \( k = \sqrt{2}x_1 \). Since \( \inf C \geq \sqrt{2}x_2 \) and in the set \( B \) we have \( \sqrt{2}x_1 \leq k \leq \sqrt{2}x_2 \) and \( k \geq x_2 + \frac{x_1}{1+\sqrt{2}} \), and also since \( \sqrt{2}x_1 \leq x_2 + \frac{x_1}{1+\sqrt{2}} \leq \sqrt{2}x_2 \) and \( x_1 + \frac{x_2}{1+\sqrt{2}} = \sqrt{2}x_1 \) it follows that \( \|(x_1, x_2)\|_\varphi = \inf B = x_2 + \frac{x_1}{1+\sqrt{2}} \) and the equality \( \|(x_1, x_2)\|_\varphi = \|(x_1, x_2)\| \) is proved.

Since the symmetric space \( X \) has the property \( \|T\|_c / \|T\| > 1 \) for \( T \in \mathcal{L}(X, X) \), then Orlicz space \( l^p_X \) which is isometric to \( X \) will inherit this property. So, for the bounded linear operator \( T(x, y) = (x + y, x - y) \) between Orlicz spaces \( l^p_X \) with \( \varphi \) defined by (3.7) real and complex norms are different.

**Remark 13.** Using the construction of Example 4 with \( 1 < p < 2 \) and the result of Grzegorek we can easily produce other examples of two-dimensional Orlicz spaces \( X \) for which the operator \( T(x, y) = (x + y, x - y) : X \to X \) has different real and complex norms.

**Problem 3.** Prove or disprove: Let \( X \) be an Orlicz space (or even a symmetric space). The equality \( \|T\|_c = \|T\| \) holds for arbitrary operator \( T \in \mathcal{L}(X, X) \) if and only if \( X = L^p(\mu) \) for some \( p > 0 \) and some positive \( \sigma \)-finite measure \( \mu \).

It is interesting to note that for any symmetric space \( X \) on \([0, 1]\) and any \( \theta \in \mathbb{R} \) we have \( \|\cos(2\pi t + \theta)\|_{X,[0,1]} = \|\cos 2\pi t\|_{X,[0,1]} \) due to equimeasurability of the functions \( \cos(2\pi t + \theta) \) and \( \cos 2\pi \theta \). Hence, the following lemma similar to Lemma 3 works.

**Lemma 6.** If \( X \) is a symmetric space on \([0, 1]\) and \( a, b \in \mathbb{R} \), then

\[ \|a \cos 2\pi t + b \sin 2\pi t\|_{X,[0,1]} = d_X \sqrt{a^2 + b^2}, \]

where \( d_X = \|\cos 2\pi t\|_{X,[0,1]} \).

**Remark 14.** Analyzing the proof of Theorem 1 we note that we applied twice Fubini theorem, namely each time we invoked Lemma 3. Applying twice Fubini theorem is essentially the same as changing the order of taking the norms in a mixed norm. The latter is possible in the case of \( L^p[L^p] \) space and by Kolmogorov-Nagumo theorem it is the only possibility among all mixed norm spaces composed of symmetric spaces (see Boccuto-Bukhvalov-Sambucini [7]).
CHAPTER FOUR

Basic Properties of Constant $\gamma_{X,Y}$

This chapter is devoted to the discussion of some basic properties of the constant $\gamma(X,Y)$, where spaces $X$ and $Y$ can be Lebesgue spaces, mixed norm Lebesgue spaces or spaces of functions of bounded $p$-variation. We also establish here that for quasi-Banach variants of these spaces the relation between complex and real norms of bounded linear operators between them is bounded by 2 as it is in the Banach case.

**Theorem 6** (cf. Verbickii [60]).

(i) If $0 < q \leq p \leq \infty$, then for arbitrary $T \in \mathcal{L}(L^p(\mu), L^q(\nu))$ we have

$$\|T_C\|_{p,q} \leq \frac{d_p}{d_q} \|T\|_{p,q}.$$  

Moreover, if $1 \leq q \leq p \leq \infty$, then $\|T_C\|_{p,q} \leq \min\{\frac{d_p}{d_q}, \frac{d_q'}{d_p'}\} \|T\|_{p,q}$.

(ii) If $0 < q \leq p \leq \infty$ and $p \geq 1$, then there exists an operator $F \in \mathcal{L}(L^p(\mu), L^q(\nu))$ such as

$$\|F_C\|_{p,q} \geq \max\{1, \frac{d_p^2}{d_p'd_q}\} \|F\|_{p,q}.$$  

**Proof.** (i) Using Lemma 3 we get

$$d_p^2 \|T_C(f + ig)\|^q_q = d_p^2 \|(Tf)^2 + |Tg|^2\|_q^q = \int_0^1 \|T(f \cos 2\pi t + g \sin 2\pi t)\|^q_q dt \leq \|T\|_{p,q}^q \int_0^1 \|f \cos 2\pi t + g \sin 2\pi t\|^q_q dt \leq \frac{d_p^2}{d_p} \|T\|_{p,q} \|f + ig\|_{p,q}.$$  

Thus, if $0 < q \leq p \leq \infty$, then $\|T_C\|_{p,q} \leq \frac{d_p}{d_q} \|T\|_{p,q}$. Moreover, considering the dual operator $T^* : L^q(\Omega_2, \nu) \rightarrow L^p(\Omega_1, \mu)$ and noticing that $\|T\| = \|T^*\|$ as well as $\|T_C\| = \|T_C^*\|$ we obtain by the same argument that $\|T_C\|_{p,q} \leq \frac{d_q}{d_p'} \|T\|_{p,q}$. Therefore,

$$\|T_C\|_{p,q} \leq \min\{\frac{d_p}{d_q}, \frac{d_q'}{d_p'}\} \|T\|_{p,q}.$$

(ii) Consider the operator $F : L^p(0,1) \rightarrow L^q(0,1)$ defined by

$$Ff(x) = a(f) \cos 2\pi x + b(f) \sin 2\pi x,$$

(4.1)
where \( a(f) = 2 \int_0^1 f(t) \cos 2\pi t \, dt \), \( b(f) = 2 \int_0^1 f(t) \sin 2\pi t \, dt \) and \( f \in L^p(0, 1) \). Applying Lemma 1 twice we get \( \|Ff\|_q = d_q (a^2(f) + b^2(f))^{1/2} = d_q/d_2 \|Ff\|_2 \). Consequently,

\[
(4.2) \quad \|F\|_{p,q} = d_q/d_2 \|F\|_{p,2}.
\]

By duality we obtain \( \|F\|_{p,2} = \|F^*\|_{2,p'} \) and \( F^* \) is defined by

\[
F^* \varphi(x) = a(\varphi) \cos 2\pi x + b(\varphi) \sin 2\pi x,
\]

where \( a(\varphi) = 2 \int_0^1 \varphi(t) \cos 2\pi t \, dt \) and \( b(\varphi) = 2 \int_0^1 \varphi(t) \sin 2\pi t \, dt \). Applying Lemma 1 gives

\[
\|F^*\varphi\|_{p'} = d_{p'}[a^2(\varphi) + b^2(\varphi)]^{1/2}.
\]

From Parseval identity it follows that the norm \( \|F^*\|_{2,p'} \) is attained on the function \( \varphi(x) = a(\varphi) \cos 2\pi x + b(\varphi) \sin 2\pi x \). The norm of this function in \( L^2(0,1) \) can be derived by using Lemma 1 again

\[
\|\varphi\|_2 = (\int_0^1 |a(\varphi) \cos 2\pi t + b(\varphi) \sin 2\pi t|^2 \, dt)^{1/2} = d_2[a^2(\varphi) + b^2(\varphi)]^{1/2}.
\]

Thus, \( \|F\|_{p,2} = \|F^*\|_{2,p'} = \|F^*\varphi\|_{p'}/\|\varphi\|_2 = d_{p'}/d_2 \). Substituting this expression into the equality \( (4.2) \) we obtain

\[
(4.3) \quad \|F\|_{p,q} = d_{p'}/d_q/d_2.
\]

Let \( F_C \) be a complexification of operator \( F \), then the expression \( (4.1) \) gives \( F_C(e^{2\pi i x}) = F(\cos 2\pi x) + iF(\sin 2\pi x) = \cos 2\pi x + i \sin 2\pi x = e^{2\pi i x} \), that implies \( \|F_C\|_{p,q} \geq \|F_C(e^{2\pi i x})\|_q/\|e^{2\pi i x}\|_p = \|e^{2\pi i x}\|_q/\|e^{2\pi i x}\|_p = 1 \). Now, since \( \|F_C\|_{p,q} \geq 1 = d_2^2/(d_{p'}d_q) \|F\|_{p,q} \) by the equality \( (4.3) \) and \( \|F_C\|_{p,q} \geq \|F\|_{p,q} \), we have

\[
\|F_C\|_{p,q} \geq \max(1, d_2^2/(d_{p'}d_q)) \|F\|_{p,q}.
\]

**Corollary 4.** If \( p \geq 2 \), then \( \gamma_{p,2} = d_2/d_{p'} \) and if \( 0 < q \leq 2 \), then \( \gamma_{2,q} = d_2/d_q \).

**Proof.** By Theorem 6(i) we have \( \gamma_{p,2} \leq \min\{d_p/d_2, d_2/d_{p'}\} \leq d_2/d_{p'} \) as well as by the preceding remark \( \gamma_{2,q} \leq d_2/d_q \). Besides, Theorem 6(ii) states that there exist operators \( F \) and \( G \) with the properties \( \|F_C\|_{2,p}/\|F\|_{2,p'} \geq d_2/d_{p'} \) and \( \|G_C\|_{2,q}/\|G\|_{2,q} \geq d_2/d_q \). These facts yield the identities of this corollary.

**Corollary 5.** If \( 1 \leq p, q \leq \infty \), then \( \gamma_{p,q} \leq \frac{\pi}{2} \).

**Proof.** By Theorem 6 and Lemma 2 we have

\[
\gamma_{p,q} \leq \min\{\frac{d_p}{d_q}, \frac{d_{p'}}{d_{p''}}\} \leq \frac{d_{\infty}}{d_1} = \frac{\pi}{2}.
\]

**Lemma 7.** If \( 1 \leq q < p \), then

\[
\min\{\frac{d_p}{d_q}, \frac{d_{p'}}{d_{p''}}\} = \begin{cases} \frac{d_q}{d_p} & \text{if } \frac{1}{p} + \frac{1}{q} \geq 1 \\ \frac{d_{p'}}{d_{p''}} & \text{if } \frac{1}{p} + \frac{1}{q} < 1 \end{cases}
\]

**Proof.** Let the operator \( F : L^p(0,1) \to L^p(0,1) \) be defined by equality \( (4.1) \) from Theorem 6. By relation \( (4.3) \) it follows

\[
(4.4) \quad \|F\|_{p,p} = d_p d_{p'}/d_2^2.
\]
4. BASIC PROPERTIES OF CONSTANT $\gamma_{X,Y}$

Consider the case when $1/p + 1/q \geq 1$. This inequality obviously implies $1/q' \leq 1/p$ and therefore $q' \geq p > p$. Taking $\theta = \frac{q' - q}{q' - p}$ we can easily check that $0 < \theta < 1$ as well as that $\theta$ satisfies the equality $\frac{1}{p} = \frac{1}{q'} + \frac{\theta}{q}$. Now we can apply Riesz-Thorin interpolation theorem (see Chapter 7) and get

$$
\|F\|_{p,p} \leq \|F\|^{\frac{1}{q'}}_{q',q'} \|F\|^{\theta}_{q,q} = \|F\|_{q,q},
$$

that due to relation (4.4) is equivalent to the inequality $d_q/d_{p'} \geq d_p/d_q$. The case when $1/p + 1/q < 1$ can be considered analogously. 

We note that by the remark which follows Lemma 2 we obtain

$$
d_p/d_q = \sqrt{\pi^{1/q-1/p}} \left( \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{q+1}{2}\right)} \right)^{1/p} \left( \frac{\Gamma\left(\frac{q+2}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)} \right)^{1/q}.
$$

Denote by $T^{p,n}$ the subspace of $L^p[0,2\pi]$ consisting of all trigonometric polynomials $t_n(x) = a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$. Using Bernstein lemma Verbickii [60] found the best constant $\gamma(T^{p,n}, T^{q,m})$ for all $1 \leq p, q \leq \infty$ and the extension of this result to $0 < p, q \leq \infty$ is straightforward.

**LEMMA 8 (Bernstein lemma).** If $1 \leq p < \infty$, then

$$
\|a_n \cos nx + b_n \sin nx\|_p \leq \|t_n\|_p
$$

for any trigonometric polynomial $t_n \in T^{p,n}$.

**PROOF.** (cf. Achieser [1, pp. 12-13]). First we assume that $1 < p < \infty$ and use the fact that under this condition the space $L^p$ is strictly convex. Then we take the trigonometric polynomial $p_n = a_0 + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx$ with fixed $a_n$ and $b_n$ and with other coefficients taken in such a way that the norm $\|p_n\|_p$ is minimal. By the periodicity of the trigonometric functions we have for all $y \in \mathbb{R}$

$$
\int_{0}^{2\pi} |p_n(x)|^p \, dx = \int_{0}^{2\pi} \sum_{k=1}^{n} |p_k(x)|^p \, dx = \int_{0}^{2\pi} |p_n(x+y)|^p \, dx.
$$

Take $y = \pi/n$ and observe that

$$
p_n(x + \pi/n) = -\alpha \cos nx - \beta \sin nx + a'_n \cos(n-1)x + b'_{n-1} \sin(n-1)x + ... + a'_1 \cos x + b'_1 \sin x + a_0.
$$

Since the space $L^p$ is strictly convex and $p_n$ was chosen with the minimal norm, then we should have

$$
p_n(x + \pi/n) = -p_n(x)
$$

(4.5)

and consequently

$$
p_n(x + 2\pi/n) = p_n(x).
$$

(4.6)

The latter expression proves that $p_n(x)$ has the period $2\pi/n$ and hence

$$
a_{n-1} = b_{n-1} = ... = a_1 = b_1 = 0,
$$

while the relation (4.5) gives that

$$
a_0 = 0.
$$
Thus, for \( p > 1 \)
\[
\|a_n \cos n x + b_n \sin n x\|_p \leq \|a_0 + \sum_{k=1}^{n} a_k \cos k x + b_k \sin k x\|_p.
\]
By continuity principle we also have that Bernstein lemma is true for \( p = 1 \) and \( p = \infty \). 

**Proposition 8.** For any \( m, n \in \mathbb{N} \) the following relation holds:
\[
\gamma(T^{p,n}, T^{q,m}) = \begin{cases} 
1, & \text{if } 1 \leq p \leq q \leq \infty \\
d_p/d_q, & \text{if } 0 < q \leq p \leq \infty \text{ and } 1 \leq p
\end{cases}
\]

**Proof.** The case when \( 1 \leq p \leq q \leq \infty \) can be derived by the same arguments as those used in the proof of Theorem 1 for Lebesgue spaces. Now, take \( p \) and \( q \) such as \( 0 < q < p \leq \infty \) and \( 1 \leq p \) and consider the operator \( F \in \mathcal{L}(T^{p,n}, T^{q,m}) \) given by
\[
Ft_n(x) = a_n \cos mx + b_n \sin mx.
\]
As usual, denote by \( F_C \) the complexification of \( F \), then
\[
F_C e^{inx} = F(\cos nx) + iF(\sin nx) = \cos mx + i \sin mx = e^{inx}.
\]
The evident equality \( |e^{i\theta}| = 1 \) for any \( \theta \in \mathbb{R} \) implies \( \|e^{i\theta}\|_p = (2\pi)^{1/p} \). Hence,
\[
(4.7) \quad \|F_C\|_{p,q} \geq \|F_C e^{i\theta}\|_{q/d_p} = (2\pi)^{1/q - 1/p}.
\]
Furthermore, by Bernstein lemma we have \( \|t_n\|_p \geq \|a_n \cos nx + b_n \sin nx\|_p \) and \( \|a \cos nx + b \sin nx\|_p = (2\pi)^{1/q} d_p(a^2 + b^2)^{1/2} \) is easily derived from Lemma 1. Therefore, for \( a_n^2 + b_n^2 > 0 \) we have
\[
\frac{\|Ft_n\|_q}{\|t_n\|_p} \leq \frac{\|a_n \cos mx + b_n \sin mx\|_q}{\|a_n \cos nx + b_n \sin nx\|_p} = (2\pi)^{1/q - 1/p} d_q/d_p
\]
and taking the supremum of the both sides of this inequality over \( t_n \in T^{p,n} \) with \( \|t_n\|_p = 1 \) we get
\[
(4.8) \quad \|F\|_{p,q} \leq (2\pi)^{1/q - 1/p} d_q/d_p.
\]
The combination of relations (4.7) and (4.8) gives \( \|F_C\|_{p,q} \|F\|_{p,q} \geq d_p/d_q \). At the same time, Theorem 6(i) states that \( \|T_C\|_{p,q} \|T\|_{p,q} \leq d_p/d_q \) for \( 0 < q < p \leq \infty \) and all \( T \in \mathcal{L}(T^{p,n}, T^{q,m}) \). Thus, \( \|T_C\|_{p,q} \|T\|_{p,q} = d_p/d_q \) for \( 0 < q \leq p \leq \infty \) and \( 1 \leq p \) and the proof is complete.

**Remark 15.** It could be expected, that \( \gamma(T^{p,n}, T^{q,m}) \) approaches \( \gamma_{p,q} \) as \( n \) and \( m \) tend to infinity, but by Theorem 6 it is not true, at least in the case when \( 1/p + 1/q < 1 \). We also note that \( d_{\infty}/d_{1} = \pi/2 > \sqrt{2} = \gamma_{\infty,1} \).

In the following theorem we consider the constants \( \gamma_{X,Y} \), where \( X \) and \( Y \) are mixed (quasi-)norm Lebesgue spaces.

**Theorem 7.** If \( 0 < \min\{r,s\} \leq \max\{p,q\} \leq \infty \), then for arbitrary operator \( T \in \mathcal{L}(L^p[L^q], L^s[L^r]) \) we have
\[
\|T_C\| \leq d_{\max\{p,q\}/d_{\min\{r,s\}}} \|T\|.
\]
Moreover, if \( 1 \leq \min\{r,s\} \leq \max\{p,q\} \leq \infty \), then \( \|T_C\| \leq \min\{d_{\max\{p,q\}/d_{\min\{r,s\}}} \|T\|, d_{\max\{r,s\}/d_{\min\{p,q\}}} \|T\| \}. \)
PROOF. Set $\tau = \min\{r, s\}$. Repeating the first step of the proof of Theorem 2 we obtain
\[
d^{\tau}_{\tau} \| T_c(f + ig) \|_{\tau, \tau}^\tau \leq \int_0^1 \| T(f \cos 2\pi t + g \sin 2\pi t) \|_{\tau, \tau}^\tau dt.
\]
Now, using boundedness of the operator $T$, Hölder inequality, Fubini theorem and assuming that $\tau \leq p$ we have
\[
d^{\tau}_{\tau} \| T_c(f + ig) \|_{\tau, \tau}^\tau \leq \| T \|_{\tau} \int_0^1 \| f \cos 2\pi t + g \sin 2\pi t \|_{q, p}^\tau dt
\]
\[
= \| T \|_{\tau} \int_0^1 \left( \int_{\Omega_2} \| f \cos 2\pi t + g \sin 2\pi t \|_q^p d\mu_2 \right)^{\tau/p} dt
\]
\[
\leq \| T \|_{\tau} \left( \int_0^1 \int_{\Omega_2} \| f \cos 2\pi t + g \sin 2\pi t \|_q^p dt \right)^{\tau/p} d\mu_2
\]
\[
= \| T \|_{\tau} \left( \int_{\Omega_2} \left( \int_{\Omega_1} \| f \cos 2\pi t + g \sin 2\pi t \|_q^p dt \right)^{\tau/p} \right) d\mu_2
\]
\[
= d^{\tau}_{\tau} \| f + ig \|_{q, p}^\tau.
\]
If $p \geq q$, then
\[
d^{\tau}_{\tau} \| T_c(f + ig) \|_{\tau, \tau}^\tau \leq \| T \|_{\tau} \left( \int_{\Omega_2} \left( \int_{\Omega_1} \| f \cos 2\pi t + g \sin 2\pi t \|_{L^p[0, 1]}^p \right)^{\tau/p} \right) d\mu_2
\]
\[
\leq \| T \|_{\tau} \left( \int_{\Omega_2} \left( \int_{\Omega_1} \| f \cos 2\pi t + g \sin 2\pi t \|_{L^p[0, 1]}^p \right)^{\tau/p} \right) d\mu_2
\]
\[
= \| T \|_{\tau} \left( \int_{\Omega_2} \left( \int_{\Omega_1} \| f \cos 2\pi t + g \sin 2\pi t \|_{L^p[0, 1]}^p \right)^{\tau/p} \right) d\mu_2
\]
\[
= d^{\tau}_{\tau} \| f + ig \|_{q, p}^\tau.
\]
If conversely, $q \geq p$, then by the Hölder inequality and Fubini theorem
\[
d^{\tau}_{\tau} \| T_c(f + ig) \|_{\tau, \tau}^\tau \leq \| T \|_{\tau} \left( \int_{\Omega_2} \left( \int_{\Omega_1} \| f \cos 2\pi t + g \sin 2\pi t \|_{L^q[0, 1]}^q \right)^{\tau/p} \right) d\mu_2
\]
\[
\leq \| T \|_{\tau} \left( \int_{\Omega_2} \left( \int_{\Omega_1} \| f \cos 2\pi t + g \sin 2\pi t \|_{L^q[0, 1]}^q \right)^{\tau/p} \right) d\mu_2
\]
\[
= \| T \|_{\tau} \left( \int_{\Omega_2} \left( \int_{\Omega_1} \| f \cos 2\pi t + g \sin 2\pi t \|_{L^q[0, 1]}^q \right)^{\tau/p} \right) d\mu_2
\]
\[
= d^{\tau}_{\tau} \| f + ig \|_{q, p}^\tau.
\]
If $q \geq \tau \geq p$, then by the integral Minkowski and Hölder inequalities we obtain
\[
d^{\tau}_{\tau} \| T_c(f + ig) \|_{\tau, \tau}^\tau \leq \| T \|_{\tau} \int_0^1 \| f \cos 2\pi t + g \sin 2\pi t \|_{q, p}^\tau dt
\]
4. BASIC PROPERTIES OF CONSTANT $\gamma_{X,Y}$

\[
= \|T\|^{\tau/p} \left( \int_{\Omega_2} \left( \int_{0}^{1} \left( \int_{0}^{1} \left| f \cos 2 \pi t + g \sin 2 \pi t \right|^q \mu_1 \right) \right) \right)^{\tau/q} \mu_2 dt \\
= \|T\|^{\tau/p} \left( \int_{\Omega_2} \left( \int_{0}^{1} \left| f \cos 2 \pi t + g \sin 2 \pi t \right|^q \mu_1 \right) \right)^{\tau/q} \mu_2 dt \\
\leq \|T\|^{\tau/p} \left( \int_{\Omega_2} \left( \int_{0}^{1} \left| f \cos 2 \pi t + g \sin 2 \pi t \right|^q \mu_1 \right) \right)^{\tau/q} \mu_2 dt \\
\leq \|T\|^{\tau/p} \left( \int_{\Omega_2} \left( \int_{0}^{1} \left| f \cos 2 \pi t + g \sin 2 \pi t \right|^q \mu_1 \right) \right)^{\tau/q} \mu_2 dt \\
= d_q^\tau \|T\|^{\tau} \|f + ig\|_{q,p}^\tau .
\]

If $1 \leq \min\{r, s\} = \max\{p, q\} \leq \infty$, then by duality we also obtain $\|T_c\|/\|T\| \leq d_{\max\{r', s'\}}/d_{\min\{p', q'\}}$.

**Remark 16.** By induction we get $\|T\| \leq d_{\max\{p_1, \ldots, p_n\}}/d_{\min\{q_1, \ldots, q_n\}}$ $\|T\|$ for any $T \in L^p([L^q], \ldots, [L^q])$, where $0 < \min\{q_1, \ldots, q_n\} \leq \max\{p_1, \ldots, p_n\} \leq \infty$.

**Problem 4.** Define the conditions on $0 < p, q, r, s \leq \infty$ such that the following equality holds

\[\gamma(L^p[L^q], L^r[L^s]) = \{d_{\max\{p, q\}}/d_{\min\{r, s\}}, d_{\min\{r', s'\}}/d_{\max\{p', q'\}}\} .\]

In the next theorem we summarize the known and just obtained properties of the constant $\gamma_{X,Y}$.

**Theorem 8.** (i) (Krivine [34]) $\gamma_{\infty, 1} = \sqrt{2}$.
(ii) (Krivine [34], Gasch-Maligranda [19]) $\gamma_{p,q}$ is increasing in $p$ and decreasing in $q$ for $1 \leq p, q \leq \infty$ and $\gamma_{p,q} \leq \gamma_{\infty, 1}$.
(iii) If $1 \leq 0 < p < q \leq \infty$, then $\gamma_{p,q} = \gamma_{q', p'}$.
(iv) $\gamma_{p,q} = 1$ if and only if $0 < p \leq q \leq \infty$.
(v) (Verbičkii [60]) If $0 < q \leq p \leq \infty$, then $1 \leq \gamma_{p,q} \leq d_p/d_q$. Moreover, if $1 \leq q \leq p \leq \infty$, then $\max\{1, d_q^p/d_p^q\} \leq \gamma_{p,q} \leq \min\{d^p_q, d^q_p\}$.
(vi) If $p \geq 2$, then $\gamma_{p,2} = d_2/d_p$ and if $0 < q \leq 2$, then $\gamma_{2,q} = d_2/d_q$.
(vii) (Defant [14]) If either $1 \leq q \leq p \leq 2$ or $2 \leq q \leq p \leq \infty$, then $\gamma_{p,q} = \min\{d^p_q, d^q_p\}$.
(viii) (Verbičkii [60]) Let $2 \leq m, n \leq \infty$. If $0 < p \leq q \leq \infty$, then $\gamma(T^{p,n}, T^{q,m}) = 1$, if conversely $0 < q \leq \infty$, then $\gamma(T^{p,n}, T^{q,m}) = d_p/d_q$.
(ix) If $0 < \min\{r, s\} \leq \max\{p, q\} \leq \infty$, then $\gamma(L^p[L^q], L^r[L^s]) \leq d_{\max\{p, q\}}/d_{\min\{r, s\}}$.
Moreover, if $1 \leq \min\{r, s\} \leq \max\{p, q\} \leq \infty$, then $\gamma(L^p[L^q], L^r[L^s]) \leq \min\{d_{\max\{p, q\}}/d_{\min\{r, s\}}, d_{\min\{r', s'\}}/d_{\max\{p', q'\}}\}$.
(x) If \(0 < p \leq q \leq \infty\) and \(X, Y\) are (quasi-)Banach lattices which are \(p\)-concave with constant one and \(q\)-convex with constant one, respectively, then \(\gamma_{X,Y} = 1\).

It is well known that if \(0 < q < 1\), then the inequality \(\|f + g\|_q \leq 2^{1/q - 1} (\|f\|_q + \|g\|_q)\) holds for arbitrary \(f, g \in L^q(\mu)\) and the constant \(2^{1/q - 1}\) is the best possible. Using this form of the triangle inequality in the proof of Proposition 3 we get for \(0 < p, q \leq \infty\)

\[
\gamma_{p,q} \leq \max\{2, 2^{1/q}\}.
\]

When \(q\) goes to zero the expression \(\max\{2, 2^{1/q}\}\) tends to infinity. The following theorem significantly improves this estimate of constant \(\gamma_{p,q}\) as well as the estimates of constant \(\gamma_{X,Y}\) for some other (quasi-)Banach spaces \(X\) and \(Y\).

**Theorem 9.** If \(0 < p, q \leq \infty\), then \(\gamma(BV_p[a,b], BV_q[a,b]) \leq 2\) and \(\gamma_{X,Y} \leq 2\), where \(X, Y\) are (quasi-)Banach lattices which are \(p\)-concave with constant one and \(q\)-convex with constant one, respectively (thus, \(\gamma_{p,q} \leq 2\)). Moreover, if \(0 < p, q, r, s \leq \infty\), then \(\gamma(L^p[L^q], L^r[L^s]) \leq 2\).

**Proof.** In all these cases we have \(\gamma_{X,Y} \leq d_\infty / \lim_{q \to 0^+} d_q\) by Remarks 10, 11 and Theorems 6 and 2. Using Lemma 2 we get that the latter expression is bounded by \(1 / \lim_{q \to 0^+} d_q\). If we show that \(\lim_{q \to 0^+} d_q = 1/2\), then the statement of the theorem will easily follows. To calculate \(\lim_{q \to 0^+} d_q = \lim_{q \to 0^+} \left(\int_0^1 |\cos 2\pi t|^q dt\right)^{1/q}\) we use the fact that \(\lim_{q \to 0^+} \|\cos 2\pi t\|_q = e^{\int_0^\pi \ln |\cos 2\pi t| dt}\) (see Rudin [51], p. 71). Denoting \(\int_0^{1/4} \ln \cos(2\pi t) dt = I_1\) and \(\int_0^{1/4} \ln \sin(2\pi t) dt = I_2\) we get

\[
I_1 = -\int_0^{1/4} \ln \sin(2\pi t) dt = \int_0^{1/4} \ln \sin(2\pi t) dt
\]

Thus,

\[
2I_1 = I_1 + I_2 = \int_0^{1/4} \ln \frac{\sin(4\pi t)}{2} dt = \int_0^{1/4} \ln \sin(4\pi t) - 2\ln 2 dt
\]

\[
= -\frac{\ln 2}{4} + \int_0^{1/4} \ln \sin(4\pi t) dt = \{2t = \tau, 2dt = d\tau\}
\]

\[
= -\frac{\ln 2}{4} + \frac{1}{2} \int_0^{1/2} \ln \sin(2\pi \tau) d\tau
\]

\[
= -\frac{\ln 2}{4} + \frac{1}{2} \left(\int_0^{1/4} \ln \sin(2\pi \tau) d\tau + \int_{1/4}^{1/2} \ln \sin(2\pi \tau) d\tau\right)
\]

\[
= -\frac{\ln 2}{4} + \int_0^{1/4} \ln \sin(2\pi \tau) d\tau = -\frac{\ln 2}{4} + I_1.
\]

Therefore, \(I_1 = -1/4 \ln 2\) and \(\lim_{q \to 0^+} d_q = e^{-\ln 2} = 1/2\).
Clarkson Inequality in the Real and the Complex Case

In this chapter we consider the so called generalized Clarkson inequality

\[(|a+b|^q + |a-b|^q)^{1/q} \leq C (|a|^p + |b|^p)^{1/p},\]

where \(0 < p, q \leq \infty\) and the scalars \(a, b\) are real or complex. This inequality is a simple generalization of the classical Clarkson inequality which is used in the proof of the uniform convexity of \(L^p\) spaces for \(1 < p < \infty\) [11]. We note that the best constant in this generalized inequality is also the norm of a non-positive operator given by the matrix \(T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\) or by the rule \(T(a, b) = (a + b, a - b)\) and acting between two-dimensional (quasi-)Banach spaces \(l^p_2\) and \(l^q_2\), i.e. \(C = C_{p,q}(\mathbb{R}) = \|T\|_{p,q}\) in the real case and \(C = C_{p,q}(\mathbb{C}) = \|T_\mathbb{C}\|_{p,q}\) in the complex case. This operator was used by Thorin to show the impossibility of extending the real Riesz convexity theorem to a convex region beyond and containing the “lower triangle”. Moreover, this operator as an operator between some other two-dimensional spaces appeared in Examples 2, 4 and we will use it again in a number of statements when we will need to know the exact value of its norms both in the real and the complex case. In this chapter we also present some other applications of the generalized Clarkson inequality, including the construction of an operator between Lebesgue spaces with non-atomic measures for which real and complex norms will differ.

5.1. The best constant in the complex case

We will look first at a generalized complex Clarkson inequality. The result is, in fact, known but our formulation of Theorem 10 and its proof will help to understand better the statement and the proof of the new result in Theorem 11 as well as the applications of both theorems presented in the third section of this chapter.

**Theorem 10.** Let \(0 < p, q \leq \infty\). Then the best constant in the generalized complex Clarkson inequality

\[(|a+b|^q + |a-b|^q)^{1/q} \leq C (|a|^p + |b|^p)^{1/p} \ \forall a, b \in \mathbb{C}\]

is \(C = C_{p,q}(\mathbb{C}) = \max\{2^{1-1/p}, 2^{1/q}, 2^{1/q-1/p+1/2}\}\).

Clarkson [11] proved that \(C_{p,p'}(\mathbb{C}) = 2^{1/p'}\) for \(1 \leq p \leq 2\), where \(p'\) is the conjugate exponent to the number \(p\), that is, \(1/p + 1/p' = 1\). Later on the best constants \(C_{p,q}(\mathbb{C})\) for the remaining pairs of \(p\) and \(q\) such that \(0 < p, q \leq \infty\) were found by Koskela [30] and Maligranda-Persson [41].
Proof. We carry out the proof in three steps. First step. For $1 \leq p \leq 2$ the following classical complex Clarkson inequality (cf. [11]) holds

\begin{equation}
(5.2) \quad \left( |a + b|^{p'} + |a - b|^{p'} \right)^{1/p'} \leq 2^{1/p'} \left( |a|^p + |b|^p \right)^{1/p} \quad \forall a, b \in \mathbb{C}.
\end{equation}

Really, we can calculate the norm of the operator $T$ in two easy cases: $T : l_2^1 \to l_2^\infty$ has the norm $\|T\|_{1, \infty} = \sup_{\|x\|=1} \max(\|y\|, \|x - y\|) = 1$ and $T : l_2^2 \to l_2^2$ has the norm $\|T\|_{2, 2} = \sup_{\|x\|^2 + \|y\|^2 = 1} \left( |x + y|^2 + |x - y|^2 \right)^{1/2} = \sqrt{2}$. Then, by the complex Riesz-Thorin interpolation theorem, $T : l_2^a(\mathbb{C}) \to l_2^b(\mathbb{C}) = l_2^{p'}(\mathbb{C})$ ($1 \leq p \leq 2$) is bounded with the norm

$$
\|T\|_{p, p'} \leq \|T\|_{1, \infty}^{1-2/p'} \|T\|_{2, 2}^{2/p'} = 2^{1/p'},
$$

since $p$ and $q$ are defined by

$$
\frac{1}{p} = \frac{1 - \theta}{2}, \quad \frac{1}{q} = \frac{1 - \theta}{\infty} + \frac{\theta}{2} \quad \text{and} \quad 0 \leq \theta \leq 1.
$$

Second step. For fixed $a, b \in \mathbb{C}$ let $A_p = (|a|^p + |b|^p)^{1/p}$ and $B_p = (|a + b|^p + |a - b|^p)^{1/p}$. Then $A_p, B_p$ are increasing and $2^{-1/p}A_p, 2^{-1/p}B_p$ are increasing in $p > 0$. The proof of these statements is standard.

Third step. Combine the first two steps by considering special four cases for $p$ and $q$:

I. $0 < q \leq 2 \leq p \Rightarrow B_q \leq 2^{1/q - 1/2}B_2 = 2^{1/q}A_2 < 2^{1/q - 1/p + 1/2}A_p$.

II. $q \geq 2, p \geq q' \Rightarrow B_q \leq 2^{1/q}A_q \leq 2^{1/q - 1/q' - 1/p}A_p = 2^{1 - 1/p}A_p$.

IIIa. $p, q \leq 2 \Rightarrow B_q \leq 2^{1/q - 1/2}B_2 = 2^{1/q}A_2 < 2^{1/q}A_p$.

IIIb. $q \geq 2, q' \geq p \Rightarrow B_q \leq 2^{1/q}A_q \leq 2^{1/q}A_p$.

The equalities hold in (I), (II) and (IIIa,b) for $(a, b) = (1, i), (a, b) = (1, 1)$ and $(a, b) = (1, 0)$, respectively.

As we see the classical complex Clarkson inequality (5.2) is an important estimate in the above proof. This estimate was of particular interest in a number of papers. After Clarkson paper [11] several different proofs of this inequality appeared in literature (see [48], [47], pp. 534–558 and [53]). All these proofs have in common that they first reduce the problem to one complex variable and then to real variables by different types of transformations.

As an example, let us assume, without loss of generality, that $|b| \leq |a|$ in (5.2). Dividing the both sides of (5.2) by $|a|$ and denoting $b/a$ by $z$ we get

\begin{equation}
|1 + z|^{p'} + |1 - z|^{p'} \leq 2(1 + |z|^p)^{p' - 1}.
\end{equation}

After replacing $z$ by $re^{i\theta}$ with $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$ inequality (5.3) becomes

\begin{equation}
|1 + re^{i\theta}|^{p'} + |1 - re^{i\theta}|^{p'} \leq 2(1 + r^p)^{p' - 1}.
\end{equation}

Now consider the function $g_r(\theta)$ for a fixed $r \in [0, 1]$ given by

$$
g_r(\theta) = |1 + re^{i\theta}|^{p'} + |1 - re^{i\theta}|^{p'} = (1 + r^2 + 2r \cos \theta)^{p'/2} + (1 + r^2 - 2r \cos \theta)^{p'/2}.
$$

Its derivative is

$$
g'_r(\theta) = -p'r \sin \theta \left[ (1 + r^2 + 2r \cos \theta)^{p'/2 - 1} - (1 + r^2 - 2r \cos \theta)^{p'/2 - 1} \right].
$$
and \( g_r'(\theta) = 0 \) if and only if \( \theta = \frac{k\pi}{2} \) for \( k = 0, 1, 2, 3, 4 \). Moreover, \( g_r \) is decreasing on intervals \([0, \frac{\pi}{2}], [\pi, \frac{3\pi}{2}] \) and increasing on \([\frac{3\pi}{2}, 2\pi] \). Therefore,

\[
\max_{\theta \in [0, 2\pi]} g_r(\theta) = g_r(0) = g_r(\pi) = (1 + r)^p + (1 - r)^p
\]

and it only remains for us to prove (5.4) for \( \theta = 0 \), that is

\[
(5.5) \quad (1 + x)^p + (1 - x)^p \leq 2(1 + x^p)^{p' - 1}
\]

for all \( 0 \leq x \leq 1 \) and \( 1 < p < 2 \) (which is called real Clarkson inequality).

Probably the shortest and the simplest among all other proofs of inequality (5.5) is one due to Friedrichs [18]. The shortness of this proof is based on the consideration of the following function

\[
f(\alpha, x) = (1 + \alpha^{1-p'} x)(1 + \alpha x)^{p'-1} + (1 - \alpha^{1-p'} x)(1 - \alpha x)^{p'-1}
\]

of real variables \( 0 \leq \alpha, x \leq 1 \). Obviously,

\[
f(1, x) = (1 + x)^p + (1 - x)^p, \quad f(x^{p-1}, x) = 2(1 + x^p)^{p' - 1}
\]

and

\[
f'_a(\alpha, x) = (p' - 1)x(1 - \alpha^{-p'}) g(\alpha, x), \quad g(\alpha, x) = (1 + \alpha x)^{p'-2} - (1 - \alpha x)^{p'-2}.
\]

Clearly, for \( p' > 2 \) we have \( g(\alpha, x) \geq 0 \), that in turn implies \( f'_a(\alpha, x) \leq 0 \), since \( 1 - \alpha^{-p'} \leq 0 \). Thus, the function \( f(\alpha, x) \) is decreasing in \( \alpha \). Noting that \( x^{p-1} \leq 1 \) we have

\[
f(1, x) \leq f(x^{p-1}, x) \quad \text{that is the same as} \quad (1 + x)^p + (1 - x)^p \leq 2(1 + x^p)^{p' - 1}.
\]

**Remark 17.** The proof of the classical complex Clarkson inequality by using the complex Riesz-Thorin interpolation theorem gives possibility to do some generalizations of the classical complex Clarkson inequality to the higher dimensions (see [63], [41], [26], [27] and [28]).

### 5.2. The best constant in the real case

Now, we consider a generalized real Clarkson inequality:

\[
(5.6) \quad |a + b|^q + |a - b|^q}^{1/q} \leq C (|a|^p + |b|^p)^{1/p} \quad \forall a, b \in \mathbb{R}.
\]

The problem is how to compute the best constant \( C = C_{p,q}(\mathbb{R}) \) in this inequality for all \( 0 < p, q \leq \infty \). As in the case of generalized complex Clarkson inequality (5.1) the best constant is also the norm of the operator \( T \) but acting now between real two-dimensional spaces \( L^p_q(\mathbb{R}) \) and \( L^q_2(\mathbb{R}) \) given by the formula \( T(a, b) = (a+b, a-b) \).

**Theorem 11.** If \( 0 < p, q \leq \infty \), then the best constant in inequality (5.6) is

\[
(5.7) \quad C_{p,q}(\mathbb{R}) = \max\{2^{1-1/p}, 2^{1/q}\} \quad \text{with} \quad B_{p,q} = \sup_{x \in [0,1]} \frac{[(1 + x)^q + (1 - x)^q]^{1/q}}{(1 + x^p)^{1/p}}.
\]

Moreover, if either \( p \leq 2 \) or \( q \geq 2 \) or \( p = \infty \) or \( 0 < q \leq 1 \), then \( C_{p,q}(\mathbb{R}) = \max\{2^{1-1/p}, 2^{1/q}\} \). Also, if \( 1 < q < 2 < p < \infty \), then \( C_{p,q}(\mathbb{R}) = B_{p,q} \) and \( \max\{2^{1-1/p}, 2^{1/q}\} < B_{p,q} \leq 2^{1/q-1/p+1/2} \).
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Proof. If either 0 < p ≤ 2 or 2 ≤ q ≤ ∞, then 1/q − 1/p + 1/2 ≤ max{1 − 1/p, 1/q} and thus C_{p,q}(C) = max{2^{1−1/p}, 2^{1/q}}. Since the constant C_{p,q}(C) is attained on real numbers we obtain that C_{p,q}(C) = C_{p,q}(R) = max{2^{1−1/p}, 2^{1/q}} (see Theorem 10).

In the case when p = ∞ we have
\[ C_{∞,q}(R) = \sup_{a,b \in \mathbb{R}, \max\{|a|,|b|\}=1} \{(|a+b|^q + |a-b|^q)^{1/q} = \max\{2,2^{1/q}\}. \]

Furthermore, if 0 < q < p < ∞, then
\[ C_{p,q}(R) = \sup_{a,b \in \mathbb{R}, |a|^p+|b|^p \neq 0} \frac{|a+b|^q + |a-b|^q}{(|a|^p + |b|^p)^{1/p}} = \sup_{x \in [0,1]} \frac{[(1+x)^q + (1-x)^q]^{1/q}}{(1+x^p)^{1/p}}. \]

For 0 < q ≤ 1 the function
\[ F(x) = \frac{[(1+x)^q + (1-x)^q]^{1/q}}{(1+x^p)^{1/p}}, \quad x \in [0,1] \]

is decreasing on [0, 1] since its derivative
\[ F'(x) = \frac{[(1+x)^q + (1-x)^q]^{1/q-1}}{(1+x^p)^{1/p}} \left\{ [(1+x)^q-1-(1-x)^q-1] (1+x^p)-x^{p-1}[(1+x)^q+(1-x)^q] \right\} \]
is strictly less than zero and thus
\[ C_{p,q}(R) = B_{p,q} = F(0) = 2^{1/q}. \]

In the case 1 < q < 2 < p < ∞ we will prove in the next theorem that there exists a unique x_0 ∈ (0, 1) at which F has its maximum. Moreover, we will see that
\[ \max\{2^{1−1/p}, 2^{1/q}\} < B_{p,q} < 2^{1/q−1/p+1/2}. \]

**Remark 18.** Let 1 ≤ p, q ≤ ∞. If dim L^p(μ) ≥ 3 and dim L^q(ν) ≥ 3, then the norm of any bounded linear operator T : L^p(μ) → L^q(ν) acting between real spaces and the norm of its natural complexification T_{C} = T + iT : L^p_{C}(μ) → L^q_{C}(ν) acting between complex spaces are the same \(∥T∥_{p,q} = ∥T_{C}∥_{p,q}\) if and only if p ≤ q (see [56], [19] and [39]). In two-dimensional case, i.e., when L^p(μ) = l^p_2 and L^q(ν) = l^q_2, we have that \(∥T∥_{p,q} = ∥T_{C}∥_{p,q}\) if and only if either 1 ≤ p ≤ 2 or 2 ≤ q ≤ ∞ (see [59], [19] and [39]).
Theorem 12. For all $1 < q < 2 < p < \infty$ there exists a unique $x_0 = x_0(p, q) \in (0, 1)$ such that

$$B_{p, q} = \sup_{x \in [0, 1]} F(x, p, q) = F(x_0, p, q),$$

where

$$F(x) = F(x, p, q) = \left[\frac{(1 + x)^q + (1 - x)^q}{(1 + x^p)^{1/p}}\right]^{1/q}, \quad x \in [0, 1].$$

Moreover, $\max\{2^{1-1/p}, 2^{1/q}\} < B_{p, q} < 2^{1/q-1/p+1/2}$.

Proof. Using standard calculus we find the maximum of function $F$ which is also the best constant in the generalized real Clarkson inequality (5.6). Since

$$F'(x) = \frac{[(1 + x)^q + (1 - x)^q]^{1/q - 1}}{(1 + x^p)^{1+1/p}} \left[(1 + x)^q - (1 - x)^q - (1 - x)^q(1 + x^p - 1)\right]$$

it follows that the derivative $F'(x)$ is equal to zero on $(0, 1)$ if and only if

$$\frac{1 - x^p}{1 + x^p} = \left(\frac{1 - x^q}{1 + x^q}\right)^{1/q - 1}.$$  \tag{5.10}

Denoting $f(x) = (1 - x)/(1 + x)$ we can rewrite (5.10) as $f(x^{p-1}) = f(x)^{q-1}$. Let $g(x) = \ln f(x)$ and define $h(x) = g(x^{p-1})/g(x)$. Then, for $x$ which satisfies (5.10) we have

$$h(x) = \frac{g(x^{p-1})}{g(x)} = \frac{\ln f(x^{p-1})}{\ln f(x)} = \frac{\ln f(x)^{q-1}}{\ln f(x)} = q - 1.$$  \tag{5.11}

With the help of the following auxiliary lemma we find that there exists a unique solution $x_0(p, q) \in (0, 1)$ of (5.11) and therefore (5.10). Note, that clearly 0 and 1 satisfies (5.10).

Lemma 9. For $g(x) = \ln(1 - x)/(1 + x)$ and $2 < p < \infty$ the function $h(x) = g(x^{p-1})/g(x)$ is strictly increasing on the interval $(0, 1)$ and $h((0, 1)) = (0, 1)$.

Proof. First we show that the function $h(x)$ has positive derivative on $(0, 1)$. Indeed,

$$h'(x) = \frac{[g(x^{p-1})]'g(x) - g(x^{p-1})g'(x)}{g^2(x)},$$

where $g'(x) = -2/(1 - x^2)$ and

$$[g(x^{p-1})]' = (p - 1)x^{p-2}g'(x^{p-1}) = -\frac{2(p - 1)x^{p-2}}{1 - x^{2(p-1)}}.$$  \tag{5.12}

Thus,

$$h'(x) = \frac{\left(-\frac{2(p - 1)x^{p-2}}{1 - x^{2(p-1)}} \ln \frac{1 - x}{1 + x} + \frac{2}{1 - x} \ln \frac{1 - x^{p-1}}{1 + x^{p-1}}\right)}{(1 - x^2)(1 - x^{2(p-1)})g^2(x)}.$$  \tag{5.12}

Using the Taylor expansion we get

$$\ln \frac{1 - x}{1 + x} = \ln(1-x) - \ln(1+x) = -2 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k + 1}$$

and

$$\ln \frac{1 - x^{p-1}}{1 + x^{p-1}} = -2 \sum_{k=0}^{\infty} \frac{x^{(2k+1)(p-1)}}{2k + 1}.$$
Now, in order to prove that $h'(x) > 0$ for all $x \in (0, 1)$ and $2 < p < \infty$ it is sufficient to show that the numerator in (5.12) is positive for all such $x$ and $p$. To do this we consider the function

$$
\phi(x) = (p - 1)(1 - x^2)x^{p-2} \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1} - (1 - x^{2(p-1)}) \sum_{k=0}^{\infty} \frac{x^{(2k+1)(p-1)}}{2k+1}
$$

$$
\implies (p - 1) \sum_{k=0}^{\infty} \frac{x^{p+2k-1} - x^{p+2k+1}}{2k+1} + \sum_{k=1}^{\infty} \frac{x^{(2k+1)(p-1)} - 1}{2k+1} = \sum_{k=1}^{\infty} \frac{2x^{p-1}}{(2k+1)(2k-1)} (p - 2 + x^{2(p-1)} - (p - 1)x^{2k}).
$$

We note that $\sum_{k=1}^{\infty} 2(2k+1)^{-1}(2k-1)^{-1} = 1$. For each $k = 1, 2, \ldots$ the function $\phi_k(x) = p - 2 + x^{2(p-1)} - (p - 1)x^{2k}$ is strictly decreasing on $(0, 1)$ since for $p \in (2, \infty)$

$$
\phi'_k(x) = 2k(p - 1)x^{2k-1} \left( x^{2k(p-2)} - 1 \right) < 0.
$$

Hence, $\phi_k(x) > \phi_k(1) = 0$. Thus, all functions $\phi_k(x)$ are positive which clearly implies that $\phi(x) > 0$ and therefore $h'(x) > 0$ for all required $x$ and $p$, i.e. the function $h(x)$ is strictly increasing on $(0, 1)$.

It remains to show that $h((0, 1)) = (0, 1)$. For $p \in (2, \infty)$ and $x \in (0, 1)$ we have that $x^{p-1} < x$ and since $g(x)$ is decreasing and negative on $(0, 1)$, then $0 < h(x) = g(x^{p-1})/g(x) < 1$. Moreover, using de l’Hospital rule we calculate the following limits

$$
\lim_{x \to 0^+} \frac{\ln(1 - x^{p-1})(1 + x^{p-1})}{\ln(1 - x)(1 + x)} = \lim_{x \to 0^+} \frac{-2(p - 1)x^{p-2}}{1 - x^{2(p-1)}} \cdot \frac{1 - x^2}{-2} = 0
$$

$$
\lim_{x \to 1^-} \frac{\ln(1 - x^{p-1})(1 + x^{p-1})}{\ln(1 - x)(1 + x)} = \lim_{x \to 1^-} \frac{(p - 1)x^{p-2}(1 - x^2)}{1 - x^{2(p-1)}} = \lim_{x \to 1^-} \frac{-2(p - 1)x}{-2(p - 1)x^{2p-3}} = 1.
$$

Thus, $h(x) \to 0$ as $x \to 0^+$ and $h(x) \to 1$ as $x \to 1^-$. All together implies that the image of $(0, 1)$ by $h$ is $(0, 1)$, i.e. $h((0, 1)) = (0, 1)$.

Now we continue our proof of Theorem 12. Since $h(x)$ is strictly increasing on $(0, 1)$ and $h((0, 1)) = (0, 1)$ as was established in Lemma 9, then for all $p \in (2, \infty)$ and $q \in (1, 2)$ there exists a unique $x_0 = x_0(p, q) \in (0, 1)$ such that $h(x_0) = q - 1$ and thus $F(x_0)$ is the extreme value of $F(x)$. Moreover, $x_0$ is the point of maximum of $F(x)$.

Namely, by (5.10) the derivative $F'(x)$ defined by (5.9) has the same sign as the $f(x^{p-1}) - f(x)^{q-1}$. Consider the functions $u(x) = f(x^{p-1})$ and $v(x) = f(x)^{q-1}$. Then

$$
u'(x) = -2(p - 1) \frac{x^{2p-2}}{(1 + x(p-1))^2} \quad \text{and} \quad v'(x) = -2(q - 1) \frac{(1 - x)^{q-2}}{(1 + x)^q}. $$
Therefore, both \( u(x) \) and \( v(x) \) are decreasing on \((0,1), \) \( u'(0) = 0 \) and \( v'(0) = -2(q-1). \) Moreover, \( u(0) = v(0) = 1. \) Hence, \( u(x) - v(x) > 0 \) which means that \( F(x) \) is increasing at this neighborhood. Thus, \( x_0 \) is the point of maximum of \( F(x) \) for \( p \in (2, \infty) \) and \( q \in (1,2). \)

It remains for us to show that \( \max\{2^{1-1/p},2^{1/q}\} < B_{p,q} < 2^{1/q-1/p+1/2}. \) Since \( F(x) \) is strictly increasing on \([0,x_0] \) and strictly decreasing on \([x_0,1] \) for \( 1 < q < 2 < p < \infty, \) then

\[
\max\{2^{1-1/p},2^{1/q}\} = \max\{F(0),F(1)\} < F(x_0) = B_{p,q}.
\]

On the other hand, using strict convexity of the functions \( u^{2/q} \) and \( u^{q/2} \) we obtain for \( x \in (0,1) \)

\[
[(1+x)^q+(1-x)^q]^{1/q} < 2^{1/q-1/2}[(1+x)^2+(1-x)^2]^{1/2} = 2^{1/q}(1+x^2)^{1/2} < 2^{1/q-1/p+1/2}(1+x^p)^{1/p}
\]

and thus \( B_{p,q} = F(x_0) < 2^{1/q-1/p+1/2}. \)

**Remark.** For \( p > 2, \) Thorin in [58] proved a stronger result, namely that the function \( F(x,p,p') \) defined in Theorem 12 is logarithmically concave in \( 1/p. \)

Theorem 12 only shows the existence of a point inside the interval \((0,1)\) at which the function \( F(x) \) given by (5.8) has its maximum. But it is even more interesting to find this point and as a consequence to obtain the best constant in the generalized real Clarkson inequality for \( 1 < q < 2 < p < \infty. \) It turns out that it is difficult, if even possible, to solve the problem for all such \( p \) and \( q. \) Here we consider only the case when \( q = p'. \)

As was established in Theorem 12, the point of maximum of \( F(x) \) is then defined by

\[
1 - x^{p-1} \over 1 + x^{p-1} = (1-x)^{p'-1} \over (1+x)^{p'-1}.
\]

Denoting \( f(x) = (1-x)/(1+x), \) we rewrite this equality in the form \( u(x) = f(x^{p-1}) = f^{p-1}(x) = v(x). \) Observe that \( u \circ v = f \circ f^{(p-1)/(p-1)} = f \circ f = id \)

and thus the point of maximum of \( F(x) \) is also defined by \( f(x^{p-1}) = x \) which is equivalent to the equation

\[
(5.13) \quad x^p + x^{p-1} + x - 1 = 0.
\]

If we denote the root of (5.13) by \( x_0, \) then \( \max\{F(x) : x \in (0,1)\} = F(x_0) \) and thus the best constant in the generalized real Clarkson inequality for \( 2 < p < \infty \) will be

\[
(5.14) \quad C_{p,p'}(\mathbb{R}) = (1+x_0)(1+x_0)^{1/p'-1/p}.
\]

We could find the exact expression for this constant when \( p = 3, 4, 8 \) and compare it with the constant \( C_{p,p}(\mathbb{C}). \)

Thus, in the case when \( p = 3 \) the equation (5.13) becomes cubic with the real root

\[
x_0 = 1/3 \left( 3/17 + 3\sqrt{33} + 3/17 - 3\sqrt{33} \right) \approx 0.544.
\]

After substitution this expression into (5.14) we get \( C_{3,3}(\mathbb{R}) = (1+x_0)(1+x_0)^{1/3} \approx 1.622. \) At the same time we have \( C_{3,3}(\mathbb{C}) = 2^{5/6} \approx 1.781 \) and so \( C_{3,3}(\mathbb{C})/C_{3,3}(\mathbb{R}) \approx 1.098. \)
The constant $C_{4,4}(\mathbb{R}) (p = 4)$ is connected to the real root of the equation
\[ x^4 + x^3 + x - 1 = (x^2 + 1)(x^2 + x - 1) = 0, \]
which is $x_0 = (\sqrt{3} - 1)/2 \in (0, 1)$ and therefore $C_{4,4}(\mathbb{R}) = \sqrt{3} \approx 1.732$. We also have that $C_{4,4}(\mathbb{C}) = 2$ and thus $C_{4,4}(\mathbb{C})/C_{4,4}(\mathbb{R}) = 2/\sqrt{3} \approx 1.154$.

When $p = 8$ the real root of the equation
\[ x^8 + x^7 + x - 1 = (x^2 + 1)(x^3 - x + 1)(x^2 + x - 1) = 0. \]
can be found from $x^3 + x^2 - 1 = 0$ and is equal to
\[ x_0 = 1/3 \left( \frac{1}{2} \left( 25 + 3\sqrt{69} \right) / 2 + \frac{1}{2} \left( 25 - 3\sqrt{69} \right) / 2 - 1 \right) \approx 0.755. \]

Therefore, $C_{8,8/7}(\mathbb{R}) \approx 1.892$ and since $C_{8,8/7}(\mathbb{C}) = 2^{5/4} \approx 2.378$ it follows that $C_{8,8/7}(\mathbb{C})/C_{8,8/7}(\mathbb{R}) \approx 1.256$.

For the remaining values of $p \in (2, \infty)$ we could not calculate exactly the constants $C_{p,p'}(\mathbb{R})$ but we noticed that (5.13) can be solved for $p$:
\[ p(x) = 1 + \log x \frac{1 - x}{1 + x}. \]

Now, taking arbitrary value of $x \in [\sqrt{2} - 1, 1)$ we find the corresponding value $p$ and therefore by (5.14) we can calculate the constant $C_{p,p}(\mathbb{R})$ for this particular $p$. The left end of the interval is determined by letting $p = 2$ in (5.15) and solving this for $x$ (limit case). It is also worth to note that the function $p(x)$ is increasing on $[\sqrt{2} - 1, 1)$. Indeed, its derivative
\[ p'(x) = \left( \frac{\ln \frac{1 - x}{1 + x}}{\ln x} \right)' = -\frac{2}{1 - x^2} \ln x - \frac{1}{x} \ln \frac{1 - x}{1 + x} \geq 0. \]

For example, if we take $x = 1/2$, then $p = 1 + \log_2 3 \approx 2.584$, $C_{p,p}(\mathbb{R}) = 7^{1/2} \approx 2.645$, and thus $C_{p,p}(\mathbb{C})/C_{p,p}(\mathbb{R}) \approx 1.256$. If $x_0 = 9/10$, then $p = 1 + \log_{10} 1/10 \approx 28.9463$, $C_{p,p}(\mathbb{R}) \approx 1.9836$, $C_{p,p}(\mathbb{C}) \approx 2.6962$ and $C_{p,p}(\mathbb{C})/C_{p,p}(\mathbb{R}) \approx 1.3592$. For $x_0 = 99/100$ we have $p = 1 + \log_{10} 1/100 \approx 527.6794$, $C_{p,p}(\mathbb{R}) \approx 1.9999$, $C_{p,p}(\mathbb{C}) \approx 2.8210$ and $C_{p,p}(\mathbb{C})/C_{p,p}(\mathbb{R}) \approx 1.4106$.

Observe that $C_{p,p}(\mathbb{C})/C_{p,p}(\mathbb{R}) \rightarrow C_{\infty,1}(\mathbb{C})/C_{\infty,1}(\mathbb{R}) = \sqrt{2}$ as $p$ approaches $\infty$ (see Krivine [34]).

### 5.3. Some applications of the Clarkson inequality

In this section we show how the real and complex generalized Clarkson inequalities can be used in calculation of the norm of a certain operator as well as in the proof of the $(p,q)$-Clarkson inequalities in Lebesgue spaces, mixed norm spaces and in normed spaces.

For $1 \leq p, q \leq \infty$ consider a linear operator $T : L^p[a, b] \rightarrow L^q[a, b]$ given by
\[
T x(t) = \begin{cases} 
  x(t) + x(t + \frac{a - b}{t}), & t \in [a, \frac{a + b}{2}] \\
  x(t - \frac{a - b}{t}) - x(t), & t \in [\frac{a + b}{2}, b].
\end{cases}
\]

As an application of Theorems 10 and 11 we calculate the norm of this operator acting between complex and real spaces.
Theorem 13. Let \(1 \leq p, q \leq \infty\). Then the operator \(T : L^p[a, b] \to L^q[a, b]\) is bounded if and only if \(p \geq q\). Moreover,

\[
\|T\|_{p \to q} = (\frac{b-a}{2})^{\frac{q-p}{pq}} C_{p,q}, \text{ where } C_{p,q} = C_{p,q}(\mathbb{R}) \text{ or } C_{p,q}(\mathbb{C})
\]

depending if the spaces are real or complex.

Proof. First, by Theorems 10, 11 (substituting \(a = x + y\) and \(b = x - y\) for \(x, y \in \mathbb{R}\) or \(x, y \in \mathbb{C}\))

\[
\|T\|_q = \left( \int_a^{(a+b)/2} |x(t) + x(t + \frac{b-a}{2})|^q dt + \int_{(a+b)/2}^b |x(t - \frac{b-a}{2}) - x(t)|^q dt \right)^{1/q}
\]

\[
= \left( \int_a^{(a+b)/2} |x(t) + x(t + \frac{b-a}{2})|^q dt + \int_a^{(a+b)/2} |x(s) - x(s + \frac{b-a}{2})|^q ds \right)^{1/q}
\]

\[
= \left[ \int_a^{(a+b)/2} \left( |x(t) + x(t + \frac{b-a}{2})|^q + |x(t) - x(t + \frac{b-a}{2})|^q \right) dt \right]^{1/q}
\]

\[
\geq \min\{2^{1/q}, 2^{1/q'}\} \left[ \int_a^{(a+b)/2} \left( |x(t)|^q + |x(t + \frac{b-a}{2})|^q \right) dt \right]^{1/q}
\]

\[
= \min\{2^{1/q}, 2^{1/q'}\} \left[ \int_a^{(a+b)/2} |x(t)|^q dt + \int_{(a+b)/2}^b |x(t + \frac{b-a}{2})|^q dt \right]^{1/q}
\]

\[
= \min\{2^{1/q}, 2^{1/q'}\} \|x\|_q.
\]

Thus, in the case \(p < q\) the operator \(T\) is unbounded, since otherwise we obtain that

\[
\min\{2^{1/q}, 2^{1/q'}\} \|x\|_q \leq \|T\|_q \leq C\|x\|_p
\]

and therefore \(L^p[a, b] \subset L^q[a, b]\) that is obviously not true.

If \(p \geq q\), then by Theorems 10, 11 and the Hölder inequality we have

\[
\|T\|_q = \left( \int_a^{(a+b)/2} |x(t) + x(t + \frac{b-a}{2})|^q dt + \int_{(a+b)/2}^b |x(t - \frac{b-a}{2}) - x(t)|^q dt \right)^{1/q}
\]

\[
\leq C_{p,q} \left[ \int_a^{(a+b)/2} \left( |x(t)|^p + |x(t + \frac{b-a}{2})|^p \right)^{q/p} dt \right]^{1/q}
\]

\[
\leq C_{p,q} \left( \int_a^{(a+b)/2} (|x(t)|^p + |x(t + \frac{b-a}{2})|^p) dt \right)^{1/p} \left( \int_a^{(a+b)/2} dt \right)^{\frac{q-p}{pq}}
\]

\[
= C_{p,q} \left( \frac{b-a}{2} \right)^{\frac{q-p}{pq}} \|x\|_p,
\]

where \(C_{p,q} = C_{p,q}(\mathbb{R})\) in the real case and \(C_{p,q} = C_{p,q}(\mathbb{C})\) in the complex case.

To verify that equality may occur in (5.16) we can take a pair \((u, v)\) of numbers from Theorem 10 and Theorem 12 (in the case \(1 < q < 2 < p < \infty\) we take \(x_0 = v/u\) at which the generalized Clarkson inequalities become equalities and
consider the following function
\[ x(t) = v \chi_{[a, (a+b)/2]}(t) + v \chi_{[(a+b)/2, b]}(t). \]

The main application of the classical complex Clarkson inequality (5.2) was for Clarkson [11] to prove that for \( 1 < p < \infty \) \( L^p \)-spaces are uniformly convex and to find the lower estimates of the modulus of uniform convexity defined by him in [11].

A normed space \((X, \|\cdot\|)\) is said to be uniformly convex if for every \( 0 < \varepsilon \leq 2 \) there exists \( \delta = \delta(\varepsilon) > 0 \) such that for all \( x, y \in X \) with \( \|x\| = \|y\| = 1 \) and \( \|x - y\| \geq \varepsilon \) it follows that \( \|(x + y)/2\| < 1 - \delta \). The function \( \delta_X(\varepsilon) : [0, 2] \to [0, 1] \) defined by the formula
\[ \delta_X(\varepsilon) = \inf \{1 - \frac{\|x + y\|}{2} : x, y \in S_X, \|x - y\| \geq \varepsilon\} \]
is called the modulus of uniform convexity of \( X \). Clearly, a normed space \( X \) is uniformly convex if \( \delta_X(\varepsilon) > 0 \) for all \( 0 < \varepsilon \leq 2 \).

Using inequality (5.2) Clarkson proved two inequalities, one for \( 1 < p < 2 \) and the other for \( 2 \leq p < \infty \), which can be written in one formula (e.g. [9], p. 21): if \( 1 < p < \infty \) and \( r = \max \{p, p'\} \), then
\[ \left( \|x + y\|_p^r + \|x - y\|_p^r \right)^{1/r} \leq 2^{1/r'} \left( \|x\|_{p'}^r + \|y\|_{p'}^r \right)^{1/r'} \text{ for all } x, y \in L^p. \]
Thus, if \( \|x\|_p = \|y\|_p = 1 \) and \( \|x - y\| \geq \varepsilon, \varepsilon \in [0, 2] \), then \( \|x + y\|_p \leq 2 \left[ 1 - (\varepsilon/2)^r \right]^{1/r} \)
and so
\[ \delta_{L^p}(\varepsilon) \geq 1 - \left[ 1 - \left( \frac{\varepsilon}{2} \right)^r \right]^{1/r} \geq \frac{1}{r} \left( \frac{\varepsilon}{2} \right)^r, \text{ where } r = \max \{p, p'\}. \]

For \( 2 \leq p < \infty \) estimate (5.17) is sharp. For \( 1 < p < 2 \) the sharp estimate \( \delta_{L^p}(\varepsilon) > \frac{\varepsilon - 1}{8} \varepsilon^2 \) was found by Hanner [21] and Kadec [24].

Later on Boas [6] obtained, by the Riesz convexity theorem, the following result: if \( 1 < p < r < q \), then
\[ \left( \|x + y\|_p^q + \|x - y\|_p^q \right)^{1/q} \leq 2^{1/q'} \left( \|x\|_{p'}^q + \|y\|_{p'}^q \right)^{1/q'} \text{ for all } x, y \in L^r(\mu). \]
Koskela [30] removed the restrictions on \( p, q \) and \( r \) and proved for \( 0 < p, q, r < \infty \) the inequality
\[ \left( \|x + y\|_p^q + \|x - y\|_p^q \right)^{1/q} \leq 2^{1/q - 1/p + 1/t} \left( \|x\|_p^p + \|y\|_p^p \right)^{1/p} \text{ for all } x, y \in L^r(\mu), \]
where \( t = \min \{p, q', r, r'\} \) with the convention that the conjugate exponent to a number is omitted if the number is not bigger than one. Moreover, if \( \dim L^r(\mu) \geq 2 \) then there exist functions \( x, y \in L^r(\mu) \) for which the equality in (5.18) holds.

For convenience we call inequality (5.18) the \((p, q)\)-Clarkson inequality in \( L^r(\mu) \).

Let us note that the generalized complex Clarkson inequality (5.1) can also be written in the form
\[ (\|a + b\|_p^q + \|a - b\|_p^q)^{1/q} \leq 2^{1/q - 1/p + 1/t} \left( |a|^p + |b|^p \right)^{1/p} \text{ for all } a, b \in \mathbb{C}, \]
where \( t = \min \{p, q', 2\} \), i.e. the sharp constants in (5.1) and (5.19) are the same. Obviously, the constant in (5.19) is not bigger than that in (5.18). At the same time this constant is the best constant in the generalized Clarkson inequality in real or complex Hilbert space \((H, \|\cdot\|)\) (put \|\cdot\| instead of \|\cdot\| in (5.19) and take \( a, b \in H \)). The latter can be proved by using arguments similar to those we used in Theorem 10.
Some results on a generalization of (5.18) to the mixed norm spaces were obtained by Boas [6], Koskela [30] and Sobolev [54] and this work was continued by Kato-Miyazaki [26] and Kato-Miyazaki-Takahashi [27], [28]. In contrast with the proofs presented in [26], [27] and [29], where the interpolation techniques were used, our proof of the (p,q)-Clarkson inequality in the mixed norm spaces will only rely on the Jensen, Minkowski and (p,q)-Clarkson inequalities in \( L^r(\mu) \) spaces. We will also complete the chain of generalizations of the (p,q)-Clarkson inequality to the mixed norm spaces.

Let us recall that the mixed norm spaces \( L^s[L^r] \) (cf. Benedek-Panzone [2]) are classes of measurable functions on \( \Omega_1 \times \Omega_2 \) generated by the quasi-norms (norms when \( r,s \geq 1 \))

\[
||x||_{r,s} = ||x||_r = \left( \int_{\Omega_2} \left( \int_{\Omega_1} |x(t_1,t_2)|^r d\mu_1 \right)^{s/r} d\mu_2 \right)^{1/s}, \quad \text{where} \ r, s > 0.
\]

**Theorem 14.** If \( 0 < p, q, r, s \leq \infty \), then

\[
(5.20) \quad \left( \|x + y\|_{r,s}^q + \|x - y\|_{r,s}^q \right)^{1/q} \leq 2^{1/q-1/p+1/t} \left( \|x\|_r^p + \|y\|_r^p \right)^{1/p}
\]

for all \( x, y \in L^t[L^r] \), where \( t = \min \{p, q', r', s', s'\} \) and the conjugate exponent is omitted if the number is not bigger than one. Moreover, if \( \dim L^t(\mu_1) \geq 2 \) and \( \dim L^s(\mu_2) \geq 2 \), then the constant \( 2^{1/q-1/p+1/t} \) is sharp.

**Proof.** We prove the inequality for real positive numbers \( p, q, r \) and \( s \). Simple modification of this proof establishes that inequality (5.20) also holds when some of these numbers are equal to infinity.

**Case 1:** If \( s \leq q \), then by applying the reverse Minkowski inequality we get

\[
(5.21) \quad \left( \|x + y\|_{r,s}^q + \|x - y\|_{r,s}^q \right)^{1/q} \leq \left( \int_{\Omega_2} (\|x + y\|_r^p + \|x - y\|_r^p)^{s/q} d\mu_2 \right)^{1/s}.
\]

Further, if \( p \leq s \), then by sequential use of the (p,q)-Clarkson and Minkowski inequalities it follows

\[
(5.22) \quad \left( \|x + y\|_{r,s}^q + \|x - y\|_{r,s}^q \right)^{1/q} \leq 2^{1/q-1/p+1/t} \left( \int_{\Omega_2} (\|x\|_r^p + \|y\|_r^p)^{s/p} d\mu_2 \right)^{1/s} \\
\leq 2^{1/q-1/p+1/t} \left[ \left( \int_{\Omega_2} \|x\|_r^p d\mu \right)^{p/s} + \left( \int_{\Omega_2} \|y\|_r^p d\mu \right)^{p/s} \right]^{1/p} \\
= 2^{1/q-1/p+1/t} \left( \|x\|_r^p + \|y\|_r^p \right)^{1/p},
\]

where \( t = \min \{p, q', r, r'\} \). Since \( s \leq q \) (or \( q' \leq s' \)) and \( p \leq s \) we can write \( t = \min \{p, q', r, r', s, s'\} \).
If we instead have $p \geq s$ and apply the $(s,q)$-Clarkson inequality to (5.21) we get
\[
\frac{|x+y|^q_{r,s} + |x-y|^q_{r,s}}{1/q} \leq 2^{1-q-1/s+1/t} \left( \int_{\Omega_2} (|x|^s_{r,s} + |y|^s_{r,s}) d\mu_2 \right)^{1/s} \\
= 2^{1-q-1/s+1/t} \left( |x|^s_{r,s} + |y|^s_{r,s} \right)^{1/s} \\
\leq 2^{1-q-1/p+1/t} \left( |x|^p_{r,s} + |y|^p_{r,s} \right)^{1/p},
\]
where $t = \min \{ s, q', r, r' \} = \min \{ p, q', r, r', s, s' \}$, since $s \leq q$ ($q' \leq s'$) and $s \leq p$.

**Case 2:** If $s \geq q$, then
\[
\frac{|x+y|^q_{r,s} + |x-y|^q_{r,s}}{1/q} \leq 2^{1-1/q-1/s} \left( |x+y|^s_{r,s} + |x-y|^s_{r,s} \right)^{1/s} \\
\leq 2^{1-1/q-1/s} \left( \int_{\Omega_2} (|x|^s_{r,s} + |y|^s_{r,s}) d\mu_2 \right)^{1/s} \\
= 2^{1-1/q-1/s} \left( \int_{\Omega_2} (|x|^s_{r,s} + |y|^s_{r,s}) d\mu_2 \right)^{1/s}.
\]

Moreover, if $s \leq p$ then by the $(s,s)$-Clarkson inequality we have
\[
\frac{|x+y|^q_{r,s} + |x-y|^q_{r,s}}{1/q} \leq 2^{1-1/q-1/s+1/t} \left( \int_{\Omega_2} (|x|^s_{r,s} + |y|^s_{r,s}) d\mu_2 \right)^{1/s} \\
= 2^{1-1/q-1/s+1/t} \left( |x|^s_{r,s} + |y|^s_{r,s} \right)^{1/s} \\
\leq 2^{1-1/q-1/p+1/t} \left( |x|^p_{r,s} + |y|^p_{r,s} \right)^{1/p},
\]
where $t = \min \{ s, s', r, r' \} = \min \{ p, q', r, r', s, s' \}$, since $s \geq q$ ($s' \leq q'$) and $p \geq s$.

If we instead have $s \geq p$, then by $(p,s)$-Clarkson and Minkowski inequalities applied to (5.22) we get
\[
\frac{|x+y|^q_{r,s} + |x-y|^q_{r,s}}{1/q} \leq 2^{1-1/q-1/s+1/s-1/p+1/t} \left( \int_{\Omega_2} (|x|^p_{r,s} + |y|^p_{r,s}) d\mu_2 \right)^{1/p} \\
\leq 2^{1-1/q-1/p+1/t} \left( \left( \int_{\Omega_2} |x|^p_{r,s} d\mu_2 \right)^{p/s} + \left( \int_{\Omega_2} |y|^p_{r,s} d\mu_2 \right)^{p/s} \right)^{1/p} \\
= 2^{1-1/q-1/p+1/t} \left( |x|^p_{r,s} + |y|^p_{r,s} \right)^{1/p},
\]
where $t = \min \{ p, s', r, r' \} = \min \{ p, q', r, r', s, s' \}$, since $s \geq q$ ($s' \leq q'$) and $s \geq p$.

Summarizing we can write
\[
\frac{|x+y|^q_{r,s} + |x-y|^q_{r,s}}{1/q} \leq 2^{1-1/q-1/p+1/t} \left( |x|^p_{r,s} + |y|^p_{r,s} \right)^{1/p},
\]
where $t = \min \{ p, q', r, r', s, s' \}$.

In order to check that the equality may occur in (5.20) let us take disjoint sets $A_1, A_2$ in $\Omega_1$ and disjoint sets $B_1, B_2$ in $\Omega_2$ with corresponding positive finite measures and define the functions
\[
\alpha(u,v) = \chi_{A_1}(u) \chi_{B_1}(v) \mu_1^{-1/s}(A_1) \mu_2^{-1/r}(B_1), \\
\beta(u,v) = \chi_{A_1}(u) \chi_{B_2}(v) \mu_1^{-1/s}(A_1) \mu_2^{-1/r}(B_2), \\
\gamma(u,v) = \chi_{A_2}(u) \chi_{B_1}(v) \mu_1^{-1/s}(A_2) \mu_2^{-1/r}(B_1).
\]
Then the required equality occurs on the following pairs of functions

1. \( x(u, v) = \alpha(u, v), \ y(u, v) = 0; \)
2. \( x(u, v) = y(u, v) = \alpha(u, v); \)
3. \( x(u, v) = \alpha(u, v), \ y(u, v) = \beta(u, v); \)
4. \( x(u, v) = \alpha(u, v), \ y(u, v) = \gamma(u, v); \)
5. \( x(u, v) = \alpha(u, v) + \gamma(u, v), \ y(u, v) = \alpha(u, v) - \gamma(u, v); \)
6. \( x(u, v) = \alpha(u, v) + \beta(u, v); \)

We note, the functions defined with the help of the sets \( A_1, A_2, B_1 \) and \( B_2 \) are real-valued, which means that the best constants in (5.20) are the same in both complex and real case.

Observe that by proving inequality (5.20) in \( L^*[L'] \) space we referred each time to the \((p, q)\)-Clarkson inequality in \( L^*[\mu] \) space. Thus, by induction, the inequality similar to (5.20) is also true in the settings of the mixed norm spaces \( L^n \) equipped with the quasi-norms (norms when \( r_1, ..., r_n \geq 1 \))

\[
\|x\|_{r_1, ..., r_n} = \left[ \sum_{i=1}^{n} \left( \|x_i\|_{r_i} \right)^{r_i} \right]^{1/r_n}, \quad \text{where} \quad r_1, ..., r_n > 0.
\]

Using inequality (5.20), it is easy to give the following estimation of the modulus of convexity of \( L^n[L'] \) spaces similar to (5.17): if \( \|x\|_{r,s} = \|y\|_{r,s} = 1 \) and \( \|x - y\|_{r,s} \geq \varepsilon, \varepsilon \in [0, 2] \), then \( \|x + y\|_{r,s} \leq 2 \left[ 1 - (\varepsilon/2)^p \right]^{1/p} (r,s) \) and

\[
\delta_{L^n[L']}(\varepsilon) \geq 1 - \left[ 1 - \left( \frac{\varepsilon}{2} \right)^p \right]^{1/p} \geq \frac{1}{2} \left( \frac{\varepsilon}{2} \right)^p, \quad \text{where} \quad p = \max \{r, s, 2\}.
\]

This estimation is sharp when \( 2 \leq r, s < \infty \). The sharp estimate \( \delta_{L^n[L']}(\varepsilon) \geq k_{r,s} \varepsilon^p \), where \( p = \max \{r, s, 2\} \) was found by Maleev-Troyanski [37].

**Theorem 15.** If \( 0 < p, q, r_1, ..., r_n \leq \infty \), then

\[
\left( \|x + y\|_{r_1, ..., r_n}^q + \|x - y\|_{r_1, ..., r_n}^q \right)^{1/q} \leq 2^{1/q-1/p+1/t} \left( \|x\|_{r_1, ..., r_n}^p + \|y\|_{r_1, ..., r_n}^p \right)^{1/p}
\]

for all \( x, y \in L^n \), where \( t = \min \{p', q', r_1, r_1', r_2, r_2', ..., r_n, r_n'\} \) and the conjugate exponent is omitted if the number is not bigger than one. Moreover, if \( \dim L^n(\mu_i) \geq 2 \) for all \( 1 \leq i \leq n \), then the constant \( 2^{1/q-1/p+1/t} \) is sharp.

Again, if \( \|x\|_{r_1, ..., r_n} = \|y\|_{r_1, ..., r_n} = 1 \) and \( \|x - y\|_{r_1, ..., r_n} \geq \varepsilon, \varepsilon \in [0, 2] \), then

\[
\|x + y\|_{r_1, ..., r_n} \leq 2 \left[ 1 - (\varepsilon/2)^p \right]^{1/p} \|x\|_{r_1, ..., r_n}, \quad \text{and so}
\]

\[
\delta_{L^n(\mu)}(\varepsilon) \geq 1 - \left[ 1 - \left( \frac{\varepsilon}{2} \right)^p \right]^{1/p} \geq \frac{1}{2} \left( \frac{\varepsilon}{2} \right)^p, \quad \text{where} \quad p = \max \{r_1, r_1', ..., r_n, r_n'\}.
\]

By induction the Maleev-Troyanski result, mentioned above, can be extended to \( L^n \) spaces. The sharp estimate of the modulus of convexity of these spaces will be \( \delta_{L^n(\mu)}(\varepsilon) \geq k_{r_1, ..., r_n} \varepsilon^p \), where \( p = \max \{r_1, ..., r_n, 2\} \). Thus, the estimation (5.23) is sharp for \( 2 \leq r_1, ..., r_n < \infty \).

We finish this part with the \((p, q)\)-Clarkson inequality in a normed space \((X, \|\cdot\|)\) for \( 0 < p, q \leq \infty \):

\[
\left( \|x + y\|^q + \|x - y\|^q \right)^{1/q} \leq C \left( \|x\|^p + \|y\|^p \right)^{1/p} \quad \text{for all} \quad x, y \in X.
\]

This inequality is always true but to find the best constant in (5.24), which we denote by \( C_{p,q}(X) \), is of interest.
Theorem 16. Let $0 < p, q \leq \infty$. Then
\[ C_{p,q} \leq C_{p,q}(X) \leq 2^{1/q} \max \left\{ 1, 2^{1/p} \right\}, \]
where $C_{p,q} = C_{p,q}(\mathbb{R})$ when $X$ is a real normed space and $C_{p,q} = C_{p,q}(\mathbb{C})$ when $X$ is a complex normed space.

Proof. For $x \in X$, $a \neq 0$ and $y = (b/a)x$ the estimate (5.24) means
\[ \left(1 + \frac{b}{a}^q \|x\|^q + \left|1 - \frac{b}{a}^q \|x\|^q\right\}^{1/q} \leq C \left(\|x\|^p + \left|\frac{b}{a}\|^p \|x\|^p\right\}^{1/p} \]
or
\[ (|a + b|)^q + |a - \bar{b}|)^{1/q} \leq C (|a|^p + |b|^p)^{1/p} . \]
Thus, $C_{p,q}(X) \geq C_{p,q}(\mathbb{R})$ when $X$ is a real space (then we take $a, b \in \mathbb{R}$) and $C_{p,q}(X) \geq C_{p,q}(\mathbb{C})$ when $X$ is a complex space (then we take $a, b \in \mathbb{C}$). Estimation from above follows by applying twice the triangle inequality for the norm and the property that $f(p) = (\|x\|^p + \|y\|^p)^{1/p}$ is decreasing in $p$.
\[ (\|x + y\|^q + \|x - y\|^q)^{1/q} \leq 2^{1/q} (\|x\| + \|y\|) \leq 2^{1/q} \max\{1, 2^{1/p}\} (\|x\| + \|y\|)^{1/p} . \]

Note that the upper estimate in the above inequality is attained. For $0 < p \leq 1$ take $y = 0$ and $x \neq 0$. For $p \geq 1$ consider $X = L^1(\mu)$ with $\dim L^1(\Omega, \mu) \geq 2$ and take $x = \frac{1}{\mu(A)} \chi_A$, $y = \frac{1}{\mu(B)} \chi_B$ with disjoint $A, B \subset \Omega$; we can also consider $L^\infty(\Omega, \mu)$ with $\dim L^\infty(\Omega, \mu) \geq 2$ and take $x = \chi_A + \chi_B$ and $y = \chi_A - \chi_B$ with disjoint $A, B \subset \Omega$.

Of course, there exist stronger versions of the $(p, q)$-Clarkson inequality in $X$, i.e. with a constant smaller than $2^{1/q} \max\{1, 2^{1/p}\}$, but then the normed space $X$ will have additional properties. Take, for $p, q > 1$ the space $L^r(\mu)$ with $1 < r < \min\{p, q\}$, then by applying (5.18) we get $C_{p,q}(L^r) = 2^{1/r - 1/p' + 1/r} < 2^{1/q - 1/p' - 1/r}$.

The Clarkson type inequality (5.24) for a Banach space with the constant $C = 2^{1/q}$ for $1 < p < 2$ and $p \leq q \leq p'$, and $C = 2^{1/p}$ for $2 \leq q \leq \infty$ and $q' \leq p \leq q$ was considered by Kato and Takahashi in [29].
Basic Properties of Constant $\gamma(l^p_n, l^q_m)$

In this chapter we consider the relation between complex and real norms of bounded linear operators from $l^p_n$ into $l^q_m$ space. The interest to this relation was mainly caused by two reasons. First, we wanted to answer the question if this relation was bounded in the case when $0 < q < 1$ and we got an affirmative answer to this question even in more general settings as Theorem 9 states. Secondly, the knowledge of the exact estimates of $\gamma(l^p_n, l^q_m)$ which were found for some values of $0 < p, q \leq \infty$ and $n, m \in \mathbb{N}$ enables us to get sharp constants in the real Riesz-Thorin interpolation theorem for these particular spaces and therefore gives us possibility to establish the best constants in the real versions of some classical inequalities when the best constants are known in the complex case.

Taking a counting measure on the set of natural numbers $\mathbb{N}$ we define $l^p$ space as a particular case of $L^p$. Namely, $l^p = L^p(\mathbb{N}, 2^n, \mu)$. This relation means that we identify functions $f : \mathbb{N} \to \mathbb{K}$ ($\mathbb{K}$ replaces $\mathbb{R}$ or $\mathbb{C}$) with sequences $x = \{x_k = f(k)\}$ and integration with summation provided that all integrable functions are everywhere finite. We supply a space $l^p_n$ where $0 < p < \infty$ with the (quasi-)norm:

$$\|x\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} = \left(\int_{\mathbb{N}} |f(k)|^p \, d\mu(k)\right)^{1/p} = \|f\|_p,$$

and for $p = \infty$ with the modified norm:

$$\|x\|_\infty = \sup_{k \in \mathbb{N}} |x_k| = \sup_{k \in \mathbb{N}} |f(k)| = \|f\|_\infty.$$

Of course, instead of $\mathbb{N}$ we can take the subset $\mathbb{N}_n$ of $\mathbb{N}$ and define a finite-dimensional space $l^p_n$ in terms of a function space.

This definition of $l^p$ spaces clarifies that all estimates of the constant $\gamma(L^p, L^q)$ obtained in the previous chapters remain valid with $L^p$ changed to $l^p$ spaces. However, we want to improve the general estimates for this particular case. We begin with two important auxiliary lemmas. The techniques we will use in this chapter are not far beyond the scope of the classical Hölder, Minkowski, Jensen inequalities and standard differential calculus.

**Lemma 10.** Let $a_k, b_k \in \mathbb{C}$ for all $k \in \mathbb{N}$ and $1 \leq p < \infty$, then

$$\left(\sum_{k=1}^{\infty} |a_k|^p\right)^{1/p} \leq \sum_{k=1}^{\infty} |a_k|. \tag{6.1}$$
The equality holds for \( p \neq 1 \) if and only if the sequence \( \{a_k\}_{k=1}^{\infty} \) has at most one element distinct from zero.

**Proof.** If \( \sum_{k=1}^{\infty} |a_k| = 0 \) or, alternatively, \( a_k = 0 \) for all \( k \in \mathbb{N} \), then we clearly have the equality in (6.1). Assume that \( \sum_{k=1}^{\infty} |a_k| > 0 \). If we take for any \( k \in \mathbb{N} \)

\[
c_k = a_k / \left( \sum_{i=1}^{\infty} |a_i|^p \right)^{1/p},
\]

then \( |c_k| = |a_k| / \left( \sum_{i=1}^{\infty} |a_i|^p \right)^{1/p} \leq 1 \) and \( \sum_{k=1}^{\infty} |c_k|^p = 1 \). Obviously, \( |c_k| \geq |c_k|^p \) whenever \( |c_k| \leq 1 \) and \( p \geq 1 \). Thus, \( \sum_{k=1}^{\infty} |c_k| \geq \sum_{k=1}^{\infty} |c_k|^p = 1 \), that is equivalent to \( \left( \sum_{k=1}^{\infty} |a_k|^p \right)^{1/p} \geq \sum_{k=1}^{\infty} |a_k|^p \).

If the equality in (6.1) is true, then by the preceding argument we should have \( |c_k| = |c_k|^p \) for all \( k \in \mathbb{N} \), that together with the relation \( \sum_{k=1}^{\infty} |c_k|^p = 1 \) implies that it should exist \( k \in \mathbb{N} \) such as \( 1 = |c_k| = |a_k| / \left( \sum_{i=1}^{\infty} |a_i|^p \right)^{1/p} \) and all other elements \( c_k \) should be equal to zero. Now it is easy to understand that in order to satisfy this condition the sequence \( \{a_k\}_{k=1}^{\infty} \) may have at most one element distinct from zero.

**Lemma 11.** If \( 0 < p \leq q \leq \infty \), then for all \( x \in \ell^p_n \)

\[
\|x\|_q \leq \|x\|_p \leq n^{1/p-1/q}\|x\|_q.
\]

**Proof.** Applying Hölder’s inequality with \( q/p \geq 1 \) we get

\[
\left( \sum_{k=1}^{n} |x_k|^p \right)^{1/p} \leq \left( \sum_{k=1}^{n} |x_k|^q \right)^{1/q} \left( \sum_{k=1}^{n} 1 \right)^{1-1/q} = n^{1-1/q} \left( \sum_{k=1}^{n} |x_k|^q \right)^{1/q}.
\]

On the other hand, Lemma 10 implies

\[
\left( \sum_{k=1}^{n} |x_k|^q \right)^{1/q} = \left( \sum_{k=1}^{n} |x_k|^p \right)^{1/p} \leq \left( \sum_{k=1}^{n} |x|^p \right)^{1/\hat{p}} = \left( \sum_{k=1}^{n} |x|^p \right)^{1/\hat{p}}.
\]

Note, for \( p < q \) the former inequality becomes equality if and only if \( |x_1| = \ldots = |x_n| \), that follows from the case when we have equality in the Hölder inequality and the latter inequality becomes equality if and only if all of \( x_k \) is nonzero (Lemma 10).

Recall, for bounded linear operators \( T \) between Lebesgue spaces \( L^p(\mu) \) and \( L^q(\nu) \), where \( \mu \) and \( \nu \) are arbitrary \( \sigma \)-finite measures we have the following statements. The equality \( \|T\| = \|T\| \) holds for all \( T \in L(L^p(\mu), L^q(\nu)) \) if \( 0 < p \leq q \leq \infty \). Moreover, the equality \( \|T\| = \|T\| \) holds for all positive operators \( T \in L(L^p(\mu), L^q(\nu)) \) without any restriction on \( p \) and \( q \), i.e. if \( 0 < p, q \leq \infty \). Taking counting measures \( \mu \) and \( \nu \) we get the same statements for the operators \( T \in L(\ell^p_n, \ell^q_n) \). We note, that in the case of finite-dimensional \( \ell^p \) spaces the notion of positivity of an operator can be specified as follows.

**Definition 7.** A linear operator \( T : \ell^p_n \to \ell^q_n \), where \( 0 < p, q \leq \infty \) is called positive if for all \( x = (x_1, x_2, \ldots, x_n) \in \ell^p_n \) with \( x_i \geq 0 \) and \( 1 \leq i \leq n \) we have

\[
T(x) = (y_1, y_2, \ldots, y_m) \quad \text{and} \quad y_k \geq 0 \quad \text{for} \quad 1 \leq k \leq m.
\]

The following proposition extends the class of operators \( T \in L(\ell^p_n, \ell^q_n) \) for which the equality of real and complex norms holds for all \( 0 < p, q \leq \infty \).
Proposition 9. Let $0 < p, q \leq \infty$. If an operator $T : l^p_n \to l^q_m$ is defined by a matrix $A = \{a_{ij}\}$ for which any two rows have the same or opposite sequence of signs of the corresponding nonzero elements, then real and complex norms of the operator are equal.

**Proof.** By the definition of the norm of a linear operator we have

$$
\|T_C\| = \sup_{\|x\| \neq 0} \|Tx\| = \sup_{\|x\| \neq 0} \left( \frac{\left( \sum_{i=1}^{m} \left| \sum_{j=1}^{n} a_{ij} x_j \right|^q \right)^{q/p}}{\left( \sum_{j=1}^{n} |x_j|^p \right)^{q/p}} \right)
$$

$$
\leq \sup_{\|x\| \neq 0} \left( \frac{\left( \sum_{i=1}^{m} \left( \sum_{j=1}^{n} |a_{ij}| \right) |x_j|^q \right)^{q/p}}{\left( \sum_{j=1}^{n} |x_j|^p \right)^{q/p}} \right).
$$

Assume that the latter expression attains its supremum at the complex-valued vector $x = (x_1, ..., x_n)$. Clearly, the supremum is also attained at $x = (|x_1|, ..., |x_n|)$. Due to special property of matrix $A$ the equality in (6) holds at $x = (\text{sgn}(a_{11}) |x_1|, ..., \text{sgn}(a_{1n}) |x_n|)$. Thus, the norm of the operator $T$ is attained at the real vector and therefore $\|T_C\| = \|T\|$. □

**Remark 20.** Proposition 9 asserts that complex and real norms of any operator defined by $2 \times 2$ matrix with nonnegative product of its elements are equal.

The results on constant $\gamma(l^p_n, l^q_m)$ which will be obtained in this chapter can be briefly summarized in the following theorem.

**Theorem 17.** Let $n, m \in \mathbb{N}$, $n, m \geq 2$.

(i) If $0 < p, q \leq \infty$, then $\gamma(l^p_n, l^q_m) \leq 2$.

(ii) If $1 \leq p, q \leq \infty$, then $\gamma(l^p_n, l^q_m) \leq \sqrt{2}$.

(iii) If $0 < p \leq q \leq \infty$, then $\gamma(l^p_n, l^q_m) = 1$. If $0 < q < p \leq \infty$ and $n, m \geq 3$, then $\gamma(l^p_n, l^q_m) > 1$.

(iv) $\gamma(l^2_n, l^2_m) = 1$ if and only if either $0 < p \leq 2$ or $q \geq 2$.

(v) If $0 < q \leq 2 \leq p$, then $\gamma(l^p_n, l^q_m) \leq \min\{2, 2^{1/q-1/2}\}$. Moreover, if $1 \leq q \leq 2 \leq p$, then $\gamma(l^p_n, l^q_m) \leq \min\{2^{1/q-1/2}, 2^{1/p'-1/2}\}$.

(vi) If $1 \leq q \leq 2$, then $\gamma(l^2_n, l^q_m) = 2^{1/q-1/2}$. If $p \geq 2$, then $\gamma(l^p_n, l^q_m) = 2^{1/p'-1/2}$.

(vii) If $q \geq 2$, then $\gamma(l^p_n, l^2_m) = 1$. If $1 \leq p \leq 2$ and $q \geq 1$, then $\gamma(l^p_n, l^2_m) = 1$.

Properties (i), (ii) and (iii) follow from the obvious relation $\gamma(l^p_n, l^q_m) \leq \gamma_{p,q}$ and the similar properties of constant $\gamma_{p,q}$ established in Theorems 8 and 9 as well as in Example 3. In the next two propositions we obtain all couples $(p, q)$ with $p, q > 0$ for which we can claim that the norm of any operator $T \in \mathcal{L}(l^p_n, l^q_m)$ coincides with the norm of its complexification $T_C$. The first proposition is a simple generalization of the statement of Verbickii-Sereda [59].

Proposition 10. Let $n \in \mathbb{N}$ and $n \geq 2$. If $q \geq 2$, then $\|T_C\|_{p,q} = \|T\|_{p,q}$ for any $T \in \mathcal{L}(l^p_n, l^q_m)$. If $1 \leq p \leq 2$ and $q \geq 1$, then $\|T_C\|_{p,q} = \|T\|_{p,q}$ for any $T \in \mathcal{L}(l^p_n, l^q_m)$. If $0 < q < 2 < p$, then there exists an operator $F \in \mathcal{L}(l^2_n, l^2_m)$ such that $\|F_C\|_{p,q} \neq \|F\|_{p,q}$. 


6. Basic Properties of Constant \( \gamma(t_0^n, t_n^0) \)

**Proof.** Assume the operator \( T \) is defined by the rule \( T(x, y) = (a_{1,1}x + a_{1,2}y, ..., a_{n,1}x + a_{n,2}y) \) and let the norm of \( T_C \) be attained at the vector \( v = (z, w) \) with \( z, w \in C \). If \( z = 0 \), then \( v = w(0,1) \) \( (w \neq 0) \) and

\[
\|T_C\|_{p,q} = \sup_{\|v\|_p \neq 0} \frac{\|T_C v\|_q}{\|v\|_p} = \frac{\|T_C(w(0,1))\|_q}{\|w(0,1)\|_p} = \frac{|w| \|T_C(0,1)\|_q}{\|w\| \|T_C(0,1)\|_p} = \|T_C(0,1)\|_q, \]

that shows the norm of \( T_C \) is also attained at the real vector \((0,1)\) and thus \( \|T_C\|_{p,q} = \|T\|_{p,q} \). The same observation says that \( w \neq 0 \). Now, for \( z \neq 0 \) take \( v = z(1,w/z) \) and note \( \frac{\|T_C(z(1,w/z))\|_q}{\|z(1,w/z)\|_p} = \frac{\|T_C(1,w/z)\|_q}{\|z\|_p} \). Assume that \( w/z = \rho(\cos \varphi + i \sin \varphi) \), then \( \|T_C(1, \rho(\cos \varphi + i \sin \varphi))\|_q \) is equal to

\[
\|\left( a_{1,1} + a_{1,2} \rho \cos \varphi + i a_{1,2} \rho \sin \varphi, ..., a_{n,1} + a_{n,2} \rho \cos \varphi + i a_{n,2} \rho \sin \varphi \right)\|_q.
\]

Consider the function \( \phi_p(\varphi) \) defined for every fixed \( p \) on the interval \([0, \pi]\) by

\[
(6.4) \quad \phi_p(\varphi) = (a_{1,1}^2 + a_{1,2}^2 \rho \cos \varphi + a_{2,1}^2 \rho^2)^{\frac{q}{2}} + ... + (a_{n,1}^2 + a_{n,2}^2 \rho \cos \varphi + a_{n,2}^2 \rho^2)^{\frac{q}{2}}.
\]

Note, that \( \|T_C(1, \rho(\cos \varphi + i \sin \varphi))\|_q = \phi_p(\varphi)^{1/q} \) and thus, the norm of \( T_C \) is attained at the same vector at which the function \( \phi_p(\varphi) \) has its maximum. Since the function \( \phi_p(\varphi) \) is even and periodic with period \( 2\pi \), it is enough to consider its values on the interval \([0, \pi]\). The derivative of \( \phi_p(\varphi) \) is

\[
\phi_p'(\varphi) = -q \rho \sin \varphi \sum_{k=1}^{n} a_{k,1} a_{k,2} \left( a_{k,1}^2 + a_{k,1} a_{k,2} \rho \cos \varphi + a_{k,2}^2 \rho^2 \right)^{q/2} + ... + a_{n,1} a_{n,2} \rho \cos \varphi + a_{n,2}^2 \rho^2)^{q/2}.
\]

We note \( \phi_p(\varphi) \) can have its maximum on \([0, \pi]\) at either \( \varphi_1 = 0 \) or \( \varphi_2 = \pi \) or \( \psi(\varphi_3) = 0 \), where \( \varphi_3 \in (0, \pi) \). Consider the functions \( \psi_k(\varphi) = a_{k,1} a_{k,2} (a_{k,1}^2 + 2 a_{k,1} a_{k,2} \cos \varphi + a_{k,2}^2 \rho^2)^{q/2} - 1 \) for \( k = 1, ..., n \). For \( q \geq 2 \) these functions are decreasing on \([0, \pi]\) and thus the sum \( \psi(\varphi) = \sum_{k=1}^{n} \psi_k(\varphi) \) is also a decreasing function. If there exists \( \varphi_1 \in (0, \pi) \) such that \( \psi(\varphi_1) = 0 \) then it is easy to check that \( \phi_p'(\varphi) \) changes its sign from minus to plus and hence \( \phi_p(\varphi) \) decreases first and then increases on \([0, \pi]\) which also means that \( \varphi_3 \) should be the point of minimum of \( \phi_p(\varphi) \). Therefore \( \phi_p(\varphi) \) may attain its maximum only at the end points of the interval \([0, \pi]\) or, in other words, at the real vector and therefore \( \|T_C\|_{p,q} = \|T\|_{p,q} \).

The second statement of this proposition can be obtained by duality from the first one which we have just proved.

As an example of the operator \( F \in L(\ell_2^n, \ell_2^n) \) for which \( \|F_C\|_{p,q} \neq \|F\|_{p,q} \) we can take \( F(x, y) = (x + y, x - y) \). Real and complex norms of this operator differ for the case when \( 0 < q < 2 < p \) (see Chapter 5). This example can be easily extended to the case of operator belonging to \( L(\ell_2^n, \ell_2^n) \).

**Remark 21.** If \( 1 \leq q < 2 \), then for \( k = 1, ..., n \) the functions \( \psi_k(\varphi) \) defined in Proposition 10 are increasing on \([0, \pi]\) and so does the function \( \psi(\varphi) = \sum_{k=1}^{n} \psi_k(\varphi) \). Assume as before that there exists \( \varphi_3 \in (0, \pi) \) such that \( \psi(\varphi_3) = 0 \), then \( \phi_p'(\varphi) \) changes its sign from plus to minus, that implies \( \phi_p(\varphi) \) increases first and then decreases on the interval \([0, \pi]\) and therefore \( \phi_p(\varphi) \) attains its maximum on \([0, \pi]\).
Problem 5. Take any operator \( T: \ell_p^2 \rightarrow \ell_2^2 \) and \( p > q \geq 2 \). Is it true that \( \| T \| = \| T \|_r \)? Recall that Riesz example 3 shows that for all \( 0 < q < p \leq \infty \) there exists an operator \( T \in \mathcal{L}(\ell_p^2, \ell_q^2) \) with \( \| T \| > \| T \|_r \).

In the proof of Proposition 10 we used the duality argument. In the next lemma we obtain this statement directly and what is the more important we extend the result on the equality of complex and operator norms to the quasi-Banach case.

Proposition 11. Let \( T \in \mathcal{L}(\ell_p^2, \ell_q^2) \). If \( 0 < p, q \leq 2 \), then \( \| T \|_{p,q} = \| T \|_{p,q} \).

Proof. Assume that the operator \( T: \ell_p^2 \rightarrow \ell_q^2 \) is given by the rule \( T(x, y) = (ax + by, cx + dy) \), where \( a, b, c, d \in \mathbb{R} \). Proposition 9 yields that the equality \( \| T \|_{p,q} = \| T \|_{p,q} \) holds for \( abcd \geq 0 \). Thus, without loss of generality, let us assume that \( a, b, c > 0 \) and \( d < 0 \). If the operator \( T \) attains its norm at the complex vector \((z_1, z_2)\). Then, \( T \) also attains its norm at \((1, z)\), where \( z = \rho \cos \theta + i \sin \theta \in \mathbb{C} \) and we will show that either \( \theta = 0 \) or \( \theta = \pi \) or \( \rho = 0 \) or \( \rho = \infty \) that will imply the equality \( \| T \|_{p,q} = \| T \|_{p,q} \).

For \( 0 < p, q \leq 2 \) consider the function on \([0, \infty] \times [0, \pi] \)

\[
F(\rho, \theta) = \left( (a^2 + 2\rho \cos \theta + b^2 \rho^2) \right)^{\frac{1}{2}} + \left( c^2 + 2d \rho \cos \theta + d^2 \rho^2 \right)^{\frac{1}{2}}
\]

It is clear that \( F \) is even in \( \theta \) and thus we can consider only those \( \theta \) which belong to \([0, \pi] \). The maximum of this function is also the norm of the operator \( T \). We show that the maximum of \( F \) is attained on the boundary of the region \([0, \infty] \times [0, \pi] \) or, in other words, at either \( \theta = 0 \) or \( \theta = \pi \) or \( \rho = 0 \) or \( \rho = \infty \). Since

\[
F'(\rho, \theta) = -\rho \sin \theta \frac{Q^{1/q-1}}{p^{2/q}} \left( ab (a^2 + 2b \rho \cos \theta + b^2 \rho^2) \right)^{1/2} + cd (c^2 + 2d \rho \cos \theta + d^2 \rho^2)^{1/2-1},
\]

then the candidates for the point of maximum are defined by

\[
\theta = 0, \theta = \pi \quad \text{or}
\]

\[
\frac{a^2 + 2ab \cos \theta + b^2 \rho^2}{c^2 + 2cd \rho \cos \theta + d^2 \rho^2} = \left( -\frac{cd}{ab} \right)^{\frac{1}{q-2}} = \alpha.
\]

If condition (6.7) is satisfied, then we get what we wanted. Thus, only relation (6.8) can contribute \( \theta \) which differs from 0 and \( \pi \). In this case if we had \( c^2 + 2cd \rho \cos \theta + d^2 \rho^2 = 0 \), then it should follow for \( abcd < 0 \) that \( a^2 + 2ab \rho \cos \theta + b^2 \rho^2 = 0 \) and therefore \( \| T \|_{(1, z)} = 0 \), that is clearly not possible. Now,

\[
F'(\rho, \theta) = \frac{Q^{1/q-1}}{p^{2/q}} \left( (a^2 + 2ab \rho \cos \theta + b^2 \rho^2)^{1/2} + (c^2 + 2cd \rho \cos \theta + d^2 \rho^2)^{1/2-1} \right)
\]

\[
+ \left( ab \cos \theta + ab \rho \cos \theta + b^2 \rho + b^2 \rho^{p-1} - a^2 \rho^{p-1} - 2ab \rho \cos \theta - b^2 \rho^{p+1} \right)
\]

\[
+ \left( cd \cos \theta + cd \rho \cos \theta + d^2 \rho^2 + d^2 \rho^{p+1} - c^2 \rho^{p-1} - 2cd \rho \cos \theta - d^2 \rho^{p+1} \right)
\]
6. BASIC PROPERTIES OF CONSTANT $\gamma(l^m_n, c_n^m)$

\[
Q^{1/q-1}_{p^{1/p}+1}((a^2 + 2abp \cos \theta + b^2 p^2)^{q/2-1}(ab(1 - \rho^p) \cos \theta + b^2 \rho - a^2 \rho^{p-1}) + (c^2 + 2cdp \cos \theta + d^2 p^2)^{q/2-1}(cd(1 - \rho^p) \cos \theta + d^2 \rho - c^2 \rho^{p-1})).
\]

Hence, $F^*_p(\rho, \theta) = 0$ if

\[
(a^2 + 2abp \cos \theta + b^2 \rho^p)^{q/2-1}(ab(1 - \rho^p) \cos \theta + b^2 \rho - a^2 \rho^{p-1}) + (c^2 + 2cdp \cos \theta + d^2 p^2)^{q/2-1}(cd(1 - \rho^p) \cos \theta + d^2 \rho - c^2 \rho^{p-1}) = 0.
\]

Therefore, the extremum points are defined by

\[
\frac{(a^2 + 2abp \cos \theta + b^2 \rho^p)^{q/2-1}}{(c^2 + 2cdp \cos \theta + d^2 p^2)^{q/2-1}} = -\frac{cd(1 - \rho^p) \cos \theta + d^2 \rho - c^2 \rho^{p-1}}{ab(1 - \rho^p) \cos \theta + b^2 \rho - a^2 \rho^{p-1}}.
\]

The combination of conditions (6.7) and (6.9) implies that the maximum of $F(\rho, \theta)$ can be attained at $\theta = 0$ or $\theta = \pi$ and $\rho$ defined by the relation (6.9), i.e. on the boundary of $[0, \infty] \times [0, \pi]$ and we are done, otherwise due relation (6.8) we should have the point of extremum $(\rho, \theta)$ defined by

\[
\frac{cd}{ab} = \frac{cd(1 - \rho^p) \cos \theta + d^2 \rho - c^2 \rho^{p-1}}{ab(1 - \rho^p) \cos \theta + b^2 \rho - a^2 \rho^{p-1}}.
\]

For $p \neq 2$ this equality is satisfied for $\rho_1 = \left(\frac{-bd}{ac}\right)^{1/p} \neq 0$ and for $1 < p \leq 2$ we have additional solution $\rho_2 = 0$. Now we prove that $(\rho_1, \theta_{\rho_1})$, where $\theta_{\rho_1}$ is defined from relation (6.8) cannot be the point of global maximum of $F(\rho, \theta)$.

In general for $\rho \neq 0$ relation (6.8) implies

\[
\cos \theta_{\rho} = \frac{(c^2 + d^2 \rho^2)\alpha - (a^2 + b^2 \rho^2)}{2\rho(ab - cd)}.
\]

After we substitute this expression in (6.5) and use relation (6.8) we get

\[
f(\rho) = \frac{1}{(1 + \rho^p)^{1/p}} \left(\frac{a^2 + ab(c^2 + d^2 \rho^2)\alpha - (a^2 + b^2 \rho^2) + b^2 \rho^2}{ab - cd}\right)^{q/2} + \left(\frac{c^2 + cd(c^2 + d^2 \rho^2)\alpha - (a^2 + b^2 \rho^2) + d^2 \rho^2}{ab - cd}\right)^{q/2} \right)^{1/q}
\]

\[
= \left(\frac{a^2 + ab(c^2 + d^2 \rho^2)\alpha - (a^2 + b^2 \rho^2)}{ab - cd}\right)^{1/2} \frac{b^2 \rho^2}{\left(1 + \rho^p\right)^{1/p}} (1 + \alpha^{-q/2})^{1/q}.
\]

The function $f(\rho)$ has the derivative

\[
f'(\rho) = \frac{a^2 + ab(c^2 + d^2 \rho^2)\alpha - (a^2 + b^2 \rho^2)}{ab - cd} b^2 \rho^2 \left(1 + \rho^p\right)^{1/p+1}(ab - cd)^{1/2} \times
\rho [bd(ad\alpha - bc) + acp^{-2}(ad - bco)].
\]

Thus, $\rho_1 = \left(\frac{-bd(\alpha a - \beta c)}{ac(ad - bc)}\right)^{1/p}$ or $\rho_2 = 0$. We note that $\rho_1$ coincides with $\left(\frac{-bd}{ac}\right)^{1/p}$ for either $\alpha = 1$ or $ad = -bc$. Moreover, for $p < 2$ and $\rho = \left(\frac{-bd(\alpha a - \beta c)}{ac(ad - bc)}\right)^{1/p}$ we have that $f'(\rho) < 0$ and if $\rho = \left(\frac{-bd(\alpha a - \beta c)}{2ac(ad - bc)}\right)^{1/p} > \rho_1$, then $f'(\rho) > 0$. Therefore, if $\rho$ and $\theta$ are connected by relation (6.8) then the function $f(\rho)$ first decreases and then increases that implies $F(\rho_1, \theta_{\rho_1}) \leq F(0, \theta)$ or $F(\rho_1, \theta_{\rho_1}) \leq$
F(∞, θ) or, in other words, there exists some point (ρ, θ) on the boundary for which $F(ρ, θ) ≥ F(ρ_1, θ_{ρ_1})$ and hence $F(ρ_1, θ_{ρ_1})$ is not the point of global maximum of $F(ρ, θ)$.

In the case when $p = 2$ and $ρ ≠ 0$ to satisfy (6.10) we should have $ac = −bd$ and thus

$$f′(ρ) = \left(\frac{a^2 + ab(\ell^2 + 2\rho^2)\cos(α^2 + b^2ρ^2) + b^2ρ^2}{(1 + \rho^2)^{1/p+1}(ab − cd)^{1/2}}\right)\rho bd (α - 1)[ad + bc].$$

Hence, if $α ≠ 1$ and $ad + bc ≠ 0$ then $f′(ρ) = 0$ if and only if $ρ = 0$. Otherwise, if $α = 1$ or $ad + bc = 0$, then $f′(ρ) = 0$ for every $ρ$ or $f(ρ)$ is constant in $ρ$ and thus $F(ρ, θ) = F(0, θ)$.

Hence, the global maximum of $F$ is attained on the boundary. Therefore the norm of the operator $T$ is attained at the real vector and $||T||_{p,q} = ||Tc||_{p,q}$. □

**Remark 22.** If $0 < q < 2 < p$, then the same arguments as were presented in Proposition 11 show that $F(ρ_1, θ_{ρ_1})$ defines the norm of the operator $T$, where

$$ρ_1 = \left(-\frac{bd}{ac}\right)^{\frac{1}{2-q}} \text{ and } \cos θ_{ρ_1} = \left(\frac{c^2 + d^2ρ_1^2 - (a^2 + b^2ρ_1^2)}{2ρ_1(ab − cd)}\right) \text{ with } α = \left(\frac{cd}{ab}\right)^{\frac{1}{2-q}}.$$

Summarizing the results of Propositions 10 and 11 we have

**Theorem 18.** If either $0 < p \leq 2$ or $q \geq 2$, then real and complex norms of any operator $T ∈ L(l_2^n, l_2^n)$ are equal, i.e. $||T||_{p,q} = ||Tc||_{p,q}$. Conversely, if $q < 2 < p$, then for the operator $F : l_2^n → l_2^n$ given by $T(x, y) = (x + y, x − y)$ we have strict inequality $||F||_{p,q} < ||Fc||_{p,q}$.

**Example 5.** Let consider an operator $T : l_2^n → l_2^n$ defined by the formula $T(x, y) = (x + ay, x − ay)$. If $0 < q ≤ 2 ≤ p ≤ ∞$, then by Remark 22 the complex norm of this operator is attained at the vector $(1, a^{\frac{1}{2-q}})$ and therefore $||Tc|| = 2^{1/q}(1 + a^{\frac{1}{2-q}})\frac{1}{2} / \left(1 + a^{\frac{1}{2-q}}\right)^{1/p}$ (if $p = 2$, then $||Tc|| = 2^{1/q}\max\{1, |a|\}$) that for $a = 1$ gives $||Tc|| = 2^{1/q−1/p+1/2}$ and this formula is in agreement with Theorem 10. On the other hand the real norm of the operator $T(x, y) = (x + ay, x − ay) : l^p → l^p$ for $0 < q ≤ 1$ and $2 ≤ p ≤ ∞$ is $2^{1/q}\max\{1, |a|\}$ (the proof is similar to that presented in Theorem 11). Hence, the relation $||Tc|| / ||T|| ≤ 2^{1/2−1/p} = \frac{||Fc||}{||F||}$, where $F(x, y) = (x + y, x − y)$. Remark 22 also helps to show that for the same values of $p$ and $q$ the operator $T(x, y) = (ax + by, bx − ay)$ attains its norm at $(1, i)$ whereof we have $||Tc|| = 2^{1/q−1/p}(a^2 + b^2)^{1/2}$.

The following proposition gives the estimate of the relation between complex and real norms of any operator $T ∈ L(l_2^n, l_2^n)$ in the case not covered by Theorem 18.

**Proposition 12.** Let $T ∈ L(l_2^n, l_2^n)$ and $n ≥ 2$. If $0 < q ≤ 2 ≤ p$, then

$$||Tc||_{p,q} ≤ \min\{2, 2^{1/q−1/2}\} ||T||_{p,q}.$$

Moreover, if $1 ≤ q ≤ 2 ≤ p$, then $||Tc||_{p,q} ≤ \min\{2^{1/q−1/2}, 2^{1/p−1/2}\} ||T||_{p,q}$. 
6. BASIC PROPERTIES OF CONSTANT $\gamma(l_n^p, l_m^q)$

Proof. Let operator $T$ be given by $n \times 2$ matrix $A = (a_{i,j})$. Then for $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ we get

$$||T_C(z_1, z_2)||_q^p = \sum_{k=1}^n |a_{k,1}z_1 + a_{k,2}z_2|^q$$

$$= \sum_{k=1}^n |a_{k,1}r_1e^{i\theta_1} + a_{k,2}r_2e^{i\theta_2}|^q$$

$$= \sum_{k=1}^n |e^{i\theta_1}| |a_{k,1}r_1 + a_{k,2}r_2e^{i(\theta_2 - \theta_1)}|^q$$

$$= \sum_{k=1}^n (a_{k,1}^2r_1^2 + 2a_{k,1}a_{k,2}r_1r_2 \cos(\theta_2 - \theta_1) + a_{k,2}^2r_2^2)^{q/2}$$

In the trigonometric identity $\cos(\theta) = \cos^2(\theta/2) - \sin^2(\theta/2)$ with $\theta = \theta_2 - \theta_1$ we denote $\cos^2(\theta/2)$ by $\alpha$ and $\sin^2(\theta/2)$ by $\beta$. Clearly, $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$. Then,

$$\sum_{k=1}^n (a_{k,1}^2r_1^2 + 2a_{k,1}a_{k,2}r_1r_2 \cos(\theta_2 - \theta_1) + a_{k,2}^2r_2^2)^{q/2}$$

$$= \sum_{k=1}^n \left( \alpha(a_{k,1}^2r_1^2 + 2a_{k,1}a_{k,2}r_1r_2 + a_{k,2}^2r_2^2) + \beta(a_{k,1}^2r_1^2 - 2a_{k,1}a_{k,2}r_1r_2 + a_{k,2}^2r_2^2) \right)^{q/2}$$

$$= \sum_{k=1}^n |a_{k,1}r_1 + a_{k,2}r_2|^2 + \beta(a_{k,1}r_1 - a_{k,2}r_2)^2|^{q/2}$$

$$\leq a^{q/2} \sum_{k=1}^n |a_{k,1}r_1 + a_{k,2}r_2|^q + \beta^{q/2} \sum_{k=1}^n |a_{k,1}r_1 - a_{k,2}r_2|^q$$

$$\leq (\alpha^{q/2} + \beta^{q/2}) ||T||_{p,q}^q ||r_1^p + r_2^p||^{q/p}$$

$$= (\alpha^{q/2} + \beta^{q/2}) ||T||_{p,q}^q (|z_1|^p + |z_2|^p)^{q/p}$$

$$\leq 2^{1-q/2} ||T||_{p,q}^q (|z_1|^p + |z_2|^p)^{q/p}.$$ 

We should mention here that inequality (1) holds for $q \leq 2$ by Lemma 11. Boundedness of $T$ implies inequality (2) and from H"older inequality it follows inequality (3). Hence, $||T_C||_{p,q}/||T||_{p,q} \leq 2^{1/q-1/2}$. By Theorem 17 (i) we also have $||T_C||_{p,q}/||T||_{p,q} \leq 2$. Therefore, $||T_C||_{p,q}/||T||_{p,q} \leq \min\{2, 2^{1/q-1/2}\}$. The second statement can be obtained from just proved by duality. 

Corollary 6. Let $n \in \mathbb{N}$, $n \geq 2$.

(i) If $1 \leq q \leq 2$, then $\gamma(l_n^p, l_m^q) = 2^{1/q-1/2}$.

(ii) If $p \geq 2$, then $\gamma(l_n^p, l_1^q) = 2^{1/p'-1/2}$.

Proof. (i) Consider the operator $F$ given by $n \times 2$ matrix $(a_{i,j})$ with $a_{11} = a_{12} = a_{21} = -a_{22} = 1$ and all other elements $a_{i,j}$ are equal to zero. Applying this operator to any vector in $l_n^p$ results in $n$-dimensional vector with at most two
nonzero components. It allows us to consider the operator $F$ as an operator between two-dimensional $l^p$ spaces and hence using the results of Theorems 10 and 11 from Chapter 5 we conclude that $\|F_c\|_{\infty,q}/\|F\|_{\infty,q} = 2^{1/q-1/2}$. That means the estimate $\|T_c\|_{\infty,q}/\|T\|_{\infty,q} \leq 2^{1/q-1/2}$ of Proposition 12 cannot be improved in this case and consequently we have $\gamma(l_\infty^p,l_n^q) = 2^{1/q-1/2}$.

The second statement follows from (i) by duality. 

**Problem 6.** The operator $T \in \mathcal{L}(l_p^2, l_q^2)$ given by $T(x, y) = (x + y, x - y)$ is extremal in the sense that for $0 < p, q \leq \infty$ the constant $\gamma(l_p^2, l_q^2)$ is attained on this operator. In other words, among all operators $\mathcal{L}(l_p^2, l_q^2)$ the operator $T$ has the biggest relation between its complex and real norms. If it is true, then for $0 < p, q \leq \infty$, then $\gamma(l_p^2, l_q^2) \leq \sqrt{2}$. 

In this chapter we collect some well known results concerning the Riesz-Thorin interpolation theorem in the complex case and show how the knowledge of the relation between complex and real operator norms helps to get the real version of this theorem from the complex one. It seems that this method of getting the real Riesz-Thorin interpolation theorem is the simplest one if we are careful of the so called Riesz constant. We also present here some new, as we hope, information concerning the real version of this theorem for the case when at least one of the spaces is quasi-Banach. Thus, we treat both real and complex versions of the Riesz-Thorin interpolation theorem in the whole first quadrant. Moreover, we draw some conclusions concerning the real Riesz-Thorin interpolation theorem for the operators between certain finite dimensional $l^p$ spaces and between mixed norm Lebesgue spaces.

There are different ways to state the Riesz-Thorin interpolation theorem and here we follow the classical one (cf. Bergh-Löfström [4], Bennett-Sharpley [3], Krein-Petunin-Semenov [32], Zygmund [65]).

**Definition 8.** A function $f$ defined on the measure space $(\Omega, \mu)$ is said to be simple if it takes only finite number of values. If, additionally, $\mu(\Omega) = \infty$, then we also assume that function $f$ vanishes outside a subset of $\Omega$ of finite measure. We will denote the set of simple functions by $S$.

**Definition 9.** Let $(\Omega_1, \mu)$ and $(\Omega_2, \nu)$ be $\sigma$-finite measure spaces and let operator $T$ be defined on all $\mu$-simple functions on $\Omega_1$ and resulting in the $\nu$-measurable functions on $\Omega_2$. Suppose $0 < p, q \leq \infty$. If there is a nonnegative constant $M$ such that

$\|Tf\|_{L^q(\nu)} \leq M\|f\|_{L^p(\mu)},$  (7.1)

for all $\mu$-simple functions $f$ on $\Omega_1$, then $T$ is said to be of strong type $(p, q)$. The least constant $M$ for which (7.1) holds is called the strong type $(p, q)$ norm of $T$ (or simply the norm of $T$ if no confusion can arise).

If $0 < p < \infty$ and $T$ is defined only for simple functions on $\Omega_1$, then $T$ can be uniquely extended to all functions of $L^p(\Omega_1, \mu)$ with the preservation of its norm, since $S$ is dense in $L^p(\Omega_1, \mu)$. We note that $S$ is also dense in $L^\infty$, if $\mu(\Omega_1) < \infty$.

For arbitrary measure spaces Calderón-Zygmund in [10] stated the following generalization of the Riesz-Thorin interpolation theorem.
Theorem 19 (Riesz-Thorin Interpolation Theorem). Let \((\Omega_1, \mu)\) and \((\Omega_2, \nu)\) be two measure spaces and \(T\) be a linear operator defined for all simple complex-valued functions \(f\) on \((\Omega_1, \mu)\). Let \(0 < p_0, p_1, q_0, q_1 \leq \infty\), \(0 < \theta < 1\) and

\[
\left( \frac{1}{p}, \frac{1}{q} \right) = (1 - \theta) \left( \frac{1}{p_0}, \frac{1}{q_0} \right) + \theta \left( \frac{1}{p_1}, \frac{1}{q_1} \right)
\]

If \(T\) is simultaneously of strong types \((p_0, q_0)\) and \((p_1, q_1)\) with strong type constants \(M_0\) and \(M_1\) respectively, then \(T\) is also of strong type \((p, q)\) and its strong type norm \(M_0\) satisfies

\[
M_0 \leq M_0^{1-\theta} M_1^\theta.
\]

For the sake of completeness we put here the proof of this theorem given by Calderón and Zygmund in [10].

Proof. Let \(k > 0\) be so small that

\[
k q_0 < 1, \quad k q_1 < 1,
\]

and let \(p\) and \(q\) be given by (7.2). We observe that relations (7.2) and (7.4) imply \(k/q < 1\). Hence,

\[
\|TF\|_q^k = \left\| |TF|^{k/q} \right\|_{q/k} = \sup_g \left\{ \int_{\Omega_2} |TF|^{k/q} d\nu : g \text{ is simple and } \|g\|_{1/(1-k/q)} = 1 \right\}.
\]

We may assume that \(\|f\|_p = 1\) and \(\theta \geq 0\). Let fix \(f\) and \(g\), write \(f = |f| e^{iu}\) and consider the integral

\[
I = \int_{\Omega_2} |TF|^k g d\nu.
\]

Consider also the functions \(p(z)\) and \(q(z)\) defined by

\[
\frac{1}{p(z)} = (1 - \theta) \frac{1}{p_0} + \theta \frac{1}{p_1} \quad \text{and} \quad \frac{1}{q(z)} = (1 - \theta) \frac{1}{q_0} + \theta \frac{1}{q_1},
\]

which for \(z = 0\) and \(z = 1\) reduce to \(p_0, q_0\) and \(p_1, q_1\) respectively. Moreover, \(p(\theta) = p\) and \(q(\theta) = q\).

Suppose \(p < \infty\). We introduce the simple functions

\[
F_z = |f|^{p/p(z)} e^{iu}, \quad G_z = g^{1-k/q(z)}
\]

depending on the parameter \(z\) and the integral

\[
\Phi(z) = \int_{\Omega_2} |TF|^k G_z d\nu,
\]

which reduces to \(I\) for \(z = \theta\) (since \(q \geq 0\)). If \(c_1, c_2, \ldots\) and \(c_1', c_2', \ldots\) be the various values taken by the functions \(f\) and \(g\) respectively, and let \(\chi_1, \chi_2, \ldots\) and \(\chi_1', \chi_2', \ldots\) be the characteristic functions of the sets where they are taken, then writing \(c_j = |c_j| e^{iu}\), we have for \(0 \leq x \leq 1\),

\[
f = \sum c_j e^{iu_j} \chi_j, \quad g = \sum c_j' \chi_j',
\]

\[
F_z = \sum e^{iu_j} |c_j|^{p/p(z)} \chi_j, \quad G_z = \sum (c_j')^{1-k/q(z)} \chi_j'
\]

and

\[
TF_z = \sum e^{iu_j} |c_j|^{p/p(z)} T \chi_j.
\]
It is easy to see that $G_z$ and $TF_z$ are linear combinations of functions $\lambda^i$ with $\lambda > 0$ and with coefficient functions defined on $\Omega_2$. Thus $|TF_z|^k|G_z|$ for every point in $\Omega_2$ is a continuous and subharmonic function in $z = x + iy$ for $0 \leq x \leq 1$ and so also $\Phi(z)$.

The function $\Phi(z)$ is also bounded there. We note,

$$|TF_z|^k = \sum e^{iz_j} |c_j|^{p/p(z)} |T\xi_j|^k \leq \text{const} \cdot \sum |T\xi_j|^k,$$

$$|G_z| = \sum (c_j)^{1 + k/p(z)} |\xi_j|^k \leq \text{const} \cdot \sum |\xi_i|,$$

(7.7) $$|\Phi(z)| \leq \text{const} \cdot \int_{\Omega_2} \sum |T\xi_j|^k |\xi_j|^k \, dv = \text{const} \cdot \sum \int_{\Omega_2} |T\xi_j|^k \, dv,$$

where $\Omega_{2,t}$ is a subset of $\Omega_2$ where $\lambda_i \neq 0$. Thus $\Omega_{2,t}$ is of finite measure. Taking $k$ so small that $k < q_1$ and applying the Hölder inequality with $q_1/k$ so as to get the integrals $\int |X_{\xi_j}|^m$, which are finite by the assumption, we see that the right hand side of (7.7) is finite, which proves the boundedness of $\Phi(z)$.

Let us consider any $z = x + 0$. The real parts of $p(z)$ and $q(z)$ are $p_0$ and $q_0$. Application of the Hölder inequality with $q_0/k$ to (7.6) gives

$$|\Phi(z)| \leq \left\| |TF_z|^k |G_z|_{1/(1-k/q_0)} \right\|_{q_0/k} \leq \left\| |TF_z|^k \right\|_{q_0/k} \left\| |G_z|_{1/(1-k/q_0)} \right\|_{q_0/k} \leq M_0^k \left\| |TF_z|^k \right\|_{q_0/k} \left\| |G_z|_{1/(1-k/q_0)} \right\|_{q_0/k}.$$

Taking into account functions $f$ and $g$ we have for all $0 < p_0, q_0 \leq \infty$

$$\left\| TF_z \right\|_{p_0} = \left\| f^{1-p/p_0} \right\|_{p_0} = \left\| f \right\|_{p_0} = 1,$$

$$\left\| G_z \right\|_{1/(1-k/q_0)} = \left\| g^{1-k/q_0} \right\|_{1/(1-k/q_0)} = \left\| g \right\|_{1/(1-k/q_0)} = 1.$$  

Hence, $|\Phi(z)| < M_0^k$ on the line $x = 0$. Similarly $|\Phi(z)| < M_1^k$ on the line $x = 1$.

Hence, using Hadamard three lines lemma we have $I = \Phi(\theta) \leq M_0^k (1-\theta) M_1^k \phi$. Applying (7.5) we get (7.3).

In the foregoing argument we used that $p < \infty$. If $p = \infty$, then also $p_0 = p_1 = \infty$ and the assumption of this theorem can then be written for $i = 0, 1$

(7.8) $$\left\| TF \right\|_{q_1} \leq M_{\text{esssup}} |f|.$$  

Then, by the Hölder inequality and (7.8) we have for all positive $q_0$ and $q_1$

$$\left\| TF \right\|_{q_0} = \int_{\Omega_2} |TF|^q \, dv = \int_{\Omega_2} \left( |TF|^{q_0} \right)^{q_0/q_0} \left( |TF|^{q_0} \right)^{q_1/q_0} \, dv \leq \left\| |TF|^{q_0} \right\|_{q_0/q_0} \left\| |TF|^{q_0} \right\|_{q_1/q_0} = \left\| |TF|^{q_0} \right\|_{q_0/q_0} \left\| |TF|^{q_0} \right\|_{q_1/q_0}$$

and the proof is complete.  

**Remark 23.** If $0 < p_i < \min \{ 1, q_i \}$ for some $i \in \overline{1, 1}$ and $\mu$ is a non-atomic measure, then any operator $T \in L^p(\Omega_i, \mu), L^q(\Omega_i, \nu)$ is trivial (cf. Remark 5) and we clearly have equality in (7.3) despite there exists no measure in this statement. However, if the measure $\mu$ is discrete, then Theorem 19 is no longer trivial.
This can be one of the reasons why we consider the Riesz-Thorin interpolation theorem in the whole first quadrant. Another reason is, the Hölder inequality shows that every operator of strong type \((p,q)\) is also of strong type \((p_i,q_i)\) for \(0 < q_i < q\) and it is a natural question about the behavior of the norm of this operator as a function of the point \((p,q)\).

Let consider the Riesz-Thorin interpolation theorem for real Lebesgue spaces. The useful notion of a Riesz constant appears. Let real Riesz constant be defined by

\[
\rho(\theta, p_i, q_i) = \inf \{ \gamma > 0 : \|T\|_{p,q} \leq \gamma \|T\|_{p_0,q_0}^{1-\theta} \|T\|_{p_1,q_1}^\theta \}
\]

for all \(T\) of strong types \((p_i,q_i)\), \(i = 0,1\), where \(p\) and \(q\) are given by (7.2), \(0 < \theta < 1\) and \(0 < p_i, q_i \leq \infty\) for \(i = 0,1\) (cf. Brudnyi-Krugljak [9]). By \(\rhoC(\theta, p_i, q_i)\) we denote the similar constant in the complex case. The connection between real and complex Riesz constants can be expressed in terms of numbers \(\gamma_{p,q}\) appearing in the relation \(\|T_c\|_{p,q} \leq \gamma_{p,q} \|T\|_{p,q}\) and which have been found before. Namely, since

\[
\|T\|_{p,q} \leq \|T_c\|_{p,q} \leq \rhoC(\theta, p_i, q_i) \|T_c\|_{p_0,q_0}^{1-\theta} \|T\|_{p_1,q_1}^\theta
\]

and

\[
\rho(\theta, p_i, q_i) \leq \rhoC(\theta, p_i, q_i) \gamma_{p_0,q_0}^{1-\theta} \gamma_{p_1,q_1}^\theta.
\]

Then \(\rho(\theta, p_i, q_i) \leq \rhoC(\theta, p_i, q_i) \gamma_{p_0,q_0}^{1-\theta} \gamma_{p_1,q_1}^\theta\). In Theorem 19 it was established that \(\rhoC(\theta, p_i, q_i) = 1\) for all \(0 < p_i, q_i \leq \infty\) and \(i = 0,1\) and thus

\[
(7.9)
\]

Theorem 20 (Riesz-Thorin interpolation theorem in the real case). Let \((\Omega_1, \mu)\) and \((\Omega_2, \nu)\) be two measure spaces and \(T\) be a linear operator defined for all simple functions \(f\) on \((\Omega_1, \mu)\). Let \(0 < p_0, p_1, q_0, q_1 \leq \infty\), \(0 < \theta < 1\) and

\[
\left( \frac{1}{p}, \frac{1}{q} \right) = (1-\theta) \left( \frac{1}{p_0}, \frac{1}{q_0} \right) + \theta \left( \frac{1}{p_1}, \frac{1}{q_1} \right).
\]

If \(T\) is simultaneously of strong types \((p_0,q_0)\) and \((p_1,q_1)\) with strong type constants \(M_0\) and \(M_1\) respectively, then \(T\) is also of strong type \((p,q)\) and its strong type norm \(M_\theta\) satisfies

\[
M_\theta \leq \rho(\theta, p_i, q_i) M_0^{1-\theta} M_1^\theta,
\]

where

\[\text{Figure 3. Real Riesz constant.}\]

(i) If \(0 < p_i, q_i \leq \infty\), \(i = 0,1\), then \(\rho(\theta, p_i, q_i) \leq 2\).

(ii) If \(1 \leq p_i, q_i \leq \infty\), \(i = 0,1\), then \(\rho(\theta, p_i, q_i) \leq \gamma_{p_0,q_0}^{1-\theta} \gamma_{p_1,q_1}^\theta \leq \gamma_{\infty,1} = \sqrt{2}.
\]
(iii) If \(1 \leq p_i, q_i \leq \infty, i = 0, 1\), then \(\rho(\theta, p_i, q_i) \leq \min\left\{\frac{d_{p_{i0}}}{d_{00}}, \frac{d_{q_{i0}}}{d_{00}}\right\}^{1-\theta}\min\left\{\frac{d_{p_{i1}}}{d_{10}}, \frac{d_{q_{i1}}}{d_{10}}\right\}^{\theta}.\)

(iv) If \(0 < p_i \leq q_i \leq \infty, i = 0, 1\), then \(\rho(\theta, p_i, q_i) = 1.\)

(v) If \(T\) is a positive operator, then \(\rho_+(\theta, p_i, q_i) = 1\) for \(0 < p_i, q_i \leq \infty\), where \(\rho_+\) is defined similar to the real Riesz constant with one additional property that the operators \(T\) are taken positive.

**Proof.** All these estimates of the constant \(\rho(\theta, p_i, q_i)\) can be obtained from inequality (7.9), Theorems 8, 9 and Proposition 7.

**Remark 24.** If \(p_i > q_i\) for some \(i \in \{0, 1\}\) (one point in the "upper triangle"), then the inequality \(\|T\|_{p,q} \leq \|T\|_{p_0,q_0}^{1-\theta}\|T\|_{p_1,q_1}^\theta\) does not hold in general.

**Example 6.** Take operator \(T(x, y)\) defined by matrix \(\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\). As it was calculated early \(\|T\|_{\infty,1} = 2\), \(\|T\|_{2,2} = \sqrt{2}\) and \(\|T\|_{4,4} = \sqrt{3}\) (see Theorem 10 and the consequence of Theorem 11 from Chapter 5). In the settings of Riesz-Thorin interpolation theorem \(3/4 = (1 - 1/2) \cdot 1/2 + 1/2 \cdot 1\) and \(\theta = 1/2.\) But \(\|T\|_{4,4} = \sqrt{3} > (\sqrt{2})^{1/2} 2^{1/2} = 2^{3/4} = \|T\|_{2,2}^{1/2} \|T\|_{\infty,1}^{1/2}\). Moreover, Thorin [58] showed that \(\|T\|_{p,q}\) is logarithmically concave as a function of \(1/p\) for \(p \geq 2.\)

**Remark 25.** It is not possible to replace the condition \(p_i \leq q_i\) for \(i = 0, 1\) by \(p \leq q\) in the real Riesz-Thorin interpolation theorem.

**Example 7 (Vogt [61]).** Take operator \(T\) given by matrix \(\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\) with \(p_0 = \infty, q_0 = 1, p_1 = 3/2, q_1 = 3\) and \(\theta = 3/4\). Then, the relation \(\|T\|_{p,q} \leq \|T\|_{p_0,q_0}^{1-\theta}\|T\|_{p_1,q_1}^\theta\) does not hold with this modified condition.

**Remark 26.** In most applications of the Riesz-Thorin theorem it is not necessary to know the exact value of the Riesz constant. However, sometimes this knowledge is very important, as it is in the interpolation proof of the classical Clarkson inequality from which the uniform convexity of \(L^p\) spaces for \(1 \leq p < \infty\) follows.

Using Theorems 8, 9, 17 and Proposition 9 we can derive in similar way the real Riesz-Thorin interpolation theorem for bounded linear operators between finite-dimensional \(l^p_0 - l^q_0\) spaces from the complex Riesz-Thorin interpolation theorem with better Riesz constant for some values of \(p, q\) and \(n, m\) than those in (7.3). We can also obtain real version of the Riesz-Thorin interpolation theorem for the operator between mixed norm Lebesgue spaces from its complex analogue proved by Benedek-Panzone [2] and Theorems 2 and 7.
Bibliography

7. RIESZ-THEORIN INTERPOLATION THEOREM


7. RIESZ-THORIN INTERPOLATION THEOREM

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