Clarkson Type Inequalities and Geometric Properties of Banach Spaces

Anna Klisinska
Clarkson Type Inequalities and Geometric Properties of Banach Spaces

Anna Klisinska
Department of Mathematics
Luleå University of Technology
S – 97 187 Luleå
SWEDEN
# TABLE OF CONTENTS

I Introduction and preliminaries

I.1. Introduction
I.2. Preliminaries to Banach spaces
I.3. Rotundity properties of Banach spaces

II Generalized Clarkson's inequalities and the concepts of type and cotype

II.1. Clarkson's inequalities
II.2. Generalized Clarkson's inequalities
II.3. Clarkson type inequalities for Banach spaces
II.4. Type and cotype of Banach spaces
II.5. Connections between generalized Clarkson's inequalities and type, cotype
II.6. The von Neumann-Jordan constant

III Modulus of convexity and modulus of smoothness

III.1. Modulus of convexity
III.2. Modulus of smoothness
III.3. Modulus of convexity in $L^p$ spaces
III.4. Examples

IV References
Acknowledgments

I would like to express the deepest gratitude to my supervisor Professor Lars-Erik Persson for sharing his great knowledge with me, his guidance, constant support and for making the dream comes true.

Further, I am indebted to my second supervisor Professor Lech Maligranda for the extensive and most detailed reading of my work.

I would also like to thank Professor Anna Kamińska from the Department of Mathematical Sciences at the University of Memphis, USA and Professor Ludmila Nikolova from the University of Sofia, Bulgaria for all help and for being my friends during those difficult days.

Thanks also go to colleagues at Department of Mathematics and my friends in the Female Graduate School for all special time that we spend together.

Finally, I would like to thank my husband Marek for his patience, support and being with me during those good and bad days.
1 Introduction and preliminaries

1.1 Introduction

Inequalities have been very important for the development of different branches of mathematics as functional analysis, interpolation theory, theory of differential and integral equations, probability, harmonic analysis, function theory, fixed point theory, etc. Nowadays the theory of inequalities may be regarded as an independent area of mathematics with two new specialized journals: Journal of Inequalities and Applications and Journal of Mathematical Inequalities and Applications and a great variety of books (see e.g. [17], [3], [37], [33], [36]).

Clarkson’s inequalities and their generalizations are the main tools in this thesis. Since they were introduced and proved in [7] in the context of uniform convexity of $L^p$-spaces, Clarkson’s inequalities have been investigated in many papers (e.g. [5], [25], [49], [22], [32], etc.). We will also investigate some concepts like strict and uniform convexity, the von Neumann-Jordan constant, type and cotype of Banach spaces. The notation of strict convexity of a Banach space as well as uniform convexity of a Banach space was introduced by James A. Clarkson in [7]. He gave also several interesting properties of this class of Banach spaces, which are useful in many areas of functional analysis, such as geometry of Banach spaces, series, fixed point theory or harmonic analysis. Uniform convexity of Lorentz spaces has been studied by I. Halperin in [15] and of Orlicz spaces with the Orlicz norm and nonatomic measure by H. W. Milnes in [35] and with Luxemburg norm in the nonatomic or counting measure cases by A. Kamińska in [21].

The first Chapter contains some basic informations from the theory of Banach spaces. In this Chapter we also present the classes of strictly convex and uniformly convex spaces. Moreover, some related information is discussed and derived.

In the second Chapter we study Clarkson’s inequalities, which were introduced by J. Clarkson in 1936. First we restrict ourselves to $L^p$-spaces and prove classical Clarkson inequality, e.g. of the form

$$\left(\|f + g\|_p^p + \|f - g\|_p^p\right)^{1/p} \leq 2^{1/p} \left(\|f\|_p^p + \|g\|_p^p\right)^{1/p}, f, g \in L^p, 1 \leq p \leq 2.$$

We use here just the classical Riesz-Thorin interpolation theorem to prove a basic inequality for complex numbers which easily implies various
generalization of Clarkson’s inequality. This technique can be used to prove
Clarkson type inequalities also in more dimensions. We present three forms
of generalized Clarkson’s inequalities: the standard form, the Kato form
and the Maligranda-Persson form, and show the difference between them.
The next step in this Chapter is to establish Clarkson type inequalities in
Banach spaces. In the last part of the second Chapter we show connections
between Clarkson’s inequalities and the concepts of type and cotype. In the
main result, it is shown that a Banach space satisfies Clarkson’s inequalities
if and only if it is of type $p$ with the type constant 1. We also prove the
similar result for cotype $q$.

The von Neumann-Jordan constant is closely related to Clarkson’s in-
equalities. It was defined by J. Clarkson in 1935 in connection with the
fact that von Neumann and Jordan had noted in [39] that for any Banach
space $X$ there exists a unique positive constant $C$, $1 \leq C \leq 2$, such that
for all $x, y \in X$, $(x, y) \neq (0, 0)$ the following inequalities always hold:

$$\frac{1}{C} \leq \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \leq C.$$ 

The smallest of such $C$ is called the von-Neumann-Jordan constant $C_{NJ}$.

In the last Chapter we discuss the concept of moduli of convexity and
smoothness for an arbitrary Banach space $X$. Clarkson used the concept of
modulus of convexity $\delta_X$ not giving however its explicit definition (cf. [7]).
He also estimated the modulus for $L^p$- spaces, when $p > 1$, showing that
these spaces are uniformly convex and in the case when $p > 2$ he gave an
explicit formula

$$\delta = 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}}.$$ 

For the case $1 < p \leq 2$ O. Hanner in 1956 established the precise formula
for the modulus of convexity $\delta = \delta_{L^p}$ of $L^p$- spaces which exact value can
be obtained from the following implicit equation

$$\left(1 + \frac{\varepsilon}{2}\right)^p + \left|1 - \delta - \frac{\varepsilon}{2}\right|^p = 2.$$ 

In the first theorem of this Chapter we formulate the fact that modulus
of convexity can be described in different but equivalent ways and in the
second theorem we formulate some properties of this function.

We also give a definition and some properties of another function con-
ected with the geometry of Banach spaces - modulus of smoothness, which
was introduced by M. M. Day in 1944 (see [8]).
In the last section of this Chapter we describe several examples of moduli of convexity and smoothness for different spaces.

1.2 Preliminaries to Banach spaces

In this section we are giving the definitions and basic notions from the theory of Banach spaces. Throughout this work $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ denotes the set of natural, real and complex numbers and $\mathbb{R}_+$ the set of non-negative reals. We shall also use $\mathbb{K}$ to denote $\mathbb{R}$ or $\mathbb{C}$.

Let $X$ be a vector space over a field $\mathbb{K}$. The norm on $X$ is a function

$$\|\cdot\|_X : X \rightarrow [0, \infty)$$

which satisfies:

(i) $\|x\|_X = 0$ if and only if $x = 0$
(ii) $\|\lambda x\|_X = |\lambda| \|x\|_X$
(iii) $\|x + y\|_X \leq \|x\|_X + \|y\|_X$

for all $x, y \in X$ and $\lambda \in \mathbb{K}$.

As a consequence of this definition we obtain the following properties:

(iv) $\|x\|_X = \|-x\|_X$
(v) $\||x|_X - \|y\|_X| \leq \min \{\|x + y\|_X, \|x - y\|_X\}$
(vi) $\||x|_X - \|y\|_X| + \|x\|_X + \|y\|_X \leq \|x + y\|_X + \|x - y\|_X$.

A vector space $X$ with a norm $\|\cdot\| = \|\cdot\|_X$ defined on it is called **normed space**. A normed space which is complete (in the natural metric $d(x, y) = \|x - y\|$) is called **Banach space**, which will be for us the most important class of normed spaces.

The unit ball and the unit sphere of the normed space $X$ are defined by

$$B = B_X = \{x \in X : \|x\| \leq 1\}, \quad S = S_X = \{x \in X : \|x\| = 1\}$$

5
respectively. From the properties of the norm we have that the unit ball $B$ is symmetric with respect to 0, convex, closed and bounded set and the unit sphere $S$ is closed, symmetric and bounded set. The ball of the normed space $X$ is the set
$$B(x, r) = \{ y \in X : \| x - y \| \leq r \}$$
and the open ball is the set
$$B^0(x, r) = \{ y \in X : \| x - y \| < r \} .$$

Finite dimensional normed spaces can be characterized by the compactness of the unit ball or the unit sphere (see e.g. [26], Th.2.5-5):

**Theorem 1.1.** (F. Riesz, 1918) The unit ball (or the unit sphere) of a normed space $X$ is compact if and only if the space $X$ is finite dimensional.

An important class of Banach spaces is the class of Hilbert spaces and which is characterized by the fact that the norm has some useful additional properties. First we define the notation of inner product (or scalar product).

An **inner product** on a vector space $X$ is a mapping
$$\langle *, * \rangle : X \times X \rightarrow \mathbb{K}$$
with the properties:

(i) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
(ii) $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$
(iii) $\langle x, y \rangle = \langle y, x \rangle$
(iv) $\langle x, x \rangle \geq 0$; $\langle x, x \rangle = 0 \iff x = 0$

for all $x, y \in X$ and $\lambda \in \mathbb{K}$.

The inner product has the following properties:

(v) $\langle x, \lambda y \rangle = \bar{\lambda} \langle x, y \rangle$
(vi) $|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle$ (the Cauchy-Schwarz inequality)
(vii) $\langle x + y, x + y \rangle^{\frac{1}{2}} \leq \langle x, x \rangle^{\frac{1}{2}} + \langle y, y \rangle^{\frac{1}{2}}$ (the Minkowski inequality)
(viii) $\| x \| = \langle x, x \rangle^{\frac{1}{2}}$ is a norm on $X$. 6
An inner product space is a vector space $X$ with an inner product defined on it. A Hilbert space is a complete inner product space. Hilbert spaces will generally be denoted by $H$.

**Theorem 1.2.** (Jordan - von Neumann, 1935) If $X$ is an inner product space, then the norm given by $\|x\| = (x,x)^{1/2}$ satisfies the parallelogram identity (parallelogram law), i.e.,

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

for all $x, y \in X$.

Conversely, if a normed space $X = (X, \|\|)$ satisfies the parallelogram identity, then it is an inner product space with the inner product given by

$$(x, y) = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2)$$

when $K = \mathbb{R}$

and

$$(x, y) = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2) + i(\|x + iy\|^2 - \|x - iy\|^2)$$

when $K = \mathbb{C}$.

Geometrically, a real two-dimensional inner product spaces have ellipses as spheres.

The set of all bounded linear functionals $f$ on a normed space $X$ constitutes a Banach space with the norm defined by

$$\|f\| = \sup_{x \in X, x \neq 0} \frac{|f(x)|}{\|x\|} = \sup_{x \in X, \|x\| = 1} |f(x)| = \sup_{x \in X, \|x\| \leq 1} |f(x)|.$$

This space is called the dual space of $X$ and it is denoted by $X^*$.

Between a normed space $X$ and its bidual $X^{**} := (X^*)^*$ there is a natural isometric injection $X \ni x \mapsto j_x \in X^{**}$ defined by

$$j_x(f) = f(x), \ f \in X^*.$$

If $X$ is a Banach space and the mapping $j$ is onto, then the space $X$ is called reflexive.

The following extension theorem is of fundamental importance in functional analysis:

**Theorem 1.3.** (Hahn - Banach extension theorem, 1929) Let $X$ be a Banach space and $Y$ a subspace of $X$. For any $f \in Y^*$ there exists $\tilde{f} \in X^*$ such that $\tilde{f}(x) = f(x)$ for any $x \in Y$ and

$$\|\tilde{f}\|_{X^*} = \|f\|_{Y^*}.$$
Corollary 1.4. For any \( x \in X \) there exists non-trivial \( f \in X^* \) such that
\[
f(x) = \|f\|_{X^*} \|x\|_X.
\]
It is not clear that every continuous functional attained the norm but we have

Theorem 1.5. (R. C. James, 1972) A Banach space \( X \) is reflexive if and only if for every \( f \in X^* \) there exists \( x \in X \setminus \{0\} \) such that
\[
f(x) = \|f\|_{X^*} \|x\|_X.
\]

Important and classical class of Banach spaces are Lebesgue \( L^p \)–spaces introduced by F. Riesz in 1910 for \( 1 \leq p < \infty \) and by H. Steinhaus in 1918 for \( p = \infty \).

Let \( (\Omega, \Sigma, \mu) \) be a measure space. We will consider everywhere only the \( \sigma \)–finite measure which means that \( \Omega = \bigcup_{n=1}^{\infty} A_n \) with \( \mu(A_n) < \infty \) for every \( n = 1, 2, \ldots \).

Let \( 1 \leq p < \infty \). The \( L^p \)–space is a set of all equivalence classes of \( \Sigma \)–measurable functions \( f : \Omega \to \mathbb{K} \), where \( f \) is equivalent to \( g \) if \( f = g \) \( \mu \)–almost everywhere on \( \Omega \), such that
\[
\int_{\Omega} |f|^p \, d\mu < \infty
\]
with the norm
\[
\|f\|_p = \left( \int_{\Omega} |f|^p \, d\mu \right)^{\frac{1}{p}}.
\]
Moreover, the space \( L^\infty \) consists of all equivalence classes of \( \Sigma \)–measurable functions such that
\[
\|f\|_\infty = \text{ess sup}_{t \in \Omega} |f(t)| < \infty
\]
where
\[
\text{ess sup}_{t \in \Omega} |f(t)| = \inf\{M > 0 : \mu\{t \in \Omega : |f(t)| > M\} = 0\}.
\]
Sometimes we can use the notation \( L^p(\mu) \) or \( L^p(\Omega) \) or \( L^p(\Omega, \Sigma, \mu) \).
Theorem 1.6. (Riesz-Thorin interpolation theorem, 1938) Let \((\Omega_1, \Sigma_1, \mu)\)
and \((\Omega_2, \Sigma_2, \nu)\) be two measure spaces, \(L^{p_i}(\Omega_i), L^{q_i}(\Omega_i)\) \((i = 0, 1; 1 \leq p_i, q_i \leq \infty)\)
be a Banach spaces of complex-valued functions and

\[
T : L^{p_0}(\mu) + L^{p_1}(\mu) \to L^{q_0}(\nu) + L^{q_1}(\nu)
\]
be a linear operator such that the restriction

\[
T \big|_{L^{p_i}(\mu)} : L^{p_i}(\mu) \to L^{q_i}(\nu)
\]
is bounded with the norm \(M_i; i = 0, 1\). Suppose that \(0 \leq \theta \leq 1\) and define \(p_\theta\) and \(q_\theta\) by

\[
\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.
\]

Then

\[
T : L^{p_\theta}(\mu) \to L^{q_\theta}(\nu)
\]
is linear and bounded with the norm

(1.1) \[M_\theta \leq M_0^{1-\theta} \cdot M_1^\theta.\]

Remark. For the real \(L^{p_i}(\mu), L^{q_i}(\nu)\)–spaces Theorem 1.6 remains
valid if \(p_i \leq q_i, i = 0, 1\). Otherwise, it holds but the exact estimate (1.1)
must be replaced by

\[
M_\theta \leq \sqrt{2} \cdot M_0^{1-\theta} \cdot M_1^\theta
\]
(cf. [4], [30]).

1.3 Rotundity properties of Banach spaces

In this chapter we investigate geometric properties of Banach spaces like
strict and uniform convexity and uniform non-squarness. These properties
are preserved under isometry but not necessarily under isomorphism of a
space.

A Banach space \((X, \|\cdot\|)\) is said to be strictly convex (or rotund) if
the mid-point of any line segment joining two different points on the unit
sphere of \(X\) does not lie on it, i.e., the following implication holds

\[
\|x\| = 1, \|y\| = 1, x \neq y \Rightarrow \|\frac{x+y}{2}\| < 1.
\]

Example 1.7. (a) The plane \(\mathbb{R}^2\) is strictly convex when it is equipped
with the Euclidean norm \(\|(x, y)\| = (x^2 + y^2)^{\frac{1}{2}}\), but it is not strictly convex
if it has the norms \(\|(x, y)\| = |x| + |y|\) or \(\|(x, y)\| = \max(|x|, |y|)\).
(b) Any Hilbert space is strictly convex.
(c) The space $C[a, b]$ of continuous real valued functions on $[a, b]$ with the
norm defined by formula
\[ \|x\| = \max \left\{ |x(t)| : t \in [a, b] \right\} \]
is not strictly convex.

Indeed, the proof of (a) is clear from the graphs of the corresponding unit
spheres.
(b) Take $\|x\| = \|y\| = 1$, $x \neq y$ and let $\|x - y\| = \alpha$, $0 < \alpha \leq 2$. Then, by
the parallelogram law, we find that
\[
\|x + y\| = -\|x - y\|^2 + 2 (\|x\|^2 + \|y\|^2)
= -\alpha^2 + 2 (1 + 1) = 4 - \alpha^2 < 4
\]
and thus $\|x + y\| < 2$.
(c) If we take the functions $x(t) = 1$ and $y(t) = \frac{b-a}{b-a} t$, $t \in [a, b]$, then $\|x\| = \|y\| = 1$, $x \neq y$, but the mid-point is on the unit sphere, i.e.,
\[
\left\| \frac{x + y}{2} \right\| = \frac{1}{2} \max_{t \in [a, b]} \left| 1 + \frac{t-a}{b-a} \right| = 1.
\]
The notation of strict convexity can be described in various equivalent ways
according to the following result:

**Theorem 1.8.** If $X$ is a Banach space, then the following properties
are equivalent:
(a) $X$ is strictly convex.
(b) If $x, y \in S_X$ and $x \neq y$ then $\|\lambda x + (1 - \lambda) y\| < 1$ for every $\lambda \in (0, 1)$,
i.e., the unit sphere $S_X$ does not contain any segment.
(c) If $x, y \in S_X$ and $x \neq y$ then $\|\lambda x + (1 - \lambda) y\| < 1$ for some $\lambda \in (0, 1)$.
(d) If for any $x, y, z \in X$, all different, $\|x - y\| = \|x - z\| + \|z - y\|$ then
there exists $\lambda \in (0, 1)$ such that $z = \lambda x + (1 - \lambda) y$.
(e) Every $f \in X^*$ has a unique maximum on the sphere.
(f) If $\|x + y\| = \|x\| + \|y\|$ for all $x \neq 0$, $y \neq 0$ then $x = cy$, for some $c > 0$.
(g) If for $x, y \in S_X$, $\|x + y\| = \|x\| + \|y\|$ then $x = y$.
(h) The function $h : X \to \mathbb{R}_+$, defined by $h(x) = \|x\|^2$ is strictly convex.

**Proof.** (a) $\Rightarrow$ (b) Take $x, y \in S_X$ and $x \neq y$. For $0 < \lambda < \frac{1}{2}$ we have
\[
z = \lambda x + (1 - \lambda) y = 2\lambda \cdot \frac{x + y}{2} + (1 - 2\lambda) y
\]
and then
\[
\|z\| \leq 2\lambda \left\| \frac{x + y}{2} \right\| + (1 - 2\lambda) \|y\| < 2\lambda + (1 - 2\lambda) = 1.
\]
When $\frac{1}{2} < \lambda < 1$ then
\[ z = \lambda x + (1 - \lambda) y = (2\lambda - 1) x + (2 - 2\lambda) \frac{x+y}{2} \]
and
\[ \|z\| \leq (2\lambda - 1) \|x\| + (2 - 2\lambda) \left\| \frac{x+y}{2} \right\| < 2\lambda - 1 + 2 - 2\lambda = 1. \]
Hence for all $\lambda \in (0, 1)$ we obtain that $\|\lambda x + (1 - \lambda) y\| < 1$.

(b) $\Rightarrow$ (a) Just take $\lambda = \frac{1}{2}$.

(a) $\Rightarrow$ (c) Obvious.

(c) $\Rightarrow$ (a) Take $x, y \in S_X$ and $x \neq y$. If $0 < \lambda_0 < \frac{1}{2}$ then
\[ \frac{x+y}{2} = \alpha x + (1 - \alpha) [\lambda_0 x + (1 - \lambda_0) y], \alpha = \frac{\lambda_0}{1 - \lambda_0} \]
and so
\[ \left\| \frac{x+y}{2} \right\| \leq \alpha \|x\| + (1 - \alpha) \|\lambda_0 x + (1 - \lambda_0) y\| < 1. \]
If $\frac{1}{2} < \lambda_0 < 1$ then
\[ \frac{x+y}{2} = \beta [\lambda_0 x + (1 - \lambda_0) y] + (1 - \beta) y, \beta = \frac{1}{2\lambda_0} \]
and so
\[ \left\| \frac{x+y}{2} \right\| \leq \beta \|\lambda_0 x + (1 - \lambda_0) y\| + (1 - \beta) \|y\| < 1. \]

(a) $\Rightarrow$ (d) Take $x, y, z \in X$, all different and such that $\|x - y\| = \|x - z\| + \|z - y\|$. Then $\|x - z\| \neq 0, \|z - y\| \neq 0$ and suppose that $\|x - z\| \leq \|z - y\|$. Then
\[
\left\| \frac{1}{2} \frac{x-z}{\|x-z\|} + \frac{1}{2} \frac{x-y}{\|x-y\|} \right\| = \left\| \frac{1}{2} \frac{x-z}{\|x-z\|} + \frac{1}{2} \frac{x-y}{\|x-y\|} - \frac{1}{2} \frac{x-y}{\|x-y\|} + \frac{1}{2} \frac{x-y}{\|x-y\|} \right\| \\
\geq \left\| \frac{1}{2} \frac{x-z}{\|x-z\|} + \frac{1}{2} \frac{x-y}{\|x-y\|} \right\| - \left\| \frac{1}{2} \frac{x-y}{\|x-y\|} - \frac{1}{2} \frac{x-y}{\|x-y\|} \right\| \\
= \left\| \frac{1}{2} \frac{x-z + x-y}{\|x-z\|} \right\| - \left\| \frac{1}{2} \frac{x-y}{\|x-y\|} (z - y) \right\| \\
= \frac{1}{2} \|x-y\| \|z-y\| + \|x-z\| \\
= \frac{1}{2} \|x-z\| + \frac{1}{2} \|z-y\| + \frac{1}{2} \|\lambda_0 x + (1 - \lambda_0) y\| \\
= 1,
\]
and according to the triangle inequality
\[
\left\| \frac{1}{2} \frac{x-z}{\|x-z\|} + \frac{1}{2} \frac{x-y}{\|x-y\|} \right\| = 1
\]
Thus, in view of the assumption (b),
\[
\frac{x-z}{\|x-z\|} = \frac{x-y}{\|z-y\|}
\]
or
\( \left( \frac{1}{\|x-z\|} + \frac{1}{\|z-y\|} \right) z = \frac{1}{\|x-z\|} x + \frac{1}{\|z-y\|} y. \)

If we take \( \lambda = \frac{1}{\|x-z\|}/\left( \frac{1}{\|x-z\|} + \frac{1}{\|z-y\|} \right) \) we get \( z = \lambda x + (1 - \lambda) y \) and (d) holds.

(d) \( \Rightarrow \) (a) Let \( x, y \in S_X, x \neq y \) and assume that \( \|\frac{x+y}{2}\| = 1 \). Thus \( \|x - (-y)\| = \|x\| + \|-y\| \) and then by (d) applied for \( z = 0 \), there exists \( \lambda \in (0, 1) \) such that \( \lambda x + (1 - \lambda) (-y) = 0 \).

Hence

\[ 0 = \|\lambda x + (1 - \lambda) (-y)\| \geq |\lambda| \|x\| - (1 - \lambda) \|-y\|| = |2\lambda - 1| \]

gives \( \lambda = \frac{1}{2} \) and so \( \frac{1}{2} x + \frac{1}{2} (-y) = 0 \) or \( x = y \), which is a contradiction with \( x \neq y \).

(b) \( \Rightarrow \) (e) Let \( X \) satisfy (b). Take any \( f \in X^* \) and suppose that there exists \( x_1 \neq x_2, 1 = \|x_1\| = \|x_2\| \) and \( f(x_1) = f(x_2) = \|f\| \). For any \( \lambda \in (0, 1) \) it yields that

\[ f(\lambda x_1 + (1 - \lambda) x_2) = \lambda f(x_1) + (1 - \lambda) f(x_2) = \|f\| \]

which implies

\[ \|\lambda x_1 + (1 - \lambda) x_2\| = 1 \]

which is a contradiction to (b). Consequently (e) holds.

(e) \( \Rightarrow \) (b) Suppose that there exists a line segment \([x_1, x_2]\) on \( S_X \). Take \( x = \frac{x_1 + x_2}{2} \). By the Corollary 1.4 there exists \( f \in X^* \) such that \( \|f\| = 1 \) and \( f(x) = \|x\| \). Then

\[ f(\frac{x_1 + x_2}{2}) = \|\frac{x_1 + x_2}{2}\| = 1 \]

and so

\[ \|f\| = f\left(\frac{x_1 + x_2}{2}\right) = 1 \]

and we have

\[ f\left(\frac{x_1 + x_2}{2}\right) = \frac{1}{2} f(x_1) + \frac{1}{2} f(x_2) = 1 \]

i.e.,

\[ f(x_1) + f(x_2) = 2. \]

Since \( f(x_1) \leq 1 \) and \( f(x_2) \leq 1 \) it follows that

\[ f(x_1) = 1, \ f(x_2) = 1. \]

This means that the norm \( \|f\| \) is attained at \( \frac{x_1 + x_2}{2}, x_1 \) and \( x_2 \) and, thus, according to the uniqueness assumption, \( \frac{x_1 + x_2}{2} = x_1 = x_2 \) and we obtain again a contradiction. The implication is proved.
(f) ⇒ (b) Take \( x, y \in S_X, x \neq y \). The equality \( \lambda x = c(1 - \lambda) y \), \( c > 0, 0 < \lambda < 1 \) can not take place because if so then \( \lambda \|x\| = c(1 - \lambda) \|y\| \) and
\[ \lambda = c(1 - \lambda) \] which means that \( x = y \) and we get a contradiction with our assumption.

If \( \lambda x \neq c(1 - \lambda) y \) and \( X \) satisfies (f), then
\[
\|\lambda x + (1 - \lambda) y\| < \|\lambda x\| + \|(1 - \lambda) y\|
\]
and hence \( i.e. (b) \) holds.

(b) ⇒ (f) Let \( X \) satisfy (b), \( x \neq 0, y \neq 0 \), \( \|x + y\| = \|x\| + \|y\| \), but \( x \neq cy \) for any \( c > 0 \). Then \( x' = \frac{x}{\|x\|}, y' = \frac{y}{\|y\|} \) satisfies \( x' \neq y' \) and \( \|x'\| = \|y'\| = 1 \).
In particular, we have:
\[
\|\lambda x' + (1 - \lambda) y'\| < 1 \quad \forall \lambda \in (0, 1).
\]

For \( \lambda = \frac{\|x\|}{\|x\| + \|y\|} \) we get:
\[
\frac{\|x\|}{\|x\| + \|y\|} x' + \frac{\|y\|}{\|x\| + \|y\|} y' = \left( \frac{\|x\|}{\|x\| + \|y\|} \right) \frac{x}{\|x\|} + \left( \frac{\|y\|}{\|x\| + \|y\|} \right) \frac{y}{\|y\|}
\]
so that
\[
\|x + y\| < \|x\| + \|y\|.
\]

From this contradiction we conclude that \( x = cy \) for some \( c > 0 \).

(a) ⇒ (g) Assume that \( x, y \in S_X, \|x + y\| = \|x\| + \|y\| \) and strict convexity of \( X \). Assume also that \( x \neq y \). Then \( \|\frac{x+y}{2}\| < 1 \).

From this contradiction we have that \( x = y \) so that (g) is satisfied.

(g) ⇒ (a) Take \( x, y \in S_X \) and \( x \neq y \). By (g) we have then:
\[
\|x + y\| < \|x\| + \|y\|,
\]
and thus
\[
\|\frac{x+y}{2}\| < 1,
\]
which means that \( X \) is strictly convex.

(b) ⇒ (h) Observe that \( h \) is a convex function. In fact,
\[
h(\lambda x + (1 - \lambda) y) =
\]
\[
= \|\lambda x + (1 - \lambda) y\|^2 \leq (\lambda \|x\| + (1 - \lambda) \|y\|)^2
\]
\[
\begin{align*}
= \lambda^2 \|x\|^2 + 2\lambda (1 - \lambda) \|x\| \|y\| + (1 - \lambda)^2 \|y\|^2 \\
\leq \lambda^2 \|x\|^2 + \lambda (1 - \lambda) (\|x\|^2 + \|y\|^2) + (1 - \lambda)^2 \|y\|^2 \\
= \lambda \|x\|^2 + (1 - \lambda) \|y\|^2 \\
= \lambda h(x) + (1 - \lambda) h(y).
\end{align*}
\]

Suppose now that \( h \) is not strictly convex, which means there exists \( \lambda_0 \in (0, 1) \) such that we have equality
\[
h(\lambda_0 x + (1 - \lambda_0) y) = \lambda_0 h(x) + (1 - \lambda_0) h(y),
\]
i.e.
\[
\lambda_0 \|x\| + (1 - \lambda_0) \|y\| = \lambda_0 \|x\|^2 + (1 - \lambda_0) \|y\|^2.
\]
Then, if we take \( x, y \in S_X \) we obtain
\[
\lambda_0 \|x\| + (1 - \lambda_0) \|y\| = 1
\]
and we have a contradiction with the strict convexity of \( X \).

(h) \( \Rightarrow \) (b) Let \( h \) be a strictly convex function and \( x, y \in X, x \neq y, \|x\| = \|y\| = 1 \). Suppose that there exists \( \lambda \in (0, 1) \) such that:
\[
\|\lambda x + (1 - \lambda) y\| = 1.
\]
Then
\[
h(\lambda x + (1 - \lambda) y) = \|\lambda x + (1 - \lambda) y\|^2 = 1
\]
\[
= \lambda \|x\|^2 + (1 - \lambda) \|y\|^2 = \lambda h(x) + (1 - \lambda) h(y)
\]
and we get a contradiction. Thus
\[
\|\lambda x + (1 - \lambda) y\| < 1
\]
and we have proved that (b) holds. \( \diamond \)

The next result is an interesting observation.

**Corollary 1.9.** The sum of an arbitrary norm and a strictly convex norm is also a strictly convex norm. More generally, if we define a new norm by the following formula
\[
\|x\|_p = (\|x\|^p + \|x\|_c^p)\frac{1}{p}, 1 \leq p < \infty,
\]
where \( \|\cdot\| \) is arbitrary norm and \( \|\cdot\|_c \) is a strictly convex norm, then \( \|\cdot\|_p \) is also a strictly convex norm.

**Proof.** Let \( \|\cdot\| \) be any norm and \( \|\cdot\|_c \) a strictly convex norm and define
\[
(1.1) \quad \|x\|' = \|x\| + \|x\|_c.
\]
Because $\|\cdot\|_c$ is strictly convex, then by Theorem 1.8, this is equivalent to

(1.2) \[ \|x + y\|_c = \|x\|_c + \|y\|_c \text{ for } x, y \neq 0 \iff x = ay, a > 0. \]

We need to show the implication

(1.3) \[ \|x + y\|' = \|x\|' + \|y\|' \text{ for } x, y \neq 0 \implies x = by, b > 0. \]

Suppose that

\[ \|x + y\|' = \|x\|' + \|y\|' \text{ for } x, y \neq 0. \]

Then, by (1.1)

\[ \|x + y\|' = \|x + y\| + \|x + y\|_c \]

and, by (1.3) and our assumption,

\[ \|x + y\|' = \|x\|' + \|y\|' = \|x\| + \|x\|_c + \|y\| + \|y\|_c, \]

which leads to

\[ \|x + y\| + \|x + y\|_c = \|x\| + \|y\| + \|x\|_c + \|y\|_c \]

for $x, y \neq 0$.

By the triangle inequality of $\|\cdot\|$ we have

\[ \|x\| + \|y\| + \|x + y\|_c \geq \|x + y\| + \|x + y\|_c \]

\[ = \|x\| + \|y\| + \|x\|_c + \|y\|_c \]

and so

\[ \|x + y\|_c \geq \|x\|_c + \|y\|_c. \]

But from the triangle inequality of $\|\cdot\|_c$ we also have

\[ \|x + y\|_c \leq \|x\|_c + \|y\|_c. \]

which gives

\[ \|x + y\|_c = \|x\|_c + \|y\|_c \]

and by (1.2): $x = ay, a > 0$. So we proved (1.3) which means that $\|\cdot\|'$ is strictly convex.

The proof that $\|\cdot\|_p$ is strictly convex is similar so we omit the details. ◦

The importance of the strict convexity can be illustrated on the uniqueness of the best approximation, on uniqueness in the Hahn-Banach theorem and on renorming Asplund theorem.
Approximation theory is concerned with the approximation of functions of a certain kind (for instance, continuous functions on some interval or functions from $L^p$) by other (probably simpler) functions (for example polynomials or step functions). Therefore if $X = (X, \| \cdot \|)$ is a Banach space and $Y$ is a fixed subspace of $X$, then the error is defined by

$$E(x, Y) = \inf_{y \in Y} \| x - y \|.$$ 

If there exists $y_0 \in Y$ such that $E(x, Y) = \| x - y_0 \|$, then $y_0$ is called a best approximation of $x$ out of $Y$.

Theorem 1.10. (uniqueness of best approximation) In a strictly convex Banach space $X$ there is at most one best approximation to an $x \in X$ out of a given subspace $Y$.

The Hahn-Banach theorem guarantee the existance of at least one minimal, norm-preserving extension $\tilde{f}$ but it can be many such extensions. The situation when the extension is unique is characterized by the strict convexity of the dual space as it follows from the next theorem.

Theorem 1.11. (Taylor-Foguel [12], 1958) Every bounded linear functional on every subspace of a Banach space $X$ has a unique norm-preserving extension to $X$ if and only if $X^*$ is strictly convex.

Theorem 1.12. (Asplund [1], 1968) If $X$ is a reflexive Banach space, then there exists a strictly convex norm on $X$ such that the dual norm is also strictly convex.

A stronger geometrical property than strict convexity is uniform convexity.

A Banach space $(X, \| \cdot \|)$ is said to be uniformly convex if, for each $\varepsilon \in (0, 2]$, there exists $\delta = \delta (\varepsilon) > 0$ such that for all $x, y \in X$

$$\| x \| = 1, \| y \| = 1, \| x - y \| \geq \varepsilon \Rightarrow \| \frac{x + y}{2} \| \leq 1 - \delta.$$ 

In terms of sequences this definition can be describe as follows: a Banach space $X$ is uniformly convex if and only if for any sequences $\{x_n\}$, $\{y_n\}$ of elements in $X$ with

$$\| x_n \| = 1, \| y_n \| = 1, \lim_{n \to \infty} \| x_n + y_n \| = 2$$  

we have

$$\lim_{n \to \infty} \| x_n - y_n \| = 0.$$  

The simplest example of a uniformly convex space is a Hilbert space. Other examples of such spaces include the sequence spaces $l^p$ and the Lebesgue spaces $L^p$ for $1 < p < \infty$. Detailed investigations of so called modulus of convexity, a function connected to uniform convexity of a space, are given further in Chapter III.

The next theorem gives one motivation for the notion of uniformly convex Banach spaces (cf. [43], see also [29], p.61).

**Theorem 1.13.** (Milman, 1938; Pettis, 1939) Every uniformly convex Banach space is reflexive.

**Remark.** There are reflexive spaces which are not even isomorphic to uniformly convex spaces. Spaces which are isomorphic to a uniformly convex space are called **super-reflexive**.

Any uniformly convex space is strictly convex. The two concepts are equivalent in finite dimensional spaces. In the case of infinite dimensional Banach spaces these concepts can be different in the sense that there are strictly convex Banach spaces which are not uniformly convex.

**Example 1.14.** (a) In $l^1$—space we take an equivalent norm

$$
\|x\|_0 = \sum_{k=1}^{\infty} |x_k| + \left( \sum_{k=1}^{\infty} |x_k|^2 \right)^{\frac{1}{2}}.
$$

Then by Corollary 1.9 this norm is strictly convex but it is not uniformly convex since $l^1$ is not a reflexive space.

If we take another norm in $l^1$ given by

$$
\|x\|_1 = \sum_{k=1}^{\infty} |x_k| + \left( \sum_{k=1}^{\infty} \frac{|x_k|^2}{k^2} \right)^{\frac{1}{2}},
$$

then, again by Corollary 1.9, this norm is strictly convex but without using Theorem 1.13 we can show that $(l^1, \|\cdot\|_1)$ is not uniformly convex. In fact, taking $x_n = \frac{n}{n+1} e_n$ and $y_n = x_{n+1}$ for $n \in \mathbb{N}$ we have

$$
\|x_n\|_1 = \frac{n}{n+1} + \frac{1}{n+1} = 1, \|y_n\|_1 = 1,
$$

$$
\|x_n + y_n\|_1 = \frac{2n+1}{n+1} + \left( \frac{1}{(n+1)^2} + \frac{(n+1)^2}{n^2(n+2)^2} \right)^{\frac{1}{2}} \to 2
$$

and

$$
\|x_n - y_n\|_1 = \|x_n + y_n\|_1 \to 2
$$
as $n \to \infty$ and it means that $(l^1, \|\cdot\|_1)$ is not uniformly convex.
(b) In the space \( C[0,1] \) with the supremum norm let the new equivalent norm be defined by
\[
\|x\|_1 = \sup_{t \in [0,1]} |x(t)| + \left( \int_0^1 |x(t)|^2 \, dt \right)^{\frac{1}{2}}.
\]
Then by Corollary 1.9 this norm is strictly convex but by Theorem 1.13 cannot be uniformly convex since \( C[0,1] \) is not a reflexive space.

2 Generalized Clarkson’s inequalities and the concepts of type and cotype

2.1 Clarkson’s inequalities

J. A. Clarkson in 1936 proved that all spaces \( L^p \), for \( 1 < p < \infty \), are uniformly convex. He used in [7] a set of inequalities which are analogous to the parallelogram identity for \( L^2 \)-spaces. A classical application of Clarkson’s inequalities is to prove that the corresponding space is uniformly convex and calculate its modulus of convexity. Another application is to find the von Neumann-Jordan constant. In this chapter we also develop the idea to use Clarkson type inequalities to describe the concepts of type and cotype of Banach spaces.

**Theorem 2.1.** If \( 1 \leq p \leq 2 \), then
\[
\left( \|f + g\|_p^p + \|f - g\|_p^p \right)^{\frac{1}{p}} \leq 2^{\frac{1}{p'}} \left( \|f\|_p^p + \|g\|_p^p \right)^{\frac{1}{p}}
\]
for any \( f, g \in L^p \).

If \( 2 \leq p \leq \infty \), then
\[
\left( \|f + g\|_p^p + \|f - g\|_p^p \right)^{\frac{1}{p}} \leq 2^{\frac{1}{p'}} \left( \|f\|_p^p + \|g\|_p^p \right)^{\frac{1}{p}}
\]
and
\[
\left( \|f + g\|_p^p + \|f - g\|_p^p \right)^{\frac{1}{p}} \leq 2^{\frac{1}{p'}} \left( \|f\|_p^p + \|g\|_p^p \right)^{\frac{1}{p'}}
\]
for any \( f, g \in L^p \). Here, as usual, \( p' \) is given by \( \frac{1}{p} + \frac{1}{p'} = 1 \).
Remark: In [7] Clarkson stated this theorem for the cases $l^p$ and $L^p[0,1]$ but it holds as well for the general case $L^p = L^p(\Omega)$ (see the proof below).

In this section we prove Theorem 2.1 and more general results by using the Riesz-Thorin interpolation theorem in a similar way as it was used in [32] (see also [31] and [42]).

**Lemma 2.2.** If $1 \leq p \leq 2$, then

\[
\left( |a + b|^p' + |a - b|^p' \right)^{\frac{1}{p'}} \leq 2^{\frac{1}{p'}} \left( |a|^p + |b|^p \right)^{\frac{1}{p}}.
\]

for any $a, b \in \mathbb{C}$.

If $2 \leq p \leq \infty$, then the reversed inequality holds.

**Proof.** Let $1 \leq p \leq 2$. We consider two-dimensional spaces $l^2_p$ as $\mathbb{C}^2$ with the norm $\|(a, b)\|_p = (|a|^p + |b|^p)^{\frac{1}{p}}$ and an easy operator given by the formula

\[
T(a, b) = (a + b, a - b).
\]

We can see that the operator $T : l^2_2 \to l^2_2$ has the norm

\[
M_0 = \|T\|_{2,2} = \sup_{|a|^2 + |b|^2 \leq 1} \left( |a + b|^2 + |a - b|^2 \right)^{\frac{1}{2}} = \sqrt{2}
\]

(by the parallelogram identity) and the operator $T : l^2_1 \to l^2_\infty$ has the norm

\[
M_1 = \|T\|_{1,\infty} = \sup_{|b| \leq 1} \max \{|a + b|, |a - b|\} = 1,
\]

(by the triangle inequality). Then by the Riesz-Thorin interpolation theorem $T : l^2_{p_\theta} \to l^2_{q_\theta}$ and

\[
M_\theta \leq M_0^{1-\theta} M_1^\theta = \left(2^{\frac{1}{2}}\right)^{1-\theta} \cdot 2^{\frac{1-\theta}{2}} = 2^{\frac{1-\theta}{2}},
\]

where $\frac{1}{p_\theta} = \frac{1-\theta}{2} + \frac{\theta}{1} = \frac{1+\theta}{2}$, $\frac{1}{q_\theta} = \frac{1-\theta}{2} + \frac{\theta}{\infty}$ and $0 \leq \theta \leq 1$. The last equalities means that $1 \leq p_\theta \leq 2$ and $\frac{1}{q_\theta} = 1 - \frac{1}{p_\theta}$. The inequality $M_\theta \leq M_0^{1-\theta} M_1^\theta$ can also be written as

\[
\|T\|_{p_\theta, q_\theta} \leq \|T\|_{2,2}^{1-\theta} \|T\|_{1,\infty}^\theta.
\]

so we can put $p = p_\theta$, $p' = q_\theta$ and get

\[
\|T\|_{p, p'} \leq \|T\|_{2,2}^{\frac{2}{p'}} \|T\|_{1,\infty}^{\frac{1-\frac{2}{p'}}{p'}} = 2^{\frac{1}{p'}}.
\]
Estimation of the norms gives
\[ \|T(a, b)\|_{l^2_p} \leq \|T\|_{p, p'} \|(a, b)\|_{l^2_p} \leq 2^{\frac{1}{p}} \|(a, b)\|_{l^2_p} \]
or, equivalently,
\[ \left(|a + b|^{p'} + |a - b|^{p'}\right)^{\frac{1}{p'}} \leq 2^{\frac{1}{p'}} \left(|a|^p + |b|^p\right)^{\frac{1}{p}}, \]
and inequality (2.4) is proved.
The case 2 \(\leq p \leq \infty\) follows from already proved estimate (2.4) by interchanging the roles of the parameters \(p\) and \(q\) and making the substitutions \(a_1 = a + b\) and \(a_2 = a - b\).

In order not to disturb our discussions later on we also formulate the following elementary lemma:

**Lemma 2.3.** (a) Let \(-\infty < \alpha \leq \beta < \infty\). Then
\[ \left(\frac{1}{n} \sum_{k=1}^{n} a_k^\alpha\right)^{\frac{1}{\beta}} \leq \left(\frac{1}{n} \sum_{k=1}^{n} a_k^\beta\right)^{\frac{1}{\beta}}. \]
(b) If \(\alpha \geq \beta > 0\), then
\[ \left(\sum_{k=1}^{n} a_k^\alpha\right)^{\frac{1}{\beta}} \left(\sum_{k=1}^{n} a_k^\beta\right)^{\frac{1}{\beta}}. \]

The statement in (a) is a consequence of the fact that the scale of power means is nondecreasing. Some historical remarks and recent developments concerning this fact (even for more general cases) can be found in [41]. The statement (b) follows from subadditivity of the function \(u \mapsto u^{\beta}\) and it is another way to write the usual imbedding between \(l^p\)-spaces.

To prove Theorem 2.1 we only need to show the inequality (2.1) because (2.3) is just another way to write (2.1) and, by Lemma 2.3 (a), (2.3) implies (2.2).

**Proof of Theorem 2.1.** Let \(1 < p \leq 2\). We have
\[
\|f + g\|_{p'}^{p'} + \|f - g\|_{p'}^{p'}
= \left(\int_{\Omega} |f(t) + g(t)|^{p'} \frac{p'}{p} \, d\mu\right)^{\frac{p'}{p}} + \left(\int_{\Omega} |f(t) - g(t)|^{p'} \frac{p'}{p} \, d\mu\right)^{\frac{p'}{p}}.
\]
We apply Lemma 2.2 with \( a = f(t) \) and \( b = g(t) \), and find that

\[
| f(t) + g(t) |^{p'} + | f(t) - g(t) |^{p'} \leq 2 \left( |f(t)|^{p} + |g(t)|^{p} \right)^{\frac{p'}{p}}.
\]

Since \( \frac{p}{p'} \leq 1 \) we can use the reversed Minkowski integral inequality and the above estimate and obtain that

\[
\| f + g \|^{p'}_{p} + \| f - g \|^{p'}_{p} \leq \left( \| f \|^{p} + \| g \|^{p} \right)^{\frac{p'}{p}}.
\]

\[
= 2 \left( \int_{\Omega} \left( |f(t)|^{p} + |g(t)|^{p} \right)^{\frac{p'}{p}} \, d\mu \right)^{\frac{p'}{p}}
\]

\[
= 2 \left( \int_{\Omega} |f(t)|^{p} \, d\mu(t) + \int_{\Omega} |g(t)|^{p} \, d\mu \right)^{\frac{p'}{p}}
\]

\[
= 2 \left( \| f \|^{p} + \| g \|^{p} \right)^{\frac{p'}{p}}.
\]

If \( p = 1 \) the estimate (2.1) follows easily from the triangle inequality. \( \diamond \)

We can also prove a more general version of the inequality (2.4).

**Proposition 2.4.** Let \( a, b \in \mathbb{C}, s \in \mathbb{R}, s \neq 0, r \in \mathbb{R}_{+} \). Then

\[
|a + b|^{s} + |a - b|^{s} \leq 2^{\gamma} \left( |a|^{r} + |b|^{r} \right)^{\frac{1}{s}}
\]

where \( \gamma = \frac{1}{s} - \frac{1}{r} + \frac{1}{t} \) and

\[
t = \begin{cases} 
\min{(2, r)} & \text{for } s \leq 2, \\
\min{(s', r)} & \text{for } s > 2, \frac{1}{s} + \frac{1}{s'} = 1.
\end{cases}
\]

**Proof.** We put

\[
A_{s} = (|a + b|^{s} + |a - b|^{s})^{\frac{1}{s}} \quad \text{and} \quad B_{r} = (|a|^{r} + |b|^{r})^{\frac{1}{s}}.
\]

According to Lemma 2.3 (a) and the parallelogram law we have that if \( s \leq 2 \), then
\[
(\left| \frac{a+b}{2} \right|^s + \left| \frac{a-b}{2} \right|^s)^{\frac{1}{s}} \leq \left( \left| \frac{a+b}{2} \right|^2 + \left| \frac{a-b}{2} \right|^2 \right)^{\frac{1}{2}}
\]

so that

\[A_s \leq 2^{\frac{1}{s}} B_2.\]

Because for \( r \leq 2 \) we also have \( B_2 \leq B_r \) so that if \( s \leq 2 \) and \( r \leq 2 \) then

\[A_s \leq 2^{\frac{1}{s}} B_2 \leq 2^{\frac{1}{s}} B_r.\]

If \( s \leq 2, r \geq 2 \), then as above \( A_s \leq 2^{\frac{1}{s}} B_2 \) and by Lemma 2.3 (a) we have

\[2^{-\frac{1}{2}} \cdot B_2 = \left( \frac{|a|^2 + |b|^2}{2} \right)^{\frac{1}{2}} \leq \left( \frac{|a|^r + |b|^r}{2} \right)^{\frac{1}{r}} = 2^{-\frac{1}{r}} \cdot B_r,
\]

so that

\[A_s \leq 2^{\frac{1}{s}} B_2 \leq 2^{\frac{1}{s} - \frac{1}{2} + \frac{1}{r}} B_r.
\]

For \( s > 2, r \leq s' \) by inequality (2.4) we have that \( A_s \leq 2^{\frac{1}{s}} B_{s'} \) and, by Lemma 2.3 (a), \( B_{s'} \leq B_r \) so that

\[A_s \leq 2^{\frac{1}{s}} B_{s'} \leq 2^{\frac{1}{s} - \frac{1}{s} + \frac{1}{s'}} B_r.
\]

If now \( s > 2, r \geq s' \) so again \( A_s \leq 2^{\frac{1}{s}} B_{s'} \) and, by Lemma 2.3,

\[2^{-\frac{1}{s'}} \cdot B_{s'} \leq 2^{-\frac{1}{r}} \cdot B_r.
\]

Hence

\[A_s \leq 2^{\frac{1}{s}} B_{s'} \leq 2^{\frac{1}{s} - \frac{1}{s'} + \frac{1}{s}} B_r.
\]

Together we have proved the following estimates:

\[
\begin{align*}
\text{if } s &\leq 2, r \leq 2 : A_s \leq 2^{\frac{1}{s}} B_r, \\
\text{if } s \leq 2, r \geq 2 : A_s \leq 2^{\frac{1}{s} - \frac{1}{r} + \frac{1}{2}} B_r, \\
\text{if } s > 2, r \leq s' : A_s \leq 2^{\frac{1}{s}} B_r, \\
\text{if } s > 2, r \geq s' : A_s \leq 2^{\frac{1}{s} - \frac{1}{r} + \frac{1}{s'}} B_r.
\end{align*}
\]

The proof follows by combining these inequalities. \( \diamond \)

If we make the substitutions \( a + b = a_1 \) and \( a - b = b_1 \) and interchange \( r \) and \( s \) we obtain the following:

**Corollary 2.5.** Let \( a,b \in \mathbb{C}, s \in \mathbb{R}_+ \) and \( r \in \mathbb{R}, r \neq 0 \). Then

\[
(|a| + |b|)^s + |a - b|^s \leq 2^s \left( |a|^r + |b|^r \right)^{\frac{1}{r}},
\]

for any \( f,g \in L^p \), where \( \gamma = \frac{1}{s} - \frac{1}{r} + \frac{1}{q} \) and

\[
q = \begin{cases} 
\min (2,s) & \text{for } r \leq 2, \\
\min (r',s) & \text{for } r > 2, \frac{1}{r} + \frac{1}{q} = 1.
\end{cases}
\]

22
Now we are ready to formulate the following generalization of Theorem 2.1.

**Theorem 2.6.** If $0 < r \leq p \leq s < \infty$, then

\[
\left( \| f + g \|_p^s + \| f - g \|_p^s \right)^{\frac{1}{s}} \leq 2\gamma \left( \| f \|_p^r + \| g \|_p^r \right)^{\frac{1}{r}},
\]

where $\gamma = \frac{1}{s} - \frac{1}{r} + \frac{1}{q}$ and

\[
q = \begin{cases} 
\min \left( r, s' \right) & \text{for } s \geq 2, \frac{1}{s} + \frac{1}{s'} = 1, \\
\frac{r}{s} & \text{for } 0 < s < 2.
\end{cases}
\]

**Proof.** Let us start with the following observation

\[
\| f + g \|_p^s + \| f - g \|_p^s = \left( \int_\Omega | f(t) + g(t) |^p \, d\mu(t) \right)^{\frac{s}{p}}
\]

\[
+ \left( \int_\Omega | f(t) - g(t) |^p \, d\mu(t) \right)^{\frac{s}{p}} = \| | f + g |^s \|_{\frac{r}{s}} + \| | f - g |^s \|_{\frac{r}{s}}.
\]

According to the reverse Minkowski inequality (since $\frac{p}{s} \leq 1$), Proposition 2.4 and the triangle inequality of the $L^r$-norm it follows that

\[
\| f + g \|_{\frac{r}{s}} + \| f - g \|_{\frac{r}{s}} \leq \| f + g \|_s^s + \| f - g \|_s^s
\]

\[
\leq \left\| 2\gamma^s (| f |^r + | g |^r)_{\frac{s}{r}}^s \right\|_{\frac{r}{s}} = \left( \left\| 2\gamma^r (| f |^r + | g |^r)_{\frac{s}{r}}^s \right\|_{\frac{r}{s}} \right)^{\frac{s}{r}}
\]

\[
= \left( \left\| 2\gamma f |^r + 2\gamma g |^r \right\|_{\frac{r}{s}} \right)^{\frac{s}{r}} \leq \left( \left\| 2\gamma f |^r \right\|_{\frac{r}{s}} + \left\| 2\gamma g |^r \right\|_{\frac{r}{s}} \right)^{\frac{s}{r}}
\]

\[
= 2\gamma^s \left( \| f \|_p^r + \| g \|_p^r \right)^{\frac{s}{r}} = 2\gamma^s \left( \| f \|_p^r + \| g \|_p^r \right)^{\frac{s}{r}}.
\]

By combining (2.7) with (2.8) we obtain (2.6) and the proof is complete. \( \diamond \)

**Remark.** The inequalities (2.1) – (2.3) in Theorem 2.1 are special cases of (2.6). In the case of $s = p', r = p, 1 < p \leq 2$ we obtain (2.1). For $s = r = p, p > 2$ we have (2.2) and finally when $s = p, r = p', p > 2$ we get (2.3).

The interpolation idea to prove Clarkson type inequalities presented in Lemma 2.2 is easy to generalize to this more general cases. In the final part
of this section will present some results which can be obtained in this way. We now consider any linear bounded operator \( T \) such that \( T : l_1^{(n)} \to l_\infty^{(m)} \) with the norm \( \| T \|_{1,\infty} = M_0 \) and \( T : l_2^{(n)} \to l_2^{(m)} \) with the norm \( \| T \|_{2,2} = M_1 \). By using the Riesz-Thorin interpolation theorem we obtain the estimate

\[
\| T(\vec{a}) \|_{i_p^{(m)}} \leq M_0^{\frac{1-\frac{2}{p}}{p'}} M_1^{\frac{2}{p'}} \| \vec{a} \|_{i_p^{(n)}}, \quad 1 \leq p \leq 2,
\]

where \( \vec{a} = (a_1, a_2, ..., a_n) \in \mathbb{C}^n \), which may be seen as a generalization of the inequality (2.4). For example we have the following generalization of Proposition 2.4 (see [32], Th.3.1):

**Theorem 2.7.** Let \( s \in \mathbb{R}, s \neq 0 \) and \( r \in \mathbb{R}_+ \). Then

\[
\left( \sum_{\varepsilon_i = \pm 1} 2^{-n} \left| \sum_{i=1}^n \varepsilon_i a_i \right|^s \right)^{\frac{1}{s}} \leq \left( \sum_{i=1}^n |a_i|^r \right)^{\frac{1}{r}}
\]

for any \( a_1, a_2, ..., a_n \in \mathbb{C} \), where \( q = \min (2, r) \) if \( s \leq 2 \) and \( q = \min (s', r) \) if \( s > 2 \).

**Proof.** Consider the operator

\[
T : \vec{a} \to \left( \sum_{1}^{n} a_i, ..., \sum_{1}^{n} \varepsilon_i a_i, ..., \sum_{1}^{n} -a_i \right),
\]

where \( \varepsilon_i = \pm 1, 1 \leq i \leq n \) (each coordinate of the vector \( T(\vec{a}) \in \mathbb{R}^m \), \( m = 2^n \) is equal to the sums of the type \( \sum_{i=1}^n \varepsilon_i a_i \)). We also note that

\[
T : l_1^{(n)} \to l_\infty^{(m)} \text{ has the norm } \| T \|_{1,\infty} = M_0 = 1 \quad \text{and} \quad T : l_2^{(n)} \to l_2^{(m)} \text{ has the norm } \| T \|_{2,2} = M_1 = 2^{\frac{n}{2}}.
\]

By using the Riesz-Thorin interpolation theorem we find that

\[
T : l_p^{(n)} \to l_p^{(m)}, \quad 1 \leq p \leq 2,
\]

with the norm \( M \leq M_0^{1-\theta} M_1^{\theta} = 1^{1-\theta} \cdot (2^{\frac{n}{2}})^{\theta} = 2^{\frac{n\theta}{2}}, \) i.e.

\[
(2.9) \quad \left( \sum_{\varepsilon_i = \pm 1} 2^{-n} \left| \sum_{i=1}^n \varepsilon_i a_i \right|^p \right)^{\frac{1}{p}} \leq \left( \sum_{i=1}^n |a_i|^p \right)^{\frac{1}{p}}, \quad 1 \leq p \leq 2.
\]

Now we let

\[
A_s = \left( \sum_{\varepsilon_i = \pm 1} 2^{-n} \left| \sum_{i=1}^n \varepsilon_i a_i \right|^s \right)^{\frac{1}{s}}, \quad \text{and} \quad B_r = \left( \frac{1}{n} \sum_{i=1}^n |a_i|^r \right)^{\frac{1}{r}}.
\]
Then, in the similar way as in the proof of Proposition 2.4 we can use (2.9) and Lemma 2.3 to obtain the following estimates:

\[ s \leq 2, r \leq 2 : A_s \leq \left( \sum_{i=1}^{n} |a_i|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{i=1}^{n} |a_i|^r \right)^{\frac{1}{r}} \leq n^{\frac{1}{r}} B_r, \]

\[ s \leq 2, r > 2 : A_s \leq \left( \sum_{i=1}^{n} |a_i|^2 \right)^{\frac{1}{2}} \leq n^{\frac{1}{2}} \cdot n^{-\frac{1}{r}} \cdot \left( \sum_{i=1}^{n} |a_i|^r \right)^{\frac{1}{r}} \leq n^{\frac{1}{2}} B_r, \]

\[ s > 2, r \leq s' : A_s \leq \left( \sum_{i=1}^{n} |a_i|^s \right)^{\frac{1}{s}} \leq \left( \sum_{i=1}^{n} |a_i|^r \right)^{\frac{1}{r}} \leq n^{\frac{1}{r}} B_r, \]

\[ s > 2, r > s' : A_s \leq \left( \sum_{i=1}^{n} |a_i|^s \right)^{\frac{1}{s}} \leq n^{\frac{1}{s'}} \cdot n^{-\frac{1}{r}} \cdot \left( \sum_{i=1}^{n} |a_i|^r \right)^{\frac{1}{r}} \leq n^{\frac{1}{s'}} B_r. \]

The proof is the consequence of these inequalities. ◦

We also formulate the following

**Corollary 2.8.** Let \( r, s \in \mathbb{R}_+ \). Then

\[ \left( \int_0^1 \left| \sum_{i=1}^{n} r_i \left( t \right) a_i \right|^s \right)^{\frac{1}{s}} \leq n^{\frac{1}{s} - \frac{1}{r}} \left( \sum_{i=1}^{n} |a_i|^r \right)^{\frac{1}{r}}, \]

for any \( a_1, a_2, ..., a_n \in \mathbb{C} \) where \( r_i \left( t \right) = \text{sign} \left( \sin \left( 2^i \pi t \right) \right) \) are the usual Rademacher functions and

\[ q = \begin{cases} \min \left( 2, r \right) & \text{for } s \leq 2, \\ \min \left( s', r \right) & \text{for } s > 2. \end{cases} \]

**Proof.** The proof follows from the very useful fact that if \( s > 0 \), then

\[ \left( \sum_{\varepsilon_i = \pm 1} 2^{-n} \left| \sum_{i=1}^{n} \varepsilon_i a_i \right|^s \right)^{\frac{1}{s}} = \left( \int_0^1 \left| \sum_{i=1}^{n} r_i \left( t \right) a_i \right|^s \right)^{\frac{1}{s}}. \]

**Remark.** For the case \( n = 1 \) the last corollary coincides with a result of Koskela ( [25], Th.1). For the case \( r = s' \), \( s > 2 \) another proof of Corollary 2.8 can be found in [49], Th.5, formula (27).

Now we will consider the Littlewood matrices \( A_{2^n} = (\varepsilon_{ij}) \), \( 1 \leq i, j \leq 2^n \), defined inductively

\[ A_{2^1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad A_{2^n} = \begin{bmatrix} A_{2^{n-1}} & A_{2^{n-1}} \\ A_{2^{n-1}} & -A_{2^{n-1}} \end{bmatrix} \text{ for } n = 2, 3, ... \]

The next result can also be obtained by using the interpolation technique described above (see [32], Th.3.3).
Theorem 2.9. Let $s \in \mathbb{R}, s \neq 0, r \in \mathbb{R}_+$. Then
\[
\left( \sum_{i=1}^{2^n} \left| \sum_{j=1}^{2^n} \varepsilon_{ij} a_j \right|^s \right)^{\frac{1}{s}} \leq 2^n \left( \sum_{j=1}^{2^n} |a_j|^r \right)^{\frac{1}{r}},
\]
for any $a_1, a_2, ..., a_{2^n} \in \mathbb{C}$, where $q = \left\{ \begin{array}{ll} \min(2, r) & \text{for } s \leq 2, \\ \min(s', r) & \text{for } s > 2. \end{array} \right.$

Proof. We consider the operator
\[
T : \tilde{a} \rightarrow \left( \sum_{i=1}^{2^n} \varepsilon_{1j} a_j, \sum_{i=1}^{2^n} \varepsilon_{2j} a_j, ..., \sum_{i=1}^{2^n} \varepsilon_{2^n j} a_j \right)
\]
and note that
\[
T : l_1^{(2^n)} \rightarrow l_\infty^{(2^n)} \text{ has the norm } M_0 = 1
\]
and
\[
T : l_2^{(2^n)} \rightarrow l_2^{(2^n)} \text{ has the norm } M_1 = 2^{\frac{n}{2}}.
\]
By using the Riesz-Thorin interpolation theorem we obtain the following estimate:
\[
\left( \sum_{i=1}^{2^n} \left| \sum_{j=1}^{2^n} \varepsilon_{ij} a_j \right|^p \right)^{\frac{1}{p}} \leq 2^{\frac{n}{q'}} \left( \sum_{j=1}^{2^n} |a_j|^p \right)^{\frac{1}{p}}, 1 \leq p \leq 2.
\]
The rest of the proof is similar to the proof for Theorem 2.7 and follows by using the last estimate and Lemma 2.4.

2.2 Generalized Clarkson’s inequality

In this part we present some natural generalization of the Clarkson inequality (GCI) to multidimensional cases.

Let $A_{2^n} = (\varepsilon_{ij}), 1 \leq i, j \leq 2^n$ denote again the Littlewood matrices.

(i) GCI, the standard form:
If $1 \leq p \leq 2$, then
\[
\left\{ \sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} \varepsilon_{ij} f_j \right\|_p^p \right\}^{\frac{1}{p}} \leq 2^n \left\{ \sum_{j=1}^{2^n} \left\| f_j \right\|_p^p \right\}^{\frac{1}{p}},
\]
for all $f_1, f_2, ..., f_{2^n} \in L^p$ and any $n \in \mathbb{N}$.

26
(ii) **GCI, the Kato form** (see [22] and also [38], [46] and [36]):

If \( 1 \leq p, r, s < \infty \), then

\[
(2.11) \quad \left\{ \sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} \varepsilon_{ij} f_j \right\|_p^s \right\}^{\frac{1}{s}} \leq 2^{n C(r,s;p)} \left\{ \sum_{j=1}^{2^n} \left\| f_j \right\|_p^r \right\}^{\frac{1}{r}},
\]

for all \( f_1, f_2, \ldots, f_{2^n} \in L^p \) and any \( n \in \mathbb{N} \), where \( t = \min (p, p') \) and

\[
C(r,s;p) = \begin{cases} 
\frac{1}{r} + \frac{1}{s} - \frac{1}{t'} & \text{for } t \leq r \leq \infty \text{ and } 1 \leq s \leq t', \\
\frac{1}{s} & \text{for } 1 \leq r \leq t \text{ and } 1 \leq s \leq r', \\
\frac{1}{r'} & \text{for } s' \leq r \leq \infty \text{ and } t' \leq s \leq \infty,
\end{cases}
\]

with usual \( r', t' \) the conjugate numbers to \( r \) and \( t \).

(iii) **GCI, the Maligranda-Persson form** (see [31], [32] and [42]):

If \( 0 < p, r, s \leq \infty \) then the inequality (2.11) holds with the constant

\[
C(r,s;p) = \frac{1}{s} - \frac{1}{r} + \frac{1}{q},
\]

where \( q = \min (p, p', r, s') \) with the convention that \( p' \) is omitted if \( p \leq 1 \) and \( s' \) is omitted if \( s \leq 1 \).

In the following figures we illustrate the difference between the set of parameters for which GCI in different forms can be applied. Note that if \( 1 \leq p \leq 2 \), then \( \frac{1}{q} = \max \left( \frac{1}{p}, \frac{1}{r}, 1 - \frac{1}{s} \right) \) if \( s \geq 1 \) and \( \frac{1}{q} = \max \left( \frac{1}{p}, \frac{1}{r} \right) \) if \( s \leq 1 \) and we consider the following five cases:

1. \( \frac{1}{s} \leq 1, \frac{1}{p} \geq \frac{1}{r}, \frac{1}{p} \geq 1 - \frac{1}{s} \). Then \( q = p \) so that \( C = \frac{1}{s} - \frac{1}{r} + \frac{1}{p} \).
2. \( \frac{1}{s} \leq 1, \frac{1}{r} \geq \frac{1}{p}, \frac{1}{r} \geq 1 - \frac{1}{s} \). Then \( q = r \) so that \( C = \frac{1}{s} \).
3. \( \frac{1}{s} \leq 1, 1 - \frac{1}{s} \geq \frac{1}{p}, 1 - \frac{1}{s} \geq \frac{1}{r} \). Then \( q = s' \) so that \( C = \frac{1}{r} \).
4. \( \frac{1}{s} \geq 1, \frac{1}{r} \geq \frac{1}{p} \). Then \( q = p \) so that \( C = \frac{1}{s} - \frac{1}{r} + \frac{1}{p} \).
5. \( \frac{1}{s} \geq 1, \frac{1}{r} \geq \frac{1}{p} \). Then \( q = r \) so that \( C = \frac{1}{s} \).

Summing up, we find that \( C = \frac{1}{s} - \frac{1}{r} + \frac{1}{p} \) in \( N_1 \), \( C = \frac{1}{s} \) in \( N_2 \) and \( C = \frac{1}{r} \) in \( N_3 \) in figure 2.1.
By making the similar investigation for the case $p \geq 2$ we obtain the analogous illustration (see figure 2.2) that $C = \frac{1}{s} - \frac{1}{r} + \frac{1}{p}$ in $N_1$, $C = \frac{1}{s}$ in $N_2$ and $C = \frac{1}{r}$ in $N_3$. 
In the last case $0 < p < 1$ we have only two possibilities: $C = \frac{1}{s} - \frac{1}{r} + \frac{1}{p}$ in $N_1$ and $C = \frac{1}{s}$ in $N_2$ (see figure 2.3).

![Figure 2.3](image)

**Remark.** Note that GCI in the form (i) corresponds to the point $\left(\frac{1}{p}, \frac{1}{p}\right)$ in figure 2.1, GCI in the form (ii) corresponds to figures 2.1 and 2.2 restricted to the squares: $\{(\frac{1}{r}, \frac{1}{s}) : \frac{1}{2} \leq \frac{1}{r}, \frac{1}{s} \leq 1\}$ and GCI in the form (iii) corresponds to the first quadrant in the figures 2.1-2.3.

To end this paragraph let us prove the following somewhat generalized form of (iii) (see [32], Th.4.1): 

**Theorem 2.10.** If $0 < p, r < \infty$ and $-\infty < s < \infty$, $s \neq 0$, then

$$
\left(\frac{2^n}{s} \sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} \varepsilon_{ij} f_j \right\|_p \right) \leq 2^n \left(\frac{1}{s} - \frac{1}{r} + \frac{1}{q}\right) \left\{\sum_{j=1}^{2^n} \left\| f_j \right\|_p \right\}^{\frac{1}{r}},
$$

for all $f_1, f_2, ..., f_{2^n} \in L^p(\Omega)$ and any $n \in \mathbb{N}$, where $q = \min(p, p', s', r)$ with the convention that $p'$ is omitted if $p \leq 1$ and $s'$ is omitted if $s \leq 1$. 

**Proof.** 1°. Let $0 < r \leq p \leq s < \infty$. We will use Theorem 2.9. First we observe that
\[
\left| \sum_{i=1}^{2^n} \sum_{j=1}^{2^n} \varepsilon_{ij} f_j(x) \right|^s \leq 2^{\frac{s}{2}} \left( \sum_{j=1}^{2^n} |f_j(x)|^r \right)^{\frac{s}{r}},
\]

so
\[
\left\| \sum_{i=1}^{2^n} \sum_{j=1}^{2^n} \varepsilon_{ij} f_j \right\|_p^s \leq 2^{\frac{s}{2}} \left( \sum_{j=1}^{2^n} |f_j|^r \right)^{\frac{s}{r}}.
\]

Hence, by also using the reversed Minkowski inequality (for \( \frac{E}{s} \leq 1 \)) and the Minkowski inequality for \( E > 1 \), we get
\[
\left\| \sum_{i=1}^{2^n} \sum_{j=1}^{2^n} \varepsilon_{ij} f_j \right\|_p^s \leq 2^{\frac{s}{2}} \left( \sum_{j=1}^{2^n} |f_j|^r \right)^{\frac{s}{r}}.
\]

where \( q = \min \left( r, s' \right) \) if \( s \geq 2 \) and \( q = \min \left( 2, r \right) \) if \( s \leq 2 \).

2°. Let \( p \leq s, r \). First we use (2.12) for the case 1° with \( r = p \). After that we use Lemma 2.3 to obtain that in this case (2.12) holds with \( q = \min \left( p, s' \right) \) if \( s \geq 2 \) and \( q = \min \left( 2, p \right) \) if \( 0 < s \leq 2 \).

3°. Let \( s, r \leq p \). We use again (2.12) for the case 1° now with \( s = p \) and Lemma 2.3 to find that in this case (2.12) holds with \( q = \min \left( r, p' \right) \) if \( p \geq 2 \) and \( q = \min \left( 2, r \right) \) if \( 0 < p \leq 2 \).

The proof follows by combining 1° – 3°. \( \ast \)

We shall close this section by presenting a generalized random Clarkson’s inequality in terms of the random matrices \( B_n \). A random matrix \( B_n = \{ b_{ij} \} \), \( n = 1, 2, ... \) is a \( n \times n \) matrix whose coefficients are independent distributed random variables taking the values +1 or -1 with equal probability. More information with some theorems concerning that type of Clarkson inequalities can be found in [47].
Generalized random Clarkson’s inequality (see [48] and also [46] and [47]) If $1 < p, r, s < \infty$ then we have

$$E \left\{ \sum_{i=1}^{n} \left\| \sum_{j=1}^{n} b_{ij} f_{j} \right\|_{p}^{s} \right\}^{\frac{1}{s}} \leq K 2^{nC(r,s;p)} \left\{ \sum_{j=1}^{n} \| f_{j} \|_{p}^{r} \right\}^{\frac{1}{r}},$$

for all $f_1, f_2, \ldots, f_{2n} \in L^p$ and any $n \in \mathbb{N}$, where $E$ denoting the mathematical expectation and the constant $C(r,s;p)$ is the same as in GCI (of the form (ii)) and $K$ is a fixed constant independent of $n, r$ and $s$.

### 2.3 Clarkson type inequalities for Banach spaces

In this section we shall discuss and prove the more general Clarkson type inequalities for Banach spaces. Concerning such ideas we refer to [24] and the references given there.

Let $X = (X, \| \cdot \|)$ denote a Banach space, $n \in \mathbb{N}$ and $1 < p \leq 2$. We say that the $(p,p') - Clarkson's inequality$ holds in $X$ if, for all $x_1, x_2 \in X$,

$$\left( \| x_1 + x_2 \|_{p'} + \| x_1 - x_2 \|_{p'} \right)^{\frac{1}{p'}} \leq 2^{\frac{1}{p'}} \left( \| x_1 \|^{p} + \| x_2 \|^{p} \right)^{\frac{1}{p}}.$$

More generally, we say that the $(p, p', n) - Clarkson's inequality$ holds in $X$ if, for all $x_1, x_2, \ldots, x_{2n} \in X$,

$$\left\{ \sum_{i=1}^{2n} \left\| \sum_{j=1}^{2n} \varepsilon_{ij} x_j \right\|_{p}^{p'} \right\}^{\frac{1}{p'}} \leq 2^{\frac{1}{p'}} \left\{ \sum_{j=1}^{2n} \| x_j \|^{p} \right\}^{\frac{1}{p}},$$

where $\varepsilon_{ij}$ are the entries of the Littlewood matrices.

First we state the following lemmas of independent interest (see [24]).

**Lemma 2.11.** If the $(p,p') - Clarkson's inequality$ holds in $X$, then, for each $n \in \mathbb{N}$, the $(p,p',n) - Clarkson's inequality$ holds in $X$.

**Proof.** The statement is true for $n = 1$ by assumption. Assume that the statement holds for a fixed $n \in \mathbb{N}$. Then, by using the $(p,p') - Clarkson's inequality$, Minkowski's inequalities and the induction assumption, we obtain the following:
\[
\left\{ \sum_{i=1}^{2n+1} \left( \sum_{j=1}^{2n+1} \varepsilon_{ij} x_j \right)^{\frac{1}{p'}} \right\}^{\frac{1}{p}}
= \left\{ \sum_{i=1}^{2n} \left( \sum_{j=1}^{2n} \varepsilon_{ij} x_j + \sum_{j=2^n+1}^{2n+1} \varepsilon_{ij} x_j \right)^{\frac{1}{p'}} + \sum_{i=2^n+1}^{2n+1} \left( \sum_{j=1}^{2n} \varepsilon_{ij} x_j + \sum_{j=2^n+1}^{2n+1} \varepsilon_{ij} x_j \right)^{\frac{1}{p'}} \right\}^{\frac{1}{p'}}
\leq \left\{ \sum_{i=1}^{2n} 2 \left( \left( \sum_{j=1}^{2n} \varepsilon_{ij} x_j \right)^{\frac{p}{p'}} + \left( \sum_{j=2^n+1}^{2n+1} \varepsilon_{ij} x_j \right)^{\frac{p}{p'}} \right) \right\}^{\frac{1}{p'}}
= 2^{\frac{1}{p'}} \left\{ \left( \sum_{i=1}^{2n} \left( \sum_{j=1}^{2n} \varepsilon_{ij} x_j \right)^{\frac{p}{p'}} \right) + \left( \sum_{i=2^n+1}^{2n+1} \left( \sum_{j=2^n+1}^{2n+1} \varepsilon_{ij} x_j \right)^{\frac{p}{p'}} \right) \right\}^{\frac{1}{p'}}
\leq 2^{\frac{1}{p'}} \left\{ \left( \sum_{i=1}^{2n} \left( \sum_{j=1}^{2n} \varepsilon_{ij} x_j \right)^{\frac{p}{p'}} \right) + 2^{\frac{np}{p'}} \left( \sum_{j=2^n+1}^{2n+1} \left( x_j \right)^{\frac{p}{p'}} \right) \right\}^{\frac{1}{p'}}
= 2^{\frac{1}{p'}} \cdot 2^{\frac{np}{p'}} \left\{ \sum_{j=1}^{2^n+1} \left( x_j \right)^{\frac{p}{p'}} \right\}^{\frac{1}{p'}}
\]

i.e., the \((p,p', n+1)\)–Clarkson’s inequality holds in \(X\). Thus, according to the induction axiom, the proof is complete. \(\diamondsuit\)

We now establish the following:

**Theorem 2.12.** Let \(1 \leq p \leq 2\), \(0 < r, s \leq \infty\), \(n \in \mathbb{N}\) and let \(X\) be a Banach space. If the \((p,p')\)–Clarkson’s or the \((p,p', n)\)–Clarkson’s inequality holds in \(X\), then, for every \(x_1, x_2, ..., x_{2^n} \in X\),
\[
\left\{ \sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} \varepsilon_{ij} x_j \right\|^s \right\}^{\frac{1}{s}} \leq 2^{nC} \left\{ \sum_{j=1}^{2^n} \|x_j\|^r \right\}^{\frac{1}{r}} ,
\]

where
\[
C = \begin{cases} 
\frac{1}{s} - \frac{1}{r} + \frac{1}{p} 
& \text{for } s \leq p' \text{ and } p \leq r, \\
\frac{1}{s} 
& \text{for } s \leq p' \text{ and } r \leq p, \\
\frac{1}{r} 
& \text{for } s \geq p' \text{ and } p \leq r, \\
\frac{1}{p} 
& \text{for } s \geq p' \text{ and } r \leq p.
\end{cases}
\]

**Proof.** We will consider the following:

\[
A_s = \left\{ \sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} \varepsilon_{ij} x_j \right\|^s \right\}^{\frac{1}{s}} \quad \text{and} \quad B_r = \left\{ \sum_{j=1}^{2^n} \|x_j\|^r \right\}^{\frac{1}{r}} .
\]

According to the assumption and Lemma 2.11 we know that

\[
A_{p'} \leq 2^n B_p .
\]

If we use Lemma 2.3, then we get that

1°. if \( s \leq p' \) and \( p \leq r \), then
\[
A_s \leq 2^{\frac{n}{s} - \frac{n}{r}} A_{p'} \leq 2^n B_p \leq 2^n 2^{\frac{n}{r} + \frac{n}{p}} B_r = 2^n (\frac{1}{s} - \frac{1}{r} + \frac{1}{p}) B_r ;
\]

2°. if \( s \leq p' \) and \( r \leq p \), then
\[
A_s \leq 2^{\frac{n}{s} - \frac{n}{r}} A_{p'} \leq 2^n B_p \leq 2^n B_r ;
\]

3°. if \( s \geq p' \) and \( p \leq r \), then
\[
A_s \leq A_{p'} \leq 2^n B_p \leq 2^n 2^{\frac{n}{r} - \frac{n}{p}} B_r = 2^n (1 - \frac{1}{r}) B_r ;
\]

4°. if \( s \geq p' \) and \( r \leq p \), then
\[
A_s \leq A_{p'} \leq 2^n B_p \leq 2^n B_r .
\]

The proof is complete. \( \diamond \)

We also state the following variant of Theorem 2.12, which in particular, implies GCI on the form \((ii)\) (see also [18], Cor. 2.4):

**Theorem 2.13.** Let \( 1 \leq p \leq 2, \ 1 \leq r, s \leq \infty, \ n \in \mathbb{N} \) and let \( X \) be a Banach space. If the \((p,p') - Clarkson's\) or the \((p,p',n) - Clarkson's\) inequality holds in \( X \), then, for every \( x_1, x_2, ..., x_{2^n} \in X \),
(2.13) \[
\left\{ \sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} \varepsilon_{ij} x_j \right\|^s \right\}^{1/s} \leq 2^{nC} \left\{ \sum_{j=1}^{2^n} \|x_j\|^r \right\}^{1/r},
\]
where
\[
C = \begin{cases} 
\frac{1}{r} + \frac{1}{s} - \frac{1}{p} & \text{for } p \leq r \leq \infty \text{ and } 1 \leq s \leq p', \\
\frac{1}{s} & \text{for } 1 \leq r \leq p \text{ and } 1 \leq s \leq r', \\
\frac{1}{r} & \text{for } s' \leq r \leq \infty \text{ and } p' \leq s \leq \infty.
\end{cases}
\]

For the proof of Theorem 2.13 we need the following lemma of independent interest:

**Lemma 2.14.** Let \( X \) be a Banach space and \( 1 < r < p \leq 2 \). If \((p, p') - Clarkinon inequality holds in \( X \), then \((r, r') - Clarkinon inequality holds in \( X \) as well.

**Proof.** Let us consider the operator
\[
T(x, y) = (x + y, x - y).
\]
The \((p, p') - Clarkinon inequality means that
\[
T : l_p^2(X) \to l_{p'}^2(X)
\]
has the norm not greater then \(2^{1/p} \). It is evident that
\[
T : l_1^2(X) \to l_{\infty}^2(X)
\]
has the norm not exceeding 1. Let \( \theta \) be such that \( \theta = \frac{p'}{r} \). Then
\[
\frac{1}{r} = \frac{1-\theta}{\infty} + \frac{\theta}{p} \quad \text{and} \quad \frac{1}{r} = \frac{1-\theta}{1} + \frac{\theta}{p}.
\]

By the complex method of interpolation (see [4]) we have
\[
\left[ l_1^2(X), l_p^2(X) \right]_{\theta} = l_r^2(X),
\]
and
\[
\left[ l_{\infty}^2(X), l_{p'}^2(X) \right]_{\theta} = l_{r'}^2(X).
\]
with equal norms. Hence
\[
T : l_r^2(X) \to l_{r'}^2(X)
\]
with the norm not bigger then \(2^{\frac{\theta}{p'}} = 2^{\frac{1}{r'}} \). This implies that \((r, r') - Clarkinon inequality holds in \( X \). \(\diamond\)
Proof of Theorem 2.13. The first case 1° is a special case of the corresponding statement in Theorem 2.12. In the second case we consider two possibilities:
1°. $s \leq p'$ and 2°. $p' \leq s$.
For $s \leq p'$ we have another special case of Theorem 2.12. If $p' \leq s$ then we use the same special case with $s = p'$ and $r = p$ so that (with the notations from the proof of Theorem 2.12)

$$A_{p'} \leq 2^{\frac{n}{r}} B_r.$$ 

By now using Lemma 2.14 we find that also

$$A_{r'} \leq 2^{\frac{n}{r}} B_r$$

and, thus, according to Lemma 2.3,

$$A_s \leq 2^{\frac{n}{r}} A_{r'} \leq 2^{\frac{n}{s}} B_r,$$

which means that (2.13) holds also in this situation. The proof of the third case is completely analogous. When $s' \leq p$ we get a special case of the third fall in Theorem 2.12 and for the case $p \leq s'$ we use Lemmas 2.14 and 2.3 to obtain that

$$A_s \leq A_{r'} \leq 2^{\frac{n}{r}} B_r,$$

which means that (2.13) holds with $C = \frac{1}{r}$ and the proof is complete. ☐

Remark. Theorem 2.3 shows that $(13, p_i) = Clarkson inequality holds in $V$ for $1 \leq p \leq 2$. Thus, Theorem 2.13 implies directly GCI in the form (ii).

2.4 Type and cotype of Banach spaces

The notions of type and cotype of Banach spaces were introduced by B. Maurey and G. Pisier in the mid-1970’s. Since then, these notions have found frequent use in the geometry of Banach spaces (cf. [2], [10], [20], [29]).

The $n$:th Rademacher function ($n \in \mathbb{N}$) is the function $r_n : [0, 1] \to \mathbb{R}$ defined by

$$r_n(t) = sign (\sin 2^n \pi t).$$

The sequence $\{r_n(t)\}$ of Rademacher functions is an orthonormal sequence in $L^2[0,1]$, that is
Moreover,
\[ \int_0^1 r_m(t) r_n(t)\,dt = \begin{cases} 0 & \text{if } m \neq n, \\ 1 & \text{if } m = n. \end{cases} \]

Moreover,
\[ \int_0^1 r_n(t)\,dt = 0 \]

and
\[ \left\| \sum_{n=1}^\infty r_n(t) a_n \right\|_2 = \left( \sum_{n=1}^\infty a_n^2 \right)^{\frac{1}{2}}. \]

Recall also the classical Khintchine's inequality (see e.g. [29]). For every 0 < r < \infty there exist positive constants \( A_r \) and \( B_r \) so that
\[ A_r \left( \sum_{k=1}^n |a_k|^2 \right)^{\frac{1}{2}} \leq \left( \int_0^1 \left| \sum_{k=1}^n a_k r_k(t) \right|^r\,dt \right)^{\frac{1}{r}} \leq B_r \left( \sum_{k=1}^n |a_k|^2 \right)^{\frac{1}{2}} \]

for every choice of scalars \( \{a_k\}_{k=1}^n \) and for every \( n \in \mathbb{N} \). This inequality can be written equivalently as follows
\[ A_r \left\| \sum_{k=1}^n a_k r_k(t) \right\|_2 \leq \left\| \sum_{k=1}^n a_k r_k(t) \right\|_r \leq B_r \left\| \sum_{k=1}^n a_k r_k(t) \right\|_2. \]

The best constant in the Khinchine's inequality have been calculated in [14].

Observe that, by Lemma 2.3 (b), Khinchine's inequality implies that for any sequence \( \{a_k\}_{k=1}^n \subset \mathbb{R} \), the following inequalities holds:
\[ \left( \int_0^1 \left| \sum_{k=1}^n a_k r_k(t) \right|^r\,dt \right)^{\frac{1}{r}} \leq B_r \left( \sum_{k=1}^n |a_k|^p \right)^{\frac{1}{p}} \]

for 1 \leq p \leq 2 and
\[ \left( \int_0^1 \left| \sum_{k=1}^n a_k r_k(t) \right|^r\,dt \right)^{\frac{1}{r}} \geq A_r \left( \sum_{k=1}^n |a_k|^q \right)^{\frac{1}{q}} \]

for \( q \geq 2 \).

Generalization of the above inequalities to Banach spaces results in the following definition (see [19] and [34]).

**Definition.** A Banach space \( X \) has **type** \( p \), where 1 \leq p \leq 2, if there is a constant \( 0 < K < \infty \) such that, for any choice of finitely many vectors \( x_1, \ldots, x_n \in X \),
(2.14) \[ \int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\| dt \leq K \left( \sum_{k=1}^n \| x_k \|^p \right)^{\frac{1}{p}}. \]

A Banach space \( X \) has **cotype** \( q \geq 2 \) if there is a constant \( C > 0 \) such that no matter how we choose finitely many elements \( x_1, ..., x_n \in X \),

(2.15) \[ \left( \sum_{k=1}^n \| x_k \|^q \right)^{\frac{1}{q}} \leq C \int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\| dt, \quad \text{for } q < \infty \]

\[
\max_{1 \leq k \leq n} \| x_k \| \leq C \int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\| dt, \quad \text{for } q = \infty.
\]

We say that the space has **trivial type** or **trivial cotype**, if it does not have any type bigger than 1 or any finite cotype, respectively.

We cite the next important result of J. P. Kahane without proof (see [29]).

**Theorem 2.15.** For every \( 1 < r < \infty \) there is a constant \( k_r < \infty \) such that for any Banach space \( X \) and every finite family \( x_1, ..., x_n \) in \( X \) we have

\[ \int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\| dt \leq \left( \int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\|^r dt \right)^{\frac{1}{r}} \leq k_r \int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\| dt. \]

In view of the above Kahane’s theorem in the definition of type or cotype we can replace \( \int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\| dt \) by \( \left( \int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\|^r dt \right)^{\frac{1}{r}} \) for any \( r \geq 1 \).

Denote by (2.14') and (2.15') the inequalities (2.14) and (2.15), where \( \int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\| dt \) have been replaced by \( \left( \int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\|^r dt \right)^{\frac{1}{r}} \).

Let \( K_{p(r)}(X) \) and \( C_{q(r)}(X) \) denote the smallest constants \( K \) and \( C \) satisfying (2.14') and (2.15') respectively.

**Remarks.** (a) By Khintchine’s inequality the space \( \mathbb{R} \) or \( \mathbb{C} \) are both of type and cotype 2.

(b) If \( X \) is of type \( p \) (respectively cotype \( q \)), then it is of type \( p_1 \) where \( 1 \leq p_1 \leq p \) (respectively cotype \( q_1 \) where \( q \leq q_1 \leq \infty \)).
(c) Any Banach space $X$ is of type 1 and cotype $\infty$ and $K_{1(r)}(X) = C_{\infty(r)}(X) = 1$ for all $1 \leq r \leq \infty$.

Indeed, if $x_i \in X$, $i = 1, \ldots, n$, then

$$\|x_i\| = \left\| \int_0^1 r_i(t) \left[ \sum_{k=1}^n r_k(t) x_i \right] dt \right\| = \left\| \int_0^1 r_i(t) \left[ \sum_{k=1}^n r_k(t) x_k \right] dt \right\|$$

$$\leq \int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\| dt$$

and

$$\max_{i=1,\ldots,n} \|x_i\| \leq \int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\| dt.$$

Type 1 of $X$ is an obvious consequence of the triangle inequality.

(d) No Banach space can be of type $p > 2$ or cotype $q < 2$.

In fact, if $x \in X$, $\|x\| = 1$ and $x_1 = x_2 = \ldots = x_n = \frac{1}{n} x$, then

$$\left( \int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\|^p dt \right)^{\frac{1}{p}} = \left( \int_0^1 \left\| \frac{1}{n} \sum_{k=1}^n r_k(t) \right\|^p dt \right)^{\frac{1}{p}}$$

$$\leq C \left( \sum_{k=1}^n \|x_k\|^p \right)^{\frac{1}{p}} = C \cdot n^{\frac{1}{p} - 1},$$

whenever $X$ is of type $p$. Moreover, by Khintchine’s inequality,

$$\left( \int_0^1 \left\| \frac{1}{n} \sum_{k=1}^n r_k(t) \right\|^p dt \right)^{\frac{1}{p}} \geq A_p \cdot \frac{1}{\sqrt{n}}.$$

and we conclude that $p \leq 2$. The proof of the second statement is similar.

(e) A Banach space $X$ is a Hilbert space if and only if it is of type 2 and cotype 2 (see [27]).

(f) Let $1 \leq p < \infty$ and $L^p = L^p(\mu)$ be an infinite-dimensional space. Then type of $L^p$ is equal to $\min(p, 2)$ and cotype of $L^p$ is equal to $\max(p, 2)$. $L^\infty$ has a trivial type and a trivial cotype, i.e. is not of type $p$ for any $p > 1$ and not of cotype $p$ for any $p < \infty$.

Indeed, if $1 \leq p \leq 2$ and $f_k \in L^p, k = 1, \ldots, n$, then in view of Fubini theorem, imbedding of $L^2[0,1]$ into $L^p[0,1]$ with the norm not bigger then 1, orthonormality of Rademacher functions, subadditivity of the function $u \mapsto u^{p/2}$ and additivity of integral we have

$$\int_0^1 \left\| \sum_{k=1}^n r_k(t) f_k \right\|^p_p dt = \int_0^1 \left( \int_0^\Omega \left\| \sum_{k=1}^n r_k(t) f_k(s) \right\|^p ds \right) dt$$

38
\[
\int_\Omega \left( \int_0^1 \left| \sum_{k=1}^n r_k(t) f_k(s) \right|^p dt \right) ds \\
\leq \int_\Omega \left( \int_0^1 \left| \sum_{k=1}^n r_k(t) f_k(s) \right|^2 dt \right)^{\frac{p}{2}} ds \\
= \int_\Omega \left( \sum_{k=1}^n |f_k(s)|^2 \right)^{\frac{p}{2}} ds \\
\leq \int_\Omega \sum_{k=1}^n |f_k(s)|^p ds = \sum_{k=1}^n \|f_k\|_p^p.
\]

If \(2 < p < \infty\), then in view of the Hölder inequality (for \(\frac{p}{2} > 1\)), Fubini theorem, Khintchine inequality and subadditivity of \(L^{\frac{p}{2}}\)-norm we obtain

\[
\int_0^1 \left\| \sum_{k=1}^n r_k(t) f_k \right\|_p^2 dt = \int_0^1 \left( \int_\Omega \left| \sum_{k=1}^n r_k(t) f_k(s) \right|^p ds \right)^{\frac{2}{p}} dt \\
\leq \left( \int_0^1 \left( \int_{\Omega} \left| \sum_{k=1}^n r_k(t) f_k(s) \right|^p ds \right) dt \right)^{\frac{2}{p}} \\
= \left( \int_\Omega \int_0^1 \left| \sum_{k=1}^n r_k(t) f_k(s) \right|^p dt ds \right)^{\frac{2}{p}} \\
\leq \left( \int_\Omega \sum_{k=1}^n \left| f_k(s) \right|^{\frac{p}{2}} ds \right)^{\frac{2}{p}} \\
= B_p^2 \left\| \sum_{k=1}^n \left| f_k \right|^{\frac{p}{2}} \right\|_2 \leq B_p^2 \sum_{k=1}^n \left\| f_k \right\|_2^{\frac{p}{2}} = B_p^2 \sum_{k=1}^n \left\| f_k \right\|_p^2.
\]

Thus \(L^p\)-space has a type \(\min(p, 2)\).

Similarly we can show that \(L^p\) has a cotype \(\max(p, 2)\).

### 2.5 Connections between generalized Clarkson’s inequalities and type, cotype

The general idea in this part is to investigate the various variants of Clarkson inequalities and their connections to the concept of type and cotype. It is easy to see that the following inequalities are equivalent:

\[
(\|x_1 + x_2\|^{\gamma} + \|x_1 - x_2\|^{\gamma})^{\frac{1}{\gamma}} \leq 2\left(\|x_1\|^p + \|x_2\|^p\right)^{\frac{1}{p}}
\]
\[
\left( \frac{1}{0} \left\| \sum_{k=1}^{2} r_k(t) x_k \right\|^r \right)^{\frac{1}{r}} \leq \left( \|x_1\|^p + \|x_2\|^p \right)^{\frac{1}{p}}
\]
where \(1 \leq p \leq 2, p \leq r \leq p'\).

Let us start with the following:

**Theorem 2.16.** A Banach space \(X = (X, \|\cdot\|)\) satisfies the Clarkson type inequality (2.16) if and only if \(X\) is of a type \(p\) and \(K_{p(r)}(X) = 1\).

**Proof.** The sufficiency condition follows from the fact that (2.17) holds as a particular case of \(n = 2\) which means that (2.16) holds.

The necessary condition will be proved by induction in \(n\) (from the definition of type).

For \(n = 2\) this implication is, of course, true. Assume now that it holds for \(n = N\). Then, by using the induction assumption and the triangle inequality for \(L^r\)-norm, we find that

\[
\begin{align*}
&\frac{1}{0} \left\| \sum_{j=1}^{N+1} r_j(t) x_j \right\|^r \leq \frac{1}{2^{N+1}} \sum_{\theta_j = \pm 1} \left\| \sum_{j=1}^{N+1} \theta_j x_j \right\|^r \\
&= \frac{1}{2} \left\{ \frac{1}{2^N} \left( \sum_{\theta_j = \pm 1} \|\theta_j x_j + x_{N+1}\|^r \right) + \frac{1}{2^N} \left( \sum_{\theta_j = \pm 1} \|\theta_j x_j - x_{N+1}\|^r \right) \right\} \\
&= \frac{1}{2} \int_0^1 \left( \left\| \sum_{j=1}^{N} r_j(t) x_j + x_{N+1} \right\|^r + \left\| \sum_{j=1}^{N} r_j(t) x_j - x_{N+1} \right\|^r \right) dt \\
&\leq \int_0^1 \left( \left\| \sum_{j=1}^{N} r_j(t) x_j \right\|^p + \|x_{N+1}\|^p \right)^{\frac{p}{r}} dt \\
&\leq \left\{ \left( \int_0^1 \left\| \sum_{j=1}^{N} r_j(t) x_j \right\|^r dt \right)^{\frac{p}{r}} + \left( \int_0^1 \|x_{N+1}\|^r dt \right)^{\frac{p}{r}} \right\}^{\frac{r}{p}} \\
&\leq \left\{ \sum_{j=1}^{N} \|x_j\|^p + \|x_{N+1}\|^p \right\}^{\frac{p}{p}} = \left( \sum_{j=1}^{N+1} \|x_j\|^p \right)^{\frac{p}{p}}
\end{align*}
\]

and the implication holds for \(N+1\) elements. ∎

Let now \(2 \leq q < \infty\) and \(q' \leq r \leq q\). Consider the inequality

\[
(\|x_1 + x_2\|^q + \|x_1 - x_2\|^q)^{\frac{1}{q}} \leq 2^{\frac{1}{q'}} (\|x_1\|^r + \|x_2\|^r)^{\frac{1}{r}}.
\]
We can easily establish the next result:

**Theorem 2.17.** A Banach space $X = (X, \| \cdot \|)$ satisfies the Clarkson type inequality (2.18) if and only if $X$ is a cotype $q$ and $C_{q(r)} = 1$.

**Proof.** If we put $x_1 + x_2 = y_1$, $x_1 - x_2 = y_2$ in (2.18) we see that this is equivalent to the following inequality:

$$
\left( \|y_1\|^q + \|y_2\|^q \right)^{\frac{1}{q}} \leq 2^{-\frac{1}{r}} \left( \|y_1 + y_2\|^r + \|y_1 - y_2\|^r \right)^{\frac{1}{r}},
$$

which is equivalent with

(2.19) $$
\left( \|y_1\|^q + \|y_2\|^q \right)^{\frac{1}{q}} \leq \left( \int_0^1 \left\| \sum_{k=1}^2 r_k(t) y_k \right\|^r dt \right)^{\frac{1}{r}}
$$

and this proves the sufficiency condition. The converse can be done by induction quite similar to the proof of Theorem 2.16 and we omitted the details. \qed

We also state the following useful result:

**Theorem 2.18.** Let $0 < r, s < \infty$ and $r \leq p \leq s$. Then

$$
\left( \int_0^1 \left\| \sum_{k=1}^n r_k(t) f_k \right\|_p^s dt \right)^{\frac{1}{s}} \leq n^{\frac{1}{s} - \frac{1}{r}} \left( \sum_{k=1}^n \left\| f_k \right\|_p^r \right)^{\frac{1}{r}},
$$

for all $f_1, f_2, \ldots, f_n \in L^p$, where $q = \begin{cases} 
\min (2, r) & \text{for } s \leq 2, \\
\min (s', r) & \text{for } s > 2.
\end{cases}$

**Proof.** In the proof we will use the corresponding result for numbers that we formulated as Theorem 2.7. By the reversed Minkowski inequality, Theorem 2.7 and the triangle inequality for $L_r^2$ — norm we find that

$$
\int_0^1 \left\| \sum_{k=1}^n r_k(t) f_k \right\|_p^s dt = \frac{1}{2^n} \sum_{\varepsilon_k = \pm 1} \left\| \sum_{k=1}^n r_k(t) f_k \right\|_p^s \leq 2^{-n} \sum_{\varepsilon_k = \pm 1} \left\| \sum_{k=1}^n r_k(t) f_k \right\|_p^s.
$$
\( \left\| n^\frac{\alpha-s}{r} \left( \sum_{k=1}^{n} |f_k|^r \right)^\frac{\alpha}{r} \right\|_p = n^\frac{\alpha-s}{r} \left( \sum_{k=1}^{n} \left\| f_k \right\|^r \right)^\frac{\alpha}{r} \)

\( = n^\frac{\alpha-s}{r} \left( \sum_{k=1}^{n} \left\| f_k \right\|^r \right)^\frac{\alpha}{r} \)

\( = n^\frac{\alpha-s}{r} \left( \sum_{k=1}^{n} \left\| f_k \right\|^r \right)^\frac{\alpha}{r} \)

\( = \left( \frac{1}{n^\frac{\alpha-s}{r}} \left( \sum_{k=1}^{n} \left\| f_k \right\|^r \right)^\frac{1}{r} \right)^s \)

and the proof is complete. ◇

**Remark.** In particular Theorem 2.18 implies that

\( \left( \int_0^1 \left\| \sum_{k=1}^{n} r_k(t) x_k \right\|^s_\infty dt \right)^\frac{1}{s} \leq \left( \sum_{k=1}^{n} \left\| x_k \right\|_p^r \right)^\frac{1}{r} \)

whenever

\[ r \leq p \leq s \leq 2 \text{ or } s > 2, \quad r < s' \text{ and } r \leq p \leq s. \]

This means that the \( L^p \)-spaces are of the type \( r \) with \( K_{r(s)} = 1 \), whenever (2.20) holds. In particular, the \( L^p \)-spaces are of type \( p \) with constant \( K_{p(s)} = 1 \) if \( 0 < p \leq 2 \) and \( s \geq p \).

**Remark.** By using Theorem 2.15 and the last remark we see that the \( L^p \)-spaces are of type \( r \), \( 0 < r \leq 2 \), whenever \( p \geq r \) and with \( K_{r(s)} = k_s \). We can say that \( L^p \) space is of type \( \min(p,2) \). See also (f) on page 38.

We can obtain the following exact relations between various variants of Clarkson’s inequalities and the notions of type and cotype with constant one (see [24]):

**Theorem 2.19.** Let \( X \) be a Banach space and \( 1 \leq p \leq 2 \). Then the following statements are equivalent:

(i) \( X \) is of type \( p \) and \( K_{p(p')} (X) = 1 \).

(ii) \( X \) is of cotype \( p' \) and \( C_{p'(p)} = 1 \).

(iii) The \((p,p')\) – Clarkson inequality holds.

(iv) The \((p,p',n)\) – Clarkson inequality holds for any \( n \in \mathbb{N} \).
(v) Let $1 \leq r, s \leq \infty$. Then, for every $x_1, x_2, ..., x_{2^n} \in X$, the inequality
\[
\left\{ \sum_{i=1}^{2^n} \left( \sum_{j=1}^{2^n} \varepsilon_{ij} x_j \right)^s \right\}^{\frac{1}{s}} \leq 2^n C \left\{ \sum_{j=1}^{2^n} \|x_j\|^r \right\}^{\frac{1}{r}},
\]
holds in $X$, where
\[
C = \begin{cases} 
\frac{1}{r} + \frac{1}{s} - \frac{1}{p} & \text{for } p \leq r \leq \infty \text{ and } 1 \leq s \leq p', \\
\frac{1}{s} & \text{for } 1 \leq r \leq p \text{ and } 1 \leq s \leq r', \\
\frac{1}{r} & \text{for } s' \leq r \leq \infty \text{ and } p' \leq s \leq \infty.
\end{cases}
\]

(vi) Let $0 < r, s \leq \infty$. For every $x_1, x_2, ..., x_{2^n} \in X$, the inequality
\[
\left\{ \sum_{i=1}^{2^n} \left( \sum_{j=1}^{2^n} \varepsilon_{ij} x_j \right)^s \right\}^{\frac{1}{s}} \leq 2^n C \left\{ \sum_{j=1}^{2^n} \|x_j\|^r \right\}^{\frac{1}{r}},
\]
holds in $X$, where
\[
C = \begin{cases} 
\frac{1}{s} - \frac{1}{r} + \frac{1}{p} & \text{for } s \leq p' \text{ and } p \leq r, \\
\frac{1}{s} & \text{for } s \leq p' \text{ and } r \leq p, \\
1 - \frac{1}{r} & \text{for } s \geq p' \text{ and } p \leq r, \\
\frac{1}{p} & \text{for } s \geq p' \text{ and } r \leq p.
\end{cases}
\]

Proof. The equivalence $(i) \iff (iii)$ is true by Theorem 2.16 for $r = p'$. 

(ii) $\iff (iii)$ This equivalence is true by Theorem 2.17 in the situation when $q = p'$ and $r = p$. 

(iii) $\implies$ (iv) This implication follows from Lemma 2.11. 

(iv) $\implies$ (iii) To prove this implication just take $n = 1$. 

(iii) $\implies$ (v) This follows from Theorem 2.13. 

(iii) $\implies$ (vi) The implication holds according to Theorem 2.12. 

(v) $\implies$ (iii) If we put $s = p'$, $r = p$ and $n = 1$ into the inequality from (v) we get $(p, p')$ — Clarkson inequality. 

(vi) $\implies$ (iii) Now we have to put $s = p'$, $r = p$ and $n = 1$ into the inequality (vi) to get the required $(p, p')$ — Clarkson inequality. 

This proves the equivalence of all relations. 

Let us also show the following interesting property of Clarkson's inequality:

**Proposition 2.20.** Let $X$ be a Banach space and $1 \leq p \leq 2$. Then the $(p, p')$ — Clarkson's inequality holds in $X$ if and only if it holds in the dual space $X^*$. 

43
Proof. Consider the operator given by the formula
\[ T(x, y) = (x + y, x - y). \]
The \((p, p')\)-Clarkson inequality means that we have
\[ \|T(x, y)\|_{l^2_p(X)} \leq 2^{\frac{1}{p'}} \|(x, y)\|_{l^2_p(X)}, \]
i.e.
\[ T: l^2_p(X) \to l^2_{p'}(X) \]
with the norm \(2^{\frac{1}{p'}}\). The duality argument gives the result. \(\diamond\)

And now we can observe that:

Corollary 2.21. Let \(1 < p \leq 2\). The following assertions are equivalent:
(i) \(X\) satisfies \((p, p')\) — Clarkson inequality.
(ii) \(X\) is of type \(p\) and \(K_{p'(p')}(X) = 1\).
(iii) \(X\) is of cotype \(p'\) and \(C_{p'}(X) = 1\).
(iv) \(X^*\) satisfies \((p, p')\) — Clarkson inequality.
(v) \(X^*\) is of type \(p\) and \(K_{p'(p')}(X^*) = 1\).
(vi) \(X^*\) is of cotype \(p'\) and \(C_{p'(p)}(X^*) = 1\).

Proof. Indeed, (i) \(\iff\) (iv) by Proposition 2.20.
(i) \(\iff\) (ii) \(\iff\) (iii) by Theorem 2.19.
(iv) \(\iff\) (v) \(\iff\) (vi) by Theorem 2.19. \(\diamond\)

We complete this section by pointing out that the definition of type of a Banach space is, in fact, equivalent to that a corresponding random Clarkson inequality holds. This result was recently proved in [24].

Theorem 2.22. Let \(1 \leq p \leq 2\). The following statements are equivalent:
(i) \(X\) is of type \(p\).
(ii) For any \(n\) the standard \((p, p', n)\) — random Clarkson inequality
\[ E \left( \sum_{i=1}^{n} \left( \sum_{j=1}^{n} b_{ij} x_j \right)^{p'} \right)^{\frac{1}{p'}} \leq K n^{\frac{1}{p'}} \left( \sum_{j=1}^{n} \|x_j\|^p \right)^{\frac{1}{p}} \]
holds in \(X\) with some constant \(K\) independent of \(n\).
(iii) For any \( n \) the random Clarkson type inequality
\[
E \left( \sum_{i=1}^{n} \left\| \sum_{j=1}^{n} b_{ij} x_j \right\| \right) \leq K n \left( \sum_{j=1}^{n} \| x_j \|^p \right)^{\frac{1}{p}}
\]
holds in \( X \) with some constant \( K \) independent of \( n \).

(iv) Let \( 1 \leq r, s \leq \infty \). For any \( n \) the random Clarkson type inequality
\[
E \left( \sum_{i=1}^{n} \left\| \sum_{j=1}^{n} b_{ij} x_j \right\|^s \right)^{\frac{1}{s}} \leq K n C(r, s; p) \left( \sum_{j=1}^{n} \| x_j \|^r \right)^{\frac{1}{r}}
\]
holds in \( X \) with some constant \( K \) independent of \( n, r \) and \( s \) and where \( C(r, s; p) \) denotes the constant defined in GCI (Chapter 2.2) of the form (ii) on page 27.

2.6 The von Neumann-Jordan constant

Let \( X = (X, \| \cdot \|) \) be a normed space. The number
\[
C_{NJ}(X) = \sup \left\{ \frac{\|x + y\|^2 + \|x - y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X \text{ not both 0} \right\}
\]
is called the von Neumann-Jordan constant of \( X \) (cf. [6]).

Classical results on \( C_{NJ} \) constant:

(i) \( 1 \leq C_{NJ}(X) \leq 2 \) for any Banach space \( X \).

(ii) (Jordan-von Neumann [39], 1935) \( C_{NJ}(X) = 1 \) if and only if \( X \) is a Hilbert space.

Let us also state another interesting property of \( C_{NJ} \) constant due to Clarkson (see [6]).

**Theorem 2.23.** If \( \dim L^p(\mu) \geq 2 \), then
\[
C_{NJ}(L^p(\mu)) = 2^{2/\min\left(p, p'\right)} - 1 = \begin{cases} 2^{\frac{2}{p} - 1} & \text{for } 1 \leq p \leq 2, \\ 2^{1 - \frac{2}{p}} & \text{for } p > 2. \end{cases}
\]

**Proof.** Since \( \dim L^p(\mu) \geq 2 \) then there are \( A_1, A_2 \in \Sigma \) so that
\( 0 < \mu A_1 < \infty, 0 < \mu A_2 < \infty \) and \( A_1 \cap A_2 = \emptyset \).

Put
\[
x = \chi_{A_1} \text{ and } y = \left(\frac{\mu A_1}{\mu A_2}\right)^{\frac{1}{p}} \chi_{A_2}.
\]

Then
\[
\frac{\|x+y\|_p^2 + \|x-y\|_p^2}{2(\|x\|_p^2 + \|y\|_p^2)} = \frac{(\mu A_1 + \mu A_1)^{2/p} + (\mu A_1 + \mu A_1)^{2/p}}{2(\mu A_1)^{2/p} + (\mu A_1)^{2/p}} \\
= \frac{2 \cdot 2^{2/p}(\mu A_1)^{2/p}}{4(\mu A_1)^{2/p}} = 2^{2/p-1}.
\]

On the other hand, if
\[u = x + y, \quad v = x - y\]
then
\[
\frac{\|u+v\|_p^2 + \|u-v\|_p^2}{2(\|u\|_p^2 + \|v\|_p^2)} = \frac{\|2x\|_p^2 + \|2y\|_p^2}{2(\|x+y\|_p^2 + \|x-y\|_p^2)} \\
= 2^{1-2/p} = 2^{2/p'-1}.
\]
Thus
\[C_{NJ}(L^p(\mu)) \geq \max \left(2^{2/p-1}, 2^{2/p'-1}\right) = 2^{\frac{2}{\min(p,p')}} - 1
\]
and
\[C_{NJ}(L^1(\mu)) = C_{NJ}(L^\infty(\mu)) = 2.
\]
Assume that \(1 < p \leq 2\). Let \(x, y \in L^p, \|x\| + \|y\| > 0\). By Hölder's inequality (with \(r = \frac{p'}{2} \geq 1\) we have
\[
\|x + y\|_p^2 + \|x - y\|_p^2 \leq \left(\|x + y\|_p^{2r} + \|x - y\|_p^{2r}\right)^{\frac{1}{r}} \cdot 2^{\frac{1}{r}} \\
\leq 2^{\frac{2}{p}-1} \left(\|x + y\|_p^{p'} + \|x - y\|_p^{p'}\right)^{\frac{2}{p'}}.
\]
If we use \((p,p') - Clarkson inequality we also get
\[
\|x + y\|_p^{p'} + \|x - y\|_p^{p'} \leq 2 \left(\|x\|_p^p + \|y\|_p^p\right)^{\frac{p'}{p}}
\]
and it follows that
\[
\|x + y\|_p^2 + \|x - y\|_p^2 \leq 2^{\frac{2}{p}-1} \cdot 2^{\frac{2}{p'}} \left(\|x\|_p^p + \|y\|_p^p\right)^{\frac{2}{p}} \\
= 2 \left(\|x\|_p^p + \|y\|_p^p\right)^{\frac{2}{p}}.
\]
If we use Hölder's inequality again (with \(s = \frac{2}{p} \geq 1\) we get
\[
\|x\|_p^p + \|y\|_p^p \leq 2^{\frac{2-p}{p}} \left(\|x\|_p^2 + \|y\|_p^2\right)^{\frac{p}{2}}
\]
and substituting it in the estimate one before gives
\[
\frac{\|x+y\|_p^2 + \|x-y\|_p^2}{2(\|x\|_p^2 + \|y\|_p^2)} \leq 2^{2/p-1}.
\]
For $p > 2$ the same computations are repeated with $p$ and $p'$ being interchanged and the proof is complete. ◊

We now formulate the following generalization of the last theorem.

**Theorem 2.24.** Let $X = (X, \| \cdot \|)$ be a normed space, $1 \leq p \leq 2$ and $(p, p')$ - Clarkson inequality holds in $X$. Then $C_{NJ}(X) \leq 2^{\frac{2}{p} - 1}$.

**Proof.** Let $\|x\| + \|y\| > 0$. Using Lemma 2.3 (a) we get

$$
\left( \frac{\|x+y\|^2 + \|x-y\|^2}{2} \right)^{\frac{1}{2}} \leq \left( \frac{\|x+y\|^{p} + \|x-y\|^{p'}}{2} \right)^{\frac{1}{p'}}.
$$

Since $(p, p')$ - Clarkson inequality holds in $X$ we have

$$
\|x + y\|^2 + \|x - y\|^2 \leq 2 \left( \frac{\|x + y\|^2 + \|x - y\|^2}{2} \right)^{\frac{2}{p'}}
$$

$$
\leq 2 \left( \frac{\|x\|^2 + \|y\|^2}{2} \right)^{\frac{2}{p'}}.
$$

Using again Lemma 2.3 (a) we get

$$
\|x + y\|^2 + \|x - y\|^2 \leq 2^\frac{2}{p} \left( \|x\|^2 + \|y\|^2 \right)
$$

so that

$$
\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \leq 2^{\frac{2}{p} - 1}.
$$

and we have proved that $C_{NJ}(X) \leq 2^{\frac{2}{p} - 1}$. ◊

Kato and Takahashi proved in [23] the following result:

**Proposition 2.25.** $C_{NJ}(X) = C_{NJ}(X^*)$.

**Proof.** We have

$$
[2C_{NJ}(X)]^{\frac{1}{2}} = \|A\|_{l_2^*(X) \rightarrow l_2^2(X)}
$$

where

$$
A = \left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right)
$$

and

$$
l_2^2(X) = \{ (x, y) \in X \times X : \|(x, y)\| = (\|x\|^2 + \|y\|^2)^{\frac{1}{2}} \}.
$$

Therfore

$$
[2C_{NJ}(X)]^{\frac{1}{2}} = \|A\|_{l_2^2(X) \rightarrow l_2^2(X)} = \|A^*\|_{l_2^2(X)^* \rightarrow l_2^2(X)^*}
$$

and $A^* = A, l_2^2(X)^* = l_2^2(X^*)$ gives

47
\[
\frac{\|A^*\|_{l_2^2(\mathbb{R})^* \to l_2^2(\mathbb{R})^*} = \|A\|_{l_2^2(\mathbb{R})^* \to l_2^2(\mathbb{R})^*} = [2C_{NJ}(X^*)]^{\frac{1}{2}}. \]

They also showed that uniform convexity for a Banach space \( X \) implies \( C_{NJ}(X) < 2 \), while conversely \( C_{NJ}(X) < 2 \) gives only the existence of an equivalent uniformly convex norm in \( X \).

3 Modulus of convexity and modulus of smoothness

In this chapter we shall investigate modulus of convexity and modulus of smoothness in a Banach space. We will also present some computations of the moduli in a number of specific spaces.

3.1 Modulus of convexity

In order to measure the degree of uniform convexity (or rotundity) of \( X \), we consider its modulus of convexity.

**Definition.** (J. A. Clarkson, 1936) The **modulus of convexity** of a Banach space \( X \) is the function \( \delta_X : [0, 2] \to [0, 1] \) defined by

\[
\delta_X(\epsilon) = \inf \{ 1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x - y\| \geq \epsilon \}.
\]

Often we suppress \( X \)’s role and denote the modulus just by \( \delta(\epsilon) \). In some sense \( \delta \) measures how deeply the mid-point of the linear segment joining two points on the unit sphere \( S \) must lie within \( B \).

The **characteristic** (or **coefficient**) of convexity of a Banach space \( X \) is the number

\[
\epsilon_0 = \epsilon_0(X) = \sup \{ \epsilon \in (0, 2) : \delta_X(\epsilon) = 0 \}.
\]

It is clear that a Banach space \( X \) is uniformly convex if and only if its modulus of convexity satisfies \( \delta_X(\epsilon) > 0 \) for any \( \epsilon > 0 \), i.e. \( \epsilon_0(X) = 0 \).

The modulus of convexity has "two-dimensional character" in the sense that for \( \epsilon \in [0, 2] \)
\[ \delta_X(\varepsilon) = \inf \{ \delta_E(\varepsilon) : E - \text{two-dimensional subspace of } X \} . \]

Indeed, if we take two-dimensional space \( Z = \text{span} \{ x, y \} \) with the norm \( \| \cdot \|_X \), then

\[
\delta_X(\varepsilon) = \inf \left\{ 1 - \frac{x+y}{2} : x, y \in S_X, \| x - y \| \geq \varepsilon \right\} \\
= \inf \left\{ 1 - \frac{x+y}{2} : x, y \in S_X, \| x - y \| \geq \varepsilon ; x, y \in Z \subset X, \dim Z = 2 \right\} \\
= \inf_{Z \subset X, \dim Z = 2} \left( \inf \left\{ 1 - \frac{x+y}{2} : x, y \in S_Z, \| x - y \| \geq \varepsilon \right\} \right) \\
= \inf_{Z \subset X, \dim Z = 2} \delta_Z(\varepsilon).
\]

The modulus of convexity can be described in various equivalent ways according to the following theorem (see [8], [9], [11] and [29]). This result is commonly used in many computations.

**Theorem 3.1.** (M. M. Day, 1943) **If** \( \dim X \geq 2 \), **then**

\[
\delta_X(\varepsilon) = \inf \left\{ 1 - \frac{x+y}{2} : \| x \| \leq 1, \| y \| \leq 1, \| x - y \| = \varepsilon \right\} \\
= \inf \left\{ 1 - \frac{x+y}{2} : \| x \| = 1, \| y \| = 1, \| x - y \| = \varepsilon \right\} \\
= \inf \left\{ 1 - \frac{x+y}{2} : \| x \| \leq 1, \| y \| \leq 1, \| x - y \| \geq \varepsilon \right\} \\
= \inf \left\{ 1 - \frac{x+y}{2} : \| x \| \leq 1, \| y \| = 1, \| x - y \| = \varepsilon \right\} \\
= \inf \left\{ 1 - \frac{x+y}{2} : \| x \| \leq 1, \| y \| = 1, \| x - y \| \geq \varepsilon \right\}
\]

for all \( 0 \leq \varepsilon \leq 2 \).

We shall use the following lemma (see [9]) in the proof of the theorem.

**Lemma 3.2.** **If** \( \varphi \in S_{X^*} \) **achieves a local maximum at** \( x \in S_X \), **then** \( |\varphi(x)| \) **is a global maximum for** \( |\varphi| \) **on** \( S_X \).

**Proof.** In view of

\[ 1 = \| \varphi \|_{X^*} = \sup \{ |\varphi(y)| : y \in S_X \}, \]

it is enough to show that \( |\varphi(x)| = 1 \). Take any \( \varepsilon > 0 \) and find \( u \in S_X \) so that \( \varphi(u) > 1 - \varepsilon \). If \( \lambda \in \mathbb{R} \) and \( \lambda \) is closed enough to 0, then

\[ \varphi \left( \frac{x + \lambda |u|}{\| x + \lambda |u| \|} \right) \leq \varphi(x) \]

because \( \varphi(x) \) is a local maximum on \( S_X \). It follows that
\[ \varphi(x + |\lambda| u) \leq \varphi(x) \|x + |\lambda| u\| \]
or
\[ \varphi(x) + |\lambda| \varphi(u) \leq \varphi(x) \|x + |\lambda| u\|. \]

Hence
\[ |\lambda| \varphi(u) \leq (\|x + |\lambda| u\| - 1) \varphi(x) \leq (\|x\| + |\lambda| \|u\| - 1) \varphi(x) \]
\[ = |\lambda| \varphi(x) \leq |\lambda| \varphi(x). \]

This means that
\[ |\varphi(x)| \geq \varphi(u) > 1 - \varepsilon, \]
and finally \( |\varphi(x)| = 1 \), because \( \varepsilon \) was arbitrarily chosen. \( \diamond \)

**Proof** of Theorem 3.1. Denote
\[ \delta_1(\varepsilon) = \inf \{1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| = \varepsilon\}, \]
\[ \delta_2(\varepsilon) = \inf \{1 - \frac{\|x + y\|}{2} : \|x\| = 1, \|y\| = 1, \|x - y\| = \varepsilon\}, \]
\[ \delta_3(\varepsilon) = \inf \{1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon\}, \]
\[ \delta_4(\varepsilon) = \inf \{1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| = 1, \|x - y\| = \varepsilon\}, \]
and
\[ \delta_5(\varepsilon) = \inf \{1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| = 1, \|x - y\| \geq \varepsilon\}. \]

It is clear that
\[ \delta_3(\varepsilon) \leq \delta_1(\varepsilon) \leq \delta_4(\varepsilon) \leq \delta_2(\varepsilon), \]
and
\[ \delta_3(\varepsilon) \leq \delta_5(\varepsilon) \leq \delta_X(\varepsilon) \leq \delta_2(\varepsilon). \]

So it is enough to show that

(3.1) \[ \delta_1(\varepsilon) \leq \delta_3(\varepsilon) \text{ and } \delta_2(\varepsilon) \leq \delta_1(\varepsilon). \]

In the presence of finite-dimensional (in fact two-dimensional) character of the modulus of convexity, assume further in the proof that \( \dim X < \infty \). Let's start by proving the first inequality in (3.1). We will show that for any \( x, y \in B_X \) with \( \|x - y\| \geq \varepsilon, 0 < \varepsilon \leq 2 \), there exist \( x_0, y_0 \in B_X \) such that
\[ \|x_0 - y_0\| = \varepsilon \text{ and } \left\| \frac{x + y}{2} \right\| \leq \left\| \frac{x_0 + y_0}{2} \right\|. \]

For \( \lambda = \frac{\varepsilon}{\|x - y\|} \leq 1 \) and \( \beta \geq 0 \) we consider the elements
\[ x_\beta = \frac{1}{2} (\lambda (x - y) + \beta (x + y)), \]
\[ y_\beta = \frac{1}{2} (\beta (x + y) - \lambda (x - y)) \]

and the function \( f(\beta) = \max(\|x_\beta\|, \|y_\beta\|) \). Since \( f(0) = \frac{\xi}{2}, \lim_{\beta \to \infty} f(\beta) = \infty \)
and \( f \) is continuous on \([0, \infty)\), there exists \( \beta_0 \geq 0 \) such that \( f(\beta_0) = \max(\|x_{\beta_0}\|, \|y_{\beta_0}\|) = 1 \). Without loss of generality assume that \( \|x_{\beta_0}\| = 1 \)
and denote \( x_{\beta_0} \) by \( x_0 \) and \( y_{\beta_0} \) by \( y_0 \). Thus \( \|x_0\| = 1, \|y_0\| \leq 1 \) and so, by
the triangle inequality,

\[
1 = \|x_0\| = \left\| \frac{1}{2} ((\lambda + \beta_0) x + y (\beta_0 - \lambda)) \right\|
\leq \frac{1}{2} (\lambda + \beta_0) \|x\| + \frac{1}{2} (\beta_0 - \lambda) \|y\|
\leq \frac{1}{2} (\lambda + \beta_0) + \frac{1}{2} (\beta_0 - \lambda) = \beta_0.
\]

Therefore

\[
\left\| \frac{1}{2} (x_0 + y_0) \right\| =
\left\| \frac{1}{2} \left[ \frac{1}{2} (\lambda (x - y) + \beta_0 (x + y)) + \frac{1}{2} (\beta_0 (x + y) - \lambda (x - y)) \right] \right\|
= \beta_0 \left\| \frac{x + y}{2} \right\| \geq \left\| \frac{x + y}{2} \right\|.
\]

It follows that

\[
1 - \left\| \frac{x_0 + y_0}{2} \right\| \leq 1 - \left\| \frac{x + y}{2} \right\|.
\]

Moreover, observe that

\[
x_0 - y_0 = \lambda (x - y)
\]

and so

\[
\|x_0 - y_0\| = \lambda \|x - y\| = \varepsilon.
\]

Thus \( \delta_1(\varepsilon) \leq \delta_3(\varepsilon) \) and it only remains to prove the second inequality in (3.1). We shall prove that infimum in \( \delta_2(\varepsilon) \) is attained on the unit sphere \( S_X \).

Let \( 0 < \varepsilon \leq 2 \). At first we show the following crucial equality:

\[
(3.2) \quad \sup \{\|x + y\| : x, y \in B_X, \|x - y\| = \varepsilon\}
= \sup \{\|x + y\| : x, y \in S_X, \|x - y\| = \varepsilon\}.
\]

There exist \( u, v \in B_X \) such that \( \|u - v\| = \varepsilon \) and

51
\[ \|u + v\| = \sup \{\|x + y\| : x, y \in B_X, \|x - y\| = \varepsilon\}. \]

Assume that \( \|u\| \leq \|v\| \leq 1 \). Then \( \|v\| \neq 0 \).

Claim : \( \|v\| = 1 \)

Let \( c = \frac{1 - \|v\|}{2} \) (then \( 0 \leq c < 1 \)) and define

\[
\begin{align*}
  u_1 &= \left[\frac{(1 - c)u + cv}{\|v\|}\right], \\
v_1 &= \left[\frac{(1 - c)v + cu}{\|v\|}\right].
\end{align*}
\]

It is clear that \( \|u_1\| \leq 1, \|v_1\| \leq 1 \) and

\[ \|u_1 - v_1\| = \|u - v - cu + cv + cu - cv\| / \|v\| = \|u(1 - 2c) - v(1 - 2c)\| / \|v\| = |1 - 2c| \cdot \|u - v\| / \|v\| = \varepsilon. \]

Thus \( u_1, v_1 \in B_X \) and \( \|u_1 - v_1\| = \varepsilon \), and so

\[ \|u_1 + v_1\| \leq \|u + v\|. \]

On the other hand, in view of \( \|v\| \leq 1 \)

\[ \|u_1 + v_1\| = \|u + v\| / \|v\| = \frac{1}{\|v\|} \|u + v\| \geq \|u + v\| . \]

It follows that \( \|v\| = 1 \), which completes the claim.

Now, if \( \|u\| = 1 \) then (3.2) holds, thus assume that \( \|u\| < 1 \). Choose \( \varphi \in S_{X^*} \) such that

\[ \varphi \left( \frac{u + v}{\|u + v\|} \right) = 1. \]

For any \( z \in B_X \) with \( \|z - v\| = \varepsilon \) we have

\[ \varphi (z + v) \leq \|\varphi\| \|z + v\| = \|z + v\| \leq \|u + v\| = \varphi (u + v). \]

Thus

\[ \varphi (z) \leq \varphi (u) \]

for every \( z \in B_X \cap (v + \varepsilon S_X) = \{z \in B_X : \|z - v\| = \varepsilon\} \).

Now, setting

\[ U = \{w \in S_X : v + \varepsilon w \in B_X \setminus S_X\} \]
\begin{align*}
&= \{ \|w\| = 1 : \|v + \varepsilon w\| < 1 \} \\
&= S_X \cap \{ w \in X : \|v + \varepsilon w\| < 1 \},
\end{align*}

notice that \( U \) is relatively open in \( S_X \). Moreover, \( \| \frac{u-v}{\varepsilon} \| = 1 \) and \( \|v + \varepsilon w\| = \|v + \varepsilon \frac{u-v}{\varepsilon}\| = \|u\| < 1 \), and thus \( (u-v)/\varepsilon \in U \).

Now we will show that \( \varphi \) attains its maximum on \( U \) at \( (u-v)/\varepsilon \). Letting \( w \in U \) and denoting \( z = v + \varepsilon w \), we see that

\[
\|z - v\| = \varepsilon \text{ and } \|z\| < 1.
\]

Hence \( z \in B_X \cap (v + \varepsilon S_X) \) and therefore

\[
\varphi(z) \leq \varphi(u).
\]

However

\[
\varphi(z) = \varphi(v) + \varepsilon \varphi(w) \leq \varphi(u)
\]

i.e.

\[
\varphi(w) \leq \varphi\left(\frac{u-v}{\varepsilon}\right)
\]

for all \( w \in U \). Since \( U \) is relatively open in \( S_X \) it means that \( \varphi \) attains its local maximum on \( S_X \) at \( (u-v)/\varepsilon \). By Lemma 3.2 it is also a global maximum on \( S_X \), i.e.

\[
(3.3) \quad |\varphi(w)| \leq |\varphi\left(\frac{u-v}{\varepsilon}\right)|,
\]

for all \( w \in S_X \), and so

\[
|\varphi\left(\frac{u-v}{\varepsilon}\right)| = 1.
\]

Hence we get two possibilities

\[
\varphi(u - v) = \|u - v\| \text{ or } \varphi(u - v) = -\|u - v\|.
\]

From the first equality we obtain

\[
\varphi(u) = \frac{1}{2} \varphi((u + v) + (u - v)) = \frac{1}{2} (\|u + v\| + \|u - v\|) \\
\geq \frac{1}{2} \|u + v + v - u\| = \|v\| = 1.
\]

It follows that \( \|u\| \geq 1 \) and we get a contradiction, in view of \( \|u\| < 1 \).

The second possibility gives

\[
\varphi(u - v) = -\|u - v\| = -\varepsilon.
\]
For any \( z \in B_x \) with \( \| z - v \| = \varepsilon \), we obtain by (3.3) that
\[
|\varphi \left( \frac{z-v}{\varepsilon} \right)| \leq |\varphi \left( \frac{u-v}{\varepsilon} \right)|
\]
and so
\[
|\varphi (z-v)| \leq |\varphi (u-v)|.
\]
Since \( \varphi (u-v) < 0 \), \( |\varphi (u-v)| = -\varphi (u-v) \). Thus \( |\varphi (z-v)| \leq -\varphi (u-v) \), and so \( \varphi (u-v) \leq -\varphi (z-v) \) or \( \varphi (u) \leq \varphi (z) \). Hence
\[
\varphi (u+v) \leq \varphi (z+v).
\]
By the choice of \( \varphi \), \( \varphi (u+v) = \| u + v \| \), and so
\[
\| u + v \| = \varphi (u+v) \leq \varphi (z+v) \leq \| z + v \|.
\]
On the other hand \( \| z + v \| \leq \| u + v \| \). Therefore \( \| u + v \| = \| z + v \| \).
Thus we have shown that for all \( z \in S_x \) such that \( \| z - v \| = \varepsilon \) it holds \( \| u + v \| = \| z + v \| \), completing the proof of equality (3.2).
Now the required equality \( \delta_1 (\varepsilon) = \delta_2 (\varepsilon) \) follows immediately. In fact
\[
\delta_1 (\varepsilon) = \inf \left\{ 1 - \frac{\| x+y \|}{2} : x, y \in B_x, \| x - y \| = \varepsilon \right\}
= 1 - \frac{1}{2} \sup \{ \| x + y \| : x, y \in B_x, \| x - y \| = \varepsilon \}
= 1 - \frac{1}{2} \sup \{ \| x + y \| : x, y \in S_x, \| x - y \| = \varepsilon \}
= \inf \left\{ 1 - \frac{\| x+y \|}{2} : x, y \in S_x, \| x - y \| = \varepsilon \right\} = \delta_2 (\varepsilon),
\]
which finishes the proof of the theorem. \( \diamond \)

**Remark.** For one-dimensional space \( X = (\mathbb{R}, |.|) \) and \( 0 \leq \varepsilon \leq 2 \) Theorem 3.1 does not hold since we have:
\[
\delta_X (\varepsilon) = 1 \text{ for } 0 \leq \varepsilon \leq 2,
\]
\[
\delta_2 (\varepsilon) = \left\{ \begin{array}{ll}
0 & \text{for } \varepsilon = 0, \\
1 & \text{for } 0 < \varepsilon \leq 2,
\end{array} \right.
\]
and \( \delta_3 (\varepsilon) = \delta_4 (\varepsilon) = \frac{\varepsilon}{2} \).

**Remark.** If \( X = \mathbb{R}^2 \) with one of the norms
\[
\|(x,y)\| = |x| + |y| \text{ or } \|(x,y)\| = \max (|x|, |y|),
\]
then \( \delta_X (\varepsilon) = 0 \) for all \( 0 \leq \varepsilon \leq 2 \) and \( \varepsilon_0 = 2 \).
In the next theorem we state a number of properties of the modulus of convexity. But at first we show the following lemma which will be useful in the proof.

**Lemma 3.3.** Let \( f_\alpha : [0, b] \to [0, \infty) \), \( f_\alpha (0) = 0 \), \( f_\alpha (x) > 0 \) for \( x \in (0, b] \) and let \( f_\alpha \) be convex for all \( \alpha \). Moreover let \( \inf_{\alpha} f_\alpha (x) > 0 \) for \( 0 < x_0 < x < b \). Then the function \( x \mapsto \inf_{\alpha} f_\alpha (x) \) is increasing on \((0, b]\), and in particular \( \inf_{\alpha} f_\alpha (x) \) is strictly increasing on \([x_0, b]\).

**Proof.** Indeed for \( 0 < x < y \leq b \)

\[
 f_\alpha (x) = f_\alpha \left( \frac{x}{y} \right) \leq \frac{x}{y} f_\alpha (y) + \left( 1 - \frac{x}{y} \right) f_\alpha (0) = \frac{x}{y} f_\alpha (y),
\]

by convexity of \( f_\alpha \). So

\[
 \inf_{\alpha} f_\alpha (x) \leq \frac{x}{y} \inf_{\alpha} f_\alpha (y) < \inf_{\alpha} f_\alpha (y). \quad \Box
\]

**Theorem 3.4.** The modulus of convexity \( \delta_X \) of a Banach space \( X \) \((\dim X \geq 2)\) has the following properties:

(a) \( \delta_X \) is continuous on \([0, 2)\) but not necessarily at \( \varepsilon = 2 \).

(b) \( \delta_X \) is an increasing function on \([0, 2]\) and strictly increasing on \([\varepsilon_0, 2]\).

(c) the function \( \varepsilon \mapsto \delta_X (\varepsilon) / \varepsilon \) is increasing on \((0, 2]\).

(d) \( \delta_X (\varepsilon) \leq \frac{\varepsilon}{2} \) for every \( \varepsilon \in [0, 2] \); \( \delta_X (2) = 1 \) if and only if \( X \) is strictly convex.

(e) \( \lim_{\varepsilon \to 2^-} \delta_X (\varepsilon) = \delta_X (2^-) = 1 - \frac{\varepsilon_0(X)}{2} \).

(f) if \( \delta_X (\varepsilon) > 0 \) then \( \delta_X (2 - 2\delta_X (\varepsilon)) = 1 - \frac{\varepsilon}{2} \).

**Proof.** Without loss of generality we assume that \( \dim X < \infty \).
For \( u, v \in S_X \) and \( 0 < \varepsilon \leq 2 \), define

\[
 \delta_{u,v} (\varepsilon) = \inf \left\{ 1 - \| \frac{x+y}{2} \| : x, y \in S_X, \| x - y \| \geq \varepsilon, x - y = \lambda u, x + y = \mu v; \lambda, \mu \in \mathbb{R} \right\}.
\]

**Claim 1.** \( \delta_{u,v} (\varepsilon) \) is a convex function.

In fact, let \( \varepsilon_1, \varepsilon_2 \in (0, 2] \) be given. Then there exist \( x_i, y_i \in S_X \), \( i = 1, 2 \), such that

\[
 \delta_{u,v} (\varepsilon_i) = 1 - \left\| \frac{x_i + y_i}{2} \right\|, \quad \| x_i - y_i \| \geq \varepsilon_i,
\]

In particular, \( \delta_{u,v} (\varepsilon) \) is increasing on \([0, 2]\), and in particular \( \inf_{\alpha} f_\alpha (x) \) is strictly increasing on \([x_0, b]\).

**Proof.** Indeed for \( 0 < x < y \leq b \)

\[
 f_\alpha (x) = f_\alpha \left( \frac{x}{y} \right) \leq \frac{x}{y} f_\alpha (y) + \left( 1 - \frac{x}{y} \right) f_\alpha (0) = \frac{x}{y} f_\alpha (y),
\]

by convexity of \( f_\alpha \). So

\[
 \inf_{\alpha} f_\alpha (x) \leq \frac{x}{y} \inf_{\alpha} f_\alpha (y) < \inf_{\alpha} f_\alpha (y). \quad \Box
\]

**Theorem 3.4.** The modulus of convexity \( \delta_X \) of a Banach space \( X \) \((\dim X \geq 2)\) has the following properties:

(a) \( \delta_X \) is continuous on \([0, 2)\) but not necessarily at \( \varepsilon = 2 \).

(b) \( \delta_X \) is an increasing function on \([0, 2]\) and strictly increasing on \([\varepsilon_0, 2]\).

(c) the function \( \varepsilon \mapsto \delta_X (\varepsilon) / \varepsilon \) is increasing on \((0, 2]\).

(d) \( \delta_X (\varepsilon) \leq \frac{\varepsilon}{2} \) for every \( \varepsilon \in [0, 2] \); \( \delta_X (2) = 1 \) if and only if \( X \) is strictly convex.

(e) \( \lim_{\varepsilon \to 2^-} \delta_X (\varepsilon) = \delta_X (2^-) = 1 - \frac{\varepsilon_0(X)}{2} \).

(f) if \( \delta_X (\varepsilon) > 0 \) then \( \delta_X (2 - 2\delta_X (\varepsilon)) = 1 - \frac{\varepsilon}{2} \).

**Proof.** Without loss of generality we assume that \( \dim X < \infty \).
For \( u, v \in S_X \) and \( 0 < \varepsilon \leq 2 \), define

\[
 \delta_{u,v} (\varepsilon) = \inf \left\{ 1 - \| \frac{x+y}{2} \| : x, y \in S_X, \| x - y \| \geq \varepsilon, x - y = \lambda u, x + y = \mu v; \lambda, \mu \in \mathbb{R} \right\}.
\]

**Claim 1.** \( \delta_{u,v} (\varepsilon) \) is a convex function.

In fact, let \( \varepsilon_1, \varepsilon_2 \in (0, 2] \) be given. Then there exist \( x_i, y_i \in S_X \), \( i = 1, 2 \), such that

\[
 \delta_{u,v} (\varepsilon_i) = 1 - \left\| \frac{x_i + y_i}{2} \right\|, \quad \| x_i - y_i \| \geq \varepsilon_i,
\]
\[ x_i - y_i = \lambda_i u, \quad x_i + y_i = \mu_i v, \]
for \( i = 1, 2 \) and some \( \lambda_i, \mu_i \in \mathbb{R} \).
Let \( 0 < \alpha < 1 \) be arbitrary. Define
\[ x = \alpha x_1 + (1 - \alpha) x_2, \quad y = \alpha y_1 + (1 - \alpha) y_2. \]
Then
\[ \| x \| = \| \alpha x_1 + (1 - \alpha) x_2 \| \leq \alpha \| x_1 \| + (1 - \alpha) \| x_2 \| = 1 \]
and similarly \( \| y \| \leq 1 \). Moreover,
\[ x - y = (\alpha \lambda_1 + (1 - \alpha) \lambda_2) u \]
and thus
\[ \| x - y \| = |\alpha \lambda_1 + (1 - \alpha) \lambda_2| \| u \| = \alpha \| \lambda_1 u \| + (1 - \alpha) \| \lambda_2 u \| \]
\[ = \alpha \| x_1 - y_1 \| + (1 - \alpha) \| x_2 - y_2 \| \geq \alpha \varepsilon_1 + (1 - \alpha) \varepsilon_2. \]
We also have that
\[ x + y = (\alpha \mu_1 + (1 - \alpha) \mu_2) v. \]
Thus by definition of \( \delta_{u,v} \) we obtain
\[ \delta_{u,v} (\alpha \varepsilon_1 + (1 - \alpha) \varepsilon_2) \leq 1 - \| \frac{x + y}{2} \|. \]
Since
\[ \| \frac{x + y}{2} \| = \| \alpha \frac{x_1 + y_1}{2} + (1 - \alpha) \frac{x_2 + y_2}{2} \| \]
\[ = |\alpha \frac{\mu_1}{2} + (1 - \alpha) \frac{\mu_2}{2}| \| u \| \]
\[ = \alpha \| \frac{x_1 + y_1}{2} \| + (1 - \alpha) \| \frac{x_2 + y_2}{2} \| , \]
so
\[ 1 - \| \frac{x + y}{2} \| = \alpha \left( 1 - \| \frac{x_1 + y_1}{2} \| \right) + (1 - \alpha) \left( 1 - \| \frac{x_2 + y_2}{2} \| \right). \]
Hence
\[ \delta_{u,v} (\alpha \varepsilon_1 + (1 - \alpha) \varepsilon_2) \leq 1 - \| \frac{x + y}{2} \| \]
\[ = \alpha \left( 1 - \| \frac{x_1 + y_1}{2} \| \right) + (1 - \alpha) \left( 1 - \| \frac{x_2 + y_2}{2} \| \right) \]
\[ = \alpha \delta_{u,v} (\varepsilon_1) + (1 - \alpha) \delta_{u,v} (\varepsilon_2) . \]
which shows that \( \delta_{u,v} (\varepsilon) \) is convex.

**Claim 2.** \( \delta_X (\varepsilon) = \inf \{ \delta_{u,v} (\varepsilon) : \| u \| = \| v \| = 1, u \neq \pm v \} . \)
We always have
\[ \delta_X(\varepsilon) \leq \inf \{ \delta_{u,v}(\varepsilon) : \|u\| = \|v\| = 1, u \neq \pm v \}. \]

By the definition of \( \delta_X(\varepsilon) \) there exist \( x_0, y_0 \) such that
\[ \|x_0\| = \|y_0\| = 1, \|x_0 - y_0\| \geq \varepsilon, x_0 \neq \pm y_0 \]
and
\[ 1 - \left\| \frac{x_0 + y_0}{2} \right\| = \delta_X(\varepsilon) \]
Defining
\[
\begin{align*}
  u &= (x_0 - y_0) / \|x_0 - y_0\|, \\
  v &= (x_0 + y_0) / \|x_0 + y_0\|, \\
  \lambda &= \|x_0 - y_0\|, \mu = \|x_0 + y_0\| .
\end{align*}
\]
it holds
\[
\begin{align*}
  x_0 - y_0 &= \lambda u, \\
  x_0 + y_0 &= \mu v .
\end{align*}
\]
We also claim that \( u \neq \pm v \). Indeed, if for example \( u = v \) then
\[
\frac{(x_0 - y_0)}{\|x_0 - y_0\|} = \frac{(x_0 + y_0)}{\|x_0 + y_0\|}
\]
from which it follows that
\[
x_0 + y_0 = \frac{\|x_0 + y_0\|}{\|x_0 - y_0\|} (x_0 - y_0)
\]
and hence \( x_0 = \gamma y_0 \) for some \( \gamma \in \mathbb{R} \). Thus \( 1 = \|x_0\| = |\gamma| \|y_0\| = |\gamma| \) and so \( \gamma = \pm 1 \), i.e., \( x_0 = \pm y_0 \). In the same way we find that if \( u = -v \), then \( x_0 = \pm y_0 \). This contradiction shows that \( u \neq \pm v \).

Hence, by the definition of \( \delta_{u,v}(\varepsilon) \), we have showed that for some \( u, v \)
\[
\delta_{u,v}(\varepsilon) \leq 1 - \left\| \frac{x_0 + y_0}{2} \right\| = \delta_X(\varepsilon)
\]
Therefore
\[
\delta_X(\varepsilon) \geq \inf \{ \delta_{u,v}(\varepsilon) : \|u\| = \|v\| = 1, u \neq \pm v \},
\]
and also Claim 2 is proved.

(a) Let \( a \in (0, 2) \) and \( \varepsilon_1, \varepsilon_2 \in [0, a] \). Since \( \delta_{u,v}(0) = 0 \) and \( \delta_{u,v}(\varepsilon) \) is a convex non-negative function,
\[
\begin{align*}
  (\delta_{u,v}(\varepsilon_2) - \delta_{u,v}(\varepsilon_1)) / (\varepsilon_2 - \varepsilon_1) \leq \\
  (\delta_{u,v}(2) - \delta_{u,v}(a)) / (2 - a) \leq \frac{1}{2-a}. 
\end{align*}
\]
Hence for all \( u, v \in S_X \) and \( 0 \leq \varepsilon_1 \leq \varepsilon_2 \leq a < 2 \)
\[ \delta_{u,v} (\varepsilon_2) - \delta_{u,v} (\varepsilon_1) \leq \frac{\varepsilon_2 - \varepsilon_1}{2 - a}. \]
Thus the family \( \{\delta_{u,v} (\varepsilon)\}_{u,v} \) is equicontinuous on \([0, a]\). Now by Claim 2,
\[ \delta_X (\varepsilon_2) - \delta_X (\varepsilon_1) \leq \frac{\varepsilon_1 - \varepsilon_2}{2 - a}, \]
and thus \( \delta_X (\varepsilon) \) is continuous on \([0, a]\). Since \( a \) was arbitrary number in \((0, 2)\), \( \delta_X \) is continuous on \([0, 2]\).

(b) Define \( A_\varepsilon = \{x, y \in X : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon\} \). Letting \( 0 \leq \varepsilon_1 \leq \varepsilon_2 \leq 2 \), we have \( A_{\varepsilon_2} \subset A_{\varepsilon_1} \) and so
\[ \delta_X (\varepsilon_2) = \inf \{1 - \frac{\|x+y\|}{2} : x, y \in A_{\varepsilon_2}\} \]
\[ \geq \inf \{1 - \frac{\|x+y\|}{2} : x, y \in A_{\varepsilon_1}\} = \delta_X (\varepsilon_1). \]
Thus \( \delta_X \) is an increasing function. Now, strict monotonicity of \( \delta_X \) on \([\varepsilon_0, 2]\) follows from the Lemma 3.3 and Claims 1 and 2.

(c) The monotonicity of \( \delta_X (\varepsilon) / \varepsilon \) on \((0, 2]\) is a direct consequence of Lemma 3.3 and Claims 1 and 2.

(d) Let \( \varepsilon \in [0, 2] \), \( \|x\| = \|y\| = 1 \) and \( \|x - y\| = \varepsilon \). Then
\[ \|x + y\| + \|x - y\| \geq \|x\| - \|y\| + \|x\| + \|y\| = 2, \]
and so
\[ \frac{\|x+y\|}{2} \geq 1 - \frac{\|x-y\|}{2} = 1 - \frac{\varepsilon}{2}. \]
Hence
\[ \delta_X (\varepsilon) \leq 1 - \frac{\|x+y\|}{2} \leq \frac{\varepsilon}{2}. \]
Let now \( \delta_X (2) = 1 \), and suppose that \( x, y \in X \) satisfy \( \|x\| = \|y\| = \|\frac{x+y}{2}\| = 1 \). Then
\[ \frac{\|x-y\|}{2} = \left| \frac{x+(-y)}{2} \right| \leq 1 - \delta (\|x - (-y)\|) = 1 - \delta (2) = 0. \]
Thus \( x = y \) and \( X \) is strictly convex.
Suppose now, that \( X \) is strictly convex and let
\[ \|x\| = \|y\| = 1 \text{ with } \|x - y\| = 2. \]
If \( x \neq -y \), then we get a contradiction.
\[ 1 = \left\| \frac{x - y}{2} \right\| = \left\| \frac{x + (-y)}{2} \right\| < 1. \]

Hence \( x = -y \) and \( \delta_X(2) = 1 - \left\| \frac{x + y}{2} \right\| = 1. \)

(e) Let \( \varepsilon \in [\varepsilon_0, 2). \) For \( \eta \in (0, 1 - \delta_X(\varepsilon)) \), there exist \( x, y \in S_X \), such that \( \|x - y\| = \varepsilon \) and

\[ \left\| \frac{x + y}{2} \right\| \geq 1 - \delta_X(\varepsilon) - \eta. \]

Then in view of monotonicity of \( \delta_X \)

\[ \frac{\varepsilon}{2} = \|x - y\|/2 \leq 1 - \delta_X(\|x - (-y)\|) \]
\[ = 1 - \delta_X(\|x + y\|) \leq 1 - \delta_X(2(1 - \delta_X(\varepsilon) - \eta)). \]

Since \( \eta \) is arbitrary small, by continuity of \( \delta_X(\varepsilon) \),

\[ (3.4) \quad \frac{\varepsilon}{2} \leq 1 - \delta_X(2(1 - \delta_X(\varepsilon))), \]

for any \( \varepsilon \in [\varepsilon_0, 2). \) Letting \( \varepsilon \to 2^- \) we obtain that \( \delta_X(2(1 - \delta(2^-))) = 0 \) and so \( \varepsilon_0 \geq 2(1 - \delta_X(2^-)) \) or equivalently \( \delta_X(2^-) \geq 1 - \frac{\varepsilon_0}{2}. \)

It is also clear that letting \( \varepsilon \to \varepsilon_0^+ \) in (3.4) we get that \( \delta_X(2^-) \leq 1 - \frac{\varepsilon_0}{2} \), which completes the proof of (d).

(f) We will use the inequality (3.4). At first, if we set \( t = \varepsilon \in [\varepsilon_0, 2) \) then by (3.4), \( \frac{\varepsilon}{2} \leq 1 - \delta_X(2(1 - \delta_X(t))). \) The second application of (3.4) with \( t = 2(1 - \delta_X(\varepsilon)) \) gives \( \frac{\varepsilon}{2} \leq 1 - \delta_X(t) \) or equivalently \( 1 - \delta_X(\varepsilon) \geq 1 - \delta_X(2(1 - \delta_X(t))). \) From both applications of (3.4) we obtain

\[ 1 - \delta_X(\varepsilon) = \frac{\varepsilon}{2} \leq 1 - \delta_X(2(1 - \delta_X(t))) \leq 1 - \delta_X(\varepsilon). \]

Thus we get equality in (3.4). Since \( t \) is arbitrary in \( [\varepsilon_0, 2) \) we get that for all \( \varepsilon \in [\varepsilon_0, 2), \)

\[ \delta_X(2(1 - \delta_X(\varepsilon))) = 1 - \frac{\varepsilon}{2}. \]

**Remark.** (a) \( \delta_X(\varepsilon) \) is not necessarily continuous at \( \varepsilon = 2 \) (see Ex. 3.16).
(b) \( \delta_X(\varepsilon) \) does not need to be convex (see Ex. 3.18 and 3.20).
**Proposition 3.5.** Let $H$ be an inner product space with $\dim H \geq 2$. Then

$$\delta_H(\varepsilon) = 1 - \sqrt{1 - \frac{1}{4}\varepsilon^2}.$$ 

In particular,

$$\frac{\varepsilon^2}{8} \leq \delta_H(\varepsilon) \leq \frac{\varepsilon^2}{4}.$$ 

Indeed, since $H$ is characterized by the parallelogram law we have that if $x, y \in H$ satisfy $\|x\| = \|y\| = 1$ and $\|x - y\| = \varepsilon$, then

$$\left\| \frac{x+y}{2} \right\|^2 = \frac{1}{2} \|x\|^2 + \frac{1}{2} \|y\|^2 - \frac{1}{4} \|x - y\|^2$$

and

$$\left\| \frac{x+y}{2} \right\| = \sqrt{1 - \frac{1}{4}\varepsilon^2}.$$ 

Consequently,

$$\delta_H(\varepsilon) = 1 - \sqrt{1 - \frac{1}{4}\varepsilon^2}.$$ 

Moreover, as it is easy to see that

$$1 - \sqrt{1 - \frac{1}{4}\varepsilon^2} = \frac{\varepsilon^2}{4(1+\sqrt{1-\frac{1}{4}\varepsilon^2})} \geq \frac{\varepsilon^2}{8},$$

and also

$$\frac{\varepsilon^2}{4(1+\sqrt{1-\frac{1}{4}\varepsilon^2})} \leq \frac{\varepsilon^2}{4},$$

we have the required inequalities. $\diamond$

We also formulate the following result of G. Nordlander from 1960 (see [40]).

**Remark.** For any Banach space $X$ with $\dim X \geq 2$,

$$\delta_X(\varepsilon) \leq \delta_H(\varepsilon) = 1 - \sqrt{1 - \frac{1}{4}\varepsilon^2} \text{ for each } \varepsilon \in [0, 2],$$

which means that a Hilbert space is "the most convex".

By using Clarkson’s methods from [7] we can estimate the modulus of convexity for $L^p$ spaces, $1 < p < \infty$. Notice that for $p = 1$, the space $L^1$ is not even strictly convex, so $\delta_{L^1}(\varepsilon) = 0$ for every $\varepsilon \in [0, 2]$. Observe also that later on O. Hanner in [16] calculated exactly the modulus of convexity in $L^p$. We will present his proof separately in Section III.3. Below we will only show the Clarkson’s proof for $p \geq 2$. 

60
Proposition 3.6. If $2 \leq p < \infty$, then

$$\delta_{L^p}(\varepsilon) \geq 1 - \left[1 - \left(\frac{\varepsilon}{2}\right)^p\right]^\frac{1}{p}.$$ 

If, additionally, measure $\mu$ is either nonatomic or counting, then

$$\delta_{L^p}(\varepsilon) = 1 - \left[1 - \left(\frac{\varepsilon}{2}\right)^p\right]^\frac{1}{p}.$$ 

**Proof.** Let $||x|| = ||y|| = 1$, then by Clarkson inequality for $p \geq 2$

$$(||x + y||^p + ||x - y||^p)^\frac{1}{p} \leq 2^\frac{1}{p} \left(||x||^p + ||y||^p\right)^\frac{1}{p}.$$ 

we obtain

$$||x + y||^p + ||x - y||^p \leq 2^p,$$

and if $||x - y|| \geq \varepsilon$, then

$$||\frac{x + y}{2}|| \leq \left[1 - \left(\frac{\varepsilon}{2}\right)^p\right]^\frac{1}{p}.$$ 

Hence

$$\delta_{L^p}(\varepsilon) \geq 1 - \left[1 - \left(\frac{\varepsilon}{2}\right)^p\right]^\frac{1}{p}.$$ 

In order to get equality we take two disjoint measurable sets of measure $\mu A_1 = \mu A_2 = 1$ and we put

$$\tilde{x}(t) = \begin{cases} 1 - \delta & \text{for } t \in A_1, \\ 0 & \text{for } t \in A_2, \end{cases}$$

$$\tilde{y}(t) = \begin{cases} 0 & \text{for } t \in A_1, \\ \frac{\varepsilon}{2} & \text{for } t \in A_2, \end{cases}$$

where $\delta = 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^\frac{1}{p}$.

Then $||\tilde{x}|| = 1 - \delta$, $||\tilde{y}|| = \frac{\varepsilon}{2}$ and for $x = \tilde{x} + \tilde{y}$ and $y = \tilde{x} - \tilde{y}$, we obtain

$$||x||^p = ||\tilde{x} + \tilde{y}||^p = \int_{A_1} (1 - \delta)^p d\mu + \int_{A_2} \left(\frac{\varepsilon}{2}\right)^p d\mu$$

$$= (1 - \delta)^p + \left(\frac{\varepsilon}{2}\right)^p = 1$$

and similarly $||y|| = 1$. Therefore
\[ \|x\| = \|y\| = 1, \|x - y\| = \varepsilon \]

and
\[ \|\frac{x+y}{2}\| = \|\frac{x}{2}\| = 1 - \delta. \]

Hence
\[ \delta_{L^p}(\varepsilon) = 1 - (1 - (\frac{\varepsilon}{2})^p)^{\frac{1}{p}}. \]

For \( 1 < p \leq 2 \) the situation is more complicated. For such \( p \) Clarkson obtained the inequality
\[ 2(\|x\|^p + \|y\|^p)^{p'-1} \geq \|x + y\|^{p'} + \|x - y\|^{p'} \]
which he used to get the following estimate
\[ \delta_{L^p}(\varepsilon) \geq 1 - \left[1 - (\frac{\varepsilon}{2})^p\right]^{\frac{1}{p}}, \]
where \( \frac{1}{p} + \frac{1}{p'} = 1. \)

In the end let us state and prove the similar result for all normed spaces.

**Theorem 3.6.** Let \( X = (X, \|\cdot\|) \) be a normed space and let the \((p', p)\)–Clarkson inequality hold in \( X \) for \( p \geq 2 \).

Then
\[ \delta_X(\varepsilon) \geq 1 - \left[1 - (\frac{\varepsilon}{2})^p\right]^{\frac{1}{p}}. \]

**Proof.** Since \( 1 < p' \leq 2 \) then the \((p', p)\)–Clarkson inequality has the form
\[ (\|x + y\|^{p'} + \|x - y\|^{p'})^{\frac{1}{p}} \leq 2^{\frac{1}{p}} \left(\|x\|^{p'} + \|y\|^{p'}\right)^{\frac{1}{p'}}. \]

Let \( \|x\| = \|y\| = 1. \) Then
\[ \|x + y\|^{p'} + \|x - y\|^{p'} \leq 2^p. \]

If \( \|x - y\| \geq \varepsilon \) then
\[ \| \frac{x+y}{2} \| \leq \left( 1 - \left( \frac{\varepsilon}{2} \right)^p \right)^{\frac{1}{p}} \]

and so

\[ 1 - \| \frac{x+y}{2} \| \geq 1 - \left( 1 - \left( \frac{\varepsilon}{2} \right)^p \right)^{\frac{1}{p}}. \]

Taking infimum over all \( \|x\| = \|y\| = 1 \) we have

\[ \delta_X (\varepsilon) \geq 1 - \left[ 1 - \left( \frac{\varepsilon}{2} \right)^p \right]^{\frac{1}{p}}. \]

### 3.2 Modulus of smoothness

In this section we discuss another parameter associated with a Banach space \( X \), its modulus of smoothness. This concept was introduced by M. M. Day in 1944 (see [8]) and in many aspects it is a dual notion to the modulus of convexity.

**Definition.** The **modulus of smoothness** of a Banach space \( X \) is a function \( \rho_X : [0, \infty) \rightarrow [0, \infty) \) defined by

\[ \rho_X (\tau) = \sup \left\{ \frac{1}{2} \left( \| x + \tau y \| + \| x - \tau y \| - 2 \right) : \| x \| = \| y \| = 1 \right\}. \]

A Banach space \( X \) is said to be **uniformly smooth** if \( \lim_{\tau \rightarrow 0^+} \frac{\rho_X (\tau)}{\tau} = 0. \)

We summarize below some basic properties of \( \rho_X \) as well as we present its connection to the modulus of convexity.

**Theorem 3.7.** The modulus of smoothness \( \rho_X \) is a continuous increasing convex function on \([0, \infty)\), \( \rho_X (0) = 0 \) and \( \max (0, \tau - 1) \leq \rho_X (\tau) \leq \tau. \)

**Proof.** The fact that \( \rho_X (0) = 0 \) is obvious. The right-hand side of the last estimation follows from the triangle inequality and the left-hand side from the inequality (vi) on page 5. We will prove that \( \rho_X \) is convex. Let \( \tau_1, \tau_2 \in [0, \infty) \) and \( \alpha, \beta \in [0, 1] \) be such that \( \alpha + \beta = 1. \) Then by the triangle inequality

\[
\rho_X (\alpha \tau_1 + \beta \tau_2) = \\
= \sup \left\{ \frac{1}{2} \left( \| \alpha (x + \tau_1 y) + \beta (x + \tau_2 y) \| + \| \alpha (x - \tau_1 y) + \beta (x - \tau_2 y) \| - 1 : \| x \| = \| y \| = 1 \right) \right\}
\]
\[ \leq \sup \left\{ \frac{1}{2} [\alpha \|x + \tau_1 y\| + \beta \|x + \tau_2 y\| + \alpha \|x - \tau_1 y\| + \beta \|x - \tau_2 y\| - 1 : \|x\| = \|y\| = 1} \right\} + \sup \left\{ \frac{1}{2} (\|x + \tau_1 y\| + 1 : \|x\| = \|y\| = 1} \right\} + \sup \left\{ \frac{1}{2} (\|x + \tau_1 y\| + \|x - \tau_1 y\| - 1 : \|x\| = \|y\| = 1} \right\} + \sup \left\{ \frac{1}{2} (\|x + \tau_1 y\| + \|x - \tau_1 y\| - 1 : \|x\| = \|y\| = 1} \right\} \\
= \alpha \sup \left\{ \frac{1}{2} (\|x + \tau_1 y\| + \|x - \tau_1 y\| - 1 : \|x\| = \|y\| = 1} \right\} + \beta \sup \left\{ \frac{1}{2} (\|x + \tau_1 y\| + \|x - \tau_1 y\| - 1 : \|x\| = \|y\| = 1} \right\} \\
= \alpha \rho_x (\tau_1) + \beta \rho_x (\tau_2), \\
\]

which shows the convexity of \( \rho_x \). Thus for \( 0 \leq \tau_1 < \tau_2 \),

\[ \rho_x (\tau_1) = \rho_x \left( \frac{\tau_1}{\tau_2} \cdot \tau_2 + \left( 1 - \frac{\tau_1}{\tau_2} \right) \cdot 0 \right) \leq \frac{\tau_1}{\tau_2} \rho_x (\tau_2), \]

consequently \( \rho_x \) is increasing. This completes the proof since \( \rho_x \) as a convex function on \([0, \infty)\) is continuous on \((0, \infty)\) (see [44], Th.3.2) and the right-continuity at 0 follows from the last estimate. \( \diamond \)

**Theorem 3.8.** For any complex Hilbert space or any real Hilbert space \( H \) with \( \dim H \geq 2 \) we have

\[ \rho_H (\tau) = \sqrt{1 + \tau^2} - 1. \]

**Proof.** Let \( H \) be either complex Hilbert space or real Hilbert space with \( \dim H \geq 2 \) and the norm be given by the inner product

\[ \|x\| = \langle x, x \rangle^{\frac{1}{2}}. \]

Then

\[ \rho_H (\tau) = \sup \left\{ \frac{1}{2} \left( \langle x + \tau y, x + \tau y \rangle^{\frac{1}{2}} + \langle x - \tau y, x - \tau y \rangle^{\frac{1}{2}} \right) - 1 : \|x\| = \|y\| = 1 \right\} \\
= \sup \left\{ \frac{1}{2} \left[ \left( \|x\|^2 + \tau \langle x, y \rangle + \tau \langle x, y \rangle + \tau^2 \|y\|^2 \right)^{\frac{1}{2}} + \left( \|x\|^2 - \tau \langle x, y \rangle + \tau \langle x, y \rangle + \tau^2 \|y\|^2 \right) \right] - 1 : \|x\| = \|y\| = 1 \right\} \\
= \sup \left\{ \frac{1}{2} \left[ (1 + \tau \cdot 2 \text{Re} \langle x, y \rangle + \tau^2)^{\frac{1}{2}} + (1 - \tau \cdot 2 \text{Re} \langle x, y \rangle + \tau^2)^{\frac{1}{2}} \right] - 1 : \|x\| = \|y\| = 1 \right\} . \\
\]

Now put \( \text{Re} \langle x, y \rangle = u \). By the Cauchy-Schwarz inequality \( |u| \leq \|x\| \cdot \|y\| \leq 1 \). The required estimate
\[
\sqrt{1 + 2\tau u + \tau^2 + \sqrt{1 - 2\tau u + \tau^2}} \leq 2\sqrt{1 + \tau^2}
\]
for all \( u \in [-1,1] \) is satisfied. Indeed, it is equivalent to
\[
\sqrt{1 + 2\tau u + \tau^2} \cdot \sqrt{1 - 2\tau u + \tau^2} \leq 1 + \tau^2
\]
or
\[
\sqrt{(1 + \tau^2)^2 - 4\tau^2u^2} \leq 1 + \tau^2,
\]
and we easily observe that the latter is true because the left-hand side of this inequality attains its biggest value at \( u = 0 \). Thus
\[
\sup_{u \in [-1,1]} \sqrt{1 + 2\tau u + \tau^2 + \sqrt{1 - 2\tau u + \tau^2}} = 2\sqrt{1 + \tau^2},
\]
and so
\[
\rho_H(\tau) = \frac{1}{2} \cdot 2\sqrt{1 + \tau^2} - 1 = \sqrt{1 + \tau^2} - 1. \circledast
\]

**Remark.** The assumption that \( \dim H \geq 2 \) is essential in Theorem 3.8. For one-dimensional Hilbert space \((\mathbb{R},|·|)\) the modulus of smoothness is a function
\[
\rho_X(\tau) = \max(0,\tau - 1).
\]
Indeed,
\[
\rho_X(\tau) = \sup \left\{ \frac{1}{2} (|x + \tau y| + |x - \tau y|) - 1 : |x| = |y| = 1 \right\}
= \frac{1}{2} \max \{|1 + \tau| + |1 - \tau|, |1 - \tau| + |1 + \tau|, |1 + \tau| + |1 - \tau|, |1 - \tau| + |1 + \tau|\} - 1
= \frac{1}{2} (1 + \tau + |1 - \tau|) - 1 = \max(0,\tau - 1).
\]

**Remark.** Modulus \( \rho_H \) of any complex Hilbert space and any real Hilbert space with \( \dim H \geq 2 \) can be estimated as follows
\[
\frac{\min(\tau,\tau^2)}{3} \leq \rho_H(\tau) \leq \min(\tau,\frac{\tau^2}{3}).
\]
In fact, we always have
\[
\sqrt{1 + \tau^2} - 1 \leq \tau
\]
and also
\[
\sqrt{1 + \tau^2} - 1 = \frac{\tau^2}{\sqrt{1 + \tau^2 + 1}} \leq \frac{\tau^2}{2}
\]
so we have the right-hand side of the required inequality. In order to show the left-hand side of this inequality we consider two cases.
If $0 \leq \tau \leq 1$, then $\sqrt{1+\tau^2} + 1 \leq 3$ which gives

$$\sqrt{1+\tau^2} - 1 = \frac{\tau^2}{\sqrt{1+\tau^2} + 1} \geq \frac{\tau^2}{3}.$$ 

Now if $\tau \geq 1$, then $\sqrt{1+\tau^2} + 1 \leq 3\tau$ which gives

$$\sqrt{1+\tau^2} - 1 = \frac{\tau^2}{\sqrt{1+\tau^2} + 1} \geq \frac{\tau}{3}.$$ 

Let us now formulate and prove the duality result for moduli of convexity and smoothness (see [28]).

**Theorem 3.9.** (Lindenstrauss, 1963) For any Banach space $X$ and $\tau \geq 0$,

$$\rho_{X^*} (\tau) = \sup \{ \frac{\tau}{2} - \delta_X (\varepsilon) : 0 \leq \varepsilon \leq 2 \},$$

and

$$\rho_X (\tau) = \sup \{ \frac{\tau}{2} - \delta_{X^*} (\varepsilon) : 0 \leq \varepsilon \leq 2 \}.$$

**Proof.** For any $\tau > 0$,

$$2\rho_{X^*} (\tau) = \sup_{f,g \in S_{X^*}} \{ \| f + \tau g \| + \| f - \tau g \| - 2 \}
\begin{align*}
&= \sup_{f,g \in S_{X^*}} \sup_{x,y \in S_X} \{ Re (f + \tau g) (x) + Re (f - \tau g) (y) - 2 \} \\
&= \sup_{x,y \in S_X} \sup_{f,g \in S_{X^*}} \{ Re f (x + y) + \tau Re g (x - y) - 2 \} \\
&= \sup_{x,y \in S_X} \sup_{f \in S_{X^*}} \left\{ \sup_{g \in S_{X^*}} \{ Re f (x + y) + \tau Re g (x - y) - 2 \} \right\} \\
&= \sup_{x,y \in S_X} \sup_{f \in S_{X^*}} \left\{ \| x + y \| + \tau \| x - y \| - 2 \right\} \\
&= \sup_{0 \leq \varepsilon \leq 2} \sup_{x,y \in S_X, \| x - y \| = \varepsilon} \{ \| x + y \| + \tau \varepsilon - 2 \} \\
&= \sup_{0 \leq \varepsilon \leq 2} \left\{ \tau \varepsilon - 2(1 - \frac{\| x + y \|}{2}) \right\} \\
&= \sup_{0 \leq \varepsilon \leq 2} \{ \tau \varepsilon - 2 \delta_X (\varepsilon) \}. 
\end{align*}$$
The other equality can be proved exactly the same way after exchanging the roles of $X$ and $X^*$.

**Corollary 3.10.** For any Banach space $X$,  
\[ \rho_X(\tau) \geq \rho_{X^*}(\tau) = \sqrt{1 + \tau^2} - 1. \]

**Proof.** By the Nordlander’s theorem (see Remark on page 60)  
\[ \delta_{X^*}(\varepsilon) \leq 1 - \sqrt{1 - \frac{\varepsilon^2}{4}} \text{ for each } \varepsilon \in [0, 2]. \]
Thus by the duality formula on $\rho_X$, for any $\tau > 0$,  
\[ \rho_X(\tau) = \sup \left\{ \frac{\varepsilon}{2} - \delta_{X^*}(\varepsilon) : 0 \leq \varepsilon \leq 2 \right\} \geq \sup_{0 \leq \varepsilon \leq 2} \left( \frac{\varepsilon}{2} - 1 + \sqrt{1 - \frac{1}{4}\varepsilon^2} \right) = \sqrt{1 + \tau^2} - 1, \]
where the last step is obtained by standard calculation.

From Theorem 3.9 easily follows the Šmuelian [45] result:

**Theorem 3.11.** A Banach space $X$ is uniformly convex (uniformly smooth) if and only if $X^*$ is uniformly smooth (uniformly convex).

**Proof.** Suppose first that $X^*$ is uniformly smooth, i.e.  
\[ \lim_{\tau \to 0} \rho_{X^*}(\tau)/\tau = 0. \]
Then for every $0 < \varepsilon < 2$ there exists $\tau_\varepsilon > 0$ so that  
\[ \rho_{X^*}(\tau_\varepsilon)/\tau_\varepsilon \leq \varepsilon/4. \]
Therefore in view of the formula on $\rho_{X^*}$ in Theorem 3.10,  
\[ \tau_\varepsilon \varepsilon/2 - \delta_X(\varepsilon) \leq \tau_\varepsilon \varepsilon/4 > 0, \]
that is  
\[ \delta_X(\varepsilon) \geq \tau_\varepsilon \varepsilon/4, \]
which shows uniform convexity of $X$.

If $X$ is uniformly smooth, then similarly from the formula on $\rho_X$ we get that $X$ is uniformly convex.

Now suppose that $X$ is uniformly convex and $t > 0$ arbitrary. Let $\varepsilon_t = \min(2, 2t)$ and $\tau_t = \delta_X(\varepsilon_t) > 0$. Then  
\[ \rho_{X^*}(\tau_t)/\tau_t = \max \left[ \sup_{0 \leq \varepsilon \leq \varepsilon_t} \left\{ \frac{\varepsilon}{2} - \delta_X(\varepsilon)/\tau_t \right\}, \sup_{\varepsilon_t < \varepsilon \leq 2} \left\{ \frac{\varepsilon}{2} - \delta_X(\varepsilon)/\tau_t \right\} \right] \]

67
\[ \leq \max \left[ \sup_{0 \leq \varepsilon \leq \varepsilon_t} \frac{\varepsilon}{2}, \sup_{\varepsilon_t < \varepsilon \leq 2} \left( 1 - \delta_X(\varepsilon) / \tau_t \right) \right] \]

\[ \leq \max \left[ \frac{\varepsilon_t}{2}, \sup_{\varepsilon_t < \varepsilon \leq 2} \left( 1 - \delta_X(\varepsilon) / \tau_t \right) \right] \]

\[ \leq \max \left( \frac{\varepsilon_t}{2}, 0 \right) = \frac{\varepsilon_t}{2} \leq t. \]

The arbitrariness of \( t > 0 \) gives that \( \lim_{\tau \to 0^+} \frac{P_{X^*}(\tau t)}{\tau t} = 0. \)

An analogous argument with the roles of \( X \) and \( X^* \) exchanged proves that \( X \) is uniformly smooth when \( X^* \) is uniformly convex. \( \diamond \)

It follows from the preceding result and the Milman-Pettis theorem that if \( X \) is uniformly smooth Banach space, then \( X^* \) is uniformly convex and therefore reflexive, which in turn implies the reflexivity of \( X \).

**Corollary 3.12.** (Šmulian [45], 1940) *Every uniformly smooth Banach space is reflexive.*

### 3.3 Modulus of convexity in \( L^p \) spaces

O. Hanner, in 1955 (see [16]), gave the following exact calculation of the moduli of convexity of \( L^p \)-spaces, which is better than Clarkson’s when \( 1 < p \leq 2. \)

**Theorem 3.12.** Let \( x, y \in L^p \). Suppose that

\[ \| x \| = \| y \| = 1, \| x - y \| \geq \varepsilon, \]

where \( 0 < \varepsilon < 2. \) Then

\[ (3.5) \quad \left\| \frac{x+y}{2} \right\| \leq 1 - \delta(\varepsilon), \]

where \( \delta = \delta(\varepsilon) = \delta_{L^p}(\varepsilon) \) is determined in the following way:

(a) \( \left( 1 - \delta + \frac{\varepsilon}{2} \right)^p + \left| 1 - \delta - \frac{\varepsilon}{2} \right|^p = 2, \text{ when } 1 < p \leq 2 \)

(b) \( \delta = 1 - \left( 1 - \left( \frac{\varepsilon}{2} \right)^p \right)^{\frac{1}{p}}, \text{ when } 2 \leq p < \infty. \)

If, additionally, measure \( \mu \) is either nonatomic or counting, then for each \( \varepsilon \) we can choose \( x \) and \( y \) such that equality holds in (3.5).

To proof this theorem we need some lemmas.
Lemma 3.13. If $1 \leq p \leq 2$, then

$$
(3.6) \ (|z_1| + |z_2|)^p + |z_1 - z_2|^p \leq |z_1 + z_2|^p + |z_1 - z_2|^p \leq 2(|z_1|^p + |z_2|^p)
$$

for all $z_1, z_2 \in \mathbb{C}$. For $2 \leq p < \infty$ these inequalities hold in the reverse directions.

Proof. The case of $p = 1$ follows from property (vi) of the norm in Section 1.2 and the triangle inequality. In the case $p = 2$ we have equality by the parallelogram identity. Let now $1 < p < 2$. The inequality (3.6) holds for $z_1 = 0$. For $z_1 \neq 0$ the inequality is equivalent to

$$(1 + |\xi|)^p + |1 - |\xi||^p \leq |1 + \xi|^p + |1 - \xi|^p \leq 2(1 + |\xi|^p).$$

Let $\xi = \rho e^{i\varphi}$ with $\rho > 0$. Then

$$|1 + \xi|^p + |1 - |\xi||^p = (1 + \rho^2 + 2\rho \cos \varphi)^{\frac{p}{2}} + (1 + \rho^2 - 2\rho \cos \varphi)^{\frac{p}{2}} = d(\varphi).$$

If we calculate extrema of $d$ we find that for $\varphi = 0$ and $\varphi = \pi$ the global minimum is $(1 + \rho)^p + |1 - \rho|^p$ and for $\varphi = \frac{\pi}{2}$ and $\varphi = \frac{3\pi}{2}$ the global maximum is $2(1 + \rho^2)^{\frac{p}{2}}$. Since $p < 2$, by Lemma 2.3, it follows that

$$2(1 + \rho^2)^{\frac{p}{2}} \leq 2(1 + \rho^p)$$

and

$$(1 + \rho)^p + |1 - \rho|^p \leq d(\varphi) \leq 2(1 + \rho^p).$$

To end the proof we need to see that for all $\varphi$ we have

$$(1 + |\xi|)^p + |1 - |\xi||^p = (1 + \rho)^p + |1 - \rho|^p$$

and

$$2(1 + |\xi|^p) = 2(1 + \rho^p).$$

On the other hand, if $p > 2$ the function $d$ has the global minimum $2(1 + \rho^2)^{\frac{p}{2}}$ for $\varphi = \frac{\pi}{2}$ and $\varphi = \frac{3\pi}{2}$ and the global maximum $(1 + \rho)^p + |1 - \rho|^p$ for $\varphi = 0$ and $\varphi = \pi$. ◦
Lemma 3.14. A function $\omega$ defined for $u, v \geq 0$ by
\[
\omega (u, v) = \left( u_p^{\frac{1}{p}} + v_p^{\frac{1}{p}} \right)^p + \left| u_p^{\frac{1}{p}} - v_p^{\frac{1}{p}} \right|^p
\]
is convex for $1 \leq p \leq 2$ and concave for $p \geq 2$.

Proof. For $p = 1$ function $\omega$ is obviously convex and for $p = 2$ we have $\omega (u, v) = 2 (u + v)$, which is both convex and concave function. Therefore let $1 < p < 2$. We need to show the convexity of $\omega$ only for $u, v > 0$ since $\omega$ is continuous on $[0, \infty) \times [0, \infty)$. Observe that

(a) $\omega (u, v) = \omega (v, u)$ for all $u, v \geq 0$,
(b) $\omega (0, 0) = 0$,
(c) $\omega (tu, tv) = t \omega (u, v)$ for $t \geq 0$.

The convexity of $\omega (u, v)$ will follow from the convexity of
\[
u \mapsto \sigma (u) = \omega (1, u).
\]

Indeed, if $u_1, v_1, u_2, v_2 > 0$ and $\alpha + \beta = 1$ with $\alpha, \beta \geq 0$, then for $s = \alpha u_1 + \beta u_2$ we have
\[
\alpha \omega (u_1, v_1) + \beta \omega (u_2, v_2) = \alpha u_1 \omega \left( 1, \frac{v_1}{u_1} \right) + \beta u_2 \omega \left( 1, \frac{v_2}{u_2} \right)
= s \left[ \frac{1}{s} \cdot \alpha u_1 \cdot \omega \left( 1, \frac{v_1}{u_1} \right) + \frac{1}{s} \cdot \beta u_2 \cdot \omega \left( 1, \frac{v_2}{u_2} \right) \right]
\geq s \omega \left( 1, \frac{1}{s u_1} \cdot \alpha u_1 v_1 + \frac{1}{s u_2} \cdot \beta u_2 v_2 \right)
= \omega \left( s, \alpha v_1 + \beta v_2 \right).
\]

We will show that the function $\sigma$ is convex. We have
\[
\sigma (u) = \left( 1 + u_p^{\frac{1}{p}} \right)^p + \left| 1 - u_p^{\frac{1}{p}} \right|^p,
\]
and the derivatives are
\[
\sigma' (u) = \left[ \left( 1 + u_p^{\frac{1}{p}} \right)^{p-1} + \epsilon (u) \left| 1 - u_p^{\frac{1}{p}} \right|^{p-1} \right] u_p^{-\frac{1-p}{p}},
\]
where $\epsilon (u) = \begin{cases} -1 & \text{for } 0 < u < 1, \\ 1 & \text{for } u > 1, \end{cases}$ and
\[
\sigma'' (u) = \frac{p-1}{p} u_p^{\frac{1}{p} - 2} \left[ \left| 1 - u_p^{\frac{1}{p}} \right|^{p-2} - \left( 1 + u_p^{\frac{1}{p}} \right)^{p-2} \right] \text{ for } u \neq 1.
\]
The second derivative $\sigma''(u)$ is strictly positive for every $u \neq 1$ and $\sigma''(1) = \infty$. The first derivative $\sigma'(u)$ is finite and continuous at $u = 1$, since
\[
\lim_{u \to 1^-} \sigma'(u) = \lim_{u \to 1^+} \sigma'(u) = 2^{p-1}.
\] Thus $\sigma(u)$ is convex. For $p > 2$ function $\sigma(u)$ is concave.

**Lemma 3.15.** If $1 \leq p \leq 2$, then
\[
(3.7) \quad (\|x\| + \|y\|)^p + \|x\| - \|y\|)^p \leq \|x + y\|^p + \|x - y\|^p \leq 2 (\|x\|^p + \|y\|^p)
\]
for all $x, y \in L^p$. For $p \geq 2$ these inequalities hold in the reverse directions.

**Proof.** Let $1 \leq p \leq 2$. By using Lemma 3.13 with $z_1 = x(t)$ and $z_2 = y(t)$ and integrating over $\Omega$ we obtain
\[
\int_{\Omega} [(|x(t)| + |y(t)|)^p + |x(t)| - |y(t)|]^p d\mu \\
\leq \int_{\Omega} (|x(t) + y(t)|^p + |x(t) - y(t)|^p) d\mu \\
= \|x + y\|^p + \|x - y\|^p.
\]
Let $\tilde{x}(t) = |x(t)|$ and $\tilde{y}(t) = |y(t)|$. To show the left-hand side inequality of (3.7) it is enough to show that
\[
(3.8) \quad \int_{\Omega} (|\tilde{x}(t) + \tilde{y}(t)|^p + |\tilde{x}(t) - \tilde{y}(t)|^p) d\mu \\
\geq \left[ \left( \int_{\Omega} \tilde{x}(t)^p d\mu \right)^{\frac{1}{p}} + \left( \int_{\Omega} \tilde{y}(t)^p d\mu \right)^{\frac{1}{p}} \right]^p \\
+ \left( \int_{\Omega} \tilde{x}(t)^p d\mu \right)^{\frac{1}{p}} - \left( \int_{\Omega} \tilde{y}(t)^p d\mu \right)^{\frac{1}{p}}
\]
since the last expression is equal to
\[
(\|\tilde{x}\| + \|\tilde{y}\|)^p + \|\tilde{x}\| - \|\tilde{y}\|)^p \\
= (\|x\| + \|y\|)^p + \|x\| - \|y\|)^p.
\]
Observe that (3.8) is a first inequality of (3.7) for nonnegative functions.
We show that the first inequality in (3.7) is true for nonnegative simple functions in $L^p$. First observe that the function $\omega$, as a convex function, satisfies inequality
\begin{equation}
\omega(u_1 + \ldots + u_n, v_1 + \ldots + v_n) \leq \omega(u_1, v_1) + \ldots + \omega(u_n, v_n)
\end{equation}

for all $u_i, v_i \geq 0$, $i = 1, 2, \ldots, n$.

We show (3.9) for $n = 2$ and for all $n \in \mathbb{N}$ it follows by induction.

For $u_1, u_2 > 0$ we have

$$
\omega(u_1 + u_2, v_1 + v_2) = (u_1 + u_2) \omega\left(1, \frac{v_1 + v_2}{u_1 + u_2}\right)
\leq (u_1 + u_2) \omega\left(1, \frac{u_1}{u_1 + u_2}\right) + \omega\left(1, \frac{u_2}{u_1 + u_2}\right)
= u_1 \omega\left(1, \frac{u_1}{u_2}\right) + u_2 \omega\left(1, \frac{v_2}{u_2}\right)
$$

In the case when either $u_1 = 0$ or $u_2 = 0$ (3.9) follows by the above estimate and the continuity of $\omega$.

Let now $x$ and $y$ be any nonnegative simple functions in $L^p$, i.e.,

$$
x(t) = \sum_{k=1}^{n} a_k \chi_{A_k}(t), \quad y(t) = \sum_{k=1}^{n} b_k \chi_{A_k}(t),
$$

where $a_k, b_k \geq 0$ and $\{A_k\}$ are disjoint, measurable sets of finite measure $\mu$, $k = 1, 2, \ldots, n$. Then, according to (3.9), we obtain

$$
(||x|| + ||y||)^p + ||x|| - ||y|| |^p =
\left[ \left( \sum_{k=1}^{n} \int_{A_k} a_k^p \mu d\mu \right)^{\frac{1}{p}} + \left( \sum_{k=1}^{n} \int_{A_k} b_k^p \mu d\mu \right)^{\frac{1}{p}} \right]^p
\geq \left[ \left( \sum_{k=1}^{n} \int_{A_k} a_k^p \mu A_k \right)^{\frac{1}{p}} + \left( \sum_{k=1}^{n} \int_{A_k} b_k^p \mu A_k \right)^{\frac{1}{p}} \right]^p
= \left[ \left( \sum_{k=1}^{n} a_k^p \mu A_k \right)^{\frac{1}{p}} + \left( \sum_{k=1}^{n} b_k^p \mu A_k \right)^{\frac{1}{p}} \right]^p
$$
Finally, if \( x \) and \( y \) are nonnegative functions in \( L^p \) then there are monotone sequences \( \{x_n\}, \{y_n\} \) of nonnegative simple functions in \( L^p \) convergent pointwise to \( x \) and \( y \), respectively. Then

\[
(\|x_n\| + \|y_n\|)^p + \|x_n\| - \|y_n\| \leq \|x_n + y_n\|^p + \|x_n - y_n\|^p
\]

and by the Levi theorem and the continuity of the \( L^p \)-norm we obtain

\[
(\|x\| + \|y\|)^p + \|x\| - \|y\| \leq \|x + y\|^p + \|x - y\|^p.
\]

Using similar arguments we can prove the right-hand side part of inequality (3.7) and also two inequalities left for \( p \geq 2 \).

Now we will show the proof of the Theorem 3.12.

**Proof.** (a) Let \( 1 < p \leq 2 \) and \( \|x\| = \|y\| = 1 \). Consider once again a function

\[
\omega(u, v) = (u + v)^p + |u - v|^p \text{ for } u, v \geq 0.
\]

Put

\[
x' = \frac{x + y}{2}, \quad y' = \frac{x - y}{2},
\]

then

\[
x = x' + y', \quad y = x' - y'.
\]

By the Lemma 3.17 we have
\[ \|x + y\|^p + \|x - y\|^p \geq \omega(\|x\|, \|y\|). \]

If we apply the above inequality to \(x\)' and \(y\)' we get

\[ \omega(\|x\|', \|y\|') \leq \|x' + y'\|^p + \|x' - y'\|^p = \|x\|^p + \|y\|^p = 2. \]

Since the function \(\omega\) is strictly increasing in both variables (because \(\sigma'(u) > 0\) for all \(u > 0\)) and

\[ \|y\'| = \|\frac{x - y}{2}\| \geq \frac{\epsilon}{2}, \]

it follows that

\[ \omega(\|x\|', \|y\|') \geq \omega(\|x\|', \frac{\epsilon}{2}) \]

and

\[ (3.10) \quad \omega(\|x\|', \frac{\epsilon}{2}) \leq \omega(\|x\|', \|y\|') \leq 2. \]

But the strict monotonicity of \(\omega\) and the fact that \(\omega(1, 0) = 2\) we obtain

\[ \omega \left(1, \frac{\epsilon}{2}\right) > \omega(1, 0) = 2 \]

and

\[ \omega \left(0, \frac{\epsilon}{2}\right) = \omega \left(\frac{\epsilon}{2}, 0\right) < \omega(1, 0) = 2. \]

Thus there exists a uniquely determined solution \(\delta \in (0, 1)\) of the equality

\[ (3.11) \quad \omega \left(1 - \delta, \frac{\epsilon}{2}\right) = 2, \]

which means that

\[ (1 - \delta + \frac{\epsilon}{2})^p + \left|1 - \delta - \frac{\epsilon}{2}\right|^p = 2. \]

Because of (3.10) we get that

\[ (3.12) \quad \|x'\| = \left\|\frac{x + y}{2}\right\| \leq 1 - \delta. \]

and (3.5) is proved.

To get equality in (3.12) we take two disjoint measurable sets \(A_1, A_2\) of measure \(\mu A_1 = \mu A_2 = 1\), \(A = A_1 \cup A_2\) and we put
\[ \overline{x}(t) = 2^{-\frac{1}{p}} (1 - \delta) \chi_A(t), \]
\[ \overline{y}(t) = 2^{-\frac{1}{p}} \left[ \frac{\delta}{2} \chi_{A_1}(t) - \frac{\delta}{2} \chi_{A_2}(t) \right], \]

where \( \delta \) is from the equality (3.11).

Then \( \|\overline{x}\| = 1 - \delta \) and \( \|\overline{y}\| = \frac{\delta}{2} \) and, as before, for \( x = \overline{x} + \overline{y}, \ y = \overline{x} - \overline{y} \) we calculate that

\[
\|x\|^p = \int A^A |1 - \delta + \frac{\epsilon}{2p} d\mu + \frac{1}{2} \int A^B |1 - \delta - \frac{\epsilon}{2p} d\mu
\]
\[
= \frac{1}{2} |1 - \delta + \frac{\epsilon}{2} \mu A_1 + |1 - \delta - \frac{\epsilon}{2} \mu A_2
\]
\[
= \frac{1}{2} |1 - \delta + \frac{\epsilon}{2} |^p + |1 - \delta - \frac{\epsilon}{2} |^p = \frac{1}{2} \omega (1 - \delta, \frac{\epsilon}{2}) = 1.
\]

Also \( \|y\|^p = 1 \) and so we have

\[
\|x\| = \|y\| = 1, \ |x - y| = 2 \|\overline{y}\| = \epsilon
\]
and

\[
\|\overline{x} + y\| = \|\overline{x}\| = 1 - \delta.
\]

(b) This part has been already proved (see Proposition 3.6).

3.4 Examples

In general it is difficult to describe both modulus of convexity and smoothness of a Banach space in explicit forms. In this section we present some computations of the moduli in specific spaces.

Example 3.16. There exist Banach spaces which are strictly convex but not uniformly convex (cf. [13], Ex.5.1 and also our Ex.1.14). Let \( X = (C[0,1], \|\cdot\|_\mu) \), where \( C[0,1] \) is a space of continuous real-valued functions on \([0,1]\) and the norm \( \|\cdot\|_\mu \) is defined by the formula

\[
\|x\|_\mu = \|x\|_0 + \mu \|x\|_2
\]
for fixed \( \mu > 0 \). Here \( \|\cdot\|_0 \) denotes the usual supremum norm on \( C[0,1] \), i.e.,

\[
\|x\|_0 = \max \{|x(t)| : t \in [0,1]\}
\]

75
and $\|\cdot\|_2$ is the usual norm in $L^2[0,1]$. We claim that the space $X$ is strictly convex for any $\mu > 0$, that is for $x, y \in X, x \neq y$ and such that $\|x\|_\mu = \|y\|_\mu = 1$ we have that $\|\frac{x+y}{2}\|_\mu < 1$. We will show at first that for all $c > 0, x \neq cy$. Indeed, if there exists $c > 0$ such that $x = cy$, then $\|x\|_\mu = \|cy\|_\mu = \|y\|_\mu = 1$ and so $c = 1$ which implies $x = y$. This is a contradiction with assumption that $x \neq y$. Now since $L^2[0,1]$ is strictly convex, in view of Theorem 1.8 (f)

$$\frac{\|x+y\|}{2} < \frac{\|x\|_2 + \|y\|_2}{2}.$$ 

Hence

$$\frac{\|x+y\|}{\mu} = \frac{\|x+y\|_0}{2} + \mu \left( \frac{\|x\|_2 + \|y\|_2}{2} \right)$$

$$= \frac{\|x\|_0 + \mu \|x\|_2}{2} + \frac{\|y\|_0 + \mu \|y\|_2}{2}$$

$$= \frac{\|x\|_\mu + \mu \|x\|_\mu}{2}$$

and it means that $X$ is strictly convex space.

On the other hand, for every $0 < \varepsilon < 2$ one can find sequences $\{x_n\}, \{y_n\}$ such that $\|x_n\|_\mu = \|y_n\|_\mu = 1, \|x_n - y_n\|_\mu \geq \varepsilon$ and $\|\frac{x_n+y_n}{2}\|_\mu \to 1$. At first let's choose sequences $a_n \in (0, \frac{1}{2})$ and $b_n \in (0, 1)$ such that $b_n \to 1$ as $n \to \infty$ and $\|u_n\|_2 \leq \frac{1-b_n}{2}$, where

$$u_n(t) = \begin{cases} b_n \left(1 - \frac{t}{a_n}\right) & \text{for } 0 \leq t \leq a_n, \\ 0 & \text{for } a_n \leq t \leq 1. \end{cases}$$

Now let $0 < \varepsilon < 2$ be given. We can find a sequence $(c_n)$ such that $\frac{1}{2} < c_n < 1$ and $\|v_n\|_2 \leq \frac{1-b_n}{2}$, where

$$v_n(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq c_n, \\ \frac{\varepsilon}{2(1-c_n)} - \frac{\varepsilon c_n}{2(1-c_n)} & \text{for } c_n \leq t \leq 1. \end{cases}$$

Finally let

$$x_n(t) = u_n(t) + v_n(t)$$

$$y_n(t) = u_n(t) - v_n(t)$$

for $n \in \mathbb{N}$ and $t \in [0,1]$. Obviously $x_n$ and $y_n$ are continuous functions. For sufficiently large $n \in \mathbb{N},$

$$\|u_n + v_n\|_0 = \|u_n - v_n\|_0 = \max_{t \in [0,1]} |u_n(t) + v_n(t)|$$
\[
= \max_{t \in [0,1]} |u_n(t)| = b_n,
\]
since \( b_n \to 1 \). Moreover for all \( n \in \mathbb{N} \)
\[
\|u_n + v_n\|_2 = \|u_n - v_n\|_2 \leq \|u_n\|_2 + \|v_n\|_2 \leq 1 - b_n.
\]
Therefore
\[
\|x_n\|_{\mu} = \|y_n\|_{\mu} = \|u_n + v_n\|_0 + \|u_n + v_n\|_2 \leq 1,
\]
for every \( n \in \mathbb{N} \). We have also
\[
1 \geq \left\| \frac{x_n + v_n}{2} \right\|_{\mu} = \|u_n\|_{\mu} \geq \|v_n\|_0 = b_n \to 1,
\]
as \( n \to \infty \) and for each \( n \in \mathbb{N} \)
\[
\|x_n - y_n\|_{\mu} = \|2v_n\|_{\mu} \geq 2 \|v_n\|_0 = 2 \cdot \frac{\varepsilon}{2} = \varepsilon.
\]
Hence \( \delta_X(\varepsilon) = 0 \) for all \( \varepsilon \in [0,2) \). However \( \delta_X(2) = 1 \) in view of strict convexity of \( X \) and thus \( \delta_X \) is discontinuous at 2.

In all remaining examples we will consider two-dimensional spaces equipped with different norms.

**Example 3.17.** (a) Let \( X = (\mathbb{R}^2, \|\cdot\|) \), where
\[
\|(x_1, x_2)\| = \max\left(\frac{2}{\sqrt{3}}|x_2|, |x_1| + \frac{1}{\sqrt{3}}|x_2|\right)
\]
or equivalently
\[
\|(x_1, x_2)\| = \max_{i=0,\ldots,5} \{|x_1 \cos \alpha_i + x_2 \sin \alpha_i|\} \text{ with } \alpha_i = (\frac{1}{2} + i) \frac{\pi}{3}.
\]
The unit sphere is a following hexagon

![Hexagon Diagram](image-url)
Modulus of convexity is a function

\[ \delta_X (\varepsilon) = \max \left( 0, \frac{\varepsilon - 1}{2} \right) \]

and modulus of smoothness is a function

\[ \rho_X (\tau) = \max \left( \frac{1}{2} \tau, \tau - \frac{1}{2} \right) \]

with the following graphs:

(b) Let \( X = (\mathbb{R}^2, \| \cdot \|) \), where

\[ \| (x_1, x_2) \| = \max \left( \frac{\sqrt{2}}{2} (|x_1| + |x_2|), |x_1|, |x_2| \right) \]

or equivalently

\[ \| (x_1, x_2) \| = \max_{i=0, \ldots, 7} \{|x_1 \cos \alpha_i + x_2 \sin \alpha_i|\} \]

with \( \alpha_i = (1 + i) \frac{\pi}{4} \).

Then the unit sphere is an octagon.
The moduli are the following functions

\[ \delta_X(\varepsilon) = \begin{cases} 
0 & \text{for } 0 \leq \varepsilon \leq 2(\sqrt{2} - 1) \\
1 - \sqrt{2} + \frac{\varepsilon}{2} & \text{for } 2(\sqrt{2} - 1) < \varepsilon \leq 2,
\end{cases} \]

\[ \rho_X(\tau) = \begin{cases} 
(\sqrt{2} - 1) \tau & \text{for } 0 \leq \tau \leq 1 \\
\tau + \sqrt{2} - 2 & \text{for } \tau > 1,
\end{cases} \]

with the following graphs:
Example 3.18. Let $X = (\mathbb{R}^2, \|\cdot\|)$ where

$$\|(x_1, x_2)\| = \max \left\{ |x_1|, \frac{2}{3} |x_2|, \frac{2}{5} (2 |x_1| + |x_2|) \right\}.$$ 

The unit sphere of this space is the octagon with the vertices $(\pm 1, \pm 1)$ and $(\pm \frac{1}{2}, \pm \frac{3}{2})$.

Moreover the function $\delta_X$

$$\delta_X(\varepsilon) = \begin{cases} 
0 & \text{for } 0 \leq \varepsilon \leq 1, \\
\frac{2}{3} \varepsilon - \frac{2}{3} & \text{for } 1 \leq \varepsilon \leq \frac{14}{11}, \\
\frac{3}{5} \varepsilon - \frac{1}{5} & \text{for } \frac{14}{11} \leq \varepsilon \leq \frac{4}{3}, \\
\frac{5}{12} \varepsilon - \frac{1}{3} & \text{for } \frac{4}{3} \leq \varepsilon \leq \frac{16}{11}, \\
\frac{5}{6} \varepsilon - 1 & \text{for } \frac{16}{11} \leq \varepsilon \leq \frac{8}{5}, \\
\frac{3}{8} \varepsilon - \frac{1}{4} & \text{for } \frac{8}{5} \leq \varepsilon \leq \frac{18}{11}, \\
\frac{7}{25} & \text{for } \frac{18}{11} \leq \varepsilon \leq 2 
\end{cases}$$

is not convex. Indeed, if we take $\varepsilon_1 = \frac{4}{3}$, $\varepsilon_2 = \frac{8}{5}$ then

$$\frac{7}{25} = \delta_X((\varepsilon_1 + \varepsilon_2)/2) > (\delta_X(\varepsilon_1) + \delta_X(\varepsilon_2))/2 = \frac{4}{15}.$$
It is also clear from the graph below that the modulus is not convex.

The graph of the modulus of smoothness, done by computer, is the following:
Example 3.19. Let $X = (\mathbb{R}^2, \| \cdot \|)$, where

$$
\|(x_1, x_2)\| = \max \{|x_1|, |x_2|, |x_1 - x_2|\}.
$$

We will call this space $l_\infty - l_1$. Then the unit sphere is

![Graph of the unit sphere in $\mathbb{R}^2$.](image)

Moduls of convexity in this space is a function

$$
\delta_X(\varepsilon) = \max (0, \frac{\varepsilon - 1}{2})
$$

and modulus of smoothness is following

$$
\rho_X(\tau) = \max \left( \frac{\tau}{2}, \tau - \frac{1}{2} \right)
$$

with the graphs which were done in Example 3.17 (a).

The dual space to $l_\infty - l_1$ is the space $l_1 - l_\infty$ with the norm

$$
\|(u_1, u_2)\|_* = \max (|u_1|, |u_2|, |u_1 + u_2|),
$$

where $u = (u_1, u_2) \in \mathbb{R}^2$. 

82
The unit sphere is

Applying duality formula for $\rho_X(\tau)$ (Theorem 3.9) we easily obtain

1°. if $0 \leq \varepsilon \leq 1$, then $\delta_X(\varepsilon) = 0$ and

$$\rho_{X^*}(\tau) = \sup \left\{ \frac{\tau \varepsilon}{2} \right\} = \frac{\tau}{2}$$

2°. if $1 < \varepsilon \leq 2$, then $\delta_X(\varepsilon) = \frac{\varepsilon - 1}{2}$ and

$$\rho_{X^*}(\tau) = \sup \left\{ \frac{\tau \varepsilon}{2} - \frac{\varepsilon - 1}{2} \right\} = \max \left( \frac{\tau}{2}, \tau - \frac{1}{2} \right) = \rho_X(\tau).$$

Example 3.20. Let $X = (\mathbb{R}^2, \| \cdot \|)$ with the norm

$$\|(x_1, x_2)\| = \begin{cases} (x_1^2 + x_2^2)^{\frac{1}{2}} & \text{for } x \in Q_1 \cup Q_3, \\ |x_1| + |x_2| & \text{for } x \in Q_2 \cup Q_4 \end{cases}$$

$$= \begin{cases} \|x\|_2 & \text{if } x_1 \cdot x_2 \geq 0, \\ \|x\|_1 & \text{if } x_1 \cdot x_2 \leq 0, \end{cases}$$

where $Q_i, i = 1, 2, 3, 4$ are quadrants in $\mathbb{R}^2$. In this situation we can talk about the space $l_2 - l_1$ and the unit sphere is
We note that the coefficient of convexity of this space is $\varepsilon_0 = \sqrt{2}$ and

$$\|x\|_2 \leq \|x\| \leq \sqrt{2} \|x\|_2.$$ 

In order to find the modulus of convexity for this space we need to take only $\varepsilon > \sqrt{2}$. Let $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$. We consider two cases:

Case 1. If $y - x \in Q_1 \cup Q_3$, then $\|y - x\| = \|y - x\|_2$.

Because $\|x\|_2 \leq \|x\| = 1$ we have

$$\|x\|_2 \leq 1, \|y\|_2 \leq 1, \|x - y\|_2 \geq \varepsilon.$$ 

By using Hanner inequality for $p = 2$ we get

$$\frac{1}{2} \|x + y\|_2 \leq \left[1 - \left(\frac{\varepsilon}{2}\right)^2\right] \frac{1}{2}.$$ 

It means that

$$\frac{1}{2} \|x + y\| \leq \frac{1}{2} \cdot \sqrt{2} \cdot \|x + y\|_2 = \sqrt{2} \cdot \left(\frac{1}{2} \|x + y\|_2\right)$$

$$\leq \sqrt{2} \cdot \left[1 - \left(\frac{\varepsilon}{2}\right)^2\right] \frac{1}{2} = (2 - \frac{\varepsilon^2}{2})^{1/2}$$

and

$$(*) \quad 1 - \frac{1}{2} \|x + y\| \geq 1 - (2 - \frac{\varepsilon^2}{2})^{1/2}.$$ 

84
Case 2. Now, if $y - x \in Q_2 \cup Q_4$, then $\frac{1}{2} (x + y) \in Q_1 \cup Q_3$ and $\frac{1}{2} \|x + y\| = \frac{1}{2} \|x + y\|_2$.

Because $\|x - y\| \leq \sqrt{2} \|x - y\|_2$ we get

$$\|x - y\|_2 \geq 2^{-\frac{1}{2}} \|x - y\| \geq \frac{\varepsilon}{\sqrt{2}}.$$  

Thus by Hanner inequality

$$\frac{1}{2} \|x + y\| \leq \left(1 - \frac{\varepsilon^2}{8}\right)^{\frac{1}{2}}$$

and

$$1 - \frac{1}{2} \|x + y\| \geq 1 - \left(1 - \frac{\varepsilon^2}{8}\right)^{\frac{1}{2}}.$$  

It means that for $\varepsilon > \sqrt{2}$ we have

$$\delta_X(\varepsilon) \geq \min \left\{1 - (2 - \frac{\varepsilon^2}{2})^{\frac{1}{2}}, 1 - \left(1 - \frac{\varepsilon^2}{8}\right)^{\frac{1}{2}}\right\}.$$  

We can show that there are such points $x, y \in \mathbb{R}^2$ that we get equality in the last condition. We will consider points $x, y$ such that $x - y$ (or $y - x$) has the direction of one of the lines: $y = x$ or $y = -x$. We have the following possibilities:

1°. $x, y \in Q_1$ and are symmetric with respect to $y = x$. Then $x - y$ (or $y - x$) satisfies our requirement. If we choose them in such way that $\|x\| = \|y\| = 1$ and $\|x - y\| = \varepsilon$, we get equality in (**).

Similarly if $x, y \in Q_3$.

2°. $x, y \in Q_2$ and are symmetric with respect to $y = -x$. But in this situation $\|x - y\| < 2^\frac{1}{2}$ and $\delta_X(\varepsilon) = 0$.

Similarly if $x, y \in Q_4$.

3°. $x \in Q_1, y \in Q_3$ and are symmetric with respect to $y = -x$. Once again if $\|x\| = \|y\| = 1$ and $\|x - y\| = \varepsilon$ equality occurs in (*).

4°. $x \in Q_2, y \in Q_4$ and are symmetric with respect to $y = x$. For $\|x\| = \|y\| = 1$ we get that $\|x - y\| = 2$ and in this situation we can take $x = (0, 1), y = (1, 0)$ which gives us equality in (**).

Thus the modulus of convexity $\delta_X(\varepsilon)$ for this space is the function

$$\delta_X(\varepsilon) = \begin{cases} 
0 & \text{for } 0 \leq \varepsilon \leq \sqrt{2}, \\
\min \left\{1 - (2 - \frac{\varepsilon^2}{2})^{\frac{1}{2}}, 1 - \left(1 - \frac{\varepsilon^2}{8}\right)^{\frac{1}{2}}\right\} & \text{for } \sqrt{2} \leq \varepsilon \leq 2.
\end{cases}$$

85
It is clear that $\delta_X(\varepsilon)$ is not a convex function, e.g. if we take $\varepsilon_1 = \frac{8}{5}$ and $\varepsilon_2 = \frac{9}{5}$ then

$$0.2 \approx \delta_X \left( \left( \varepsilon_1 + \varepsilon_2 \right) / 2 \right) > \frac{(\delta_X(\varepsilon_1) + \delta_X(\varepsilon_2))}{2} \approx 0.19.$$ 

The calculations of modulus of convexity (or modulus of smoothness) are really complicated. We finish this part with another examples of two-dimensional Banach spaces for which the spheres are easy to draw but we could compute modulus of convexity numerically.
Example 3.21. a) For the space $l_1 - l_3$ we have the sphere as follows

and the graph of the modulus of convexity is following
b) If we look on the space $l_2 - l_3$, then the sphere is the following:

and the graph of the modulus of convexity
References


