Some new Results concerning Boundedness and Compactness for Embeddings between Spaces with Multiweighted Derivatives

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by

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To the memory of my father
Abdikalikov TBarsynbay
Abstract

This Doctoral Thesis consists of five chapters, which deal with a new Sobolev type function space called the space with multiweighted derivatives. This space is a generalization of the usual one dimensional Sobolev space. As basis for this space serves some differential operators containing weight functions.

Chapter 1 is an introduction, where, in particular, the importance to study function spaces with weights is discussed and motivated.

In Chapter 2 we prove some new estimates for each function in a Tchebychev system. In order to be able to study compactness of the embeddings from Chapter 3 such estimates are crucial.

In Chapter 3 we rewrite and present some results of L. D. Kudryavtsev, where he investigated one dimensional Sobolev spaces. Moreover, in this chapter we rewrite and discuss some analogous results by B. L. Baïdeldinov for generalized Sobolev spaces. These results are not available in the Western literatures in this way and they are crucial for the proofs of the main results in Chapter 4.

In Chapter 4 we prove some embedding theorems for these new generalized Sobolev spaces. The main results of Kudryavtsev and Baïdeldinov about characterization of the behavior of functions at a singularity take place in weak degeneration of the spaces. However, with the help of our new embedding theorems we can extend these results to the case of strong degeneration.

The main aim of Chapter 5 is to establish boundedness and compactness of the embedding considered in Chapter 4. In Chapter 4 basically only sufficient conditions for boundedness of this embedding were obtained. In Chapter 5 we obtain necessary and sufficient conditions for boundedness and compactness of this embedding and the main results are proved in a different way.
Preface

This Doctoral thesis is written as a monograph and it is mainly devoted to introduce and study a new Sobolev type function space called the space with multiweighted derivatives. The main content is described in the abstract on the previous page.

Moreover, it includes the contributions of the author in the following papers:


*Remark*: Paper [4] is an improved version of the following paper:

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Chapter 1

Introduction

1.1 The role of weighted spaces in general theory

At the end of the 19th and the beginning of the 20th centuries the notion of function space or set of functions with defined properties was introduced and studied. Some of the main properties, equipped with these spaces, were properties like continuity and continuous differentiability of functions.

The idea to study function spaces with the purpose to apply them to different problems concerning differential equations appeared in papers by S.L. Sobolev [61]-[63] in the thirties of the last century. In his investigations Sobolev for the first time in an essential way used integral representation of functions by their derivatives. Similar representations became one of the main tools for investigating embedding theorems between function spaces. The method of integral representation, developed by Sobolev in this way, contains estimates of potential type integrals.

From this time the theory of Sobolev spaces has been developed to be a very powerful instrument for solving boundary value problems of differential equations, some problems in potential theory and for investigating integral equations. In view of various types of questions connected to differential equations a great number of new function spaces have appeared and this, in its turn, has implied the appearance of new directions in Analysis such as the theory of generalized functions, relations between these spaces (e.g., embedding theorems), the problem of traces and extension theory.

The weighted spaces appeared naturally during the study of boundary value problems for equations involving partial derivatives with variable coef-
ficients. For example, we can consider the equation
\[
\sum_{|\nu| \leq l} (-1)^{|\nu|} D^\nu(a_\nu(x) D^\nu u(x)) = 0, \quad x \in G, \tag{1.1.1}
\]
as the Euler equation for extremizing the functional
\[
\int_G \sum_{|\nu| = l} a_\nu(x) |D^\nu u(x)|^2 dx, \quad \nu \in \mathbb{N}_0^n, \tag{1.1.2}
\]
where \(G\) is a bounded domain in the \(n\) - dimensional Euclidean space \(\mathbb{R}^n\), \(\mathbb{N}_0^n\) is a \(n\) - dimensional set of nonnegative integers and \(|\nu|\) is the length of the vector \(\nu\).

It is necessary to look for a solution of the equation (1.1.1) in the class of functions, for which the functional (1.1.2) ("the energy integral") takes finite values. The set of functions, in which the functional (1.1.2) is finite, coincides with some weighted function space. This space consists of functions, for which their partial derivatives are quadratic summable on the domain \(G\) only after multiplying them with the multipliers \(a_\nu(x)\), the so-called weights.

The study of weighted spaces started with spaces with weights of power type. Let \(G\) be an open set in the \(m\) - dimensional Euclidean space \(\mathbb{R}^m\), \(r = r(x)\) be the distance from a point \(x \in G\) to the boundary \(\partial G\) of the open set \(G\), \(p \geq 1\) and \(\alpha \in \mathbb{R}\). For the function \(u : G \to \mathbb{R}\) we put
\[
||u||_p := ||u||_{L^p(G)}
\]
and
\[
||u||_{p,\alpha} := |||\rho^\alpha u|||_p.
\]
We define \(w^{(r)}_{p,\alpha} = w^{(r)}_{p,\alpha}(G)\) to be a seminormed space consisting of functions \(u : G \to \mathbb{R}\), which have weak partial derivatives \(f^k\), \(k \in \mathbb{N}_0^n\), of order \(r\) on the set \(G\), and
\[
||u||_{w^{(r)}_{p,\alpha}} := \sum_{|k|=r} |||f^k|||_{p,\alpha} < \infty.
\]
If the power \(\alpha\) is small enough, then the spaces \(w^{(r)}_{p,\alpha}\) keep a lot of the properties of the unweighted spaces \(w^{(r)}_{p,0} = w^{(r)}_{p,0}\) and the methods of proofs concerning various properties can be similar. In this case the spaces \(w^{(r)}_{p,\alpha}\) are called the spaces with weak degeneration. For example, if \(\alpha < r - \frac{n-m}{p}\), then in any parts of the boundary \(\partial G\) it yields that the function \(u \in w^{(r)}_{p,\alpha}\) has a stable trace in terms of convergence in average, where the boundary \(\partial G\) is smooth enough. If the exponent \(\alpha\) is big enough, then the function \(u \in w^{(r)}_{p,\alpha}\) can not
have the traces in any parts of the boundary of the domain $G$. This is very important because when solving boundary value problems with the help of such embedding theorems for the spaces $W_{p,a}$ we can decide which parts of the boundary will be free from setting a priori boundary conditions.

A systematic study of weighted spaces was started at the end of the fifties of the 20th century in the papers by L. D. Kudryavtsev (see e.g. [35]-[37]) and was continued in the papers by R. A. Adams [9], T. I. Amanov [10], O. B. Besov [17] - [18], V. I. Burenkov [19] - [23], W. I. Gilderman [24], A. Kufner [42] - [44], P. I. Lizorkin [48] - [50], V. G. Maz’ya [51], S. M. Nikol’skiĭ [54], [41], J. Nečas [53], M. O. Otelbaev [52], [57], R. Oinarov [55], J. Peetre [58], E. Poulsen [59], V. N. Sedov [60], V. D. Stepanov [64] - [65], H. Triebel [67] - [69], G. N. Yakovlev [70] and many others.

1.2 The theory of spaces with multiweighted derivatives

In general the study of multi-dimensional weighted Sobolev spaces with power weights may be based on weighted spaces of functions with only one variable. Correspondingly, in the papers [35]-[40] L. D. Kudryavtsev presented a fairly complete theory of one-dimensional Sobolev spaces with power weights. He considered the space $L_{p,\gamma}^n = L_{p,\gamma}(I)$ of functions $f : I \rightarrow R$, $I = (0, +\infty)$, which on the interval $I$ have a weak derivative of $n$th order with the finite semi-norm

$$\|f\|_{L_{p,\gamma}^n} := \|x^{\gamma} f^{(n)}\|_p,$$

where $\gamma \in R$, $1 \leq p \leq \infty$, and $n$ is a natural number. For the finite interval $I = (0, 1)$ it was shown that: if $\gamma < 1 - \frac{1}{p}$, then for every $x \in [0, 1]$ the limit values $f^{(j)}(x)$, $j = 0, 1, \ldots, n - 1$, exist, and if $\gamma > n - \frac{1}{p}$, then the function $f$, in general, has not finite limit values when $x = 0$. On the infinite interval $(1, +\infty)$ it yields that: if $\gamma < 1 - \frac{1}{p}$, then for functions from this class we have that neither the function $f$ nor its derivatives $f^{(k)}$, $k = 1, 2, \ldots, n - 1$, in general, have limit values when $x \rightarrow +\infty$. If $\gamma > n - \frac{1}{p}$, then $f^{(n-1)}(\infty)$ = \lim_{x \rightarrow +\infty} f^{(n-1)}(x)$ exists, but \lim_{x \rightarrow +\infty} f^{(i)}(x) = \infty when $i = 0, 1, \ldots, n - 2$, i.e., the limit values of less order derivatives are, in general, infinite. Therefore, at infinity we have such a singularity that can not be handled even with a weight. For this case Kudryavtsev [38] - [40] proved the existence of a unique
polynomial \( P_{n-1} = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} \) such that
\[
\lim_{x \to +\infty} [f(x) - P_{n-1}(x)]^{(k)} = 0, \; k = 0, 1, \ldots, n - 1.
\] (1.2.1)

This theory gives us the possibility to say that all functions \( f \in L^2_{p, \gamma} \) at a singular point in weak degeneration approaches to some unique polynomial \( P_{n-1} \) in the sense of limits (1.2.1).

Assume that on the interval \( I \) the differential expression
\[
(ly)(t) = \sum_{i=0}^{n} a_i(t)y^{(i)}(t)
\]
is given with continuous coefficients \( a_i(\cdot), \; i = 0, 1, \ldots, n \). The endpoints of the interval \( I \) are called singular points (endpoints) of the differential expression \( l \), if they are infinite. In the case when some endpoint is finite we call it a singular point if only one function from \( a_i^{-1}(t) = \frac{1}{a_i(t)}, \; a_i(t), \; i = 0, 1, \ldots, n - 1 \), is not summable in a neighbourhood of this endpoint.

If the endpoint of the interval \( I \) is not singular for \( l \), then it is called a regular endpoint for \( l \). In the case when the differential expression is regular on \( I \), then for the differential equation \( ly = f \) it is possible to put different correct boundary values and nowadays this theory is developed in a satisfactory way.

However, if one or both of the endpoints are singular, then the solutions of the homogeneous equation \( ly(x) = 0 \) or their derivatives of order up to \( n - 1 \) can not have boundary values at the singular endpoints of the interval \( I \) and the formulation of ordinary boundary conditions becomes problematical. In this case the question to find an analogue of the boundary conditions is a very important problem in the theory of the corresponding singular equations.

In view of the foregoing theory Kudryavtsev proposed to put analogies of the boundary conditions at infinity, as those to which the solutions of the equation \( ly = 0 \) go to, when \( t \to \infty \), i.e., the coefficients of a polynomial \( P_{n-1} \) of order \( n - 1 \) (see (1.2.1)). Moreover, for the equation \( ly = f \) Kudryavtsev put the Cauchy condition at infinity and proved existence and uniqueness of the solutions. He also proved such results with other boundary conditions.

In [16] B. L. Baidel’dinov and R. Oinarov pointed out the possibility to characterize the behavior of a function not at infinity but at the finite singular point zero by making the variable transformation \( x = \frac{1}{t} \). This transformation of variable implies that the operation of derivation is transferred to the following differential operations:
\[
\frac{df}{dx} = \frac{df}{dt} \frac{dt}{dx} = -x^{-2} \frac{df}{dt} = (-1)^{1/2} \frac{df}{dt},
\]
\[
\frac{d^2 f}{dx^2} = \frac{d}{dx} \left( -t^2 \frac{df}{dt} \right) = -t^2 \frac{d}{dx} \left( \frac{df}{dt} \right) = -t^2 \frac{d}{dt} \left( \frac{df}{dx} \right) = (-1)^2 t^2 \frac{d}{dt} \frac{df}{dt},
\]
and in general
\[
\frac{d^n f}{dx^n} = (-1)^n t^2 \frac{d}{dt} t^2 \frac{d}{dt} \cdots t^2 \frac{df}{dt}.
\]
If in the derived expressions we put any real numbers instead of "2" and multiply \( f \) by some power function before the first operation of derivation, then, by using the corresponding notions, we have that
\[
D^0_{\bar{a}} f(t) = t^{\alpha_0} f(t),
\]
\[
D^i_{\bar{a}} f(t) = t^{\alpha_i} \frac{d}{dt} t^{\alpha_{i-1}} \frac{d}{dt} \cdots t^{\alpha_1} \frac{d}{dt} t^{\alpha_0} f(t), \quad i = 1, 2, \ldots, n,
\]
where \( \bar{a} = (\alpha_0, \alpha_1, \ldots, \alpha_n) \) and \( \alpha_i \in R, \ i = 0, 1, \ldots, n. \)

We call this differential operation or operator \( D^i_{\bar{a}} \) the \( \bar{a} \) - multaweighted derivative of the function \( f \) of order \( i, \ i = 0, 1, \ldots, n. \)

To solve a boundary value problem of differential equations with singular coefficients I.A. Kiprijanov (see [31] - [32]) used the space \( W^{(i)}_{x_0, 2, \nu}(R^n) \) in the halfspace \( x_n \geq 0 \) of the \( n \) - dimensional Euclidean space \( R^n \) with finite norm:
\[
\|f\|_{W^{(i)}_{x_0, 2, \nu}(R^n)} = \int_{R^n} |f|^2 2^\nu dx + \sum_{k=1}^{l} \int_{R^n} |B^k_{x_n} f|^2 2^\nu dx,
\]
where
\[
B^k_{x_n} = \frac{\partial^2}{\partial x_n^2} + \frac{2\nu}{x_n} \frac{\partial}{\partial x_n}, \quad (\nu > 0).
\]
If \( n = 1 \) Bessel's singular operator \( B_{x_n} \) has the following form:
\[
B^1_{x_n} f(t) = \left[ \frac{d^2}{dt^2} + \frac{2\nu}{t^2} \frac{d}{dt} \right] f(t),
\]
\[
B^k_{x_n} f(t) = B_t[B^{k-1}_{x_n} f(t)], \quad k = 1, 2, \ldots.
\]
At a specially chosen set of numbers \( \alpha_j, \ j = 0, 1, \ldots, 2k, \ k \) is a natural number, the differential operation \( D^{2k}_{\bar{a}} f(t) \) coincides with the operator \( B^k_{x_n} f(t) \).

For example, if \( k = 1 \), then
\[
D^2_{\bar{a}} f(t) = t^{\alpha_2} \frac{d}{dt} t^{\alpha_1} \frac{d}{dt} t^{\alpha_0} f(t) = t^{\alpha_2 + \alpha_1 + \alpha_0} \left[ \frac{d^2}{dt^2} + \frac{\alpha_1 + 2\alpha_0}{t^2} \frac{d}{dt} + \frac{\alpha_0(\alpha_1 + \alpha_0 - 1)}{t^2} \right] f(t).
\]
Therefore, $D_α^2 f(t) = B_1^1 f(t)$, if $α_0 = 0$, $α_1 = 2v$, $α_2 = -2v$ or $α_0 = 2v - 1$, $α_1 = 2 - 2v$, $α_2 = -1$.

In the same way, for the case $k = 2$ we find that

$$D_α^4 f(t) = t^{α_4} \frac{d}{dt} t^{α_3} \frac{d}{dt} t^{α_2} \frac{d}{dt} t^{α_1} \frac{d}{dt} t^{α_0} f(t)$$

$$= t^{α_4} \frac{d}{dt} t^{α_3} \frac{d}{dt} t^{α_2} \frac{d}{dt} t^{α_1} \frac{d}{dt} t^{α_0} f(t) = t^{α_4} \frac{d}{dt} t^{α_3} \frac{d}{dt} t^{α_1} B_1^1 f(t)$$

$$= t^{α_4 + α_3 + α_1} \left[ \frac{d^2}{dt^2} + \frac{α_3 + 2α_1}{t} \frac{d}{dt} + \frac{α_1(α_3 + α_1 + 1)}{t^2} \right] B_1^1 f(t) = B_1^2 f(t),$$

where $α_2 = α_1^2 + α_2^2$, if we choose $α_i$, $i = 1, 2, 3, 4$, as follows:
1) $α_0 = 0$, $α_1 = 2v$, $α_2 = -2v$, $α_3 = 2v$, $α_4 = -2v$; or
2) $α_0 = 0$, $α_1 = 2v$, $α_2 = -1$, $α_3 = 2 - 2v$, $α_4 = -1$; or
3) $α_0 = 2v - 1$, $α_1 = 2 - 2v$, $α_2 = -1$, $α_3 = 2v$, $α_4 = -2v$; or
4) $α_0 = 2v - 1$, $α_1 = 2 - 2v$, $α_2 = 2v$, $α_3 = 2 - 2v$, $α_4 = -1$, etc.

With help of the operator $D_α^i$, $i = 0, 1, \ldots, n$, we define a new space $W_{p,α}^n = W_{p,α}^n(I)$, $1 ≤ p < ∞$, $I = (0, 1)$ or $I = (1, +∞)$, of functions $f : I → R$ with the following norm:

$$\|f\|_{W_{p,α}^n} := \|D_α^n f\|_p + \sum_{i=0}^{n-1} |D_α^i f(1)|. \quad (1.2.2)$$

In the papers [13] - [16] BaieVdinov proved analogous results for the spaces $W_{p,α}^n$ as those obtained by Kudryavtsev for the space $L_{p,γ}^n$. In particular, BaieVdinov proved in the case of weak degeneration the existence of a generalized unique polynomial, to which each function from the space with multiweighted derivatives goes at the singular point $x = 0$.

In Chapter 3 the results of Kudryavtsev and BaieVdinov for the considered Sobolev spaces are presented and fitted to the frame of this Doctoral Thesis. In particular, it is given a description of the behavior of the functions from these spaces at the singular points, and different equivalent norms are presented and discussed. These results have independent interest but are also crucial for the proofs of the main results in Chapter 4.
1.3 Some applications of the theory of weighted spaces and embedding of spaces

In Chapter 2 on the intervals \([0, 1]\) and \([1, +\infty)\), respectively, we consider the following systems of functions:

\[
\begin{align*}
  u_0(t) &= t^{\alpha_0} \\
  u_1(t) &= t^{\alpha_0} \int_{t_1}^t t_1^{\alpha_1} dt_1 \\
  u_2(t) &= t^{\alpha_0} \int_{t_1}^t t_1^{\alpha_1} \int_{t_2}^{t_1} t_2^{\alpha_2} dt_2 dt_1 \\
  \vdots & \\
  u_n(t) &= t^{\alpha_0} \int_{t_1}^t t_1^{\alpha_1} \int_{t_2}^{t_1} \cdots \int_{t_n}^{t_{n-1}} t_n^{\alpha_n} dt_n dt_{n-1} \cdots dt_1
\end{align*}
\] (1.3.1)

and

\[
\begin{align*}
  v_0(t) &= t^{\beta_0} \\
  v_1(t) &= t^{\beta_0} \int_{t_1}^t t_1^{\beta_1} dt_1 \\
  v_2(t) &= t^{\beta_0} \int_{t_1}^t t_1^{\beta_1} \int_{t_2}^{t_1} t_2^{\beta_2} dt_2 dt_1 \\
  \vdots & \\
  v_n(t) &= t^{\beta_0} \int_{t_1}^t t_1^{\beta_1} \int_{t_2}^{t_1} \cdots \int_{t_n}^{t_{n-1}} t_n^{\beta_n} dt_n dt_{n-1} \cdots dt_1.
\end{align*}
\] (1.3.2)

These functional systems \(\{u_i(\cdot)\}_{i=0}^n\) and \(\{v_i(\cdot)\}_{i=0}^n\) form Tchebyshev systems or \(T\) - systems on the intervals where they are defined. Moreover, the second system \(\{v_i(\cdot)\}_{i=0}^n\) is an extended complete Tchebyshev system or ECT system.

\(T\) - systems are very important in different areas of analysis, the theory of differential equations and statistics, e.g., in the theory of approximation (interpolation methods, cubature formulas), in boundary-value problems and problems with oscillation properties of zeros of the solutions of differential equations, and in the theory of statistical inequalities. In the monographs [30] and [34] we can find an almost complete presentation of applications of Tchebyshev systems.

Let us illustrate at least one classical problem with a solution given in terms of a Tchebysheff system.

**Problem definition.** Let \(P_n(\cdot)\) be a generalized polynomial \(P_n(\cdot) = \sum_{i=0}^n c_i u_i(\cdot)\), where \(\{u_i(\cdot)\}_{i=0}^n\) are real functions from a function space \(\Phi[a,b]\),
In the case when $\Phi[a,b]$ is a space of continuous functions and $u(\cdot) \in \Phi[a,b]$, it is required to find a polynomial $P_n(\cdot)$ such that $\|u - P_n\| = \inf_{P_n} \|u - P_n\|$. This polynomial is called the polynomial of the best uniform approximation.

Haar Theorem. For any continuous function $u(\cdot)$ there exists a unique polynomial of the best uniform approximation if and only if the system of functions $\{u_i(\cdot)\}_{i=0}^n$ is a Tchebyshev system on $[a,b]$.

In our turn we study $T$-systems $\{u_i(\cdot)\}_{i=0}^n$ and $\{v_i(\cdot)\}_{i=0}^n$ in connection with problems of correct posing of some boundary-value problems for $(n + 1)$-th order differential equations, which have singularities at zero and infinity. In the case when a solution of the differential equation and its derivatives do not have traces at these singular points, we consider the following generalized conditions: at zero

$$
\lim_{t \to 0} D_i u(t) = D_i u(0), \ i = 0, 1, \ldots, n,
$$

and at infinity

$$
\lim_{t \to +\infty} D_i u(t) = D_i u(\infty), \ i = 0, 1, \ldots, n,
$$

where

$$
D_0 u(t) = \frac{u(t)}{t^\alpha}, \ D_i u(t) = \frac{1}{t^{\alpha+i}} D_{i-1} u(t), \ i = 1, 2, \ldots, n. \quad (1.3.3)
$$

It is obvious that the introduced differential operator (1.3.3) and the systems $\{u_i(\cdot)\}_{i=0}^n$ and $\{v_i(\cdot)\}_{i=0}^n$ have the following close connection: these two systems (here for the second system consider $\alpha$ instead of $\beta$) are the fundamental systems of the solutions of the equation:

$$
D_{n+1} u(t) = 0.
$$

In order to be able to solve the boundary-value problems for equations with singularities at zero and infinity, we are faced with the problem to find conditions under which the functions from (1.3.1) and (1.3.2) belong to the Lebesgue space $L_p(I)$, where $I = (0, 1)$ and $I = (1, +\infty)$, respectively, with the norm:

$$
\|u\|_{L_p} := \left( \int_I |u(t)|^p dt \right)^{\frac{1}{p}}, \ 1 \leq p < \infty.
$$

In Chapter 2 we present a complete solution of this problem. In particular, our result can be useful to solve the approximation problem given above or in many other applications of Tchebyshev systems.
Embedding theorems for weighted spaces, as for ordinary unweighted spaces, can be used to solve some boundary value problems. One of the main reasons to develop the theory of weighted function spaces was to be able to solve such problems in more general situations. By using an embedding theorem it is possible to prove the existence and uniqueness of traces and to describe its properties not only for the function, but also for its derivatives.

The main aim of Chapter 4 is to prove necessary and sufficient conditions for the continuous embedding

$$W_{p,\alpha}^n(I) \hookrightarrow W_{q,\beta}^m(I),$$

(1.3.4)

where $1 \leq p, q < \infty$, $0 \leq m < n$, $I = (0, 1)$ or $I = (1, \infty)$ and $\tilde{\beta} = (\beta_0, \beta_1, \ldots, \beta_m)$, $\beta_i \in \mathbb{R}$, $i = 0, 1, \ldots, m$, when the space $W_{p,\alpha}^n(I)$ is strong degenerated or not strong degenerated. The embedding (1.3.4) has been considered in works by A. A. Kalybay (see [27] - [29]) when $1 < q < p < \infty$.

In this Chapter we prove the same results but with another proof, where we in particular will use Hardy type inequalities in a crucial way. Moreover, for the case $1 < p \leq q < \infty$ we prove necessary and sufficient conditions for the embedding (1.3.4) to hold. The results, given in Chapter 3, about the characterization of the behavior of functions at a singularity point concerns the case of weak degeneration. However, with help of the new embedding theorems, proved in Chapter 4, we can extend these results to the case of strong degeneration.

The close connection between the spaces $W_{p,\alpha}^n(0, 1)$ and $W_{p,\alpha}^n(1, +\infty)$ can be seen by making the transformation $x = \frac{1}{t}$. In Section 4.4 we have rewritten the embedding theorems from Section 4.3 for the spaces $W_{p,\alpha}^n(0, 1)$ to the case with the spaces $W_{p,\alpha}^n(1, +\infty)$.

In Chapter 5 we consider the problem of bounded and compact embedding (1.3.4) when $1 \leq q < p < \infty$ and $1 < p \leq q < \infty$, separately, without dependence on powers of degeneration of the space $W_{p,\alpha}^n(I)$, i.e. for which relations between the parameters $\alpha, \beta, p$ and $q$ bounded and compact embedding (1.3.4) holds. In this Chapter we improve the results of Chapter 4. More exactly, in our main result of Chapter 5 we find equivalent conditions to the boundedness and compactness of the embedding (1.3.4) when $1 \leq q < p < \infty$. We remark that for the proof of the main results we use some powerful results about boundedness and compactness for integral operators by A.O. Balayrastanov [12] and R. Oinarov [35]. Moreover, as a corollary of the main results we obtain some new results concerning the relations between the Kudryavtsev spaces and the spaces with multiweighted derivatives.
1.4 Weighted Hardy type inequalities

In this Doctoral Thesis Hardy type inequalities for functions defined on finite and infinite intervals \((a, b), 0 \leq a < b \leq +\infty\), are used repeatedly. Inequalities of this kind have always been of great importance for the development of many branches of mathematics (e.g. functional analysis, differential and integral equations etc.) and of other sciences (e.g. mechanics, physics, signal processing etc.). From 1925 Hardy type inequalities have been investigated by many authors, and their generalizations and applications have been considered in many publications, for example, in the classical Hardy - Littlewood - Polya book [25] and in the books [47] and [56]. The newest developments of importance for this Doctoral Thesis concerning Hardy type inequalities and more general weighted norm inequalities can be found in the books [33], [45] and [46].

We now present some useful results from this theory, which we will need later on.

**Theorem 1.4.1** Let \(1 \leq p \leq q < \infty\) and let \(v\) and \(w\) be weight functions. Then the Hardy inequality

\[
\left( \int_{a}^{b} w(x) \left| \int_{a}^{x} f(t) dt \right|^q dx \right)^{\frac{1}{q}} \leq H_l \left( \int_{a}^{b} v(t)|f(t)|^p dt \right)^{\frac{1}{p}} \quad (1.4.1)
\]

holds if and only if

\[
B_l = \sup_{a \leq x \leq b} \left( \int_{a}^{b} w(t)dt \right)^{\frac{1}{q}} \left( \int_{a}^{x} v^{-p'}(t)dt \right)^{\frac{1}{p'}} < \infty.
\]

Moreover, the best constant \(H_l\) in (1.4.1) satisfies the estimates \(B_l \leq H_l \leq (1 + \frac{q}{p'})^{\frac{1}{q}}(1 + \frac{p'}{q})^{\frac{1}{p'}} B_l\).

**Theorem 1.4.2** Let \(1 \leq q < p < \infty\), \(\frac{1}{r} = \frac{1}{q} - \frac{1}{p}\) and let \(v\) and \(w\) be weight functions. Then the Hardy inequality (1.4.1) holds if and only if

\[
A_l = \left\{ \int_{a}^{b} \left[ \left( \int_{x}^{b} w(t)dt \right)^{\frac{1}{q}} \left( \int_{a}^{x} v^{1-p'}(t)dt \right)^{\frac{1}{p'}} \right]^{\frac{r}{q}} v^{1-p'}(x) dx \right\}^{\frac{1}{r}} < \infty.
\]
Moreover, the best constant $H_l$ in (1.4.1) satisfies the estimates 
\[ \frac{1}{q^q} \left( \frac{p'q}{r} \right)^{\frac{1}{q'}} A_l \leq H_l \leq \frac{1}{q^q(p')^{\frac{1}{q'}}} A_l. \]

Moreover, the dual versions of Theorems 1.4.1 and 1.4.2 reads, respectively:

**Theorem 1.4.3** Let $1 \leq p \leq q < \infty$ and let $v$ and $w$ be weight functions. Then the Hardy inequality 
\[ \left( \int_a^b w(x) \left( \int_x^b f(t) dt \right)^{q} dx \right)^{\frac{1}{q'}} \leq H_r \left( \int_a^b |f(t)|^p dt \right)^{\frac{1}{p'}} \]
\[ (1.4.2) \]

holds if and only if 
\[ B_r = \sup_{a \leq x \leq b} \left( \int_a^x w(t) dt \right)^{\frac{1}{q'}} \left( \int_x^b v^{-p'}(t) dt \right)^{\frac{1}{p'}} < \infty. \]

Moreover, the best constant $H_r$ in (1.4.2) satisfies the estimates $B_r \leq H_r \leq (1 + \frac{q}{p'})^{\frac{1}{q'}} (1 + \frac{p'}{q})^{\frac{1}{p'}} B_r$.

**Theorem 1.4.4** Let $1 \leq q < p < \infty$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ and let $v$ and $w$ be weight functions. Then the Hardy inequality (1.4.2) holds if and only if 
\[ A_r = \left\{ \int_a^b \left[ \left( \int_a^x w(t) dt \right)^{\frac{1}{q'}} \left( \int_x^b v^{-p'}(t) dt \right)^{\frac{1}{p'}} \right]^{r} v^{1-p'(x)} dx \right\}^{\frac{1}{r}} < \infty. \]

Moreover, the best constant $H_r$ in (1.4.2) satisfies the estimates 
\[ q^q \left( \frac{p'q}{r} \right)^{\frac{1}{q'}} A_r \leq H_r \leq q^q(p')^{\frac{1}{q'}} A_r. \]

Furthermore, we need the following special cases of Theorems 1.4.3 and 1.4.4 (with $w^q(x) = x^\mu$ and $v^{p'}(x) = x^\gamma$), respectively:
Corollary 1.4.5 Let $1 \leq p \leq q < \infty$. Then the inequality
\[
\left( \int_0^1 |t^\mu[f(t) - f(1)]^q dt \right)^{\frac{1}{q}} \leq H \left( \int_0^1 |t^\gamma \frac{df(t)}{dt}|^p dt \right)^{\frac{1}{p}}
\] (1.4.3)
holds if and only if $\mu > -\frac{1}{q}$ and $\gamma \leq 1 - \frac{1}{p} + \frac{1}{q} + \mu$.

Corollary 1.4.6 Let $1 \leq q < p < \infty$. Then the inequality
\[
\left( \int_0^1 |t^\mu[f(t) - f(1)]^q dt \right)^{\frac{1}{q}} \leq H \left( \int_0^1 |t^\gamma \frac{df(t)}{dt}|^p dt \right)^{\frac{1}{p}}
\] (1.4.4)
holds if and only if $\mu > -\frac{1}{q}$ and $\gamma < 1 - \frac{1}{p} + \frac{1}{q} + \mu$.

In this Doctoral Thesis we use the following conventions: if $i > j$, then the sum $\sum_{k=i}^j$ is considered to be equal to zero; and in inequalities by the letter $c$, sometimes with either one or more indexes, we denote constants, which are independent of the functions, for which the considered inequalities have sense.
Chapter 2

Summability of a Tchebysheff system of functions

In this Chapter we consider a special type of Tchebysheff systems of functions defined on the intervals $(0, 1]$ and $[1, +\infty)$, respectively. We find necessary and sufficient conditions under which functions from the investigated systems belong to the corresponding Lebesgue spaces $L_p(0, 1)$ and $L_p(1, +\infty)$.

This Chapter consists of three Sections. It is organized as follows: In Section 2.1 we present some necessary notation. In Section 2.2 we state and prove our main result (Theorem 2.2.1) concerning a characterization of the functions in the studied system, which belong to the space $L_p(0, 1)$. Moreover, we present some lemmas concerning lower and upper estimates of functions from the investigated system. These lemmas are of independent interest but also necessary for the proof of main theorem. Finally, in Section 2.3 we present and discuss some further remarks and results.
2.1 Necessary notions

Let \( \alpha_i \) and \( \beta_i, \) \( i = 0, 1, \ldots, n, \) be real numbers. On the intervals \( (0, 1] \) and \([1, +\infty), \) respectively, we consider the following systems of functions:

\[
\begin{align*}
    u_0(t) &= t^{\alpha_0} \\
    u_1(t) &= t^{\alpha_0} \int \frac{1}{t} \, t_1^{\alpha_1} \, dt_1 \\
    u_2(t) &= t^{\alpha_0} \int \frac{1}{t} \, t_1^{\alpha_1} \int \frac{1}{t_2} \, t_2^{\alpha_2} \, dt_2 \, dt_1 \\
    \vdots \\
    u_n(t) &= t^{\alpha_0} \int \frac{1}{t} \, t_1^{\alpha_1} \int \frac{1}{t_2} \cdots \int \frac{1}{t_{n-1}} \, t_n^{\alpha_n} \, dt_n \, dt_{n-1} \cdots dt_1.
\end{align*}
\]

(2.1.1)

and

\[
\begin{align*}
    v_0(t) &= t^{\beta_0} \\
    v_1(t) &= t^{\beta_0} \int \frac{1}{t} \, t_1^{\beta_1} \, dt_1 \\
    v_2(t) &= t^{\beta_0} \int \frac{1}{t} \, t_1^{\beta_1} \int \frac{1}{t_2} \, t_2^{\beta_2} \, dt_2 \, dt_1 \\
    \vdots \\
    v_n(t) &= t^{\beta_0} \int \frac{1}{t} \, t_1^{\beta_1} \int \frac{1}{t_2} \cdots \int \frac{1}{t_{n-1}} \, t_n^{\beta_n} \, dt_n \, dt_{n-1} \cdots dt_1.
\end{align*}
\]

(2.1.2)

If in the system (2.1.2) we make a change variables \( t = \frac{1}{x}, \) then we get the system of functions \( \{\tilde{v}_i(x)\}_{i=0}^n, \) which are defined on the interval \( (0, 1] \) and have the same forms as functions from \( \{u_i(t)\}_{i=0}^n, \) i.e.:

\[
\begin{align*}
    \tilde{v}_0(x) &= x^{-\beta_0} \\
    \tilde{v}_1(x) &= x^{-\beta_0} \int x_1^{-(-\beta_1+2)} \, dx_1 \\
    \tilde{v}_2(x) &= x^{-\beta_0} \int x_1^{-(-\beta_1+2)} \int x_2^{-(-\beta_2+2)} \, dx_2 \, dx_1 \\
    \vdots \\
    \tilde{v}_n(x) &= x^{-\beta_0} \int x_1^{-(-\beta_1+2)} \int x_2^{-(-\beta_2+2)} \cdots \int x_{n-1}^{-(-\beta_{n-1}+2)} \, dx_n \, dx_{n-1} \cdots dx_1.
\end{align*}
\]

(2.1.3)

Therefore, we consider the system (2.1.1) as the main object of our investigation.

Conventions. Here and in the sequel we suppose that \( \sum_{j=1}^k \beta_j = 0 \) if \( i > k. \) The symbol \( X \ll Y \) means \( X \leq cY \) with some constant \( c > 0. \)
2.2 The main results

Our main result in this Section reads:

**Theorem 2.2.1** The functions \( u_i, \ i = 0, 1, \ldots, n, \) from the system (2.1.1) belong to \( L_p(0, 1), \ 1 \leq p < \infty, \) if and only if

\[
\min_{0 \leq s \leq t} (\alpha_0 + \sum_{j=1}^{s} (\alpha_j + 1)) > -\frac{1}{p}.
\]

**Remark 2.2.2** A corresponding result for the functions \( v_i, \ i = 0, 1, \ldots, n, \) from the system (2.1.2) is given in the next Section (see Theorem 2.3.2).

For the proof of Theorem 2.2.1 we need two Lemmas of independent interest.

Assume that \( i_0 = \min_k \) where

\[
M_i = \{ k : 0 \leq k \leq i, \ \alpha_0 + \sum_{j=1}^{k} (\alpha_j + 1) = \min_{0 \leq s \leq i} (\alpha_0 + \sum_{j=1}^{s} (\alpha_j + 1)) \}.
\]

**Lemma 2.2.3** For the functions \( u_i, \ i = 1, 2, \ldots, n, \) from the system (2.1.1) and for all \( 0 < \delta < 1 \) there exists \( 0 < \delta_1 \leq \delta \) such that for any \( t \in (0, \delta_1] \) the following estimate

\[
c_i(\delta) t^{\min_{0 \leq s \leq t} (\alpha_0 + \sum_{j=1}^{s} (\alpha_j + 1))} \leq u_i(t)
\]

holds, where \( c_i(\delta) \rightarrow 0 \) when \( \delta \rightarrow 1, \ i = 1, 2, \ldots, n. \)

**Lemma 2.2.4** For the functions \( u_i, \ i = 1, 2, \ldots, n, \) from the system (2.1.1) for any \( t \in (0, 1] \) the following estimate

\[
u_i(t) \ll t^{\min_{0 \leq s \leq t} (\alpha_0 + \sum_{j=1}^{s} (\alpha_j + 1))} |\ln t|^{l_i}
\]

holds, where \( l_i \) is the number of \( k, \ i_0 + 1 \leq k \leq i, \) such that \( \sum_{j=i_0+1}^{k} (\alpha_j + 1) = 0, \) if \( i_0 < i, \) and \( l_i = 0 \) if \( i_0 = i. \)

**Proof of Lemma 2.2.3.** It is obvious that \( 0 \leq i_0 \leq i. \) We divide the proof into three cases: (1) \( i_0 = i, \) (2) \( 0 < i_0 < i, \) and (3) \( i_0 = 0. \) Let \( t \in (0, 1]. \)
(1) \( i_0 = i \). Then

\[
 u_i(t) = u_{i_0}(t) = t^{\alpha_0} \int_1^t \int_{t_1}^{t_2} \int_{t_{i_0}}^{t_{i_0-1}} \int_{t_{i_0-1}}^{t_{i_0-2}} \cdots dt_{i_0} dt_{i_0-1} \ldots dt_1.
\]

By changing the order of integration, we obtain that

\[
 u_i(t) = u_{i_0}(t) = t^{\alpha_0} \int_t^{t_{i_0}} \int_{t_{i_0-1}}^{t_{i_0-2}} \int_{t_{i_0-2}}^{t_{i_0-3}} \cdots dt_{i_0} dt_{i_0-1} \ldots dt_1.
\]

Moreover, by changing variables \( t_j = t\tau_j, j = 1, 2, \ldots, i_0 \), in the obtained integrals, we have that

\[
 u_i(t) = u_{i_0}(t) = t^{\alpha_0} \int_1^{t_{i_0}} \int_{t_{i_0-1}}^{t_{i_0-2}} \int_{t_{i_0-2}}^{t_{i_0-3}} \cdots \int_{t_{i_0-i}}^{t_{i_0-i-1}} dt_{i_0} dt_{i_0-1} \ldots dt_1. \tag{2.2.3}
\]

It is easy to see that the function

\[
 \int_{t_{i_0-i}}^{t_{i_0-i-1}} dt_1 dt_2 \ldots dt_{i_0}
\]

is non-increasing with respect to the variable \( t \), \( 0 < t < 1 \). Hence, for any \( t \in (0, \delta] \), where \( 0 < \delta < 1 \), we get that

\[
 \int_{t_{i_0-i}}^{t_{i_0-i-1}} dt_1 dt_2 \ldots dt_{i_0} \geq \int_{1}^{t_1^{\alpha_0}} \int_{t_{i_0-i-1}}^{t_{i_0-i}} \int_{t_{i_0-i-2}}^{t_{i_0-i-3}} \cdots \int_{t_{i_0-i}}^{t_{i_0-i-1}} dt_{i_0} dt_{i_0-1} \ldots dt_1. \tag{2.2.4}
\]

By denoting the right-hand side of the last inequality by \( c_i^{(1)}(\delta) \), from (2.2.3) and (2.2.4) for any \( t \in (0, \delta] \) we obtain that

\[
 u_i(t) = u_{i_0}(t) \geq c_i^{(1)}(\delta) t^{\alpha_0} \int_{t_{i_0-i}}^{t_{i_0-i-1}} dt_{i_0} dt_{i_0-1} \ldots dt_1 = c_i^{(1)}(\delta) t^{\alpha_0} \int_{t_{i_0-i}}^{t_{i_0-i-1}} dt_{i_0} dt_{i_0-1} \ldots dt_1, \tag{2.2.5}
\]

where it is obvious that \( c_i^{(1)}(\delta) \to 0 \), when \( \delta \to 1 \).
(2) $0 < i_0 < i$. For $0 < t \leq \delta < 1$ we have that

$$u_i(t) = t^{\alpha_0} \int_{t_1}^{t_1} \int_{t_2}^{t_2} \cdots \int_{t_{i_0-1}}^{t_{i_0-1}} \int_{t_{i_0+1}}^{t_{i_0+1}} \cdots \int_{t_{i-1}}^{t_{i-1}} t^{\alpha_i} dt_i dt_{i-1} \cdots dt_1$$

$$\geq t^{\alpha_0} \int_{t_1}^{\delta} \int_{t_2}^{\delta} \cdots \int_{t_{i_0-1}}^{\delta} \int_{t_{i_0+1}}^{\delta} \cdots \int_{t_{i-1}}^{\delta} t^{\alpha_i} dt_i dt_{i-1} \cdots dt_1$$

$$\times \int_{\delta}^{1} \int_{t_{i_0+1}}^{1} \int_{t_{i_0+2}}^{1} \cdots \int_{t_{i_0-1}}^{1} \int_{t_{i-1}}^{1} t^{\alpha_i} dt_i dt_{i-1} \cdots dt_{i_0+1}. \quad (2.2.6)$$

By denoting the last line by $c_i^{(2)}(\delta)$, from (2.2.6) we get that

$$u_i(t) \geq c_i^{(2)}(\delta) t^{\alpha_0} \int_{t_1}^{\delta} \int_{t_2}^{\delta} \cdots \int_{t_{i_0-1}}^{\delta} \int_{t_{i_0+1}}^{\delta} \cdots \int_{t_{i-1}}^{\delta} t^{\alpha_i} dt_i dt_{i-1} \cdots dt_1, \quad (2.2.7)$$

where $c_i^{(2)}(\delta) \to 0$, when $\delta \to 1$.

Arguing as before for the first case, in (2.2.7) we change the order of integration:

$$u_i(t) \geq c_i^{(2)}(\delta) t^{\alpha_0} \int_{t_1}^{t_{i_0}} \int_{t_{i_0}}^{t_{i_0-1}} \cdots \int_{t_1}^{t_2} \int_{t_1}^{t_{i_0}} t^{\alpha_i} dt_i dt_2 \cdots dt_{i_0}. \quad (2.2.8)$$

Next we change variables $t_j = t \tau_j$, $j = 1, 2, \ldots, i_0$, in the integrals from (2.2.8):

$$u_i(t) \geq c_i^{(2)}(\delta) t^{\alpha_0 + \sum_{j=1}^{i_0} (a_j + 1)} \int_{1}^{\tau_{i_0}} \int_{1}^{\tau_{i_0-1}} \cdots \int_{1}^{\tau_{1}} \int_{1}^{\tau_{1}} t^{\alpha_i} d\tau_1 d\tau_2 \cdots d\tau_{i_0}. \quad (2.2.9)$$

Due to the fact that the function

$$\int_{1}^{\tau_{i_0}} \int_{1}^{\tau_{i_0-1}} \cdots \int_{1}^{\tau_{1}} \int_{1}^{\tau_{1}} t^{\alpha_i} d\tau_1 d\tau_2 \cdots d\tau_{i_0}$$

is non-increasing with respect to the variable $t$, $0 < t \leq \delta$, for any $t \in (0, \frac{1}{2} \delta)$, we have that

$$\int_{1}^{\tau_{i_0}} \int_{1}^{\tau_{i_0-1}} \cdots \int_{1}^{\tau_{1}} \int_{1}^{\tau_{1}} t^{\alpha_i} d\tau_1 d\tau_2 \cdots d\tau_{i_0} \geq$$
\[
\alpha_0 + \sum_{j=1}^{i_0} (\alpha_j + 1) = \min_{0 \leq s \leq i} (\alpha_0 + \sum_{j=1}^{s} (\alpha_j + 1)).
\]
Therefore, for $i_0 < s \leq i$ we have that
\[
(\alpha_0 + \sum_{j=1}^{i_0} (\alpha_j + 1)) - (\alpha_0 + \sum_{j=1}^{s} (\alpha_j + 1)) = - \sum_{j=s+1}^{i_0} (\alpha_j + 1) \leq 0, \quad (2.2.13)
\]
and, if $i_0 > 0$, for $0 \leq s \leq i_0 - 1$ we have that
\[
(\alpha_0 + \sum_{j=1}^{i_0} (\alpha_j + 1)) - (\alpha_0 + \sum_{j=1}^{s} (\alpha_j + 1)) = \sum_{j=s+1}^{i_0} (\alpha_j + 1) < 0. \quad (2.2.14)
\]
Let us again divide the proof of Lemma 2.2.4 into three cases: (1) $i_0 = i$, (2) $0 < i_0 < i$, and (3) $i_0 = 0$. Let $t \in (0, 1]$.

(1) $i_0 = i$. Then $t_i = t_{i_0} = 0$. According to (2.2.14) we have that
\[
u_i(t) = u_{i_0}(t) = t^{\alpha_0} \int_t^1 t_{l_1}^{\alpha_1} \int_{t_{l_1}}^1 t_{l_2}^{\alpha_2} \ldots \int_{t_{l_{i_0} - 1}}^1 t_{l_{i_0}}^{\alpha_{i_0}} dt_{l_{i_0}} dt_{l_{i_0} - 1} \ldots dt_1
\]
\[
\leq t^{\alpha_0} \int_t^1 t_{l_1}^{\alpha_1} \int_{t_{l_1}}^1 t_{l_2}^{\alpha_2} \ldots \int_{t_{l_0} - 1}^1 t_{l_0}^{\alpha_0} dt_{l_0} dt_{l_{i_0} - 1} \ldots dt_1
\]
\[
\leq c_i^{(5)} t^{\alpha_0 + \sum_{j=1}^{i_0} (\alpha_j + 1)} = c_i^{(5)} \left( \prod_{k=1}^{i_0} \sum_{j=k}^{i_0} (\alpha_j + 1) \right)^{-1} \left| \ln t \right|^l, \quad (2.2.15)
\]
where
\[
c_i^{(5)} = \left( \prod_{k=1}^{i_0} \sum_{j=k}^{i_0} (\alpha_j + 1) \right)^{-1}.
\]

(2) $0 < i_0 < i$. By changing the order of integration in the inter integral, we can present $u_i$ in the following form:
\[
u_i(t) = t^{\alpha_0} \int_t^1 \int_{t_{l_1}}^{t_{l_0}} t_{l_2}^{\alpha_2} \ldots \int_{t_{l_{i_0} + 1}}^{t_{l_{i_0}}} t_{l_{i_0} + 1}^{\alpha_{i_0}} dt_{l_{i_0}} dt_{l_{i_0} - 1} \ldots dt_1
\]
\[
= t^{\alpha_0} \int_t^1 \int_{t_{l_1}}^{t_{l_0}} t_{l_2}^{\alpha_2} \ldots \int_{t_{l_{i_0} - 1}}^{t_{l_{i_0}}} t_{l_{i_0}}^{\alpha_{i_0}} dt_{l_{i_0}} dt_{l_{i_0} - 1}
\]
\[
\times \left( \int_{l_{i_0} - 1}^{l_{i_0}} \int_{l_{i_0} - 2}^{l_{i_0}} \ldots \int_{l_{i_0} + 2}^{l_{i_0}} dt_{l_{i_0} + 1} dt_{l_{i_0} + 2} \ldots dt_i \right) dt_{l_{i_0}} dt_{l_{i_0} - 1} \ldots dt_1. \quad (2.2.16)
\]
We separately consider the following integral:

\[
I_{(i, t_0)}(t_0) = \int_{t_0}^{t_i} \int_{t_0}^{t_{i-1}} \cdots \int_{t_0}^{t_{i-2}} \int_{t_{i-1}}^{t_{i+2}} dt_{i_0+1} dt_{i_0+2} \cdots dt_i. \tag{2.2.17}
\]

According to (2.2.13) for \( s = i_0 + 1 \) we have that \( \alpha_{i_0 + 1} + 1 \geq 0 \). There are two possible cases: either \( \alpha_{i_0 + 1} + 1 = 0 \) or \( \alpha_{i_0 + 1} + 1 > 0 \). For the first case \( \alpha_{i_0 + 1} + 1 = 0 \), by taking into account that \( t \leq t_{i_0} \leq t_{i_0 + 2} \leq 1 \), we get that

\[
\int_{t_0}^{t_{i_0 + 2}} t_{i_0 + 1}^{\alpha_{i_0} + 1} dt_{i_0 + 1} = \int_{t_0}^{t_{i_0 + 2}} t_{i_0 + 1}^{-1} dt_{i_0 + 1} \leq \int_{t}^{t_{i_0 + 1}} t^{-1} dt_{i_0 + 1} = |\ln t|.
\]

This gives that

\[
I_{(i, t_0)}(t_0) \leq |\ln t| \int_{t_0}^{t_i} \int_{t_0}^{t_{i-1}} \cdots \int_{t_0}^{t_{i-2}} \int_{t_{i-1}}^{t_{i+2}} dt_{i_0+1} dt_{i_0+2} dt_{i_0+3} \cdots dt_i \tag{2.2.18}
\]

for \( 0 < t \leq t_0 < 1 \).

If \( \alpha_{i_0 + 1} + 1 > 0 \), then

\[
\int_{t_0}^{t_{i_0 + 2}} t_{i_0 + 1}^{\alpha_{i_0} + 1} dt_{i_0 + 1} \leq \int_{0}^{t_{i_0 + 2}} t_{i_0 + 1}^{\alpha_{i_0} + 1} dt_{i_0 + 1} = \frac{1}{\alpha_{i_0 + 1} + 1} t_{i_0 + 2}^{\alpha_{i_0} + 1 + 1}.
\]

Hence,

\[
I_{(i, t_0)}(t_0) \leq \frac{1}{\alpha_{i_0 + 1} + 1} \int_{t_0}^{t_i} \int_{t_0}^{t_{i-1}} \cdots \int_{t_0}^{t_{i-2}} \int_{t_{i-1}}^{t_{i+2}} dt_{i_0+1} dt_{i_0+2} dt_{i_0+3} \cdots dt_i. \tag{2.2.19}
\]

In (2.2.18) and (2.2.19) the integral expressions are equal, and the number of iterated integrals is one less than in (2.2.17). From (2.2.13), by continuing to reduce the number of iterated integrals, for \( 0 < t \leq t_0 \leq 1 \), we have that

\[
I_{(i, t_0)}(t_0) \leq c_i^{(6)} |\ln t|^{i}, \tag{2.2.20}
\]

where \( c_i^{(6)} \), \( i = 0, 1, \ldots, n \), depend only on \( \alpha_i \), \( i = 0, 1, \ldots, n \).
Hence, from (2.2.16) and (2.2.20) it follows that
\[ u_i(t) \leq c_i^{(6)} |\ln t|^l t^{\alpha_0} \int_{t_1}^{1} \int_{t_2}^{1} \ldots \int_{t_{n-1}}^{1} t_i^{\alpha_i} dt_1 dt_2 \ldots dt_{n-1} = c_i^{(6)} |\ln t|^l u_{i0}(t). \]

In view of (2.2.15) this estimate yields
\[ u_i(t) \leq c_i^{(5)} c_i^{(6)} t^{\min_{0 \leq s \leq i} (\alpha_0 + \sum_{j=1}^{s} (\alpha_j + 1))} |\ln t|^l. \tag{2.2.21} \]

(3) \( i_0 = 0 \). By changing the order of integration, the function \( u_i \) can be presented in the following form:
\[ u_i(t) = t^{\alpha_0} \int_{t}^{1} \int_{t_1}^{1} \ldots \int_{t_{i-1}}^{1} t_i^{\alpha_i} dt_1 dt_2 \ldots dt_{i-1} dt_i = t^{\min_{0 \leq s \leq i} (\alpha_0 + \sum_{j=1}^{s} (\alpha_j + 1))} I_{(i,0)}(t). \] \tag{2.2.22}

Since in this case, due to (2.2.13), \( \sum_{j=1}^{s} (\alpha_j + 1) \geq 0 \) for all \( 1 \leq s \leq i \), then in the same way as in the previous case it is easy to prove that
\[ I_{(i,0)}(t) \leq c_i^{(7)} |\ln t|^l, \]
where \( c_i^{(7)}, i = 0, 1, \ldots, n, \) depend only on \( \alpha_i, i = 0, 1, \ldots, n. \)

This estimate together with (2.2.22) give that
\[ u_i(t) \leq c_i^{(7)} t^{\min_{0 \leq s \leq i} (\alpha_0 + \sum_{j=1}^{s} (\alpha_j + 1))} |\ln t|^l. \tag{2.2.23} \]

By combining (2.2.15), (2.2.21) and (2.2.23) we obtain (2.2.2).

The proof of Lemma 2.2.4 is complete. \( \blacksquare \)

**Remark 2.2.5** The assumptions of Lemma 2.2.3 and Lemma 2.2.4 hold also for the functions \( u_0 \), since in this case \( i_0 = 0 \) and \( l_0 = 0 \), and thus,
\[ u_0 = t^{\alpha_0} = t^{\min_{0 \leq s \leq 0} (\alpha_0 + \sum_{j=1}^{s} (\alpha_j + 1))} |\ln t|^l. \]

**Proof of Theorem 2.2.1.** Let \( u_i \in L_p(0,1) \). Then, due to Lemma 2.2.3 (see (2.2.1)) and Remark 2.2.5, we have that
\[ t^{\min_{0 \leq s \leq i} (\alpha_0 + \sum_{j=1}^{s} (\alpha_j + 1))} \in L_p(0,1). \]

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This yields that
\[ \min_{0 \leq s \leq t} (\alpha_0 + \sum_{j=1}^{s} (\alpha_j + 1)) > -\frac{1}{p}. \]

Now suppose that \( \min_{0 \leq s \leq t} (\alpha_0 + \sum_{j=1}^{s} (\alpha_j + 1)) > -\frac{1}{p} \). Then, for every \( i \) there exists \( \delta_i > 0 \) such that
\[ \min_{0 \leq s \leq t} (\alpha_0 + \sum_{j=1}^{s} (\alpha_j + 1)) - \delta_i > -\frac{1}{p}. \] (2.24)

Moreover, according to Lemma 2.2.4 (see (2.2.2)) and Remark 2.2.5, it yields that
\[ u_i(t) \ll t^{\min_{0 \leq s \leq t} (\alpha_0 + \sum_{j=1}^{s} (\alpha_j + 1))} |\ln t|^{\delta_i} \ll t^{\min_{0 \leq s \leq t} (\alpha_0 + \sum_{j=1}^{s} (\alpha_j + 1)) - \delta_i} \sup_{0 \leq t \leq 1} t^{\delta_i} |\ln t|^{\delta_i}. \]
(2.25)

Since \( \sup_{0 \leq t \leq 1} t^{\delta_i} |\ln t|^{\delta_i} < \infty \) for any \( l_i > 0 \), then from (2.2.5) and (2.2.4) we conclude that \( u_i \in L_p(0, 1) \).

The proof of Theorem 2.2.1 is complete.

\section*{2.3 Further results and remarks}

The following Corollary follows from Lemma 2.2.3 and Lemma 2.2.4:

\textbf{Corollary 2.3.1} If \( \alpha_0 + \sum_{j=1}^{m} (\alpha_j + 1) \neq \alpha_0 + \sum_{j=1}^{l} (\alpha_j + 1) \) for \( m \neq l \), \( m, l = 0, 1, \ldots, n \), then for the functions \( u_i \), \( i = 1, 2, \ldots, n \), from the system (2.1.1) for \( 0 < \delta < 1 \) the following estimate
\[ u_i(t) \approx t^{\min_{0 \leq s \leq t} (\alpha_0 + \sum_{j=1}^{s} (\alpha_j + 1))}, \ t \in (0, \delta), \]
holds.

Let us now turn to the problem of summability of functions from (2.1.2) and state the following main result of the Section:

\textbf{Theorem 2.3.2} The functions \( v_i \), \( i = 0, 1, \ldots, n \), from the system (2.1.2) belong to \( L_p(1, +\infty) \), \( 1 \leq p < \infty \), if and only if
\[ \max_{0 \leq s \leq t} (\beta_0 + \sum_{j=1}^{s} (\beta_j + 1)) < \frac{1}{p}. \]
It was mentioned above that if in (2.1.2) we change variables \( t = \frac{1}{x} \),
then we get the system (2.1.3) of functions defined on \((0, 1]\). Moreover,
these functions have the same forms as functions from (2.1.1). By comparing
powers of functions from (2.1.1) and (2.1.3), we have that

\[
\alpha_0 = -\beta_0, \quad \alpha_i = -(\beta_i + 2), \quad i = 1, 2, \ldots, n.
\]

Therefore, all conditions we introduced for (2.1.1) can easily be rewritten for
the system (2.1.3). Thus,

\[
\min_{0 \leq s \leq i} (\alpha_0 + \sum_{j=1}^{s} (\alpha_j + 1)) = \min_{0 \leq s \leq i} (-\beta_0 + \sum_{j=1}^{s} (-\beta_j + 2) + 1))
\]

\[= - \max_{0 \leq s \leq i} (\beta_0 + \sum_{j=1}^{s} (\beta_j + 1)). \tag{2.3.1}\]

Now, \( i_0 = \min_{k \in K_i} \), where

\[
L_i = \{k: 0 \leq k \leq i, \beta_0 + \sum_{j=1}^{k} (\beta_j + 1) = \max_{0 \leq s \leq i} (\beta_0 + \sum_{j=1}^{s} (\beta_j + 1))\}.
\]

Moreover, the conditions \( \sum_{j=1}^{k} (\alpha_j + 1) = 0 \) and \( \sum_{j=1}^{k} (\beta_j + 1) = 0 \) are
equivalent since

\[
\sum_{j=1}^{k} (\alpha_j + 1) = \sum_{j=1}^{k} (\beta_j + 1) = 0 \tag{2.3.2}
\]

Finally, to prove Theorem 2.3.2 we state and prove the following corresponding Lemmas (to Lemmas 2.2.3 and Lemma 2.2.4).

**Lemma 2.3.3** For the functions \( v_i, i = 1, 2, \ldots, n \), from the system (2.1.2)
and for all \( \lambda > 1 \) there exists \( \lambda_1 \geq \lambda > 1 \) such that for any \( t \in [\lambda_1, +\infty) \) the
following estimate

\[
c_i(\lambda)t^{\max_{0 \leq s \leq i} (\beta_0 + \sum_{j=1}^{s} (\beta_j + 1))} \leq v_i(t)
\]

holds, where \( c_i(\lambda) \to 0 \) when \( \lambda \to 1 \).
Lemma 2.3.4 For the functions $v_i$, $i = 1, 2, \ldots, n$, from the system (2.1.2) for any $t \in [1, +\infty)$ the following estimate

$$v_i(t) \ll t^{\max \{\beta_0 + \sum_{j=1}^{k} (\beta_j + 1)\} |\ln t|^l_i}$$

holds, where $l_i$ is the number of $k$, $i_0 + 1 \leq k \leq i$, such that $\sum_{j=i_0+1}^{k} (\beta_j + 1) = 0$, if $i > i_0$, and $l_i = 0$, if $i = i_0$.

Proof of Lemma 2.3.3. In view of Lemma 2.2.3 and (2.3.1) for functions from (2.1.3) and for any $0 < \delta < 1$ there exists $0 < \delta_1 \leq \delta$ such that

$$\tilde{c}_i(\delta) x^{-\max \{\beta_0 + \sum_{j=1}^{k} (\beta_j + 1)\} \leq \tilde{v}_i(x), \text{ for } x \in (0, \delta_1], \quad (2.3.3)$$

where $\tilde{c}_i(\delta) \to 0$ when $\delta \to 1$.

Let $\lambda = \frac{1}{\delta}$, $\lambda_1 = \frac{1}{\delta_1}$. Since $\tilde{v}_i(x) = v_i(t)$ for $x = \frac{1}{t}$, we can substitute them into (2.3.3) and then, for any $t \in [\lambda_1, +\infty)$, we get that

$$\tilde{c}_i\left(\frac{1}{\lambda}\right) \left(\frac{1}{t}\right)^{-\max \{\beta_0 + \sum_{j=1}^{k} (\beta_j + 1)\} \leq v_i(t),$$

where $\tilde{c}_i\left(\frac{1}{\lambda}\right) \to 0$ when $\lambda \to 1$, i.e. that

$$c_i(\lambda)^{\max \{\beta_0 + \sum_{j=1}^{k} (\beta_j + 1)\} \leq v_i(t),$$

where $c_i(\lambda) \equiv \tilde{c}_i\left(\frac{1}{\lambda}\right) \to 0$ when $\lambda \to 1$.

The proof of Lemma 2.3.3 is complete.

Proof of Lemma 2.3.4. Due to Lemma 2.2.4 and (2.3.1) for functions from (2.1.3) for any $x \in (0, 1]$ we find that

$$\tilde{v}_i(x) \ll x^{-\max \{\beta_0 + \sum_{j=1}^{k} (\beta_j + 1)\} |\ln x|^l_i}, \quad (2.3.4)$$

where, due to (2.3.2), $l_i$ is the number of $k$, $i_0 + 1 \leq k \leq i$, such that $\sum_{j=i_0+1}^{k} (\beta_j + 1) = 0$. 

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In (2.3.4) we substitute \( \tilde{v}_i(x) = v_i(t) \) and \( x = \frac{1}{t} \) and then, for any \( t \in [1, +\infty) \), we get that

\[
v_i(t) \leq \left( \frac{1}{t} \right)^{-\max_{0 \leq s \leq s_i} \left( \beta_0 + \sum_{j=1}^{s} (\beta_j + 1) \right)} \left| \ln \frac{1}{t} \right|^s = \left( \frac{1}{t} \right)^{\max_{0 \leq s \leq s_i} \left( \beta_0 + \sum_{j=1}^{s} (\beta_j + 1) \right)} \left| \ln t \right|^s.
\]

The proof of Lemma 2.3.4 is complete.

\[ \blacksquare \]

**Remark 2.3.5** The assumptions of Lemma 2.3.3 and Lemma 2.3.4 hold also for the functions \( v_0 \), since in this case \( v_0 = 0 \) and \( l_0 = 0 \), and thus,

\[
v_0 = t^{\beta_0} = t^{\max_{0 \leq s \leq s_0} \left( \beta_0 + \sum_{j=1}^{s} (\beta_j + 1) \right)} \left| \ln t \right|^0.
\]

*Proof of Theorem 2.3.2.* Assume that \( \max_{0 \leq s \leq s_i} \left( \beta_0 + \sum_{j=1}^{s} (\beta_j + 1) \right) < -\frac{1}{p} \). Then, for every \( i \) there exists \( \gamma_i > 0 \) such that

\[
\max_{0 \leq s \leq s_i} \left( \beta_0 + \sum_{j=1}^{s} (\beta_j + 1) \right) + \gamma_i < -\frac{1}{p},
\]

(2.3.5)

On the other hand from Lemma 2.3.3 and Remark 2.3.5 it follows that

\[
v_i(t) \leq \left( \frac{1}{t} \right)^{\max_{0 \leq s \leq s_i} \left( \beta_0 + \sum_{j=1}^{s} (\beta_j + 1) \right)} \left| \ln t \right|^s \leq \left( \frac{1}{t} \right)^{\max_{0 \leq s \leq s_i} \left( \beta_0 + \sum_{j=1}^{s} (\beta_j + 1) \right) + \gamma_i} \sup_{1 \leq t \leq \infty} t^{-\gamma_i} \left| \ln t \right|^s.
\]

(2.3.6)

Since \( \sup_{1 \leq t \leq \infty} t^{-\gamma_i} \left| \ln t \right|^s < \infty \) for any \( l_i \geq 0 \), then from (2.3.5) and (2.3.6) we have that \( v_i \in L_p(1, +\infty) \).

Let now \( v_i \in L_p(1, +\infty) \). Then, in view of Lemma 2.3.4 and Remark 2.3.5, we find that

\[
\max_{0 \leq s \leq s_i} \left( \beta_0 + \sum_{j=1}^{s} (\beta_j + 1) \right) \in L_p(1, +\infty).
\]

This gives that \( \max_{0 \leq s \leq s_i} \left( \beta_0 + \sum_{j=1}^{s} (\beta_j + 1) \right) < -\frac{1}{p} \).

The proof of Theorem 2.3.2 is complete.

\[ \blacksquare \]
Next consider the following system:

\[
\begin{align*}
  w_0(t) &= t^{\alpha_0} \\
  w_1(t) &= t^{\alpha_0} \int_{t_i}^{t_1} t^{\alpha_1} dt_1 \\
  w_2(t) &= t^{\alpha_0} \int_{t_i}^{t_1} t^{\alpha_1} \int_{t_2}^{t_1} t^{\alpha_2} dt_2 dt_1 \\
  \vdots \\
  w_n(t) &= t^{\alpha_0} \int_{t_i}^{t_1} t^{\alpha_1} \int_{t_2}^{t_1} \cdots \int_{t_n}^{t_{n-1}} t^{\alpha_n} dt_n dt_{n-1} \cdots dt_1.
\end{align*}
\]

(2.3.7)

In this system the first functions \( w_0 \) and \( w_1 \) coincide with the functions \( u_0 \) and \( u_1 \) from (2.1.1), respectively. But, as we can see, the lower and upper integral bounds of the functions \( w_i \) and \( u_i \), \( i = 2, 3, \ldots, n \), are correspondingly different.

By changing the order of integration in each function from (2.3.7), for \( w_i \) we have that

\[
  w_i(t) = t^{\alpha_0} \int_{t_i}^{t_1} t^{\alpha_1} \int_{t_i}^{t_{i-1}} \cdots \int_{t_i}^{t_1} t^{\alpha_i} dt_1 dt_2 \cdots dt_i.
\]

Now the function \( w_i \) has the same form as the function \( u_i \) from (2.1.1) with the difference in order of powers of the functions in the integrals. Hence, it is easy to formulate lower and upper estimates for \( w_i \) and conditions of its summability to the power \( p \) on the interval \((0, 1)\). In particular, the next two Lemmas follow from Lemmas 2.2.3 and 2.2.4, respectively.

**Lemma 2.3.6** For the functions \( w_i \), \( i = 1, 2, \ldots, n \), from the system (2.3.7) and for all \( 0 < \delta < 1 \) there exists \( 0 < \delta_i \leq \delta \) such that for any \( t \in (0, \delta_i) \) the following estimate

\[
c_i(\delta) t^{\min_{1 \leq s \leq i+1} (\alpha_0 + \sum_{j=s}^{i} (\alpha_j + 1))} \leq w_i(t)
\]

holds, where \( c_i(\delta) \to 0 \) when \( \delta \to 1 \), \( i = 1, 2, \ldots, n \).

**Lemma 2.3.7** For the functions \( w_i \), \( i = 1, 2, \ldots, n \), from the system (2.3.7) for any \( t \in (0, 1) \) the following estimate

\[
w_i(t) \ll t^{\min_{1 \leq s \leq i+1} (\alpha_0 + \sum_{j=s}^{i} (\alpha_j + 1))} |\ln t|^{l_i}
\]

holds, where \( l_i \) is the number of \( k \), \( 1 \leq k \leq i_0 - 1 \), such that \( \sum_{j=k}^{i_0-1} (\alpha_j + 1) = 0 \), when \( 1 < i_0 \), and \( l_i = 0 \) when \( i_0 = 1 \).
Here $i_0 = \max_{k \in \mathbb{N}_i} k$, where

$$N_i = \{ k : 1 \leq k \leq i + 1, \alpha_0 + \sum_{j=k}^{i} (\alpha_j + 1) = \min_{1 \leq s \leq i+1} (\alpha_0 + \sum_{j=s}^{i} (\alpha_j + 1)) \}.$$

**Remark 2.3.8** The assumptions of Lemma 2.3.6 and Lemma 2.3.7 hold also for the functions $w_0$, since in this case $t_0 = 0$, and, thus,

$$w_0 = t^{\alpha_0} = t^{\min_{1 \leq s \leq i+1} (\alpha_0 + \sum_{j=s}^{i} (\alpha_j + 1))} |\ln t|^{t_0}.$$

Moreover, Theorem 2.2.1 implies the following result:

**Theorem 2.3.9** The functions $w_i$, $i = 0, 1, \ldots, n$, from the system (2.3.7) belong to $L_p(0, 1)$, $1 \leq p < \infty$, if and only if

$$\min_{1 \leq s \leq i+1} (\alpha_0 + \sum_{j=s}^{i} (\alpha_j + 1)) > -\frac{1}{p}.$$

**Remark 2.3.10** Consider the system $\{y_i(\cdot)\}_{i=0}^{n}$ with $i$:th function in the form:

$$y_i(t) = t^{\beta_0} \int_{1}^{t} \int_{\tau_1}^{t} \int_{\tau_2}^{t} \cdots \int_{\tau_{i-1}}^{t} t_{i-1}^{\beta_i} dt_{i-1} \cdots dt_1.$$

It is possible to formulate two corresponding Lemmas and a theorem as above also for this case. However, these results can be derived by only making a change of variables as discussed before so we leave out both the formulations and the proofs.
Chapter 3

Weighted Sobolev type function spaces

In this Chapter we consider the weighted Sobolev type spaces $L^n_{p,\gamma}(I)$, the so-called Kudryavtsev spaces, and $W^n_{p,\alpha}$, which we call the spaces with multi-weighted derivatives. This Chapter is written on the base of some important papers of L. D. Kudryavtsev ([35] - [40]) and B. L. Baidel'dinov ([13] - [16]). These results are formulated and presented in a form so that they can be used in the proofs of the main results in Chapter 4. However, since the results are essentially known we include them without proofs (for the complete proofs see e.g. [4]). We pronounce that the results from this Chapter establish the existence of the boundary values of the functions from these spaces at the singular points and different equivalent norms in different degenerations.

3.1 Weighted Kudryavtsev spaces

Let $R$ be the set of real numbers and $n$ be a natural number, $\gamma \in R$, $1 \leq p \leq \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $I = (a, b)$, $0 \leq a < b \leq \infty$.

The Kudryavtsev space $L^n_{p,\gamma} = L^n_{p,\gamma}(I)$ consists of the functions $f : I \to R$, for which the following quantity has sense and is finite:

$$
\|f\|_{L^p_{p,\gamma}} = \left( \int_I |t^\gamma f^{(n)}(t)|^p dt \right)^{\frac{1}{p}},
$$

if $1 \leq p < \infty$, and

$$
\|f\|_{L^n_{\infty,\gamma}} = \text{ess sup}_{t \in I} |t^\gamma f^{(n)}(t)|,
$$

if $p = \infty$. The spaces $W^k_{p,\alpha}$, $k = 0, 1, 2, \ldots$, $0 < \alpha < n$, are defined by the following norms:

$$
\|f\|_{W^k_{p,\alpha}} = \left( \int_I \left( \sum_{j=0}^{k} \|f^{(j)}\|_{L^p} \right)^p dt \right)^{\frac{1}{p}},
$$

if $1 \leq p < \infty$, and

$$
\|f\|_{W^k_{\infty,\alpha}} = \text{ess sup}_{t \in I} \left( \sum_{j=0}^{k} |f^{(j)}(t)|^\alpha \right)^{\frac{1}{\alpha}},
$$

if $p = \infty$.
if $p = +\infty$.

Here with derivatives we mean weak derivatives (see e.g. [20]), so the spaces $L^{n}_{p,\gamma}$ are complete normed spaces. We denote $L^{n}_{p,\gamma} = L^{0}_{p,\gamma}$; $L^{n}_{p} = L^{n}_{p,0}$: $\|f\|_{p,\gamma} = \|f\|_{L^{p,\gamma}}$ and $\|f\|_{p} = \|f\|_{p,0}$.

**Definition 3.1.1** Let $I = [0, +\infty)$, If $f: I \rightarrow R$, $t_{0} \in [0, +\infty)$ and there exists an equivalent function $f_{1}: I \rightarrow R$ to $f$ (in the sense of Lebesgue measure), for which the finite limit $\lim_{t\to t_{0}} f_{1}(t) = f_{1}(t_{0})$ exists, then it is called the boundary value or trace of the function $f$ at the point $t = t_{0}$ and it is denoted $f(t_{0})$.

**Definition 3.1.2** Let $I = [0, +\infty)$. If for the function $f$ there exists an equivalent function $f_{1}$, which has finite or infinite limit $\lim_{t\to+\infty} f_{1}(t)$, then it is called the boundary value of the function $f$ when $t \to +\infty$ and it is denoted $f(\infty)$, that is $f(\infty) = \lim_{t\to+\infty} f_{1}(t)$.

These definitions do not depend on the choice of the function $f_{1}$.

**Definition 3.1.3** Let $I = (0, +\infty)$ and $t_{0} \in I$. If, for the function $f: I \rightarrow R$, there exists a unique polynomial $P_{n-1}(t) = \sum_{\nu=0}^{n-1} a_{\nu} t^{\nu}$, such that

$$\lim_{t\to t_{0}} [f(t) - P_{n-1}(t)]^{(k)} = 0, \quad k = 0, 1, \ldots, n - 1,$$

then we say that the function $f$ is stabilized with order $n$ to the unique polynomial $P_{n-1}(t)$ when $t \to t_{0}$.

The spaces $L^{n}_{p,\gamma}(0,1)$ and $L^{n}_{p,\gamma}(1, +\infty)$ are investigated separately. When $1 - \frac{1}{p} < \gamma < n - \frac{1}{p}$ we assume that the number $\gamma + \frac{1}{p}$ is not an integer. In some cases the results are first given for $n = 1$, i.e. for the space $L^{1}_{p,\gamma}(0,1)$, and after that extended for all $n > 1$.

First we investigate the space $L^{n}_{p,\gamma}(0,1)$, where three cases are considered separately:

1. $\gamma < 1 - \frac{1}{p}$ — weak degeneration,

2. $1 - \frac{1}{p} < \gamma < n - \frac{1}{p}$ — mixed case,

3. $\gamma > n - \frac{1}{p}$ — strong degeneration.

1. **The case** $\gamma < 1 - \frac{1}{p}$. When $n = 1$, i.e. for the space $L^{1}_{p,\gamma}(0,1)$ in the case of weak degeneration the following statement holds:
Lemma 3.1.4 If \( f \in L^1_{p,\gamma}(0, 1), \gamma < 1 - \frac{1}{p} \), then for each point \( t_0 \in [0, 1] \):

(i) A finite boundary value \( f(t_0) \) exists;

(ii) \( f(t) = f(t_0) + \int_{t_0}^t f'(t) dt, 0 \leq t \leq 1; \)

(iii) \( ||f||_p - |f(t_0)| \leq c_1(\gamma, p)||f'||_{p,\gamma}; \)

(iv) \( ||f(t_0)| - |f(0)|| \leq c_2(\gamma, p)||f'||_{p,\gamma}; \)

and if \( p = 1 \), then the properties (i) - (iv) hold even when \( \gamma = 0. \)

Theorem 3.1.5 Let \( f \in L^n_{p,\gamma}(0, 1), \gamma < 1 - \frac{1}{p} \). Then, for every \( t_0 \in [0, 1] \), the boundary values \( f^{(j)}(t_0), j = 0, 1, \ldots, n - 1 \), exist and

\[
||f||_p \leq c \left( \sum_{j=0}^{n-1} ||f^{(j)}(t_0)|| + ||f^{(n)}||_{p,\gamma} \right).
\]

Moreover, if \( P_{n-1}(t) = \sum_{j=0}^{n-1} \frac{f^{(j)}(0)}{j!} t^j \) (the Taylor polynomial of order \( n - 1 \) of the function \( f \)), then

\[
||f(t) - P_{n-1}(t)||^{(n-k)}_{p,\gamma-k} \leq c ||f^{(n)}||_{p,\gamma}, \ k = 1, 2, \ldots, n.
\]

2. The case \( \gamma > n - \frac{1}{p} \). If \( f \in L^n_{p,\gamma}, \gamma > n - \frac{1}{p} \), then there exists not, in general, a finite boundary value of the function \( f \) when \( t = 0 \), i.e., here it is a situation which is different from the situation with ordinary unweighted spaces. In this case the following estimate holds:

Lemma 3.1.6 If \( f \in L^1_{p,\gamma}, \gamma > 1 - \frac{1}{p} \), then

\[
||f||_{p,\gamma-1} - c|f(1)| \leq \frac{1}{\gamma - 1 + 1/p} ||f'||_{p,\gamma},
\]

where \( c = ||t^{\gamma-1}||_p. \)

Corollary 3.1.7 If \( f \in L^n_{p,\gamma}, \gamma > n - \frac{1}{p} \), then

\[
||f^{(n-k)}||_{p,\gamma-k} \leq c \left( \sum_{j=1}^{k} ||f^{(n-j)}(1)|| + ||f^{(n)}||_{p,\gamma} \right), \ k = 1, 2, \ldots, n.
\]
3. The case

\[ 1 - \frac{1}{p} < \gamma \leq n - \frac{1}{p}, \quad n \geq 2, \quad (3.1.3) \]

i.e., we consider the problem about the behavior of functions in the space \( L_{p,\gamma}^n(0, 1) \) in the mixed case.

We denote by \( \tilde{\gamma} \) the integer such that the following inequality holds:

\[ \tilde{\gamma} - \frac{1}{p} \leq \gamma < \tilde{\gamma} + 1 - \frac{1}{p}. \]

According to (3.1.3) we have that \( 1 \leq \tilde{\gamma} \leq n - 1 \) and \( \tilde{\gamma} \leq \gamma + \frac{1}{p} < \tilde{\gamma} + 1 \), i.e.,
\( \tilde{\gamma} = \lfloor \gamma + \frac{1}{p} \rfloor \) is the integer part of the number \( \gamma + \frac{1}{p} \). For all \( \gamma \) we define the number \( \tilde{\gamma} \) in the following way:

\[ \tilde{\gamma} = \begin{cases} \lfloor \gamma + \frac{1}{p} \rfloor, & \text{if } \gamma > 1 - \frac{1}{p}, \\ 0, & \text{if } \gamma < 1 - \frac{1}{p}. \end{cases} \]

**Remark 3.1.8** For every \( \gamma \in \mathbb{R} \), \( \gamma \neq 1 - \frac{1}{p} \), the following inequality holds:

\[ \gamma - \tilde{\gamma} < 1 - \frac{1}{p} \]

and for \( \gamma > -\frac{1}{p} \) and for non integer \( \gamma + \frac{1}{p} \) the following inequality holds:

\[ \gamma - \tilde{\gamma} + 1 > 1 - \frac{1}{p}. \]

**Lemma 3.1.9** Let \( \gamma + \frac{1}{p} \) be non integer and \( 1 - \frac{1}{p} < \gamma < n - \frac{1}{p} \). Then every function \( f \in L_{p,\gamma}^n \) has finite boundary values

\[ f(0), \quad f'(0), \quad f^{(n-\gamma-1)}(0) \]

and the following inequality holds:

\[ \| f^{(j)} \|_{p,\gamma_j} < +\infty, \quad j = 0, 1, \ldots, n - 1. \]
Remark 3.1.10 The conclusion about existence of finite boundary values of the function \( f \in L^p_{p,\gamma} \) at the point zero is sharp in the sense that there exist functions \( f \in L^p_{p,\gamma} \), \( \gamma + \frac{1}{p} \) is not integer, \( 1 - \frac{1}{p} < \gamma < n - \frac{1}{p} \), for which all derivatives \( f^{(n-\gamma)} \), \( f^{(n-\gamma+1)} \), \ldots, \( f^{(n)} \) do not have finite boundary values at \( t = 0 \).

Example: Let \( f(t) = t^\beta \), where \( \beta = n - \tilde{\gamma} + \eta - \varepsilon \), \( \varepsilon = \gamma - \tilde{\gamma} + \frac{1}{p} \) and \( 0 < \eta < \varepsilon < 1 \). Then \( f \in L^p_{p,\gamma}(0,1) \) since

\[
\|f\|_{L^p_{p,\gamma}} = \left( \int_0^1 |t^\gamma f^{(n)}(t)|^p dt \right)^{\frac{1}{p}} = c \left( \int_0^1 t^{(n+\eta-\frac{1}{p})p} dt \right)^{\frac{1}{p}} < +\infty.
\]

However, since \( \eta < \gamma - \tilde{\gamma} + \frac{1}{p} \), it yields that

\[
\lim_{t \to 0} f^{(n-\gamma)}(t) = c \lim_{t \to 0} t^{\beta-n+\tilde{\gamma}} = c \lim_{t \to 0} t^{\gamma-\gamma+\frac{1}{p}} = +\infty.
\]

Next we consider the case of infinite interval \( I = (1, +\infty) \), so we will denote \( L^p_{p,\gamma} = L^p_{p,\gamma}(1, +\infty) \). On the infinite interval \( I = (1, +\infty) \) we consider only the cases \( \gamma < 1 - \frac{1}{p} \) (strong degeneration) and \( \gamma > 1 - \frac{1}{p} \) (weak degeneration).

1. The case \( \gamma < 1 - \frac{1}{p} \), i.e., the case of strong degeneration. If \( f \in L^1_{p,\gamma} \) and \( \gamma < 1 - \frac{1}{p} \), then the function \( f \), in general, need not to have finite boundary value when \( t \to +\infty \).

Example: If \( \gamma < 1 - \frac{1}{p} \), then for the function \( f(t) = \ln t \) the integral

\[
\int_{t=1}^{+\infty} t^{(\gamma-1)p} dt < +\infty \text{ and, therefore, } f'(t) = \frac{1}{t} \in L^p_{p,\gamma}, \text{ i.e. } f \in L^1_{p,\gamma}, \text{ but } \lim_{t \to +\infty} f(t) = \lim_{t \to +\infty} \ln t = +\infty.
\]

Moreover, if \( \gamma < 1 - \frac{1}{p} \) and \( f \in L^p_{p,\gamma} \), \( n \geq 1 \), then neither the function \( f \) nor its derivatives \( f^{(k)} \), \( k = 1, 2, \ldots, n-1 \), in general, need to have limit values when \( t \to +\infty \).
Example: Let \( f(t) = t^\beta, \gamma = 1 - \frac{1}{p} - \varepsilon, \varepsilon > 0 \), and choose \( \beta \) such that
\[
0 \leq n - 1 < \beta < n - 1 + \varepsilon.
\]
Then the condition
\[
\int_1^{+\infty} |t^n f^{(n)}(t)|^p dt < +\infty
\]
is equivalent to the condition \((n - \gamma - \beta)p > 1\), that is
\[
\beta < n - \frac{1}{p} = n - 1 + \varepsilon.
\]
Therefore, according to the choice of \( \beta \), \( f(t) = t^\beta \in L^p_{\gamma} \). Since \( f^{(k)}(t) = (t^\beta)^{(k)} = \beta!(\beta - 1)\cdots(\beta - k + 1)t^{\beta-k} \) and, for every \( k = 0, 1, \ldots, n - 1 \), \( \beta - k > 0 \) we conclude that \( f^{(k)}(\infty) = c \lim_{t \to +\infty} t^{\beta-k} = +\infty \).

2. The case \( \gamma > 1 - \frac{1}{p} \), i.e. the case of weak degeneration. First we state the following result:

**Lemma 3.1.11** If \( f \in L^1_{p,\gamma} \) and \( \gamma > 1 - \frac{1}{p} \), then
\[
\lim_{t \to +\infty} f(t) = f(+\infty) < \infty,
\]
and, for arbitrary \( \delta > 0 \), the following inequality holds:
\[
||f(+\infty) - |f(1)||| \leq c||f'||_{p,\gamma},
\]
and, for arbitrary \( \delta > 0 \), the following inequality holds:
\[
||f||_{p,\gamma} - \delta p^{\frac{1}{p} - s} - (\delta p)^{\frac{1}{p}} |f(1)| \leq \delta ||f'||_{p,\gamma}.
\]

**Remark 3.1.12** If \( \gamma > n - \frac{1}{p} \) and \( f \in L^n_{p,\gamma} \), then, according to Lemma 3.1.11, the finite boundary value \( f^{(n-1)}(+\infty) \) exists. However, when \( t \to +\infty \) the boundary value of other derivatives of the function \( f \), in general, need not to exist. For example, \( f(t) = t^{n-1} \in L^n_{p,\gamma} \), since \( ||f||_{L^n_{p,\gamma}} = 0 \), and \( f^{(n-1)}(+\infty) = c < +\infty \), but \( f^{(j)}(+\infty) = +\infty \), \( j = 0, 1, \ldots, n-2 \).

**Lemma 3.1.13** Let \( f \in L^n_{p,\gamma} \) and \( \gamma > m - \frac{1}{p}, 1 \leq m \leq n \). Then there exist numbers \( a_0, a_1, \ldots, a_{m-1} \), such that, for \( s = 1, 2, \ldots, m \), it yields that
\[
\lim_{t \to +\infty} f^{(n-s)}(t) - \sum_{\mu=0}^{s-1} \frac{a_{m-s+\mu} t^{\mu}}{\mu!} = a_{m-s}, \tag{3.1.4}
\]
and
\[
||f^{(n-s)}(t) - \sum_{\mu=0}^{s-1} \frac{a_{m-s+\mu} t^{\mu}}{\mu!}||_{p,\gamma-s} \leq c||f^{(s)}||_{p,\gamma}.
\]
Theorem 3.1.14 Let $f \in L^p_{p, \gamma}$ and $\gamma > m - \frac{1}{p}$, $1 \leq m \leq n$. If

$$b_\nu = \frac{a_\nu}{\nu!}, \quad \nu = 0, 1, \ldots, m - 1,$$

and

$$P_{m-1}(t) = \sum_{\nu=0}^{m-1} b_\nu t^\nu,$$

then, for all $f \in L^p_{p, \gamma}$,

$$\lim_{t \to +\infty} [f^{(n-m)}(t) - P_{m-1}(t)]^{(k)} = 0, \quad k = 0, 1, \ldots, m - 1,$$

and there exists a constant $c > 0$ such that

$$||f^{(n-m)} - P_{m-1}||^{(k)}_{p, \gamma - m + k} \leq c||f^{(n)}||_{p, \gamma}, \quad k = 0, 1, \ldots, m - 1,$$

where $a_\nu$ are inductively defined by formula (3.1.4).

3.2 Function spaces with multiweighted derivatives

Let $I = (0, 1)$ or $I = (1, +\infty)$, $\bar{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_n)$, where $\alpha_i \in \mathbb{R}$, $i = 0, 1, \ldots, n$, $n$ is a natural number, $|\bar{\alpha}| = \sum_{i=0}^{n} \alpha_i$ and $1 \leq p < \infty$.

For the function $f: I \to \mathbb{R}$ we define the differential operations

$$D_0^0 f(t) = t^{\alpha_0} f(t),$$

$$D_0^i f(t) = t^{\alpha_0} t^{\alpha_1} \frac{d}{dt} \cdots t^{\alpha_i} \frac{d}{dt} f(t), \quad i = 1, 2, \ldots, n,$$  (3.2.1)

where each derivative is understood as a weak derivative (see e.g. [20]).

Definition 3.2.1 The operation $D_0^i f(t)$ is called $\alpha$ - multiweighted derivative of the function $f$ of order $i$, $i = 0, 1, \ldots, n$.

By $W^m_{p, \bar{\alpha}} = W^m_{p, \bar{\alpha}}(I)$ we denote the space of functions $f: I \to \mathbb{R}$, which have $\alpha$ - multiweighted $n$:th order derivatives and for which the following norm is finite:

$$||f||_{W^m_{p, \bar{\alpha}}} = ||D_0^n f||_p + \sum_{i=0}^{n-1} |D_0^i f(1)|,$$  (3.2.2)
where \( \| \cdot \|_p \) denotes the usual norm of the space \( L_p(I), \ 1 \leq p < \infty \).

When \( \alpha_i = 0, \ i = 0, 1, \ldots, n-1 \), and \( \alpha_n = \gamma \) the space \( W_{p,\gamma}^n \) coincides with the usual Kudryavtsev space \( L_{p,\gamma}^n = L_{p,\gamma}(I) \) with the finite norm \( \| f \|_{p,\gamma} = \| t^\gamma f^{(n)} \|_p + \sum_{i=0}^{n-1} |f^{(i)}(1)| \) (see [38]).

For \( n > 1 \) let:

\[
\gamma_n^\alpha = 1, \ \gamma_{n-1}^\alpha = \alpha_n, \ \gamma_i^\alpha = \alpha_n + \sum_{k=i+1}^{n-1} (\alpha_k - 1), \ i = 0, 1, \ldots, n-1, \quad (3.2.3)
\]

and

\[
\gamma_i^{\max} = \max_{i \leq j \leq n-1} \gamma_j^\alpha, \ \gamma_i^{\min} = \min_{i \leq j \leq n-1} \gamma_j^\alpha, \ i = 0, 1, \ldots, n-1.
\]

It is easy to see that

\[
\alpha_i = \gamma_i^{\alpha} - \gamma_i^{\alpha} + 1, \ i = 1, 2, \ldots, n-1. \quad (3.2.4)
\]

In the paper [14] it was shown that the properties of the space \( W_{p,\alpha}^n \) are dependent of the values \( \gamma_i, \ i = 0, 1, \ldots, n-1 \), and the following three cases were considered:
1) \( \gamma_i^{\max} = \max_{0 \leq j \leq n-1} \gamma_j^\alpha < 1 - \frac{1}{p} \) (the case of weak degeneration),
2) \( \gamma_i^{\min} = \min_{0 \leq j \leq n-1} \gamma_j^\alpha > 1 - \frac{1}{p} \) (the case of strong degeneration),
3) \( \gamma_i^{\min} < 1 - \frac{1}{p} < \gamma_i^{\max} \) (mixed case).

Moreover, for \( i, j = 0, 1, \ldots, n-1 \) we define the following set of functions \( K_{i+1,j}(t, x) \equiv K_{i+1,j}(t, x, \alpha) \) and \( K_{i+1,j}(x, t) \equiv K_{i+1,j}(x, t, \alpha) \):

\[
K_{i+1,j}(t, x) = \frac{x}{t} t_i^{\alpha_i+1} \int_{t_{i+1}}^x t_i^{\alpha_i+2} \ldots \int_{t_{j-1}}^x t_j^{\alpha_j} dt_j dt_{j-1} \ldots dt_{i+1} \text{ when } i < j,
\]

\[
K_{i+1,j}(t, x) \equiv 1 \text{ when } i = j, \ K_{i+1,j}(t, x) \equiv 0 \text{ when } i > j \text{ for } 0 < t \leq x < \infty;
\]

\[
K_{i+1,j}(x, t) = \frac{x}{t} t_i^{\alpha_i+1} \int_{t_{i+1}}^x t_i^{\alpha_i+2} \ldots \int_{t_{j-1}}^x t_j^{\alpha_j} dt_j dt_{j-1} \ldots dt_{i+1} \text{ when } i < j,
\]

\[
K_{i+1,j}(x, t) \equiv 1 \text{ when } i = j, \ K_{i+1,j}(x, t) \equiv 0 \text{ when } i > j \text{ for } 0 < t \leq x < \infty.
\]

We denote

\[
w_j(t, x) = t^{-\alpha_j} K_{1,j}(t, x), \quad \bar{w}_j(x, t) = t^{-\alpha_j} \bar{K}_{1,j}(t, x), \quad j = 0, 1, \ldots, n-1.
\]
The systems \( \{w_j(t, x)\}_{j=0}^{n-1} \) and \( \{\bar{w}_j(x, t)\}_{j=0}^{n-1} \) of functions with fixed \( x \in R \) are the fundamental systems of the solutions of the equation

\[
D_\alpha^n w(t) = 0
\]

on the intervals \((0, x)\) and \((x, +\infty)\), \(x > 0\), respectively.

Next result concerns the behavior of functions in a neighborhood of the point \( t = 0 \) in the case of weak degeneration.

**Theorem 3.2.2** Let \( I = (0, 1) \) and

\[
\gamma_\text{max}^\alpha < 1 - \frac{1}{p}.
\]

Then \( \forall f \in W^n_{p, \alpha} \) there exists a unique generalized polynomial based on the system \( \{w_j(t, 1) = w_j(t)\}_{j=0}^{n-1} \) defined by

\[
P_n(t; f; \alpha) = \sum_{j=0}^{n-1} a_j(f)w_j(t),
\]

where the coefficients are recursively defined by the formulas

\[
a_{n-1}(f) = (-1)^{n-1} \lim_{t \to 0} D_\alpha^{n-1} f(t),
\]

\[
a_i(f) = (-1)^i \lim_{t \to 0} D_\alpha^i \left[ f(t) - \sum_{j=i+1}^{n-1} a_j(f)w_j(t) \right], \quad i = 0, 1, \ldots, n - 2,
\]

and for all \( x \in (0, 1) \) the following estimates hold:

\[
\|D_\alpha^i(f - P_n)\|_{p, \gamma_\text{max}^\alpha - 1, (0,x)} \leq c_1\|D_\alpha^n f\|_{p, (0,x)}, \quad i = 0, 1, \ldots, n - 1,
\]

\[
\sup_{0 \leq t \leq x} |t^{-(1-1/p-\gamma_\text{max}^\alpha)} D_\alpha^i[f(t) - P_n(t; f; \alpha)]| \leq c_1\|D_\alpha^n f\|_{p, (0,x)}, \quad i = 0, 1, \ldots, n - 1.
\]

Moreover, for \( i = 0, 1, \ldots, n - 1 \) the following representation formula holds

\[
D_\alpha^i f(t) = \sum_{j=i}^{n-1} a_j(f)D_\alpha^i w_j(t) + \int_0^t s^{-\alpha n} D_\alpha^n f(s) \cdot D_\alpha^i \bar{w}_{n-1}(s, t)ds.
\]

Theorem 3.2.2 shows that in the case of weak degeneration each function \( f \in W^n_{p, \alpha} \) is stabilized to the unique polynomial \( P_n(t; f; \alpha) \) when \( t \to 0 \), i.e.

\[
\lim_{t \to 0} t^{-(1-1/p-\gamma_\text{max}^\alpha)} D_\alpha^i \left[ f(t) - \sum_{j=0}^{n-1} a_j(f)w_j(t) \right] = 0, \quad i = 0, 1, \ldots, n - 1.
\]

Moreover, for this case in [29] the following result was proved:
Theorem 3.2.3 Let $1 < p < \infty$ and $t_0 > 0$. If, for all $f \in W^\gamma_{p,\alpha}$, there exist numbers $a_i = a_i(t, f)$, $i = 0, 1, \ldots, n - 1$, such that
\[
\lim_{t \to t_0} D_{\alpha}^i(f(t) - \tilde{P}_n(t, f, \alpha)) = 0, \quad i = 0, 1, \ldots, n - 1,
\]
where $\tilde{P}_n(t, f, \alpha) = \sum_{i=0}^{n-1} (-1)^i a_i \omega_i(t, t_0)$, then the finite limits
\[
\lim_{t \to t_0} B_{\alpha}^i f(t) = B_{\alpha}^i f(0),
\]
where $B_{\alpha}^i f(t) = \sum_{j=i}^{n-1} K_{i+1,j}(t, t_0) D_{\alpha}^j f(t)$, exist for $i = 0, 1, \ldots, n - 1$ and, moreover, $B_{\alpha}^i f(0) = a_i$, $i = 0, 1, \ldots, n - 1$.

But in the case of strong degeneration statements of these results do not hold, that is there exists not, in general, a unique polynomial, which $f \in W^\gamma_{p,\alpha}$ is stabilized when $t \to 0$.

Our next aim is to derive alternative (equivalent) descriptions of the norm of the space $W^\gamma_{p,\alpha}$, in different cases of degenerations.

Theorem 3.2.4 Let $I = (0, 1)$. If $\gamma_{\alpha} < 1 - \frac{1}{p}$, then for arbitrary $\delta$, which satisfies the condition $\delta > 1 - \frac{1}{p} - \gamma_{\alpha}^0$, the functional
\[
\|f\|_{W^\gamma_{p,\alpha}}^{(1)} = \|D_{\alpha}^0 f\|_p + \sum_{i=0}^{n-1} \|D_{\alpha}^i f\|_{p, \gamma_{\alpha}^0 - 1 + \delta}
\]
is equivalent to the norm (3.2.2) of the space $W^\gamma_{p,\alpha}$.

Theorem 3.2.5 Let $I = (0, 1)$, In the normed space $W^\gamma_{p,\alpha}$, if $\gamma_{\alpha}^0 > 1 - \frac{1}{p}$, then the functional
\[
\|f\|_{W^\gamma_{p,\alpha}}^{(2)} = \sum_{i=0}^{n-1} \|D_{\alpha}^i f\|_{p, \gamma_{\alpha}^0 - 1} + \|D_{\alpha}^n f\|_p
\]
is equivalent to the norm (3.2.2).

Theorem 3.2.6 Let $I = (0, 1)$. In the normed space $W^\gamma_{p,\alpha}$, $\gamma_{\alpha}^0 < 1 - \frac{1}{p}$, the functional
\[
\|f\|_{W^\gamma_{p,\alpha}}^{(3)} = \sum_{i=0}^{n-1} \|D_{\alpha}^i f\|_{p, \gamma_{\alpha}^0 - 1} + \sum_{i=0}^{n_1-1} \|D_{\alpha}^i f\|_{p, \gamma_{\alpha}^{n_1-1 + \delta}} + \|D_{\alpha}^n f\|_p,
\]
where $n_1 = \max\{k = i + 1: 0 \leq i \leq n - 1, \gamma_{\alpha}^i < 1 - \frac{1}{p}\}$, $\delta > 1 - \frac{1}{p} - \gamma_{\alpha}^0$, is equivalent to the norm (3.2.2).
Chapter 4
Boundedness of the Embedding between Spaces with Multiweighted Derivatives

For $\beta = (\beta_0, \beta_1, \ldots, \beta_m)$, $\beta_i \in R$, $i = 0, 1, \ldots, m$, analogously to the space $W_{p, \beta}^m(I)$ we define the space $W_{q, \beta}^m(I)$.

In this Chapter we will investigate the question concerning continuous embedding of multiweighted spaces $W_{p, \beta}^n(I) \hookrightarrow W_{q, \beta}^m(I)$, where $1 \leq p, q < \infty$, $0 \leq m < n$ and $I = (0, 1)$ or $I = (1, +\infty)$. With help of this theory it is possible to solve the question about characterization of the behavior of functions at singular points in the strong degeneration case. The main results in this Chapter are proved with help of Hardy type inequalities. This Chapter consists of three Sections. In Sections 4.1 and 4.2 the embedding $W_{p, \alpha}^n(I) \hookrightarrow W_{q, \beta}^m(I)$ is considered when $I = (0, 1)$. More exactly, in Section 4.1 embedding theorems are established when $p \leq q$, and in Section 4.2 when $q < p$. In Section 4.3 we rewrite the corresponding embedding theorems for the space $W_{p, \alpha}^n(1, +\infty)$.

For the reader's convenience we recall the following definition (see [66]):

**Definition 4.0.1** The normed space $X$ is continuously embedded into the normed space $Y$ and written $X \hookrightarrow Y$, if

i) $X \subset Y$,

ii) there exists a constant $c > 0$, such that, for every $x \in X$, the following inequality holds:

$$\|x\|_Y \leq c\|x\|_X.$$  

The best constant $c > 0$ in this inequality is called the embedding constant.
4.1 Embedding of multiweighted spaces when $1 < p \leq q < \infty$

For the proofs of the main results below we use the following lemma, which is also of independent interest (see [27]):

**Lemma 4.1.1** Let the function $f: (0, 1) \rightarrow R$ have weak derivatives up to order $n$. Then

$$D^k_{\beta}f(t) = \sum_{i=0}^{k} c_{k,i,t^\gamma_0 - \gamma_\beta + \beta_0 - \alpha_0 + \gamma_\alpha - \gamma_0^\beta} D^i_{\alpha}f(t), \quad k = 0, \ldots, m,$$  \hspace{1cm} \text{(4.1.1)}

where $c_{k,k} = 1$, $k = 0, 1, \ldots, m$, and the coefficients $c_{k,i}$, $i = 0, 1, \ldots, k - 1$, $k = 0, 1, \ldots, m$, are defined by the recurrent formula

$$c_{k,0} = c_{k-1,0} (\gamma_0^\beta - \gamma_\beta_{k-1} + \beta_0 - \alpha_0),$$

$$c_{k,i} = c_{k-1,i-1} + c_{k-1,i} (\gamma_0^\beta - \gamma_\beta_0 + \beta_0 - \alpha_0 + \gamma_\alpha - \gamma_\beta_{k-1}), \quad i = 1, 2, \ldots, k - 1.$$

1. The case of strong degeneration, i.e. $\gamma_{\min}^\alpha > 1 - \frac{1}{p}$.

**Theorem 4.1.2** Let $I = (0, 1)$, $1 < p \leq q < \infty$, $0 \leq m < n$, $\gamma_{\min}^\alpha > 1 - \frac{1}{p}$.

Then the continuous embedding (1.3.4) holds if and only if

$$\gamma_\alpha^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0 \geq \frac{1}{p} - \frac{1}{q}.$$  \hspace{1cm} \text{(4.1.2)}

**Proof. Sufficiency.** Assume that (4.1.2) holds. First we note that the norm of the space $W_{q,\beta}^m$ has the form

$$\|f\|_{W_{q,\beta}^m} = \|D_{\beta}^m f\|_q + \sum_{i=0}^{m-1} |D_{\beta}^i f(1)|,$$  \hspace{1cm} \text{(4.1.3)}

and, by Definition 4.0.1, the continuous embedding between the spaces (1.3.4) holds if there exists a constant $c > 0$ such that, for each $f \in W_{p,\alpha}^n$, the following inequality holds:

$$\|f\|_{W_{q,\beta}^m} \leq c \|f\|_{W_{p,\alpha}^n}.$$  \hspace{1cm} \text{(4.1.4)}

Hence, in view of (4.1.3) and (4.1.4), the continuous embedding (1.3.4) is equivalent to that the following two inequalities hold:

$$\|D_{\beta}^m f\|_q \leq c_1 \|f\|_{W_{p,\alpha}^n}, \quad \forall f \in W_{p,\alpha}^n,$$  \hspace{1cm} \text{(4.1.5)}

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and

\[ \sum_{i=0}^{m-1} |D_i^f(1)| \leq c_2 \| f \|_{W_{p,\alpha}^n}, \quad \forall f \in W_{p,\alpha}^n, \quad (4.1.6) \]

with constants \( c_1, c_2 > 0 \), which do not depend on \( f \).

Proof of (4.1.5): We assume that \( k = m \) in (4.1.1). Since \( \gamma_i^\beta = 1 \), due to (3.2.3); we find that

\[ D_m^\beta f(t) = \sum_{i=0}^{m-1} c_{m,i}(\alpha, \beta) \gamma_0^\alpha - \gamma_0^\alpha + \beta_0 - \alpha_0 + \gamma_i^\alpha - 1 D_i^\beta f(t). \quad (4.1.7) \]

Now by taking the \( q \)-th norm and using the Minkowski inequality, from (4.1.7) we get that

\[
\| D_m^\beta f \|_q \leq c_3 \sum_{i=0}^{m-1} \left( \int_0^1 |t^{\gamma_0^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0 + \gamma_i^\alpha - 1} D_i^\beta f(t)|^q dt \right)^{\frac{1}{q}} \]

\[ \leq c_3 \left[ \sum_{i=0}^{m-1} \left( \int_0^1 |t^{\gamma_0^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0 + \gamma_i^\alpha - 1} |D_i^\beta f(t) - D_i^\beta f(1)|^q dt \right)^{\frac{1}{q}} + \right. \]

\[ \left. + \sum_{i=0}^{m-1} \left( \int_0^1 |t^{\gamma_0^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0 + \gamma_i^\alpha - 1} D_i^\beta f(1)|^q dt \right)^{\frac{1}{q}} \right], \quad (4.1.8) \]

where \( c_3 = \max_{0 \leq i \leq m} |c_{m,i}| \).

According to (4.1.2) we have that

\[ \gamma_i^\alpha - 1 + \gamma_\alpha^0 - \gamma_\alpha^0 + \beta_0 - \alpha_0 \geq \frac{1}{p} - \frac{1}{q} + \gamma_i^\alpha - 1, \quad (4.1.9) \]

and, since \( \gamma_\alpha^\alpha > 1 - \frac{1}{p} \),

\[ \gamma_0^\beta - \gamma_\alpha^0 + \beta_0 - \alpha_0 + \gamma_i^\alpha - 1 > - \frac{1}{q}, \quad i = 0, 1, \ldots, n - 1. \quad (4.1.10) \]

Consequently,

\[ \int_0^1 |t^{\gamma_0^\beta - \gamma_\alpha^0 + \beta_0 - \alpha_0 + \gamma_i^\alpha - 1}|^q dt \leq c_3^*, \quad i = 0, 1, \ldots, n - 1, \quad (4.1.11) \]
where \( c_3 = \frac{1}{(\gamma_0^\alpha - \gamma_0^\alpha + \beta_0 - \alpha_0 + \gamma_i^\alpha - 1)q + 1} \).

Hence according to (4.1.8) and (4.1.11), we have that

\[
\|D^m_{\gamma} f\|_q \leq c_3' \left[ \sum_{i=0}^m \left( \int_0^1 |t^{\alpha - \gamma_0^\alpha + \beta_0 - \alpha_0 + \gamma_i^\alpha - 1} (D^i_{\alpha} f(t) - D^i_{\alpha} f(1))|^q dt \right)^{\frac{1}{q}} + \sum_{i=0}^m |D^i_{\alpha} f(1)| \right],
\]

where \( c_3' = \max\{c_3, c_3^\ast\} \).

In view of (3.2.4), \( \gamma_i^\alpha = \alpha_{i+1} + \gamma_i^\alpha - 1 \). Hence, from (4.1.9) it follows that

\[
\gamma_i^\alpha - 1 + \alpha_{i+1} \leq 1 - \frac{1}{p} + \frac{1}{q} + \gamma_0^\alpha - \gamma_0^\alpha + \beta_0 - \alpha_0 + \gamma_i^\alpha - 1, \quad i = 0, 1, \ldots, n - 1.
\]

By now applying the Hardy inequality (1.4.3) to (4.1.12), and, according to (4.1.10) and (4.1.13), we get that

\[
\|D^m_{\gamma} f\|_q \leq c_4 \left[ \sum_{i=0}^m H_i \left( \int_0^1 |t^{\gamma_i^\alpha - 1} D^i_{\alpha} f(t)|^p dt \right)^{\frac{1}{p}} + \sum_{i=0}^m |D^i_{\alpha} f(1)| \right]
\]

\[
\leq c_4 \left[ \sum_{i=0}^m \left( \int_0^1 |t^{\gamma_i^\alpha - 1} D^i_{\alpha} f(t)|^p dt \right)^{\frac{1}{p}} + \sum_{i=0}^m |D^i_{\alpha} f(1)| \right],
\]

where \( c_4 = c_3' \max\{1, H_i, i = 0, 1, \ldots, m\} \).

Since \( n > m \geq 0 \), from (4.1.14) it follows that

\[
\|D^m_{\gamma} f\|_q \leq c_4 \left[ \sum_{i=0}^n |D^i_{\alpha} f(1)| + \sum_{i=0}^m |D^i_{\alpha} f(1)| \right].
\]

Moreover, the norm (3.2.2) and the functional (3.2.5) are equivalent, so from (4.1.15) it follows that

\[
\|D^m_{\gamma} f\|_q \leq c_4 \left[ \|f\|_{W_p^m} + \sum_{i=0}^m |D^i_{\alpha} f(1)| + \|D^m_{\alpha} f\|_p \right] = 2c_4 \|f\|_{W_p^m} = c_1 \|f\|_{W_p^m},
\]
where \( c_1 = 2c_4 \), and (4.1.5) is proved.

Proof of (4.1.6): From (4.1.1) we obtain that

\[
D^k_\beta f(1) = \sum_{i=0}^{k} D^i_\beta f(1), \quad k = 0, 1, \ldots, m.
\]

Hence,

\[
\sum_{k=0}^{m} |D^k_\beta f(1)| \leq \sum_{k=0}^{m} \sum_{i=0}^{k} |c_{k,i}| |D^i_\alpha f(1)| = \sum_{i=0}^{m} \sum_{k=i}^{m} |c_{k,i}| |D^i_\alpha f(1)|
\]

\[
= \sum_{i=0}^{m} |D^i_\alpha f(1)| \sum_{k=i}^{m} |c_{k,i}|.
\]

Since, by the assumption of Theorem 4.1.2, \( n > m \geq 0 \), then from (4.1.16) it follows that

\[
\sum_{k=0}^{m} |D^k_\beta f(1)| \leq c_5 \sum_{i=0}^{n-1} |D^i_\alpha f(1)| \leq c_2 \|f\|_{W_p^{m,n}},
\]

where \( c_2 = \max_{0 \leq i \leq m} \sum_{k=i}^{m} |c_{k,i}| \) and also (4.1.6) is proved. The sufficiency is proved.

**Necessity.** Assume that (1.3.4) holds. Consider the function \( f_0(t) = t^{-\gamma_0 - \alpha_0 + \frac{1}{p} + \varepsilon} \), where \( \varepsilon > 0 \) and \( \frac{1}{p} + \frac{1}{\rho} = 1 \). Then, by using (3.2.1), (3.2.3) and (3.2.4), we have that

\[
D^i_\alpha f_0(t) = t^{\alpha_i} \frac{d}{dt} t^{\alpha_{i-1}} \frac{d}{dt} \cdots \frac{d}{dt} t^{\alpha_1} \frac{d}{dt} (t^{\alpha_0 - \gamma_0 - \alpha_0 + \frac{1}{p} + \varepsilon})
\]

\[
= (-\gamma_0 + \frac{1}{p} + \varepsilon) t^{\alpha_0} \frac{d}{dt} t^{\alpha_{i-1}} \frac{d}{dt} \cdots \frac{d}{dt} t^{\alpha_1} \frac{d}{dt} t^{\alpha_0 - \gamma_0 - \frac{1}{p} + \varepsilon - 1}
\]

\[
= (-\gamma_0 + \frac{1}{p} + \varepsilon) (-\gamma_1 + \frac{1}{p} + \varepsilon) t^{\alpha_1} \frac{d}{dt} t^{\alpha_{i-1}} \frac{d}{dt} \cdots \frac{d}{dt} t^{\alpha_0 - \gamma_0 + \frac{1}{p} + \varepsilon - 1}
\]

\[
= \prod_{j=0}^{i-1} (-\gamma_j + \frac{1}{p} + \varepsilon) t^{\alpha_0 - \gamma_0 - \frac{1}{p} + \varepsilon - 1} = \prod_{j=0}^{i-1} (-\gamma_j + \frac{1}{p} + \varepsilon) t^{\alpha_0 - \gamma_0 + \frac{1}{p} + \varepsilon - 1},
\]

since \( \alpha_i - \gamma_{i-1} - 1 = -\gamma_i, i = 1, 2, \ldots, n - 1, \)
If, for any \( i = 1, 2, \ldots, n-1 \), the expression \((-\gamma_i^\alpha + \frac{1}{p'} + \varepsilon)\) vanishes, then \( f_0 \in W_{p,\alpha}^m \). If such \( i \) does not exist, then from (4.1.17) we obtain that

\[
D_{\alpha}^n f_0(t) = \sum_{j=0}^{n-1} (-\gamma_j^\alpha + \frac{1}{p'} + \varepsilon) t^{-1+\frac{1}{p'}+\varepsilon}.
\]

Since \((-1 + \frac{1}{p'} + \varepsilon)p + 1 = \varepsilon p > 0\) it yields that

\[
\int_0^1 t^{(-1+\frac{1}{p'}+\varepsilon)p} dt < \infty,
\]

and, hence, \( f_0 \in W_{p,\alpha}^m \). Now, by using the fact that \( \beta_i = \gamma_i^\beta - \gamma_i^\alpha + 1, \)
\( \gamma_i^\beta_m = 1, \) and sequentially calculating the \( \beta \)-multiweighted derivatives of the function \( f_0(t) \) by (3.2.1), we find that

\[
D_{\beta}^m f_0(t) = t^{\beta_0} \frac{d}{dt} t^{\beta_m-1} \beta_m \beta_{m-1} \ldots \frac{d}{dt} t^{\beta_2} \frac{d}{dt} t^{\beta_1} \frac{d}{dt} t^{\beta_0-\gamma_0^\alpha-\alpha_0+\frac{1}{p'}+\varepsilon}
\]

\[
= (\beta_0 - \gamma_0^\alpha - \alpha_0 + \frac{1}{p'} + \varepsilon) t^{\beta_0} \frac{d}{dt} t^{\beta_m-1} \beta_m \beta_{m-1} \ldots \frac{d}{dt} t^{\beta_2} \frac{d}{dt} t^{\beta_1+\beta_0-\gamma_0^\alpha-\alpha_0+\frac{1}{p'}+\varepsilon-1}
\]

\[
= (\beta_0 - \gamma_0^\alpha - \alpha_0 + \frac{1}{p'} + \varepsilon) (\gamma_0^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0 - \gamma_0^\beta + \frac{1}{p'} + \varepsilon) t^{\beta_0} \frac{d}{dt} t^{\beta_m-1} \ldots
\]

\[
\ldots \frac{d}{dt} t^{\beta_0+\gamma_0^\beta-\gamma_0^\alpha+\beta_0-\alpha_0-\gamma_0^\beta+\frac{1}{p'}+\varepsilon-1} = \ldots
\]

\[
= \prod_{i=0}^{m-1} (\gamma_0^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0 - \gamma_i^\beta + \frac{1}{p'} + \varepsilon) t^{\beta_0+\gamma_0^\beta-\gamma_0^\alpha+\beta_0-\alpha_0-\gamma_0^\alpha-1+\frac{1}{p'}+\varepsilon-1}
\]

\[
= \prod_{i=0}^{m-1} (\gamma_0^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0 - \gamma_i^\beta + \frac{1}{p'} + \varepsilon) t^{\beta_0-\gamma_0^\alpha+\beta_0-\alpha_0-1+\frac{1}{p'}+\varepsilon}.
\]

Since we have finitely many factors in the product, then there exists \( \varepsilon_0 > 0 \) such that, for each \( \varepsilon \in (0, \varepsilon_0) \),

\[
\prod_{i=0}^{m-1} (\gamma_0^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0 - \gamma_i^\beta + \frac{1}{p'} + \varepsilon) \neq 0.
\]

Consequently, since \( f_0 \in W_{q,\beta}^m \) it is necessary that

\[
\int_0^1 t^{(\gamma_0^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0 + \frac{1}{p'} + \varepsilon)q} dt < \infty,
\]

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and this, in its turn, is equivalent to that

$$(\gamma_0^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0 - 1 + \frac{1}{p'}) + \varepsilon q + 1 > 0.$$  

Moreover, since $\varepsilon \in (0, \varepsilon_0)$ is arbitrary, it follows that $\gamma_0^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0 \geq 1 - \frac{1}{p'} - \frac{1}{q}$.

The proof is complete.

**Theorem 4.1.3** Let $I = (0, 1)$, $1 \leq p < \infty$, $n \geq 1$ and $\gamma_{\min} > 1 - \frac{1}{p}$. Then the continuous embedding $W^{n, \alpha}(I) \hookrightarrow W^{n, \beta}(I)$ holds if and only if $\gamma_0^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0 \geq 0$.

**Proof.** The necessity follows from the corresponding part of the proof of Theorem 4.1.2 when $p = q$. In fact, during this proof the condition $m < n$ was not used anywhere.

**Sufficiency.** From (3.2.1) when $k = n$ we obtain that

$$D^n_\beta f(t) = \sum_{i=0}^n c_{n,i}(\alpha, \beta) t^{\beta-i} \gamma_0^\alpha + \beta_0 - \alpha_0 + \gamma_i^\alpha - 1 D^i_\alpha f(t).$$

We take the $p$th norm of both parts of this identity and use the Minkowski inequality to find that

$$\|D^n_\beta f\|_p \leq c_3 \left[ \left( \int_0^1 |t^{\beta-i} \gamma_0^\alpha + \beta_0 - \alpha_0 + \gamma_i^\alpha - 1 D^i_\alpha f(t)|^p dt \right)^{\frac{1}{p}} \right]$$

$$+ \sum_{i=0}^{n-1} \left( \int_0^1 |t^{\beta-i} \gamma_0^\alpha + \beta_0 - \alpha_0 + \gamma_i^\alpha - 1 D^i_\alpha f(t)|^p dt \right)^{\frac{1}{p}}.$$

where $c_3 = \max_{0 \leq i \leq n} |c_{n,i}|$.

Since, $\gamma_0^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0 \geq 0$ and $\gamma_{\min} > 1 - \frac{1}{p}$, then

$$\gamma_0^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0 + \gamma_i^\alpha > 1 - \frac{1}{p}, \quad i = 0, 1, \ldots, n.$$  

Consequently,

$$\int_0^1 |t^{\beta-i} \gamma_0^\alpha + \beta_0 - \alpha_0 + \gamma_i^\alpha - 1| dt \leq c_3^i, \quad i = 0, 1, \ldots, n - 1.$$  

(4.1.19)
where \( c_3' = \frac{1}{(\gamma_0^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0 + \gamma_1^\alpha - 1)p + 1} \).

By using (4.1.18), (4.1.19) and the Minkowski inequality we find that
\[
\|D^n_\beta f\|_p \leq c_3' \left[ \left( \int_0^1 |t^{\gamma_0^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0 + \gamma_1^\alpha - 1}(D^n_\beta f(t) - D^n_\beta f(1))|^p dt \right)^{\frac{1}{p}} + \sum_{i=0}^{n-1} \left( \int_0^1 |t^{\gamma_0^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0 + \gamma_1^\alpha - 1}(D^n_\beta f(t) - D^n_\beta f(1))|^p dt \right)^{\frac{1}{p}} \right],
\]
\[
\text{(4.1.20)}
\]

where \( c_3' = \max\{c_3, c_4'\} \).

By now using the Hardy inequality (1.4.3) when \( p = q \) to the second part of the right hand side of (4.1.20) and using the estimate \( |t^{\gamma_0^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0}| \leq 1 \) (which is valid due to the fact that \( \gamma_0^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0 \geq 0 \) and \( 0 < t < \infty \)) to the first part, we find that
\[
\|D^n_\beta f\|_p \leq c_4 \left[ \left( \int_0^1 |t^{\gamma_0^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0 + \gamma_1^\alpha - 1}(D^n_\beta f(t) - D^n_\beta f(1))|^p dt \right)^{\frac{1}{p}} + \sum_{i=0}^{n-1} |D^n_\beta f(1)| \right],
\]
where \( c_4 = c_3' \max\{1, H_i, \ i = 0, 1, \ldots, n - 1\} \). Due to the equivalence of (3.2.2) and (3.2.5) it follows an inequality of the form (4.1.5):
\[
\|D^n_\beta f\|_p \leq c_1 \|f\|_{W_{p,\alpha}^n},
\]
which together with the inequality (4.1.6) when \( m = n \) show that the embedding \( W_{p,\alpha}^n(I) \hookrightarrow W_{p,\beta}^n(I) \) holds.

The proof is complete. 

Theorem 4.1.3 implies that it is always possible continuously embed a strong degenerated space \( W_{p,\alpha}^n \) into the weak degenerated space \( W_{p,\beta}^n(I) \) even for the case when \( |\bar{\alpha}| = |\bar{\beta}| \).

Indeed, \( W_{p,\alpha}^n = L_{p,\gamma}^n \), where \( \gamma = |\bar{\alpha}| > n - \frac{1}{p} \). We choose \( \bar{\beta} \) by the following conditions: \( \beta_n < 1 - \frac{1}{p} \), \( \beta_i < n - i + 1 - \frac{1}{p} - \sum_{k=i+1}^{n} \beta_k, \ i = 1, 2, \ldots, n - 1, \) and \( \beta_0 = \gamma - \sum_{k=1}^{n} \beta_k \).
Then $\gamma^\beta_{\max} < 1 - \frac{1}{p}$, $|\hat{\beta}| = \gamma$ and according to Theorem 4.1.3 we get that

$$W^n_{p, \alpha} = L^n_{p, \gamma} \hookrightarrow W^n_{p, \beta}.$$  

Consequently, due to Theorem 3.2.2 for each $f \in W^n_{p, \alpha} = L^n_{p, \gamma}$ there exists a unique polynomial $P_n(t; f, \beta)$ such that

$$\lim_{t \to 0} t^{-\frac{\alpha}{p}} D^\beta_{\min} [f(t) - P_n(t; f; \beta)] = 0, \quad i = 0, 1, \ldots, n - 1,$$

i.e. it is possible to describe the function $f \in W^n_{p, \alpha} = L^n_{p, \gamma}$ in a neighborhood of a singular point $t = 0$ in terms of the space $W^n_{p, \beta}$.

**Corollary 4.1.4** Let $I = (0, 1)$, $1 \leq p < \infty$, $\gamma^\alpha_{\min} > 1 - \frac{1}{p}$ and $\gamma^\beta_{\min} > 1 - \frac{1}{p}$. Then the space $W^n_{p, \alpha}(I)$ coincides with the space $W^n_{p, \beta}(I)$ with respect to equivalence of norms if and only if

$$|\bar{a}| = |\hat{\beta}|.$$

(4.1.21)

**Proof.** Indeed, in view of Theorem 4.1.3, the embeddings $W^n_{p, \alpha}(I) \hookrightarrow W^n_{p, \beta}(I)$ and $W^n_{p, \beta}(I) \hookrightarrow W^n_{p, \alpha}(I)$ are valid if and only if the conditions $\gamma^\beta_{0 - \gamma^\alpha_{0}} + \beta_{0} - \alpha_{0} \geq 0$ and $\gamma^\alpha_{0} - \gamma^\beta_{0} + \alpha_{0} - \beta_{0} \geq 0$, respectively, hold. Therefore both embeddings hold simultaneously if and only if

$$\gamma^\beta_{0} - \gamma^\alpha_{0} + \beta_{0} - \alpha_{0} = 0.$$  

(4.1.22)

Since $\gamma^\beta_{i-1} = \beta_{i} + \gamma^\beta_{i} - 1$, $\gamma^\alpha_{i-1} = \alpha_{i} + \gamma^\alpha_{i} - 1$, $i = 0, 1, \ldots, n - 1$ and $\gamma^\beta_{n} = 1$, $\gamma^\alpha_{n} = 1$, then from (4.1.22) we obtain that:

$$\beta_{0} + \gamma^\beta_{0} = \alpha_{0} + \gamma^\alpha_{0} \Rightarrow \beta_{0} + \beta_{1} + \gamma^\beta_{1} = \alpha_{0} + \alpha_{1} + \gamma^\alpha_{1} \Rightarrow$$

$$\Rightarrow \beta_{0} + \beta_{1} + \beta_{2} + \ldots + \beta_{n} + \gamma^\beta_{n} = \alpha_{0} + \alpha_{1} + \alpha_{2} + \ldots + \alpha_{n} + \gamma^\alpha_{n}.$$  

We conclude that $\sum_{i=0}^{n} \beta_{i} = \sum_{i=0}^{n} \alpha_{i}$, so that (4.1.21) holds.

The proof is complete.  

Let $\beta_{n} = \gamma$, $\beta_{i} = 0$, $i = 0, 1, \ldots, n - 1$. Then, from the condition

$$\gamma^\beta_{\min} > 1 - \frac{1}{p},$$

by using (3.2.3) and (3.2.4), we obtain that

$$\gamma^\beta_{\min} = \gamma^\beta_{0} = \beta_{1} + \gamma^\beta_{1} - 1 = \gamma_{1}^\beta - 1 = \beta_{2} + \gamma^\beta_{2} - 2 = \ldots = \beta_{n-1} + \gamma^\beta_{n-1} - (n-1) =$$

$$= \gamma^\beta_{n-1} - (n-1) = \beta_{n} - (n-1) = \gamma - (n-1) > 1 - \frac{1}{p}.$$  

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Therefore, the condition $\gamma_{\min}^\beta > 1 - \frac{1}{p}$ is equivalent to the condition $\gamma > n - \frac{1}{p}$, and the space $W_{p,\beta}^n(I)$ transforms to the Kudryavtsev space $L_{p,\gamma}^n(I)$.

Hence, in view of Corollary 4.1.4, we also have the following information:

**Corollary 4.1.5** Let $I = (0, 1)$, $1 \leq p < \infty$, $\gamma_{\min}^\alpha > 1 - \frac{1}{p}$ and $\gamma > n - \frac{1}{p}$. Then the space $W_{p,\beta}^n(I)$ coincides with the space $L_{p,\gamma}^n(I)$ with respect to equivalence of norms if and only if $\sum_{i=0}^n \alpha_i = \gamma$.

Every strong degenerated space $W_{p,\beta}^n$ coincides isometrically with the strong degenerated space $L_{p,\gamma}^n(I)$.

2. **The case of weak and mixed degeneration, i.e. $\gamma_{\min}^\alpha < 1 - \frac{1}{p}$**

**Theorem 4.1.6** Let $I = (0, 1)$, $1 < p \leq q < \infty$, $0 \leq m < n$ and $\gamma_{\min}^\alpha < 1 - \frac{1}{p}$. If

$$\gamma_0^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0 > 1 - \frac{1}{q} - \gamma_{\min}^\alpha, \quad (4.1.23)$$

then (1.3.4) holds.

**Proof.** From the proof of Theorem 4.1.2 it follows that to prove the continuous embedding (1.3.4) it is enough to show that (4.1.5) holds.

We will carry out the proof for the possible two cases separately: $m + 1 \leq n_1 - 1$ and $n \geq m + 1 > n_1 - 1$, where $n_1 = \max\{k = i + 1 : 0 \leq i \leq n - 1, \gamma_i^\alpha < 1 - \frac{1}{p}\}$.

First let $m + 1 \leq n_1 - 1$. Due to equivalence of the norm (3.2.2) and the functional (3.2.6), in order to prove that (4.1.5) holds it is enough to show that

$$\|D^{m+1}_\beta f\|_q \leq c_4 \left( \sum_{i=0}^{m+1} \|D^i f\|_{p,\gamma_{i-1}^\alpha + 1 + \delta} + \sum_{i=0}^{m} |D^i f(1)| \right), \quad (4.1.24)$$

for any $\delta > 1 - \frac{1}{p} - \gamma_{\min}^\alpha$.

In fact, our assumptions imply that

$$\gamma_0^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0 + \frac{1}{q} - \frac{1}{p} > 1 - \frac{1}{p} - \gamma_{\min}^\alpha.$$
We now choose a number $\delta$ such that
\[
\gamma_0^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0 + \frac{1}{q} - \frac{1}{p} \geq \delta > 1 - \frac{1}{p} - \gamma_{\min}^\alpha. \tag{4.1.25}
\]
From the first part of inequality (4.1.25) and (3.2.4) we have that
\[
\gamma_{i+1}^\alpha - 1 + \alpha_{i+1} + \delta \leq 1 - \frac{1}{p} + \frac{1}{q} + (\gamma_0^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0 + \gamma_i^\alpha - 1), \quad i = 0, 1, \ldots, n - 1. \tag{4.1.26}
\]
Moreover, since
\[
\gamma_0^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0 + \gamma_{\min}^\alpha - 1 > -\frac{1}{q}, \tag{4.1.27}
\]
then from (4.1.8) and the Minkowski inequality it follows that
\[
\|D_{\beta}^m f\|_q \leq c_3 \left[ \sum_{i=0}^{m} \left( \int_0^1 |t^{\gamma_0^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0 + \gamma_0^\alpha - 1} (D_{\alpha}^i f(t) - D_{\alpha}^i f(1))|^{\frac{1}{q}} dt \right)^{\frac{1}{q}} \right] +
\]
\[
\quad \quad \quad + \sum_{i=0}^{m} |D_{\alpha}^i f(1)|. \tag{4.1.28}
\]
By now using the Hardy inequality (1.4.3) in (4.1.28) and due to (4.1.27) and (4.1.26), we obtain that
\[
\|D_{\beta}^m f\|_q \leq c_3' \left[ \sum_{i=0}^{m} H_i \left( \int_0^1 |t^{\gamma_0^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0 + \gamma_0^\alpha - 1} \frac{d}{dt} D_{\alpha}^i f(t)|^{\frac{1}{p}} dt \right)^{\frac{1}{p}} + \sum_{i=0}^{m} |D_{\alpha}^i f(1)| \right]
\]
\[
\leq c_4 \left[ \sum_{i=0}^{m} \left( \int_0^1 |D_{\alpha}^i f(t)|^{\frac{1}{p}} dt \right)^{\frac{1}{p}} + \sum_{i=0}^{m} |D_{\alpha}^i f(1)| \right]
\]
\[
= c_4 \left[ \sum_{i=1}^{m+1} \|D_{\alpha}^i f\|_{p, \gamma_0^\alpha - 1 + \delta} + \sum_{i=0}^{m} |D_{\alpha}^i f(1)| \right],
\]
where $c_4 = c_3' \max\{1, H_i, i = 0, 1, \ldots, m\}$ and (4.1.24) is proved.
Now we consider the case $n \geq m + 1 > n_1 - 1$. We rewrite the inequality (4.1.28) in the following form:

$$
||D^m f||_q \leq c' \left[ \sum_{i=n_1-1}^m \left( \int_0^1 |t^{\gamma_0^+ - \gamma_0^+ + \gamma_0^+ - 1}(D^i_\alpha f(t) - D^i_\alpha f(1))|^q dt \right)^{1/q} \right]
$$

$$
+ \sum_{i=0}^{n_1-2} \left( \int_0^1 |t^{\gamma_0^+ - \gamma_0^+ + \gamma_0^+ - 1}(D^i_\alpha f(t) - D^i_\alpha f(1))|^q dt \right)^{1/q} + \sum_{i=0}^m |D^i_\alpha f(1)|
$$

where the sum $\sum_{i=0}^{n_1-2}$ is considered as zero, if $n_1 < 2$.

By using the Hardy inequality (1.4.3) to the first and to the second sum of the right hand side of (4.1.29), in view of (4.1.10) and (4.1.13), (4.1.26) and (4.1.27), respectively, we have that

$$
||D^m f||_q \leq c_3 \left[ \sum_{i=n_1-1}^m H_i \left( \int_0^1 |t^{\gamma_0^+ + \alpha_{i+1} - 1} \frac{d}{dt} D^i_\alpha f(t)|^p dt \right)^{1/p} \right]
$$

$$
+ \sum_{i=0}^{n_1-2} H_i \left( \int_0^1 |t^{\gamma_0^+ + \alpha_{i+1} - 1 + \delta} \frac{d}{dt} D^i_\alpha f(t)|^p dt \right)^{1/p} + \sum_{i=0}^m |D^i_\alpha f(1)|
$$

$$
\leq c_4 \left[ \sum_{i=n_1-1}^m \left( \int_0^1 |t^{\gamma_0^+ + \alpha_{i+1} - 1} D^i_\alpha f(t)|^p dt \right)^{1/p} + \sum_{i=0}^{n_1-2} \left( \int_0^1 |t^{\gamma_0^+ + \alpha_{i+1} - 1 + \delta} D^i_\alpha f(t)|^p dt \right)^{1/p} \right]
$$

$$
+ \sum_{i=0}^m |D^i_\alpha f(1)| = c_4 \left[ \sum_{i=n_1}^{m+1} ||D^i_\alpha f||_{p, \gamma_0^+ - 1} + \sum_{i=1}^{n_1-1} ||D^i_\alpha f||_{p, \gamma_0^+ + \delta} + \sum_{i=0}^m |D^i_\alpha f(1)| \right]
$$

$$
\leq c_4 \left[ \sum_{i=n_1}^{m+1} ||D^i_\alpha f||_{p, \gamma_0^+ - 1} + \sum_{i=0}^{n_1-1} ||D^i_\alpha f||_{p, \gamma_0^+ + \delta} + \sum_{i=0}^m |D^i_\alpha f(1)| \right].
$$

Finally, by again using the equivalence of the norm (3.2.2) and the functional (3.2.6), we obtain (4.1.5) and the proof is complete.  

Theorem 4.1.6 yields only a sufficient condition for the embedding (1.3.4), but, in general, it is impossible to weaken it, as can be seen in our next result:
Proposition 4.1.7 Let \( l = (0,1) \), \( 1 < p \leq q < \infty \), \( 0 \leq m < n \), \( \gamma_{\min}^{\alpha} < 1 - \frac{1}{p} \), \( \gamma_{\min}^{\alpha} - \gamma_{0}^{\alpha} + \beta_{0} - \alpha_{0} \neq 0 \) and \( \gamma_{\min}^{\alpha} - \gamma_{0}^{\alpha} + \beta_{0} - \alpha_{0} + \gamma_{i}^{\alpha} - \gamma_{0}^{\alpha} \neq 0 \), \( i = 1, 2, \ldots, m-1 \). Then the continuous embedding (1.3.4) yields that \( \gamma_{0}^{\beta} - \gamma_{0}^{\alpha} + \beta_{0} - \alpha_{0} > 1 - \frac{1}{q} - \gamma_{\min}^{\alpha} \).

Proof. We consider the function \( f_{0}(t) = t^{\gamma_{\min}^{\alpha} - \gamma_{0}^{\alpha} - \alpha_{0}} \) and note that, by using (4.1.1), (3.2.3) and (3.2.4), we have that

\[
D_{\alpha}^{i} f_{0}(t) = t^{\alpha_{0}} \frac{d}{dt} t^{\alpha_{1}} \frac{d}{dt} \ldots t^{\alpha_{2}} \frac{d}{dt} t^{\gamma_{\min}^{\alpha} - \gamma_{0}^{\alpha} - \alpha_{0}} \\
= (\gamma_{\min}^{\alpha} - \gamma_{0}^{\alpha}) t^{\alpha_{0}} \frac{d}{dt} t^{\alpha_{1}} \frac{d}{dt} \ldots t^{\alpha_{2}} \frac{d}{dt} t^{\gamma_{\min}^{\alpha} - \gamma_{0}^{\alpha} - 1} \\
= (\gamma_{\min}^{\alpha} - \gamma_{0}^{\alpha}) t^{\alpha_{0}} \frac{d}{dt} t^{\alpha_{1}} \frac{d}{dt} \ldots \frac{d}{dt} t^{\gamma_{\min}^{\alpha} - \gamma_{1}^{\alpha}} \\
= (\gamma_{\min}^{\alpha} - \gamma_{0}^{\alpha}) (\gamma_{\min}^{\alpha} - \gamma_{1}^{\alpha}) \frac{d}{dt} t^{\gamma_{\min}^{\alpha} - \gamma_{2}^{\alpha}} = \ldots \\
= \prod_{j=0}^{i-1} (\gamma_{\min}^{\alpha} - \gamma_{j}^{\alpha}) t^{\gamma_{\min}^{\alpha} - \gamma_{i}^{\alpha}}, \; i = 1, 2, \ldots, n. 
\]

(4.1.30)

Let \( \gamma_{i}^{\alpha} = \gamma_{\min}^{\alpha} \), \( 0 \leq i \leq n - 1 \). Even if \( D_{\alpha}^{i} f_{0}(t) \neq 0 \), \( \forall i \leq i_{0} \), from (4.1.30) it follows that \( D_{\alpha}^{i+1} f_{0}(t) = 0 \), so that \( D_{\alpha}^{n} f_{0}(t) = 0 \). Therefore \( f_{0} \in W_{p,\alpha}^{n} \).

Hence, due to the embedding (1.3.4) the function \( f_{0} \in W_{q,\beta}^{m} \).

Moreover, by sequentially calculating the \( \beta \)-multiweighted derivatives of the function \( f_{0}(t) \), we obtain that

\[
D_{\beta}^{m} f_{0}(t) = t^{\beta_{0}} \frac{d}{dt} t^{\beta_{1}} \frac{d}{dt} \ldots t^{\beta_{m}} \frac{d}{dt} t^{\gamma_{\min}^{\alpha} - \gamma_{0}^{\alpha} - \alpha_{0}} \\
= (\beta_{0} - \alpha_{0} + \gamma_{\min}^{\alpha} - \gamma_{0}^{\alpha}) t^{\beta_{0}} \frac{d}{dt} t^{\beta_{1}} \frac{d}{dt} \ldots t^{\beta_{2}} \frac{d}{dt} t^{\gamma_{\min}^{\alpha} - \gamma_{0}^{\alpha} - 1} \\
= (\beta_{0} - \alpha_{0} + \gamma_{\min}^{\alpha} - \gamma_{0}^{\alpha}) t^{\beta_{0}} \frac{d}{dt} t^{\beta_{1}} \frac{d}{dt} \ldots \frac{d}{dt} t^{\gamma_{\min}^{\alpha} - \gamma_{1}^{\alpha}} \\
= (\beta_{0} - \alpha_{0} + \gamma_{\min}^{\alpha} - \gamma_{0}^{\alpha}) (\gamma_{\min}^{\alpha} - \gamma_{0}^{\alpha} + \gamma_{0}^{\beta} + \beta_{0} - \alpha_{0} + \gamma_{1}^{\beta}) t^{\beta_{2}} \frac{d}{dt} t^{\gamma_{\min}^{\alpha} - \gamma_{2}^{\alpha} - \gamma_{0}^{\alpha} - \alpha_{0} - \gamma_{1}^{\beta} - 1} = \ldots \\
= (\beta_{0} - \alpha_{0} + \gamma_{\min}^{\alpha} - \gamma_{0}^{\alpha}) \prod_{i=1}^{m-1} (\gamma_{\min}^{\alpha} - \gamma_{0}^{\alpha} + \gamma_{0}^{\beta} + \beta_{0} - \alpha_{0} + \gamma_{i}^{\beta}) t^{\gamma_{\min}^{\alpha} - \gamma_{0}^{\alpha} - \gamma_{0}^{\beta} - \alpha_{0} - \gamma_{1}^{\beta} - 1}.
\]

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By now using the assumptions, we find that $D^\alpha_{\beta} f_0(t) \neq 0$ for $0 < t \leq 1$. Consequently, the condition $f_0 \in W^m_{q, \beta}$ requires that the following condition holds
\[
\int_0^1 t^{(\gamma^\alpha_{\min} - 1 + \beta_0 - \alpha_0 + \gamma^\beta_0 - \gamma^\alpha_0)q} dt < \infty,
\]
i.e., that $\gamma^\beta_0 - \gamma^\alpha_0 + \beta_0 - \alpha_0 > 1 - \frac{1}{q} - \gamma^\alpha_{\min}$.

The proof is complete.  

\textbf{Remark 4.1.8} Note that the conditions of Theorem 4.1.6, i.e. that $\gamma^\alpha_{\min} < 1 - \frac{1}{p}$, $\gamma^\beta_0 - \gamma^\alpha_0 + \beta_0 - \alpha_0 > 1 - \frac{1}{q} - \gamma^\alpha_{\min}$, imply that $\gamma^\beta_0 - \gamma^\alpha_0 + \beta_0 - \alpha_0 > \frac{1}{p} - \frac{1}{q}$, which, in fact, is necessary for the continuous embedding (1.3.4). However, this condition, in general, is not sufficient for the embedding (1.3.4) when $\gamma^\alpha_{\min} < 1 - \frac{1}{p}$. Indeed, since $1 - \frac{1}{q} - \gamma^\alpha_{\min} > \frac{1}{p} - \frac{1}{q}$, then choosing the numbers $\gamma^\alpha_i$, $i = 0, 1, \ldots, n - 1$, $\gamma^\beta_0$, $\beta_0$ and $\alpha_0$ such that $\gamma^\alpha_{\min} < 1 - \frac{1}{p}$,

$1 - \frac{1}{q} - \gamma^\alpha_{\min} > \gamma^\beta_0 - \gamma^\alpha_0 + \beta_0 - \alpha_0 > \frac{1}{p} - \frac{1}{q}$, and the assumptions in Proposition 4.1.7 holds we see from the proof of this Proposition that the function $f_0(\cdot)$ belongs to $W^m_{p, \alpha}$, but it does not belong to $W^m_{q, \beta}$.

\section{4.2 Embedding of multiweighted spaces when $1 < q < p < \infty$}

1. The case of strong degeneration, i.e. $\gamma^\alpha_{\min} > 1 - \frac{1}{p}$.

\textbf{Theorem 4.2.1} Let $I = (0, 1)$, $1 < q < p < \infty$, $0 \leq m < n$ and $\gamma^\alpha_{\min} > 1 - \frac{1}{p}$. If

\[
\gamma^\beta_0 - \gamma^\alpha_0 + \beta_0 - \alpha_0 > \frac{1}{p} - \frac{1}{q},
\]

then the continuous embedding (1.3.4) holds. On the other hand, if the continuous embedding (1.3.4) holds, then $\gamma^\beta_0 - \gamma^\alpha_0 + \beta_0 - \alpha_0 \geq \frac{1}{p} - \frac{1}{q}$.

\textbf{Proof}. The proof of the first statement it literally the same as that of the sufficiency part of Theorem 4.1.2. It is sufficient to prove the inequalities (4.1.5) and (4.2.5) and the only difference is that in this case we must use the Hardy inequality (1.4.4) instead of (1.4.3).
Moreover, the second part of Theorem 4.2.1 can be proved analogously as the necessity part of Theorem 4.1.2 was proved. In fact, during the proof the condition \(1 < p \leq q < \infty\) was not used and, hence, the arguments hold for the case \(1 \leq q < p < \infty\) as well.

The proof is complete.

\[\]

2. The case of weak and mixed degeneration, i.e. \(\gamma_{\text{min}}^\alpha < 1 - \frac{1}{p}\).

**Theorem 4.2.2** Let \(I = (0,1)\), \(1 \leq q < p < \infty\), \(0 \leq m < n\) and \(\gamma_{\text{min}}^\alpha < 1 - \frac{1}{p}\). If

\[
\gamma_0^\alpha - \gamma_0^\alpha + \beta_0 - \alpha_0 > 1 - \frac{1}{q} - \gamma_{\text{min}}^\alpha, 
\] (4.2.2)

then the continuous embedding (1.3.4) holds.

**Proof.** To prove the continuous embedding (1.3.4) it is sufficient to prove that the inequality (4.1.5) holds i.e. that

\[
\|D_{\beta}^m f\|_q \leq c_1 \|f\|_{W_{p,\alpha}^n}, \quad \forall f \in W_{p,\alpha}^n.
\]

We consider two possible cases: \(m + 1 \leq n_1 - 1\) and \(n_1 - 1 < m + 1 \leq n\), where \(n_1 = \max\{k = i + 1 : 0 \leq i \leq n - 1, \gamma_i^\alpha < 1 - \frac{1}{p}\}\).

Let \(m + 1 \leq n_1 - 1\). Analogously as in the proof of Theorem 4.1.4 we note that, due to the equivalence of the norm (3.2.2) and the functional (3.2.6), to prove inequality (4.1.5) it is enough to prove that

\[
\|D_{\beta}^m f\|_q \leq c_k \left( \sum_{i=0}^{m+1} \|D_{\alpha}^i f\|_{p,\gamma^{\alpha-1+i}+\delta} + \sum_{i=0}^{m} |D_{\alpha}^i f(1)| \right)
\]

for any \(\delta > 1 - \frac{1}{p} - \gamma_{\text{min}}^\alpha\).

The assumptions in Theorem 4.2.2 imply that we can choose a number \(\delta\) such that

\[
\gamma_0^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0 + \frac{1}{q} - \frac{1}{p} > \delta > 1 - \frac{1}{p} - \gamma_{\text{min}}^\alpha. 
\] (4.2.3)

By using (3.2.4) and the first part of (4.2.3), we find that

\[
\gamma_i^\alpha + \delta < 1 - \frac{1}{p} + \frac{1}{q} + (\gamma_0^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0 + \gamma_i^\alpha - 1), \quad i = 0, 1, \ldots, n - 1,
\]

i.e. that

\[
\gamma_{i+1}^\alpha + \delta < 1 - \frac{1}{p} + \frac{1}{q} + (\gamma_0^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0 + \gamma_i^\alpha - 1), \quad i = 0, 1, \ldots, n - 1.
\] (4.2.4)
Moreover, since
\[ \gamma - \gamma_0 + \beta_0 - \alpha_0 + \gamma_{\min} - 1 > -\frac{1}{q}, \]
then from (4.1.7) it follows that
\[
\|D^m f\|_q \leq c_3 \left[ \sum_{i=0}^{m} \left( \int_0^1 |t^\alpha f(t) - D^i f(1)|^q dt \right)^{\frac{1}{q}} \right] + \\
+ \sum_{i=0}^{m} |D^i f(1)|. \tag{4.2.5}
\]

Hence, by using the Hardy inequality (1.4.4) in (4.2.5), and in view of (4.1.27) and (4.1.26), we obtain that
\[
\|D^m f\|_q \leq c_3 \left[ \sum_{i=0}^{m} H_i \left( \int_0^1 |t^{\gamma_i - 1 + \alpha_i + \delta} D^i f(t)|^q dt \right)^{\frac{1}{q}} \right] + \\
+ \sum_{i=0}^{m} |D^i f(1)| \]

\[
= c_4 \left[ \sum_{i=1}^{m+1} \|D^i f\|_{p, \gamma_i - 1 + \delta} + \sum_{i=0}^{m} |D^i f(1)| \right],
\]

where \( c_4 = c_3 \max\{1, H_i, i = 0, 1, \ldots, m\} \).

Therefore
\[
\|D^m f\|_q \leq c_4 \left[ \sum_{i=0}^{m+1} \|D^i f\|_{p, \gamma_i - 1 + \delta} + \sum_{i=0}^{m} |D^i f(1)| \right].
\]

Now we consider the case \( n_1 - 1 < m + 1 \leq n \). It is possible to write inequality (4.2.5) in the following form:
\[
\|D^m f\|_q \leq c_3 \left[ \sum_{i=0}^{n_1-2} \left( \int_0^1 |t^\alpha f(t) - D^i f(1)|^q dt \right)^{\frac{1}{q}} \right]
\]

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+ \sum_{i=n_1-1}^{n_1-2} \left( \int_0^1 |t^\alpha_i + \beta_i \alpha_i \gamma_i^{\alpha_i - 1} (D^i_{\alpha_i} f(t) - D^i_{\alpha_i} f(1))|^{\frac{1}{q}} dt \right)^{\frac{1}{q}} + \sum_{i=0}^{m} |D^i_{\alpha_i} f(1)|
\right],

(4.2.6)

where the sum \( \sum_{i=0}^{n_1-2} \) is considered to be equal to zero, if \( n_1 < 2 \).

Using the Hardy inequality (1.4.4) to the first and to the second sum of the right part of (4.2.6), and after that (4.2.3) and (4.2.4), respectively, we obtain that

\[
\|D^m_{\beta_i} f\|_q \leq c_3 \left[ \sum_{i=0}^{n_1-2} H_i \left( \int_0^1 |t^\alpha_i + \beta_i \alpha_i \gamma_i^{\alpha_i - 1} \frac{d}{dt} D^i_{\alpha_i} f(t)|^p dt \right)^{\frac{1}{p}} + \sum_{i=n_1-1}^{m} H_i \left( \int_0^1 |t^\alpha_i + \beta_i \alpha_i \gamma_i^{\alpha_i - 1} \frac{d}{dt} D^i_{\alpha_i} f(t)|^p dt \right)^{\frac{1}{p}} + \sum_{i=0}^{m} |D^i_{\alpha_i} f(1)| \right]
\]

\[
\leq c_4 \left[ \sum_{i=0}^{n_1-2} \left( \int_0^1 |t^\alpha_i + \beta_i \alpha_i \gamma_i^{\alpha_i - 1} D^i_{\alpha_i} f(t)|^p dt \right)^{\frac{1}{p}} + \sum_{i=n_1-1}^{m} \left( \int_0^1 |t^\alpha_i + \beta_i \alpha_i \gamma_i^{\alpha_i - 1} D^i_{\alpha_i} f(t)|^p dt \right)^{\frac{1}{p}} + \sum_{i=0}^{m} |D^i_{\alpha_i} f(1)| \right]
\]

\[
+ \sum_{i=0}^{m} |D^i_{\alpha_i} f(1)| = c_4 \left[ \sum_{i=1}^{n_1-1} \|D^i_{\alpha_i} f\|_{p, \gamma_i^{\alpha_i - 1} + \beta_i \alpha_i \gamma_i^{\alpha_i - 1} + \sum_{i=1}^{m} \|D^i_{\alpha_i} f\|_{p, \gamma_i^{\alpha_i - 1} + \sum_{i=0}^{m} |D^i_{\alpha_i} f(1)|} \right]
\]

\[
\leq c_4 \left[ \sum_{i=0}^{n_1-2} \|D^i_{\alpha_i} f\|_{p, \gamma_i^{\alpha_i - 1} + \beta_i \alpha_i \gamma_i^{\alpha_i - 1} + \sum_{i=1}^{m} \|D^i_{\alpha_i} f\|_{p, \gamma_i^{\alpha_i - 1} + \sum_{i=0}^{m} |D^i_{\alpha_i} f(1)|} \right].
\]

Hence, due to the equivalence of the norm (3.2.2) and the functional (3.2.6) the estimate (4.1.5) follows.

The proof is complete.

We finish this Section by giving two examples.

**Example 3.1.** Let \( \alpha_k = 1, k = 0, 1, \ldots, n-1, \alpha_n = n \) and \( \beta_k = 1, k = 0, 1, \ldots, m-1, \beta_m = m, 0 \leq m < n \). Then, due to the formula (3.2.3), we have that

\[
\gamma_k^\alpha = \alpha_n + \sum_{i=k+1}^{n-1} (\alpha_i - 1) = n, k = 0, 1, \ldots, n-1, \gamma_0^\beta = \beta_m + \sum_{i=1}^{m-1} (\beta_i - 1) = m.
\]

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Therefore,
\[ \gamma^\alpha_{\min} = \min_{0 \leq k \leq n-1} \gamma^\alpha_k = n > 1 - \frac{1}{p}. \]

Since
\[ \gamma^\beta_0 - \gamma^\alpha_0 + \beta_0 - \alpha_0 = m - n + 1 - 1 = m - n < 0, \]
in view of the results of this Chapter, we conclude that the continuous embedding (1.34) holds only when \( 1 \leq q < p < \infty \).

**Example 3.2.** Let now \( \alpha_0 = 1 - n^2, \alpha_k = 2k + 1, k = 1, 2, \ldots, n - 1, \)
\( \alpha_n = n - \frac{1}{p}; \beta_0 = -m^2, \beta_k = 2k, k = 1, 2, \ldots, m - 1, \beta_m = 2m - \frac{1}{q}, \)
\( 0 \leq m < n \). By formula (3.2.3) we see that
\[ \gamma^\alpha_k = n - \frac{1}{p} + \sum_{i=k+1}^{n-1} (2i) = n^2 - \frac{1}{p} - k(k + 1), k = 0, 1, \ldots, n - 1, \]
so that
\[ \gamma^\alpha_{\min} = \gamma^\alpha_{n-1} = n - \frac{1}{p} > 1 - \frac{1}{p}. \]

Moreover, this implies that
\[ \gamma^\beta_0 = 2m - \frac{1}{q} + \sum_{i=1}^{m-1} (2i - 1) = m^2 + 1 - \frac{1}{q}, \]
\[ \gamma^\beta_0 - \gamma^\alpha_0 + \beta_0 - \alpha_0 = m^2 + 1 - \frac{1}{q} - n^2 + \frac{1}{p} - n^2 + m^2 - 1 + n^2 = \frac{1}{p} - \frac{1}{q}. \]
Hence, in view of the theorems of this Chapter, we conclude that the continuous embedding (1.34) can hold only when \( 1 < p \leq q < \infty \).

### 4.3 The corresponding results for the spaces \( W^n_{p,\tilde{\alpha}} (1, \infty) \)

The connection between the spaces \( W^n_{p,\tilde{\alpha}} (0, 1) \) and \( W^n_{p,\tilde{\alpha}} (1, \infty) \) can be seen by making the variable transformation \( x = \frac{1}{t} \). In this way every function \( f \in W^n_{p,\tilde{\alpha}} (1, \infty) \) can be transformed to a function \( \tilde{f}(x) = f(\frac{1}{x}) \) from the space \( W^n_{p,\tilde{\alpha}} (0, 1) \), where \( \tilde{\alpha} = (\tilde{\alpha}_0, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_n), \tilde{\alpha}_n = -\alpha_n + 2 - \frac{1}{p}, \tilde{\alpha}_i = -\alpha_i + 2, \)
\[ i = 1, 2, \ldots, n - 1, \tilde{\alpha}_0 = -\alpha_0. \] Moreover,
\[
\|D^n_{\alpha}f\|_{p,(1, +\infty)} = \left(\int_1^{+\infty} |D^n_{\alpha}f(t)|^p dt\right)^{\frac{1}{p}} = \\
= \left(\int_1^{+\infty} \left|\frac{d}{dt}^{\alpha_n} \frac{d}{dt}^{\alpha_{n-1}} \cdots \frac{d}{dt}^{\alpha_1} f(t)\right|^p dt\right)^{\frac{1}{p}} = \\
= \left(\int_0^1 \left|x^{-\alpha_n} \frac{d}{dx} x^{-\alpha_{n-1}} \cdots x^{-\alpha_1} \frac{d}{dx} x^{-\alpha_0} f(x)\right|^p dx\right)^{\frac{1}{p}} = \\
= \left(\int_0^1 \left|x^{\tilde{\alpha}_n} \frac{d}{dx} x^{\tilde{\alpha}_{n-1}} \cdots x^{\tilde{\alpha}_1} \tilde{f}(x)\right|^p dx\right)^{\frac{1}{p}} = \|D^n_{\alpha}\tilde{f}\|_{p,(0,1)},
\]
and \(D^i_{\alpha}f(1) = D^i_{\alpha}f(1), i = 0, 1, \ldots, n - 1,\)

Analogously, from the space \(W^{m,q}_{\alpha,\beta}(1, +\infty)\) we can pass to the space \(W^{m,q}_{\alpha,\beta}(0,1)\),

Therefore, the embedding (1.3.4) is equivalent to the embedding:
\[
W^{n,q}_{p,\alpha}(0,1) \hookrightarrow W^{m,q}_{p,\alpha}(0,1),
\]
and all notions and statements for the space \(W^{n,q}_{p,\alpha}(0,1)\) can be rewritten for the space \(W^{n,q}_{p,\alpha}(1, +\infty)\).

Moreover,
\[
\gamma_i^\alpha = -\alpha_n + 2 - \frac{2}{p} + \sum_{k=i+1}^{n-1} (-\alpha_i + 2 - 1) = -(\alpha_n + \sum_{k=i+1}^{n-1} (\alpha_i - 1)) + 2 - \frac{2}{p} = \\
-\gamma_i^\alpha + 2 - \frac{2}{p}, i = 0, 1, \ldots, n - 1.
\]

Consequently, the condition \(\gamma_{\min}^\alpha > 1 - \frac{1}{p}\) is equivalent to the condition
\(\gamma_{\max}^\alpha < 1 - \frac{1}{p}\),
Furthermore, the condition
\[ \gamma_0^\beta - \gamma_0^\alpha + \bar{\beta}_0 - \bar{\alpha}_0 \geq \frac{1}{p} - \frac{1}{q} \]
will be changed to the condition
\[ \gamma_0^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0 \leq \frac{1}{q} - \frac{1}{p}. \]

It also yields that
\[ \gamma_0^\beta - \gamma_0^\alpha + \bar{\beta}_0 - \bar{\alpha}_0 = -\gamma_0^\beta + 2 \frac{2}{p} + \gamma_0^\alpha - 2 \frac{2}{p} - \beta_0 + \alpha_0 = -\gamma_0^\beta + \gamma_0^\alpha - \beta_0 + \alpha_0 \leq \frac{1}{p} - \frac{1}{q}. \]

The next two results follow from Theorem 4.1.2 and Theorem 4.1.3, respectively.

**Theorem 4.3.1** Let \( I = (1, +\infty) \), \( 1 < p \leq q < \infty \), \( 0 \leq m < n \) and \( \gamma_{\max}^\alpha < 1 - \frac{1}{p} \). Then the continuous embedding (1.3.4) holds if and only if
\[ \gamma_0^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0 \leq \frac{1}{q} - \frac{1}{p}. \]

In the case \( m = n \) when \( p = q \) we have the following statement:

**Theorem 4.3.2** Let \( I = (1, \infty) \), \( 1 \leq p < \infty \) and \( \gamma_{\max}^\alpha < 1 - \frac{1}{p} \). Then the continuous embedding \( W_p^\alpha(I) \hookrightarrow W_p^\beta(I) \) yields if and only if \( \gamma_0^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0 = 0 \).

Due to the fact that the condition \( \gamma_0^\beta - \gamma_0^\alpha + \beta_0 - \alpha_0 = 0 \) is equivalent to the condition \( \sum_{i=0}^{n} \alpha_i = \sum_{i=0}^{n} \beta_i \) we obtain the following result:

**Corollary 4.3.3** Let \( I = (1, \infty) \), \( 1 \leq p < \infty \), \( \gamma_{\max}^\alpha < 1 - \frac{1}{p} \) and \( \gamma_{\max}^\beta < 1 - \frac{1}{p} \). Then the space \( W_p^\alpha(I) \) coincides with the space \( W_p^\beta(I) \) with respect to equivalence of norms if and only if
\[ |\tilde{\alpha}| = |\tilde{\beta}|. \]
Let now $\beta_n = \mu, \beta_i = 0, i = 0, 1, \ldots, n - 1$. Then the condition $\gamma^\beta_{\max} < 1 - \frac{1}{p}$ is equivalent to the condition $\mu < n - \frac{1}{p}$, and the space $W^n_{p,\beta}(I)$ will be transformed to the Kudryavtsev space $L^n_{p,\mu}(I)$. Then, Corollary 4.3.3 implies the following result:

**Corollary 4.3.4** Let $I = (1, \infty), 1 \leq p < \infty, \gamma^\alpha_{\max} < 1 - \frac{1}{p}$ and $\mu > n - \frac{1}{p}$. Then the space $W^n_{p,\alpha}(I)$ coincides with the space $L^n_{p,\mu}(I)$ with respect to equivalence of norms if and only if $\sum_{i=0}^{n-1} \alpha_i = \mu$.

After passing to the case with an infinite interval, the condition $\gamma^a_{\min} < 1 - \frac{1}{p}$ will be changed to the condition $\gamma^\alpha_{\max} > 1 - \frac{1}{p}$, and the condition $\gamma_0^\beta - \gamma_0^a + \beta_0 - \alpha_0 > 1 - \frac{1}{q} - \gamma_0^a$ will be changed to the condition $\gamma_0^\beta - \gamma_0^a + \beta_0 - \alpha_0 < \gamma^\alpha_{\max} + \frac{1}{q} - 1$. Hence, the following result follows from Theorem 4.1.6:

**Theorem 4.3.5** Let $I = (1, \infty), 1 < p \leq q < \infty, 0 \leq m < n$ and $\gamma^\alpha_{\max} > 1 - \frac{1}{p}$. If $\gamma_0^\beta - \gamma_0^a + \beta_0 - \alpha_0 < \gamma^\alpha_{\max} + \frac{1}{q} - 1$, then the continuous embedding (1.3.4) holds.

This theorem yields only a sufficient condition of the continuous embedding (1.3.4). However, in general, it is impossible to weaken it, which can be seen from our next result:

**Proposition 4.3.6** Let $I = (1, +\infty), 1 < p \leq q < \infty, 0 \leq m < n$, $\gamma^\alpha_{\max} > 1 - \frac{1}{p}, \gamma^\alpha_{\max} - \gamma_0^a + \beta_0 - \alpha_0 \neq 0$, and $\gamma^\alpha_{\max} - \gamma_0^a + \beta_0 - \alpha_0 + \gamma_0^\beta - \gamma_0^a \neq 0$, $\beta_i = 0, i = 1, 2, \ldots, m-1$. Then the continuous embedding $W^n_{p,\alpha}(I) \hookrightarrow W^m_{q,\beta}(I)$ yields that $\gamma_0^\beta - \gamma_0^a + \beta_0 - \alpha_0 < \gamma^\alpha_{\max} + \frac{1}{q} - 1$.

This result follows directly from Proposition 4.1.7 and the transformations above.
Chapter 5

Compactness of the Embedding between Spaces with Multiweighted Derivatives

The main aim of this Chapter is to establish boundedness and compactness of the embedding

\[ W^m_{p,q_j}(I) \hookrightarrow W^m_{q,j}(I) \]

when \( 1 \leq p, q < \infty, 0 \leq m < n, I = (0, 1) \) or \( I = (1, +\infty) \). This Chapter consists of four Sections. In order not disturb our proof of the main results in Sections 5.2 and 5.3 we use Section 5.1 to present some necessary notations and Lemmas. In Section 5.4 the embedding Theorems from Sections 5.2 and 5.3 for the spaces \( W^m_{p,\alpha}(0, 1) \) are rewritten to the case with the spaces \( W^m_{p,\alpha}(1, +\infty) \).

5.1 Necessary notions and preliminaries

For \( i, j = 0, 1, \ldots, n - 1 \) we define the following set of functions:

\[ K_{i,j}(t, x) \equiv K_{i,j}(t, x, \alpha) \equiv 1 \text{ when } i = j, \]

\[ K_{i,j}(t, x) \equiv K_{i,j}(t, x, \alpha) \equiv 0 \text{ when } i > j \text{ for } 0 < t \leq x. \]
By changing variables, when \( i < j \) the following properties of uniformity of the functions \( K_{i+1,j} \) can be established:

\[
K_{i+1,j}(zt, zx) = \int_{t_{i+1}}^{x} t_{i+1}^{\alpha_{i+1}-1} t_{i+2}^{\alpha_{i+2}} \ldots t_{j}^{\alpha_{j}-1} dt_{j} dt_{j-1} \ldots dt_{i+1} =
\]

\[
[t_{k} = z\tau_{k}, \ dt_{k} = zd\tau_{k}] = \int_{\tau_{i+1}}^{x} (z\tau_{i+1})^{\alpha_{i+1}} (z\tau_{i+2})^{\alpha_{i+2}} \ldots
\]

\[
\ldots \int_{\tau_{j-1}}^{x} (z\tau_{j})^{\alpha_{j}z^{-i}} d\tau_{j} d\tau_{j-1} \ldots d\tau_{i+1} = z^{k_{i+1}}(1-\alpha_{k}) K_{i+1,j}(t, x).
\]

In particular, when \( x = 1 \) and \( t = 1 \) we have that

\[
K_{i+1,j}(zt, z) = \sum_{k=1}^{j} (1-\alpha_{k}) K_{i+1,j}(t, 1),
\]

\[
K_{i+1,j}(z, zx) = \sum_{k=1}^{j} (1-\alpha_{k}) K_{i+1,j}(1, x),
\]

(5.1.1)

respectively.

For \( 0 \leq i \leq j \leq n - 1 \) we define:

\[
k_{i,j} = \min \{ k : i \leq k \leq j, \ \sum_{s=i+1}^{k} \alpha_{s} - k = \max_{\xi \leq \xi \leq \xi \leq +1} \sum_{s=i+1}^{\xi} \alpha_{s} - \xi \},
\]

and

\[
M_{i,j} = \max_{i \leq s \leq j} (j - s + 1 - \sum_{k=s+1}^{j+1} \alpha_{k}).
\]

By the definitions of \( M_{i} \), \( \gamma_{i}^{\min} \) (see Chapter 3) and (3.2.3) it can be established connection between \( M_{i} \) and \( \gamma_{i}^{\min} \), i.e. we have that

\[
M_{i} = \max_{i \leq s \leq n-1} (n - s - \sum_{k=s+1}^{n} \alpha_{k}) = - \min_{i \leq s \leq n-1} (\alpha_{n} + \sum_{k=s+1}^{n-1} (\alpha_{k} - 1)) + 1 = 1 - \gamma_{i}^{\min}
\]

(5.1.2)

For convenience, we denote \( k_{i} \equiv k_{i,n-1}, \ M_{i} = M_{i,n-1} \). Note that \( M_{i} \geq M_{i+1} \) and \( M_{0} = \max_{0 \leq i \leq n-1} M_{i} \).

Furthermore, we need upper and lower estimates for the functions \( K_{i+1,j}(t, 1) \) when \( 0 < t \leq 1 \) and \( K_{i+1,n-1}(1, t) \) when \( 1 \leq t < \infty, \ 0 \leq i \leq j \leq n - 1 \).
\( n - 1 \). In Chapter 2 we derived upper and lower estimates for the functions \( u_i(t) = t^{\alpha_i} K_{1,i}(t,1, -\alpha_i), i = 0, 1, \ldots, n - 1 \). Below we give three statements about estimates for the functions \( K_{i+1,j}(t,1) \) and \( K_{i+1,j}(1,t) \), which follow from these results. Moreover, for convenience we use the following equalities:

\[
\min_{i \leq s \leq j} \left( \alpha_0 + \sum_{k=i+1}^{s} (1 - \alpha_k) \right) = \min_{i \leq s \leq j} \left[ \alpha_0 + j - i + 1 - \sum_{k=i+1}^{j+1} \alpha_k - (j - s + 1 - \sum_{k=s+1}^{j+1} \alpha_k) \right] = \alpha_0 + j - i + 1 - \sum_{k=i+1}^{j+1} \alpha_k - M_{i,j}.
\]

**Lemma 5.1.1** Let \( 0 \leq i \leq j \leq n - 1 \). Then

\[
K_{i+1,j}(t,1) < < t^{j-i+1-\sum_{k=i+1}^{j+1} \alpha_k-M_{i,j} \lvert\ln t\rvert^{l_{i,j}}, \ t \in (0,1],
\]

where \( l_{i,j} \) is the number of \( k \), \( k_{i,j} + 1 \leq k \leq j \), such that \( \sum_{s=k_{i,j}+1}^{k} (\alpha_s - 1) = 0 \)
if \( k_{i,j} < j \), and \( l_{i,j} = 0 \) if \( k_{i,j} = j \).

**Lemma 5.1.2** Let \( 0 \leq i \leq n - 1 \). Then there exists \( \delta, 0 < \delta < 1 \), such that for any \( t \in (0,\delta] \) the following estimate

\[
K_{i+1,n-1}(t,1) > > t^{n-i-\sum_{k=i+1}^{n} \alpha_k - M_i}
\]

holds.

**Lemma 5.1.3** Let \( 0 \leq i \leq n - 1 \). Then

\[
t^{-\alpha_i} K_{i+1,i-1}(1,t) < < t^{M_i-1} \lvert\ln t\rvert^{l_i}, \ t \geq 1,
\]

where \( l_i \) is the number of \( k \), \( i + 1 \leq k \leq k_i - 1 \), such that \( \sum_{s=k}^{k_i-1} (\alpha_s - 1) = 0 \)
when \( k_i > i + 1 \), and \( l_i = 0 \) when \( k_i = i + 1 \).

Moreover, for the proof of our main results we also need the following Lemma:

**Lemma 5.1.4** The functions \( f_s(t) = t^{-\alpha_s} K_{1,s}(t,1, \alpha), 0 \leq m \leq s \leq n \), are not solutions of the equation

\[
D_{j}^{m} f(t) = 0, \ \forall t \in (0,1].
\]  

(5.1.3)
Proof. We assume the opposite, i.e. that \( f_s(t) = t^{-\alpha_0}K_{1,s}(t, 1, \alpha), \) \( 0 \leq m \leq s \leq n, \) are the solutions of the equation (5.1.3). Then they can be written as the linear combinations of the fundamental solutions (see [26]):

\[
f_i(t) = t^{-\beta_0}K_{1,i}(t, 1, \bar{\beta}), \quad i = 0, 1, \ldots, m - 1,
\]

of the homogeneous equation (5.1.3), i.e.

\[
f_s(t) = \sum_{i=0}^{m-1} c_i t^{-\beta_0} K_{1,i}(t, 1, \bar{\beta}), \quad \forall t \in (0, 1], \tag{5.1.4}
\]

where \( \sum_{i=0}^{m-1} c_i^2 \neq 0, c_i \in R, \) \( i = 0, 1, \ldots, m - 1. \)

Taking \( \bar{\alpha} - \) multiweighted derivative of order \( k, \) \( k = 0, 1, \ldots, m - 1, \) from both parts of (5.1.4), we have that

\[
D^k_{\bar{\alpha}} f_s(t) = \sum_{i=0}^{m-1} c_i D^k_{\bar{\alpha}} (t^{-\beta_0} K_{1,i}(t, 1, \bar{\beta})), \quad \forall t \in (0, 1]. \tag{5.1.5}
\]

Since \( \bar{\alpha} - \) multiweighted derivative of order \( k, \) \( 0 \leq k < m, \) can be written by using \( \bar{\beta} - \) multiweighted derivatives (see Lemma 4.1.2), i.e.

\[
D^k_{\bar{\alpha}} f(t) = \sum_{j=0}^{k} d_{k,j} t^{\gamma_{k,j}} D^j_{\bar{\beta}} f(t),
\]

where \( \gamma_{k,j} = \sum_{i=0}^{k} \alpha_i - \sum_{i=0}^{j} \beta_i + j - k \) and \( d_{k,k} \equiv 1, \) \( 0 \leq j \leq k < m, \) then from (5.1.5) for \( k, \) \( 0 \leq k < m, \) it follows that

\[
D^k_{\bar{\alpha}} f_s(t) = \sum_{i=0}^{m-1} c_i \sum_{j=0}^{k} d_{k,j} t^{\gamma_{k,j}} D^j_{\bar{\beta}} (t^{-\beta_0} K_{1,i}(t, 1, \bar{\beta})) =
\[
= \sum_{j=0}^{k} (-1)^j d_{k,j} t^{\gamma_{k,j}} \sum_{i=j}^{m-1} c_i K_{j+1,i}(t, 1, \bar{\beta}). \tag{5.1.6}
\]

On the other hand a straightforward calculation shows that

\[
D^k_{\bar{\alpha}} f_s(t) = D^k_{\bar{\alpha}} (t^{-\alpha_0} K_{1,s}(t, 1, \alpha)) = (-1)^k K_{k+1,s}(t, 1, \bar{\alpha}), \tag{5.1.7}
\]

\( k = 0, 1, \ldots, m - 1; \) \( s = m, m + 1, \ldots, n. \)
Then, from (5.1.6) and (5.1.7) we obtain that
\[
(-1)^k K_{k+1,s}(t, 1, \bar{\alpha}) = \sum_{j=0}^{k} (-1)^j d_{k,j} t^{\gamma_{k,j}} \sum_{i=j}^{m-1} c_i K_{j+1,i}(t, 1, \bar{\beta}) = \\
= \sum_{j=0}^{k} (-1)^j d_{k,j} c_j t^{\gamma_{k,j}} K_{j+1,j}(t, 1, \bar{\beta}) + \sum_{j=0}^{k} (-1)^j d_{k,j} t^{\gamma_{k,j}} \sum_{i=j+1}^{m-1} c_i K_{j+1,i}(t, 1, \bar{\beta}),
\]
\[k = 0, 1, \ldots, m - 1; \ s = m, m + 1, \ldots, n.\]

Since \(K_{j+1,j}(t, 1, \bar{\beta}) = 1\) and \(K_{j+1,i}(t, 1, \bar{\beta}) = 0, i = j + 1, j + 2, \ldots, m - 1,\) from (5.1.8) it follows that
\[
(-1)^k K_{k+1,s}(t, 1, \bar{\alpha}) = \sum_{j=0}^{k} (-1)^j d_{k,j} c_j t^{\gamma_{k,j}},
\]
\[k = 0, 1, \ldots, m - 1; \ s = m, m + 1, \ldots, n.\]

In particular, when \(t = 1\) we get the following system of equations of order \(m:\)
\[
\sum_{j=0}^{k} (-1)^j d_{k,j} c_j = 0, \ k = 0, 1, \ldots, m - 1.
\]

Solving this system of equations when \(k = 0,\) we have that \(d_{0,0} c_0 = 0.\) Since \(d_{0,0} = 1,\) it yields that \(c_0 = 0.\) Furthermore, by sequentially solving the system for \(k = 1, 2, \ldots, m - 1\) (note that \(d_{k,k} \neq 0\)) we get that \(c_k = 0, \ k = 0, 1, \ldots, m - 1.\) However, by our assumption, \(c_k, k = 0, 1, \ldots, m - 1,\) can not be equal to zero simultaneously. This contradiction completes the proof.

We also recall the following Lemma by T. Andô [11]:

**Lemma 5.1.5** Every linear integral operator, acting from \(L_p\) to \(L_q,\) where \(1 \leq q < p < \infty,\) is compact.

According to (5.1.2), we can give Lemma 4.1.2 in the following form, which will be useful later on:

**Lemma 5.1.6** Let the function \(f: (0, 1) \to R\) have weak derivatives up to order \(n,\) then
\[
D_{\beta}^k f(t) = \sum_{i=0}^{k} c_{k,i} t^{\mu_{k,i}} D_{\alpha}^i f(t), \ k = 0, 1, \ldots, m, \ \ \ (5.1.9)
\]

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where $\mu_{k,i} = \sum_{j=0}^{k} \beta_j - \sum_{j=0}^{i} \alpha_j + i - k$, $i = 0, 1, \ldots, k$, $k = 0, 1, \ldots, m$; and the coefficients $c_{k,i}$, $i = 0, 1, \ldots, k - 1$, $k = 0, 1, \ldots, m$, are defined by the recurrent formula:

$$c_{k,k} = 1, \quad c_{k,0} = c_{k-1,0} \left( \sum_{j=0}^{k-1} \beta_j - \alpha_0 - k + 1 \right),$$

$$c_{k,i} = c_{k-1,i-1} + c_{k-1,i} \left( \sum_{j=0}^{k-1} \beta_j - \sum_{j=0}^{i} \alpha_j + i - k + 1 \right), \quad i = 1, 2, \ldots, k - 1,$$

For the proof of our main result we also need the following statement from [27]:

**Lemma 5.1.7** For all $f \in W_{p, \alpha}^n$ we have the following:

$$D^i_\alpha f(t) = \sum_{j=i}^{n-1} (-1)^{j-i} K_{i+1,j}(t,1) D^j_\alpha f(1) + \int_{t}^{1} x^{-\alpha_n} K_{i+1,n-1}(t,x) D^n_\alpha f(x) dx,$$

$$i = 0, 1, \ldots, n - 1. \quad (5.1.10)$$

### 5.2 Compactness of the Embedding when $q < p$

Denote $i_0 = \min \{ i : 0 \leq i \leq m, c_{m,i} \neq 0 \}$, where $c_{m,i}$, $i = 0, 1, \ldots, m$, are defined as in Lemma 5.1.6.

Our main result in this Section reads:

**Theorem 5.2.1** Let $I = (0,1)$, $1 \leq q < p < \infty$ and $0 \leq m < n$. Then the following conditions are equivalent:

i) The embedding (1.3.4) is bounded;

ii) The embedding (1.3.4) is compact;

iii) $|\tilde{\beta}| - |\tilde{\alpha}| + n - m + \frac{1}{q} > \max \{ \frac{1}{p}, M_{\alpha} \}$. \quad (5.2.1)

**Proof.** First we prove that i) $\Rightarrow$ ii).

Assume that i) holds, i.e., for all $f \in W_{p, \alpha}^m$ the following estimate

$$\|f\|_{W_{p, \alpha}^m} \leq c \|f\|_{W_{p, \alpha}^n}$$
holds. Then, by the definition of the norm in the space $W^m_{q,0}$, the following estimate
\[ \| D^m \beta f \|_q \leq c \| f \|_{W^m_{p,0}} \] (5.2.2)
holds, where $c > 0$ does not depend on $f \in W^m_{p,0}$. By inserting (5.1.10) into (5.1.9) when $k = m$ we find that
\[
D^m \beta f(t) = \sum_{i=0}^{m} c_{m,i} t^{\mu_{m,i}} \sum_{j=0}^{n-1} (-1)^{j-i} K_{i+1,j}(t,1) D^j \alpha f(1) + \\
\quad + \sum_{i=0}^{m} c_{m,i} t^{\mu_{m,i}} \int_{1}^{t} x^{-\alpha n} K_{i+1,n-1}(t,x) D^i \alpha f(x) dx. \tag{5.2.3}
\]
Now we take a set $L$ of functions from $W^n_{p,0}$ such that for all $f \in L$:
\[ D^j \alpha f(1) = 0, \quad j = 0, 1, \ldots, n - 1. \tag{5.2.4} \]

It is obvious that $L$ is a subset of the space $W^n_{p,0}$. For any $F \in L_p(0,1)$ there exists a unique function $f \in L$ as a solution of the equation $D^m \alpha f(t) = F(t)$ with initial condition (5.2.4). Therefore, due to the fact that $\| f \|_{W^n_{p,0}} = \| F \|_{p}$, the operator $D^m \alpha$, establishes an isometry between the subspace $L \subset W^n_{p,0}$ and the space $L_p(0,1)$.

Let
\[
\sum_{i=0}^{m} c_{m,i} t^{\mu_{m,i}} K_{i+1,n-1}(t,x) = \hat{K}(t, x).
\]
Then, for all $f \in L$, the expression (5.2.3) has the following form:
\[
D^m \beta f(t) = \int_{1}^{t} \hat{K}(t,x) D^i \alpha f(x) dx = \hat{K} D^i \alpha f(t).
\]
Using this expression in (5.2.2), for all $f \in L$ we have that
\[ \| \hat{K} D^i \alpha f \|_q \leq c \| D^i \alpha f \|_p, \]
or
\[ \| \hat{K} f \|_q \leq c \| f \|_p, \]
which means that the operator $\hat{K}$ is bounded from $L_p$ to $L_q$. In our case $1 \leq q < p < \infty$, and, thus, by Lemma 5.1.5, the integral operator $\hat{K}$ is compact from $L_p$ to $L_q$. Since the first sum in (5.2.3) is finite-dimensional,
then the expression (5.2.3), as operator, is compact from $W^m_{p,\tilde{\alpha}}$ to $L_q$. Hence, the embedding (1.3.4) is compact, i.e., $ii$) holds.

Next we prove that $iii) \Rightarrow i)$. Let $iii)$ hold. According to (5.1.9) for $f \in W^m_{p,\tilde{\alpha}}$ when $t = 1$ we have that

$$\sum_{k=0}^{n-1} |D^k_{\tilde{\alpha}} f(1)| << \sum_{k=0}^{n-1} |D^k_{\alpha} f(\tilde{\alpha})|.$$  \hspace{1cm} (5.2.5)

From (5.2.3) and (5.2.5) it follows that the embedding (1.3.4) is bounded whenever

$$\int_0^1 |t^{\mu_{m,i}} K_{i+1,j}(t,1)|^q dt < \infty, \ i = i_0, i_0 + 1, \ldots, m; \ j = i, i + 1, \ldots, n - 1,$$

and the integral operators

$$K_i D^m_{\alpha} f(t) = t^{\mu_{m,i}} \int_0^1 x^{-\alpha_n} K_{i+1,n-1}(t,x) D^m_{\alpha} f(x) dx, \ i = i_0, i_0 + 1, \ldots, m,$$

are bounded from $L_p(0,1)$ to $L_q(0,1)$.

By using Lemma 5.1.1 for $0 \leq i \leq j \leq n - 1$ we find that

$$\int_0^1 |t^{\mu_{m,i}} K_{i+1,j}(t,1)|^q dt << \int_0^1 t^{q|\mu_{m,i} - \max\left(\frac{\sum_{k=i+1}^{n} \alpha_k + i - s}{n - s - s} \right)|} |\ln t|^q dt.$$  \hspace{1cm} (5.2.6)

The last integral converges, if, for $i_0 \leq i \leq j \leq m \leq n - 1$, the following conditions hold:

$$\mu_{m,i} - \max\left(\frac{\sum_{k=i+1}^{n} \alpha_k + i - s}{n - s - \sum_{k=s+1}^{n} \alpha_k}\right) + \frac{1}{q} > 0,$$

i.e.

$$|\tilde{\beta}| - |\tilde{\alpha}| + n - m + \frac{1}{q} > \max\left(\sum_{k=i+1}^{n} \alpha_k - s\right) - \sum_{k=i+1}^{n} \alpha_k + n =$$

$$= \max\left(n - s - \sum_{k=s+1}^{n} \alpha_k\right).$$  \hspace{1cm} (5.2.8)

Since $M_{i_0} \geq \max\left(n - s - \sum_{k=s+1}^{n} \alpha_k\right)$ for $i_0 \leq i \leq j \leq n - 1$, then due to (5.2.1) the conditions (5.2.8) hold for all $i = 0, 1, \ldots, m, \ j = i, i + 1, \ldots, n - 1$, and we conclude that (5.2.6) holds.
By using the results in [55] we know that when \(1 \leq q < p < \infty\) the integral operators (5.2.7) are bounded from \(L_p(0, 1)\) to \(L_q(0, 1)\) if and only if

\[
B_n = \max_{i_0 \leq i \leq m} \max_{i_j \leq n-1} B_{i,j}^n < \infty,
\]

where

\[
B_{i,j}^n = \left\{ \int_0^1 \left( \int_0^1 |x^{-\alpha_n} K_{j+1,n-1}(t, x)|^{p'} dx \right)^{\frac{q(p-1)}{p-q}} \times \times \left( \int_0^t |s^{\mu_{i,j}} K_{i+1,j}(s, t)|^q ds \right)^{\frac{p-q}{pq}} d \left( \int_0^t |s^{\mu_{i,j}} K_{i+1,j}(s, t)|^q ds \right) \right\}^{\frac{1}{pq}}.
\]

(5.2.9)

To prove boundedness of the integral operators (5.2.7) we estimate each integral in \(B_{i,j}\). By using the properties (5.1.1) of uniformity of the functions \(K_{i+1,j}\); we find that

\[
\int_0^t |s^{\mu_{i,j}} K_{i+1,j}(s, t)|^q ds = [s = tz, \ ds = tdz] =
\]

\[
= t^{\mu_{i,j}+1} \left( \int_0^1 |z^{\mu_{i,j}} K_{i+1,j}(tz, t)|^q dz \right) =
\]

\[
= t^{\mu_{i,j}+1} \sum_{k=i+1}^{j} (1-\alpha_k) \left( \int_0^1 |z^{\mu_{i,j}} K_{i+1,j}(z, 1)|^q dz \right).
\]

(5.2.10)

Moreover, due to (5.2.6), we know that the last integral converges. By now using the assumptions of our theorem, we find that

\[
|\beta| - |\alpha| + n - m + \frac{1}{q} = \sum_{k=0}^{m} \beta_k - \sum_{k=0}^{i} \alpha_k + i - m + n - i - \sum_{k=i+1}^{n} \alpha_k + \frac{1}{q} >
\]

\[
> M_{i_0} \geq n - j - \sum_{k=j+1}^{n} \alpha_k.
\]

Then

\[
\mu_{i,j} + j - i - \sum_{k=i+1}^{j} \alpha_k + \frac{1}{q} > 0
\]

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or

\[ 1 + q\mu_{m,i} + q \sum_{k=i+1}^{j} (1 - \alpha_k) > 0, \]

and, consequently,

\[
d \left( \int_{0}^{t} |s^{\mu_{m,i}} K_{i+1,j}(s, t)|^q ds \right) =
\]

\[
= c \cdot d \left( t^{1+q\mu_{m,i}+q \sum_{k=i+1}^{j} (1-\alpha_k)} \right) = c_1 \cdot t^{q(\mu_{m,i}+\sum_{k=i+1}^{j} (1-\alpha_k))} \ dt, \tag{5.2.11}
\]

where

\[
c = \int_{0}^{1} |s^{\mu_{m,i}} K_{i+1,j}(s, 1)|^q ds, \quad c_1 = c \left( 1 + q\mu_{m,i} + q \sum_{k=i+1}^{j} (1 - \alpha_k) \right),
\]

\[ i = i_0, i_0 + 1, \ldots, m, \quad j = i, i + 1, \ldots, n - 1. \]

Putting (5.2.10) and (5.2.11) into (5.2.9), we find that

\[ B^n_{i,j} < \left\{ \int_{0}^{1} t^{\left( \mu_{m,i} + \frac{j}{p} \right) + \frac{q(\mu_{m,i} + \sum_{k=i+1}^{j} (1-\alpha_k))}{p-q} \times} \rightleftharpoons \left( \int_{1}^{t} |x^{-\alpha_n} K_{j+1,n-1}(t, x)|^p dx \right)^{\frac{q(\mu_{m,i} + \sum_{k=i+1}^{j} (1-\alpha_k))}{p-q}} \ dt \right\} =
\]

\[ = \left\{ \int_{0}^{1} t^{\left( \mu_{m,i} + \frac{j}{p} \right) + \frac{q(\mu_{m,i} + \sum_{k=i+1}^{j} (1-\alpha_k))}{p-q} \times} \left( \int_{1}^{t} |x^{-\alpha_n} K_{j+1,n-1}(t, x)|^p dx \right)^{\frac{q(\mu_{m,i} + \sum_{k=i+1}^{j} (1-\alpha_k))}{p-q}} \ dt \right\}. \]

Since \( \frac{p-1}{p} = \frac{1}{p'} \) we conclude that

\[ B^n_{i,j} < \left\{ \int_{0}^{1} t^{\mu_{m,i} + \frac{j}{p} \frac{1}{p'} \times} \rightleftharpoons \left( \int_{1}^{t} \right) \left( \int_{1}^{t} \right) \frac{q(\mu_{m,i} + \sum_{k=i+1}^{j} (1-\alpha_k))}{p-q} \ dt \right\}. \]
\[
\times \left( \int_1^\beta \left| x^{-\alpha_n} K_{j+1,n-1}(t, x) \right|^{\eta'} \, dx \right)^{\frac{1}{\eta'}} \bigg\{ \int \left( \frac{p}{p-q} \right)^{\frac{p-q}{p-q}} dt \bigg\} \quad . \quad (5.2.12)
\]

Using again the properties (5.1.1) of uniformity of the functions \( K_{i+1,j} \) and Lemma 5.1.3, we obtain that

\[
\left( \int_1^\beta \left| x^{-\alpha_n} K_{j+1,n-1}(t, x) \right|^{\eta'} \, dx \right)^{\frac{1}{\eta'}} = t^{-\alpha_n + \frac{1}{p} \sum_{k=j+1}^{n-1} (1-\alpha_k)} \left( \int_1^\beta \left| x^{-\alpha_n} K_{j+1,n-1}(1, x) \right|^{\eta'} \, dx \right)^{\frac{1}{\eta'}} < <
\]

\[
<< t^{-\frac{1}{p} + \frac{1}{p} \sum_{k=j+1}^{n-1} (1-\alpha_k)} \left( \int_1^\beta \left| x^{(M_j-1)} \ln x \right|^{\eta'} \, dx \right)^{\frac{1}{\eta'}} \quad , \quad j = i_0, i_0 + 1, \ldots, n - 1.
\]

Since

\[
\int_1^\beta x^{(M_j-1)} \ln x \, dx < \infty \quad \text{when} \quad M_j < \frac{1}{p}, \quad j = i_0, i_0 + 1, \ldots, n - 1,
\]

then from (5.2.13) for small enough \( t > 0 \) we have that

\[
\left( \int_1^\beta \left| x^{-\alpha_n} K_{j+1,n-1}(t, x) \right|^{\eta'} \, dx \right)^{\frac{1}{\eta'}} < < \begin{cases} \sum_{k=j+1}^{n-1} (1-\alpha_k)^{-M_j} |\ln t|^j & \text{if} \quad M_j > \frac{1}{p}, \\ \sum_{k=j+1}^{n-1} (1-\alpha_k)^{-\frac{1}{p}} & \text{if} \quad M_j < \frac{1}{p}, \\ \sum_{k=j+1}^{n-1} (1-\alpha_k)^{-\frac{1}{p}} |\ln t|^j + \frac{1}{p} & \text{if} \quad M_j = \frac{1}{p}. \end{cases}
\]

(5.2.14)
From (5.2.12) and (5.2.14) we get that

\[ B_{i,j}^n \leq \begin{cases} 
\left( \int_0^1 (\mu_{m,i} + \sum_{k=i+1}^{j} (1-\alpha_k) + \frac{1}{p} - M_j) \frac{pq}{p-q} \right)^{p-q/p} \frac{1}{p} t^i \left( \int_0^1 (\mu_{m,i} + \sum_{k=i+1}^{j} (1-\alpha_k)) \frac{pq}{p-q} \ dt \right) 
& \text{if } M_j > \frac{1}{p}, \\
\left( \int_0^1 (\mu_{m,i} + \sum_{k=i+1}^{j} (1-\alpha_k)) \frac{pq}{p-q} \ dt \right)^{p-q/p} \frac{1}{p} t^i \left( \int_0^1 (\mu_{m,i} + \sum_{k=i+1}^{j} (1-\alpha_k)) \frac{pq}{p-q} \ dt \right) 
& \text{if } M_j < \frac{1}{p}, \\
\left( \int_0^1 (\mu_{m,i} + \sum_{k=i+1}^{j} (1-\alpha_k)) \frac{pq}{p-q} \ dt \right)^{p-q/p} \frac{1}{p} t^i \left( \int_0^1 (\mu_{m,i} + \sum_{k=i+1}^{j} (1-\alpha_k)) \frac{pq}{p-q} \ dt \right) 
& \text{if } M_j = \frac{1}{p}.
\end{cases} \]

(5.2.15)

From (5.2.15) it follows that \( B_{i,j}^n \), \( i_0 \leq i \leq m, i \leq j \leq n-1 \), will be finite if

\[ \mu_{m,i} + \frac{\sum_{k=i+1}^{n} (1-\alpha_k)}{n} + \frac{1}{p} - M_j > \frac{q-p}{pq}, \]

or

\[ |\bar{\beta}| - |\bar{\alpha}| + n - m + \frac{1}{q} > M_j \text{ when } M_j > \frac{1}{p}, \]

(5.2.16)

and

\[ \mu_{m,i} + \frac{\sum_{k=i+1}^{n} (1-\alpha_k)}{n} > \frac{q-p}{pq}, \]

or

\[ |\bar{\beta}| - |\bar{\alpha}| + n - m + \frac{1}{q} > \frac{1}{p} \text{ when } M_j \leq \frac{1}{p}. \]

(5.2.17)

Since the left hand sides of (5.2.16) and (5.2.17) are the same and do not depend on \( i, j \), and the quantities \( M_i \) do not increase by index \( i = i_0, i_0 + 1, \ldots, n-1 \), then the quantity \( B_n = \max_{i_0 \leq i \leq m} \max_{i \leq j \leq n-1} B_{i,j}^n \) will be finite, if (5.2.1) holds. Consequently, iii) implies i).

To complete the proof it is sufficient to prove that ii) \( \Rightarrow \) iii), so we assume that ii) holds. Then the embedding (1.3.4) is bounded and (5.2.2) holds for every \( f \in W^n_{p,\bar{\alpha}} \).

Let us put \( f_0(t) = t^{-\alpha_0} K_{1,n-1}(t,1) \). Then \( D_{\alpha}^n f_0(t) = 0 \) when \( t \in (0,1) \) and \( D_{\alpha}^1 f_0(1) = 0, i = 0, 1, \ldots, n-2, |D_{\alpha}^{n-1} f_0(1)| = 1 \). Consequently, \( f_0 \in W^n_{p,\bar{\alpha}} \) and \( \|f_0\|_{W^n_{p,\bar{\alpha}}} = 1 \). Hence, (5.2.2) implies that

\[ \|D_{\beta}^n f_0\|_{q} \leq c. \]
Due to Lemma 5.1.4 it yields that $\|D^n_{\beta} f_0\|_q > 0$. By using (5.1.9), we have that
\[
\int_0^1 \left| \sum_{i=0}^m (-1)^i c_{m,i} t^{\mu_{m,i}} K_{i+1,n-1}(t,1) \right|^q dt \leq c^q. \tag{5.2.18}
\]
Since, due to Lemma 5.1.2, $K_{i+1,n-1}(t,1) \gg t^{n-1-\sum_{k=i+1}^n \alpha_k - M_i}$, $0 \leq i \leq n-1$, for small enough $t > 0$, then
\[
t^{\mu_{m,i}} K_{i+1,n-1}(t,1) \gg t^{\beta_i - |\bar{\alpha}| + n - m - M_i}, \quad i = i_0, i_0 + 1, \ldots, m,
\]
for small enough $t > 0$. By our condition $c_{m,i_0} \neq 0$ and $M_{i_0} \geq M_i, i_0 \leq i \leq m$, it yields that when $M_{i_0} > \frac{1}{p}$ the order of the underintegral function in (5.2.18) is not less than $t^{\beta_i - |\bar{\alpha}| + n - m - M_{i_0}}$. Therefore, the function $t^{(|\bar{\beta}| - |\bar{\alpha}| + n - m - M_{i_0})q}$ is integrable in a neighbourhood of $t = 0$ and this is equivalent to the following condition
\[
|\bar{\beta}| - |\bar{\alpha}| + n - m + \frac{1}{q} > M_{i_0}. \tag{5.2.19}
\]
Now let us take the function $f_1(t) = t^{n-|\bar{\alpha}| - \frac{\varepsilon}{p}}$, where $0 < \varepsilon < 1$. Then
\[
D^n_{\bar{\alpha}} f_1(t) = \prod_{j=0}^{n-1} (n-j - \sum_{k=j+1}^n \alpha_k - \frac{\varepsilon}{p}) t^{-\frac{\varepsilon}{p}}.
\]
Consequently, $f_1 \in W^n_{\bar{\alpha}}$. By making some calculations we find that
\[
D^n_{\beta} f_1(t) = \prod_{i=0}^{m-1} \left( \sum_{k=0}^i \beta_k - |\bar{\alpha}| + n - i - \frac{\varepsilon}{p} \right) t^{\beta_i - |\bar{\alpha}| + n - m - \frac{\varepsilon}{p}}.
\]
Since we have finite many factors in the product, then there exists $\varepsilon_0 > 0$ such that, for each $\varepsilon \in (\varepsilon_0, 1)$,
\[
\prod_{i=0}^{m-1} \left( \sum_{k=0}^i \beta_k - |\bar{\alpha}| + n - i - \frac{\varepsilon}{p} \right) \neq 0.
\]
Due to the continuous embedding (1.3.4) it must hold that $D^n_{\beta} f_1 \in L_q(0,1)$, but this is possible if and only if
\[
|\bar{\beta}| - |\bar{\alpha}| + n - m - \frac{\varepsilon}{p} + \frac{1}{q} > 0 \text{ for all } \varepsilon \in (\varepsilon_0, 1).
\]
Hence, by letting $\varepsilon \to 1$, we have that
\[
|\bar{\beta}| - |\bar{\alpha}| + n - m + \frac{1}{q} \geq \frac{1}{p}. \tag{5.2.20}
\]
Let \( M_{i_0} < \frac{1}{p} \). We suppose that

\[
|\beta| - |\alpha| + n - m + \frac{1}{q} - \frac{1}{p} = 0. \quad (5.2.21)
\]

We consider the following set of the functions:

\[
f_{\varepsilon}(t) = c_{\varepsilon} t^{-\alpha_0} \int_{t}^{1} K_{1,n-1}(t, x)x^{\alpha} \chi_{0,\varepsilon}(x) x^{-\frac{\varepsilon}{p}} dx, \quad \varepsilon_0 < \varepsilon < 1,
\]

where \( c_{\varepsilon} \) is a constant and \( \chi_{0,\varepsilon}(\cdot) \) denotes the characteristic function of the interval \((0, \varepsilon)\).

Since \( D^\alpha_{\omega} f_{\varepsilon}(t) = c_{\varepsilon} (-1)^n \chi_{(0,\varepsilon)}(t) t^{-\frac{\varepsilon}{p}} \), then \( f_{\varepsilon} \in W^{n}_{p,\alpha} \) for all \( \varepsilon \in (0, 1) \).

We choose a constant \( c_{\varepsilon} \) such that \( ||f_{\varepsilon}||_{W^n_p} = ||D^\alpha_{\omega} f_{\varepsilon}||_p = 1 \). Then

\[
c_{\varepsilon} = (1 - \varepsilon)^{\frac{1}{p} - \frac{1}{p}}.
\]

We now prove that the set of functions \( f_{\varepsilon} \), \( 0 < \varepsilon < 1 \), converges weakly to zero when \( \varepsilon \to 0 \). By definition of the space \( W^{n}_{p,\alpha} \) it follows that it is isometric to the space \( L^p(I) \times R^n \). Therefore, \( (W^{n}_{p,\alpha})^* = (L^p(I) \times R^n)^* = L^{p'}(I) \times R^n \).

Since \( D^\alpha_{\omega} f_{\varepsilon}(1) = 0 \), \( i = 0, 1, \ldots, n-1 \), then, according to Hölder’s inequality, for each \( G = (g, a) \in L^{p'}(I) \times R^n \):

\[
|<f_{\varepsilon}, G>| = |\int_{0}^{1} D^\alpha_{\omega} f_{\varepsilon}(t) g(t) dt| = c_{\varepsilon} |\int_{0}^{1} t^{-\frac{\varepsilon}{p}} g(t) dt| \leq
\]

\[
\leq c_{\varepsilon} \left( \int_{0}^{\varepsilon} t^{-\frac{\varepsilon}{p}} dt \right)^{\frac{1}{p'}} \left( \int_{0}^{\varepsilon} |g(t)|^{p'} dt \right)^{\frac{1}{p'}} = \left( \int_{0}^{\varepsilon} |g(t)|^{p'} dt \right)^{\frac{1}{p'}}.
\]

Hence, it follows that \( <f_{\varepsilon}, G> \to 0 \) when \( \varepsilon \to 0 \) for all \( G \in (W^{n}_{p,\alpha})^* \). Therefore, due to the compactness of the embedding \((1.3.4)\), the set of functions \( f_{\varepsilon} \), \( 0 < \varepsilon < 1 \), when \( \varepsilon \to 0 \) converges strongly to zero in \( W^{m}_{q,\beta} \). Moreover, by using \((5.1.9), (5.1.10) \) and \((5.2.3)\), we have that

\[
D^\alpha_{\omega} f_{\varepsilon}(t) = \sum_{i=i_0}^{m} c_{m,i} t^{\mu_{m,i}} D^\alpha_{\omega} f_{\varepsilon}(t) =
\]

\[
= \sum_{i=i_0}^{m} (-1)^i c_{m,i} t^{\mu_{m,i}} \int_{t}^{1} K_{i+1,n-1}(t, x)x^{\alpha} \chi_{0,\varepsilon}(x) x^{-\frac{\varepsilon}{p}} dx. \quad (5.2.22)
\]

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Now we prove that for \( i = i_0, i_0 + 1, \ldots, m \) and for all \( \varepsilon \in (0, 1) \), the estimate
\[
\int_0^1 |t^\mu_{m,i} \int t K_{i+1,n-1}(t, x) x^{-\alpha_n} \chi_{0,\varepsilon}(x) x^{-\frac{\varepsilon}{p}} dx|^q dt < \infty \tag{5.2.23}
\]
holds.

By changing variables, due to Lemma 5.1.3 we get that
\[
\int_0^1 |t^\mu_{m,i} \int t K_{i+1,n-1}(t, x) x^{-\alpha_n-\frac{\varepsilon}{p}} dx|^q dt <<
\]
\[
<< \int_0^1 |t^\mu_{m,i} \int t K_{i+1,n-1}(t, x) x^{-\alpha_n-\frac{\varepsilon}{p}} dx|^q dt <<
\]
\[
<< \int_0^1 t^\mu_{m,i} \int K_{i+1,n-1}(t, x) x^{-\alpha_n-\frac{\varepsilon}{p}} dx|^q dt < < \int_0^1 t^{\mu_{m,i} - \alpha_n - \frac{\varepsilon}{p} + \sum_{k=i+1}^{n-1} (1 - \alpha_k)} \int_1^t z^{M_i - \frac{\varepsilon}{p}} |lnz|^l dz |dt. \tag{5.2.24}
\]
Since \( M_{i_0} < \frac{1}{p} \) and \( M_i \leq M_0 \), \( i = i_0, i_0 + 1, \ldots, m \), then for all \( \varepsilon \in (0, 1) \) we have that \( M_i - 1 - \frac{\varepsilon}{p} < 0, \) \( i = 0, 1, \ldots, m \). Therefore,
\[
\int_1^t z^{M_i - \frac{\varepsilon}{p}} |lnz|^l dz \leq \int_1^t |lnz|^l dz \leq \frac{1}{t} |ln t|^l,
\]
and, hence, from (5.2.24) it follows that
\[
\int_0^1 |t^\mu_{m,i} \int K_{i+1,n-1}(t, x) x^{-\alpha_n-\frac{\varepsilon}{p}} dx|^q dt << \int_0^1 t^{\mu_{m,i} - \alpha_n - \frac{\varepsilon}{p} + \sum_{k=i+1}^{n-1} (1 - \alpha_k)} |ln t|^l dt. \tag{5.2.25}
\]
Moreover, according to (5.2.21) we have that
\[
\mu_{m,i} - \alpha_n - \frac{\varepsilon}{p} + \sum_{k=i+1}^{n-1} (1 - \alpha_k) > -\frac{1}{q}, \quad \forall \varepsilon \in (0, 1).
\]
Consequently, the last integral in (5.2.25) is converge and this fact yields the estimate (5.2.23).

Further, by taking the norm in (5.2.22) we get that
\[
\| D_{\mu}^n f_c \|_q = c_z \left( \int_0^1 \left( \sum_{i=0}^m (-1)^i c_{m,i} t^\mu_{m,i} \int K_{i+1,n-1}(t, x) x^{-\alpha_n-\frac{\varepsilon}{p}} \chi_{0,\varepsilon}(x) dx \right)^q dt \right)^{\frac{1}{q}} =
\]
\[
= c_z \left( \int_0^1 \left( \sum_{i=0}^m (-1)^i c_{m,i} t^\mu_{m,i} \int K_{i+1,n-1}(t, x) x^{-\alpha_n-\frac{\varepsilon}{p}} dx \right)^q dt \right)^{\frac{1}{q}}. \tag{5.2.26}
\]
In (5.2.26) first we change variables $t \to \varepsilon t$ in the outer integral, next we change variables $x \to \varepsilon x$ in the inner integral, and, taking into account the relation (5.2.21), and a result we find that
\[
\|D_{\beta}^{m}f_{\varepsilon}\|_{q} = \varepsilon^{[|\beta| - |\alpha| + n - m + \frac{1}{q} - \frac{1}{p} T_{\varepsilon}} = T_{\varepsilon},
\]
(5.2.27)
where
\[
T_{\varepsilon} = (1 - \varepsilon)^{\frac{1}{p}} \left( \int_{0}^{1} | \sum_{i=0}^{m} (-1)^{i} c_{m,i} t^{m,i} \int_{t}^{1} K_{i+1,n-1}(t,x)x^{-\alpha_n - \frac{\varepsilon}{p}} dx |^{q} dt \right)^{\frac{1}{q}}.
\]

Due to (5.2.23) it yields that $T_{\varepsilon} < \infty$ for all $\varepsilon \in (0, 1)$. Moreover,
\[
T_{0} = \lim_{\varepsilon \to 0} T_{\varepsilon} =
\]
\[
\lim_{\varepsilon \to 0} (1 - \varepsilon)^{\frac{1}{p}} \left( \int_{0}^{1} | \sum_{i=0}^{m} (-1)^{i} c_{m,i} t^{m,i} \int_{t}^{1} K_{i+1,n-1}(t,x)x^{-\alpha_n - \frac{\varepsilon}{p}} dx |^{q} dt \right)^{\frac{1}{q}} =
\]
\[
= \left( \int_{0}^{1} | D_{\beta}^{m}(t^{-\alpha_0} K_{1,n}(t,1)) |^{q} dt \right)^{\frac{1}{q}} \neq 0,
\]
since, according to Lemma 5.1.4, $D_{\beta}^{m}(t^{-\alpha_0} K_{1,n}(t,1)) \neq 0$ for almost every $t \in (0, 1)$. Consequently, $\|D_{\beta}^{m}f_{\varepsilon}\|_{q} \neq 0$ when $\varepsilon \to 0$, that is $f_{\varepsilon}$ does not converge to zero in $W_{q,\beta}^{m}$ when $\varepsilon \to 0$. The obtained contradiction shows that in (5.2.20) when $M_{i_{0}} < \frac{1}{p}$ it will be strong inequality, that is
\[
|\beta| - |\alpha| + n - m + \frac{1}{q} > \frac{1}{p},
\]
which together with (5.2.19) gives (5.2.1).

The proof is complete.

Now on the interval $I = (0, 1)$ when $\alpha_k = 0$, $k = 0, 1, \ldots, n - 1$, $\alpha_n = \gamma$, $\beta_i = 0$, $i = 0, 1, \ldots, m - 1$, and $\beta_n = \nu$ we consider the Kudryavtsev spaces $L_{p,\gamma}^{n}$ and $L_{q,\nu}^{m}$, respectively. Then $M_{i_{0}} = \max_{i_{0} \leq s \leq n-1} (n-s-\gamma) = n-\gamma-i_{0}$. Hence, Theorem 5.2.1 implies the following new information about the embedding between these spaces and the spaces with multiweighted derivatives:
Corollary 5.2.2 Let \( I = (0, 1) \), \( 0 \leq m < n \) and \( 1 \leq q < p < \infty \). Then the following conditions are equivalent:

i) The embedding \( L^n_{p,\gamma}(I) \hookrightarrow W^m_{q,\beta}(I) \) is bounded;

ii) The embedding \( L^n_{p,\gamma}(I) \hookrightarrow W^m_{q,\beta}(I) \) is compact;

\[ m \| \beta \| - \gamma + n - m + \frac{1}{q} > \max \{ n - \gamma - i_0, \frac{1}{p} \} . \]

Corollary 5.2.3 Let \( I = (0, 1) \), \( 0 \leq m < n \) and \( 1 \leq q < p < \infty \). Then the following conditions are equivalent:

i) The embedding \( W^n_{p,\alpha}(I) \hookrightarrow L^m_{q,\alpha}(I) \) is bounded;

ii) The embedding \( W^n_{p,\alpha}(I) \hookrightarrow L^m_{q,\alpha}(I) \) is compact;

\[ m \| \alpha \| - n - m + \frac{1}{q} > \max \{ M_{i_0}, \frac{1}{p} \} . \]

5.3 Compactness of the Embedding when \( p \leq q \)

Our main result in this section reads:

Theorem 5.3.1 Let \( I = (0, 1) \), \( 1 \leq p \leq q < \infty \), \( 0 \leq m < n \) and \( M_{i_0} \geq \frac{1}{p} \).

Then the following conditions are equivalent:

i) The embedding (1.3.4) is bounded;

ii) The embedding (1.3.4) is compact;

iii) \[ |\beta| - |\alpha| + n - m + \frac{1}{q} > M_{i_0}. \] (5.3.1)

Proof. Let us first prove that i) \( \Rightarrow \) iii). Assume that the embedding (1.3.4) is bounded. Then

\[ \| D^n_\beta f \|_q \leq c \| f \|_{W^n_{p,\alpha}}, \quad \forall f \in W^n_{p,\alpha}. \] (5.3.2)

Let us take the function \( f_0(t) = t^{-\alpha_0} K_{1,n-1}(t, 1) \). Since \( D^n_\alpha f_0(t) = 0 \), \( \forall t \in (0, 1) \), and \( D^n_\alpha f_0(1) = 0, i = 0, 1, \ldots, n - 2, |D^n_{\alpha-1} f_0(1)| = 1 \), then \( f_0 \in W^n_{p,\alpha} \) and \( \| f_0 \|_{W^n_{p,\alpha}} = 1 \). Hence, from (5.3.2) we have that

\[ \| D^n_\beta f_0 \|_q \leq c. \] (5.3.3)

Moreover, due to Lemma 5.1.4 we get that \( \| D^n_\beta f_0 \|_q > 0 \). From (5.1.9) and (5.3.3) it follows that

\[ \int_0^1 \left| \sum_{i=i_0}^m (-1)^i c_{m,t} t^{\mu_{m,i}} K_{i+1,n-1}(t, 1) \right|^q dt < \infty. \] (5.3.4)
According to Lemma 5.1.2 for small enough $t > 0$ we obtain that

$$K_{i+1,n-1}(t,1) >> t^{\frac{n-i-\sum_{k=i+1}^{n} m_k - M_i}{p}} \quad i = 0, 1, \ldots, n - 1.$$ 

Therefore,

$$t^{\mu_{m,i}} K_{i+1,n-1}(t,1) >> t^{\beta - \alpha + n - m - M_i} \quad i = i_0, i_0 + 1, \ldots, m_i,$$

in a neighbourhood of $t = 0$. Since $c_{m,i_0} \neq 0$ and $M_{i_0} \geq M_i$, $i_0 \leq i \leq m_i$, then for $M_{i_0} > \frac{1}{p}$ the order of the underintegral function in (5.3.4) in a neighbourhood of $t = 0$ is not less than $t^{\beta - \alpha + n - m - M_{i_0}}$. Consequently, the function $t^{\beta - \alpha + n - m - M_{i_0}}$ is integrable in a neighbourhood of $t = 0$ and this is equivalent to the condition

$$\beta - \alpha + n - m + \frac{1}{q} > M_{i_0}.$$ 

Hence, the implication $i) \Rightarrow iii)$ is proved.

Obviously, it is sufficient to prove that $iii) \Rightarrow ii)$. Assume that $iii)$ hold and let $f \in W_{p,\alpha}$. According to (5.1.9) and (5.1.10) we have that

$$D_{\beta}^{m}f(t) = \sum_{i=i_0}^{m} c_{m,i} t^{\mu_{m,i}} \sum_{j=i}^{n-1} (-1)^{j-i} K_{i+1,j}(t,1) D_{\alpha}^{j}f(1) +$$

$$+ \sum_{i=i_0}^{m} c_{m,i} t^{\mu_{m,i}} \int_{t}^{1} x^{-\alpha} K_{i+1,n-1}(t,x) D_{\alpha}^{n-1}f(x) dx.$$ 

(5.3.5)

Moreover, from (5.1.9) it follows that

$$\sum_{k=0}^{m-1} |D_{\beta}^{k}f(1)| << \sum_{k=i_0}^{n-1} |D_{\alpha}^{k}f(1)|,$$

and, therefore, for the boundedness of the embedding (1.3.4) it is sufficient that the conditions

$$\int_{0}^{1} |t^{\mu_{m,i}} K_{i+1,j}(t,1)|^{q}dt < \infty, \quad i = i_0, i_0 + 1, \ldots, m_i, \quad j = i, i + 1, \ldots, n - 1,$$

(5.3.6)
hold and that the following integral operators

\[ K_i F(t) = t_\mu \int_0^1 x^{-\alpha_n} K_{i+1,n-1}(t, x) F(x) dx, \quad i = i_0, i_0 + 1, \ldots, m, \quad (5.3.7) \]

are bounded from \( L_p(0, 1) \) to \( L_q(0, 1) \). Moreover, for the compactness of the embedding (1.3.4), due to the finiteness of the first sum on the right hand side in (5.3.5), it is sufficient to prove that the operators (5.3.7) from \( L_p(0, 1) \) to \( L_q(0, 1) \) are compact.

First we prove that (5.3.6) holds. Let \( i_0 \leq i \leq m \), according to Lemma 5.1.1, we have that

\[ \int_0^1 |t^{\mu_{m,i}} K_{i+1,j}(t, 1)|^q dt << \int_0^1 t^{q(\mu_{m,i} + j - i + 1 - \sum_{k=i+1}^{j+1} \alpha_k - M_{i,j})} |\ln t|^q dt. \]

The last integral converges if and only if

\[ \mu_{m,i} + j - i + 1 - \sum_{k=i+1}^{j+1} \alpha_k - M_{i,j} + \frac{1}{q} > 0, \quad i = i_0, i_0 + 1, \ldots, m, \quad j = i, i+1, \ldots, n-1, \]

i.e.

\[ |\tilde{\beta} - |\tilde{\alpha}| + n - m + \frac{1}{q} > \max_{i \leq s \leq j} \left( j - s + 1 - \sum_{k=s+1}^{j+1} \alpha_k \right) - \sum_{k=j+2}^{n} \alpha_k + n = \]

\[ \max_{i \leq s \leq j} \left( n - s + 1 - \sum_{k=s+1}^{n} \alpha_k \right). \]

But by the definition \( M_{i_0} \geq \max_{i \leq s \leq j} \left( n - s + 1 - \sum_{k=s+1}^{n} \alpha_k \right) \) when \( i_0 \leq i \leq m \) and \( i \leq j \leq n - 1 \), then from (5.3.1), it follows that (5.3.8) holds and, hence, that (5.3.6) holds.

According to the results in [12] the integral operators (5.3.7) are compact from \( L_p(0, 1) \) to \( L_q(0, 1) \) when \( 1 \leq p \leq q < \infty \) if and only if

\[ \max_{i \leq j \leq n-1} \sup_{0 < z < 1} A_{i,j}(z) < \infty, \quad i = i_0, i_0 + 1, \ldots, m, \quad (5.3.9) \]

and

\[ \lim_{z \to 0} A_{i,j}(z) = \lim_{z \to 1} A_{i,j}(z) = 0, \quad i = i_0, i_0 + 1, \ldots, m, \quad j = i, i+1, \ldots, n - 1; \quad (5.3.10) \]
where

\[ A_{i,j}(z) = \left( \int_0^z |t^{\alpha_m} K_{i+1,j}(t,z)|^q dt \right)^{\frac{1}{q}} \left( \int_1^1 |x^{-\alpha_n} K_{0,0}(z,x)|^p dx \right)^{\frac{1}{p}}. \]  

(5.3.11)

Due to (5.3.6), the first integral in (5.3.11) converges for all \( 0 \leq z \leq 1 \) and the underintegral function of the second integral is continuous on \((0,1]\), we find that the function \( A_{i,j}(z) \) is continuous on \((0,1]\) and \( \lim_{z \to 0} A_{i,j}(z) = 0 \) for all \( i = i_0, i_0 + 1, \ldots, m, \ j = i, i + 1, \ldots, n - 1 \). Therefore, the fulfilment of (5.3.9) and (5.3.10) depends on the behavior of the function \( A_{i,j}(z) \) when \( z \to 0 \).

In the integrals (5.3.11), by changing variables \( t \to tz, x \to xz \), respectively, and, using the property of uniformity (5.1.1), we find that

\[
\left( \int_0^z |t^{\alpha_m} K_{i+1,j}(t,z)|^q dt \right)^{\frac{1}{q}} = z^{\mu_{m,i} + \sum_{k=0}^{i+1} (1-\alpha_k) + \frac{1}{q}} \left( \int_0^1 |t^{\alpha_m} K_{i+1,j}(t,1)|^q dt \right)^{\frac{1}{q}} = c_{i,j} z^{|\bar{\alpha}| - \sum_{k=0}^{i+1} \alpha_k + j - m + \frac{1}{q}},
\]

(5.3.12)

where, due to (5.3.6), \( c_{i,j} = \left( \int_0^1 |t^{\alpha_m} K_{i+1,j}(t,1)|^q dt \right)^{\frac{1}{q}} < \infty \) when \( j = i, i + 1, \ldots, n - 1, \ i = i_0, i_0 + 1, \ldots, m, \) and

\[
\left( \int_1^1 |x^{-\alpha_n} K_{0,0}(z,x)|^p dx \right)^{\frac{1}{p}} = \]

\[
= z^{-\alpha_n + \frac{1}{p} + \sum_{k=j+1}^{n-1} (1-\alpha_k)} \left( \int_1^1 |x^{-\alpha_n} K_{0,0}(1,x)|^p dx \right)^{\frac{1}{p}} <<
\]

[according to Lemma 5.1.3]

\[
<< z^{-\frac{1}{p} - \sum_{k=j+1}^{n} \alpha_k + n - j} \left( \int_1^1 x^{(M-j)-1} |\ln x|^p dx \right)^{\frac{1}{p}}. \]  

(5.3.13)
Since
\[
\int_1^\infty x^{p'(M_j-1)}|\ln x|^{p'j}dx < \infty \quad \text{when} \quad M_j < \frac{1}{p}, \quad j = 0, 1, \ldots, n - 1,
\]
then for small enough \( z > 0 \) we have that
\[
\int_1^z x^{p'(M_j-1)}|\ln x|^{p'j}dx \ll \begin{cases} 
    z^{-p'(M_j-1)-1}|\ln z|^{p'j} & \text{when} \quad M_j > \frac{1}{p}, \\
    1 & \text{when} \quad M_j < \frac{1}{p}, \\
    |\ln z|^{p'j+1} & \text{when} \quad M_j = \frac{1}{p}.
\end{cases}
\]

From (5.3.11) - (5.3.14) for small \( z > 0 \) we get that
\[
A_{i,j}(z) \ll z^{[\bar{\beta}]-[\bar{\alpha}]+n-m+\frac{1}{q}-M_j}|\ln z|^{j} \quad \text{when} \quad M_j > \frac{1}{p}, \quad (5.3.15)
\]
\[
A_{i,j}(z) \ll z^{[\bar{\beta}]-[\bar{\alpha}]+n-m+\frac{1}{q}-\frac{1}{p}} \quad \text{when} \quad M_j < \frac{1}{p}, \quad (5.3.16)
\]
\[
A_{i,j}(z) \ll z^{[\bar{\beta}]-[\bar{\alpha}]+n-m+\frac{1}{q}-\frac{1}{p}}|\ln z|^{j+\frac{1}{p}} \quad \text{when} \quad M_j = \frac{1}{p}. \quad (5.3.17)
\]

Moreover, by the assumptions of Theorem 5.3.1, it yields that \( M_{i_0} \geq \frac{1}{p} \), and by the definition we have that \( M_{i_0} \geq M_j \) when \( i_0 \leq j \leq n - 1 \). Therefore, from (iii) it follows that
\[
[\bar{\beta}] - [\bar{\alpha}]+n-m+\frac{1}{q}-\max\{M_j, \frac{1}{p}\} > 0
\]
for all \( j = i_0, i_0 + 1, \ldots, n - 1 \). Hence, (5.3.15) - (5.3.17) imply that
\[
\lim_{z \to 0} A_{i,j}(z) = 0 \quad \text{for all} \quad i = i_0, i_0 + 1, \ldots, m, \quad j = i, i+1, \ldots, n-1, \quad \text{i.e.} \quad (5.3.9) \quad \text{and} \quad (5.3.10) \quad \text{hold}.
\]

Thus (iii) implies (5.3.6) and the compactness of the integral operators (5.3.7). Consequently, also the implication (iii) \( \Rightarrow \) (i) is proved. The proof is complete.

**Theorem 5.3.2** Let \( I = (0,1) \), \( 1 < q \leq q < \infty \), \( 0 \leq m < n \) and \( M_{i_0} < \frac{1}{p} \).

Then
a) the embedding (1.5.4) is bounded if and only if
\[
[\bar{\beta}] - [\bar{\alpha}]+n-m+\frac{1}{q} \geq \frac{1}{p}. \quad (5.3.18)
\]
b) the embedding (1.3.4) is compact if and only if

$$|\beta| - |\alpha| + n - m + \frac{1}{q} > \frac{1}{p}. \quad (5.3.19)$$

Proof. Let us first prove a). Let the embedding (1.3.4) be bounded. Consider the function $f_0(t) = t^{n-|\alpha| - \frac{1}{p} + \varepsilon}$, where $\varepsilon > 0$. Then

$$D_\alpha^n f_0(t) = \prod_{j=0}^{n-1} (n - j - \sum_{k=j+1}^{n} \alpha_k - \frac{1}{p} + \varepsilon) t^{\frac{1}{p} + \varepsilon},$$

and, consequently, $f_0 \in W_{p,\bar{\alpha}}^n$. A direct calculation implies that

$$D_\beta^m f_0(t) = \prod_{i=0}^{m-1} \left( \sum_{k=0}^{i} \beta_k - |\alpha| + n - i - \frac{1}{p} + \varepsilon \right) t^{\frac{|\beta| - |\alpha| + n - m - \frac{1}{p} + \varepsilon}}.$$

Since we have only finite many factors in the product, then there exists $\varepsilon_0 > 0$ such that, for each $\varepsilon \in (0, \varepsilon_0)$:

$$\prod_{i=0}^{m-1} \left( \sum_{k=0}^{i} \beta_k - |\alpha| + n - i - \frac{1}{p} + \varepsilon \right) \neq 0.$$

Due to the boundedness of the embedding (1.3.4) it must hold that $D_\beta^m f_0 \in L_q(0,1)$, but this is possible if and only if

$$|\beta| - |\alpha| + n - m - \frac{1}{p} + \varepsilon + \frac{1}{q} > 0 \quad \text{when} \quad \varepsilon \in (0, \varepsilon_0).$$

Hence, by letting $\varepsilon \to 0$, we have (5.3.18).

On the contrary, assume that (5.3.18) hold. In Theorem 5.3.1 it was shown that the embedding (1.3.4) is bounded if (5.3.6) holds and the integral operators (5.3.7) are bounded from $L_p(0,1)$ to $L_q(0,1)$, and this is equivalent to the condition (5.3.9). By the assumptions of Theorem 5.3.2 it yields that $M_0 < \frac{1}{p}$ and, therefore, from (5.3.18) it follows that (5.3.1) holds, which, in its turn, implies (5.3.6), as it was in Theorem 5.3.1. Since $\frac{1}{p} > M_0 \geq M_j, \ i_0 \leq j \leq n - 1$, then from (5.3.16) and (5.3.18) it follows that (5.3.9) holds. Thus a) is proved.

Let us now prove b). Assume that the embedding (1.3.4) is compact. Then (5.3.18) holds. We suppose that in (5.3.18) it will be equality, i.e. that

$$|\beta| - |\alpha| + n - m + \frac{1}{q} = \frac{1}{p}. \quad (5.3.20)$$
We consider the following set of functions:

\[ f_\varepsilon(t) = c_\varepsilon t^{-\alpha_0} \int_1^t K_{1,n-1}(t,x)x^{-\alpha n} \chi_{(\varepsilon,\varepsilon)}(x)x^{-\frac{\varepsilon}{p}} dx, \quad 0 < \varepsilon < 1, \]

where \( c_\varepsilon \) is a constant and \( \chi_{(\varepsilon,\varepsilon)}(\cdot) \) denotes the characteristic function of the interval \((0, \varepsilon)\).

Since \( D_\alpha^n f_\varepsilon(t) = c_\varepsilon (-1)^n \chi_{(\varepsilon,\varepsilon)}(t)t^{-\frac{\varepsilon}{p}} \), then \( f_\varepsilon \in W_{p,\alpha}^n \) for all \( \varepsilon \in (0, 1) \).

We now choose the constant \( c_\varepsilon \) such that \( \|f_\varepsilon\|_{W_{p,\alpha}^n} = \|D_\alpha^n f_\varepsilon\|_p = 1 \), i.e.

\[ c_\varepsilon = (1 - \varepsilon)^\frac{1}{p} \varepsilon^{\frac{1}{p}}. \]

Let us show that the set of functions \( f_\varepsilon \), \( 0 < \varepsilon < 1 \), converges weakly to zero when \( \varepsilon \to 0 \). By definition of the space \( W_{p,\alpha}^n \) it follows that it is isometric to the space \( L_p(I) \times R^n \). Therefore, \((W_{p,\alpha}^n)^* = (L_p(I) \times R^n)^* = L_{p'}(I) \times R^n\).

Since \( D_\alpha^n f_\varepsilon(1) = 0, i = 0, 1, \ldots, n - 1 \), then, according to Hölder’s inequality, for each \( G \equiv (g, a) \in L_{p'}(I) \times R^n \) we have that

\[ |< f_\varepsilon, G >| = \left| \int_0^1 D_\alpha^n f_\varepsilon(t)g(t)dt \right| = c_\varepsilon \left| \int_0^1 t^{-\frac{\varepsilon}{p}} g(t)dt \right| \leq c_\varepsilon \left( \int_0^1 s^{-\varepsilon} ds \right)^\frac{1}{p} \left( \int_0^1 |g(t)|^{p'} dt \right)^\frac{1}{p'} = \left( \int_0^1 |g(t)|^{p'} dt \right)^\frac{1}{p'}. \]

It yields that \(< f_\varepsilon, G > \to 0\) when \( \varepsilon \to 0 \) for all \( G \in (W_{p,\alpha}^n)^* \). Then, according to the compactness of the embedding (1.3.4) the set of functions \( f_\varepsilon \), \( 0 < \varepsilon < 1 \), when \( \varepsilon \to 0 \) converges strongly to zero in \( W_{q,\beta}^m \). By using (5.1.9), (5.1.10) and (5.3.5) we find that

\[ D_\beta^n f_\varepsilon(t) = \sum_{i=i_0}^m c_{m,i} t^{\mu_{m,i}} D_\alpha^n f_\varepsilon(t) = c_\varepsilon \sum_{i=i_0}^m (-1)^i c_{m,i} t^{\mu_{m,i}} \int_1^t K_{i+1,n-1}(t,x)x^{-\alpha n} \chi_{(\varepsilon,\varepsilon)}(x)x^{-\frac{\varepsilon}{p}} dx. \]  

Now we show that for \( i = i_0, i_0 + 1, \ldots, m \) it holds that

\[ \int_0^1 |t^{\mu_{m,i}} \int_1^t K_{i+1,n-1}(t,x)x^{-\alpha n} \chi_{(\varepsilon,\varepsilon)}(x)x^{-\frac{\varepsilon}{p}} dx|^q dt < \infty, \]
for all $\varepsilon \in (0, 1)$.

By changing variables, due to Lemma 5.1.3, we get that

$$
\int_0^1 \int_0^{1/ \mu_{m,i}} K_{i+1,n-1}(t, x) x^{-\alpha_n - \frac{\varepsilon}{p}} dx |q| dt <<
$$

$$
<< \frac{1}{t} \int_0^{\mu_{m,i}} \int_0^{1/ \mu_{m,i}} K_{i+1,n-1}(t, x) x^{-\alpha_n - \frac{\varepsilon}{p}} dx |q| dt.
$$

(5.3.23)

Since $M_{i_0} < \frac{1}{p}$ and $M_i \leq M_{i_0}$ for $i = i_0, i_0 + 1, \ldots, n$, then, for all $\varepsilon \in (0, 1)$, it yields that $M_i - 1 - \frac{\varepsilon}{p} < 0, i = i_0, i_0 + 1, \ldots, n$. Consequently,

$$
\int_0^{1/ t} \int_0^{M_i - 1 - \frac{\varepsilon}{p}} |q| dz \leq \int_0^{1/ t} |q| dz \leq \frac{1}{t} | q | dz.
$$

Therefore, (5.3.23) implies that

$$
\int_0^1 \int_0^{1/ \mu_{m,i}} K_{i+1,n-1}(t, x) x^{-\alpha_n - \frac{\varepsilon}{p}} dx |q| dt << \int_0^{1/ t} \int_0^{1/ \mu_{m,i}} K_{i+1,n-1}(t, x) x^{-\alpha_n - \frac{\varepsilon}{p}} dx |(\mu_{m,i} - \alpha_n - \frac{\varepsilon}{p} + \sum_{k=i+1}^{n-1} (1 - \alpha_k))| |q| dz dt.
$$

(5.3.24)

From (5.3.20) we obtain that

$$
\mu_{m,i} - \alpha_n - \frac{\varepsilon}{p} + \sum_{k=i+1}^{n-1} (1 - \alpha_k) > - \frac{1}{q}
$$

for all $\varepsilon \in (0, 1)$ and, consequently, the last integral in (5.3.24) converges, which means that (5.3.22) holds.

We take the $q$ - norm in both sides of (5.3.21) and find that

$$
|D_\beta^m f_\varepsilon|_q = c_\varepsilon \left( \int_0^1 \left| \sum_{i=i_0}^m (-1)^i c_{m,i} \int_0^{1/ \mu_{m,i}} K_{i+1,n-1}(t, x) x^{-\alpha_n - \frac{\varepsilon}{p}} \chi_0(x) dx |q| dt \right|^{1/ q} \right)^{1/ q}
$$

$$
= c_\varepsilon \left( \int_0^1 \left| \sum_{i=i_0}^m (-1)^i c_{m,i} \int_0^{1/ \mu_{m,i}} K_{i+1,n-1}(t, x) x^{-\alpha_n - \frac{\varepsilon}{p}} dx |q| dt \right|^{1/ q} \right)^{1/ q}.
$$

(5.3.25)
In (5.3.25) we first change variables \( t \rightarrow \varepsilon t \) in the outer integral and next we change variables \( x \rightarrow \varepsilon x \) in the inner integral. Then, by taking into account the relation (5.3.20), we get that

\[
\|D_j^m f_\varepsilon\|_q = \varepsilon^{j-\lfloor \alpha \rfloor + n-m+1} T_\varepsilon = T_\varepsilon,
\]

where

\[
T_\varepsilon = (1 - \varepsilon)^p \left( \int_0^1 \left| \sum_{i=i_0}^m (-1)^i c_{m,i} t^{\mu_{m,i}} \int t K_{i+1,n-1}(t,x) x^{-\alpha_n-\varepsilon} dx \right|^q dt \right)^{1/q}.
\]

Due to (5.3.22) we have that \( T_\varepsilon < \infty \) for all \( \varepsilon \in (0,1) \). Moreover,

\[
T_0 = \lim_{\varepsilon \to 0} T_\varepsilon = \lim_{\varepsilon \to 0} (1 - \varepsilon)^p \left( \int_0^1 \left| \sum_{i=i_0}^m (-1)^i c_{m,i} t^{\mu_{m,i}} \int t K_{i+1,n-1}(t,x) x^{-\alpha_n} dx \right|^q dt \right)^{1/q} = \left( \int_0^1 \left| D_j^m (t^{-\alpha_n} K_{1,n}(t,1)) \right|^q dt \right)^{1/q} \neq 0,
\]

since, according to Lemma 5.1.4, \( D_j^m (t^{-\alpha_n} K_{1,n}(t,1)) \neq 0 \) for all \( t \in (0,1) \). Consequently, \( \|D_j^m f_\varepsilon\|_q \neq 0 \) when \( \varepsilon \to 0 \), that is \( f_\varepsilon \) does not converge to zero in \( W_m^{q,3} \) when \( \varepsilon \to 0 \). Thus contradiction shows that in (5.3.18) it will be strong inequality, i.e. that (5.3.19) holds.

Conversely, assume that (5.3.19) holds. Then (5.3.16) and (5.3.18) yield that \( \lim_{\varepsilon \to 0} A_{i,j}(z) = 0 \) for all \( i = i_0, i_0 + 1, \ldots, m \), \( j = i, i + 1, \ldots, n - 1 \), i.e. the integral operators (5.3.7) are compact from \( L_p(0,1) \) to \( L_q(0,1) \) and, thus, the embedding (1.3.4) is compact.

The proof is complete.

Now we consider the following embedding

\[
W_m^{n,p}(I) \hookrightarrow W_m^{m,q}(I), \quad 0 \leq m < n.
\] (5.3.26)

In this case \( i_0 = m \). In particular, Theorems 5.3.1 and 5.3.2 imply the following:
Corollary 5.3.3 Let \( I = (0, 1) \), \( 0 \leq m < n \) and \( 1 < p \leq q < \infty \).

a) If \( M_m \geq \frac{1}{p} \), then the following conditions are equivalent:

i) The embedding (5.3.26) is bounded;
ii) The embedding (5.3.26) is compact;

iii) \( \frac{1}{q} > M_m - \left( n - m - \sum_{k=m+1}^{n} \alpha_k \right) \).

b) If \( M_m < \frac{1}{p} \), then the embedding (5.3.26) is bounded if and only if

\[
\frac{1}{q} \geq \frac{1}{p} - \left( n - m - \sum_{k=m+1}^{n} \alpha_k \right)
\]

and the embedding (5.3.26) is compact if and only if

\[
\frac{1}{q} > \frac{1}{p} - \left( n - m - \sum_{k=m+1}^{n} \alpha_k \right).
\]

In particular, from Corollary 5.3.3 it follows that the estimate of the intermediate derivatives

\[
\|D^n f\|_p \leq c \left( \|D^m f\|_p + \sum_{i=0}^{n-1} |D^i f(1)| \right), \quad 0 \leq m < n,
\]

holds for functions \( f \in W^{n,\alpha} \) if and only if

\[
n - m - \sum_{k=m+1}^{n} \alpha_k > 0 \quad \text{when} \quad M_m \geq \frac{1}{p},
\]

and

\[
n - m - \sum_{k=m+1}^{n} \alpha_k \geq 0 \quad \text{when} \quad M_m < \frac{1}{p}.
\]

Moreover, Theorems 5.3.1 and 5.3.2 yield the following statements:

Corollary 5.3.4 Let \( I = (0, 1) \), \( 0 \leq m < n \) and \( 1 < p \leq q < \infty \).

a) If \( n - i_0 - \gamma \geq \frac{1}{p} \), then the following conditions are equivalent:

i) The embedding \( L^n_{p,\gamma}(I) \hookrightarrow W^m_{q,\beta}(I) \) is bounded;

ii) The embedding \( L^n_{p,\gamma}(I) \hookrightarrow W^m_{q,\beta}(I) \) is compact;

iii) \( |\beta| - m + i_0 + \frac{1}{q} > 0 \).

b) If \( n - i_0 - \gamma < \frac{1}{p} \), then the embedding \( L^n_{p,\gamma}(I) \hookrightarrow W^m_{q,\beta}(I) \) is bounded if and only if

\[
|\beta| - \gamma + n - m + \frac{1}{q} \geq \frac{1}{p}
\]

and the embedding \( L^n_{p,\gamma}(I) \hookrightarrow W^m_{q,\beta}(I) \) is compact if and only if

\[
|\beta| - \gamma + n - m + \frac{1}{q} > \frac{1}{p}.
\]
Corollary 5.3.5 Let $I = (0, 1)$, $0 \leq m < n$ and $1 < p \leq q < \infty$.

a) If $M_{io} \geq \frac{1}{p}$, then the following conditions are equivalent:

i) The embedding $W^{n,q}_{p,\alpha}(I) \hookrightarrow L^{m,q}_{q,v}(I)$ is bounded;

ii) The embedding $W^{n,q}_{p,\alpha}(I) \hookrightarrow L^{m,q}_{q,v}(I)$ is compact;

iii) $\nu - |\bar{\alpha}| + n - m + \frac{1}{q} > M_{io}$.

b) If $M_{io} < \frac{1}{p}$, then the embedding $W^{n,q}_{p,\alpha}(I) \hookrightarrow L^{m,q}_{q,v}(I)$ is bounded if and only if $\nu - |\bar{\alpha}| + n - m + \frac{1}{q} \geq \frac{1}{p}$ and the embedding $W^{n,q}_{p,\alpha}(I) \hookrightarrow L^{m,q}_{q,v}(I)$ is compact if and only if $\nu - |\bar{\alpha}| + n - m + \frac{1}{q} > \frac{1}{p}$.

Remark 5.3.6 In Chapters 4 and 5 in the conditions of the embedding we used the terms $\gamma_{min}^{\alpha}$ and $M_{io}$, respectively. According to (5.1.2) the following connections between these quantities hold:

$$\gamma_{min}^{\alpha} = \min_{0 \leq i \leq n-1} \gamma_{i} \leq \min_{i_{0} \leq i \leq n-1} \gamma_{i} = 1 - M_{io}$$

and $\gamma_{min}^{\alpha} = 1 - M_{0}$.

Moreover, according to (5.2.3) we have that

$$\gamma_{0}^{\beta} - \gamma_{0}^{\alpha} + \beta_{0} - \alpha_{0} + \frac{1}{q} = \beta - |\bar{\alpha}| + n - m + \frac{1}{q}.$$

Hence, the statements in Theorems 4.1.2 and 4.1.6 fit well to the statements in Theorems 5.3.2 and 5.3.1. Moreover, $\gamma_{min}^{\alpha} \leq 1 - \frac{1}{p}$ implies that $M_{io} \leq 1 - \frac{1}{p}$, and from the condition $M_{io} \geq \frac{1}{p}$ it follows that $\gamma_{min}^{\alpha} \leq 1 - \frac{1}{p}$.

Therefore, in Theorem 4.1.6 only a sufficient condition in terms $\gamma_{min}^{\alpha}$ is given.

The conditions of Proposition 4.1.7 means that $i_{0} = 0$. Then $M_{0} = 1 - \gamma_{min}^{\alpha}$ and in this case the condition (4.1.23) coincides with the condition (5.2.1).

We finish this Section by considering the following example:

Example. Let $1 \leq p \leq q < \infty$, $0 \leq m < n$, $\bar{\alpha} = (0, 0, \ldots, \alpha_{k_{0}}, 0, \ldots, 0)$ and $\bar{\beta} = (0, 0, \ldots, \beta_{0}, 0, \ldots, 0)$, where $0 \leq k_{0} \leq n$ and $0 \leq I_{0} \leq m$. Then the differential operations $D_{\bar{\alpha}}^{i}$ and $D_{\bar{\beta}}^{i}$ have the following forms:

$$D_{\bar{\alpha}}^{i} f(t) = \frac{d^{k_{0}}}{dt^{k_{0}}} \frac{d^{\alpha_{k_{0}}}}{dt^{k_{0}}} \frac{d^{k_{0}}}{dt^{k_{0}}} f(t), \quad 0 \leq k_{0} \leq i \leq n,$$
\[ D_t^i f(t) = \frac{d^{i-l_0}}{dt^{i-l_0}} M_{l_0} \frac{d^{l_0}}{dt^{l_0}} f(t), \quad 0 \leq l_0 \leq i \leq m. \]

We investigate for which \( \alpha_{k_0} \) and \( \beta_{k_0} \) the embedding (1.3.4) is bounded and compact.

Consider two cases:

1) If \( k_0 \leq l_0 \), then by the definitions of \( i_0 \) and \( M_i \) we obtain that:

\[
M_{i_0} = M_{k_0} = \max_{k_0 \leq s \leq n-1} (n - s - \sum_{j=s+1}^{n} \alpha_j) = n - k_0.
\]

Therefore, for all \( k_0 = 0, 1, \ldots, n - 1 \) we have that \( M_{i_0} = n - k_0 > \frac{1}{p} \).

According to Theorem 5.3.1, in this case the embedding (1.3.4) is bounded and compact if

\[
|\beta| - |\alpha| + n - m + \frac{1}{q} = \beta_{l_0} - \alpha_{k_0} + n - m + \frac{1}{q} > n - k_0
\]

holds. Therefore, (1.3.4) is bounded and compact if \( \beta_{l_0} - \alpha_{k_0} + n - m + \frac{1}{q} > 0 \).

In particular, this yields if \( \alpha_{k_0} = k_0 \) and \( \beta_{l_0} > m - \frac{1}{q} \).

2) If \( l_0 < k_0 \), then \( M_0 = M_{i_0} = \max_{k_0 \leq s \leq n-1} (n - s - \alpha_{k_0}) = n - l_0 - \alpha_{k_0} \).

Hence, it yields that:

a) if \( \alpha_{k_0} > n - l_0 - \frac{1}{p} \), then \( M_{i_0} < \frac{1}{p} \). Therefore, by Theorem 5.3.2 the embedding (1.3.4) is bounded if \( \beta_{l_0} - \alpha_{k_0} + n - m + \frac{1}{q} \geq \frac{1}{p} \), and compact if \( \beta_{l_0} - \alpha_{k_0} + n - m + \frac{1}{q} > \frac{1}{p} \). In particular, this yields if \( \alpha_{k_0} = n - l_0 - \frac{1}{p} + 1 \), \( \beta_{l_0} \geq m - l_0 - \frac{1}{q} + 1 \) and \( b_{l_0} > m - l_0 - \frac{1}{q} + 1 \), respectively.

b) if \( \alpha_{k_0} \leq n - l_0 - \frac{1}{p} \), then \( M_{i_0} \geq \frac{1}{p} \). Therefore, due to Theorem 5.3.1 the embedding (1.3.4) is bounded and compact if \( \beta_{l_0} - \alpha_{k_0} + n - m + \frac{1}{q} > M_{i_0} \). In particular, this holds if \( \alpha_{k_0} = n - l_0 - \frac{1}{p}, \beta_{l_0} > m - l_0 - \frac{1}{q} \) or if \( \alpha_{k_0} = n - l_0 - 1 \), \( \beta_{l_0} > m - l_0 - \frac{1}{q} \).
5.4 The corresponding results for the spaces
\( W_{p,\tilde{\alpha}}^n(1, \infty) \)

By using the connection between the spaces \( W_{p,\tilde{\alpha}}^n(0, 1) \) and \( W_{p,\tilde{\alpha}}^n(1, \infty) \), which has been discussed in Section 4.4, we can establish the corresponding results (to the results for the space \( W_{p,\tilde{\alpha}}^n(0, 1) \)) for the case with the space \( W_{p,\tilde{\alpha}}^n(1, +\infty) \).

We note that
\[
\tilde{M}_i = \max_{i \leq s \leq n-1} \left( n - s - \sum_{k=s+1}^{n} \tilde{\alpha}_k \right) = \\
= \max_{i \leq s \leq n-1} \left( n - s - \sum_{k=s+1}^{n-1} (-\alpha_k + 2) + \alpha_n - 2 + \frac{2}{p} \right) = \\
= \max_{i \leq s \leq n-1} \left( -(n - s - \sum_{k=s+1}^{n} \alpha_k) + \frac{2}{p} \right) = -M_i + \frac{2}{p},
\]

where \( M_i = \min_{i \leq s \leq n-1} (n - s - \sum_{k=s+1}^{n} \alpha_k) \), \( i = 0, 1, \ldots, n - 1 \).

Since \( |\tilde{\beta}| = \sum_{i=1}^{m-1} (-\beta_i + 2) - \beta_0 - \beta_m + 2 - \frac{2}{q} = -|\tilde{\beta}| + 2m - \frac{2}{q} \) and \( |\tilde{\alpha}| = -|\alpha| + 2n - \frac{2}{p} \), then from (5.3.1) and (5.3.18), respectively, we have that
\[
|\tilde{\beta}| - |\tilde{\alpha}| + n - m + \frac{1}{q} = |\tilde{\alpha}| - |\tilde{\beta}| + 2m - 2n + n - m + \frac{1}{q} - \frac{2}{q} + \frac{2}{p} = \\
= |\tilde{\alpha}| - |\tilde{\beta}| + m - n - \frac{1}{q} + \frac{2}{p} > \tilde{M}_{io} \quad (5.4.1)
\]

and
\[
|\tilde{\beta}| - |\tilde{\alpha}| + n - m + \frac{1}{q} = |\tilde{\alpha}| - |\tilde{\beta}| + m - n - \frac{1}{q} + \frac{2}{p} \geq \frac{1}{p}. \quad (5.4.2)
\]

When \( \tilde{M}_{io} = -M_{io} + \frac{2}{p} \geq \frac{1}{p} \) this is equivalent to that \( M_{io} \leq \frac{1}{p} \) and from (5.4.1) it follows that
\[
|\tilde{\alpha}| - |\tilde{\beta}| + m - n - \frac{1}{q} + \frac{2}{p} > -M_{io} + \frac{2}{p},
\]

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that is
\[ |\beta| - |\alpha| + n - m + \frac{1}{q} < M_{\alpha} \text{ when } M_{\alpha} \leq \frac{1}{p}. \]

When \( M_{\alpha} < \frac{1}{p} \), that is \( M_{\alpha} > \frac{1}{p} \), from (5.4.2) we get that
\[ |\alpha| - |\beta| + m - n - \frac{1}{q} + \frac{2}{p} \geq \frac{1}{p}, \]
i.e.
\[ |\beta| - |\alpha| + n - m + \frac{1}{q} \leq \frac{1}{p} \text{ when } M_{\alpha} > \frac{1}{p}. \]

Correspondingly, the condition (5.3.1) will be changed to the condition
\[ |\beta| - |\alpha| + n - m + \frac{1}{q} < \min\{\frac{1}{p}, M_{\alpha}\}. \]

1. The case \( 1 \leq q < p < \infty \).

The following statements follow from Theorem 5.2.1, Corollary 5.2.2 and Corollary 5.2.3, respectively:

**Theorem 5.4.1** Let \( I = (1, +\infty) \), \( 1 \leq q < p < \infty \) and \( 0 \leq m < n \). Then the following conditions are equivalent:

i) The embedding (1.3.4) is bounded;

ii) The embedding (1.3.4) is compact;

iii) \( |\beta| - |\alpha| + n - m + \frac{1}{q} < \min\{\frac{1}{p}, M_{\alpha}\}. \)

Note that, in the space \( L^p_{\mu, \gamma}(1, +\infty) \) we have that \( M_{\alpha} = 1 - \gamma \).

**Corollary 5.4.2** Let \( I = (1, +\infty) \), \( 0 \leq m < n \) and \( 1 \leq q < p < \infty \). Then the following conditions are equivalent:

i) The embedding \( L^p_{\mu, \gamma}(I) \hookrightarrow W^q_{\mu, \beta}(I) \) is bounded;

ii) The embedding \( L^p_{\mu, \gamma}(I) \hookrightarrow W^q_{\mu, \beta}(I) \) is compact;

iii) \( |\beta| - \gamma + n - m + \frac{1}{q} < \min\{1 - \gamma, \frac{1}{p}\}, \)

**Corollary 5.4.3** Let \( I = (1, +\infty) \), \( 0 \leq m < n \) and \( 1 \leq q < p < \infty \). Then the following conditions are equivalent:

i) The embedding \( W^p_{\mu, \alpha}(I) \hookrightarrow L^q_{\mu, \beta}(I) \) is bounded;

ii) The embedding \( W^p_{\mu, \alpha}(I) \hookrightarrow L^q_{\mu, \beta}(I) \) is compact;

iii) \( v - |\alpha| + n - m + \frac{1}{q} < \min\{M_{\alpha}, \frac{1}{p}\}, \)

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2. The case $1 < p \leq q < \infty$.

Theorems 5.3.1 and 5.3.2, and Corollaries 5.3.3-5.3.5, respectively, imply the following results:

**Theorem 5.4.4** Let $I = (1, +\infty)$, $1 < p \leq q < \infty$, $0 \leq m < n$ and $M_{\alpha} \leq \frac{1}{p}$. Then the following conditions are equivalent:

i) The embedding (1.3.4) is bounded;

ii) The embedding (1.3.4) is compact;

\[ m|\bar{\beta}| - |\bar{\alpha}| + n - m + \frac{1}{q} < M_{\alpha}. \]

**Theorem 5.4.5** Let $I = (1, +\infty)$, $1 < p \leq q < \infty$, $0 \leq m < n$ and $M_{\alpha} > \frac{1}{p}$.

a) The embedding (1.3.4) is bounded if and only if

\[ |\bar{\beta}| - |\bar{\alpha}| + n - m + \frac{1}{q} \leq \frac{1}{p}. \]

b) The embedding (1.3.4) is compact if and only if

\[ |\bar{\beta}| - |\bar{\alpha}| + n - m + \frac{1}{q} < \frac{1}{p}. \]

**Corollary 5.4.6** Let $I = (1, +\infty)$, $0 \leq m < n$ and $1 < p \leq q < \infty$.

a) If $M_{\alpha} \leq \frac{1}{p}$, then the following conditions are equivalent:

i) The embedding (5.3.26) is bounded;

ii) The embedding (5.3.26) is compact;

\[ mM_{\alpha} - \left( n - m - \sum_{k=m+1}^{n} \alpha_k \right) + \frac{1}{q} > 0. \]

b) If $M_{\alpha} > \frac{1}{p}$, then the embedding (5.3.26) is bounded if and only if

\[ \frac{1}{p} + \frac{1}{q} - \left( n - m - \sum_{k=m+1}^{n} \alpha_k \right) \geq 0 \] and the embedding (5.3.26) is compact if and only if

\[ \frac{1}{p} + \frac{1}{q} - \left( n - m - \sum_{k=m+1}^{n} \alpha_k \right) > 0. \]

In particular, Corollary 5.4.6 yields that for the functions $f \in W_{p, \alpha}^{n}(I)$ the following estimate of intermediate derivatives holds

\[ ||D_{\alpha}^{m}f||_{p} \leq c \left( ||D_{\alpha}^{m}f||_{p} + \sum_{i=0}^{n-1} |D_{\alpha}^{i}f(1)| \right), \quad 0 \leq m < n, \]
if and only if
\[ n - m - \sum_{k=m+1}^{n} \alpha_k < 0 \text{ when } M_m \leq \frac{1}{p}, \]
and
\[ n - m - \sum_{k=m+1}^{n} \alpha_k \leq 0 \text{ when } M_m > \frac{1}{p}. \]

**Corollary 5.4.7** Let \( I = (1, +\infty), 0 \leq m < n \) and \( 1 < p \leq q < \infty \).

a) If \( 1 - \gamma \leq \frac{1}{p} \), then the following conditions are equivalent:

i) The embedding \( L^m_{p,\gamma}(I) \hookrightarrow W^m_{q,\beta}(I) \) is bounded;

ii) The embedding \( L^m_{p,\gamma}(I) \hookrightarrow W^m_{q,\beta}(I) \) is compact;

iii) \(|\tilde{\beta}| + n - m + \frac{1}{q} - 1 < 0.\)

b) If \( 1 - \gamma > \frac{1}{p} \), then the embedding \( L^m_{p,\gamma}(I) \hookrightarrow W^m_{q,\beta}(I) \) is bounded if and only if \(|\tilde{\beta}| - \gamma + n - m + \frac{1}{q} \leq \frac{1}{p} \) and the embedding \( L^m_{p,\gamma}(I) \hookrightarrow W^m_{q,\beta}(I) \) is compact if and only if \(|\tilde{\beta}| - \gamma + n - m + \frac{1}{q} < \frac{1}{p} \).

**Corollary 5.4.8** Let \( I = (1, +\infty), 0 \leq m < n \) and \( 1 < p \leq q < \infty \).

a) If \( M_m \leq \frac{1}{p} \), then the following conditions are equivalent:

i) The embedding \( W^m_{p,\alpha}(I) \hookrightarrow L^m_{q,v}(I) \) is bounded;

ii) The embedding \( W^m_{p,\alpha}(I) \hookrightarrow L^m_{q,v}(I) \) is compact;

iii) \( \nu - |\tilde{\alpha}| + n - m + \frac{1}{q} < M_m. \)

b) If \( M_m > \frac{1}{p} \), then the embedding \( W^m_{p,\alpha}(I) \hookrightarrow L^m_{q,v}(I) \) is bounded if and only if \( \nu - |\tilde{\alpha}| + n - m + \frac{1}{q} \leq \frac{1}{p} \) and the embedding \( W^m_{p,\alpha}(I) \hookrightarrow L^m_{q,v}(I) \) is compact if and only if \( \nu - |\tilde{\alpha}| + n - m + \frac{1}{q} < \frac{1}{p}. \)
Bibliography


