Weighted Hardy-Type Inequalities on the Cones of Monotone and Quasi-Concave Functions

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and Quasi-concave Functions

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Key words and phrases. Weights, Hardy operator, measures, Hardy-type inequalities, monotone functions, Volterra operator, Lorentz space, maximal operator, concave functions, quasi-concave functions.
Abstract

This PhD thesis deals with weighted Hardy-type inequalities restricted to the cones of monotone functions and quasi-concave functions.

The thesis consists of four papers (papers A, B, C and D) and an introduction, which gives an overview of this specific field of functional analysis and also serves to put these papers into a more general frame. The papers A and B are devoted to characterizing some weighted Hardy-type inequalities on the cones of monotone functions while in the papers C and D we solve the similar problems for the cones of quasi-concave and $\psi$–quasi-concave functions.

In paper A some two-sided inequalities for Hardy operators on the cones of monotone functions are proved. The idea to study such equivalences follows from the Hardy inequality

\[ \left( \int_{[0,\infty)} f^p d\lambda \right)^{\frac{1}{p}} \leq \left( \int_{[0,\infty)} \left( \frac{1}{\Lambda(x)} \int_{[0,x]} f(t) d\lambda(t) \right)^p d\lambda(x) \right)^{\frac{1}{p}} \]

\[ \leq \frac{p}{p-1} \left( \int_{[0,\infty)} f^p d\lambda \right)^{\frac{1}{p}}, \]

which holds for any $f \in \mathcal{M}$ and $1 < p < \infty$. In paper A similar equivalences are found for some other Hardy-type operators for the full range of parameter $-\infty < p < \infty$. As an application we obtain a new characterization of the discrete Hardy inequality for the most difficult case of parameters, namely when $0 < q < p \leq 1$.

The equivalences we proved in paper A are used in paper B to obtain necessary and sufficient conditions for some other Hardy-type inequalities on cones of monotone functions. In particular, we give a complete description for inequalities with Volterra integral operators involving Oinarov’s kernel for parameters $0 < p < \infty$, $1 \leq q < \infty$. We also study inequalities of the
type
\[
\left( \int_{[0,\infty)} \left( \int_{(r,\infty)} f d\lambda \right)^q d\lambda \right)^{\frac{1}{q}} \leq C \left( \int_{[0,\infty)} f^p d\mu \right)^{\frac{1}{p}}, \quad f \in M \downarrow, \ f \neq 0
\]
and
\[
\left( \int_{[0,\infty)} \left( \int_{[0,x]} f d\lambda \right)^q d\lambda \right)^{\frac{1}{q}} \leq C \left( \int_{[0,\infty)} f^p d\mu \right)^{\frac{1}{p}}, \quad f \in M \uparrow, \ f \neq 0
\]
and find necessary and sufficient conditions not only for positive, but also for negative parameters of summation.

Paper C is motivated by the well-known characterization problem of the boundedness of the classical Hardy-Littlewood maximal operator in weighted Lorentz $\Gamma-$spaces. Here we introduce the classes of quasi-concave and $\psi-$quasi-concave functions and derive criteria for the inequality of the form
\[
\left( \int_{[0,\infty)} (Af)^q d\gamma \right)^{\frac{1}{q}} \leq C \left( \int_{[0,\infty)} f^p d\beta \right)^{\frac{1}{p}}
\]
with
\[
Af(t) = \left( \int_{[0,t]} f^p d\mu \right)^{\frac{1}{p}},
\]
and also for the one with the operator (0.2) replaced by
\[
Bf(t) = \left( \int_{(t,\infty)} f^p d\mu \right)^{\frac{1}{p}},
\]
to hold on the cone of $\psi-$quasi-concave functions for the range of parameters $0 < p < \infty$, $1 \leq q < \infty$. For the rest of the range we come up with the sufficient conditions for the inequality (0.1) with operator (0.2) to hold.

In paper D we extend the method of characterization we have used in paper C for the cone of quasi-concave functions to a cone of $\psi-$quasi-concave functions to obtain yet another characterization for the inequality (0.1) with the operator (0.3) to hold on this cone. As an application we obtain some new results concerning mapping properties of the Hilbert transform and Riesz potentials in weighted Lorentz spaces.
Preface

This PhD thesis consists of four papers (papers A, B, C and D) and an introduction, which puts these papers into a more general frame.


The results of paper A have also been announced in the following paper:

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Acknowledgment

I want to express my deep gratitude to my main supervisors Professor Lars-Erik Persson and Professor Vladimir D. Stepanov for their valuable remarks and attention to my work and their incessant support.

I want to thank my co-supervisor Docent Thomas Strömberg for his help in editing my work and also Doctor John Fabricius and Doctor Yulia Koroleva for their help in many practical things.

I thank Luleå University of Technology and Peoples’ Friendship University of Russia for giving me an opportunity to participate in a partnership program in research and postgraduate education in mathematics. I also thank both universities for financial support which made my studies possible.

Finally, I am grateful to my family and friends helping me in every possible way.
Introduction

Hardy-type inequalities and their applications compose a great part of Classical Analysis concentrated around the characterization problems for various integral and differential inequalities to hold. In particular, the inequalities restricted to the cone of monotone functions have been intensively studied during the last two decades.

The study of Hardy-type inequalities in weighted Lebesgue spaces began in 1915 with the work of G.H. Hardy and his contemporaries. In 1925 G.H. Hardy proved the following result [31]:

Suppose that \( f(x) > 0, p > 1, f \) is integrable over any finite interval \((0, X)\) and \( f^p \) is integrable over \((0, \infty)\). Then

\[
\int_0^\infty \left( \frac{1}{x} \int_0^x f(t)dt \right)^p dx \leq \left( \frac{p}{p - 1} \right)^p \int_0^\infty f^p(x)dx.
\]

The dramatic period of almost 10 years of research, which involved such mathematicians as M. Riesz, E. Landau and G. Polya, was recently described in the book [42].

Then the first weighted modifications appeared (see [32]), especially the ones with power weights, which more frequently can be seen in applications. Finally the inequality appeared in the form

\[
\left( \int_a^b \left( \int_a^x f(t)dt \right)^q u(x)dx \right)^{\frac{1}{q}} \leq C \left( \int_a^b f^p(x)v(x)dx \right)^{\frac{1}{p}},
\]

where \(-\infty \leq a < b \leq \infty, 0 < q < \infty, 1 \leq p < \infty\) and \( u, v \) are measurable functions (weights), positive a.e. in \((a, b)\).

Let us mention some important results on the inequality (0.5). We notate \( p' := \frac{p}{p - 1} \) for \( 0 < p < \infty, p \neq 1 \), \( \frac{1}{r} := \frac{1}{q} - \frac{1}{p} \) for \( 0 < q < p < \infty \).

**Theorem 1.** (i) If \( 1 \leq p \leq q < \infty \), then (0.5) holds for all measurable
functions \( f(x) \geq 0 \) on \((a, b)\) iff

\[
A_1 := \sup_{a < x < b} \left( \int_x^b u(t) dt \right)^{\frac{1}{p}} \left( \int_a^x v^{1-p'}(t) dt \right)^{\frac{1}{p'}} < \infty.
\]

(ii) If \( 1 \leq q < p < \infty \), then (0.5) holds iff

\[
A_2 := \left( \int_a^b \left( \int_x^b u(t) dt \right)^{\frac{q}{r}} \left( \int_a^x v^{1-p'}(t) dt \right)^{\frac{q}{r'}} v^{1-p'}(x) dx \right)^{\frac{1}{r}} < \infty.
\]

(iii) If \( 0 < q < 1 < p < \infty \), then (0.5) holds iff

\[
A_3 := \left( \int_a^b \left( \int_x^b u(t) dt \right)^{\frac{q}{r}} \left( \int_a^x v^{1-p'}(t) dt \right)^{\frac{q}{r'}} u(x) dx \right)^{\frac{1}{r}} < \infty.
\]

(iv) If \( 0 < q < 1 = p \), then (0.5) holds iff

\[
A_4 := \left( \int_a^b \left[ \bar{v}(x) \int_x^b u(t) dt \right]^{\frac{q}{r-q}} u(x) dx \right)^{\frac{1}{r-1}} < \infty,
\]

where \( \bar{v}(x) = \text{ess sup}_{a < t < x} v(t) \).

The proof of (i) can be found in [18], [48], [49], [54, Theorem 1.14], [86], [3], [93], [94] and [41, Theorem 1.1], the proof of (ii) in [48], [49], [54, Theorem 1.15], [33] and [41, Theorem 1.2]. For the proof of (iii) see [75] and [54, Chapter 9]. The proof of the case \( 0 < q < p, 1 < p < \infty \) was given in [81, Theorem 2.4], and the proof of (iv) in [81, Theorem 3.3]. The case \( 0 < q < p, p \geq 1 \) was also proved by E. Sawyer in [72], where he presented an equivalent condition, and by G. Sinnamon in [75], [76], [77].

The problem of the best constant has been studied in several papers. For example, for \( p < q \) it was investigated in [15] in the classical case and in [46] for (0.5).

**Remark 1.** It is well known that the conditions \( A_1 < \infty \) and \( A_2 < \infty \) can be replaced by different (equivalent) conditions. For example, the condition \( A_2 < \infty \) can be replaced by the following one (see the paper [60] by L.-E. Persson and V.D. Stepanov):

\[
A^* := \int_0^\infty \left( \int_0^x \left( \int_0^t v(s)^{1-p'} ds \right)^q dt \right)^{\frac{1}{r}} \left( \int_0^x v(t)^{1-p'} dt \right)^{\frac{1}{r'}} v(x)^{1-p'} dx < \infty.
\]
More generally, the condition $A_2 < \infty$ can be replaced by four different scales of conditions (see [62]) and the condition $A_1 < \infty$ can be replaced by fourteen different scales of conditions (see [22] and the references given there).

The analogous criteria were found for the dual inequality which has the form

$$\left( \int_a^b \left( \int_x^b f(t) \, dt \right)^q \, u(x) \, dx \right)^{\frac{1}{q}} \leq C \left( \int_a^b f^p(x)v(x) \, dx \right)^{\frac{1}{p}}.$$

The inequality (0.5) can also be written in a short way

$$\|Hf\|_{q,u} \leq C\|f\|_{p,v},$$

where the Hardy operator is given by

$$(Hf)(x) := \int_a^x f(t) \, dt.$$  (0.7)

We use here the notion of a norm (quasi-norm, if $p < 1$) in a weighted Lebesgue space $L^p(a,b,u)$, where $0 < p \leq \infty$, $u$ is a weight function on $(a,b)$. The space $L^p(a,b,u)$ consists of all measurable functions on $(a,b)$ such that

$$\|f\|_{p,u} := \left( \int_a^b |f(x)|^p u(x) \, dx \right)^{\frac{1}{p}} < \infty, \quad 0 < p < \infty,$$

and

$$\|f\|_{\infty,u} := \text{ess sup}_{a < x < b} |f(x)| < \infty.$$

It is obvious that we can also write

$$\|\tilde{H}f\|_{q,u} \leq C\|f\|_{p,v},$$

for the dual inequality, where $\tilde{H}$ is a conjugate Hardy operator:

$$\tilde{(Hf)}(x) := \int_x^b f(t) \, dt.$$  (0.8)

The inequalities (0.6) and (0.8) can be extended to a more generalized form

$$\|Tf\|_{q,u} \leq C\|f\|_{p,v},$$

where $T$ is an integral operator. The validity of this inequality is equivalent to the continuity of the mapping $T : L^p(v) \to L^q(u)$.

Also the weighted Hardy operator of the form

$$(H_{u,v}f)(x) := u(x) \int_a^x v(t)f(t) \, dt$$  (0.9)
is used. Clearly, the inequality
\[ \|H_{u,v}f\|_q \leq C\|f\|_p \]
can be easily reduced to the original inequality (0.6).

The reverse Hardy-type inequality
\[ \left( \int_0^\infty f(x)^p dx \right)^{1/p} \leq C \left( \int_0^\infty \left( \int_0^x f(y)v(y)dy \right)^q dx \right)^{1/q}, \quad f > 0 \quad (0.10) \]
for \(-\infty < p, q < 0\) and \(0 < p, q < 1\) was studied in [4], where P.R. Beesack and H.P. Heinig obtained sufficient and necessary conditions for (0.10) to hold. However, the authors pointed out that the necessary conditions required some restrictions on the weight functions and therefore were weaker than the sufficient conditions.

D.V. Prokhorov in [69] gave the precise characterization of (0.10) for the same range of parameters \(p, q\), as Beesack and Heinig. He also established a duality between the cases \(p, q < 0\) and \(0 < p, q < 1\). Specifically, he proved that the Hardy-type inequality with negative indices of integration is equivalent to the same inequality with the dual operator for conjugate indices. This principle of duality allowed to extend the conditions obtained for (0.10) in cases \(-\infty < p < q < 0\) and \(-\infty < q \leq p < 0\) to the case \(0 < p, q < 1\). The principle of duality reads as follows.

**Theorem 2** [69]. Let \(-\infty < p, q < 0\), \(0 < C < +\infty\), \(k\) be a non-negative measurable function on \((a, b) \times (a, b)\) and
\[
(Tf)(x) := \int_a^b k(x, y)f(y)dy,
\]
\[
(T^*g)(x) := \int_a^b k(y, x)g(y)dy
\]
for \(x \in (a, b)\). Then the inequality
\[ \left( \int_a^b f(x)^p dx \right)^{1/p} \leq C \left( \int_a^b [(Tf)(x)]^q dx \right)^{1/q} \]
holds for all \(f \in M^+(a, b)\) iff the inequality
\[ \left( \int_a^b g(x)^q dx \right)^{1/q} \leq C \left( \int_a^b [(T^*g)(x)]^p dx \right)^{1/p} \]
holds for all \( g \in \mathfrak{M}^+(a, b) \). In particular, it follows that the least possible constants in these inequalities are equal.

Here \( \mathfrak{M}^+(a, b) \) denotes the class of all measurable functions \( f : (a, b) \rightarrow [0, +\infty] \).

The case \(-\infty < q \leq p < 0\) for (0.10) was described in [39]. The curious cases \( q < 0, p > 0 \) and \( q > 0, p < 0 \) were studied in [40]. We summarize the results of [40] as follows.

**Theorem 3** [40].

(i) Let \( p < 0 \) and \( q > 0 \). Then the inequality

\[
\left( \int_a^b \left( \int_a^t f(t)v(t)dt \right)^q u(x)dx \right)^{\frac{1}{q}} \geq C \left( \int_a^b f(x)^p dx \right)^{\frac{1}{p}} \tag{0.11}
\]

holds for all \( f \geq 0 \) iff there exists \( \tau \in (a, b) \) such that

\[ A(\tau) > 0, \]

where

\[ A(t) := \left( \int_a^t v(x)dx \right)^{\frac{p}{p'}} \left( \int_a^b u(x)dx \right)^{\frac{1}{q}}, \quad p' = \frac{p}{p-1}. \]

(ii) Let \( 0 < p < 1 \) and \( q < 0 \). Then (0.11) holds for all \( f \geq 0 \) iff

\[ A_* := \inf_{(a,b)} A(t) > 0. \]

(iii) Let \( p \geq 1 \) and \( q < 0 \). Then (0.11) holds for all \( f \geq 0 \) iff there exists \( \tau \in (a, b) \) such that

\[ B(\tau) < \infty, \]

where

\[ B(t) := \begin{cases} \left( \text{ess sup}_{[a,t]} v(x) \right) \left( \int_a^t u(x)dx \right)^{\frac{1}{q}}, & \text{if} \quad p = 1, \\ \left( \int_a^t v(x)dx \right)^{\frac{1}{p'}} \left( \int_a^t u(x)dx \right)^{\frac{1}{q}}, & \text{if} \quad p > 1. \end{cases} \]

For the Hardy inequality with arbitrary \( \sigma \)-finite Borel measures there are less results than there are for the case of Lebesgue measures. In the general case the inequality has the form

\[
\left( \int_{[a,b]} \left( \int_{[a,x]} fud\lambda \right)^q v(x)d\mu(x) \right)^{\frac{1}{q}} \leq C \left( \int_{[a,b]} f^p d\nu \right)^{\frac{1}{p}}, \tag{0.12}
\]
where \( f \in \mathcal{M}^+ \). The case \( u = v = w \equiv 1 \), where \( \lambda \) is a Lebesgue measure, was described in [49]. The more general cases for two (when \( \lambda = \nu \) and three measures for \( p > 1, q > 0 \) were in full characterized in [66]. The criterion for three measures is given in [66] in terms which include \((\nu_a, \nu_s)\) — the Lebesgue decomposition of measure \( \nu \) relative to \( \lambda \), where \( d\nu_a/d\lambda \) denotes the Radon-Nikodym derivative of \( \nu_a \) with respect to \( \lambda \).

Similar to the weighted Hardy inequalities in Lebesgue spaces, they can also be studied in the Lorentz spaces \( \Lambda^p(w), 0 < p < \infty \), where

\[
\Lambda^p(w) := \left\{ f : \| f^* \|_{p,w} = \left( \int_0^\infty (f^*(t))^p w(t) dt \right)^{\frac{1}{p}} < \infty \right\}.
\]

Here \( f^* \) is the equimeasurable decreasing rearrangement of \(|f|\) defined by

\[
f^*(t) := \inf \left\{ y > 0 : \lambda_f(y) \leq t \right\},
\]

where \( \lambda_f \) is the distribution function:

\[
\lambda_f(y) := \text{mes} \{ x \in X : |f(x)| > y \}.
\]

These spaces were first introduced by G. Lorentz in 1951 in the paper [44]. A weight characterization of classical operators in Lorentz spaces brought up the necessity to study operators defined on the cone of decreasing functions.

For instance, let the Hardy-Littlewood maximal function \( Mf \) be defined by

\[
(Mf)(x) := \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(z)| dz, \quad x \in \mathbb{R}^n,
\]

where \( Q \) is a cube in \( \mathbb{R}^n \) with sides parallel to the coordinate axes and \( |Q| \) is its Lebesgue measure. It is well known that

\[
(Mf)^*(t) \approx \frac{1}{t} \int_0^t f^*(s) ds, \quad t > 0
\]

(see, for example, [2] or [5] for the proofs and historical remarks concerning this estimate). Thus, the characterization of weight functions \( u \) and \( v \), for which the mapping

\[
M : \Lambda^p(v) \to \Lambda^q(u), \quad 1 < p, q < \infty,
\]

is bounded between Lorentz spaces, is equivalent to the characterization of weight functions \( u \) and \( v \), for which the Hardy operator \( P \) defined by

\[
(Pf)(t) := \frac{1}{t} \int_0^t f(s) ds, \quad t \geq 0,
\]
is bounded from $L^p(v)$ to $L^q(u)$, $0 < p, q < \infty$ on the cone of non-negative decreasing functions. That means, we need to characterize the weight functions, for which the Hardy inequality
\[
\left( \int_0^\infty \left( \frac{1}{t} \int_0^t f(s) \, ds \right)^q w(t) \, dt \right)^{\frac{1}{q}} \leq C \left( \int_0^\infty f^p(t) v(t) \, dt \right)^{\frac{1}{p}}
\] (0.14)
holds for all decreasing functions $f \geq 0$.

This case clearly differs from the non-monotone one, because the inequality makes sense for all values of positive parameters of integration $p$ and $q$ and, naturally, the study breaks up into four cases: (I) $1 \leq p \leq q < \infty$; (II) $0 < q < p < \infty, p > 1$; (III) $0 < p \leq q, 0 < p \leq 1$ and (IV) $0 < q < p \leq 1$.

The above problem has been investigated by many authors. One of the first results was obtained by D.W. Boyd in 1967 (see [17]). He received conditions on the weight $w$, for which the inequality $\|Hf\|_{p,w} \leq A\|f\|_{p,w}$ holds for all $0 \leq f \downarrow$. His criterion reads as follows:

There exist positive constants $C > 0$ and $0 \leq \gamma < 1$ such that $\frac{w(t)}{W(st)} \leq Cs^{-\gamma p}$ for all $0 < s \leq 1$, where $W(x) = \int_0^x w(t) \, dt$.

The same case was also studied in [38].

In 1990 the necessary and sufficient condition for (0.14) in the case $1 \leq p = q < \infty$ and $u(t) = v(t)$ was proved in [1] by M. Ariño and B. Muckenhoupt. The result is the following:

Let $1 \leq p < \infty$. Then the inequality
\[
\left( \int_0^\infty \left( \frac{1}{t} \int_0^t f(s) \, ds \right)^p w(t) \, dt \right)^{\frac{1}{p}} \leq C \left( \int_0^\infty f^p(t) w(t) \, dt \right)^{\frac{1}{p}}
\] (0.15)
holds for all $0 \leq f \downarrow$ if and only if there is a constant $D > 0$ such that
\[
\int_t^\infty s^{-p} w(s) \, ds \leq Dt^{-p} \int_0^t w(s) \, ds \text{ for all } t > 0.
\]

This condition is sometimes called the $B_p$-condition. It is known nowadays that the $B_p$-condition can be replaced by infinitely many equivalent conditions (see [23] and the references given there).

Observe that the result is valid also for $0 < p < 1$. This was proved in [83] and [21].

If we introduce
\[
\|w\|_{B_p} := \sup_{r > 0} \frac{\int_r^\infty w(t) \, dt + r^p \int_r^\infty \frac{w(t)}{t^p} \, dt}{\int_0^r w(t) \, dt} < \infty,
\]
then, due to C.J. Neugebauer \[52\], for the constant $C$ in (0.15) it yields that

$$C = \|w\|_{B_p}^{\frac{1}{p}}, \quad p \leq 1,$$

and

$$\|w\|_{B_p}^{\frac{1}{p}} \leq C \leq \|w\|_{B_p}, \quad p > 1.$$

The study of such inequalities on the cone of decreasing functions clearly differs from investigating the similar problem on the set of positive functions and has some peculiarities which require careful consideration. Note, that there is an example of weight \(w\), for which (0.15) is true on the cone of monotone functions, but invalid for all \(f \geq 0\) (see \[1\]).

Let \(1 \leq q < \infty\). As an example we consider the function

$$W(x) := \begin{cases} 0, & 1 < x < 2, \\ x^{-\frac{1}{q}}, & 0 \leq x \leq 1 \text{ or } 2 \leq x. \end{cases}$$

This function does not satisfy the condition

$$\sup_{r > 0} \left[ \int_{r}^{\infty} \frac{W(x)}{x^q} \, dx \right] \left[ \int_{0}^{r} W(x)^{-\frac{q'}{q}} \, dx \right]^{\frac{q}{q'}} < \infty,$$

where the second factor is taken to be \(\text{ess sup}_{[0,r]} \frac{1}{W(x)}\) in case \(q = 1\) and \(q' = \frac{q}{q-1}\).

It was proved by B. Muckenhoupt in \[50, \text{Theorem 1}\], that this condition is equivalent to the validity of inequality (0.15) with the weight \(W\) on the set all non-negative functions \(f\) on \([0, \infty)\). However, with the restriction that \(f\) is non-negative and non-increasing it can be shown that the inequality (0.15) holds.

The result of M. Ariño and B. Muckenhoupt was extended by E. Sawyer \[71\] to the case of different weights \(v\) and \(w\) and \(1 < p, q < \infty\). In his proof he used the duality principle which is also of great importance as an independent result.

The Sawyer duality principle. Let \(1 < p < \infty\), \(g, v\) be non-negative measurable functions on \((0, \infty)\) with \(v\) locally integrable. Then

$$\sup_{0 \leq f \leq 1} \frac{\int_{0}^{\infty} f(x)g(x) \, dx}{\left( \int_{0}^{\infty} f^{p}(x)v(x) \, dx \right)^{\frac{1}{p}}} \approx \left( \int_{0}^{\infty} \left( \int_{0}^{x} g(t) \, dt \right)^{p'} \left( \int_{0}^{x} v(t) \, dt \right)^{-p'} v(x) \, dx \right)^{\frac{1}{p'}} + \frac{\int_{0}^{\infty} g(x) \, dx}{\left( \int_{0}^{\infty} v(x) \, dx \right)^{\frac{1}{p}}} \quad (0.16)$$
We note, that if $\int_0^\infty v(x)dx = \infty$, then the second term on the right hand side of (0.16) can be omitted.

**Theorem 4** [71]. (i) Let $1 < p \le q < \infty$, $V(t) := \int_0^t v(s)ds$. Then (0.14) holds for all non-increasing functions $f \ge 0$ iff

$$A_0 := \sup_{x>0} \left( \int_x^\infty u(t)dt \right)^{\frac{1}{q}} \left( \int_0^x v(t)dt \right)^{-\frac{1}{p}} < \infty,$$

$$A_1 := \sup_{x>0} \left( \int_x^\infty t^{-q}u(t)dt \right)^{\frac{1}{q}} \left( \int_0^x t^p V^{-p'}(t)v(t)dt \right)^{\frac{1}{p}} < \infty,$$

and $C \approx A_0 + A_1$.

(ii) Let $1 < q < p < \infty$, $V(t) := \int_0^t v(s)ds$. Then (0.14) holds for all non-increasing functions $f \ge 0$ iff

$$B_0 := \left( \int_0^\infty \left[ \left( \int_0^x u(t)dt \right)^{\frac{1}{p}} \left( \int_0^x v(t)dt \right)^{-\frac{1}{p}} \right]^r u(x)dx \right)^{\frac{1}{r}} < \infty,$$

$$B_1 := \left( \left[ \left( \int_x^\infty t^{-q}u(t)dt \right)^{\frac{1}{q}} \left( \int_0^x t^p V^{-p'}(t)v(t)dt \right)^{\frac{1}{p}} \right]^r x^p V^{-p'}(x)v(x)dx \right)^{\frac{1}{r}} < \infty,$$

and $C \approx B_0 + B_1$.

The paper [71] is also important because of the fact that in it, while working on the problem of characterizing the boundedness of the maximal operator between the $\Lambda-$spaces, E. Sawyer introduced the Lorentz $\Gamma-$spaces. More exactly,

$$\Gamma_p(v) := \left\{ f : \|f^{**}\|_{p,w} := \left( \int_0^\infty (f^{**}(t))^p w(t)dt \right)^{\frac{1}{p}} < \infty \right\},$$

where $f^{**}(x) := \frac{1}{x} \int_{[0,x]} f^*(t)dt$. The Lorentz $\Gamma-$spaces have gained much popularity after this work was published, although it was not the first paper where such spaces were mentioned. They had appeared earlier in the works of Calderon, Hunt and some other authors. Similar to $\Lambda-$analysis, describing the mapping properties of classical operators in the Lorentz $\Gamma-$spaces became a challenging task of numerous investigations of the weighted inequalities on the cones of monotone functions.

V.D. Stepanov [83] gave an alternative proof of the theorem above and also extended it to the cases $0 < q < 1 < p < \infty$ and $0 < p \le q < \infty$, $0 < p < 1$. 
Theorem 5 [83]. (i) Assertion (ii) of Theorem 4 holds if $0 < q < 1 < p < \infty$.

(ii) Let $0 < p \leq q < \infty$, $0 < p < 1$. Then (0.14) holds for all non-increasing functions $f \geq 0$ iff $A_0 < \infty$

and

$$A_1 := \sup_{x>0} x \left( \int_x^\infty t^{-q} u(t) dt \right)^{\frac{1}{q}} \left( \int_0^x v(t) dt \right)^{-\frac{1}{p}} < \infty$$

and also $C \approx A_0 + A_1$.

The duality principle for the case $0 < p \leq 1$ was proved in [21, Theorem 3.2], [35] and [83].

The duality principle (the case $0 < p \leq 1$). Let $0 < p \leq 1$, $g, v$ be non-negative measurable functions on $(0, \infty)$ with $v$ locally integrable. Then

$$\sup_{0 \leq f \leq 1} \frac{\int_0^\infty f(x)g(x)dx}{\left( \int_0^\infty f^p(x)v(x)dx \right)^{\frac{1}{p}}} = \sup_{t>0} \left( \int_0^t g(x)dx \right) \left( \int_0^t v(x)dx \right)^{-\frac{1}{p}}.$$

The case $0 < q < p < 1$ of (0.14) was studied in [26] and later in [9]. Here we cite the following criterion.

Theorem 6 [9]. Let $0 < q < p \leq 1$. Then the inequality

$$\left( \int_0^\infty \left( \int_0^x f(t) dt \right)^q w(x)dx \right)^{\frac{1}{q}} \leq C \left( \int_0^\infty f(x)^p v(x)dx \right)^{\frac{1}{p}}$$

holds for all non-increasing functions $f \geq 0$ on $(0, \infty)$ iff

$$B_0 := \int_0^\infty x^q w(x) \left( \frac{1}{V(x)} \int_0^x t^p w(t) dt \right)^{\frac{1}{p}} dx < \infty$$

and

$$B_1 := \int_0^\infty w(x) \left( \int_0^x w(t) dt \right)^{\frac{1}{p}} \sup_{0 < t \leq x} \left( \frac{t^p}{V(t)} \right)^{\frac{1}{p}} dx < \infty.$$

Numerous publications dedicated to inequalities on the cones of monotone functions also include [10], [12], [13], [19], [20], [29], [34], [36], [37], [41], [51], [61], [73], [74], [79], [80], [81], [82], [84] and most recently [25] and [28].

The next step in the process of generalization of inequalities restricted to the cones of monotone functions is inequalities with measures. As we proceed
from Lebesgue measures to arbitrary positive Borel $\sigma$–finite measures, in general case we receive the inequality with three measures

$$\left(\int_{[0,\infty)} (H_s f)^q \, v d\mu\right)^{\frac{1}{q}} \leq C \left(\int_{[0,\infty)} f^p w d\nu\right)^{\frac{1}{p}}$$

which holds for all $f \in \mathcal{M} \downarrow (f \in \mathcal{M} \uparrow)$, where

$$(H_s f)(x) := \left(\int_{[0,x]} f^s u d\lambda\right)^{\frac{1}{s}}$$

and $\mathcal{M} \downarrow (\mathcal{M} \uparrow)$ is a subclass of all non-increasing (non-decreasing) functions $f \in \mathcal{M}^+$. Note that by making the substitution $f^s \to f$ this inequality can be reduced to the equivalent one with new parameters $p$ and $q$. Thus, we need only to study the inequality

$$\left(\int_{[0,\infty)} (H f)^q \, v d\mu\right)^{\frac{1}{q}} \leq C \left(\int_{[0,\infty)} f^p w d\nu\right)^{\frac{1}{p}}, \quad (0.17)$$

where $(H f)(x) := \int_{[0,x]} f u d\lambda$.

It is important to mention that this inequality, unlike the inequality on the set of positive measurable functions, cannot be reduced to the one with two measures because of the condition of monotonicity, and the general case includes three measures.

The inequality (0.17) on monotone functions for $d\lambda = dv$ and $u \equiv v \equiv w \equiv 1$ was investigated by G. Sinnamon [74]. Specifically, in this case he established the equivalence of (0.17) for $f \in \mathcal{M}^+$ to the similar inequality restricted to $f \in \mathcal{M} \downarrow$.

**Theorem 7** [74]. Let $1 < p < \infty$, $0 < q < \infty$, $\lambda, \mu$ be measures on $[0,\infty)$, $\Lambda(x) := \int_{[0,x]} d\lambda$. Then the inequality

$$\left(\int_0^\infty \left(\int_0^x f d\lambda\right)^q \, d\mu(x)\right)^{\frac{1}{q}} \leq C \left(\int_0^\infty f^p d\lambda\right)^{\frac{1}{p}} \quad (0.18)$$

holds for functions $f \geq 0$ if and only if the inequality

$$\left(\int_0^\infty F^p \Lambda^q d\mu\right)^{\frac{1}{q}} \leq C \left(\int_0^\infty F^p d\lambda\right)^{\frac{1}{p}} \quad (0.19)$$

holds for all non-increasing functions $F \geq 0$. 
The similar theorem was proved for non-decreasing functions, but it requires introducing the following notion.

**Definition 1** [74]. Let $1 < p < \infty$. We say that $\lambda \in T_p(\infty)$ provided

$$\Lambda(x)^{1-p} - \Lambda(\infty)^{1-p} \leq C \int_{[x,\infty)} \Lambda^{-p} d\lambda$$

for some constant $C$.

**Theorem 8** [74]. Let $1 < p < \infty$, $0 < q < \infty$, $\lambda, \mu$ be measures on $[0,\infty)$, $\Lambda(x) := \int_{[0,x]} d\lambda$. Then the inequality (0.18) holds for all functions $f \geq 0$ whenever

$$\left( \int_0^\infty F^q d\mu \right)^{\frac{1}{q}} \leq C \left( \int_0^\infty F^p \Lambda^{-p} d\lambda \right)^{\frac{1}{p}}$$

holds for all non-decreasing functions $F \geq 0$. If $\lambda \in T_p(\infty)$ and $\Lambda(\infty) = \infty$, then the condition (0.20) is also necessary for (0.18).

In [74] G. Sinnamon also obtained the characterization of the inequality

$$\left( \int_0^\infty F^q d\mu \right)^{\frac{1}{q}} \leq C \left( \int_0^\infty F^p d\lambda \right)^{\frac{1}{p}}, \quad F \downarrow$$

for $p, q > 0$. This result has been used in paper B of this thesis and obviously it can be extended to the case of non-decreasing functions $F$.

**Theorem 9** [74]. (i) If $0 < p \leq q < \infty$, then

$$\sup_{F^\downarrow} \frac{\left( \int_0^\infty F^q d\mu \right)^{\frac{1}{q}}}{\left( \int_0^\infty F^p d\lambda \right)^{\frac{1}{p}}} = \sup_{x \geq 0} \frac{\left( \int_0^x d\mu \right)^{\frac{1}{q}}}{\left( \int_0^x d\lambda \right)^{\frac{1}{p}}}.$$

(ii) If $0 < q < p < \infty$ and $\frac{1}{r} := \frac{1}{q} - \frac{1}{p}$, then

$$\sup_{F^\downarrow} \frac{\left( \int_0^\infty F^q d\mu \right)^{\frac{1}{q}}}{\left( \int_0^\infty F^p d\lambda \right)^{\frac{1}{p}}} \approx \left( \int_0^\infty \left( \int_{[x,\infty]} \frac{d\mu}{\Lambda(x)} \right)^{\frac{1}{r}} d\lambda(x) \right)^{\frac{1}{r}}.$$

(iii) Suppose that $\lambda \in T_2(\infty)$. If $0 < q < p < \infty$, then

$$\sup_{F^\downarrow} \frac{\left( \int_0^\infty F^q d\mu \right)^{\frac{1}{q}}}{\left( \int_0^\infty F^p d\lambda \right)^{\frac{1}{p}}} \approx \left( \int_0^\infty \left( \frac{1}{\Lambda(x)} \int_{[0,x]} d\mu + \frac{1}{\Lambda(\infty)} \int_0^\infty d\mu \right)^{\frac{1}{r}} d\lambda(x) \right)^{\frac{1}{r}}.$$
Note that assertion (ii) of Theorem 9 with \( q = 1 \) gives an analog of the Sawyer duality principle with general Borel measures.

The general case of (0.17) was studied in [36]. Here we summarize the main results of this paper, and first we introduce the following definition.

**Definition 2** [36]. Let \( f \in \mathcal{M} \downarrow \) and be continuous on the left. It is known (see [70]) that there exists a Borel measure \( \eta_w \), such that
\[
w(x) = \int_{[x, \infty)} d\eta_w + w(+\infty).
\]
We say that \( w \in \mathcal{S}(0) \) if there exists a constant \( C \geq 1 \) such that
\[
\frac{1}{w(x)} - \frac{1}{w(0)} \leq C \int_{[0,x]} \frac{d\eta_w}{w^2}, \quad x > 0.
\]

**Theorem 10** [36]. (i) Let \( 0 < p \leq q < \infty, \ 0 < p \leq 1 \) and \( \Lambda(x) := \int_{[0,x]} d\lambda \). Then (0.17) holds for all \( f \in \mathcal{M} \downarrow \) iff
\[
A_0 := \sup_{t \in [0,\infty)} \left( \int_{[0,t]} w(t) \right)^{-\frac{1}{p}} \left( \int_{[0,t]} \Lambda^q v d\mu \right)^{\frac{1}{q}} < \infty,
\]
\[
A_1 := \sup_{t \in [0,\infty)} \Lambda(t) \left( \int_{[0,t]} w(t) \right)^{-\frac{1}{p}} \left( \int_{[t,\infty)} v d\mu \right)^{\frac{1}{q}} < \infty,
\]
and \( C \approx A_0 + A_1 \).

(ii) Let \( 0 < q < 1 = p \) and \( W(x) := \int_{[0,x]} w d\nu \). Then (0.17) holds for all \( f \in \mathcal{M} \downarrow \) iff
\[
B_0 := \left( \int_{[0,\infty)} w(y) \left( \int_{[y,\infty)} W^{-1} \Lambda^q v d\mu \right)^{\frac{1}{q}} \right)^{\frac{1}{1-q}} < \infty,
\]
\[
B_1 := \left( \int_{[0,\infty)} \left( \int_{[0,x]} \esssup_{s \in [0,t]} \Lambda(s) W(s) v(t) d\mu(t) \right)^{\frac{q}{p}} v(x) d\mu(x) \right)^{\frac{1}{1-q}} < \infty,
\]
and \( C \approx B_0 + B_1 \).

(iii) Let \( 0 < q < p < 1 \) and \( V_p(t) := \esssup_{s \in [0,t]} \Lambda^p(s) W(s) \). Then (0.17) holds for all \( f \in \mathcal{M} \downarrow \) if
\[
B_0 := \left( \int_{[0,\infty)} w(y) \left( \int_{[y,\infty)} W^{-1} \Lambda^q v d\mu \right)^{\frac{1}{q}} \right)^{\frac{p}{p-q}} < \infty,
\]
\[ B_1 := \left( \int_{[0, \infty)} \left( \int_{[0, x]} V_p(t)v(t)d\mu(t) \right) \frac{v(x)}{p - 1} \frac{d\mu(x)}{x} \right)^{\frac{p}{p - 1}} < \infty \]

and only if \( B_0 + B_1 < \infty \), provided \( V_p(t) \) is continuous on \((0, \infty)\) and \( \frac{1}{V_p(t)} \in S(0) \). Then \( C \approx B_0 + B_1 \).

(iv) Let \( 1 < p \leq q < \infty \), \( k(x, y) := \int_{[y, x]} W^{-1}d\lambda \) and \( f \in \mathcal{M} \downarrow \). Then (0.17) holds iff \( \Lambda = \max \{ A_0 + A_1 \} < \infty \), where

\[ A_0 := \sup_{t \in [0, \infty)} \left( \int_{[0, t]} w(y)k(t, y)^{d\nu(y)} \right)^{\frac{1}{p'}} \left( \int_{[t, \infty)} v(t)d\mu(t) \right)^{\frac{1}{q'}}, \]

\[ A_1 := \sup_{t \in [0, \infty)} \left( \int_{[0, t]} w(t)d\nu \right)^{\frac{1}{p'}} \left( \int_{[t, \infty)} v(t)k(t, x)^{d\mu(x)} \right)^{\frac{1}{q'}} \cdot w(t)d\nu(t) \]

Moreover, if \( C \) is the best constant in (0.17), then \( C = \Lambda \).

(v) Let \( 1 < q < p < \infty \) and \( f \in \mathcal{M} \downarrow \). Then (0.17) holds iff \( \Lambda = \max \{ B_0 + B_1 \} < \infty \), where

\[ B_0 := \left( \int_{[0, \infty)} \left( \int_{[0, t]} w(y)k(t, y)^{d\nu(y)} \right)^{\frac{1}{p'}} \left( \int_{[t, \infty)} v(t)d\mu(t) \right)^{\frac{1}{q'}} \right), \]

\[ B_1 := \left( \int_{[0, \infty)} \left( \int_{[0, t]} w(t)d\nu \right)^{\frac{1}{p'}} \left( \int_{[t, \infty)} v(t)k(t, x)^{d\mu(x)} \right)^{\frac{1}{q'}} \right)^{\frac{1}{q}} \cdot w(t)d\nu(t) \cdot \]

Moreover, if \( C \) is the best constant in (0.17), then \( C = \Lambda \).

Paper A of this thesis (see [58] and also [92]) is dedicated to studying Hardy-type operators on the cones of monotone functions with general positive \( \sigma \)-finite measure. This study was strongly influenced by the paper [74] by G. Sinnamon, where he obtained the inequality

\[ \left( \int_{[0, \infty)} \left( \frac{1}{A(x)} \int_{[0, x]} f(t)d\lambda(t) \right)^{p} d\lambda(x) \right)^{\frac{1}{p}} \leq \frac{p}{p - 1} \left( \int_{[0, \infty)} f^{p}d\lambda \right)^{\frac{1}{p}} \]

for all \( \lambda \)-measurable functions \( f \geq 0 \) in the case \( 1 < p < \infty \). It can be easily checked that this inequality also holds on the cone of non-increasing functions. Therefore, in this case we obtain the equivalence

\[ \left( \int_{[0, \infty)} \left( \frac{1}{A(x)} \int_{[0, x]} f(t)d\lambda(t) \right)^{p} d\lambda(x) \right)^{\frac{1}{p}} \approx \left( \int_{[0, \infty)} f^{p}d\lambda \right)^{\frac{1}{p}}. \]
The aim of paper A is to establish the similar equivalences for some other Hardy-type operators and also for the full range of parameter $p$ by means of proving two-sided inequalities with measures. For example, for $p > 0$, $f \in M \downarrow$ we prove the following result.

**Theorem 11** [58]. Let $p > 0$ and $\Lambda(x) := \int_{[0,x]} d\lambda$. Then for all $f \in M \downarrow$ we have

\[
\left( \int_{[0,\infty)} \left( \frac{1}{\Lambda(x)} \int_{[0,x]} \Lambda(t)^{\frac{1}{p}} f(t) d\lambda(t) \right)^p \frac{d\lambda(x)}{\Lambda(x)} \right)^{\frac{1}{p}} \approx \left( \int_{[0,\infty)} f^p d\lambda \right)^{\frac{1}{p}}.
\]

The next equivalence is concerned with the dual Hardy operator.

**Theorem 12** [58]. Let $1 \leq p < \infty$. Then for all $f \in M \downarrow$, $f \not\equiv 0$ it holds that

\[
\frac{1}{p+1} \leq \frac{\left( \int_{[0,\infty)} \left( \int_{[x,\infty)} f(y)\Lambda^p(y) \right)^p d\lambda(y) \right)^{\frac{1}{p}}}{\left( \int_{[0,\infty)} f^p \Lambda^p d\lambda \right)^{\frac{1}{p}}} \leq p.
\]

In the case $0 < p < 1$ we have

\[
0 < c(p) \leq \frac{\left( \int_{[0,\infty)} \left( \int_{[x,\infty)} f(y)\Lambda^p(y) \right)^p d\lambda(y) \right)^{\frac{1}{p}}}{\left( \int_{[0,\infty)} f^p \Lambda^p d\lambda \right)^{\frac{1}{p}}} \leq \frac{1}{p}.
\]

Proving an analogous result for the operator involving suprema instead of the integral, we obtain the following theorem.

**Theorem 13** [58]. Let $0 < p < \infty$. Then for all $f \in M \downarrow$, $f \not\equiv 0$

\[
1 \leq \frac{\left( \int_{[0,\infty)} \left( \text{ess sup}_{y \geq x} f^p(y)\Lambda^p(y) \right) d\lambda(x) \right)^{\frac{1}{p}}}{\left( \int_{[0,\infty)} f^p \Lambda^p d\lambda \right)^{\frac{1}{p}}} \leq c(p),
\]

where

\[
c^p(p) = \max(1, 2^{p-1}) \left( 1 + \frac{\max(1, p)}{\min(1, p)} \right).
\]

For an arbitrary function $f \in M \downarrow$, which is continuous on the right (denote by $M \downarrow (+0)$), we introduce a Borel measure $\mu_f$ such that $\frac{1}{f(x)} =$
\[ \int_{[0,x]} d\mu_f. \] Such measure exists and it is unique, because the function \( \frac{1}{f} \in \mathcal{M} \) is also continuous on the right (see [70]). The following two-sided inequality for the measure \( \mu_f \) was proved in [58].

**Theorem 14** [58]. Let \( 0 < p < \infty \). Then for all \( f \in \mathcal{M} \) \((+0)\)

\[
1 \leq \left( \frac{\int_{[0,\infty)} \left( \int_{[x,\infty]} f d\lambda \right)^{p+1} d\mu_f(x)}{\left( \int_{[0,\infty)} \left( \int_{[x,\infty]} f d\lambda \right)^p d\lambda(x) \right)^{\frac{1}{p}}} \right)^{\frac{1}{p}} \leq (p+1)^{\frac{1}{p}}.
\]

Finally, combining the results of Theorems 13 and 14, we obtain the following result.

**Theorem 15** [58]. Let \( 0 < p < \infty \). Then for all \( f \in \mathcal{M} \) \((+0)\) it yields that

\[
c(p) \leq \left( \frac{\int_{[0,\infty)} \left( \text{ess sup}_{y \geq x} f(y) \Lambda(y) \right)^{p+1} d\mu_f(x)}{\left( \int_{[0,\infty)} \left( \text{ess sup}_{y \geq x} f(y) \Lambda(y) \right)^p d\lambda(x) \right)^{\frac{1}{p}}} \right)^{\frac{1}{p}} \leq \left( \frac{\max(1,p)}{\min(1,p)} \right)^{\frac{1}{p}},
\]

where \( c(p) = \left( \frac{p}{(p+1)^{2p+1}} \right)^{\frac{1}{p}} \).

**Remark 2.** Note that Theorems 11-15 have complete analogues for non-decreasing functions in case we replace \( \Lambda(x) \) by \( \Lambda_* (x) := \int_{[x,\infty)} d\lambda \).

For negative values of parameter \( p \) the following result was proved in [58].

**Theorem 16** [58]. Let \( -1 < p < 0 \). Then for all \( f \in \mathcal{M} \) \((+0)\), \( f(x) > 0 \) \( \lambda - \text{a.e.} \)

\[
(-p)^{\frac{1}{p}} \leq \left( \frac{\int_{[0,\infty)} \left( \int_{[0,x]} f d\lambda \right)^p d\lambda}{\left( \int_{[0,\infty)} f^p \Lambda d\lambda \right)^{\frac{1}{p}}} \right)^{\frac{1}{p}} \leq 1.
\]

For \( p \leq -1 \) it yields that

\[
0 < c_p \leq \left( \frac{\int_{[0,\infty)} \left( \int_{[0,x]} f d\lambda \right)^p d\lambda}{\left( \int_{[0,\infty)} f^p \Lambda d\lambda \right)^{\frac{1}{p}}} \right)^{\frac{1}{p}} \leq 1.
\]
Remark 3. Theorem 16 has complete analogue for non-increasing functions with the changes mentioned above.

As an application, using Theorem 2, we obtain a new characterization of the discrete Hardy inequality

\[ \left( \sum_n u_n^q \left( \sum_{k \geq n} \alpha_k \right) \right)^{\frac{1}{q}} \leq C \left( \sum_n \phi_n^p \alpha_n^r \right)^{\frac{1}{p}}, \]

which was also studied in [26], in the case \( 0 < q < p \leq 1 \) and thus establish its equivalence to a simpler inequality on monotone sequences.

The equivalences we have derived in paper A are used in paper B of this thesis (see [63]) to obtain necessary and sufficient conditions for some other Hardy-type inequalities on cones of monotone functions for the full range of parameters of summation \( p \) and \( q \). In particular, in Section 3.3 of paper B we give a full characterization of the inequalities

\[ \left( \int_{[0, \infty)} \left( \int_{[x, \infty)} f d\lambda \right)^q d\lambda \right)^{\frac{1}{q}} \leq C \left( \int_{[0, \infty)} f^p d\mu \right)^{\frac{1}{p}}, \quad f \in \mathcal{M} \downarrow, \quad f \not\equiv 0 \quad (0.21) \]

and

\[ \left( \int_{[0, \infty)} \left( \int_{[0, x]} f d\lambda \right)^q d\lambda \right)^{\frac{1}{q}} \leq C \left( \int_{[0, \infty)} f^p d\mu \right)^{\frac{1}{p}}, \quad f \in \mathcal{M} \uparrow, \quad f \not\equiv 0 \]

both when \( 0 < p, q < \infty \) and \( -\infty < p, q < 0 \).

For instance, for (0.21) in the case \( 0 < p, q < \infty \) we obtain the following criteria:

\[ C = \sup_{x \in [0, \infty)} \frac{\left( \int_{[0, x]} \Lambda_q d\lambda \right)^{\frac{1}{q}}}{\left( \int_{[0, x]} d\mu \right)^{\frac{1}{p}}}, \quad 0 < p \leq q < \infty, \]

and

\[ C \approx \left( \int_{[0, \infty)} \left( \int_{[0, x]} \Lambda_q d\lambda \right)^{\frac{1}{q}} \left( \int_{[0, x]} d\mu \right)^{\frac{-1}{q}} d\mu(x) \right)^{\frac{1}{q}}, \quad 0 < q < p < \infty, \]

provided

\[ \int_{[0, \infty)} d\mu = \infty, \quad \left( \int_{[0, x]} d\mu \right)^{-1} \leq c \int_{[x, \infty)} \left( \int_{[0, t]} d\mu \right)^{-2} d\mu(x). \]
As the inequality (0.21) has been studied in [26] by M.L. Goldman, we also compare our criteria with the criteria from [26] (see paper B, Section 3.2).

Another question we investigate in paper B is Hardy-type inequalities with kernels restricted to the cone of monotone functions. In particular, we give a complete description for the inequalities with Volterra integral operators involving so called Oinarov’s kernels. Specifically, we study the inequality of the form

\[
\left( \int_{[0, \infty)} (Kf)^q d\mu \right)^{\frac{1}{q}} \leq C \left( \int_{[0, \infty)} f^p d\lambda \right)^{\frac{1}{p}},
\]

where \(0 < p, q < \infty\),

\[
Kf(x) := \int_{[0,x]} k(x, y)f(y)d\nu(y),
\]

\(\lambda, \mu\) and \(\nu\) are positive \(\sigma\)-finite Borel measures on \(\mathbb{R}_+ := [0, \infty)\) and a measurable kernel \(k(x, y) \geq 0\) satisfies the Oinarov condition, i.e. that there exists a constant \(D \geq 1\), for which

\[
D^{-1}(k(x, z) + k(z, y)) \leq k(x, y) \leq D(k(x, z) + k(z, y)), x \geq z \geq y. \tag{0.24}
\]

The problem is to obtain a sharp two-sided estimate of the least possible positive constant \(C\) provided the inequality (0.22) is fulfilled for any function \(f\) from a prescribed set.

The study of inequalities for the class of Volterra-Hardy operators and also the class of general integral operators with a given non-negative kernel \(k(x, t)\) has initially started from investigating the Riemann-Liouville operator and naturally included only Lebesgue measures. The two-weighted inequality

\[
\| (Kf)u \|_q \leq C \| f v \|_p, \ f \leq 0, \tag{0.25}
\]

where

\[
Kf(x) := \int_{[0,x]} k(x, y)f(y)dy, \tag{0.26}
\]

has been studied by many authors. As we have shown, the problem (0.25), (0.26) with \(K\) replaced by the Hardy operator having kernel \(k(x, y) \equiv 1\) has now been almost completely characterized.

V.D. Stepanov obtained necessary and sufficient conditions for (0.25) to hold and for the operator (0.26) to be compact in the case of Riemann-Liouville operator with kernel

\[
k(x, y) = \frac{1}{\Gamma(r)}(x - y)^{r-1}, \ r \geq 1
\]
for $1 < p, q < \infty$ (see, for example, [87], [88], [89]).

During the same period P.J. Martin-Reyes and E. Sawyer in [47] by different methods characterized (0.25) in the case $1 < p \leq q < \infty$ for a class of integral operators given by

$$K_\phi f(x) := \int_{[0,x]} \phi \left( \frac{t}{x} \right) f(t)dt,$$

where $\phi : (0, 1) \to (0, \infty)$ is non-increasing and satisfies the condition $\phi(ab) \leq D(\phi(a) + \phi(b))$ for all $0 < a, b < 1$.

Similar criteria were obtained by V.D. Stepanov ([90], [91]) for Volterra convolution operators of the form

$$K_k f(x) := \int_{[0,x]} k(x-t) f(t)dt$$

(0.27)

for the cases $1 < p \leq q < \infty$ and $1 < q < p < \infty$, but with different conditions on the kernel. For instance, in [91] he studied the operator with the kernel $k$ which satisfies the following conditions:

(i) $k(x) \geq 0$ is non-decreasing on $(0, \infty)$,

(ii) $k(x + y) \leq D(k(x) + k(y))$ for all $x, y \in (0, \infty)$.

Also, the case $0 < q < 1 < p < \infty$ with a kernel satisfying only condition (i) was investigated, and necessary and sufficient conditions were obtained, but they did not coincide. Moreover, the compactness criteria for the operator (0.27) for the cases $1 < p \leq q < \infty$ and $1 < q < p < \infty$ were proved.

In 1991 S. Bloom and R. Kerman [16] obtained criteria for (0.25) to hold for the operators (0.26) when $1 < p \leq q < \infty$ under assumption that not only the condition (0.24) is satisfied, but also $k(x, y) \geq 0$ for $0 < y < x$ and the kernel $k(x, y)$ is both non-decreasing in $x$ and non-increasing in $y$.

Specifically, they considered kernels $\phi(x, y)$ on $\mathbb{R}_+ \times \mathbb{R}_+$ with the following properties:

(i) $\phi(x, y) > 0$ if $x > y$,

(ii) $\phi(x, y)$ is non-decreasing in $x$ and non-increasing in $y$,

(iii) $\phi(x, y) \approx \phi(x, z) + \phi(z, y)$ if $y < z < x$.

Moreover, the following operators were defined for $r \geq 0$:

$$(T_r f)(x) := \int_{[0,x]} \phi(x, y)^r f(y)dy,$$

$$(T^*_r g)(x) := \int_{[x,\infty)} \phi(y, x)^r g(y)dy,$$

$$(T f)(x) := (T_1 f)(x)$$
and

\[(I f)(x) := (T_0 f)(x)\].

Here we cite the principal result obtained in [16].

**Theorem 17** [16]. Let \(1 < p \leq q < \infty\), \(u\) and \(v\) be non-negative, measurable functions on \((0, \infty)\) with \(0 < u, v < \infty\) a.e. Then

\[
\left( \int_{[0,\infty)} (uT f)^q \right)^{\frac{1}{q}} \leq C \left( \int_{[0,\infty)} (v f)^p \right)^{\frac{1}{p}} \text{ for all } f \geq 0
\]

if and only if

\[
I^* \left[ (v^{-1} T^* u^q)^p \right] \leq C \left( I^* u^q \right)^{\frac{p'}{p}} < \infty
\]

and

\[
I^* \left[ (v^{-1} T^* u^q)^p \right] \leq C \left( T^* q u^q \right)^{\frac{p'}{q}} < \infty
\]
a.e. on \((0, \infty)\).

In 1993 R. Oinarov [53] introduced the final form of condition for the kernel \(k\) and proved corresponding necessary and sufficient conditions for (0.25) with operator of the form (0.26) to hold. The condition he used reads as follows: \(k(x, y) > 0\) for \(x > y\) and there exists a constant \(D > 0\) such that (0.24) holds. This condition is usually called the Oinarov condition.

The results obtained in papers [16], [47], [90], [91] were generalized in [85] by V.D. Stepanov for the cases \(1 < p \leq q < \infty\), \(1 < q < p < \infty\) with less strict conditions on the kernel. The criteria below keep the symmetric form as in [87] and differ from the criteria in [16] and [47].

Before turning to the main theorem of [85], we need to introduce the following notations. For \(1 < p, q < \infty\) we assume that \(u(x) \geq 0\) and \(v(x) \geq 0\) are such that the dual spaces \((L^p_x)^*\) and \((L^q_x)^*\) can be identified with \(L^p_{1/v}\) and \(L^q_{1/u}\). We also define for \(s \geq 0\)

\[
K_s f(x) := \int_{[0,x]} k(x, y)^s f(y) dy,
\]

\[
K^*_s g(y) := \int_{[y,\infty)} k(x, y)^s g(x) dx,
\]

\[
K_0 f(x) := \int_{[0,x]} f(y) dy
\]
Let $1 < p \leq q < \infty$ and an operator $K$ be of the form (0.26) with kernel $k$ satisfying the following conditions:

1) $k(x, y) \geq 0$ for $0 < y < x$ and is non-decreasing in $x$ or non-increasing in $y$,
2) \[
D^{-1}(k(x, z) + k(z, y)) \leq k(x, y) \leq D(k(x, z) + k(z, y)),
\]
where $0 < x < z < y$.

Then the inequality (0.25) holds iff $A = \max(A_0, A_1) < \infty$, where

$$A_0 := \sup_{t>0} \left( \int_{[0, \infty)} \left( K_0^* u^q(t) \right)^{\frac{1}{q}} \left( K_0 v^{-\varrho}(t) \right)^{\frac{1}{\varrho}} v(t)^{-\varrho'} dt \right)^{\frac{1}{r}},$$

$$A_1 := \sup_{t>0} \left( \int_{[0, \infty)} \left( K_0^* u^q(t) \right)^{\frac{1}{q}} \left( K_0 v^{-\varrho}(t) \right)^{\frac{1}{\varrho}} v(t)^{-\varrho'} dt \right)^{\frac{1}{r}}.$$

(ii) Let $1 < q < p < \infty$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. If operator $K$ has the form (0.26) with kernel $k$ satisfying (0.28), then the inequality (0.25) holds iff $B = \max(B_0, B_1) < \infty$, where

$$B_0 := \left( \int_{[0, \infty)} \left( K_0^* u^q(t) \right)^{\frac{1}{q}} \left( K_0 v^{-\varrho}(t) \right)^{\frac{1}{\varrho}} v(t)^{-\varrho'} dt \right)^{\frac{1}{r}},$$

$$B_1 := \left( \int_{[0, \infty)} \left( K_0^* u^q(t) \right)^{\frac{1}{q}} \left( K_0 v^{-\varrho}(t) \right)^{\frac{1}{\varrho}} v(t)^{-\varrho'} dt \right)^{\frac{1}{r}}.$$

Remark 4. The criteria for compactness of operator $K$ for the mentioned range of $p$ and $q$ are also obtained in [85].

In 1999 Q. Lai [43] gave a characterization of (0.25), (0.26) which had an essentially different form. He imposed the following conditions on the kernel $k(x, y)$:

(i) $k(x, y) > 0$ if $x > y$,
(ii) $k(x, y)$ is non-decreasing in $x$ and non-increasing in $y$,
(iii) there exists a constant $D > 0$ such that $k(x, y) \leq D \left( k(x, z) + k(z, y) \right)$ for all $0 \leq y \leq z \leq x$.

The case of (0.22) with arbitrary Borel measures was fractionally presented in [50], [49], for discrete inequalities in [6], [7], [8] and for three measures in [66]. The full characterization of the inequality (0.22) for the operators with the Oinarov kernels in the case $1 < p, q < \infty$ was obtained by
D.V. Prokhorov in [68]. Here we summarize the results of [68] in Theorem 19 for the case of two measures and in Theorem 20 for the case of three measures.

**Theorem 19** [68]. (i) Let \(1 < p \leq q < +\infty\), \(\lambda, \mu\) be positive Borel \(\sigma\)-finite measures on \([a, b]\), \(u, v \in \mathcal{M}^+\), \(w \equiv 1\), \(k\) a \(\mu \times \lambda\)-measurable non-negative function satisfying condition (0.24). Then for the existence of a constant \(C \geq 0\), such that inequality

\[
\left( \int_{[a,b]} (Kf)^q \, v d\mu \right)^{\frac{1}{q}} \leq C \left( \int_{[a,b]} f^p w d\lambda \right)^{\frac{1}{p}}
\]  

(0.29)

with \((Kf)(x) := \int_{[a,x]} k(x, y)u(y)f(y)\, d\lambda(y)\)

holds for all \(f \in \mathcal{M}^+\), it is necessary and sufficient that

\[
A := \max \{ A_1, A_2 \} < +\infty,
\]

where

\[
\begin{align*}
A_1 &:= \sup_{t \in [a,b]} \left( \int_{[t,b]} v(x)k(x, t)^q d\mu(x) \right)^{\frac{1}{q}} \left( \int_{[a,t]} u^p d\lambda \right)^{\frac{1}{p}}, \\
A_2 &:= \sup_{t \in [a,b]} \left( \int_{[t,b]} v d\mu \right)^{\frac{1}{q}} \left( \int_{[a,t]} k(t, y)^p u(y)^p d\lambda(y) \right)^{\frac{1}{p}}.
\end{align*}
\]

(ii) Let \(1 < q < p < +\infty\), \(\lambda, \mu\) be positive Borel \(\sigma\)-finite measures on \([a, b]\), \(u, v \in \mathcal{M}^+\), \(w \equiv 1\), then for the existence of a constant \(C \geq 0\) such that inequality (0.29) holds, it is necessary and sufficient, that

\[
B := \max \{ B_1, B_2 \} < +\infty,
\]

where

\[
\begin{align*}
B_1 &:= \left( \int_{[a,b]} \left( \int_{[t,b]} v(x)k(x, t)^q d\mu(x) \right)^{\frac{1}{q}} \left( \int_{[a,t]} u^p d\lambda \right)^{\frac{1}{p}} u(t)^p d\lambda(t) \right)^{\frac{1}{p}}, \\
B_2 &:= \left( \int_{[a,b]} \left( \int_{[t,b]} v d\mu \right)^{\frac{1}{q}} \left( \int_{[a,t]} k(t, y)^p u(y)^p d\lambda(y) \right)^{\frac{1}{p}} v(t)^p d\mu(t) \right)^{\frac{1}{p}}.
\end{align*}
\]

**Theorem 20** [68]. (i) Let \(1 < p \leq q < +\infty\), \(\mu, \lambda\) and \(\nu\) be positive Borel \(\sigma\)-finite measures on \([a, b]\), \(u, v, w \in \mathcal{M}^+\); \((\nu_a, \nu_s)\) the Lebesgue decomposition of \(\nu\) relative to \(\lambda\), and \(d\nu_a/d\lambda\) the Radon-Nikodym derivative of \(\nu_a\) with respect
to $\lambda, k$ a $\mu \times \lambda$-measurable non-negative function satisfying condition (0.24).

Put

$$\omega := w^{p^{'}} \left( w \frac{d\nu_{a}}{d\lambda} \right)^{1-p^{'}}.$$ 

Then for the existence of a constant $C \geq 0$ such that inequality

$$\left( \int_{[a,b]} (Kf)^{q} vd\mu \right)^{\frac{1}{q}} \leq C \left( \int_{[a,b]} f^{p}wd\nu \right)^{\frac{1}{p}}$$

(0.30)

with

$$(Kf)(x) := \int_{[a,x]} k(x,y)u(y)f(y)d\lambda(y)$$

holds, it is necessary and sufficient, that $\mathfrak{A} := \max \{ \mathfrak{A}_{1}, \mathfrak{A}_{2} \} < \infty$, where

$$\mathfrak{A}_{1} := \sup_{t \in [a,b]} \left( \int_{[t,b]} v(x)k(x,t)^{q}d\mu(x) \right)^{\frac{1}{q}} \left( \int_{[a,t]} \omega d\lambda \right)^{\frac{1}{p^{'}}},$$

$$\mathfrak{A}_{2} := \sup_{t \in [a,b]} \left( \int_{[t,b]} v d\mu \right)^{\frac{1}{q}} \left( \int_{[a,t]} k(t,y)^{p^{*}}\omega(y)d\lambda(y) \right)^{\frac{1}{p^{*}}}.$$

(ii) Let $1 < q < p < +\infty$, then for the existence of a constant $C \geq 0$ such that inequality (0.30) holds, it is necessary and sufficient, that $\mathfrak{B} := \max \{ \mathfrak{B}_{1}, \mathfrak{B}_{2} \} < +\infty$, where

$$\mathfrak{B}_{1} := \left( \int_{[a,b]} \left( \int_{[t,b]} v(x)k(x,t)^{q}d\mu(x) \right)^{\frac{1}{q}} \left( \int_{[a,t]} \omega d\lambda \right)^{\frac{1}{p^{'}}} \omega(t)d\lambda(t) \right)^{\frac{1}{p}},$$

$$\mathfrak{B}_{2} := \left( \int_{[a,b]} \left( \int_{[t,b]} v d\mu \right)^{\frac{1}{q}} \left( \int_{[a,t]} k(t,y)^{p^{*}}\omega(y)d\lambda(y) \right)^{\frac{1}{p^{*}}} v(t)d\mu(t) \right)^{\frac{1}{p}}.$$

The results of Theorem 20 are used in [68] to characterize the inequality

$$\left( \int_{[a,b]} \left( \int_{[a,x]} fud\lambda \right)^{q} v(x)d\mu(x) \right)^{\frac{1}{q}} \leq C \left( \int_{[a,b]} f^{p}wd\nu \right)^{\frac{1}{p}}, \quad f \in \mathfrak{M}^{+}$$

for $1 < p, q < \infty$.

The case $0 < p < 1, 0 < q < \infty$ was in full characterized by D.V. Prokhorov in [67]. We omit the details here.
Inequalities of the form (0.22) for operator (0.23) on the cones of monotone functions have not been studied as actively as classical inequalities of Hardy type.

The weighted inequalities with Hardy-type operators restricted to the cones of monotone functions were investigated in [20], [21], [84]. In particular, in [84] a number of classical operators was explicitly described and criteria of boundedness for the Hardy-Littlewood maximal operator in the Lorentz space $\Gamma_p(v)$ were obtained.

In [61] some of the classical results obtained for Hardy-type inequalities, especially results from [74], were generalized for Volterra integral operators with the kernel satisfying some conditions of monotonicity. Moreover, some equivalences between inequalities on the sets of non-negative and non-increasing functions for different forms of integral operators were established.

In Section 3.1 of paper B we characterize the Hardy-type inequality (0.22) on the cone of non-increasing functions $\mathcal{M}^\downarrow$.

**Theorem 21** [63]. Let $0 < p < \infty$, $1 \leq q < \infty$. Let

$$
\bar{k}(x) := \int_{[0,x]} k(x, y) d\nu(y),
$$

$$
\bar{k}(x) := \int_{[x,\infty)} k(y, x) d\mu(y),
$$

$$
\bar{\nu}(x) := \int_{[0,x]} d\nu(y).
$$

Then the inequality (0.22) holds for all functions $f \in \mathcal{M}^\downarrow$ iff

$$
A_0 < \infty, \text{ if } p \leq q = 1,
$$

$$
B_0 < \infty, \text{ if } 1 = q < p,
$$

$$
A_1 + A_{2,1} + A_{2,1}^* + A_{2,2} < \infty, \text{ if } 1 < p \leq q,
$$

$$
B_1 + B_{2,1} + B_{2,1}^* + B_{2,2} < \infty, \text{ if } 1 < q < p,
$$

$$
D_1 + D_2 + D_3 < \infty, \text{ if } 0 < p \leq 1 < q,
$$

where

$$
A_0 := \sup_{x \geq 0} \left( \int_{[0,x]} \bar{k}(y) d\nu(y) \right) \left( \int_{[0,x]} d\lambda \right)^{-\frac{1}{p}},
$$

$$
B_0 := \left( \int_{[0,\infty)} \left( \int_{[x,\infty)} \frac{\bar{k}(t) d\nu(t)}{\int_{[0,t]} d\lambda} \right)^{p'} d\lambda(x) \right)^{\frac{1}{p'}}.
$$
\[
A_1 := \sup_{t \geq 0} \left( \int_{|t, \infty)} \Lambda(x)^{-\nu'} d\lambda(x) \right)^{\frac{1}{p'}} \left( \int_{[0,t]} \tilde{k}(z)^q d\mu(z) \right)^{\frac{1}{q}},
\]
\[
A_{2,1} := \sup_{t \geq 0} \left( \int_{[0,t]} \Lambda(x)^{-\nu'} \tilde{\nu}(x)^{\nu'} k(x,t)^{\nu'} d\lambda(x) \right)^{\frac{1}{p'}} \left( \int_{[t, \infty)} d\mu(x) \right)^{\frac{1}{q}},
\]
\[
A_{2,1}^* := \sup_{t \geq 0} \left( \int_{[0,t]} \Lambda(x)^{-\nu'} \tilde{\nu}(x)^{\nu'} d\lambda(x) \right)^{\frac{1}{p'}} \left( \int_{[t, \infty)} k(t,y)^{\nu'} d\mu(y) \right)^{\frac{1}{q}},
\]
\[
A_{2,2} := \sup_{t \geq 0} \left( \int_{[0,t]} \Lambda(x)^{-\nu'} \tilde{k}(x)^{\nu'} d\lambda(x) \right)^{\frac{1}{p'}} \left( \int_{[t, \infty)} d\mu(z) \right)^{\frac{1}{q}};
\]
\[
B_1^r := \int_{[0,\infty)} \left( \int_{[0,x]} \tilde{k}(z)^q d\mu(z) \right)^{\frac{1}{q'}} \left( \int_{[x, \infty)} \Lambda(x)^{-\nu'} d\lambda(x) \right)^{\frac{1}{p'}} \Lambda(x)^{-\nu'} d\lambda(x),
\]
\[
B_{2,1}^r := \int_{[0,\infty)} \left( \int_{[0,t]} \Lambda(x)^{-\nu'} \tilde{\nu}(x)^{\nu'} k(x,t)^{\nu'} d\lambda(x) \right)^{\frac{1}{p'}} \left( \int_{[t, \infty)} d\mu(t) \right)^{\frac{1}{q'}} \Lambda(t)^{-\nu'} \tilde{\nu}(t)^{\nu'} d\lambda(t),
\]
\[
B_{2,1}^* := \int_{[0,\infty)} \left( \int_{[0,t]} \Lambda(x)^{-\nu'} \tilde{\nu}(x)^{\nu'} d\lambda(x) \right)^{\frac{1}{p'}} \left( \int_{[t, \infty)} k(t,y)^{\nu'} d\mu(y) \right)^{\frac{1}{q'}} \Lambda(t)^{-\nu'} \tilde{\nu}(t)^{\nu'} d\lambda(t),
\]
\[
B_{2,2}^r := \int_{[0,\infty)} \left( \int_{[0,x]} d\mu(z) \right)^{\frac{1}{q'}} \left( \int_{[x, \infty)} \Lambda(t)^{-\nu'} \tilde{k}(t)^{\nu'} d\lambda(t) \right)^{\frac{1}{p'}} \Lambda^{-\nu'} (x) \tilde{k}(x)^{\nu'} d\lambda(x),
\]
where \( \frac{1}{r} = \frac{1}{q'} = \frac{1}{p'} \);

\[
D_1 := \sup_{x \geq 0} \left( \int_{[0,x]} \tilde{k}^q(z) d\mu(z) \right)^{\frac{1}{q'}} \Lambda^{-\frac{1}{p'}} (x),
\]
\[
D_2 := \sup_{x \geq 0} \left( \int_{[x, \infty)} k^q(y,x) d\mu(y) \right)^{\frac{1}{q'}} \tilde{\nu}(x) \Lambda^{-\frac{1}{p'}} (x),
\]
\[
D_3 := \sup_{x \geq 0} \left( \int_{[x, \infty)} d\mu(z) \right)^{\frac{1}{q'}} \tilde{k}(x) \Lambda^{-\frac{1}{p'}} (x).
\]

**Remark 5.** In paper B we also obtain a characterization of the inequality
\[
\left( \int_{[0, \infty)} (Kf)^q d\mu \right)^{\frac{1}{q'}} \leq C \left( \int_{[0, \infty)} \left( \int_{[x, \infty)} f d\lambda \right)^p d\lambda \right)^{\frac{1}{p'}}
\]
for all \( f \in \mathfrak{M} \) with \( 0 < p, q < \infty, q \geq 1 \) (see [63, Theorem 8]).
Instead of monotonicity condition, a function \( f(x) \) can also satisfy the condition of quasi-monotonicity, which means that for some \( a \in \mathbb{R} \) \( f(x)x^a \) is non-increasing or non-decreasing. The functions that satisfy two different conditions of quasi-monotonicity has been of particular interest recently. Different types of inequalities for such functions have been studied in [12], [13], [45], [56]. Some results for quasi-monotone weight functions were obtained in [59].

The partial case of a function satisfying two quasi-monotonicity conditions is a function \( u(t) \), such that \( u(t) \) is non-decreasing and \( \frac{u(t)}{t} \) is non-increasing. Since it is proved that such functions are equivalent to concave functions, they are called quasi-concave.

The importance of studying concave and quasi-concave functions is explained by the fact that many of the central objects of harmonic analysis, interpolation theory, homogenization theory, operator theory and some other areas of mathematics are in fact quasi-concave.

Let us mention a few examples of such objects:

(i) We have already mentioned the operator \( f^{**}(t) = \frac{1}{t} \int_{[0,t]} f^*(s)ds \) used in describing the norm of a function \( f(t) \) in Lorentz \( \Gamma \)-spaces. It is easy to see that \( tf^{**}(t) \) is quasi-concave.

(ii) Another example is the Peetre \( K \)–functional

\[
K(t,x;A_0,A_1) = \inf_{x_0+x_1=x} (\|x_0\|_{A_0} + t\|x_1\|_{A_1}),
\]

where \((A_0,A_1)\) are two Banach spaces, \(0 < t < \infty\) and \(x \in A_0 + A_1\). It is known (see [14]) that \(K\) is also quasi-concave.

(iii) The modulus of continuity

\[
\omega = \omega_{p,m}(t,f) = \sup_{|h| \leq t} \|\Delta^m f\|_{L^p},
\]

where

\[
\Delta^m f(x) = \sum_{k=0}^{m} \binom{m}{k} (-1)^k f(x + kh),
\]

is non-decreasing, which is in fact a partial case of quasi-concavity.

(iv) The fundamental function \( \phi_X(t) = \|\chi_E\|_X \), where \(X\) is a rearrangement-invariant Banach functional space over a resonant measure space \((R,\mu)\), \(E\) is subset of \(R\), such that \(\mu(E) = t\), is also proved to be quasi-concave (see e.g. [11]).

The paper C of this thesis (see [64]) is devoted to the study of Hardy-type inequalities on a more extended set of functions then the cone of quasi-concave functions.
Definition 3 [24]. Let $\psi$ be a continuous strictly increasing function on $[0, \infty)$ such that $\psi(0) = 0$ and $\lim_{t \to \infty} \psi(t) = \infty$. Then we say that $\psi$ is admissible.

Definition 4 [24]. Let $\psi$ be an admissible function. Then a function $f$ is $\psi$–quasi-concave if $f$ is equivalent to a non-decreasing function on $[0, \infty)$ and $\frac{f}{\psi}$ is equivalent to a non-increasing function on $(0, \infty)$.

It is obvious that the class of functions with two conditions of quasi-monotonicity along with the class of quasi-concave functions are partial cases of $\psi$–quasi-concave functions.

The question of characterizing the weights $v$ and $w$ and indices $p$ and $q$, for which the embedding from $L_{p,v}$ to $L_{q,u}$ takes place for quasi-concave functions, is the main question for understanding their properties. For the case $0 < p \leq q < \infty$ it was solved in [45], and sufficient conditions for the case $0 < q = 1 < p < \infty$ were obtained in [84]. Since a quasi-concave function $u(t)$ can be presented as $u(t) \approx \int_{[0,t]} v(s)ds$ for some non-increasing function $v(t)$, it is easy to see that characterizing the embedding $L_{p,v} \hookrightarrow L_{q,u}$ on the cone of quasi-concave functions is equivalent to characterizing the embedding $\Gamma_{p,v} \hookrightarrow \Gamma_{q,u}$. The necessary and sufficient conditions on the weight functions $v$ and $u$ and indices $0 < p, q < \infty$ for the embeddings $\Gamma_{p,v} \hookrightarrow \Gamma_{q,u}$ and $\Gamma_{p,v} \hookrightarrow \Lambda_{1,u}$ to hold were obtained in [29]. This result was obtained by using the discretization technique and, thus, the answer was given in terms of discretizing sequences, which makes the obtained conditions quite hard to verify. The same applies to the criteria obtained by M.L. Goldman and M.V. Sorokina in the paper [30] for the weighted Hardy-type inequalities on the cone of $\psi$–quasi-concave functions. Several years later G. Sinnamon was able to obtain the integral conditions on the weights in the paper [78] using a completely different method. He came up with a reduction principle for the operator acting on the cones of quasi-concave function (more specifically, on the cones of functions satisfying two different conditions of quasi-monotonicity). Almost at the same time A. Gogatishvili and L. Pick in [24] introduced their approach based not only on discretization, but, more important, on anti-discretization methods, which made it possible to present the boundedness criteria for embeddings between Lorentz spaces, in particular, $\Gamma_{p,v} \hookrightarrow \Lambda_{q,u}$ and $\Gamma_{p,v} \hookrightarrow \Gamma_{q,u}$ for $0 < p, q < \infty$, in integral terms. Let us cite the result concerning the latter embedding.

Theorem 22 [24, Theorem 5.1]. Let $p, q \in (0, \infty)$, $u, v, w$ be locally integrable weights on $[0, \infty)$, $U(t) = \int_{[0,t]} u(s)ds$, $V(t) = \int_{[0,t]} v(s)ds$, $W(t) = \int_{[0,t]} w(s)d$, and for $p > q$ let $r := \frac{pq}{p-q}$. Assume that $u$ is such that $U^p$ is
admissible and \( v(t) \) satisfies the condition

\[
\int_{(0, \infty)} \frac{v(s)ds}{U^p(s) + U^p(t)} < \infty, \quad \int_{[0,1]} \frac{v(s)ds}{U^p(s)} = \int_{[1,\infty)} v(s)ds = \infty.
\]

(i) Let \( 0 < p \leq q < \infty \). Then the inequality

\[
\left( \int_{[0, \infty)} \left( \frac{1}{U(t)} \int_{[0,t]} f^*(s)u(s)ds \right)^q w(t)dt \right)^{\frac{1}{q}} \leq C \left( \int_{[0, \infty)} \left( \frac{1}{U(t)} \int_{[0,t]} f^*(s)u(s)ds \right)^p v(t)dt \right)^{\frac{1}{p}}
\]

(0.32)

holds if and only if

\[
A_1 := \sup_{t > 0} \left( \frac{W(t) + U(t)^q \int_{[t,\infty)} U(s)^{-q}w(s)ds}{(V(t) + U(t)^p \int_{[t,\infty)} U(s)^{-p}v(s)ds)^{\frac{q}{p}}} \right)^{\frac{1}{q}} < \infty.
\]

(ii) Let \( 0 < q < p < \infty \). Then (0.32) holds if only if

\[
A_2 := \left( \int_{[0, \infty)} \left( \frac{W(t) + U(t)^q \int_{[t,\infty)} U(s)^{-q}w(s)ds}{(V(t) + U(t)^p \int_{[t,\infty)} U(s)^{-p}v(s)ds)^{\frac{q}{p}+2}} \right)^{\frac{1}{q}} \times V(t) \int_{[t,\infty)} U(s)^{-p}v(s)dsd(U^p(t)) \right)^{\frac{1}{q}} < \infty.
\]

Moreover, \( A_2 \approx A_3 \), where

\[
A_3 := \left( \int_{[0, \infty)} \frac{(W(t) + U(t)^q \int_{[t,\infty)} U(s)^{-q}w(s)ds)^{\frac{q-1}{p}} v(t)}{(V(t) + U(t)^p \int_{[t,\infty)} U(s)^{-p}v(s)ds)^{\frac{q}{p}}} dt \right)^{\frac{1}{q}} < \infty.
\]

The discretization technique used in the papers [29], [30] and [24] is based on building for a quasi-concave (or, in general case, \( \psi \)-quasi-concave) function \( u(t) \) a discretizing sequence \( \{ \mu_k \}_{k \in \mathbb{Z}} \), which separates the sections, where \( u(t) \) strongly increases, from the sections, where \( \frac{u(t)}{\psi(t)} \) (in case of a \( \psi \)-quasi-concave function \( \frac{u(t)}{\psi(t)} \)) strongly decreases, and then using the methods for
discrete inequalities. Discretization methods has been used by several authors, e.g. G. Kalyabin, V. Kolyada, I. Netrusov, M. Goldman (e.g. [27]), N. Krugljak, J. Brudnij, S. Janson and V. Ovchinnikov, but it is believed that it first appeared in the paper [55], cf. also the PhD thesis of L.-E. Persson [57], where the elementary form of this idea can be found.

Above we have given the historical overview of characterizing the boundedness of the Hardy-Littlewood maximal operator between the Lorentz \( \Lambda \)-spaces, which is equivalent to finding the criteria of boundedness for the Hardy operator between the Lebesgue spaces for \( 0 < p, q < \infty \) on the cone of non-increasing functions (see (0.13)). The most recent results concerning the boundedness of the Hardy-Littlewood maximal operator between the Lorentz \( \Gamma \)-spaces are obtained in [29], [30] and [78]. As we have mentioned, in the papers [29] and [30] the answers are given in terms of implicit sequences, while in [78] the criteria have integral form for \( 1 < p, q < \infty \) and are obtained using the reduction principle for the inequalities on the cone of quasi-concave functions.

For convenience in paper C we introduce the cone of functions \( \Omega_\psi \) - the subset of \( f \in \mathbb{R}^+ \) such that \( f(t) \) is non-increasing and \( \psi(t)f(t) \) is non-decreasing. Thus, the function \( \psi(t)f(t) \) is \( \psi \)-quasi-concave. In this paper we study the inequality of the type

\[
\left( \int_{[0, \infty)} (Af)^q d\gamma \right)^{\frac{1}{q}} \leq C \left( \int_{[0, \infty)} f^p d\beta \right)^{\frac{1}{p}},
\]

(0.33)

where

\[
Af(t) = \left( \int_{[0, t]} f^p d\mu \right)^{\frac{1}{p}}, \quad f \in \Omega_\psi.
\]

(0.34)

We obtain necessary and sufficient conditions for this inequality to hold for all parameters \( q \geq 1, p > 0 \). For the rest of the range of parameters \( p \) and \( q \) we find the sufficient conditions using one of the reduction theorems for the operators on the cones of monotone functions that were proved in the paper [25].

The main result of paper C reads as follows:

**Theorem 23** [64]. Let \( q \geq 1, p > 0 \). Then the inequality

\[
\left( \int_{[0, \infty)} \left( \int_{[0, t]} f(s)u(s)ds \right)^q u(t)dt \right)^{\frac{1}{q}} \leq C \left( \int_{[0, \infty)} f^p v(t)dt \right)^{\frac{1}{p}}
\]

(0.35)

holds for all functions \( f \in \Omega_\psi \) if and only if
(i) $A_1 < \infty$, if $q = 1$, $0 < p \leq 1$, where

$$A_1 := \sup_{t \geq 0} \left( \int_{[0,t]} \left( \int_{[y,\infty)} U(z,y)w(z)dz \right) d\psi(y) \right)^\frac{1}{p} V(t)^{-\frac{1}{p}};$$

(ii) $A_2 < \infty$, if $q = 1$, $1 < p < \infty$, where

$$A_2 := \left( \int_{[0,\infty)} \left( \int_{[0,t]} U(t,z)w(t)dt \right) d\psi(z) \right)^\frac{q'}{q} \frac{1}{p} V(t)d(\psi^p(t))^{\frac{1}{p}};$$

(iii) $A_3 < \infty$, if $q > 1$, $0 < p \leq 1$, where

$$A_3 := \sup_{t \geq 0} \left( \int_{[0,\infty)} \left( \int_{0}^{\min(s,t)} U(s,y)d\psi(y) \right)^q w(s)ds \right)^\frac{1}{p} V(t)^{-\frac{1}{p}};$$

(iv) $A_{4,1} + A_{4,2} + A_{4,3} + A_{4,4} < \infty$, if $1 < p \leq q < \infty$, where

$$A_{4,1} := \sup_{t \geq 0} \left( \int_{[t,\infty)} V(s)d(\psi^p(s)) \right)^\frac{1}{p} \left( \int_{[0,t]} U^q(s)w(s)ds \right)^{\frac{1}{q}},$$

$$A_{4,2} := \sup_{t \geq 0} \left( \int_{[0,t]} U(t,s)^p V(s)d(\psi^p(s)) \right)^\frac{1}{p} \left( \int_{[t,\infty)} w^q(s)ds \right)^{\frac{1}{q}},$$

$$A_{4,3} := \sup_{t \geq 0} \left( \int_{[0,t]} V(s)d(\psi^p(s)) \right)^\frac{1}{p} \left( \int_{[t,\infty)} U(y,t)^q w(y)^q dy \right)^{\frac{1}{q}},$$

$$A_{4,4} := \sup_{t \geq 0} \left( \int_{[0,t]} V(s)d(\psi^p(s)) \right)^\frac{1}{p} \left( \int_{[t,\infty]} w \right)^{\frac{1}{q}},$$

(v) $A_{5,1} + A_{5,2} + A_{5,3} + A_{5,4} < \infty$, if $1 < q < p < \infty$, where

$$A_{5,1} := \left( \int_{[0,\infty)} \left( \int_{[0,x]} U^q(s)w(s)ds \right)^{\frac{1}{q}} \left( \int_{[x,\infty)} V(s)d(\psi^p(s)) \right)^{\frac{1}{q}} V(x)d(\psi^p(x)) \right)^\frac{1}{p},$$

$$A_{5,2} := \left( \int_{[0,\infty)} \left( \int_{[0,t]} U(t,x)^p V(x)d(\psi^p(x)) \right)^{\frac{1}{p}} \left( \int_{[t,\infty)} w^q \right)^{\frac{1}{p}} w(t)^q dt \right)^{\frac{1}{q}},$$

$$A_{5,3} := \left( \int_{[0,\infty)} \left( \int_{[0,t]} V(s)d(\psi^p(s)) \right)^{\frac{1}{p}} \left( \int_{[t,\infty)} U(y,t)^q w(y)dy \right)^{\frac{1}{q}} \times V(t)d(\psi^p(t)) \right)^{\frac{1}{q}},$$

$$A_{5,4} := \left( \int_{[0,\infty)} \left( \int_{[0,t]} V(s)d(\psi^p(s)) \right)^{\frac{1}{p}} \left( \int_{[t,\infty]} w \right)^{\frac{1}{q}} \times V(t)d(\psi^p(t)) \right)^{\frac{1}{q}}.$$
\[ A_{5,4} := \left( \int_{[0,\infty)} \left( \int_{[x,\infty)} w(t) dt \right)^\frac{p}{q} \left( \int_{[0,x]} V(t) U(t)^q d(\psi^q(t)) \right)^\frac{1}{q} \right)^\frac{1}{q} \times \left( \int_{[x,\infty)} V(x)^p d(\psi^p(x)) \right)^\frac{1}{p}. \]

Here

\[ V(t) := \int_{[0,t]} v, \quad U(t) := \int_{[0,t]} u, \]

\[ V(t) := V(t) + \psi(t)^p \int_{[t,\infty)} \psi(s)^{-p} v(s) ds, \]

\[ V(t) := V(t)^{-p-1} V(t) \int_{[t,\infty)} \psi(s)^{-p} v(s) ds \]

and

\[ U(t, y) := \int_{[y,t]} \frac{u(s)}{\psi(s)} ds. \]

We also study the inequality (0.33), where the operator (0.34) is replaced with the complementary operator

\[ Bf(t) = \left( \int_{[t,\infty)} f^p d\mu \right)^\frac{1}{p}. \] (0.36)

In this case we obtain necessary and sufficient conditions for the same range of \( p \) and \( q \), but for a more special class of functions \( \Omega \), i.e. for usual quasi-concave functions.

**Remark 6.** In paper C we also show that the criteria obtained for (0.33) with the operator (0.34) can be used to obtain implicit characterization of (0.33) with the operator (0.36) in the general case \( f \in \Omega_\psi \).

The results obtained in paper C for the cone \( \Omega_\psi \) are extended to the cone \( \Omega_\psi \) in paper D (see [65]). In order to do this, in paper D we also extend some results of the paper [78] to be valid for the cone of \( \psi \)-quasi-concave functions. More precisely, we extend the estimate of the \( L_{p,v} \leftarrow L_{q,u} \) embedding for quasi-concave functions proved by G. Sinnamon in [78, Theorem 2.6] for \( 0 < q < p < \infty \) to \( \psi \)-quasi-concave functions and also obtain the analogous estimate for the case \( 0 < p < q < \infty \).

For some admissible function \( \psi(t) \) let \( \Omega^{\psi}_{0,1} \) be the set of \( \psi \)-quasi-concave functions. We obtain the estimate for the norm of the embedding \( L_{p,v} \leftarrow L_{q,u} \) on \( \Omega^{\psi}_{0,1} \) for \( 0 < p, q < \infty \):
Theorem 24 [65]. Let

\[ H_{\psi,\alpha}^{\beta}h(x) = \psi(x)^{-\alpha} \int_{[0,x]} \psi(t)^{\alpha} h(t) dt + \psi(x)^{\beta} \int_{[x,\infty)} \psi(t)^{-\beta} h(t) dt. \]

(i) If \(0 < q < p < \infty, u, v \in M^+,\) then

\[ \sup_{f \in \Omega_{0,1}^v} \frac{\|f\|_{q,u}}{\|f\|_{p,v}} \approx \left( \int_{[0,\infty)} \left( H_{\psi,0}^{\psi} \right)^{-\frac{\tau}{q}} \left( H_{\psi,q}^{\psi} \right)^{\frac{\tau}{q}} u \right)^{\frac{1}{\tau}}. \] (0.37)

(ii) If \(0 < p \leq q < \infty, u, v \in M^+,\) then

\[ \sup_{f \in \Omega_{0,1}^v} \frac{\|f\|_{q,u}}{\|f\|_{p,v}} \approx \sup_{\tau > 0} \left( H_{\psi,0}^{\psi} \right)^{-\frac{1}{\tau}} \left( H_{\psi,q}^{\psi} \right)^{\frac{1}{\tau}} u. \] (0.38)

As we have seen, the study of Hardy-type inequalities on the cones of monotone functions in fact give an insight into the characterization of quasi-concave functions. Furthermore, since the integral \(f^{**}(t) = \frac{1}{t} \int_{|y|<t} f^*(s) ds\) of the decreasing rearrangement \(f^*(t)\) is a central object in building the Lorentz \(\Gamma\) spaces, we can see that the problem of characterizing these spaces is of the same origin as the problem of characterizing Hardy-type inequalities on the cone of quasi-monotone functions. This point is illustrated in the paper D, where we obtain the criteria of boundedness between the Lorentz \(\Gamma\) spaces for the Hilbert transform

\[ Hf(x) := \lim_{\epsilon \to 0} \int_{|x-y| < \epsilon} \frac{f(x-y)}{y} dy \]

and the Riesz potentials

\[ I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad 0 < \alpha < n, \]

for the range of parameters \(p > 0, q \geq 1.\) The criteria we derive are quite important for applications because both of these operators are used extensively in different areas of functional analysis and also in the applications in other areas outside the mathematical sciences.
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Paper A
Two-sided Hardy-type inequalities for monotone functions†

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Communicated by M. Lanza de Cristoforis

(Received 11 April 2009; final version received 17 June 2009)

We consider Hardy-type operators on the cones of monotone functions with general positive \( \sigma \)-finite Borel measure. Some two-sided Hardy-type inequalities are proved for the parameter \( 0 < q < p < 1 \). It is pointed out that such equivalences, in particular, imply a new characterization of the discrete Hardy inequality for the (most difficult) case \( 0 < q < p < 1 \).

Keywords: integral inequalities; weights; Hardy operator; monotone functions; measures

AMS Subject Classifications: Primary 26D10; Secondary 26D15; 26D07

1. Introduction

For the last three decades, the Hardy-type inequalities have been extensively studied, see, e.g. the monographs [1–3] and references given there. In particular, much attention was paid to the inequalities restricted to the cones of monotone functions. Here we just mention the papers [4–16], which have been sources of inspiration to study the questions presented in this article.

Suppose that \( \lambda \) is a positive Borel \( \sigma \)-finite measure on \( \mathbb{R}_+ := [0, \infty) \). Let \( \mathcal{M}^+ \) be the class consisting of all Borel functions \( f : [0, \infty) \rightarrow [0, +\infty] \) and \( \mathcal{M}^\downarrow (\mathcal{M}^\uparrow) \) be a subclass of \( \mathcal{M}^+ \) which consists of all non-increasing (non-decreasing) functions \( f \in \mathcal{M}^+ \). Put \( \Lambda(x) := \int_{[0,x]} d\lambda \) and suppose \( \Lambda(x) < \infty \) for all \( x \in \mathbb{R}_+ \). It follows from the Hardy inequality [13] that for any \( f \in \mathcal{M}^\downarrow \) and \( 1 < p < \infty \)

\[
\left( \int_{[0,\infty)} f^p \, d\lambda \right)^{\frac{1}{p}} \leq \left( \int_{[0,\infty)} \left( \frac{1}{\Lambda(x)} \int_{[0,x]} f(t) \, d\lambda(t) \right)^p \, d\lambda(x) \right)^{\frac{1}{p}} \leq \frac{p}{p-1} \left( \int_{[0,\infty)} f^p \, d\lambda \right)^{\frac{1}{p}}. \tag{1}
\]

The aim of this article is to extend the equivalence (1) for similar Hardy-type operators and the full range of parameter \( p \). The case of a positive parameter is studied in Section 2 and a negative in Section 3. In particular, in Section 2 we extend the discrete result of [5, Theorem 1] for any positive \( \sigma \)-finite Borel measures. As an

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application, we obtain a new characterizations of the discrete Hardy inequality in the most difficult case of parameters $0 < q < p \leq 1$ (see Section 4 and cf. [4,8]).

Throughout this article products of the form $0 \cdot \infty$ are taken to be equal to $0$. The relation $A \ll B$ means that $A \leq cB$ with some constant $c$ depending only on the parameter of summation. We write $A \approx B$ instead of $A \ll B \ll A$ or $A = cB$. $\mathbb{Z}$ denotes the set of all integers and $x_E$ stands for the characteristic function (indicator) of a subset $E \subseteq \mathbb{R}_+$. Moreover, we make use of the marks $:=$ and $=$: for introducing new quantities and denote $p':=p/(p-1)$ for $0 < p < \infty$, $p \neq 1$.

2. Positive parameter

Let $0 \leq x < y \leq \infty$. We put $[x, y):=[x, y]$ if $y < \infty$ and $[x, y):=[x, y)$ if $y = \infty$. For $f \in \mathfrak{M}$, we denote

$$\Lambda_f(x, y) := \int_{[x, y)} f \, d\lambda. \quad (2)$$

We need the following statement which easy follows from [17, Lemmas 1 and 2]):

**Lemma 1** If $y > 0$ and $\Lambda_f$ is defined by (2), then the inequalities

$$\frac{\Lambda_f^{p+1}(x, y)}{\max\{1, y + 1\}} \leq \int_{[x, y)} f(z) \Lambda_f^p(z, z) \, d\lambda(z) \leq \frac{\Lambda_f^{p+1}(x, y)}{\min\{1, y + 1\}} \quad (3)$$

and

$$\frac{\Lambda_f^{p+1}(x, y)}{\max\{1, y + 1\}} \leq \int_{[x, y)} f(z) \Lambda_f^p(z, y) \, d\lambda(z) \leq \frac{\Lambda_f^{p+1}(x, y)}{\min\{1, y + 1\}} \quad (4)$$

hold. If $y \in (-1, 0)$ and $\Lambda_f(x, y) < +\infty$, then (3) and (4) also hold.

**Theorem 1** Let $p > 0$. Then for all $f \in \mathfrak{M}$

$$\left(\int_{[0, \infty)} \frac{1}{\Lambda(x)} \int_{[0, x]} \Lambda(t) \hat{f}(t) \, d\lambda(t) \right) \frac{p \, d\lambda(x)}{\Lambda(x)} \approx \left(\int_{[0, \infty)} f^p \, d\lambda\right)^{\frac{1}{p}}. \quad (5)$$

**Proof** For the lower bound applying the left-hand side of (3), we find that

$$\left(\int_{[0, \infty)} \frac{1}{\Lambda(x)} \int_{[0, x]} \Lambda(t) \hat{f}(t) \, d\lambda(t) \right) \frac{p \, d\lambda(x)}{\Lambda(x)} \geq \int_{[0, \infty)} f^p(x) \int_{[0, x]} \Lambda(t) \, d\lambda(t) \frac{p \, d\lambda(x)}{\Lambda(x)} = \left(\frac{p}{p+1}\right)^p \int_{[0, \infty)} f^p \, d\lambda.$$

To prove the upper bound, we observe that the characterization of the inequality

$$\left(\int_{[0, \infty)} \frac{1}{\Lambda(x)} \int_{[0, x]} \Lambda(t) \hat{f}(t) \, d\lambda(t) \right) \frac{p \, d\lambda(x)}{\Lambda(x)} \leq C \left(\int_{[0, \infty)} f^p \, d\lambda\right)^{\frac{1}{p}},$$

where $C = C(p) > 0$, with $C(p)$ defined as in (5). This completes the proof.
Hence, sup\(\sup_{i>0}\) and \(\inf\) with a least possible constant \(C\) governed by Theorems 3.4 and 4.2 in [9] for the cases \(0<p<1\) and \(1<p<\infty\), respectively. The case \(p=1\) is trivial. Let \(0<p<1\). According to [9, Theorem 3.4], it yields
\[
C \ll \sup_{i>0} A_0(t) + \sup_{i>0} A_1(t),
\]
where
\[
A_0(t) := \Lambda(t)^{\frac{1}{p}} \left( \left( \int_{|\xi|,\lambda} \Lambda^\frac{1}{p} \lambda \right) \Lambda(\lambda) \right)^{\frac{1}{2}}
\]
and
\[
A_1(t) := \left( \int_{|\xi|,\lambda} \Lambda^\frac{1}{p} \lambda \right)^{\frac{1}{2}} \left( \int_{|\xi|,\lambda} \Lambda(\lambda) \right)^{\frac{1}{2}}.
\]
Moreover, by Lemma 1, we see that \(\sup_{t>0} A_0(t) \approx 1\). Observe that for \(a>0\)
\[
\int_{|\xi|,\lambda} \Lambda(\lambda)^{-\frac{a}{p}-1} \lambda \lambda(x) = \int_{|\xi|,\lambda} \left( \int_{|\xi|,\lambda} (a+1)^{\frac{a}{2}-2} ds \right) d\lambda(x)
\]
\[
\quad = \int_{|\xi|,\lambda} (a+1)^{\frac{a}{2}-2} ds \int_{|\xi|,\lambda} d\lambda(x) \leq \frac{a+1}{a} \Lambda(t)^{-a}. \tag{6}
\]
It follows from this and Lemma 1 that \(\sup_{t>0} A_1(t) \ll 1\).

Now let \(1<p<\infty\). According to [9, Theorem 4.2], we have
\[
C \approx \sup_{i>0} A_3(t) + \sup_{i>0} A_4(t),
\]
where
\[
A_3(t) := \left( \int_{|\xi|,\lambda} \Lambda^\frac{1}{p} \lambda \right)^{\frac{1}{2}} \left( \int_{|\xi|,\lambda} \Lambda^{-\frac{1}{p}-1} \lambda \lambda(x) \right)^{\frac{1}{2}}
\]
and
\[
A_4(t) := \Lambda^\frac{1}{p} \left( \int_{|\xi|,\lambda} \Lambda^{-\frac{1}{p}-1} \lambda \lambda(x) \right)^{\frac{1}{2}}.
\]
By (6) and Lemma 1, it yields
\[
\left( \int_{|\xi|,\lambda} \Lambda^{-\frac{1}{p}-1} \lambda \lambda(x) \right)^{\frac{1}{2}} \ll \Lambda(t)^{-1}, \quad \int_{|\xi|,\lambda} \Lambda^{-\frac{1}{p}-1} \lambda \lambda(x) \approx \Lambda^{-\frac{1}{p}+1}(t).
\]
Hence, \(\sup_{t>0} A_3(t) \ll 1\). Analogously, \(\sup_{t>0} A_4(t) \ll 1\) and we conclude that (5) holds. The proof is complete.

We make use of the following result which follows from [12, Lemma 1.2] (see also [13, Proposition 1.5]).

**Lemma 2** ([9, Lemma 2.4]) *Let \(f \in \mathfrak{M} \downarrow\) and \(f(\infty) = 0\). Then there exist \(f_0 \in \mathfrak{M} \downarrow\), \(|\delta_n| \subset \mathfrak{M}^+\) such that*

(a) \(f_0(x) \leq f(x)\) for all \(x \in \mathbb{R}_+\), \(f_0(x) = f(x)\) a.e.,

(b) \(f_0(x) := \int_{|\xi|,\lambda} \delta_n \lambda \lambda(x) \leq f_0(x)\) for all \(x \in \mathbb{R}_+\).
(c) \( \{f_n(x)\} \) is non-decreasing in \( n \) for every \( x \in \mathbb{R}^+ \) and \( f_0(x) = \lim_{n \to \infty} f_n(x) \) \( \lambda \) a.e.

The next result is concerned with the dual Hardy operator.

**Theorem 2** Let \( 1 \leq \gamma < \infty \). Then for all \( f \in \mathcal{M} \downarrow \), \( f \neq 0 \)

\[
\frac{1}{(\gamma + 1)^\gamma} \leq \left( \frac{\left( \int_{(0, \infty)} f(x)^\gamma d\lambda(x) \right)^\frac{1}{\gamma}}{\left( \int_{(0, \infty)} f(x)^\gamma d\lambda(x) \right)^\frac{1}{\gamma}} \right)^\gamma \leq \gamma.
\]

(7)

If \( 0 < \gamma < 1 \), then \( \forall f \in \mathcal{M} \downarrow \), \( f \neq 0 \)

\[
0 < c(\gamma) \leq \left( \frac{\left( \int_{(0, \infty)} f(x)^\gamma d\lambda(x) \right)^\frac{1}{\gamma}}{\left( \int_{(0, \infty)} f(x)^\gamma d\lambda(x) \right)^\frac{1}{\gamma}} \right)^\gamma \leq \frac{1}{\gamma}.
\]

(8)

**Proof** The right-hand side of (7) is the Hardy inequality dual to (1). For the left-hand side, without loss of generality assume that \( f(\infty) = 0 \). Then by Lemma 2 we can find a function \( f_0 \in \mathcal{M} \downarrow \) and a sequence \( \{h_n\} \subset \mathcal{M}^+ \) such that

(1a) \( f'_0(x) \leq f'(x) \) for all \( x \in \mathbb{R}^+ \), \( f'_0(x) = f'(x) \) \( \lambda \) a.e.,

(1b) \( f'_0(x) := \int_{(0, \infty)} h_n d\lambda \leq f'_0(x) \) for all \( x \in \mathbb{R}^+ \),

(1c) \( f'_0(x) = \lim_{n \to \infty} f'_0(x) \) \( \lambda \) a.e. and \( \{f'_0(x)\} \) is non-decreasing in \( n \) for every \( x \in \mathbb{R}^+ \).

Now we write, using these properties, Fatou’s lemma, Lemma 1 and Fubini’s theorem

\[
\int_{(0, \infty)} f'(x)^\gamma d\lambda(x) \overset{(1a)}{=} \int_{(0, \infty)} f'_0(x)^\gamma d\lambda(x) \overset{(1b)}{=} \lim_{n \to \infty} \int_{(0, \infty)} f_n(x)^\gamma d\lambda(x) \overset{(1c)}{=} \lim_{n \to \infty} \left( \int_{(0, \infty)} h_n(y) d\lambda(y) \right)^\gamma \left( \int_{(0, \infty)} f_n(x)^\gamma d\lambda(x) \right) \leq (\gamma + 1) \lim_{n \to \infty} \left( \int_{(0, \infty)} h_n(y) d\lambda(y) \right)^\gamma \left( \int_{(0, \infty)} f_n(x)^\gamma d\lambda(x) \right)
\]

[by Minkowski’s inequality]

\[
\leq (\gamma + 1) \lim_{n \to \infty} \left( \int_{(0, \infty)} d\lambda(y) \left( \int_{(0, \infty)} h_n(y) d\lambda(y) \right)^\gamma \left( \int_{(0, \infty)} f_n(x)^\gamma d\lambda(x) \right) \right)^\gamma \overset{(1b)}{=} (\gamma + 1) \left( \gamma + 1 \int_{(0, \infty)} f(x)^\gamma d\lambda(x) \right)^\gamma (\gamma + 1) \]

and (7) is proved.
For the proof of the right-hand side of (8), we suppose without loss of generality that \( \int_{[0,\infty)} (\int_{[x,\infty)} f(x)\,d\lambda) \, dx(x) < \infty \), otherwise we use truncation. By Lemma 1

\[
\int_{[0,\infty)} \left( \int_{[x,\infty)} f(x)\,d\lambda \right)^y \, dx(x) \leq \int_{[0,\infty)} \left( \int_{[x,\infty)} f(x)\,d\lambda \right)^{y-1} f(y)\,d\lambda(x) \, dx(x)
\]

\[
\leq \int_{[0,\infty)} \left( \int_{[x,\infty)} \frac{f'(y)}{y} \, d\lambda(y) \right) \, dx(x)
\]

\[
\leq \frac{1}{y} \int_{[0,\infty)} f''(y) \Lambda'(y) \, d\lambda(y),
\]

and the right-hand side of (8) is proved.

For the left-hand side of (8), we note that \( \Lambda(x) \) is a non-decreasing right continuous function. Without loss of generality we can suppose that \( \Lambda(0)=0 \), \( \Lambda(x) > 0 \) for \( x > 0 \), \( \Lambda(\infty)=\infty \). For \( x > 0 \) we define

\[
k(x) := \max\{k \in \mathbb{Z} : \Lambda(x) \geq 2^k \}.
\]

Then \( k : (0, \infty) \rightarrow \mathbb{Z} \) is an integer-valued non-decreasing right-continuous function and

\[
2^{k(x)} \leq \Lambda(x) < 2^{k(x)+1}
\]

(9) for all \( x \in (0, \infty) \). Let the range of \( k(x) \) be \( \{k_i\} \), \( k_i < k_{i+1} \) and

\[
\Delta_i := \{x \in \mathbb{R}_+ : k(x) = k_i \}.
\]

Then \( \Delta_i=[x_i, x_{i+1}] \), where \( x_i := \inf \Delta_i \), and (9) implies that

\[
2^{k_i} \leq \Lambda(x) < 2^{k_{i+1}}, \quad x \in \Delta_i.
\]

(10)

Put

\[
I_1 := \{i \in \mathbb{Z} : k_i - k_{i-1} \geq 2\}, \quad I_2 := \{i \in \mathbb{Z} : k_i = k_{i-1} + 1\}.
\]

It is clear that \( \{x_i\} \) is an atom of \( \lambda \), when \( i \in I_1 \) and \( \lambda[x_i] = \Lambda(x_i) - \Lambda(x_i-0) \). It follows from (10) that \( \lambda[x_i] \leq 2 \cdot 2^{k_i} \) and \( \Lambda(x_i) \geq 2^{k_i}, \quad \Lambda(x_i-0) \leq 2^{k_{i+1}}, \quad \Lambda(x_i) - \Lambda(x_i-0) \geq 2^{k_i} \). Hence,

\[
\frac{1}{2} \cdot 2^{k_i} \leq \lambda[x_i] \leq 2 \cdot 2^{k_i}, \quad i \in I_1.
\]

This and (10) imply that

\[
\frac{1}{2} \cdot 2^{k_i} \leq \lambda[x_i] \leq \int_{\Lambda_i} d\lambda \leq \sup_{x \in [x_i, x_{i+1}]} \Lambda(x) \leq 2 \cdot 2^{k_i}.
\]

(11)

Now we write

\[
\int_{[0,\infty)} f'' \Lambda' \, d\lambda = \sum_{i \in I_1} \int_{\Lambda_i} f'' \Lambda' \, d\lambda + \sum_{i \in I_2} \int_{\Lambda_i} f'' \Lambda' \, d\lambda =: J_1 + J_2
\]
and begin with the estimation of $J_1$. By (11) we have
\[ J_1 \ll \sum_{i \in I_1} f^\gamma(x_i) 2^{k_i} \]
as well as
\[ f(x_i) 2^{k_i} \leq \int_{x_i} f(x) \, d\lambda, \quad i \in I_1. \]
Hence,
\[ J_1 \ll \sum_{i \in I_1} \left( \int_{a_i} f \, d\lambda \right)^{\gamma} \ll \sum_{i \in I_1} \left( \sum_{m \geq 2} \left( \int_{a_m} f \, d\lambda \right)^{\gamma} \right)^{2^{k_i}} \begin{aligned} &\approx \sum_{i \in I_1} \left( \int_{(x_{i-2}, x_i]} f \, d\lambda \right)^{\gamma} \lambda(x_i) \leq \int_{[0, \infty)} \left( \int_{(x_{i-2}, x_i]} f \, d\lambda \right)^{\gamma} d\lambda(x). \end{aligned} \]
By using (10) and (11), it yields
\[ J_2 \leq \sum_{i \in I_2} f^\gamma(x_i) \int_{a_i} \Lambda^\gamma \, d\lambda \leq 2^{\gamma+1} \sum_{i \in I_2} f^\gamma(x_i) 2^{k_i+1}. \]
Moreover, by (10),
\[ \int_{(x_{i-2}, x_i]} f \, d\lambda = \Lambda(x_i) - \Lambda(x_{i-2}) \geq 2^{k_i} \geq 2^{k_{i-2}+1} = \frac{1}{2} \cdot 2^{k_i}. \]
By this and Lemma 1 we find that
\[ 2^{k_i+1} \ll \left( \int_{(x_{i-2}, x_i]} f \, d\lambda \right)^{\gamma+1} \approx \int_{[x_{i-2}, x_i]} \left( \int_{(x_{i-2}, x_i]} f \, d\lambda \right)^{\gamma} d\lambda. \]
Hence,
\[ J_2 \ll \sum_{i \in I_2} f^\gamma(x_i) \int_{(x_{i-2}, x_i]} \left( \int_{(x_{i-2}, x_i]} f \, d\lambda \right)^{\gamma} d\lambda(x) \]
\[ \leq \sum_{i \in I_2} \left[ \int_{(x_{i-2}, x_{i-1}]} f \, d\lambda \right] \left( \int_{(x_{i-2}, x_i]} f \, d\lambda \right)^{\gamma} d\lambda(x) \]
\[ \leq \sum_{i \in I_2} \left[ \int_{(x_{i-2}, x_{i-1}]} f \, d\lambda \right] \left( \int_{(x_{i-2}, x_{i-1}]} f \, d\lambda \right)^{\gamma} d\lambda(x) \]
\[ \leq 3 \int_{[0, \infty)} \left( \int_{(x_{i-2}, x_i]} f \, d\lambda \right)^{\gamma} d\lambda(x) \]
and also (8) is proved. So the proof is complete.

Now we deal with operators involving suprema.

**Theorem 3** Let $0 < \gamma < \infty$. Then for all $f \in \mathcal{M}$, $f \neq 0$
\[ 1 \leq \left( \frac{\int_{[0, \infty)} \text{ess sup}_{y \geq 1} f^\gamma(y) \Lambda^\gamma(y) \, d\lambda(x)}{\left( \int_{[0, \infty)} f^\gamma \Lambda^\gamma \, d\lambda \right)^{\gamma}} \right)^{\gamma} \leq c(\gamma), \quad (12) \]
where
\[ e^y(y) = \max(1, 2y^{-1}) \left( 1 + \frac{\max(1, y)}{\min(1, y)} \right). \]

**Proof** Since the product of two monotone functions is a Borel function, elementary arguments show that 
\[ f(x) \leq \text{ess sup}_{y \geq x} f(y) \Lambda(y) \] for \( \lambda \)-a.e. \( x \in \mathbb{R}_+ \) and the left-hand side of (12) follows. For the proof of the right-hand side of (12), we suppose that 
\[ \int_{(0, \infty)} f'^r \Lambda^r \, d\lambda < \infty \]
and without loss of generality assume that \( f(\infty) = 0 \). Then, by Lemma 2, there exist a function \( f_0 \in \mathbb{R}_+ \) and a sequence \( \{h_n\} \subset \mathbb{R}_+ \) such that

1. \( f_0(x) \leq f'(x) \) for all \( x \in \mathbb{R}_+ \),
2. \( f'(x) := \int_{(0, \infty)} h_n \, d\lambda \leq f_0(x) \) for all \( x \in \mathbb{R}_+ \),
3. \( f_0(x) = \lim_{n \to \infty} f_n(x) \) \( \lambda \)-a.e. and \( \{f_n(x)\} \) is non-decreasing in \( n \) for every \( x \in \mathbb{R}_+ \).

Moreover, we make use of the formula
\[ \text{ess sup}_{E} f = \sup_{E, 0 < \lambda < \infty} \frac{1}{\lambda} \int_{E} F \, d\lambda, \]
which is valid for any \( \sigma \)-finite measure \( \lambda \) and a \( \lambda \)-measurable function \( F \) and a set \( E \).

Now we write
\[ \text{ess sup}_{x \geq y} f'^r(y) \Lambda^r(y) = \text{ess sup}_{x \geq y} f_0(y) \Lambda^r(y) \]
which by the Monotone Convergence Theorem
\[ = \sup_{x \in \mathbb{R}_+, 0 < \lambda < \infty} \frac{1}{\lambda} \int_{x} f_0 \Lambda^r \, d\lambda \]
\[ = \sup_{x \in \mathbb{R}_+, 0 < \lambda < \infty} \frac{1}{\lambda} \int_{x} f_0 \Lambda^r \, d\lambda \]
\[ = \sup_{x \in \mathbb{R}_+, 0 < \lambda < \infty} \left( \int_{(0, \infty)} h_n \, d\lambda \right) \Lambda^r(y) \]
\[ \leq \sup_{x \geq y} \text{ess sup}_{n} \left( \int_{(0, \infty)} h_n \, d\lambda \right) \Lambda^r(y) \]
\[ = \sup_{x \geq y} \int_{(0, \infty)} h_n \, d\lambda. \] (14)
Now, by Lemma 1 and an elementary inequality with \( c_0(\gamma) := \max(1, 2^{\gamma - 1}) \), we have
\[
\int_{(0, \infty)} h_\gamma \Lambda^\gamma d\lambda \leq c_0(\gamma) \left( \Lambda^\gamma(x) \int_{(0, \infty)} h_\gamma d\lambda + \int_{[1, \infty)} h_\gamma(y) \left( \int_{[1, \infty)} \Lambda^\gamma d\lambda(y) \right)^\gamma d\lambda(y) \right)
\leq c_0(\gamma) \left( \int_{(0, \infty)} \Lambda^\gamma f d\lambda + \max(1, \gamma) \int_{[1, \infty)} \Lambda^{\gamma-1} f d\lambda \right)
= c_0(\gamma) \left( \int_{(0, \infty)} \Lambda^\gamma f d\lambda + \max(1, \gamma) \int_{[1, \infty)} \Lambda^{\gamma-1} f d\lambda \right)
\leq (2^{\gamma-2}) c_0(\gamma) \left( \int_{(0, \infty)} \Lambda^\gamma f d\lambda \right).
\]
By (14), this implies that
\[
\text{ess sup}_{\gamma \geq x} f^\gamma(\gamma) \Lambda^\gamma(\gamma) \leq c_0(\gamma) \left( \int_{(0, \infty)} f^\gamma \Lambda^\gamma d\lambda + \max(1, \gamma) \int_{[1, \infty)} \Lambda^{\gamma-1} f d\lambda \right)
\leq c_0(\gamma) \left( 1 + \frac{\max(1, \gamma)}{\min(1, \gamma)} \right) \int_{(0, \infty)} f^\gamma \Lambda^\gamma d\lambda
\]
and also the right-hand side of (12) is proved. The proof is complete.

Denote by \( \mathcal{M} \downarrow (+0) \) the cone of all non-increasing functions \( f \in \mathcal{M} \downarrow \) continuous on the right and such that \( f(x) > 0 \) for all \( x \in \mathbb{R}_+ \). Then \( \frac{1}{f} \in \mathcal{M} \uparrow \) is continuous on the right and there exists a unique Borel measure \( \mu_f \) such that
\[
\frac{1}{f(x)} = \int_{[0, x]} d\mu_f
\]
(see, e.g. [18, Chapter 12, Section 3]).

**Theorem 4** Let \( 0 < \gamma < \infty \). Then, for all \( f \in \mathcal{M} \downarrow \) \(+0\),
\[
1 \leq \left( \int_{(0, \infty)} \left( \int_{[1, \infty)} f d\lambda \right)^{\gamma+1} d\mu_f(\gamma) \right)^\frac{1}{\gamma+1} \leq (\gamma+1)^{\frac{1}{\gamma}}.
\]

**Proof** Using (15) for the left-hand side of (16) we write
\[
\int_{(0, \infty)} \left( \int_{[1, \infty)} f d\lambda \right)^{\gamma} d\lambda(x) = \int_{(0, \infty)} \left( \int_{[1, \infty)} f d\lambda \right)^{\gamma} \left( \int_{[0, x]} d\mu_f \right) f(x) d\lambda(x)
\leq \int_{(0, \infty)} \left( \int_{[1, \infty)} f d\lambda \right)^{\gamma} d\mu_f(y).
\]
For the right-hand side of (16), by Lemma 1 we find
\[
\int_{[0,\infty)} \left( \int_{[0,\infty)} f dx \right)^{\gamma + 1} d\mu(x) \leq (\gamma + 1) \int_{[0,\infty)} d\mu(x) \left( \int_{[0,\infty)} f dy \right)^{\gamma} d\lambda(y).
\]
\[
= (\gamma + 1) \int_{[0,\infty)} \left( \int_{[0,\infty)} f dy \right)^{\gamma} d\lambda(y).
\]
This completes the proof.

**Theorem 5** Let \( 0 < \gamma < \infty \). Then for all \( f \in \mathcal{M} \downarrow (-\infty)
\]
\[
\alpha(\gamma) = \frac{\int_{[0,\infty)} \left( \text{ess sup}_{y \geq 1} f(y) \Lambda(y) \right)^{1/\gamma} d\lambda(x)}{\left( \int_{[0,\infty)} \left( \text{ess sup}_{y \geq 1} f(y) \Lambda(y) \right)^{1/\gamma} d\mu(x) \right)^{\gamma}} \leq \left( \frac{\max(1, \gamma)}{\min(1, \gamma)} \right)^{1/\gamma},
\]
where \( \alpha(\gamma) = \left( \frac{1}{\gamma + 1} \right)^{1/\gamma} \).

**Proof** Denote
\[
W(x) := \text{ess sup}_{y \geq 1} f(y) \Lambda(y)
\]
and without loss of generality assume that \( W(\infty) = 0 \). Then, by Lemma 2, there exist a function \( W_0 \in \mathcal{M} \downarrow \) and a sequence \( \{w_n\} \subset \mathcal{M}^+ \) such that
\begin{enumerate}
  \item \( W_0(x) \leq W(x) \) for all \( x \in \mathcal{R}_+ \), \( W_0(x) = W(x) \) \( \lambda \)-a.e.,
  \item \( W(x) := \int_{[0,\infty)} w_n d\lambda \leq W_0(x) \) for all \( x \in \mathcal{R}_+ \),
  \item \( W_0(x) = \lim_{n \to \infty} W_n(x) \) \( \lambda \)-a.e. and \( \{W_n(x)\} \) is non-decreasing in \( n \) for every \( x \in \mathcal{R}_+ \).
\end{enumerate}

Using these properties for the right-hand side of (17), by Lemma 1 we write
\[
\int_{[0,\infty)} W^\gamma d\lambda \\overset{(3a)}{=} \int_{[0,\infty)} W^\gamma_0 d\lambda \\overset{(3b)}{=} \lim_{n \to \infty} \int_{[0,\infty)} W^\gamma_n d\lambda
\]
\[
\leq \max(1, \gamma) \lim_{n \to \infty} \int_{[0,\infty)} \left( \int_{[0,\infty)} W^{\gamma - 1} w_n d\lambda \right) d\lambda(x)
\]
\[
\leq \max(1, \gamma) \lim_{n \to \infty} \int_{[0,\infty)} \left( \int_{[0,\infty)} W^{\gamma - 1} \frac{W}{w_n} d\lambda \right) d\lambda(x)
\]
\[
= \max(1, \gamma) \lim_{n \to \infty} \int_{[0,\infty)} W^{\gamma - 1} (y) \frac{W(y)}{W_0(y)} w_n(y) d\lambda(y)
\]
\[
= \max(1, \gamma) \lim_{n \to \infty} \int_{[0,\infty)} d\mu(z) \int_{[0,\infty)} W^{\gamma - 1} w_n d\lambda
\]
\[
\leq \max(1, \gamma) \lim_{n \to \infty} \int_{[0,\infty)} W(z) d\mu(z) \int_{[0,\infty)} W^{\gamma - 1} w_n d\lambda
\]
\[
\leq \max(1, \gamma) \lim_{n \to \infty} \int_{[0,\infty)} W^{\gamma + 1} d\mu_f
\]
For the left-hand side, we use an idea from the proof [5, Theorem 1]. Suppose
\[ \int_{[0,\infty)} W^{\gamma}(x) \, d\lambda(x) < \infty. \]
Without loss of generality we assume that \( 0 < f(0) < \infty, \ \Lambda(\infty) = \infty. \) Then \( f(0)\Lambda(0) = 0, \ W(x) \) is bounded and \( W(\infty) = 0. \) Let \( t_0 = 0 \) and suppose that we find a sequence \( 0 < t_1 < t_2 < \cdots < t_{k-1}. \) Then put
\[ t_k := \sup \left\{ t \geq t_{k-1} : f(t)\Lambda(t) \geq \frac{1}{2} \text{ ess sup}_{y \geq t_k} f(y)\Lambda(y) \right\}. \]
For the sake of brevity, we omit analysis of pathological cases and assume that \( t_k > t_{k-1} \) and \( \lim_{k \to \infty} f(t_k) = \infty. \) Put \( f(t_0 - 0)\Lambda(t_0 - 0) := 0, \) and for \( k > 0 \) we have
\[ f(t_k - 0)\Lambda(t_k - 0) \geq \frac{1}{2} \text{ ess sup}_{y \geq t_k} f(y)\Lambda(y). \tag{18} \]
Since \( f \) and \( \Lambda \) are monotone, the left limits \( f(t_k - 0) \) and \( \Lambda(t_k - 0) \) exist. Moreover,
\[ \int_{[t_k, t_{k+1})} d\mu_f = \frac{1}{f(t_k - 0)} - \frac{1}{f(t_{k+1} - 0)} \cdot \int_{[t_k, t_{k+1})} d\lambda = \Lambda(t_k - 0) - \Lambda(t_{k+1} - 0). \]
Then
\[
\int_{[0,\infty)} W^{\gamma+1} d\mu_f = \sum_{k=1}^{\infty} \int_{[t_k, t_{k+1})} W^{\gamma+1} d\mu_f \\
\leq \sum_{k=1}^{\infty} \int_{[t_k, t_{k+1})} \left[ \text{ ess sup}_{y \geq t_k} f(y)\Lambda(y) \right]^{\gamma+1} d\mu_f \\
\leq 2^{\gamma+1} \sum_{k=1}^{\infty} \int_{[t_k, t_{k+1})} f(t_k - 0)\Lambda(t_k - 0) f^{\gamma+1} d\mu_f\\
\leq \int_{[0,\infty)} f^{\gamma+1} d\lambda \\
\leq 2^{\gamma+1} \sum_{k=1}^{\infty} \int_{[t_k, t_{k+1})} W^\gamma d\lambda = 2^{\gamma+1} \int_{[0,\infty)} W^\gamma d\lambda
\]
and the proof is completed.

Remark 1 For the counting measure \( \lambda, \) Theorems 2–4 have been established in [5, Theorem 1]. In analogy with this result, it follows from Theorems 2–4 that the following inequalities are equivalent:

1. \( \int_{[0,\infty)} (f(x,\infty) f \, d\lambda)^\gamma \, d\lambda(x) < \infty, \)
2. \( \int_{[0,\infty)} (f(x,\infty) f \, d\lambda)^{\gamma+1} \, d\mu_f < \infty, \)
3. \( \int_{[0,\infty)} \text{ ess sup}_{y \geq x} f(y)\Lambda(y) \, d\lambda(x) < \infty, \)
4. \( \int_{[0,\infty)} (f(x,\infty) f \, d\lambda)^{\gamma+1} \, d\mu_f(x) < \infty, \)
5. \( \int_{[0,\infty)} f^\gamma \, \Lambda^\gamma d\lambda < \infty \)
in a sense that
\[
\begin{align*}
\text{(2) } & \iff \text{(1) } \iff \text{(5) } \iff \text{(3) } \iff \text{(4).} 
\end{align*}
\]
(19)

**Remark 2** Theorems 1–5 have the complete analogues for non-decreasing functions

**Definition** [15]: Let \(0 < \gamma < \infty\) and a function \(f \in \mathcal{M} \downarrow\) be continuous on the right.

We say that \(f\) belongs to \(\mathcal{T}_\gamma\), if
\[
f^\gamma(x) - f^\gamma(\infty) \leq C \int_{[x,\infty)} f^{\gamma+1} \, d\mu_f
\]
for some constant \(C > 0\).

We can supplement (19) by an additional equivalence, which replaces the exact equality
\[
\int_0^\infty [f\Lambda]^\gamma \, d\lambda = \frac{1}{\gamma + 1} f^\gamma(\infty)\Lambda^{\gamma+1}(\infty) + \frac{\gamma}{\gamma + 1} \int_0^\infty [f\Lambda]^{\gamma+1} \left( \frac{1}{f(x)} \right) \, d\lambda
\]
for absolutely continuous \(\Lambda\) and \(f\) [7, Theorem 3.1(ii)].

**Proposition** Let \(0 < \gamma < \infty\) and \(f \in \mathcal{T}_\gamma\). Then
\[
f^\gamma(\infty)\Lambda^{\gamma+1}(\infty) + \int_{[0,\infty)} [f\Lambda]^{\gamma+1} \, d\mu_f \approx \int_{[0,\infty)} [f\Lambda]^\gamma \, d\lambda.\]
(21)

**Proof** Since the reverse inequality to (20) is always true [13], we have
\[
f^\gamma(x) - f^\gamma(\infty) \approx \int_{[x,\infty)} f^{\gamma+1} \, d\mu_f,\]
(22)
provided \(f \in \mathcal{T}_\gamma\). Using this, Lemma 1 and (22), we write
\[
\int_{[0,\infty)} [f\Lambda]^{\gamma+1} \, d\mu_f \approx \int_{[0,\infty)} \left( \int_{[0,\infty)} \Lambda^\gamma \, d\lambda \right) f^{\gamma+1}(x) \, d\mu_f(x)
\]
\[
= \int_{[0,\infty)} \left( \int_{[0,\infty)} f^{\gamma+1} \, d\mu_f \right) \Lambda^\gamma(y) \, d\lambda(y)
\]
\[
\approx \int_{[0,\infty)} (f^\gamma(y) - f^\gamma(\infty))\Lambda^\gamma(y) \, d\lambda(y)
\]
\[
= \int_{[0,\infty)} [f\Lambda]^\gamma \, d\lambda - f^\gamma(\infty)\Lambda^{\gamma+1}(\infty).
\]

**Remark 2** Theorems 1–5 have the complete analogues for non-decreasing functions and \(\Lambda\) replaced by \(\Lambda^*(t) := \int_{[t,\infty)} \, d\lambda\). For instance, the following inequalities:

1. \(\int_{[0,\infty)} (\int_{[0,\infty)} f \, d\lambda)^\gamma \, d\lambda(x) < \infty\),
2. \(\int_{[0,\infty)} (\int_{[t,\infty)} f \, d\lambda)^{\gamma+1} \, d\mu_f < \infty\),
3. \(\int_{[0,\infty)} [\text{ess sup}_{y \leq x} f(y)\Lambda^*(y)]^\gamma \, d\lambda(x) < \infty\),
4. \(\int_{[0,\infty)} [\text{ess sup}_{y \leq x} f(y)\Lambda^*(y)]^{\gamma+1} \, d\mu_f(x) < \infty\),
5. \(\int_{[0,\infty)} f^\gamma\Lambda^*_y \, d\lambda < \infty\)
are equivalent, that is

\[
(\text{5c}) \quad f(x) \equiv f(\text{t}(-0))
\]

\[
(\text{2}) \iff (\text{1}) \iff (\text{5}) \iff (\text{3}) \iff (\text{4})
\]

with the corresponding changes in definitions.

3. **Negative parameter**

Let \( \gamma < 0 \). Then it obviously yields

\[
\int_{[0, \infty)} f^\gamma \Lambda' \, d\lambda \leq \int_{[0, \infty)} \left( \int_{[0, 1]} f \, d\lambda \right) \gamma \, d\lambda.
\]

for all \( f \in \mathcal{M} \uparrow \), \( f(x) > 0 \lambda - \text{a.e.} \).

**Theorem 6** Let \(-1 < \gamma < 0\). Then for all \( f \in \mathcal{M} \uparrow \), \( f(x) > 0 \lambda - \text{a.e.} \).

\[
(-\gamma)^{\frac{1}{2}} \leq \frac{\left( \int_{[0, \infty)} \left( \int_{[0, 1]} f \, d\lambda \right) \gamma \, d\lambda \right)^{\frac{1}{2}}}{\left( \int_{[0, \infty)} f^\gamma \Lambda' \, d\lambda \right)^{\frac{1}{2}}} \leq 1.
\]

**Proof** The right-hand side of (24) follows from (23). For the left-hand side, by Lemma 2, we can find a function \( f_0 \in \mathcal{M} \uparrow \) and a sequence \( \{\delta_n\} \in \mathcal{M} \), such that

\(\begin{align*}
(5a) & \quad f_0^\gamma(x) \leq f^\gamma(x) \text{ for all } x \in \mathbb{R}_+,
(5b) & \quad f_0^\gamma(x) := f_0 |_{[1, \infty)} h_0 \, d\lambda \leq f_0^\gamma(x) \text{ for all } x \in \mathbb{R}_+,
(5c) & \quad f_n^\gamma(x) = \lim_{n \to \infty} f_n^\gamma(x) \lambda - \text{a.e. and } \{f_n^\gamma(x)\} \text{ is non-decreasing in } n \text{ for every } x \in \mathbb{R}.
\end{align*}\)

Now using the properties (5a)–(5c), Lemma 1, Fatou's lemma and Fubini's theorem, we find that

\[
\int_{[0, \infty)} f^\gamma \Lambda' \, d\lambda \overset{(5a)}{=} \int_{[0, \infty)} f_0^\gamma \Lambda' \, d\lambda \overset{(5c)}{=} \lim_{n \to \infty} \int_{[0, \infty)} f_n^\gamma \Lambda' \, d\lambda \\
\overset{(5b)}{=} \lim_{n \to \infty} \int_{[0, \infty)} \left( \int_{[0, 1]} h_0(y) \, d\lambda(y) \right) \Lambda' \, d\lambda(y) \\
= \lim_{n \to \infty} \int_{[0, \infty)} h_0(y) \left( \int_{[0, 1]} \Lambda' \, d\lambda(y) \right) \, d\lambda(y) \\
\geq \lim_{n \to \infty} \int_{[0, \infty)} h_0(y) \Lambda^{y+1}(y) \, d\lambda(y) \\
= \lim_{n \to \infty} \int_{[0, \infty)} f_0^{y+1}(y) \Lambda^{y+1}(y) \, d\lambda(y) \\
\overset{(5a)}{=} \lim_{n \to \infty} \int_{[0, \infty)} \left( \int_{[0, 1]} f \, d\lambda \right) \gamma \, d\lambda \\
\overset{(5c)}{=} \lim_{n \to \infty} \int_{[0, \infty)} \left( \int_{[0, 1]} f \, d\lambda \right) \gamma \, d\lambda \\
\geq \lim_{n \to \infty} \int_{[0, \infty)} \left( \int_{[0, 1]} f \, d\lambda \right) \gamma \, d\lambda.
\]
Hence, also the left-hand side of (24) is proved, so the proof is complete.

**Theorem 7** Let \( \gamma \leq -1 \). Then, for all \( f \in \mathfrak{M} \), \( f(x) > 0 \), \( \lambda \) a.e. it yields

\[
0 < c_\gamma \leq \frac{\left( \int_{[0,\infty]} f \, d\lambda \right)^\gamma}{\left( \int_{[0,\infty]} f^\gamma \, d\lambda \right)^{\frac{\gamma}{\gamma - 1}}} \leq 1. \tag{25}
\]

**Proof** The right-hand side of (25) follows from (23). For the left-hand side of (25), we utilize the sequences of points \( \{x_k\} \) and intervals \( \Delta_k \) constructed in the proof of Theorem 2. We have

\[
J := \int_{[0,\infty]} \left( \int_{[0,\infty]} f \, d\lambda \right)^\gamma \, d\lambda = \sum_i \int_{\Delta_i} \left( \int_{[0,\infty]} f \, d\lambda \right)^\gamma \, d\lambda(x) \\
\leq \sum_i \int_{\Delta_i} \left( \int_{[x_{k-2},x_k]} f \, d\lambda \right)^\gamma \, d\lambda(x) \leq \sum_i \int_{\Delta_i} \left( \int_{[x_{k-2},x_k]} f \, d\lambda \right)^\gamma \, d\lambda(x).
\]

If \( x \in \Delta_n \), then by (10)

\[
\int_{[x_{k-2},x_k]} \, d\lambda = \Lambda(x) - \Lambda(x_{k-2}) \geq 2^{k_{l-1}} - 2^{k_{l-1}+1} \geq 2^{k_{l-1}}.
\]

Applying this and (11), we find that

\[
\int_{\Delta_i} \left( \int_{[x_{k-2},x_k]} d\lambda \right)^\gamma \, d\lambda(x) \leq 2^{1+\gamma k_{l-1} (p+1)} \leq 2^{1+\gamma k_{l-1} (p+1)}.
\]

Hence,

\[
J \leq 2^{1+\gamma} \sum_i \int_{[x_{k-2},x_k]} 2^{k_{l-1} (1+p)}.
\]

Observe that if \( x \in [x_{k-5}, x_{k-2}) \), then by (10) we have \( \Lambda(x) \leq 2 \cdot 2^{k_{l-1}} \). This implies that

\[
\int_{[x_{k-5},x_{k-2})} \Lambda^\gamma \, d\lambda \geq 2^{2^{k_{l-1}}} \int_{[x_{k-5},x_{k-2})} \, d\lambda \geq 2^{2^{k_{l-1}}}. \tag{26}
\]
Applying this and (26), we obtain
\[
J \leq 2^{2(1-\gamma)} \sum_i f'(x_i) \int_{[x_i-\epsilon,x_i+\epsilon]} \Lambda^\gamma d\lambda \leq 2^{2(1-\gamma)} \sum_i \int_{[x_i-\epsilon,x_i+\epsilon]} f''(x_i) \Lambda^\gamma d\lambda
\]
\[
\leq 2^{2(1-\gamma)} \sum_i \left( \int_{[x_i, x_{i+1}]} f''(x_i) \Lambda^\gamma d\lambda + \int_{(x_{i+1}, x_{i+2}]} f''(x_i) \Lambda^\gamma d\lambda + \int_{(x_{i+2}, x_{i+3}]} f''(x_i) \Lambda^\gamma d\lambda \right)
\]
\[
\leq 3 \cdot 2^{2(1-\gamma)} \int_{[0,\infty)} f''(x_i) \Lambda^\gamma d\lambda
\]
and also the left-hand side of (16) is proved, so the proof is complete.

**Remark 3** Theorems 6 and 7 have the complete analogues for non-increasing functions with the changes mentioned in Remark 2.

### 4. Applications

Let \(0 < q < p \leq 1\), \(\frac{1}{p} := \frac{1}{q} - \frac{1}{r}\) and

\[
H(p,q) := \sup_{\alpha \geq 0} \left( \sum_n w_n^p \left( \sum_{k \geq n} \alpha_k \right)^p \right)^{\frac{1}{p}} \left( \sum_n w_n^p \alpha_n \right)^{-\frac{1}{p}}.
\]  
(27)

Put

\[
\Phi_n := \inf_{k \geq n} \phi_k, \quad W_n := \left( \sum_{k \geq n} w_k^p \right)^{\frac{1}{p}}.
\]  
(28)

and

\[
G_0(p,q) := \left( \sum_n \Phi_n^{-\gamma} (W_n^p - W_{n-1}^p) \right)^{\frac{1}{p}}.
\]

The following result is due to Goldman [8, Proof of Theorem 2.1, page 128].

**Theorem 8** If \(0 < q < p \leq 1\), then

\[
H(p,q) \approx G_0(p,q).
\]

Observe that if \(\gamma > 0\), \(0 \leq a \leq b\), then

\[
b^\gamma - a^\gamma \approx b^\gamma - b^\gamma - (b - a).
\]  
(29)

According to (29), we have

\[
W_n^p - W_{n-1}^p \approx \left( \sum_{k \geq n} w_k^p \right)^{\frac{1}{p}} w_n^p.
\]

Hence,

\[
G_0(p,q) \approx \left( \sum_n \Phi_n^{-\gamma} W_n^p W_{n-1}^p \right)^{\frac{1}{p}} =: G(p,q)
\]
and, thus, by Theorem 8,

$$H(p, q) \approx G(p, q).$$

Since $\Phi_n \uparrow$ there exists $z_k \geq 0$ such that

$$\Phi_n^p = \sum_{k \geq n} z_k,$$

we have with $p \in (0, 1)$

$$\sum_n w_n^p a_n^p \geq \sum_n \Phi_n^p a_n^p \geq \sum_n \left( \sum_{k \geq n} z_k \right) \alpha_n^p \geq \sum_k z_k \sum_{n \geq k} \alpha_n^p \geq \sum_{n \geq k} \left( \sum_{n \geq k} \alpha_n \right)^p z_k. \quad (30)$$

Hence,

$$H(p, q) \leq \varkappa(p, q) : = \sup_{0 \leq h \leq 1} \frac{\left( \sum_n w_n^p r_n \right) \left( \sum_n r_n z_n \right)^p}{\left( \sum_n r_n z_n \right)^p}. \quad (31)$$

Moreover, it is known [13] that

$$\varkappa(p, q) \approx \left( \sum_n \left( \sum_{k \geq n} \Phi_k^{-p} w_k^p \right)^{\frac{p}{q}} z_n \right)^{\frac{1}{p}} =: F_0(p, q).$$

By Lemma 1, it yields

$$F_0(p, q) \approx \sum_n \left[ \sum_{k \geq n} \left( \sum_{i \geq k} \Phi_i^{-p} w_i^p \right)^{\frac{p}{q}} \Phi_k^{-p} w_k^p \right] z_n = \sum_k \left( \sum_{i \geq k} \Phi_i^{-p} w_i^p \right)^{\frac{p}{q}} w_k^p =: F'(p, q). \quad (32)$$

If we put

$$\lambda[k] = w_k^p; \ f(k) = \Phi_k^{-p}; \ \gamma = \frac{r}{p},$$

then, by Theorem 2,

$$F(p, q) \approx G(p, q).$$

Summing up, we see that

$$G(p, q) \approx H(p, q) \leq \varkappa(p, q) \approx F(p, q) \approx G(p, q).$$

Thus, we have proved the following new characterization of the discrete Hardy inequality.

**Theorem 9** Let $0 < q < p \leq 1$ and $H(p, q)$ is defined by (27). Then

$$H(p, q) \approx F(p, q),$$

where $F(p, q)$ is determined by (32).
In particular, the above results imply the equivalence of the discrete Hardy inequality to a simpler inequality on monotone sequences.

**Corollary** The discrete Hardy inequality

\[
\left( \sum_n \left( \sum_{k \geq n} \alpha_k^n \right)^{\frac{q}{p}} w_n^{\frac{1}{q}} \right)^{\frac{1}{q}} \leq C \left( \sum_n \varphi_n^q \omega_n^p \right)^{\frac{1}{p}}, \quad \alpha_n \geq 0
\]

holds if and only if

\[
\left( \sum_n (\beta_n w_n)^{q} \right)^{\frac{1}{q}} \leq C \left( \sum_n \beta_n^{q} z_n^{p} \right)^{\frac{1}{p}}, \quad 0 \leq \beta_n \downarrow,
\]

where \( z_n \) is given by (28) and (30).

**Acknowledgements**

The authors express their deep gratitude to Dr. Dmitry Prokhorov and Prof. Gordon Sinnamon for helpful remarks. The works of O.V. Popova and V.D. Stepanov were partially supported by the Russian Fund for Basic Research (Projects 09-01-00093 and 09-01-00586) and V.D. Stepanov was also supported by grants of Far Eastern Branch of the Russian Academy of Sciences.

**References**


Paper B
HARDY-TYPE INEQUALITIES ON THE CONES OF MONOTONE FUNCTIONS

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Abstract: We establish necessary and sufficient conditions for various Hardy-type inequalities on the cones of monotone functions.

Keywords: integral operator, Hardy inequality

1. Introduction

Hardy’s inequalities, including those on the cones of monotone functions, are actively studied in the last two decades. In particular, we can point out the articles [1–17] devoted to this topic.

Let us introduce some notation. Denote by \( \lambda, \mu, \) and \( \nu \) positive \( \sigma \)-finite Borel measures on \( \mathbb{R}^+ := [0, \infty) \); by \( \mathcal{M}^+ \), the class of all Borel functions \( f : \mathbb{R}^+ \to [0, \infty) \), and by \( \mathcal{M}_\downarrow \) and \( \mathcal{M}_\uparrow \) the subclasses of \( \mathcal{M}^+ \) consisting of all nonincreasing and all nondecreasing functions. Put \( \Lambda(x) = \int_0^x d\lambda \) and assume henceforth that \( \Lambda(x) < \infty \) for all \( x \in \mathbb{R}^+ \).

In Subsection 3.1 we consider the inequality

\[
\left( \int_{[0,\infty)} (Kf)^q d\mu \right)^{\frac{1}{q}} \leq C \left( \int_{[0,\infty)} \left( \int_{[0,\infty)} f d\lambda \right)^p d\lambda \right)^{\frac{1}{p}} \tag{1}
\]

for the operator

\[
Kf(x) = \int_{[0,x]} k(x,y) f(y) \, d\nu(y). \tag{2}
\]

where \( k(x,y) \geq 0 \) is a measurable kernel satisfying the Oinarov condition: there exists a constant \( D \geq 1 \) such that

\[
D^{-1}(k(x,z) + k(z,y)) \leq k(x,y) \leq D(k(x,z) + k(z,y)), \quad x \geq z \geq y. \tag{3}
\]

The inequality in (1) with absolutely continuous measures for all \( f \in \mathcal{M}^+ \) for \( 1 < p, q < \infty \) is studied in [18] (also see [19–22]) while in the case of Borel measures, in [23, 24]. On the cone of monotone functions for \( k(x,y) \equiv 1 \) the inequality in (1) is studied in [17] for \( 0 < p, q \leq 1, 1 < p, q < \infty \), and \( 0 < p \leq 1 \leq q \).

In Subsection 3.1 we give criteria for the fulfillment of (1) as well as the inequality

\[
\left( \int_{[0,\infty)} (Kf)^q d\mu \right)^{\frac{1}{q}} \leq C \left( \int_{[0,\infty)} \left( \int_{[x,\infty)} f d\lambda \right)^p d\lambda \right)^{\frac{1}{p}} \tag{4}
\]

for all \( f \in \mathcal{M}_\downarrow \), \( 0 < p, q < \infty \), and \( q \geq 1 \).

In Subsection 3.2, basing on the results of [25], we obtain criteria for the fulfillment of

\[
\left( \int_{[0,\infty)} \left( \int_{[x,\infty)} f d\lambda \right)^q d\lambda \right)^{\frac{1}{q}} \leq C \left( \int_{[0,\infty)} f^p d\mu \right)^{\frac{1}{p}}, \quad f \in \mathcal{M}_\downarrow, \tag{5}
\]

and compare them with the criteria of [12].

The author was supported by the Russian Foundation for Basic Research (Grant 09–01–00093).
In Subsection 3.3 we consider the inequality for negative exponents \( p \) and \( q \). The weighted Hardy inequalities with negative exponents are characterized in [26]. We give criteria for the fulfillment of the inequalities with measures

\[
\left( \int_{[0,\infty)} \left( \int f \, d\lambda \right)^q \, d\lambda \right)^{\frac{1}{q}} \leq C \left( \int_{[0,\infty)} f^p \, d\mu \right)^{\frac{1}{p}}, \quad f \in \mathcal{M}^+,
\]

for both positive and negative values of \( p \) and \( q \). This result complements and generalizes the criteria of [24, 14].

The expression \( A \ll B \) stands for \( A \leq cB \) with some constant \( c \) depending only on the exponent. We write \( A \approx B \) instead of \( A \ll cB \ll A \) or \( A = cB \).

The exponent \( p' \) equals \( \frac{p}{p-1} \) for \( 0 < p < \infty \) and \( p \neq 1 \), while \( L^p(\lambda) \) means the collection of all \( \lambda \)-measurable functions \( f \) on \( \mathbb{R}_+ \) with

\[
\|f\|_{p,\lambda} := \left( \int_{[0,\infty)} |f|^p \, d\lambda \right)^{\frac{1}{p}}.
\]

2. Auxiliary Facts

In order to prove the main results, we need the following statements:

**Theorem 1** [14, Theorem 2.1]. For \( 0 < p \leq q < \infty \) we have

\[
\sup_{0 \leq F \leq F^+} \left( \int_{[0,\infty)} F^q \, d\omega \right)^{\frac{1}{q}} \approx \sup_{x \geq 0} \left( \int_{[0,x]} d\omega \right)^{\frac{1}{q}} \left( \int_{[0,\infty)} f^p \, d\nu \right)^{\frac{1}{p}}.
\]

**Theorem 2** [14, Theorem 2.2]. If \( 0 < q < p < \infty \) and \( \frac{1}{r} = \frac{1}{q} - \frac{1}{p} \) then

\[
\sup_{0 \leq F \leq F^+} \left( \int_{[0,\infty)} F^q \, d\omega \right)^{\frac{1}{q}} \approx \left( \int_{[0,\infty)} \left( \int_{[0,x]} \frac{d\omega(t)}{d\nu} \right)^\frac{r}{q} \, d\nu(x) \right)^{\frac{1}{r}}.
\]

**Theorem 3** [14, Theorem 2.3]. If \( 0 < q < p < \infty \) and \( \frac{1}{r} = \frac{1}{q} - \frac{1}{p} \) then

\[
\sup_{0 \leq F \leq F^+} \left( \int_{[0,\infty)} F^q \, d\omega \right)^{\frac{1}{q}} \approx \left( \int_{[0,\infty)} \left( \int_{[0,x]} \frac{1}{d\nu} \, d\omega + \frac{1}{d\nu} \int_{[0,\infty)} d\omega \right)^\frac{r}{q} \, d\nu(x) \right)^{\frac{1}{r}}
\]

provided that

\[
\left( \int_{[0,x]} d\nu \right)^{-1} - \left( \int_{[0,\infty)} d\nu \right)^{-1} \leq C \int_{[x,\infty)} \left( \int_{[0,\infty)} d\nu \right)^{-2} \, d\nu
\]

for some constant \( C \).

**Theorem 4** [25, 27, Theorem 2]. If \( 1 \leq \gamma < \infty \) then

\[
\frac{1}{(\gamma + 1)^{\frac{1}{\gamma}}} \leq \frac{1}{\left( \frac{\int_{[0,\infty)} f \, d\lambda}{\gamma + 1} \right)^{\frac{1}{\gamma}}} \leq \gamma
\]

(6)
for all \( f \in \mathcal{M} \downarrow \), \( f \not\equiv 0 \). If \( 0 < \gamma < 1 \) then
\[
0 < c(\gamma) \leq \frac{\left( \int_{[0,\infty)} \left( \int_{x,\infty} f \, d\lambda \right)^\gamma \, d\lambda(x) \right)^{\frac{1}{\gamma}}}{\left( \int_{[0,\infty)} f^\gamma \, d\lambda \right)^{\frac{1}{\gamma}}} \leq \frac{1}{\gamma}
\] (7)

for all \( f \in \mathcal{M} \uparrow \), \( f \not\equiv 0 \).

Theorem 5 [25; 27, Theorem 6]. If \( -1 < \gamma < 0 \) then
\[
(-\gamma)^{\frac{1}{\gamma}} \leq \frac{\left( \int_{[0,\infty)} \left( \int_{x,\infty} f \, d\lambda \right)^\gamma \, d\lambda(x) \right)^{\frac{1}{\gamma}}}{\left( \int_{[0,\infty)} f^\gamma \, d\lambda \right)^{\frac{1}{\gamma}}} \leq 1
\] (8)

\( \lambda \)-almost everywhere for all \( f \in \mathcal{M} \uparrow \), \( f(x) > 0 \).

Theorem 6 [25; 27, Theorem 7]. If \( \gamma \leq -1 \) then the two-sided inequality
\[
0 < c_\gamma \leq \frac{\left( \int_{[0,\infty)} \left( \int_{x,\infty} f \, d\lambda \right)^\gamma \, d\lambda(x) \right)^{\frac{1}{\gamma}}}{\left( \int_{[0,\infty)} f^\gamma \, d\lambda \right)^{\frac{1}{\gamma}}} \leq 1
\] (9)

holds \( \lambda \)-almost everywhere for all \( f \in \mathcal{M} \uparrow \), \( f(x) > 0 \).

3. The Main Results

3.1. On the cone of nonincreasing functions, consider (1) with \( 0 < p < \infty \) and \( 1 \leq q < \infty \) for the operator (2) whose kernel satisfies the Oinarov condition (3) with the constant \( C > 0 \) chosen as small as possible.

Define the functions
\[
\bar{k}(x) := \int_{[0,x]} k(x, y) \, d\nu(y), \quad \check{k}(x) := \int_{[x,\infty)} k(y, x) \, d\mu(y), \quad \nu(x) := \int_{[0,x]} d\nu(y).
\]

Theorem 7. Suppose that \( 0 < p < \infty \), \( 1 \leq q < \infty \), and
\[
\int_{[0,\infty)} d\lambda = \infty, \quad \left( \int_{[0,x]} d\lambda \right)^{-1} \leq c \int_{[x,\infty)} \left( \int_{[0,t]} d\lambda \right)^{-2} d\lambda(t)
\]
for \( p > 1 \). Then (1) for all \( f \in \mathcal{M} \downarrow \) if and only if
\[
A_0 < \infty, \quad p \leq q = 1, \quad B_0 < \infty, \quad 1 = q < p,
\]
\[
A_1 + A_{2.1} + A_{2.1}^2 + A_{2.2} < \infty, \quad 1 < p \leq q,
\]
\[
B_1 + B_{2.1} + B_{2.1}^2 + B_{2.2} < \infty, \quad 1 < q < p,
\]
\[
D_1 + D_2 + D_3 < \infty, \quad 0 < p \leq 1 < q,
\]
where
\[
A_0 := \sup_{x \geq 0} \left( \int_{[0,x]} \check{k} \, d\nu \right) \left( \int_{[0,x]} d\lambda \right)^{-\frac{1}{p}}, \quad B_0 := \left( \int_{[0,\infty)} \left( \int_{[x,\infty)} \check{k}(t) \, d\nu(t) \right)^{p'} d\lambda(x) \right)^{\frac{1}{p}},
\]

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For the smallest constant $C$ we obtain

$$C = \sup_{0 \leq f \downarrow} \left( \frac{\int \left( \int k(x,y) f(y) d\nu(y) \right) d\mu(x)}{\int f^p d\lambda} \right)^{\frac{1}{q'}} = \sup_{0 \leq f \downarrow} \left( \frac{\int \hat{k}(y) f(y) d\nu(y)}{\int f^p d\lambda} \right)^{\frac{1}{q'}}.$$
By Theorems 1 and 2,

$$C = \sup_{x \geq 0} \frac{\int_0^x \hat{k} \, d\nu}{\int_0^x d\lambda}$$

for \( p \leq 1 \) and

$$C \approx \left( \int_{[0,\infty)} \left( \int_{[x,\infty)} \left( \frac{\hat{k}(t) \, d\nu(t)}{\int_0^t d\lambda} \right)^p \, d\lambda(x) \right)^{\frac{1}{p}} \right)^{\frac{1}{p'}}$$

for \( p > 1 \).

In order to calculate \( C \) in the case \( q > 1 \), we use the duality principle in the \( L^q \) space with the measure \( \mu \):

$$C = \sup_{0 \leq f \in L^q} \left( \frac{\int_{[0,\infty)} (Kf)^p \, d\mu}{\int_{[0,\infty)} f^p \, d\lambda} \right)^{\frac{1}{p}} = \sup_{0 \leq f \in L^q} \frac{\int_{[0,\infty)} (Kf)^p \, d\mu}{\int_{[0,\infty)} f^p \, d\lambda} = \sup_{g \geq 0} \frac{1}{\|g\|_{q,\mu}} \int_0^1 fK^*(g) \, d\nu.$$

Applying Theorems 1 and 3 to this expression, we deduce that

$$C \approx \sup_{g \geq 0} \frac{1}{\|g\|_{q,\mu}} \left( \int_{[0,\infty)} \left( \frac{1}{\Lambda(x)} \int_{[0,\infty)} K^*(g) \, d\nu \right)^{\frac{1}{p'}} \, d\lambda(x) \right)^{\frac{1}{p}} \quad (10)$$

for \( p > 1 \) and

$$C = \sup_{g \geq 0} \frac{1}{\|g\|_{q,\mu}} \sup_{x \geq 0} \frac{\int_{[0,x]} K^*(g) \, d\nu}{\int_{[0,x]} d\lambda} \quad (11)$$

for \( 0 < p \leq 1 \).

Since the adjoint operator \( K^*(g) \) is

$$K^*(g)(y) = \int g(x,y) \, d\mu(x),$$

we obtain

$$\int_{[0,x]} K^*(g) \, d\nu = \int_{[0,x]} \left( \int_{[x,\infty)} \hat{k}(z)g(z) \, d\mu(z) \right) \, d\nu(y)$$

$$\approx \int_{[0,x]} \left( \int_{[x,\infty)} \hat{k}(z,y)g(z) \, d\mu(z) + \int_{[x,\infty)} \hat{k}(z,y)g(z) \, d\mu(z) \right) \, d\nu(y) =: I_1 + I_2,$$

$$I_1 = \int_{[0,x]} \left( \int_{[x,\infty)} \hat{k}(z,y)g(z) \, d\mu(z) \right) \, d\nu(y) = \int_{[0,x]} g(z) \, d\mu(z) \left( \int_{[x,\infty)} \hat{k}(z,y) \, d\nu(y) \right),$$

$$I_2 = \int_{[0,x]} \left( \int_{[x,\infty)} \hat{k}(z,y)g(z) \, d\mu(z) \right) \, d\nu(y)$$

$$\approx \int_{[0,x]} \left( \int_{[x,\infty)} \hat{k}(z,x)g(z) \, d\mu(z) + \int_{[x,\infty)} \hat{k}(z,x)g(z) \, d\mu(z) \right) \, d\nu(y)$$

$$= \left( \int_{[0,x]} \, d\nu(y) \right) \int_{[x,\infty)} \hat{k}(z,x)g(z) \, d\mu(z) + \int_{[0,x]} \hat{k}(z,x) \, d\nu(y) \int_{[x,\infty)} g(z) \, d\mu(z).$$
Thus,
\[
\int_{[0,x]} K^*(g) \, dv = I_1 + I_{2,1} + I_{2,2},
\]
where
\[
I_1 = \int_{[0,x]} k g \, d\mu, \quad I_{2,1} := \nu(x) K^* g(x), \quad I_{2,2} := \bar{k}(x) \int_{[x,\infty)} g \, d\mu.
\]

Consider the case \( p > 1 \). Inserting (12) into (10), we reduce the problem to the three inequalities:
\[
\left( \int_{[0,\infty)} A(x)^{-p'} \left( \int_{[0,x]} k g \, d\mu \right)^{p'} \, d\mu(x) \right)^{\frac{1}{p'}} \leq C_1 \|g\|_{q', \mu},
\]
\[
\left( \int_{[0,\infty)} A(y)^{-p'} \left( \int_{[0,\infty)} k(x, y) g(x) \, d\mu(x) \right)^{p'} \, d\mu(y) \right)^{\frac{1}{p'}} \leq C_{2,1} \|g\|_{q', \mu},
\]
and
\[
\left( \int_{[0,\infty)} A(x)^{-p'} \left( \int_{[x,\infty)} g \, d\mu \right)^{p'} \, d\mu(x) \right)^{\frac{1}{p'}} \leq C_{2,2} \|g\|_{q', \mu}.
\]

We have the following criteria for [23, 24]:
\[
C_1 \approx A_1, \quad 1 < p \leq q, \quad C_1 \approx B_1, \quad 1 < q < p,
\]
\[
C_{2,1} \approx A_{2,1} + A_{1,2} \approx B_{2,1} + B_{1,2}, \quad 1 < p \leq q, \quad C_{2,2} \approx A_{2,2}, \quad 1 < p \leq q, \quad C_{2,2} \approx B_{2,2}, \quad 1 < q < p.
\]

In the case \( 0 < p \leq 1 \) we obtain
\[
C = \sup_{q \geq 0} \frac{1}{\|g\|_{q', \mu}} \sup_{x \geq 0} \left( \int_{[0,x]} K^*(g) \, dv \right) =: J_1 + J_{2,1} + J_{2,2},
\]
where
\[
J_1 = \sup_{x \geq 0} \sup_{g \geq 0} \left( \int_{[0,x]} k g \, d\mu \right) \|g\|_{q', \mu}, \quad J_{2,1} = \sup_{x \geq 0} \sup_{g \geq 0} \left( \int_{[x,\infty)} g \, d\mu \right) \|g\|_{q', \mu},
\]
and
\[
J_{2,2} = \sup_{x \geq 0} \sup_{g \geq 0} \left( \int_{[x,\infty)} g \, d\mu \right) \|g\|_{q', \mu}.
\]

Consider the constant \( J_1 \). It is the norm of the operator
\[
g(z) \rightarrow A^{-\frac{1}{p'}}(z) \int_{[0,x]} k(z) g(z) \, d\mu(z) = \int_{[0,x]} K(x, z) g(z) \, d\mu(z) =: \overline{K}(x, z) g(z).
\]
from $L^q(\mu)$ into $L^\infty$:
\[
\|\tilde{K} g\|_\infty \leq J_1 \|g\|_{q', \mu}.
\]
The duality principle in $L^q$ implies that
\[
J_1 = \sup_{x \geq 0} \left( \int_{[0, \infty)} \tilde{K}(x, z)^q d\mu(z) \right)^{\frac{1}{q}} = \sup_{x \geq 0} \left( \int_{[0, x]} \tilde{k}(z)^q d\mu(z) \right)^{\frac{1}{q}} \Lambda(x)^{-\frac{1}{p}}.
\]
Similarly,
\[
J_{2,1} = \sup_{x \geq 0} \tilde{\nu}(x) \Lambda(x)^{-\frac{1}{p}} \left( \int_{[x, \infty)} k(y, x)^q d\mu(y) \right)^{\frac{1}{q}}
\]
and
\[
J_{2,2} = \sup_{x \geq 0} \tilde{k}(x) \Lambda(x)^{-\frac{1}{p}} \left( \int_{[x, \infty)} d\mu \right)^{\frac{1}{q}}.
\]
Therefore, for $0 < p \leq 1$ we obtain $C \approx D_1 + D_2 + D_3$.

On the cone of nonincreasing functions, we similarly consider (4) with $0 < p < \infty$ and $1 \leq q < \infty$ for the operator (2) whose kernel satisfies (3).

Theorem 8. Suppose that $0 < p < \infty$, $1 \leq q < \infty$, and
\[
\int \Lambda^p d\lambda = \infty, \quad \left( \int \Lambda^p d\lambda \right)^{-1} \leq c \int \left( \int \Lambda^p d\lambda \right)^{-2} \Lambda(t)^p d\lambda(t)
\]
for $p > 1$. Then (4) holds for all $f \in \mathfrak{M}_1$ if and only if
\[
A_0 < \infty, \quad p \leq q = 1, \quad B_0 < \infty, \quad 1 = q < p,
\]
\[
A_1 + A_{2.1} + A_{2.2} < \infty, \quad 1 < p \leq q,
\]
\[
B_1 + B_{2.1} + B_{2.2} < \infty, \quad 1 < q < p,
\]
\[
D_1 + D_2 + D_3 < \infty, \quad 0 < p \leq 1 < q,
\]
where
\[
A_0 := \sup_{x \geq 0} \left( \int_{[0, x]} \tilde{\nu}(x) \right)^{\frac{1}{q}} \left( \int_{[0, x]} \Lambda^p d\lambda \right)^{-\frac{1}{p}},
\]
\[
B_0 := \int_{[0, \infty)} \left( \int_{[x, \infty)} \frac{\tilde{k}(t) d\nu(t)}{\Lambda^p d\lambda} \right)^{\frac{1}{p}} \Lambda(x)^p d\lambda(x)^{\frac{1}{p}},
\]
\[
A_1 := \sup_{t \geq 0} \left( \int_{[0, \infty)} \Lambda^{-2q'} d\lambda \right)^{\frac{1}{p}} \left( \int_{[0, \infty)} \tilde{\nu} d\mu \right)^{\frac{1}{q}},
\]
\[
A_{2.1} := \sup_{t \geq 0} \left( \int_{[0, \infty)} \Lambda(x)^{-2q'} \tilde{\nu}(x)^{q'} k(t, x)^{q'} d\lambda(x) \right)^{\frac{1}{p}} \left( \int_{[0, \infty)} d\mu \right)^{\frac{1}{q}},
\]
and
Suppose that $i = C$ holds on the cone of nonincreasing functions. Assume that the constant $C$ is the smallest possible; thus, it equals $J_{pq}$. Theorem 4 yields

$$J_{pq} = \sup_{0 \leq f \leq 1} \left( \int_{[0, \infty)} f^p \, d\mu \right)^{\frac{1}{p}} \approx \sup_{0 \leq f \leq 1} \left( \int_{[0, \infty)} f^p \, d\lambda \right)^{\frac{1}{p}}.$$
Therefore,

\[
J_{pq} = \sup_{x \geq 0} \left( \int \frac{d\mu}{[0,x]} \right)^{\frac{1}{\tau}} \int \Lambda^q d\lambda(x)^{\frac{q-1}{\tau}}, \quad 0 < q < p < \infty,
\]

for

\[
J_{pq} \approx \left( \int_{[0,\infty)} \left( \int_{[0,x]} d\mu \right)^{\frac{1}{\tau}} \left( \int_{[0,x]} \Lambda^q d\lambda(x) \right)^{\frac{q-1}{\tau}} \Lambda(x)^q d\lambda(x) \right)^{\frac{q}{\tau}}, \quad 0 < p < q < \infty.
\]

In particular, for \( p = 1 \) we have

\[
J_{1q} \approx \left( \int_{[0,\infty)} \left( \int_{[0,x]} d\mu \right)^{\frac{1}{\tau}} \left( \int_{[0,x]} \Lambda^q d\lambda(x) \right)^{\frac{q-1}{\tau}} \Lambda(x)^q d\lambda(x) \right)^{\frac{q}{\tau}}, \quad 0 < q < \infty
\]

and so

\[
J_{1q} = \sup_{x \geq 0} \left( \int \frac{d\mu}{[0,x]} \right)^{\frac{1}{\tau}} \int \Lambda^q d\lambda(x)^{\frac{q-1}{\tau}},
\]

for \( 0 < q \leq 1 \).

For \( q = 1 \) we infer from (17) and (18) that

\[
C = \sup_{x \geq 0} \left( \int \tilde{k} d\nu \right) \left( \int \Lambda^p d\lambda(x) \right)^{\frac{1}{\tau}}, \quad 0 < p \leq 1,
\]

and

\[
C \approx \left( \int_{[0,\infty)} \left( \int \tilde{k}(t) d\nu(t) \right)^{\frac{p}{q}} \Lambda(x)^p d\lambda(x) \right)^{\frac{q}{p}} \Lambda(x)^p d\lambda(x), \quad p > 1.
\]

In the case \( q > 1 \) we use the duality principle in \( L^q(\mu) \) and then

\[
C = \sup_{0 \leq f \leq 1} \left( \int \left( f^q \right)^p \mu d\lambda(x) \right)^{\frac{1}{p}} = \sup_{0 \leq f \leq 1} \sup_{g \geq 0} \left[ \left. \frac{1}{p} \left( \int f d\mu \right)^p \right| \left( \int f d\lambda(x) \right)^{\frac{1}{p}} \right],
\]

\[
= \sup_{g \geq 0} \sup_{0 \leq f \leq 1} \left( \int \left( f^q \right)^p \mu d\lambda(x) \right)^{\frac{1}{p}} \left( \int f d\nu \right)^{\frac{q}{p}}.
\]
Inserting (16) and (18) into (19), in the case $p > 1$ we find
\[
C \approx \sup_{g \geq 0} \frac{1}{\|g\|_{q',\mu}^p} \left( \int_{[0,\infty)} \int_{[0,x]} \Lambda^p d\lambda \right)^{-\frac{1}{p'}} \left( \int_{[0,x]} K^*(g) d\nu \right)^{\frac{1}{p}} \Lambda(x)^p d\lambda(x) \right)^{\frac{1}{p}}.
\]
For $0 < p \leq 1$ we have
\[
C = \sup_{g \geq 0} \frac{1}{\|g\|_{q',\mu}^p} \sup_{x \geq 0} \left( \int_{[0,x]} \Lambda^p d\lambda \right)^{\frac{1}{p}} \int_{[0,x]} K^*(g) d\nu.
\]
The dual operator $K^*(g)$ is
\[
\int_{[0,x]} K^*(g) d\nu = I_1 + I_{2,1} + I_{2,2},
\]
where
\[
I_1 := \int_{[0,x]} k g d\mu, \quad I_{2,1} := \check{\nu}(x) K^*(g(x)), \quad I_{2,2} := \check{k}(x) \int_{[x,\infty)} g d\mu.
\]
Consider the case $p > 1$. In the same fashion as in the previous theorem, the problem reduces to the three inequalities:
\[
\left( \int_{[0,\infty)} \Lambda(x)^{-2p'} \left( \int_{[0,x]} k g d\mu \right)^{p'} d\lambda(x) \right)^{\frac{1}{p'}} \leq C_1 \|g\|_{q',\mu},
\]
\[
\left( \int_{[0,\infty)} \Lambda(y)^{-2p'} \check{\nu}(y)^{p'} \left( \int_{[0,\infty)} k(x,y) g(x) d\mu(x) \right)^{p'} d\lambda(y) \right)^{\frac{1}{p'}} \leq C_2 \|g\|_{q',\mu},
\]
\[
\left( \int_{[0,\infty)} \Lambda(x)^{-2p'} \check{k}^{p'}(x) \left( \int_{[x,\infty)} g d\mu \right)^{p'} d\lambda(x) \right)^{\frac{1}{p'}} \leq C_2 \|g\|_{q',\mu}.
\]
We obtain the criteria
\[
C_1 + C_{2,1} + C_{2,2} \approx A_1 + A_{2,1} + A_{2,2}, \quad 1 < p \leq q,
\]
\[
C_1 + C_{2,1} + C_{2,2} \approx B_1 + B_{2,1} + B_{2,2}, \quad 1 < q < p.
\]
In the case $0 < p \leq 1 < q$ we obtain
\[
C = \sup_{g \geq 0} \frac{1}{\|g\|_{q',\mu}^p} \sup_{x \geq 0} \left( \int_{[0,x]} \Lambda^p d\lambda \right)^{\frac{1}{p}} = J_1 + J_{2,1} + J_{2,2},
\]
where
\[
J_1 := \sup_{g \geq 0} \sup_{x \geq 0} \frac{\int_{[0,x]} k g d\mu}{\left( \int_{[0,x]} \Lambda^p d\lambda \right)^{\frac{1}{p}}}, \quad J_{2,1} := \sup_{g \geq 0} \sup_{x \geq 0} \frac{\check{\nu}(x) K^*(g(x))}{\left( \int_{[0,x]} \Lambda^p d\lambda \right)^{\frac{1}{p}}}, \quad J_{2,2} := \sup_{g \geq 0} \sup_{x \geq 0} \frac{\check{k}(x) \int_{[x,\infty)} g d\mu}{\left( \int_{[0,x]} \Lambda^p d\lambda \right)^{\frac{1}{p}}}.\]
and
\[ J_{2,2} := \sup_{g \geq 0} \sup_{x \geq 0} \frac{\bar{k}(x) \int_{[x,\infty)} g \, d\mu}{\int_{[0,x]} |L^p \, d\lambda|^{\frac{1}{q}} \|g\|_{q',\mu}}. \]

By analogy with the previous result,
\[ J_1 = \|\tilde{K}\| = \sup_{x \geq 0} \left( \int_{[0,\infty)} \tilde{K}(x,z)^q \, d\mu(z) \right)^{\frac{1}{q}} = \sup_{x \geq 0} \left( \int_{[0,x]} \tilde{k}(z) \, d\mu(z) \right)^{\frac{1}{q}} \Lambda(x)^{-\frac{1}{q}}, \]

\[ J_{2,1} = \sup_{x \geq 0} \tilde{\nu}(x) \Lambda(x)^{-\frac{1}{q}} \left( \int_{[x,\infty)} k(y,x)^q \, d\mu(y) \right)^{\frac{1}{q}}, \]

and
\[ J_{2,2} = \sup_{x \geq 0} \tilde{k}(x) \Lambda(x)^{-\frac{1}{p}} \left( \int_{[x,\infty)} d\mu \right)^{\frac{1}{q}}. \]

Therefore, for \(0 < p \leq 1\) we obtain the criterion \(C \approx D_1 + D_2 + D_3\).

3.2. Suppose that \(0 < p, q < \infty\). Consider the inequality
\[ \left( \int_{[0,\infty)} \left( \int_{[x,\infty)} f \, d\lambda \right)^q \, d\lambda(x) \right)^{\frac{1}{q}} \leq C \left( \int_{[0,\infty)} f^p \, d\mu \right)^{\frac{1}{p}}, \quad f \in \mathcal{M}_+\quad f \neq 0, \tag{20} \]
where the constant \(C\) is assumed to be as small as possible; thus, it equals \(J_{pq}\), where
\[ J_{pq} := \sup_{0 \leq f \leq 1} \left( \int_{[0,\infty)} \left( \int_{[x,\infty)} f \, d\lambda \right)^q \, d\lambda(x) \right)^{\frac{1}{q}} \left( \int_{[0,\infty)} f^p \, d\mu \right)^{\frac{1}{p}}. \]

Theorem 4 yields
\[ J_{pq} = \sup_{0 \leq f \leq 1} \left( \int_{[0,\infty)} \left( \int_{[x,\infty)} f \, d\lambda \right)^q \, d\lambda(x) \right)^{\frac{1}{q}} \left( \int_{[0,\infty)} f^p \, d\mu \right)^{\frac{1}{p}} \approx \sup_{0 \leq f \leq 1} \left( \int_{[0,\infty)} f^q \Lambda^q \, d\lambda \right)^{\frac{1}{q}} \left( \int_{[0,\infty)} f^p \, d\mu \right)^{\frac{1}{p}}. \tag{21} \]

In order to characterize the right-hand side, we use Theorems 1 and 3. Then \(J_{pq} \approx S_{pq}\), where
\[ S_{pq} = \sup_{x \geq 0} \left( \int_{[0,x]} \Lambda^q \, d\lambda \right)^{\frac{1}{q}} \left( \int_{[0,x]} d\mu \right)^{\frac{1}{p}}, \quad 0 < p \leq q < \infty, \]
and
\[ S_{pq} \approx \left( \int_{[0,\infty)} \left( \int_{[x,\infty)} \Lambda^q \, d\lambda \right)^{\frac{1}{q}} \left( \int_{[0,x]} d\mu \right)^{\frac{1}{p}} \, d\mu(x) \right)^{\frac{1}{q}}, \quad 0 < q < p < \infty, \]
provided that
\[ \int_{[0, \infty)} d\mu = \infty, \quad \left( \int_{[0, x]} d\mu \right)^{-1} \leq c \int_{[0, t]} \left( \int_{[0, t]} d\mu \right)^{-2} d\mu(t). \]

The value
\[ H_{pq} := H_{\text{pt}}(B; p, q, \gamma, \mu, \lambda) := \sup_{0 \leq f \leq 1} \left[ \left( \int_{0}^{\infty} (B f)^{q} d\gamma \right)^{\frac{1}{q}} \left( \int_{0}^{\infty} f^{p} d\mu \right)^{-\frac{1}{p}} \right], \]

where
\[ (B f)(t) = \left( \int_{t}^{\infty} f d\lambda \right), \quad t \in \mathbb{R}^+, \]

is considered in [12] and characterized in Theorem 1.1 of [12] under the assumption that the integrals \( (\int_{0}^{t} d\mu) \) and \( (\int_{0}^{t} d\gamma) \) are continuous and the measure \( \lambda \) is nondegenerate,
\[ \lambda \in N_p \Leftrightarrow \left( \int_{0}^{1} d\lambda \right) = 1, \quad \left( \int_{1}^{\infty} d\lambda \right) = \infty, \]
as \( H_{pq} \approx E_{pq} + F_{pq} \); furthermore, the functionals \( E_{pq} \) and \( F_{pq} \), are independent in general. Apart from the above conditions, the finiteness of \( H_{pq} \) follows from the alternative pairs of conditions [28].

Together with that, in many problems with additional restrictions on the measure it is important to have a characterization of \( H_{pq} \) by the finiteness of just one constant \( E_{pq} \) or \( F_{pq} \). We show then that for \( d\lambda = d\gamma \) the finiteness of \( H_{pq} \) is equivalent to the finiteness of \( E_{pq} \). Furthermore, observe that condition (1.21) of [12] can be violated even if we impose the additional restriction (23) on \( \mu \).

It is obvious that for \( d\lambda = d\gamma \) the finiteness of \( H_{pq} \) is equivalent to the fulfillment of (20). Applying Theorem 1.1 of [12] to this particular case, we find that \( J_{pq} \approx E_{pq} + F_{pq} \), where
\[ E_{pq} = \sup_{t \in [0, \infty)} \left[ \left( \int_{[0, t]} d\lambda \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} \left( \int_{[0, t]} d\mu \right)^{-\frac{1}{p}} \]
for \( 0 < p \leq q < \infty \),

\[ E_{pq}^{r} = \int_{[0, \infty)} \left( \int_{[0, x]} d\lambda \right)^{q} \left( \int_{[0, x]} d\mu \right)^{-2} \left( \int_{[0, x]} d\lambda \right)^{r}\left( \int_{[0, x]} d\mu \right)^{-\frac{r}{p}} \]
for \( 0 < q < p < \infty \).

Let us compare the criteria obtained for (20) using Theorems 1, 3, and 4. Lemma 1 of [23] yields
\[ \left( \int_{[0, t]} d\lambda \right)^{\frac{1}{q}} \left( \int_{[0, t]} d\mu \right)^{-\frac{1}{p}} \approx \left( \int_{[0, t]} \Lambda(x) d\lambda(x) \right)^{\frac{1}{p}}. \]

Hence, for \( 0 < p \leq q < \infty \) we have
\[ \left( \int_{[0, t]} d\lambda \right)^{\frac{1}{q}} \left( \int_{[0, t]} d\mu \right)^{-\frac{1}{p}} \approx \left( \int_{[0, t]} \Lambda(x) d\lambda(x) \right)^{\frac{1}{p}} \left( \int_{[0, t]} d\mu \right)^{-\frac{1}{p}}. \]
In this case, using (22) again, we obtain

\[ \int \left( \int_{[0, t]} \lambda^\lambda \mu(x) \right)^{-1} \mu(x) \approx \left( \int_{[0, t]} \mu \right)^{-1}, \quad \lambda > 1. \] (23)

In this case, using (22) again, we obtain

\[ E_{pq} = \int_{[0, \infty]} \left( \int_{[0, t]} \Lambda^\lambda \lambda \Lambda^\lambda(t) \right) \left( \int_{[0, t]} \mu \right) \approx \left( \int_{[0, \infty]} \mu \right) \approx \left( \int_{[0, \infty]} \Lambda^\lambda \lambda \Lambda^\lambda(t) \right) \left( \int_{[0, t]} \mu \right), \]

therefore, \( S_{pq} \approx E_{pq} \) and \( E_{pq} \gg F_{pq} \).

3.3. For \( p, q > 0 \), consider the integral inequality

\[ \left( \int_{[0, \infty]} \left( \int_{[0, t]} f \lambda \Lambda^\lambda(x) \right)^q \right)^{\frac{1}{q}} \leq C \left( \int_{[0, \infty]} f \lambda \right)^{\frac{1}{q}}, \quad f \in \mathcal{M}^+, f \neq 0, \] (24)

on the cone of nondecreasing functions. Use a corollary to Theorem 4:

\[ \frac{1}{(\gamma + 1)^{\frac{1}{q}}} \leq \left( \int_{[0, \infty]} f \lambda \Lambda^\lambda \right)^{\frac{1}{q}} \leq \gamma, \quad f^{\uparrow}, \quad p, q > 0, \]

where \( \Lambda^\lambda(t) = \int_{[0, t]} \lambda \). Theorems 1 and 3 once again yield \( C = J_{pq} \), where

\[ J_{pq} = \sup_{0 \leq f^{\uparrow} \in \mathcal{M}^+} \left( \int_{[0, \infty]} f \lambda \Lambda^\lambda \right)^{\frac{1}{q}} \left( \int_{[0, \infty]} f \lambda \right)^{\frac{1}{q}} \approx \sup_{0 \leq f^{\uparrow} \in \mathcal{M}^+} \left( \int_{[0, \infty]} f \lambda \Lambda^\lambda \right)^{\frac{1}{q}} \left( \int_{[0, \infty]} f \lambda \right)^{\frac{1}{q}}. \]

Hence,

\[ J_{pq} = \sup_{x \geq 0} \left( \int_{[x, \infty]} \lambda \right)^{\frac{1}{q}}, \quad p \leq q, \]

and

\[ J_{pq} = \left( \int_{[0, \infty]} \left( \int_{[0, \infty]} \lambda \right)^{\frac{1}{q}} \right)^{\frac{1}{q}}, \quad q < p, \]

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provided that
\[ \int_{[0,\infty)} \, d\mu = \infty, \quad \left( \int_{[0,\infty)} \, d\mu \right)^{-1} \leq c \int_{[x,\infty)} \, d\mu(t) \left( \int_{[0,\infty)} \, d\mu \right)^{-2} d\mu(t). \]

Similarly using corollaries to Theorems 5 and 6, we characterize (24) in the case \( p, q < 0 \). We have
\[
\left( \int_{[0,\infty)} \left( \int_{[0,x]} \, d\lambda(x) \right)^{pq} d\lambda(x) \right)^{\frac{1}{pq}} \leq C\left( \int_{[0,\infty)} f^p \, d\mu \right)^{\frac{1}{p}} \left( \int_{[0,\infty)} f^q \Lambda^q \, d\lambda \right)^{\frac{1}{q}}, \quad f \in M^\dagger, \ f \neq 0,
\]
where \( J_{pq} = \sup_{a \leq f \downarrow} \left( \int_{[0,\infty)} \left( \int_{[0,a]} \, d\mu \right)^{\frac{1}{pq}} \left( \int_{[a,\infty)} f^q \Lambda^q \, d\lambda \right)^{\frac{1}{q}} \right) \approx \sup_{a \leq f \downarrow} \left( \int_{[0,\infty)} f^p \, d\mu \right)^{\frac{1}{p}} \left( \int_{[0,\infty)} f^q \Lambda^q \, d\lambda \right)^{\frac{1}{q}} \]

Put \( \tilde{f}(x) := f(x)^{-1} \). Then
\[
J_{pq} = \sup_{a \leq f \downarrow} \left( \int_{[0,\infty)} \left( \int_{[0,a]} \, d\mu \right)^{\frac{1}{pq}} \left( \int_{[a,\infty)} \tilde{f}^{-q} \Lambda^q \, d\lambda \right)^{\frac{1}{q}} \right) = \sup_{a \leq f \downarrow} \left( \int_{[0,\infty)} \left( \int_{[0,a]} \, d\mu \right)^{\frac{1}{pq}} \left( \int_{[a,\infty)} \tilde{f}^{-q} \Lambda^q \, d\lambda \right)^{\frac{1}{q}} \right),
\]
where
\[
J_{pq} \approx \sup_{x \geq 0} \left( \int_{[0,\infty)} \left( \int_{[0,x]} \, d\mu \right)^{\frac{1}{pq}} \left( \int_{[x,\infty)} \Lambda^q \, d\lambda \right)^{\frac{1}{q}} \Lambda(x)^q \, d\lambda(x) \right)^{-\frac{1}{q}}, \quad -\infty < p < 0,
\]
for
\[
\int_{[0,\infty]} \Lambda^q \, d\lambda = \infty, \quad \left( \int_{[0,\infty)} \Lambda^q \, d\lambda \right)^{-1} \leq c \int_{[x,\infty]} \left( \int_{[0,\infty)} \Lambda^q \, d\lambda \right)^{-2} \Lambda(t)^q \, d\lambda(t).
\]
Consider a similar inequality on the cone of nonincreasing functions. For the inequality
\[
\left( \int_{[0,\infty)} \left( \int_{[x,\infty)} f \, d\lambda \right)^q \, d\lambda(x) \right)^{\frac{1}{q}} \leq C\left( \int_{[0,\infty)} f^p \, d\mu \right)^{\frac{1}{p}}, \quad f \downarrow, \, p, q < 0,
\]
we obtain \( C = J_{pq} \), where
\[
J_{pq} = \sup_{a \leq f \downarrow} \left( \int_{[0,\infty)} \left( \int_{[0,a]} \, d\mu \right)^{\frac{1}{pq}} \left( \int_{[a,\infty)} f^q \Lambda^q \, d\lambda \right)^{\frac{1}{q}} \right) \approx \sup_{a \leq f \downarrow} \left( \int_{[0,\infty)} \left( \int_{[0,a]} \, d\mu \right)^{\frac{1}{pq}} \left( \int_{[a,\infty)} f^q \Lambda^q \, d\lambda \right)^{\frac{1}{q}} \right) = \sup_{a \leq f \downarrow} \left( \int_{[0,\infty)} \left( \int_{[0,a]} \, d\mu \right)^{\frac{1}{pq}} \left( \int_{[a,\infty)} \tilde{f}^{-q} \Lambda^q \, d\lambda \right)^{\frac{1}{q}} \right).
\]
Therefore,

\[ J_{pq} \approx \sup_{x \geq 0} \left( \frac{\int_{[x, \infty]} \Lambda_q^2 \, d\lambda}{\int_{[x, \infty]} d\mu} \right)^{\frac{1}{2}} \]

for \(-\infty < p \leq q < 0\) and

\[ J_{pq} \approx \left( \frac{\int_{[0, \infty]} (\int_{[x, \infty]} \Lambda_q \, d\mu)^{\frac{q}{p}} (\int_{[x, \infty]} \Lambda_q \, d\lambda)^{-\frac{q}{p}} \Lambda_*(x)^q \, d\lambda(x)}{\int_{[0, \infty]} (\int_{[x, \infty]} \Lambda_q \, d\mu)^{\frac{q}{p}} (\int_{[x, \infty]} \Lambda_q \, d\lambda)^{-\frac{q}{p}} \Lambda_*(t)^q \, d\lambda(t)} \right)^{\frac{1}{q}} \]

for \(-\infty < q < p < 0\) provided that

\[ \int_{[0, \infty]} \Lambda_p^2 \, d\lambda = \infty, \quad \left( \int_{[0, x]} \Lambda_p^2 \, d\lambda \right)^{-1} \leq c \int_{[x, \infty]} \left( \int_{[0, t]} \Lambda_p \, d\lambda \right)^{-2} \Lambda_*(t)^q \, d\lambda(t). \]

Finally, using the argument above, we characterize the inequality

\[ \left( \int_{[0, \infty]} (\int_{[x, \infty]} f \, d\mu) \, d\mu(x) \right)^{\frac{1}{q}} \leq C \left( \int_{[0, \infty]} (\int_{[x, \infty]} f \, d\lambda) \, d\lambda(x) \right)^{\frac{1}{p}} \]

for \(p, q > 0\), where \(f \in \mathcal{M}_*, f \not\equiv 0\). Theorem 4 yields

\[ C = \sup_{0 \leq f \in \mathcal{M}_*, f \not\equiv 0} \left( \frac{\int_{[0, \infty]} (\int_{[x, \infty]} f \, d\mu) \, d\mu(x)}{\int_{[0, \infty]} (\int_{[x, \infty]} f \, d\lambda) \, d\lambda(x)} \right)^{\frac{1}{q}} \approx \sup_{0 \leq f \in \mathcal{M}_*, f \not\equiv 0} \left( \frac{\int_{[0, \infty]} f^q \, d\mu}{\int_{[0, \infty]} f^p \, d\lambda} \right)^{\frac{1}{q}}. \]

Hence,

\[ J_{pq} \approx \sup_{0 \leq f \in \mathcal{M}_*, f \not\equiv 0} \left( \frac{\int_{[0, \infty]} f^q \Lambda^q \, d\mu}{\int_{[0, \infty]} f^p \Lambda^p \, d\lambda} \right)^{\frac{1}{q}} \]

Therefore,

\[ J_{pq} = \sup_{x \geq 0} \left( \frac{\int_{[x, \infty]} \Lambda^q \, d\mu}{\int_{[x, \infty]} \Lambda^p \, d\lambda} \right)^{\frac{1}{q}}, \quad 0 < p \leq q < \infty, \]

and

\[ J_{pq} \approx \left( \int_{[0, \infty]} \left( \int_{[x, \infty]} \Lambda^q \, d\mu \right)^{\frac{q}{p}} \left( \int_{[x, \infty]} \Lambda^p \, d\lambda \right)^{-\frac{q}{p}} \Lambda(x)^q \, d\lambda(x) \right)^{\frac{1}{q}}, \quad 0 < q < p < \infty, \]

provided that

\[ \int_{[0, \infty]} \Lambda^p \, d\lambda = \infty, \quad \left( \int_{[0, x]} \Lambda^p \, d\lambda \right)^{-1} \leq c \int_{[x, \infty]} \left( \int_{[0, t]} \Lambda^p \, d\lambda \right)^{-2} \Lambda_*(t)^p \, d\lambda(t). \]

**Remark.** Some inequalities similar to those considered above hold for nondecreasing functions, as well as for \(p\) and \(q\) negative.

The author is grateful to the referee for valuable remarks.

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References

Paper C
WEIGHTED HARDY-TYPE INEQUALITIES ON THE CONE OF QUASI-CONCAVE FUNCTIONS

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Abstract. This paper is devoted to the study of weighted Hardy-type inequalities on the cone of quasi-concave functions, which is equivalent to investigating the boundedness of operators between Lorentz $\Gamma-$spaces. For described inequalities we obtain necessary and sufficient conditions to hold for parameters $q \geq 1$, $p > 0$ and sufficient conditions for the rest of the range of possible parameters.

2010 Mathematics Subject Classification. 26D10, 26D15, 26D07, 44A05, 45D99.

Key words and phrases. Hardy-type inequalities, monotone functions, quasi-concave functions, Lorentz spaces.

Note. This report will be submitted for publication elsewhere.
1. Introduction

In the remarkable paper [20] G.G.Lorentz introduced and characterized the basic properties of new function spaces \( \Lambda^p(\phi) \) defined in terms of rearrangement

\[
f^*(t) := \inf \{ s > 0 : \text{mes} \{ x : |f(x)| > s \} \leq t \}
\]

of a function \( f \) on the semiaxis in decreasing order. More exactly, we say that \( f \in \Lambda^p(\phi) \), if

\[
\|f\|_{\Lambda^p(\phi)} := \left( \int_0^\infty \left[ f^* \phi \right]^p \right)^{1/p} < \infty.
\]

Since then the Lorentz spaces have become an important tool in various branches of analysis and its applications. In particular, the study of mapping properties of operators of classical analysis in Lorentz spaces has started. First of all, it concerns Interpolation Theory [4], where the Lorentz spaces play a crucial role in the extension of the classical Marcinkiewicz theorem. In fact, they appear naturally in the real interpolation method as intermediate spaces for \( L^p \)-spaces (see e.g. [4] and [6]). In spite of the fact that the original definition of \( \Lambda^p(\phi) \) contains an arbitrary weight function \( \phi \), for a long period only the special case of power functions \( \phi \) was used. A breakthrough happened in 1990, when M. Ariño and B. Muckenhoupt [1] characterized \( \Lambda^p(\phi) \hookrightarrow \Lambda^q(\psi) \) property of the Hardy-Littlewood maximal operator \( Mf \) in the case \( 1 < p < \infty \). It took about 15 years to find precise characterization of \( \Lambda^p(\phi) \hookrightarrow \Lambda^q(\psi) \) mapping properties of the maximal operator for all \( 0 < p, q < \infty \) and arbitrary weight functions \( \phi \) and \( \psi \). In the course of the research of this problem E.T.Sawyer [29] introduced Lorentz \( \Gamma \)-spaces \( \Gamma^p(v) \) such that \( f \in \Gamma^p(v) \), if

\[
\|f\|_{\Gamma^p(v)} := \left( \int_0^\infty \left[ f^{**} \right]^p v \right)^{1/p} < \infty,
\]

where

\[
f^{**}(t) := \frac{1}{t} \int_0^t f^*(s) ds.
\]

It is now a well-known fact that \( (Mf)^* \approx f^{**} \) (see e.g. [2] for some simple proofs and historical facts concerning this estimate).

Similar to \( \Lambda \)-analysis, describing the mapping properties of classical operators in the Lorentz \( \Gamma \)-spaces became a challenging task of numerous investigations of the weighted inequalities on the cones of monotone functions (see, for example, [1], [3], [7], [8], [9], [12], [13], [16], [17], [19], [21], [26], [29], [31], [32], [33], [34], [35], [36], [37]).

It has also become convenient to use functions with two different conditions of monotonicity ([5], [15], [22], [30]), in particular, a function \( u(t) \), such that \( u(t) \) is non-decreasing and \( u(t)^2 \) is non-increasing. It has been proved that such functions are equivalent to concave functions,
that is why they are called quasi-concave. Some new information and historical remarks concerning such functions can be found in the paper [25].

Quasi-concave functions are used in Harmonic Analysis, interpolation theory, approximation theory, homogenization theory and some other areas of mathematics. Among the central objects of the analysis that have quasi-concave property are the Peetre $K$–functional $K(f, x) = K(f, x; A_0, A_1)$, which is the most central notion in real interpolation theory (here $A_0$ and $A_1$ are two quasi-Banach spaces), the integral modulus of continuity $\omega_p(x)$, the fundamental function $u_E(t) = \|\chi_{(0,t)}\|_E$ (here $E$ is a symmetric space) and, finally, the averaging operator $\frac{1}{t} \int_0^t f^*(s)ds$, which, as we have shown above, is used in building the Lorentz $\Gamma$–spaces.

The main motivation of this paper is to find the integral criteria of $\Gamma_p(v) \hookrightarrow \Gamma_q(w)$ boundedness for the Hardy–Littlewood maximal operator. This problem was first studied in [36] in the diagonal case $1 < p = q < \infty, v = w$. The next result was obtained in the paper [14] with answers in terms of implicit sequences as well as in the paper [15]. Moreover, G.Sinnamon [30] gave integral criteria for the case $1 < p, q < \infty$ using a reduction principle for the inequalities on the cone of quasi-concave functions. In this paper we extend Sinnamon’s result to cover also the case $0 < p < 1$ and give an alternative approach for the other cases.

In this paper we obtain some results not only for quasi-concave functions, but also for a more generalized class of functions. Let $\psi$ be a continuous strictly increasing function on $[0, \infty)$ such that $\psi(0) = 0$ and $\lim_{t \to \infty} \psi(t) = \infty$. Such functions are called admissible. A function $u(t)$, such that $u(t)$ is non-decreasing and $\frac{u(t)}{\psi(t)}$ is non-increasing, is called $\psi$–quasi-concave. For admissible function $\psi$ we let $\Omega_\psi$ be the subset of $f \in \mathfrak{M}^+$ such that $f(t)$ is non-increasing and $\psi(t)f(t)$ is non-decreasing.

Let $\beta, \gamma, \mu$ be nonnegative Borel measures on $\mathbb{R}_+ := [0, \infty)$, $\rho \in \mathbb{R}_+, \mathfrak{M}^+$ be the class of all measurable functions $f : \mathbb{R}_+ \to [0, +\infty]$. We study the inequality of the type

\begin{equation}
\left( \int_{[0, \infty)} (Af)^q d\gamma \right)^{\frac{1}{q}} \leq C \left( \int_{[0, \infty)} f^p d\beta \right)^{\frac{1}{p}},
\end{equation}

where

\begin{equation}
Af(t) = \left( \int_{[0,t]} f^\rho d\mu \right)^{\frac{1}{\rho}}, \quad f \in \Omega_\psi.
\end{equation}

In the main section of the paper we obtain the necessary and sufficient conditions for this inequality to hold for parameters $q \geq 1, p > 0$. We also study the inequality (3), where the operator (4) is replaced with
the complementary operator

\[(5) \quad Bf(t) = \left( \int_{[t, \infty)} f^p \, d\mu \right)^{\frac{1}{p}}.\]

In this case we obtain necessary and sufficient conditions for the same range of \(p\) and \(q\), but for a more special class of functions \(\Omega_t\).

In the last section of the paper we make several additional observations. Remark 1 gives an alternative proof of Theorem 4. In Remark 2 we show that we can characterize the inequality (3) for the operator (5) by using the criteria obtained earlier for the operator (4). Note also that this way we obtain criteria for a more general of functions \(\Omega\).

In the last part we obtain sufficient conditions for the inequality (3) with operator (4) to hold for \(0 < p \leq q < 1\), \(0 < q < p \leq 1\), \(0 < q < 1 < p < \infty\) and \(f \in \Omega\).

Throughout the paper expressions of the type \(0 \cdot \infty\) are taken to be equal to 0. The relation \(A \ll B\) means that \(A \leq cB\) with a constant \(c\) depending only on the parameter of summation. We write \(A \approx B\) instead of \(A \ll B \ll A\) or \(A = cB\) and use the sign \(Z\) for the set of all integers. We set \(p' = \frac{p}{p-1}\) for \(0 < p < \infty\), \(p \neq 1\) and use the notation \(\frac{1}{r} = \frac{1}{q} - \frac{1}{p}\) for \(0 < q < p < \infty\). \(L^p(\lambda)\) denotes the set of all \(\lambda\)-measurable functions \(f\) on \(\mathbb{R}_+\) such that \(\|f\|_{p, \lambda} := \left( \int_{(0, \infty)} |f|^p \, d\lambda \right)^{\frac{1}{p}} < \infty\). The constant \(C\) may be different at different occurrences.

2. Preliminaries

In this paper we use the following results by A. Gogatishvili and L. Pick.

We say that a measure \(d\nu\) is non-degenerate with respect to function \(\phi\), if for every \(t \in (0, \infty)\)

\[
\int_{[0, \infty)} \frac{d\nu(s)}{\phi(s) + \phi(t)} < \infty, \quad \int_{[0, 1]} \frac{d\nu(s)}{\phi(s)} = \int_{[1, \infty)} d\nu(s) = \infty.
\]

**Theorem 1** [10, Theorem 4.2]. Let \(u, v, w\) be locally integrable weights on \([0, \infty)\), \(p, q \in (0, \infty)\) and denote \(U(t) := \int_0^t u(s) \, ds\). Assume that \(u\) is such that \(U^p\) is admissible and the measure \(v(t) dt\) is non-degenerate with respect to \(U^p\).

(i) If \(0 < p \leq q < \infty\) and \(1 \leq q < \infty\), then the inequality

\[(6) \quad \left( \int_0^\infty f(t)^q w(t) \, dt \right)^{\frac{1}{q}} \leq C \left( \int_0^\infty \left( \frac{1}{U(t)} \int_0^t f(s) u(s) \, ds \right)^p v(t) \, dt \right)^{\frac{1}{p}}\]

holds.
holds for some $C > 0$ and all non-increasing functions $f$ if and only if

$$
A_1 := \sup_{t \in (0, \infty)} \frac{W(t)^{\frac{1}{q}}}{(V(t) + U(t))^p \int_t^\infty U(s)^{-p} v(s) \, ds}^{\frac{1}{q}} < \infty.
$$

(ii) If $1 \leq q < p < \infty$, then (6) holds for some $C > 0$ and all non-increasing functions $f$ if and only if

$$
A_2 := \left( \int_0^\infty \frac{U(t)^r \left[ \sup_{y \in [t, \infty)} U(y)^{-r} W(y)^{\frac{1}{q}} \right]}{(V(t) + U(t))^p \int_t^\infty U(s)^{-p} v(s) \, ds}^{\frac{1}{q}+2} \times V(t) \int_t^\infty U(s)^{-p} v(s) \, ds (U^p(t))^{\frac{1}{r}} \right) < \infty.
$$

(iii) If $0 < p \leq q < 1$, then (6) holds for some $C > 0$ and all non-increasing functions $f$ if and only if

$$
A_3 := \sup_{t \in (0, \infty)} \frac{W(t)^{\frac{1}{q}} + U(t) \left( \int_t^\infty W(s)^{\frac{1}{q}} w(s) U(s)^{-\frac{1}{q}} \, ds \right)^{\frac{1}{q}}} {(V(t) + U(t))^p \int_t^\infty U(s)^{-p} v(s) \, ds}^{\frac{1}{q}} < \infty.
$$

(iv) If $0 < q < 1$ and $0 < q < p$, then (6) holds for some $C > 0$ and all non-increasing functions $f$ if and only if

$$
A_4 := \left( \int_0^\infty \frac{W(t)^{\frac{1}{q}} + U(t) \frac{q}{p} \int_t^\infty W(s)^{\frac{1}{q}} w(s) U(s)^{-\frac{1}{q}} \, ds} {(V(t) + U(t))^p \int_t^\infty U(s)^{-p} v(s) \, ds}^{\frac{1}{q}} \times W(t)^{\frac{1}{q}} w(t) \, dt \right)^{\frac{1}{q}} < \infty.
$$

Moreover, $A_4 \approx A_5$, where

$$
A_5 := \left( \int_0^\infty \frac{W(t)^{\frac{1}{q}} + U(t) \frac{q}{p} \int_t^\infty W(s)^{\frac{1}{q}} w(s) U(s)^{-\frac{1}{q}} \, ds} {(V(t) + U(t))^p \int_t^\infty U(s)^{-p} v(s) \, ds}^{\frac{1}{q}} \times V(t) \int_t^\infty U(s)^{-p} v(s) \, ds (U^p(t))^{\frac{1}{r}} \right) < \infty.
$$

**Theorem 2** [10, Theorem 5.1]. Let $p, q \in (0, \infty)$ and let $u, v, w$ be weights. Assume that $v(t)\, dt$ is a non-degenerate measure with respect to $U^p$.

(i) Let $0 < p \leq q < \infty$. Then the inequality

$$
\left( \int_0^\infty \left( \frac{1}{U(t)} \int_0^t f(s) u(s) \, ds \right)^q w(t) \, dt \right)^{\frac{1}{q}}
$$


\[
\leq C \left( \int_0^\infty \left( \frac{1}{U(t)} \int_0^t f(s)u(s)ds \right)^p v(t) dt \right)^{\frac{1}{p}}
\]
holds for all non-increasing functions \( f \) if and only if
\[
A_6 := \sup_{t \in (0, \infty)} \frac{(W(t) + U(t)^q \int_t^\infty U(s)^{-q}w(s)ds)^{\frac{q}{q-p}}}{{(V(t) + U(t)^p \int_t^\infty U(s)^{-p}v(s)ds)^{\frac{p}{q-p}}}} < \infty.
\]
(ii) Let \( 0 < q < p < \infty \). Then (7) holds for all non-increasing functions if and only if
\[
A_7 := \left( \int_0^\infty \frac{(W(t) + U(t)^q \int_t^\infty U(y)^{-q}w(y)dy)^{\frac{q}{q-1}}}{{(V(t) + U(t)^p \int_t^\infty U(s)^{-p}v(s)ds)^{\frac{p}{q-1}}}dt} \right)^{\frac{1}{p}} < \infty,
\]
where, again, \( r = \frac{pq}{p-q} \). Moreover, \( A_7 \approx A_8 \), where
\[
A_8 := \left( \int_0^\infty \frac{(W(t) + U(t)^q \int_t^\infty U(s)^{-q}w(s)ds)^{\frac{q-1}{q-2}w(t)}}{{(V(t) + U(t)^p \int_t^\infty U(s)^{-p}v(s)ds)^{\frac{p}{q-2}}}dt} \right)^{\frac{1}{p}} < \infty.
\]
We also use the following general theorem about the norm of integral operators.

**Theorem 3** [18, Chapter XI, §1.5, Theorem 4]. Let the operator \( K \) be given by
\[
Kf(x) := w(x) \int_a^x k(x, y)f(y)\nu(y)dy, \ x \in (a, b).
\]
Then, if \( 1 \leq q < \infty \), we have
\[
\|K\|_{L_l[a, b] \to L_q[a, b]} = \text{ess sup}_{t > 0} \|\chi_{[a, \cdot]}(t)k(\cdot, t)w(\cdot)\nu(t)\|_q.
\]
If \( 1 < p \leq \infty \) and \( \frac{1}{p} + \frac{1}{q} = 1 \), then
\[
\|K\|_{L_p[a, b] \to L_\infty[a, b]} = \text{ess sup}_{t > 0} \|\chi_{[a, \cdot]}(t)k(t, \cdot)w(\cdot)\nu(t)\|_p.
\]

3. **The main results**

First of all, we note that the inequality (3) for both operators (4) and (5) can be reduced to the case \( \rho = 1 \) if we substitute \( f^\rho \) by \( f \).

We let \( d\mu(x) = u(x)dx \), \( d\beta(x) = v(x)dx \) and \( d\gamma(x) = w(x)dx \). Throughout the paper the measure \( v(x)dx \) is assumed to be non-degenerate with respect to the function \( \psi^p(x) \).

The main result of this paper reads as follows:
Theorem 4. Let $q \geq 1, p > 0$. Then the inequality

\[
\left( \int_{[0,\infty)} \left( \int_{[0,t]} f(s)w(s)ds \right)^q w(t)dt \right)^{\frac{1}{q}} \leq C \left( \int_{[0,\infty)} f^p v(t)dt \right)^{\frac{1}{p}}
\]

holds for all functions $f \in \Omega_v$ if and only if

(i) $A_1 < \infty$, if $q = 1$, $0 < p \leq 1$, where

\[
A_1 := \sup_{t \geq 0} \left( \int_{[0,t]} \left( \int_{[0,\infty)} U(z,y)w(z)dz \right) d\psi(y) \right)^{\frac{1}{q}} V(t)^{-\frac{1}{q}};
\]

(ii) $A_2 < \infty$, if $q = 1$, $1 < p < \infty$, where

\[
A_2 := \left( \int_{[0,\infty)} \left( \int_{[0,t]} \left( \int_{[t,\infty)} U(s,z)w(s)ds \right)^{p'} d\psi(z) \right)^{\frac{1}{p'}} \right)^{\frac{1}{q}} V(t)^{-\frac{1}{q}};
\]

(iii) $A_3 < \infty$, if $q > 1$, $0 < p \leq 1$, where

\[
A_3 := \sup_{t \geq 0} \left( \int_{[0,t]} \left( \int_{[0,\infty)} \min(s,t) U(s,y)d\psi(y) \right)^q w(s)ds \right)^{\frac{1}{q}} V(t)^{-\frac{1}{q}};
\]

(iv) $A_{4,1} + A_{4,2} + A_{4,3} + A_{4,4} < \infty$, if $1 < p \leq q < \infty$, where

\[
A_{4,1} := \sup_{t \geq 0} \left( \int_{[0,t]} \left( \int_{[0,\infty)} U(s)w(s)ds \right)^q \right)^{\frac{1}{q}} V(t)^{-\frac{1}{q}};
\]

\[
A_{4,2} := \sup_{t \geq 0} \left( \int_{[0,t]} U(t,s)^{p'} \left( \int_{[t,\infty)} \psi(p'(s)) \right)^{\frac{1}{p'}} \left( \int_{[t,\infty)} w(s)ds \right)^{\frac{1}{q}} \right)^{\frac{1}{q}} V(t)^{-\frac{1}{q}};
\]

\[
A_{4,3} := \sup_{t \geq 0} \left( \int_{[0,t]} \left( \int_{[0,\infty)} \psi(p'(s)) \right)^{\frac{1}{p'}} \left( \int_{[0,\infty)} w(s)ds \right)^{\frac{1}{q}} \right)^{\frac{1}{q}} V(t)^{-\frac{1}{q}};
\]

\[
A_{4,4} := \sup_{t \geq 0} \left( \int_{[0,t]} \left( \int_{[0,\infty)} \psi(p'(s)) \right)^{\frac{1}{p'}} \left( \int_{[0,\infty)} w(s)ds \right)^{\frac{1}{q}} \right)^{\frac{1}{q}} V(t)^{-\frac{1}{q}};
\]

(v) $A_{5,1} + A_{5,2} + A_{5,3} + A_{5,4} < \infty$, if $1 < q < p < \infty$, where

\[
A_{5,1} := \left( \int_{[0,\infty)} \left( \int_{[0,\infty)} U^q(s)w(s)ds \right)^{\frac{1}{q}} \left( \int_{[0,\infty)} \psi(s) \right) \right)^{\frac{1}{q}} V(t)^{-\frac{1}{q}};
\]

\[
A_{5,2} := \left( \int_{[0,\infty)} \left( \int_{[0,\infty)} \psi \left( \int_{[0,\infty)} U^q(s)w(s)ds \right) \right)^{\frac{1}{q}} \left( \int_{[0,\infty)} w(t)dt \right)^{\frac{1}{q}} \right)^{\frac{1}{q}} V(t)^{-\frac{1}{q}};
\]
Here

\[ V(t) := \int_{[0,t]} v, \quad U(t) := \int_{[0,t]} u, \]

\[ \mathcal{V}(t) := V(t) + \psi(t)^p \int_{[t,\infty)} \psi(s)^{-p} v(s) ds, \]

\[ \mathcal{V}(t) := V(t) - \psi^{-1} V(t) \int_{[t,\infty)} \psi(s)^{-p} v(s) ds \]

and

\[ U(t, y) := \int_{[y,t]} \frac{u(s)}{\psi(s)} ds. \]

**Proof.** Let \( f(t) \in \Omega_\psi \). Then by Lemma 2.8 in [10] there exists a non-increasing function \( g(t) \) such that

\[ f(t) \approx \frac{1}{\psi(t)} \int_{[0,t]} g(s) d\psi(s). \]

We denote the least possible constant \( C \) in (8) by \( H_{A(p, q)} \), that is

\[ H_{A(p, q)} := \sup_{f \in \Omega_\psi} \left( \int_{[0,\infty)} \left( \int_{[0,t]} f(s) u(s) ds \right)^q w(t) dt \right)^{\frac{1}{q}} \left( \int_{[0,\infty)} f(t)^p v(t) dt \right)^{\frac{1}{p}}. \]

Then we have

\[ H_{A(p, q)} \approx \sup_{g^1} \left( \int_{[0,\infty)} \left( \frac{1}{\psi(t)} \int_{[0,\infty)} g d\psi \right) u(s) ds \right)^q w(t) dt)^{\frac{1}{q}} \left( \int_{[0,\infty)} \left( \frac{1}{\psi(t)} \int_{[0,\infty)} g d\psi \right)^p v(t) dt \right)^{\frac{1}{p}}. \]

We write

\[ \int_{[0,t]} \left( \frac{1}{\psi(s)} \int_{[0,s]} g d\psi \right) u(s) ds = \int_{[0,t]} U(t, y) g(y) d\psi(y). \]
Thus,

\begin{equation}
H_A(p, q) \approx \sup_{g_1} \frac{\left( \int_{[0, \infty)} \left( \int_{[0, t]} U(t, y)g(y) d\psi(y) \right)^q w(t) dt \right)^{\frac{1}{q}}}{\left( \int_{[0, \infty)} \left( \frac{1}{\psi(t)} \int_{[0, t]} g d\psi \right)^p v(t) dt \right)^{\frac{1}{p}}}.
\end{equation}

For $q = 1$ we obtain

\begin{equation}
H_A(p, 1) \approx \sup_{g_1} \frac{\int_{[0, \infty)} \left( \int_{[0, t]} U(t, y)w(t) dt \right) g(y) d\psi(y)}{\left( \int_{[0, \infty)} \left( \frac{1}{\psi(t)} \int_{[0, t]} g d\psi \right)^p v(t) dt \right)^{\frac{1}{p}}},
\end{equation}

and Theorem 1 yields

\begin{equation}
H_A(p, 1) \approx \sup_{t \geq 0} \frac{\int_{[0, t]} \left( \int_{[0, \infty)} U(z, y) w(z) dz \right) d\psi(y)}{\left( V(t) + \psi(t)p \int_{[t, \infty)} \psi(s)^{-p} v(s) ds \right)^{\frac{1}{p}}}
\end{equation}

for $0 < p \leq 1$ and $H_A(p, 1) \approx$

\begin{equation}
\approx \left[ \int_{[0, \infty)} \psi(t)^p \left[ \sup_{y \geq t} \frac{1}{\psi(y)} \left( \int_{[0, y]} \left( \int_{[z, \infty)} U(t, z) w(t) dt \right) d\psi(z) \right) \right]^{\frac{1}{p'}}
\times V(t) d(\psi^p(t)) \right]^{\frac{1}{p'}}
\end{equation}

for $1 < p < \infty$. Note that $\int_{[z, \infty)} U(t, z) w(t) dt$ is non-increasing. Hence,

\begin{equation}
\sup_{y \geq t} \frac{1}{\psi(y)} \left( \int_{[0, y]} \left( \int_{[z, \infty)} U(t, z) w(t) dt \right) d\psi(z) \right) = \frac{1}{\psi(t)} \left( \int_{[0, t]} \left( \int_{[z, \infty)} U(t, z) w(t) dt \right) d\psi(z) \right),
\end{equation}

and we obtain

\begin{equation}
H_A(p, 1) \approx \left( \int_{[0, \infty)} \left( \int_{[0, t]} \left( \int_{[z, \infty)} U(t, z) w(t) dt \right) d\psi(z) \right)^{\frac{1}{p'}} V(t) d(\psi^p(t)) \right)^{\frac{1}{p'}}
\end{equation}

for $1 < p < \infty$. In the case $1 < q < \infty$ we have

\begin{equation}
H_A(p, q) \approx \sup_{g_1} \sup_{h \geq 0} \frac{\int_{[0, \infty)} \left( \int_{[0, t]} U(t, y)g(y) d\psi(y) \right) h(t) w(t) dt}{\left( \int_{[0, \infty)} \left( \frac{1}{\psi(t)} \int_{[0, t]} g d\psi \right)^p v(t) dt \right)^{\frac{1}{p}}} \left( \int_{[0, \infty)} h' w \right)^{\frac{1}{q}}.
\end{equation}

\begin{equation}
= \sup_{h \geq 0} \frac{1}{\left( \int_{[0, \infty)} h' w \right)^{\frac{1}{q}}} \sup_{g_1} \frac{\int_{[0, \infty)} \left( \int_{[0, \infty)} U(t, y) h(t) w(t) dt \right) g(y) d\psi(y)}{\left( \int_{[0, \infty)} \left( \frac{1}{\psi(t)} \int_{[0, t]} g d\psi \right)^p v(t) dt \right)^{\frac{1}{p}}},
\end{equation}
If $0 < p \leq 1$, then, according to Theorem 1 (i),

$$H_A(p, q) \approx \sup_{h \geq 0} \frac{\|V^{-\frac{1}{p}}(t) \int_{[0,t]} \left( \int_{[y,\infty]} U(s, y) h(s) w(s) ds \right) d\psi(y) \|_\infty}{\left( \int_{[0,\infty]} h^{p'} w \right)^{\frac{1}{p'}}} \approx \sup_{h \geq 0} \frac{\|V^{-\frac{1}{p}}(t) \int_{[0,\infty]} \left( \int_{t}^{\min(s,t)} U(s, y) d\psi(y) \right) h(s) w(s) ds \|_\infty}{\left( \int_{[0,\infty]} h^{p'} w \right)^{\frac{1}{p'}}}.$$  

By using Theorem 3 we get

$$H_A(p, q) \approx \sup_{t > 0} \left( \int_{[0,\infty]} \left( \int_{t}^{\min(s,t)} U(s, y) d\psi(y) \right) h(s) w(s) ds \right)^{\frac{1}{q'}}.$$  

If $1 < p < \infty$, then by using Theorem 1 (ii) we find

$$H_A(p, q) \approx \sup_{h \geq 0} \left( \int_{[0,\infty]} \psi(t) \left[ \sup_{y \geq t} \frac{1}{\psi(y)} \int_{[0,\psi]} \Phi(s) d\psi(s) \right]^{p'} V(t) d(\psi^p(t)) \right)^{\frac{1}{p'}},$$

where

$$\Phi(z) := \int_{[z,\infty]} U(t, z) h(t) w(t) dt.$$  

It is easy to see that $\Phi(z) \downarrow$. Hence,

$$\sup_{y \geq t} \frac{1}{\psi(y)} \int_{[0,\psi]} \Phi(s) d\psi(s) = \frac{1}{\psi(t)} \int_{[0,t]} \Phi(s) d\psi(s),$$

and (14) is equivalent to

$$H_A(p, q) \approx \sup_{h \geq 0} \left( \int_{[0,\infty]} \left( \int_{[z,\infty]} U(t, z) h(t) w(t) dt \right) d\psi(z) \right)^{p'} V(s) d(\psi^p(t)) \left( \int_{[0,\infty]} h^{p'} w \right)^{\frac{1}{p'}}.$$

Since

$$\int_{[0,t]} \left( \int_{[z,\infty]} U(t, z) h(t) w(t) dt \right) d\psi(z) = \int_{[0,t]} \left( \int_{[z,\infty]} U(t, z) h(t) w(t) dt \right) d\psi(z) + \int_{[0,t]} \left( \int_{[z,\infty]} U(t, z) h(t) w(t) dt \right) d\psi(z) =: I_1 + I_2,$$
we have

\[ I_1 = \int_{[0,s]} \left( \int_{[t,s]} U(t, z) d\psi(z) \right) h(t) w(t) dt \]

\[ = \int_{[0,s]} h(t) w(t) \left( \int_{[0,t]} \left( \int_{[z,t]} \frac{u(\tau)}{\psi(\tau)} d\tau \right) d\psi(z) \right) dt = \int_{[0,s]} U(t) h(t) w(t) dt \]

and

\[ I_2 = \int_{[0,s]} \left( \int_{[s,\infty]} [U(t, s) + U(s, z)] h(t) w(t) dt \right) d\psi(z) \]

\[ = \psi(s) \int_{[s,\infty]} U(t, s) h(t) w(t) dt + U(s) \int_{[s,\infty]} h(t) w(t) dt. \]

Thus, the characterization of (14) is equivalent to the following inequalities restricted to the set of non-negative functions:

\[ \left( \int_{[0,\infty]} \left( \int_{[0,s]} U hw \right)^{p'} d\psi(s) d(\psi^{p'}(s)) \right)^{\frac{1}{p'}} \leq B_1 \left( \int_{[0,\infty]} h^{q'} w \right)^{\frac{1}{q'}}, \]

\[ \left( \int_{[0,\infty]} \left( \int_{[s,\infty]} \left( \int_{[0,t]} U(t, s) h(t) w(t) dt \right)^{p'} d\psi(s) d(\psi^{p'}(s)) \right)^{\frac{1}{p'}} \leq B_2 \left( \int_{[0,\infty]} h^{q'} w \right)^{\frac{1}{q'}}, \]

\[ \left( \int_{[0,\infty]} \left( U(s) \int_{[s,\infty]} h w \right)^{p'} d\psi(s) d(\psi^{p'}(s)) \right)^{\frac{1}{p'}} \leq B_3 \left( \int_{[0,\infty]} h^{q'} w \right)^{\frac{1}{q'}}, \]

and

\[ H_A(p, q) \approx B_1 + B_2 + B_3. \]

Applying well-known criteria for the weighted Hardy and Hardy-type inequalities [23, 24, 27, 28], we obtain

\[ B_1 + B_2 + B_3 \approx A_{4,1} + A_{4,2} + A_{4,3} + A_{4,4} \]

for \( 1 < p \leq q < \infty \) and

\[ B_1 + B_2 + B_3 \approx A_{5,1} + A_{5,2} + A_{5,3} + A_{5,4} \]

for \( 1 < q < p < \infty \).

For investigating the inequality (3) for the operator (5) we use a different approach, which is based on the result obtained by G. Sinnammon ([30], Theorem 2.6) for the class of quasi-concave functions \( \Omega_{0,1} = \left\{ f \in \mathfrak{M}^+ : f(t) \uparrow, \frac{f(t)}{t} \downarrow \right\} \). We obtain the characterization for the particular case \( f \in \Omega_5 = \left\{ f \in \mathfrak{M}^+ : f(t) \downarrow, \frac{f(t)}{t} \uparrow \right\} \).
To prove the main result we need the following lemma of independent interest:

**Lemma.** Let $u, v$ be non-negative measurable functions. Assume that the measure $v(t)dt$ is non-degenerate with respect to the function $t^p$.

(i) If $0 < p \leq 1$, then for every function $f \in \Omega_t$
\[
\sup_{f \in \Omega_t} \|f\|_{\frac{1}{u}, \frac{1}{p}, v} \approx \sup_{x \geq 0} \left( \frac{\int_{[0,x]} u(t)dt + x \int_{[x,\infty)} \frac{u(t)}{t} dt}{\sqrt[x]{v(x)}} \right)^{-\frac{1}{p}},
\]
where
\[
V(x) := \int_{[0,x]} v(t)dt + x^p \int_{[x,\infty)} \frac{v(t)}{tp^p} dt.
\]

(ii) If $p > 1$, then for every function $f \in \Omega_t$
\[
\sup_{f \in \Omega_t} \|f\|_{\frac{1}{u}, \frac{1}{p}, v} \approx \left( \int_{[0,\infty)} \int_{[0,x]} u(t)dt \left[ \int_{[0,x]} t^{p'} v(t)dt + \int_{[x,\infty)} v(t)dt \right]^{-\frac{p'}{p}} \right) \left( \int_{[0,\infty)} \int_{[x,\infty)} u(t)dt \left[ \int_{[0,x]} t^{p'} v(t)dt + \int_{[x,\infty)} v(t)dt \right]^{-\frac{p'}{p}} \right)^{\frac{1}{p}}
\]
where
\[
V_1(x)dx \approx d \left( V(x)^{-\frac{p'}{p}} \right), \quad V_2(x)dx \approx d \left( -x^{p'} V(x)^{-\frac{p'}{p}} \right).
\]

**Proof.** (i) For the case $p > 1$ we use [30, Theorem 2.6] and set $q = 1$. We have
\[
\sup_{f \in \Omega_{t,1}} \|f\|_{\frac{1}{u}, \frac{1}{p}, v} \approx \left( \int_{[0,\infty)} \int_{[0,x]} u(t)dt \left[ \int_{[0,x]} t^{p'} v(t)dt + \int_{[x,\infty)} v(t)dt \right]^{-\frac{p'}{p}} \right)
\]
\[
\times \left( \int_{[0,\infty)} \int_{[x,\infty)} u(t)dt \left[ \int_{[0,x]} t^{p'} v(t)dt + \int_{[x,\infty)} v(t)dt \right]^{-\frac{p'}{p}} \right)^{\frac{1}{p}}.
\]

Put $V^{*}(x) := \frac{1}{x^p} \int_{[0,x]} t^p v(t)dt + \int_{[x,\infty)} v(t)dt$, then we have
\[
\sup_{f \in \Omega_{t,1}} \|f\|_{\frac{1}{u}, \frac{1}{p}, v} \approx \left( \int_{[0,\infty)} \int_{[0,x]} u(t)dt \left[ \int_{[0,x]} t^{p'} V^{*}(x)^{-\frac{p'}{p}} \left( \int_{[0,x]} t^{p'} v(t)dt + \int_{[x,\infty)} v(t)dt \right) u(x)dx \right] \right)^{\frac{1}{p}}
\]
\[
+ \left( \int_{[0,\infty)} \int_{[x,\infty)} u(t)dt \left[ \int_{[0,x]} t^{p'} V^{*}(x)^{-\frac{p'}{p}} u(x)dx \right] \right)^{\frac{1}{p}}.
\]
\[
= \left( \int_{[0,\infty)} \frac{1}{x^\beta} V^*(x)^{-\frac{\nu}{p}} d\left( \int_{[0,x]} tu(t)dt \right)^\nu \right)^\frac{1}{\nu}
\]
\[
+ \left( \int_{[0,\infty)} V^*(x)^{-\frac{\nu}{p}} d\left( - \int_{[x,\infty)} u(t)dt \right)^\nu \right)^\frac{1}{\nu}
\]
\[
\approx \left( \int_{[0,\infty)} \left( \int_{[0,x]} tu(t)dt \right)^\nu V_1^*(x)dx \right)^\frac{1}{\nu}
\]
\[
+ \left( \int_{[0,\infty)} \left( \int_{[x,\infty)} u(t)dt \right)^\nu V_2^*(x)dx \right)^\frac{1}{\nu},
\]
where
\[
V_1^*(x)dx \approx d \left( V^*(x)^{-\frac{\nu}{p}/x^\nu} \right), \quad V_2^*(x)dx \approx d \left( -V^*(x)^{-\frac{\nu}{p}} \right).
\]

For the class \( \Omega_t \) we have
\[
\sup_{f \in \Omega_t} \|f\|_{L^1_{tu,v}} = \sup_{f \in \Omega_{t,1}} \|f\|_{L^{1/p}_{tu,v}}.
\]

We make the substitutions \( u(t) \to \frac{u(t)}{\nu} \) and \( v(t) \to \frac{v(t)}{\nu} \) to obtain \( V(t)/\nu, V_1(t) \) and \( V_2(t) \) instead of \( V^*(t), V_1^* \) and \( V_2^* \), respectively, and thus derive the proposition of the lemma.

(ii) The corresponding result for the case \( 0 < p \leq 1 \) follows directly from [36, Theorem 3.3] once we write \( f(t) \approx \frac{1}{T} \int_0^t g(s)ds \) for \( f \in \Omega_t \), where \( g \) is some non-increasing function.

Our result for the inequality (3) with the operator (5) for \( \psi(t) = t \) reads as follows.

**Theorem 5.** Let \( q \geq 1, p > 0 \). Then the inequality
\[
\left( \int_{[0,\infty)} \int_{[t,\infty)} f(s)u(s)ds w(t)dt \right)^\frac{q}{2} \leq C \left( \int_{[0,\infty)} f^{p,v}v(t)dt \right)^\frac{1}{p}
\]
holds for all functions \( f \in \Omega_t \) if and only if

(i) \( A_1 < \infty \), if \( q = 1 \), \( 0 < p \leq 1 \), where
\[
A_1 := \sup_{x \geq 0} \left( \int_{[0,x]} u(t)W(t)dt + x \int_{[x,\infty)} \frac{u(t)}{t}W(t)dt \right) V(x)^{-\frac{1}{p}};
\]

(ii) \( A_2 < \infty \), if \( q = 1 \), \( 1 < p < \infty \), where
\[
A_2 := \left( \int_{[0,\infty)} \left( \int_{[0,x]} u(t)W(t)dt \right)^\nu V_1(x)dx \right.
\]
\[
+ \left. \int_{[0,\infty)} \left( \int_{[x,\infty)} \frac{u(t)}{t}W(t)dt \right)^\nu V_2(x)dx \right)^\frac{1}{\nu};
\]
(iii) $A_3 < \infty$, if $q > 1$, $0 < p \leq 1$, where

$$A_3 := \sup_{t \geq 0} \left[ \left( \int_{[0,t]} U^*(t, s)^p w(s) ds \right)^{\frac{1}{p}} \right]$$

$$+ t \left( \int_{[0,\infty]} U^*_1(\max(s, t)) w(s) ds \right)^{\frac{1}{q}} V(t)^{-\frac{1}{q}};$$

(iv) $A_{4,1} + A_{4,2} + A_{4,3} + A_{4,4} < \infty$, if $1 < p \leq q < \infty$, where

$$A_{4,1} := \sup_{t \geq 0} \left( \int_{[t,\infty]} V_1(x) U(x, t)^{p^*} dx \right)^{\frac{1}{p^*}} \left( \int_{[0,t]} w(x) dx \right)^{\frac{1}{q}};$$

$$A_{4,2} := \sup_{t \geq 0} \left( \int_{[t,\infty]} V_1(x) dx \right)^{\frac{1}{p}} \left( \int_{[0,t]} U(t, y)^q w(y) dy \right)^{\frac{1}{q}};$$

$$A_{4,3} := \sup_{t \geq 0} \left( \int_{[t,\infty]} U_1(x)^p V_2(x) dx \right)^{\frac{1}{p}} \left( \int_{[0,t]} w(x) dx \right)^{\frac{1}{q}};$$

$$A_{4,4} := \sup_{t \geq 0} \left( \int_{[0,t]} V_2(x) dx \right)^{\frac{1}{p}} \left( \int_{[t,\infty]} U_1^q(s) w(s) ds \right)^{\frac{1}{q}};$$

(v) $A_{5,1} + A_{5,2} + A_{5,3} + A_{5,4} < \infty$, if $1 < q < p < \infty$, where

$$A_{5,1} := \left( \int_{[0,\infty]} \left( \int_{[t,\infty]} V_1(x) U(x, t)^{p^*} dx \right)^{\frac{1}{p^*}} \left( \int_{[0,t]} w(x) dx \right)^{\frac{1}{q}} w(t) dt \right)^{\frac{1}{q}};$$

$$A_{5,2} := \left( \int_{[0,\infty]} \left( \int_{[t,\infty]} V_1(x) dx \right)^{\frac{1}{p}} \left( \int_{[0,t]} U(t, y)^q w(y) dy \right)^{\frac{1}{q}} V_1(t) dt \right)^{\frac{1}{q}};$$

$$A_{5,3} := \left( \int_{[0,\infty]} \left( \int_{[0,t]} w(t) dt \right)^{\frac{1}{q}} \left( \int_{[t,\infty]} U_1(t)^p V_2(t) dt \right)^{\frac{1}{p}} w(x) dx \right)^{\frac{1}{q}};$$

$$A_{5,4} := \left( \int_{[0,\infty]} \left( \int_{[t,\infty]} U_1(x)^q w(s) ds \right)^{\frac{1}{q}} \right) \times \left( \int_{[0,\infty]} V_2(s) ds \right)^{\frac{1}{p}} U_1(x)^q w(x) dx)^{\frac{1}{q}};$$

$$U^*(x, s) = \int_{[s,\infty]} u(t) dt, \quad U_1(t) := \int_{[t,\infty]} \frac{u(s)}{s} ds.$$
Proof. Let $q = 1$, then

$$H_B(p, 1) = \sup_{f \in \Omega_1} \frac{\int_{[0, \infty)} \left( \int_{[t, \infty)} f u(t) w(t) dt \right)^{\frac{q}{p}}}{\left( \int_{[0, \infty)} f^{\frac{p}{q}} u^{\frac{q}{p}} \right)^{\frac{1}{p}}} = \sup_{f \in \Omega_1} \frac{\int_{[0, \infty)} f(t) u(t) W(t) dt}{\left( \int_{[0, \infty)} f^{\frac{p}{q}} u^{\frac{q}{p}} \right)^{\frac{1}{p}}},$$

where $W(t) = \int_{[0, t]} w(s) ds$. We use Lemma and obtain

$$H_B(p, 1) \approx \left( \int_{[0, \infty)} \left( \int_{[0, x]} u(t) W(t) dt \right)^{\frac{p'}{p}} V_1(x) dx \right)^{\frac{1}{p'}} + \int_{[0, \infty)} \left( \int_{[x, \infty)} \frac{u(t)}{t} W(t) dt \right)^{\frac{p'}{p}} V_2(x) dx^{\frac{1}{p'}}$$

for $1 < p < \infty$ and

$$H_B(p, 1) \approx \sup_{x \geq 0} \left( \int_{[0, x]} u(t) W(t) dt + x \int_{[x, \infty]} \frac{u(t)}{t} W(t) dt \right) V(x)^{\frac{1}{p'}}$$

for $0 < p \leq 1$. Let now $q > 1$. We have

$$H_B(p, q) = \sup_{f \in \Omega_1} \frac{\left( \int_{[0, \infty)} \left( \int_{[t, \infty)} f u(t) w(t) dt \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}}{\left( \int_{[0, \infty)} f^{\frac{p}{q}} u^{\frac{q}{p}} \right)^{\frac{1}{p}}}$$

$$= \sup_{f \in \Omega_1} \sup_{g \geq 0} \frac{\int_{[0, \infty)} \left( \int_{[t, \infty)} f u(t) g(t) dt \right)^{\frac{1}{p'}}}{\|g\|_{q', w^{1 - q'}} \|f\|_{p, w^{1 - q'}}}$$

$$= \sup_{g \geq 0} \frac{1}{\|g\|_{q', w^{1 - q'}}} \sup_{f \in \Omega_1} \frac{\int_{[0, \infty)} f(t) u(t) \left( \int_{[0, t]} g \right) dt}{\|f\|_{p, w^{1 - q'}}}.$$

By using Lemma for $p > 1$ we find

$$H_B(p, q) \approx \sup_{g \geq 0} \frac{1}{\|g\|_{q', w^{1 - q'}}} \left( \int_{[0, \infty)} \left( \int_{[0, x]} u(t) \left( \int_{[0, t]} g \right) dt \right)^{\frac{p'}{p}} \right)^{\frac{1}{p'}} V_1(x) dx$$

$$+ \int_{[0, \infty)} \left( \int_{[x, \infty)} \frac{u(t)}{t} \left( \int_{[0, t]} g \right) dt \right)^{\frac{p'}{p}} V_2(x) dx^{\frac{1}{p'}}.$$

We can write

$$\int_{[x, \infty)} \frac{u(t)}{t} \left( \int_{[0, t]} g(s) ds \right) dt$$

$$= \left( \int_{[0, x]} g(s) ds \right) \left( \int_{[x, \infty]} \frac{u(t)}{t} dt \right) + \int_{[x, \infty)} \left( \int_{[0, \infty)} \frac{u(t)}{t} dt \right) g(s) ds,$$

and, thus, the initial inequality (15) is equivalent to the following three inequalities on the set of non-negative functions:
\begin{align}
(16) \quad \left( \int_{[0, \infty)} \left( \int_{[0, x]} U(x, s) g(s) ds \right)^{p'} \right)^{\frac{1}{p'}} V_1(x) dx \leq C_1 \| g \|_{q', w^{1-q'}},
\end{align}

\begin{align}
(17) \quad \left( \int_{[0, \infty)} \left( \int_{[0, x]} g(s) ds \right)^{p'} U_1(x)^{q'} V_2(x) dx \right)^{\frac{1}{p'}} \leq C_2 \| g \|_{q', w^{1-q'}}
\end{align}

and

\begin{align}
(18) \quad \left( \int_{[0, \infty)} \left( \int_{[x, \infty]} g(s) ds \right)^{p'} U_2(x) dx \right)^{\frac{1}{p'}} \leq C_3 \| g \|_{q', w^{1-q'}}
\end{align}

where

\[ U^*(x, s) = \int_{[s, x]} u(t) dt, \quad U_1(s) = \int_{[s, \infty)} \frac{u(t)}{t} dt. \]

Criteria from [23,24] yield the result of the theorem for

\[ 1 < p \leq q < \infty \]

and

\[ 1 < q < p < \infty \]

In case \( 0 < p \leq 1 \) we have

\[ H_B(p, q) \approx \sup_{g \geq 0} \frac{1}{\| g \|_{q', w^{1-q'}}} \]

\begin{align}
\times \sup_{t \geq 0} \left[ \left( \int_{[0, t]} u(x) \left( \int_{[0, x]} g \right) dx + t \int_{[t, \infty)} \frac{u(x)}{x} \left( \int_{[0, x]} g \right) dx \right) V(t)^{-\frac{1}{p'}} \right]
\end{align}

\begin{align}
= \sup_{g \geq 0} \| g \|_{q', w^{1-q'}} \sup_{t \geq 0} \left[ \left( \int_{[0, t]} U^*(t, s) g(s) ds 
+ t U_1(t) \int_{[0, t]} g(s) ds + t \int_{[t, \infty)} U_1(s) g(s) ds \right) V(t)^{-\frac{1}{p'}} \right]
\end{align}

\begin{align}
= \sup_{g \geq 0} \| g \|_{q', w^{1-q'}} \sup_{t \geq 0} \left[ \left( \int_{[0, t]} U^*(t, s) g(s) ds + t \int_{[0, \infty)} U_1(s) g(s) ds \right) \| V(t) \|_{p'} \right]
\end{align}

Then Theorem 3 yields

\[ H_B(p, q) \approx \sup_{t \geq 0} V(t)^{-\frac{1}{p'}} \left[ \left( \int_{[0, t]} U^*(t, s)^q w(s) ds \right)^{\frac{1}{q}} + t \left( \int_{[0, \infty)} U_1^q(\max(s, t)) w(s) ds \right)^{\frac{1}{q}} \right]. \]
4. Final remarks and results.

Remark 1 (an alternative proof of Theorem 4). In the proof of Theorem 4 $H_{A}(p,q)$ is presented as the norm of the embedding $\Gamma \hookrightarrow \Lambda$. The same criteria can be obtained alternatively by using the $\Gamma \hookrightarrow \Gamma$ embedding. For instance, in the case $q = 1$ we have

$$H_{A}(p,1) \approx \sup_{g \in G} \frac{\int_{[0,\infty)} W_{1}(s) u(s) \left(\frac{1}{\psi(t)} \int_{[0,t]} gd\psi\right) ds}{\left(\int_{[0,\infty)} \left(\frac{1}{\psi(t)} \int_{[0,t]} gd\psi\right)^{p} v(t) dt\right)^{\frac{1}{p}}},$$

where $W_{1}(t) = \int_{(t,\infty)} w(s) ds$, and Theorem 2 yields

$$H_{A}(p,1) \approx \sup_{t \geq 0} \left(\int_{[0,t]} W_{1}(s) u(s) ds + \psi(t) \int_{(t,\infty)} \frac{u(s)}{\psi(s)} W_{1}(s) ds \right) V(t)^{-\frac{1}{p}}$$

for $0 < p \leq 1$ and

$$H_{A}(p,1) \approx \left(\int_{[0,\infty)} \left(\int_{[0,t]} W_{1}(s) u(s) ds \right. \right.$$

$$\left. + \psi(t) \int_{(t,\infty)} \frac{u(s)}{\psi(s)} W_{1}(s) ds \right)^{\frac{1}{p}}$$

for $1 < p < \infty$. We have

$$\int_{[0,t]} W_{1}(s) u(s) ds + \psi(t) \int_{(t,\infty)} \frac{u(s)}{\psi(s)} W_{1}(s) ds$$

$$= \int_{[0,t]} \left(\int_{[s,\infty)} \frac{u(y)}{\psi(y)} W_{1}(y) dy\right) d\psi(s)$$

$$= \int_{[0,t]} \left(\int_{[s,\infty)} U(y,s) w(y) dy\right) d\psi(s)$$

and thus obtain the needed criteria.

In the case $q > 1$ we have

$$H_{A}(p,q) \approx \sup_{h \geq 0} \frac{1}{\|h\|_{q',w}} \sup_{g \in G} \frac{\int_{[0,\infty)} \left(\frac{1}{\psi(t)} \int_{[0,t]} gd\psi\right) u(t) \left(\int_{[t,\infty)} w(s) h(s) ds\right) dt}{\left(\int_{[0,\infty)} \left(\frac{1}{\psi(t)} \int_{[0,t]} gd\psi\right)^{p} v(t) dt\right)^{\frac{1}{p}}},$$

We use again Theorem 2 to get

$$H_{A}(p,q) \approx \sup_{h \geq 0} \frac{1}{\|h\|_{q',w}} \sup_{t \geq 0} \left(\int_{[0,t]} u(s) \left(\int_{[s,\infty)} wh\right) ds \right.$$

$$+ \psi(t) \int_{(t,\infty)} \frac{u(s)}{\psi(s)} \left(\int_{[s,\infty)} wh\right) ds \right) V(t)^{-\frac{1}{p}}.$$
Thus, we have

\[ H_A(p, q) \approx \sup_{h \geq 0} \frac{1}{\|h\|_{q', w}} \left( \int_{[0, \infty)} \left( \int_{[0, t]} u(s) \left( \int_{[s, \infty)} wh \right) ds \right. \right. \\
\left. + \psi(t) \int_{[t, \infty)} \frac{u(s)}{\psi(s)} \left( \int_{[s, \infty)} wh \right) ds \right)^{p'} \left( \frac{\psi(t)^p}{\psi(t)^p} \right)^{\frac{1}{p'}} \]

for \( 1 < p < \infty \). Therefore, for \( 0 < p \leq 1 \) we have

\[ H_A(p, q) \approx \sup_{h \geq 0} \frac{\|V(t)^{-\frac{1}{p}} \int_{[0, t]} \left( \int_{[s, \infty)} U(y, s)w(y)h(y)dy \right) d\psi(s)\|_\infty}{\|h\|_{q', w}}, \]

and for \( 1 < p < \infty \) we have

\[ H_A(p, q) \approx \sup_{h \geq 0} \frac{1}{\|h\|_{q', w}} \left( \int_{[0, \infty)} \left( \int_{[0, t]} \left( \int_{[s, \infty)} U(y, s)w(y)h(y)dy \right) d\psi(s) \right)^{p'} \left( \frac{\psi(t)^p}{\psi(t)^p} \right)^{\frac{1}{p'}} \right)^{\frac{1}{p'}}. \]

and applying again criteria for the weighted Hardy and Hardy-type inequalities we can finish as in the proof of Theorem 4.

**Remark 2** (another characterization of (15) via Theorem 4). To characterize the inequality (3) for the operator (5) we can reduce it to one for the operator (4) and then use the result of Theorem 4. We need to describe the criteria for

\[ H_B(p, q) = \sup_{f \in \Omega_v} \frac{\left( \int_{[0, \infty)} f u \right)^q \left( \int_{[0, \infty)} f t \right)^\frac{1}{q}}{\left( \int_{[0, \infty)} f v \right)^\frac{1}{q}}. \]

We have

\[ \int_{[t, \infty)} f(s)u(s)ds = \int_{[0, \frac{1}{t}]} \frac{f(\frac{1}{s})}{s} \cdot \frac{u(\frac{1}{s})}{s} ds =: \int_{[0, \frac{1}{t}]} \tilde{f}(s)\tilde{u}(s)ds. \]

Observe that \( f \in \Omega_v \Leftrightarrow \tilde{f} \in \Omega_v \) for \( \tilde{f}(s) = \frac{1}{s} f \left( \frac{1}{s} \right) \). Then

\[ \int_{[0, \infty)} \left( \int_{[t, \infty)} f u \right)^q w(t)dt = \int_{[0, \infty)} \left( \int_{[0, \frac{1}{t}]} \tilde{f} \tilde{u} \right)^q \tilde{w}(t)dt \]

\[ = \int_{[0, \infty)} \left( \int_{[0, \frac{1}{t}]} \tilde{f} \tilde{u} \right)^q \tilde{w}(t)dt, \]

\[ \int_{[0, \infty)} \int_{[0, \infty)} \tilde{f} \tilde{u} \tilde{w}(t)dt = \int_{[0, \infty)} \left[ \frac{1}{s} f \left( \frac{1}{s} \right) \right]^p \cdot s^{p-2} v \left( \frac{1}{s} \right) ds =: \int_{[0, \infty)} \tilde{f}^{p}\tilde{v}, \]

Thus,

\[ H_B(p, q) = H_A(p, q)[\tilde{u}, \tilde{v}, \tilde{w}], \]
where

\( \tilde{u}(s) = \frac{1}{s} u \left( \frac{1}{s} \right), \quad (20) \)

\( \tilde{w}(s) = \frac{1}{s^2} w \left( \frac{1}{s} \right), \quad (21) \)

\( \tilde{v}(s) = s^{p-2} v \left( \frac{1}{s} \right), \quad (22) \)

In particular, for \( \psi(t) = t \) we have

\[ \tilde{U}(t, y) \approx \int_{\left[ \frac{1}{t}, \frac{1}{s} \right]} u(s) ds, \quad \tilde{V}(t) = t^p V \left( \frac{1}{t} \right), \quad \tilde{V}(t) \approx t^{-p(p+1)} V \left( \frac{1}{t} \right). \]

Thus, the finiteness of \( H_B(p, q) \) is equivalent to the finiteness of the following constants:

(i) \( \tilde{A}_1 < \infty \), if \( q = 1, 0 < p \leq 1 \), where

\[ \tilde{A}_1 := \sup_{t \geq 0} \left( \int_{[0, t]} \left( \int_{[s, \frac{1}{s}]} u \left( \frac{1}{s} \right) d \left( \frac{1}{s} \right) \right) w(s) ds \right) \frac{1}{t} V \left( \frac{1}{t} \right)^{-\frac{1}{p}}; \]

(ii) \( \tilde{A}_2 < \infty \), if \( q = 1, 1 < p < \infty \), where

\[ \tilde{A}_2 := \left( \int_{[0, \infty]} \left( \int_{[0, t]} \left( \int_{[y, \frac{1}{y}]} u \left( \frac{1}{y} \right) d \left( \frac{1}{y} \right) \right) w(s) ds \right) dz \right)^{p'} \]

\[ \times V \left( \frac{1}{t} \right)^{\frac{1}{p'}} \left( \int_{[t, \infty]} U_1(s) ds \right)^{\frac{1}{q}}; \]

(iii) \( \tilde{A}_3 < \infty \), if \( q > 1, 0 < p \leq 1 \), where

\[ \tilde{A}_3 := \sup_{t \geq 0} \left( \int_{[0, \infty]} \left( \int_{[z, \frac{1}{z}]} u(z) dz \right)^q W(s) ds \right)^{\frac{1}{q}} \frac{1}{t} V \left( \frac{1}{t} \right)^{-\frac{1}{p}}; \]

(iv) \( \tilde{A}_{4,1} + \tilde{A}_{4,2} + \tilde{A}_{4,3} + \tilde{A}_{4,4} < \infty \), if \( 1 < p \leq q < \infty \), where

\[ \tilde{A}_{4,1} := \sup_{t \geq 0} \left( \int_{[0, t]} V(s) ds \right)^{\frac{1}{p'}} \left( \int_{[t, \infty]} U_1(s) s^p w(s) ds \right)^{\frac{1}{q}}; \]
\[ \tilde{A}_{4,2} := \sup_{t \geq 0} \left( \int_{[\frac{1}{t}, \infty)} \left( \int_{[s,t]} u \left( \frac{1}{y} \right) d \left( \frac{1}{y} \right) \right)^{p'} \mathcal{V}(s) ds^{p} \right)^{\frac{1}{p}} \times \left( \int_{[0, \frac{1}{t}]} w(s) ds \right)^{\frac{1}{q}} , \]

\[ \tilde{A}_{4,3} := \sup_{t \geq 0} \left( \int_{[0, \frac{1}{t}]} \left( \int_{[t,y]} u \left( \frac{1}{s} \right) d \left( \frac{1}{s} \right) \right)^{q} w(y) dy \right)^{\frac{1}{q'}} \times \left( \int_{[\frac{1}{t}, \infty)} \mathcal{V}(s) ds^{p} \right)^{\frac{1}{p}}, \]

\[ \tilde{A}_{4,4} := \sup_{t \geq 0} \left( \int_{[t, \infty)} \mathcal{V}(s) U_{1}(s)^{p' ds^{p}} \right)^{\frac{1}{p}} \left( \int_{[0, t]} w(s) ds \right)^{\frac{1}{q'}} ; \]

(v) \[ \tilde{A}_{5,1} + \tilde{A}_{5,2} + \tilde{A}_{5,3} + \tilde{A}_{5,4} < \infty, \text{if } 1 < q < p < \infty, \] where

\[ \tilde{A}_{5,1} := \left( \int_{[t, \infty)} \left( \int_{[s, \infty)} U_{1}(s)^{q} w(s) ds \right)^{\frac{1}{q}} \left( \int_{[0, x]} \mathcal{V}(s) ds^{p} \right)^{\frac{1}{p'}} \times \mathcal{V}(x) dx^{p'} \right)^{\frac{1}{q}}, \]

\[ \tilde{A}_{5,2} := \left( \int_{[t, \infty)} \left( \int_{[s, \infty)} \left( \int_{[0, \frac{1}{t}]} u(y) dy \right)^{p'} \mathcal{V}(s) ds^{p} \right)^{\frac{1}{p'}} \times \left( \int_{[0, \frac{1}{t}]} w(s) ds \right)^{\frac{1}{q}} w(t) dt \right)^{\frac{1}{q'}}, \]

\[ \tilde{A}_{5,3} := \left( \int_{[t, \infty)} \left( \int_{[0, t]} \left( \int_{[s, \frac{1}{t}]} u(s) ds \right)^{q} w(y) dy \right)^{\frac{1}{q}} \times \left( \int_{[\frac{1}{t}, \infty)} \mathcal{V}(s) ds^{p} \right)^{\frac{1}{p'}} \mathcal{V}(t) dt^{p} \right)^{\frac{1}{p}}, \]

\[ \tilde{A}_{5,4} := \left( \int_{[t, \infty)} \left( \int_{[0, \infty]} \left( \int_{[s, \infty)} \mathcal{V}(s) ds^{p} \right)^{\frac{1}{p'}} \mathcal{V}(t) dt^{p} \right)^{\frac{1}{p'}} \times \mathcal{V}(x) U_{1}(x)^{p' dx^{p'}} \right)^{\frac{1}{p}} . \]
Remark 3 (sufficient conditions for remaining range of parameters). In the last part of the paper we study the cases $0 < p \leq q < 1$, $0 < q < p \leq 1$, $0 < q < 1 < p < \infty$ and find the sufficient conditions for the inequality (8) to hold for $f(t) \in \Omega_{\psi}$. As in Theorem 4, we denote the least possible constant in (8) by $H_A(p, q)$ and obtain (10). We note that

$$
\left( \int_{[0, \infty)} g^p v \right)^{\frac{1}{p}} \leq \left( \int_{[0, \infty)} \left( \frac{1}{\psi(t)} \int_{[0, t]} g d\psi \right)^p v(t) dt \right)^{\frac{1}{p}}
$$

for any non-increasing function $g(t)$. Hence, the validity of the inequality (23)

$$
\left( \int_{[0, \infty)} \left( \int_{[0, t]} u(s) ds \right)^q w(t) dt \right)^{\frac{1}{q}} \leq C_1 \left( \int_{[0, \infty)} g^p v \right)^{\frac{1}{p}}
$$

for any non-increasing function $g$ is sufficient for the validity of the inequality (8). It is easy to see that (23) is equivalent to the inequality (24)

$$
\left( \int_{[0, \infty)} \left( \int_{[0, t]} U(t, y) g(y) d\psi(y) \right)^q w(t) dt \right)^{\frac{1}{q}} \leq C_1 \left( \int_{[0, \infty)} g^p v \right)^{\frac{1}{p}},
$$

where

$$
U(x, y) = \int_{[y, x]} \frac{u(s)}{\psi(s)} ds.
$$

We use [11, Theorem 5.7] to estimate the constant $C_1$ in (24) for the range of parameters $p$ and $q$ mentioned above. Hence, we obtain the following result for (8):

**Theorem 6.** Let $p > 0$, $0 < q < 1$. For the inequality (8) to hold for all functions $f \in \Omega_{\psi}$ it is sufficient that $C_1 < \infty$, where $C_1$ is defined as follows:

(i) if $0 < p \leq q < 1$, then

$$
C_1 = \sup_{x \geq 0} \left( \int_{[0, \infty)} U^q(\min(x, y)) w(y) dy \right)^{\frac{1}{q}} V(x)^{-\frac{1}{r}},
$$

(ii) if $0 < q < p \leq 1$, then

$$
C_1 \approx \left( \sup_{(x_k)} \sum_{k \in \mathbb{Z}} \left( \int_{x_k}^{x_{k+1}} (U(x_k) + \psi(x_k) U(y, x_k))^q w(y) dy \right)^{\frac{1}{q}} V(x_k)^{-\frac{1}{p}} \right)^{\frac{1}{p}},
$$

(iii) if $0 < q < 1 < p < \infty$, then

$$
C_1 \approx C_{1,1} + C_{1,2} + C_{1,3},
$$

where

$$
C_{1,1} := \left( \int_{[0, \infty)} \left( \int_{[0, x]} U(y)^q w(y) dy \right)^{\frac{1}{q}} U(x)^{q} w(x) V(x)^{-\frac{1}{p}} dx \right)^{\frac{1}{p}},
$$
\begin{align*}
C_{1,2} &:= \left( \sup_{\{x_k\}} \sum_{k \in \mathbb{Z}} \left( \int_{x_k}^{x_{k+1}} w \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \\
& \quad \times \left( \int_{x_k}^{x_{k-1}} \left( U(y) + \psi(y)U(x_k, y) \right)^{p'} V(y)^{-p'} v(y) dy \right)^{\frac{1}{p'}} .
\end{align*}

\begin{align*}
C_{1,3} &:= \left( \sup_{\{x_k\}} \sum_{k \in \mathbb{Z}} \left( \int_{x_k}^{x_{k+1}} \left( \int_{[x_k,y]} \frac{u(s)}{\psi(s)} ds \right)^q w(y) dy \right)^{\frac{1}{q}} \right)^{\frac{1}{q'}} \\
& \quad \times \left( \int_{x_k}^{x_{k-1}} U(y)^{p'} V(y)^{-p'} v(y) dy \right)^{\frac{1}{p'}} .
\end{align*}

Note that the case (i) is valid for the bigger range $0 < p \leq 1$, $p \leq q < \infty$. However, the sufficient condition for $0 < p \leq 1$, $q \geq 1$ is of no interest to us here, as we have obtained the necessary and sufficient conditions for this range of parameters in Section 3.

\textbf{References}


Paper D
ON THE REDUCTION PRINCIPLE FOR WEIGHTED INEQUALITIES ON THE CONE OF QUASI-CONCAVE FUNCTIONS AND APPLICATIONS

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Abstract. We obtain necessary and sufficient conditions for a new Hardy-type inequality to hold on the cone of \( \psi \)–quasi-concave functions for \( q \geq 1, p > 0 \) and we also establish the criteria of boundedness for the Hilbert transform and the Riesz potentials between the Lorentz \( \Gamma \)–spaces for the same range of parameters.

2010 Mathematics Subject Classification. 26D10, 26D15, 26D07, 44A05, 45D99.

Key words and phrases. Hardy-type inequalities, monotone functions, quasi-concave functions, Lorentz spaces, the Hilbert transform, Riesz potentials.

Note. This report will be submitted for publication elsewhere.
1. Introduction

In this paper we establish the criteria for the boundedness of two classical operators – the Hilbert transform and the Riesz potentials – between Lorentz $\Gamma$-spaces. We also obtain necessary and sufficient conditions for a new Hardy-type inequality to hold on the cone of $\psi$-quasi-concave functions.

The spaces $\Lambda_p(v)$ were first introduced by G.G. Lorentz in [20]. They were defined in terms of decreasing rearrangement

$$f^*(t) := \inf\{s > 0 : \text{mes}\{x : |f(x)| > s\} \leq t\}$$

of a function $f$ on the semiaxis as follows:

$$\Lambda_p(v) = \{f \text{ measurable on } \mathbb{R}^n : \left(\int_{[0,\infty)} f^*(x)^p v(x) dx\right)^\frac{1}{p} < \infty\}.$$ 

The fact that the spaces $\Lambda_p(v)$ have been extensively studied since then is explained by the crucial role they play in various branches of analysis and its applications. For instance, in Interpolation theory they appear as intermediate spaces for $L^p-$spaces (see e.g. [5], [7]). The study of the spaces $\Lambda_p(v)$ began with a special case of the power functions $v$ and only in 1990 M. Arinio and B. Muckenhoupt [1] succeeded in characterizing $\Lambda_p(v) \hookrightarrow \Lambda_q(w)$ property of the Hardy-Littlewood maximal operator $Mf$ in the case $1 < p < \infty$. In general, it took about 15 years to find precise characterization of $\Lambda_p(v) \hookrightarrow \Lambda_q(w)$ mapping properties of the maximal operator for all $0 < p, q < \infty$ and arbitrary weight functions $v$ and $w$. The connection between the maximal operator and Lorentz $\Gamma$-spaces is a well-known relation $$(Mf)^* (x) \approx \frac{1}{x} \int_{[0,x]} f^*(t) dt =: f^{**}(x)$$ (see e.g. [2] for some simple proofs and historical facts concerning this estimate).

The Lorentz $\Gamma$-spaces were introduced by E.T. Sawyer [27] while working on the problem of characterizing the boundedness of the maximal operator between the $\Lambda$-spaces. More exactly,

$$\Gamma_p(v) = \{f \text{ measurable on } \mathbb{R}^n : \left(\int_{[0,\infty)} f^{**}(x)^p v(x) dx\right)^\frac{1}{p} < \infty\}.$$ 

Similar to $\Lambda-$analysis, describing the mapping properties of classical operators in the Lorentz $\Gamma$-spaces became a challenging task of numerous investigations of the weighted inequalities on the cones of monotone functions (see, for example, [1], [3], [8], [9], [10], [13], [16], [17], [21], [25], [27], [29], [30], [31], [32], [33], [34], [35]). See also the books [18, Ch.10] and [19, Ch.6] and the references given there.

The first part of the present paper serves as an extension of the paper [26]. The initial motivation was to find the integral criteria of $\Gamma_p(v) \hookrightarrow \Gamma_q(w)$ boundedness for the Hardy–Littlewood maximal operator. This problem was first studied in [34] in the diagonal case.
\[ 1 < p = q < \infty, \, v = w. \] The next result was obtained in the paper [14] with answers in terms of implicit sequences as well as in the paper [15]. G. Sinnamon [28] gave integral criteria for the case \( 1 < p, q < \infty \) using a reduction principle for the inequalities on the cone of quasi-concave functions, and in [26] we extended Sinnamon’s result to cover also the case \( 0 < p < 1 \) and, moreover, gave an alternative approach for the other cases.

In particular, in [26] we studied the Hardy-type inequality, which characterizes the boundedness of the operator that is adjoint to the Hardy–Littlewood maximal operator. We obtained criteria for the inequality

\[ \left( \int_{[0, \infty)} (Af)^q \, d\gamma \right)^{\frac{1}{q}} \leq C \left( \int_{[0, \infty)} f^p \, d\beta \right)^{\frac{1}{p}}, \]

where

\[ Af(t) = \left( \int_{(t, \infty)} f^p \, d\mu \right)^{\frac{1}{p}}, \]

\( q \geq 1, \, p > 0, \) to hold on the cone of quasi-concave functions, i.e. such functions \( u(t) \) that \( u(t) \) is non-decreasing and \( \frac{u(t)}{t} \) is non-increasing. It has been proved that such functions are equivalent to concave functions, that is why they are called quasi-concave. The results concerning quasi-concave functions can be found in papers [6], [12], [15], [22], [28], and some historical remarks along with new regularization results and applications in the paper [24].

It is important to note the crucial role quasi-concave functions play in Harmonic Analysis, interpolation theory, approximation theory, homogenization theory and some other areas of mathematics. Among the central objects of the analysis that have quasi-concave property are the Peetre \( K \)-functional, the integral modulus of continuity \( \omega_p(f, x) \), the fundamental function and, finally, the averaging operator \( \frac{1}{t} \int_0^t f^*(s) \, ds \), which, as we have shown above, is used in building the Lorentz \( \Gamma \)-spaces.

Let \( \psi \) be a continuous strictly increasing function on \( \mathbb{R}_+ \) such that \( \psi(0) = 0 \) and \( \lim_{t \to -\infty} \psi(t) = \infty. \) Such a function is called admissible. Then the function \( f(t) \) is called a \( \psi \)-quasi-concave function, if \( f(t) \) is non-decreasing and \( \frac{t \psi(t)}{\psi(t)} \) is non-increasing. In the present paper we extend the criteria obtained for (1) to the cone of \( \psi \)-quasi-concave functions. For our convenience we characterize (1) on the cone of functions \( \Omega_\psi \) consisting of all measurable functions \( f: \mathbb{R}_+ \to [0, +\infty] \) such that \( f(t) \) is non-increasing and \( \psi(t)f(t) \) is non-decreasing, which means that functions of the form \( \psi(t)f(t) \) are \( \psi \)-quasi-concave.
The second section of this paper is devoted to investigating the mapping properties of the Hilbert transform

\[ H_f(x) := \lim_{\epsilon \to 0} \int_{\epsilon < |y|} \frac{f(x-y)}{y} dy, \]

which arises from the study of boundary values of the real and imaginary parts of analytic functions. The Hilbert transform plays the decisive role in questions of norm-convergence of Fourier series and Fourier integrals. We obtain necessary and sufficient conditions for this operator to be bounded from \( \Gamma^p(v) \) to \( \Gamma^q(w) \) for the range of parameters \( q \geq 1, p > 0 \).

In the last section we solve the same problem for the Riesz potentials

\[ I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad 0 < \alpha < n, \]

which play a crucial role in describing the embedding of the Sobolev spaces and also in several applications outside the mathematical sciences.

Throughout the paper expressions of the type \( 0 \cdot \infty \) are taken to be equal to 0. The relation \( A \ll B \) means that \( A \leq cB \) with a constant \( c \) depending only on the parameter of summation. We write \( A \approx B \) instead of \( A \ll cB \ll A \). We set \( p' := \frac{p}{p-1} \) for \( 0 < p < \infty, p \neq 1 \) and use the notation \( \frac{1}{r} := \frac{1}{q} - \frac{1}{p} \) for \( 0 < q < p < \infty \).

\[ L^p(\lambda) \] denotes the set of all \( \lambda \)-measurable functions \( f \) on \( \mathbb{R}_+ \) such that

\[ \|f\|_{p,\lambda} := \left( \int_{[0,\infty)} |f|^p d\lambda \right)^{\frac{1}{p}} < \infty. \]

The constant \( C \) may be different at different occurrences.

2. On the Hardy-type inequality on the cone of generalized quasi-concave functions

In the paper [28] G. Sinnamon obtained necessary and sufficient conditions for the cone of quasi-concave functions in \( L^p_v \) to be embedded in \( L^q_u \) when \( 0 < q < p < \infty \). In particular, he found the estimate for the embedding norm \( L^p_v \hookrightarrow L^q_u \) in integral form. In this paper we extend his technique to derive such estimate for the cone of \( \psi \)-quasi-concave functions and also include the case when \( 0 < p \leq q < \infty \).

For admissible function \( \psi(t) \) we introduce the class of functions \( \Omega_{\alpha,\beta}^\psi \), consisting of the functions \( f(t) \in \mathfrak{M}^+ \), such that \( \psi(t)^\alpha f(t) \) is non-decreasing and \( \psi(t)^{-\beta} f(t) \) is non-increasing. In particular, the class \( \Omega_{0,1}^\psi \) consists of functions \( f(t) \), such that \( f(t) \) is non-decreasing and \( \frac{f(t)}{\psi(t)} \) is non-increasing.

We also define the operators

\[ H_{\psi,\alpha} h(x) := \psi(x)^{-\alpha} \int_{[0,x]} \psi(t)^\alpha h(t) dt \]
and
\[ H^{\psi,\beta}h(x) := \psi(x)^\beta \int_{[x,\infty)} \psi(t)^{-\beta} h(t) dt. \]

For \( \alpha + \beta > 0 \) we also use the operator
\[ H^{\psi,\beta}_{\psi,\alpha} h(x) = H_{\psi,\alpha} h(x) + H^{\psi,\beta} h(x), \]
which can be rewritten as
\[ H^{\psi,\beta}_{\psi,\alpha} h(x) = \int_{[0,\infty)} \min \left( \frac{(\psi(t))^{\alpha}}{\psi(x)}, \frac{(\psi(x))^{\beta}}{\psi(t)} \right) h(t) dt. \]

It is easy to see that \( \psi(x)^\alpha H^{\psi,\beta}_{\psi,\alpha} h(x) \) is non-decreasing and \( \psi(x)^{-\beta} H^{\psi,\beta}_{\psi,\alpha} h(x) \) is non-increasing for any function \( h(t) \in \mathfrak{M}^+ \). Thus, \( H^{\psi,\beta}_{\psi,\alpha} \mathfrak{M}^+ \subseteq \Omega^\psi_{\alpha,\beta} \).

It is also easy to check that
\[ \int_{[0,\infty)} \left( H^{\psi,\beta}_{\psi,\alpha} h_1 \right) h_2 = \int_{[0,\infty)} h_1 \left( H^{\psi,\alpha}_{\psi,\beta} h_2 \right). \]

The following theorem characterizes the embedding \( L^p_v \rightarrow L^q_u \) on \( \Omega^\psi_{0,1} \) for \( 0 < p, q < \infty \).

**Theorem 1.** (i) If \( 0 < q < p < \infty, \ u, v \in \mathfrak{M}^+ \), then
\[ \left( \frac{1}{q} \right)^\frac{1}{p} \left( \int_{[0,\infty)} \left( H^{\psi,\beta}_{\psi,\alpha} h_1 \right) h_2 \right) \leq \sup_{f \in \Omega^\psi_{0,1}} \left( \frac{\| f \|_{q,u}}{\| f \|_{p,v}} \right)^\frac{1}{q}. \]

(ii) If \( 0 < p \leq q < \infty, \ u, v \in \mathfrak{M}^+ \), then
\[ \sup_{f \in \Omega^\psi_{0,1}} \left( \frac{\| f \|_{q,u}}{\| f \|_{p,v}} \right)^{\frac{1}{q}} \leq \sup_{t > 0} \left( H^{\psi,0}_{\psi,\beta} \right)^\frac{1}{q}. \]

**Proof.** (i) The first part of the theorem is proved in several steps:
1. First we obtain the norm of the embedding \( L^1_v \rightarrow L^q_u \) for the functions from \( H^{\psi,0}_{\psi,\beta} \mathfrak{M}^+ \).
2. Then we extend the result of step 1 to be valid on \( \Omega^\psi_{0,1} \).
3. Finally, we obtain the norm of the embedding \( L^p_v \rightarrow L^q_u \) on \( \Omega^\psi_{0,1} \).

**Step 1.** For \( 0 < q < 1 \) we need to show that
\[ \sup_{f \in H^{\psi,0}_{\psi,\beta} \mathfrak{M}^+} \left( \frac{\| f \|_{q,u}}{\| f \|_{1,v}} \right)^\frac{1}{q} \leq \left( \int_{[0,\infty)} \left( H^{\psi,0}_{\psi,\beta} h \right)^q u \right)^\frac{1}{q}. \]

We denote the left part by \( C \). Then we have
\[ \left( \int_{[0,\infty)} \left( H^{\psi,0}_{\psi,\beta} h \right)^q u \right)^\frac{1}{q} \leq C \left( \int_{[0,\infty)} \left( H^{\psi,0}_{\psi,\beta} h \right)^p u \right)^\frac{1}{p}. \]
Since $\int_{[0,\infty)} \left( H_{\psi,0}^{\psi,1} \right)^{\alpha} \, d\tau = \int_{[0,\infty)} h \left( H_{\psi,1}^{\psi,0} \right)^{\alpha}$, the inequality can be rewritten as follows:

\[
\left( \int_{[0,\infty)} \left( \int_{[0,x]} h(t) \, dt + \psi(x) \int_{[x,\infty)} \frac{h(t)}{\psi(t)} \, dt \right)^{q} u(x) \, dx \right)^{\frac{1}{q}} \leq C \int_{[0,\infty)} h(t) H_{\psi,1}^{\psi,0} v(t) \, dt.
\]

By the Minkowsky inequality, $C \approx C_1 + C_2$, where $C_1$ and $C_2$ are the least constants for the following inequalities for $h \in \mathcal{M}^+$:

\[
\left( \int_{[0,\infty)} \left( \int_{[0,x]} h(t) \, dt \right)^{q} u(x) \, dx \right)^{\frac{1}{q}} \leq C_1 \int_{[0,\infty)} h(t) H_{\psi,1}^{\psi,0} v(t) \, dt,
\]

\[
\left( \int_{[0,\infty)} \left( \psi(x) \int_{[x,\infty)} \frac{h(t)}{\psi(t)} \, dt \right)^{q} u(x) \, dx \right)^{\frac{1}{q}} \leq C_2 \int_{[0,\infty)} h(t) H_{\psi,1}^{\psi,0} v(t) \, dt.
\]

It is easy to check that $H_{\psi,1}^{\psi,0} v$ is a non-increasing function. Using [33, Theorem 3.3] with $V = H_{\psi,1}^{\psi,0} v$ and $U = u$, we obtain

\[
C_1 \approx \left( \int_{[0,\infty)} \left( H_{\psi,1}^{\psi,0} v \right)^{\frac{q}{q-1}} \left( H_{\psi,0}^{\psi,0} u \right)^{\frac{q}{q-1}} u \right)^{\frac{1}{q}}.
\]

For estimating $C_2$ we replace $\frac{h(t)}{\psi(t)}$ by $h(t)$, and the second part of [33, Theorem 3.3] with non-decreasing function $V(t) = \psi(t) H_{\psi,0}^{\psi,0} v(t)$ and $U(t) = \psi(t)^q u(t)$ yields

\[
C_2 \approx \left( \int_{[0,\infty)} \left( H_{\psi,1}^{\psi,0} v \right)^{\frac{q}{q-1}} \left( H_{\psi,0}^{\psi,0} u \right)^{\frac{q}{q-1}} u \right)^{\frac{1}{q}}.
\]

Since $C \approx C_1 + C_2$, we obtain the needed estimate.

**Step 2.** Let $f$ be a function from $\Omega_{\psi,1}$. It was shown in [5, Proposition 2.5.10] that for a quasi-concave function $\phi$ there exists the least concave majorant $\tilde{\phi}$ and that it satisfies the estimate $1/2 \tilde{\phi} \leq \phi \leq \tilde{\phi}$. The same estimate holds for the case of $\psi$—quasi-concave functions. More precisely, we can write out the function $F(t) = \frac{f(t)}{\psi(t)} \left( \psi(1) + \psi(t) \right)$ that dominates the $\psi$—quasi-concave function $f(t)$. Since $F$ is quasi-concave, it is dominated by the concave function $2f(1) \left[ 1 + F^{-1} \left( \frac{f(1)}{\psi(1)} \left( \psi(1) + \psi(t) \right) \right) \right]$. Thus, there exists the smallest concave function $\tilde{f}$ defined as the pointwise infimum of all dominating concave functions, which dominates $f$. In order to prove that $H_{\psi,0}^{\psi,1} \mathcal{M}^+$ is dense in $\Omega_{\psi,1}$, we show that $\tilde{f}$ is a pointwise limit of an increasing sequence of functions in $H_{\psi,0}^{\psi,1} \mathcal{M}^+$.

Let $a = \lim_{x \to -\infty} f(x)$, $b = \lim_{x \to \infty} \frac{f(x)}{\psi(x)}$. Since $\tilde{f}$ is non-negative and concave, these limits exist and are non-negative. Then we write $\tilde{f} = a+$
to the cone \( \Omega \rightarrow \infty \), we get.

Theorem, the least concave majorant \( \tilde{h} \) is a sequence from \( \Omega \rightarrow \infty \). If we take \( h_n(t) := b\psi(t)\chi_{n,n+1}(t) \), then \( H_{\psi,0}^{0,1}h_n(x) \) converges to \( b\psi(x) \) as \( n \rightarrow \infty \). Now we show that \( g(x) \) is also a pointwise limit of functions in \( H_{\psi,0}^{0,1} \). We have \( g(x) = \int_{[0,\infty]} g(t)dt \). Now, setting

\[
h_n(t) := \frac{\psi(t)}{t \ln \left( \frac{n+1}{n} \right)} \left( \frac{g'(t)}{\psi(t)} - \frac{g'(\frac{(n+1)t}{n})}{\psi'(\frac{(n+1)t}{n})} \right) dt,
\]

we get

\[
\int_{[y,\infty]} \frac{h_n(t)}{\psi(t)} dt = \left( \int_{y}^{n+1/n} \frac{g'(t)}{\psi'(t)} dt \right) / \left( \int_{y}^{n+1/n} \frac{dt}{t} \right).
\]

The sequence \( \int_{[y,\infty]} \frac{h_n(t)}{\psi(t)} dt \) converges to \( \frac{g'(y)}{\psi'(y)} \) for almost every \( y \), when \( n \rightarrow \infty \). This implies that

\[
H_{\psi,0}^{0,1}h_n(x) = \int_{[0,\infty]} \left( \frac{h_n(t)}{\psi(t)} dt \right) d\psi(y)
\]

converges to \( \int_{[0,\infty]} g'(y)dy = g(x) \), when \( n \rightarrow \infty \).

We use the observation above to extend the result of the first step to the cone \( \Omega_{0,1} \). On the one hand, \( H_{\psi,0}^{0,1} \subseteq \Omega_{0,1} \). On the other hand, if \( f_n \) is a sequence from \( H_{\psi,0}^{0,1} \), which converges pointwise to the least concave majorant \( \tilde{f} \) of \( f \), then, by the Monotone Convergence Theorem,

\[
\|f\|_{q,u} \leq \|\tilde{f}\|_{q,u} = \lim_{n \rightarrow \infty} \|f_n\|_{q,u} \approx \lim_{n \rightarrow \infty} \|f_n\|_{1,v} = \|\tilde{f}\|_{1,v} \leq 2\|f\|_{1,v}.
\]

Thus, for \( 0 < q < 1 \) we have

\[
\sup_{f \in \Omega_{0,1}} \frac{\|f\|_{q,u}}{\|f\|_{1,v}} \approx \left( \int_{[0,\infty]} \left( H_{\psi,0}^{0,1} \right)^{\frac{q}{n}} \left( H_{\psi,0}^{0,1} \right)^{\frac{q}{n}} u \right)^{\frac{1-q}{n}}.
\]

Step 3. To extend this result to the case \( 0 < q < p < \infty \) we make the following observation:

\[
f \in \Omega_{0,1} \iff g \in \Omega_{0,1}^p
\]

where \( f(x)^p = g(x)^p \) and \( \psi(x)^p = \phi(x)^p \). Then for \( 0 < p, q < \infty \) we have

\[
\sup_{f \in \Omega_{0,1}} \frac{\|f\|_{q,u}}{\|f\|_{p,v}} = \left( \sup_{g \in \Omega_{0,1}^p} \frac{\|g\|_{q,p,v}}{\|g\|_{1,v}} \right)^{\frac{1}{p}},
\]

where

\[
U(x^{p})d(x^{p}) = u(x)dx, \quad V(x^{p})d(x^{p}) = v(x)dx.
\]
Thus, for $0 < q < p < \infty$ we obtain
\[
\sup_{f \in \Omega_{0,1}^\psi} \frac{\|f\|_{q,u}}{\|f\|_{p,v}} \approx \left( \int_{[0,\infty)} \left( H_{\phi,1}^{\psi,0} V \right)^{-\frac{q}{p}} \left( H_{\phi,q/p}^{\psi,0} U \right)^{\frac{q}{p}} u(t) dt \right)^{\frac{1}{q}}.
\]

Making the substitution $t \to t^p$, we have
\[
\sup_{f \in \Omega_{0,1}^\psi} \frac{\|f\|_{q,u}}{\|f\|_{p,v}} \approx \left( \int_{[0,\infty)} \left( H_{\phi,1}^{\psi,0} V(t^p) \right)^{-\frac{q}{p}} \left( H_{\phi,q/p}^{\psi,0} U(t^p) \right)^{\frac{q}{p}} u(t) dt \right)^{\frac{1}{q}}.
\]

Next we make the substitution $x \to x^p$ to get
\[
H_{\phi,1}^{\psi,0} V(t^p) = \int_{[0,\infty)} \min \left( \frac{\phi(x)}{\phi(t^p)}, 1 \right) V(x) dx = \int_{[0,\infty)} \left( \frac{\phi(x^p)}{\phi(t^p)}, 1 \right) v(x) dx = H_{\psi,0}^{\psi,p} v(t).
\]

Similarly, we have
\[
H_{\phi,q/p}^{\psi,0} U(t^p) = H_{\psi,q}^{\psi,p} u(t).
\]

Hence, we obtain the needed estimate
\[
\sup_{f \in \Omega_{0,1}^\psi} \frac{\|f\|_{q,u}}{\|f\|_{p,v}} \approx \left( \int_{[0,\infty)} \left( H_{\psi,0}^{\psi,p} V \right)^{-\frac{q}{p}} \left( H_{\psi,q}^{\psi,p} U \right)^{\frac{q}{p}} u(t) dt \right)^{\frac{1}{q}}.
\]

(ii) We can represent any function $f(t) \in \Omega_{0,1}^\psi$ as the integral $f(t) = \int_{[0,t]} g d\psi(s)$, where function $g(t)$ is non-increasing. Therefore, we have
\[
\sup_{f \in \Omega_{0,1}^\psi} \frac{\|f\|_{q,u}}{\|f\|_{p,v}} = \sup_{g \in \mathcal{G}_1} \left( \int_{[0,\infty)} \left( \frac{1}{\psi(t)} \int_{[0,t]} g d\psi \right)^q u(t) \psi(t) dt \right)^{\frac{1}{q}}.
\]

and by using [11, Theorem 5.1 (i)] we obtain the result for $0 < p < q < \infty$. 

\[\square\]

**Remark 1.** The expression in the right part of (3) coincides with the constant $A(8)$ from [11, Theorem 5.1, ii], which describes the embedding norm between Lorentz $\Gamma$–spaces. Note, however, that the alternative constant $A(7)$ cannot be obtained by the technique we use to prove Theorem 1.

By using Theorem 1 we extend [26, Theorem 5] to the cone of $\psi$–quasi-concave functions.

**Theorem 2.** Let $q \geq 1, p > 0$ and
\[
W(t) := \int_{[0,t]} w(s) ds,
\]


Then the inequality
\[ A_i := \sup_{x \geq 0} \left( \int_{[0,x]} u(t) dt + \psi(x) \right) \]
holds for all functions \( f \in \Omega_{\psi} \) if and only if
(i) \( A_1 < \infty \), if \( q = 1, 0 < p \leq 1 \), where
\[ A_1 := \sup_{x \geq 0} \left( \int_{[0,x]} u(t) W(t) dt + \psi(x) \right) \left( \int_{[x,\infty]} \frac{u(t)}{\psi(t)} W(t) dt \right) V(x)^{-\frac{1}{p}}; \]
(ii) \( A_2 < \infty \), if \( q = 1, 1 < p < \infty \), where
\[ A_2 := \left( \int_{[0,\infty]} \left( \int_{[0,x]} u(t) W(t) dt \right)^{p'} V_1(x) dx \right)^\frac{1}{p'} \]
\[ + \int_{[0,\infty]} \left( \int_{[x,\infty]} \frac{u(t)}{\psi(t)} W(t) dt \right)^{p'} V_2(x) dx \right)^\frac{1}{p'}; \]
(iii) \( A_3 < \infty \), if \( q > 1, 0 < p \leq 1 \), where
\[ A_3 := \sup_{t \geq 0} \left[ \left( \int_{[0,t]} U(t,s)^{q} w(s) ds \right)^\frac{1}{q} \right. \]
\[ + \left( \int_{[0,\infty]} U_1(s) \right) U_1^{q}(\max(s,t)) w(s) ds \right] V(t)^{-\frac{1}{p}}; \]
(iv) \( A_{4,1} + A_{4,2} + A_{4,3} + A_{4,4} < \infty \), if \( 1 < p \leq q < \infty \), where
\[ A_{4,1} := \sup_{t \geq 0} \left( \int_{[t,\infty]} V_1(x) U(x,t)^{p'} dx \right)^{\frac{1}{p'}} \left( \int_{[0,t]} w(x) dx \right)^{\frac{1}{q}}; \]
\[ A_{4,2} := \sup_{t \geq 0} \left( \int_{[t,\infty]} V_1(x) dx \right)^{\frac{1}{p'}} \left( \int_{[0,t]} U(t,x)^{q} w(x) dx \right)^{\frac{1}{q}}; \]
\[ A_{4,3} := \sup_{t \geq 0} \left( \int_{[t,\infty]} U_1(x)^{q} V_2(x) dx \right)^{\frac{1}{p'}} \left( \int_{[0,t]} w(x) dx \right)^{\frac{1}{q}}; \]
\[ A_{4,4} := \sup_{t \geq 0} \left( \int_{[0,t]} V_2(x) dx \right)^{\frac{1}{p'}} \left( \int_{[t,\infty]} U_1^{q}(x) w(x) dx \right)^{\frac{1}{q}}; \]
(v) $A_{5,1} + A_{5,2} + A_{5,3} + A_{5,4} < \infty$, if $1 < q < p < \infty$, where

$$A_{5,1} = \left( \int_{[0,\infty)} \left( \int_{[t,\infty)} V_1(x) U(x, t)^p \, dx \right)^{\frac{q}{p}} \left( \int_{[0,t]} w(x) \, dx \right)^{\frac{q}{p}} \, w(t) \, dt \right)^{\frac{1}{q}},$$

$$A_{5,2} = \left( \int_{[0,\infty)} \left( \int_{[t,\infty)} V_1(x) \, dx \right)^{\frac{q}{p}} \left( \int_{[0,t]} U(t, x)^q w(x) \, dx \right)^{\frac{q}{p}} \, V_1(t) \, dt \right)^{\frac{1}{q}},$$

$$A_{5,3} = \left( \int_{[0,\infty)} \left( \int_{[0,x]} w(t) \, dt \right)^{\frac{q}{p}} \left( \int_{[x,\infty)} U_1(t)^p V_2(t) \, dt \right)^{\frac{q}{p}} \times U_1(x)^p V_2(x) \, dx \right)^{\frac{1}{q}},$$

$$A_{5,4} = \left( \int_{[0,\infty)} \left( \int_{[x,\infty)} U_1(s)^q w(s) \, ds \right)^{\frac{q}{p}} \left( \int_{[0,x]} V_2(s) \, ds \right)^{\frac{q}{p}} \, V_2(x) \, dx \right)^{\frac{1}{q}}.$$

3. **The Mapping Properties of the Hilbert Transform**

In this section we derive criteria for the boundedness of the Hilbert transform between weighted Lorentz $\Gamma$-spaces.

**Theorem 3.** Let $0 < p < \infty$, $1 \leq q < \infty$ and the weight $v$ satisfy the conditions

$$\int_{[0,\infty)} \frac{v(s) \, ds}{s^p + t^p} < \infty, \quad \int_{[0,1]} \frac{v(s) \, ds}{s^p} = \int_{[1,\infty)} v(s) \, ds = \infty$$

for every $t > 0$. Let

$$V(t) := \int_{[0,t]} v,$$

$$V(t) := V(t) + t^p \int_{[t,\infty)} \frac{v(s) \, ds}{s^p},$$

$$\mathbb{V}(t) := V(t)^{p' - 1} V(t) \left( \int_{[t,\infty)} \frac{v(s) \, ds}{s^p} \right)^{p - 1}.$$

Then the Hilbert transform is bounded from $\Gamma_p(v)$ to $\Gamma_q(w)$ if and only if

(i) $A := \max A_i < \infty$, $i = 1, 2, 3$, if $1 < p < q < \infty$, where

$$A_1 := \sup_{t > 0} \left( \int_{[0,t]} w \right)^{\frac{1}{q}} \left( \int_{[t,\infty)} \mathbb{V} \right)^{\frac{1}{p}},$$

$$A_2 := \sup_{t > 0} \left( \int_{[0,t]} \frac{w(s) \, ds}{s^q} \right)^{\frac{1}{q}} \left( \int_{[t,\infty)} s^p \mathbb{V}(s) \, ds \right)^{\frac{1}{p}},$$

$$A_3 := \sup_{t > 0} \left( \int_{[t,\infty)} \ln^q \left( \frac{s}{t} \right) \frac{w(s) \, ds}{s^q} \right)^{\frac{1}{q}} \left( \int_{[t,\infty)} s^p \mathbb{V}(s) \, ds \right)^{\frac{1}{p}}.$$
\( A_4 := \sup_{t > 0} \left( \int_{(t, \infty)} \frac{w(s)}{s^q} ds \right)^{\frac{1}{q}} \left( \int_{(t, \infty)} s^{p'} \ln s \left( \frac{s}{t} \right) V(s) ds \right)^{\frac{1}{p'}} \),

\( A_3 := \sup_{t > 0} \left( \int_{[0, t]} w \right)^{\frac{1}{q}} \left( \int_{(t, \infty)} \ln s \left( \frac{s}{t} \right) V(s) ds \right)^{\frac{1}{p'}} \),

\( A_6 := \sup_{t > 0} \left( \int_{[0, t]} \ln s \left( \frac{s}{t} \right) w(s) ds \right)^{\frac{1}{q}} \left( \int_{(t, \infty)} V \right)^{\frac{1}{p'}} \).

(ii) \( B := \max B_i < \infty, \ i = \overline{1, 6}, \) if \( 1 < q < p < \infty, \) where

\( B_1 := \left( \int_{[0, \infty)} \left( \int_{[0, t]} w \right)^{\frac{1}{q}} \left( \int_{[t, \infty)} V \right)^{\frac{1}{p'}} w(t) dt \right)^{\frac{1}{q'}} \),

\( B_2 := \left( \int_{[0, \infty)} \left( \int_{[0, t]} \frac{w(s)}{s^q} ds \right)^{\frac{1}{q'}} \left( \int_{[0, t]} s^{p'} V(s) ds \right)^{\frac{1}{p'}} w(t) \left( \frac{t^q}{s^q} \right) dt \right)^{\frac{1}{q'}} \),

\( B_3 := \left( \int_{[0, \infty)} \left( \int_{[0, t]} s^{p'} \ln s \left( \frac{t}{s} \right) V(s) ds \right)^{\frac{1}{p'}} \times \left( \int_{[t, \infty)} \frac{w(s)}{s^q} ds \right)^{\frac{1}{q'}} w(t) \left( \frac{t^q}{s^q} \right) dt \right)^{\frac{1}{q'}} \),

\( B_4 := \left( \int_{[0, \infty)} \left( \int_{[0, t]} s^{p'} \ln s \left( \frac{t}{s} \right) V(s) ds \right)^{\frac{1}{p'}} \times \left( \int_{[t, \infty)} \frac{w(s)}{s^q} ds \right)^{\frac{1}{q'}} t^p V(t) dt \right)^{\frac{1}{q'}} \),

\( B_5 := \left( \int_{[0, \infty)} \left( \int_{[0, t]} w \right)^{\frac{1}{q'}} \left( \int_{[t, \infty)} \ln s \left( \frac{s}{t} \right) V(s) ds \right)^{\frac{1}{p'}} w(t) dt \right)^{\frac{1}{q'}} \),

\( B_6 := \left( \int_{[0, \infty)} \left( \int_{[0, t]} \ln s \left( \frac{s}{t} \right) w(s) ds \right)^{\frac{1}{q'}} \left( \int_{[t, \infty)} V(s) ds \right)^{\frac{1}{p'}} V(t) dt \right)^{\frac{1}{q'}} \).

(iii) \( D := \max D_i < \infty, \ i = \overline{1, 4}, \) if \( 0 < p \leq 1 < q < \infty, \) where

\( D_1 := \sup_{t > 0} \left( \int_{[0, t]} w \right)^{\frac{1}{q'}} V(t)^{-\frac{1}{p'}} \),

\( D_2 := \sup_{t > 0} \left( \int_{[t, \infty)} \frac{w(s)}{s^q} ds \right)^{\frac{1}{q'}} t^q V(t)^{-\frac{1}{p'}} \),

\( D_3 := \sup_{t > 0} \left( \int_{[t, \infty)} \ln s \left( \frac{s}{t} \right) \frac{w(s)}{s^q} ds \right)^{\frac{1}{q'}} t^q V(t)^{-\frac{1}{p'}} \),
\[ D_4 := \sup_{t>0} \left( \int_{[0,t]} \ln^q \left( \frac{t}{s} \right) w(s) ds \right)^{\frac{1}{q}} V(t)^{-\frac{1}{q}}; \]

(iv) If \( 0 < p < 1 = q \), where

\[ F := \sup_{t>0} \left( \frac{1}{t} \int_{[0,t]} \ln \left( \frac{t}{y} \right) w(y) dy + \int_{[t,\infty)} \ln \left( \frac{y}{t} \right) \frac{w(y)}{y} dy \right) \]

\[ + \frac{1}{t} \int_{[0,t]} w(y) dy + \int_{[t,\infty)} \frac{w(y)}{y} dy \right) V(t)^{-\frac{1}{p}}; \]

(v) If \( q = 1 < p < \infty \), where

\[ G := \left( \int_{[0,\infty)} \left( \frac{1}{t} \int_{[0,t]} \ln \left( \frac{t}{y} \right) w(y) dy + \int_{[t,\infty)} \ln \left( \frac{y}{t} \right) \frac{w(y)}{y} dy \right) \]

\[ + \frac{1}{t} \int_{[0,t]} w(y) dy + \int_{[t,\infty)} \frac{w(y)}{y} dy \right) v'(V(t) dt)^{\frac{1}{p}}. \]

**Proof.** We denote

\[ Pf(t) := \frac{1}{t} \int_{[0,t]} f(s) ds, \quad Qf(t) := \int_{[t,\infty)} \frac{f(s)}{s} ds. \]

It is well-known ([5], [27]) that the Hilbert transform satisfies the inequality

\[ (Hf)^*(x) \leq C_1 [Pf^*(x) + Qf^*(x)] \leq C_2 (Hf^*)^*(x). \]

Put

\[ C := \|H\|_{\Gamma_p(v) \rightarrow \Gamma_q(w)}, \]

then \( C \) is the least possible constant in the inequality

\[ \left( \int_{[0,\infty)} ((Hf)^* (t))^q w(t) dt \right)^{\frac{1}{q}} \leq C \left( \int_{[0,\infty)} (f^*(t))^p v(t) dt \right)^{\frac{1}{p}}. \]

We have

\[ (Hf)^* (t) \approx P (Pf^* + Qf^*) (t) \]

due to (6), so we deduce that \( H \) is bounded from \( \Gamma_p(v) \) to \( \Gamma_q(w) \) if and only if the inequality

\[ \left( \int_{[0,\infty)} P (P + Q) g) q w \right)^{\frac{1}{q}} \leq C \left( \int_{[0,\infty)} (Pg)^p v \right)^{\frac{1}{p}} \]

holds for all non-increasing functions \( g \). Since

\[ Pg(t) + Qg(t) = PG(t), \]

(7) is equivalent to

\[ \left( \int_{[0,\infty)} (P^2 Q) q w \right)^{\frac{1}{q}} \leq C \left( \int_{[0,\infty)} (Pg)^p v \right)^{\frac{1}{p}}. \]
Let $q > 1$. The adjoint operator of $P^2Q$ is $PQ^2$, then, by the reverse Hölder’s inequality, (8) is equivalent to the inequality

\[
\sup_{0 \leq g \leq 1} \frac{\int_{[0, \infty)} g \left( PQ^2 h \right)}{\left( \int_{[0, \infty)} (Pg)^p \, v \right)^{\frac{1}{p}}} \leq C \left( \int_{[0, \infty)} h^{q'} \, w^{1-q'} \right)^{\frac{1}{q'}},
\]

which holds for all non-negative functions $h$.

We start by characterizing the supremum from the left part of (9). Note that $PQ^2 h$ is non-increasing for any non-negative function $h$ and, therefore, it can be rewritten as

\[
PQ^2 h(t) = \int_{[t, \infty)} d\mu(s),
\]

where $d\mu$ is some positive Borel measure. Then we have

\[
C_1 := \sup_{0 \leq g \leq 1} \frac{\int_{[0, \infty)} g \left( PQ^2 h \right)}{\left( \int_{[0, \infty)} (Pg)^p \, v \right)^{\frac{1}{p}}} = \sup_{0 \leq g \leq 1} \frac{\int_{[0, \infty)} Pg(s) \, d\mu(s)}{\left( \int_{[0, \infty)} (P)^p \, v \right)^{\frac{1}{p}}},
\]

and [11, Theorem 5.1] yields

\[
C_1 = \sup_{t > 0} \left( \int_{[0, t]} s \, d\mu(s) + t \int_{[t, \infty)} d\mu(s) \right) V(t)^{-\frac{1}{q'}}
\]

when $0 < p \leq 1$ and

\[
C_1 \approx \left( \int_{[0, \infty)} \left( \int_{[0, t]} s \, d\mu(s) + t \int_{[t, \infty)} d\mu(s) \right)^{q'} V(t)^{\frac{1}{p'}} \right)^{\frac{1}{q'}}
\]

when $1 < p < \infty$. Simple calculations show that

\[
\int_{[0, t]} s \, d\mu(s) + t \int_{[t, \infty)} d\mu(s) = \int_{[0, t]} \left( \int_{[0, \infty)} d\mu(s) \right) dy = \int_{[0, t]} PQ^2 h(s) \, ds = P^2 h(t) + 2 \left( Ph(t) + Qh(t) \right) + Q^2 h(t).
\]

Thus, the initial inequality (7) is equivalent to

\[
\sup_{t > 0} \left( P^2 h(t) + 2 \left( Ph(t) + Qh(t) \right) + Q^2 h(t) \right) V(t)^{-\frac{1}{q'}} \leq C \left( \int_{[0, \infty)} h^{q'} \, w^{1-q'} \right)^{\frac{1}{q'}}
\]

when $0 < p \leq 1$ and

\[
\left( \int_{[0, \infty)} \left( P^2 h(t) + 2 \left( Ph(t) + Qh(t) \right) + Q^2 h(t) \right)^{q'} V(t) \right)^{\frac{1}{q'}} \leq C \left( \int_{[0, \infty)} h^{q'} \, w^{1-q'} \right)^{\frac{1}{q'}}
\]
when $1 < p < \infty$. It is easy to calculate that
\[
P^2h(t) = \frac{1}{t} \int_{[0,t]} \ln \left( \frac{t}{y} \right) h(y) dy
\]
and
\[
Q^2h(t) = \int_{(t,\infty)} \ln \left( \frac{y}{t} \right) h(y) dy.
\]
Therefore, we need to find the criteria for the inequalities
\[
\sup_{x>0} \left( \int_{[0,x]} Ph \right) V(x)^{-\frac{1}{p}} \leq C \left( \int_{[0,\infty]} h^{p'} w^{1-q'} \right)^{\frac{1}{q'}}, \tag{10}
\]
\[
\sup_{x>0} \left( \int_{[0,x]} Qh \right) V(x)^{-\frac{1}{p}} \leq C \left( \int_{[0,\infty]} h^{p'} w^{1-q'} \right)^{\frac{1}{q'}}, \tag{11}
\]
\[
\sup_{x>0} \left( \int_{[0,x]} \frac{1}{t} \int_{[0,t]} \ln \left( \frac{t}{s} \right) h(s) ds \right) V(x)^{-\frac{1}{p}} \leq C \left( \int_{[0,\infty]} h^{p'} w^{1-q'} \right)^{\frac{1}{q'}} \tag{12}
\]
and
\[
\sup_{x>0} \left( \int_{[0,x]} \left( \int_{[t,\infty]} \ln \left( \frac{s}{x} \right) h(s) ds \right) dt \right) V(x)^{-\frac{1}{p}} \leq C \left( \int_{[0,\infty]} h^{p'} w^{1-q'} \right)^{\frac{1}{q'}} \tag{13}
\]
for the case $0 < p \leq 1 < q$ and for the inequalities
\[
\left( \int_{[0,\infty]} (Ph)^{p'} \right)^{\frac{1}{p'}} \leq C \left( \int_{[0,\infty]} h^{p'} w^{1-q'} \right)^{\frac{1}{q'}}, \tag{14}
\]
\[
\left( \int_{[0,\infty]} (Qh)^{p'} \right)^{\frac{1}{p'}} \leq C \left( \int_{[0,\infty]} h^{p'} w^{1-q'} \right)^{\frac{1}{q'}}, \tag{15}
\]
\[
\left( \int_{[0,\infty]} \left( \int_{[0,x]} \ln \left( \frac{x}{s} \right) h(s) ds \right) ^{p'} w^{1-q'} V(x) dx \right)^{\frac{1}{p'}} \leq C \left( \int_{[0,\infty]} h^{p'} w^{1-q'} \right)^{\frac{1}{q'}} \tag{16}
\]
and
\[
\left( \int_{[0,\infty]} \left( \int_{[x,\infty]} \ln \left( \frac{s}{x} \right) h(s) ds \right) ^{p'} V(x) dx \right)^{\frac{1}{p'}} \leq C \left( \int_{[0,\infty]} h^{p'} w^{1-q'} \right)^{\frac{1}{q'}} \tag{17}
\]
for the case $1 < p < \infty$.

The characterizations of weights for the Hardy-type inequalities (14)-(17) are well known (see [23, 36]), and this proves (i) and (ii).

To prove the case $0 < p \leq 1 < q < \infty$ we use the reverse Hölder’s inequality. For instance, the inequality (10) expresses the boundedness
of the operator \( h(s) \rightarrow Pb(s) \mathbf{V}(t)^{-\frac{1}{p}} \) from \( L^p_{w_{1-q}}(0, \infty) \) to \( L^1(0, t) \), \( t > 0 \). This is equivalent to the estimate

\[
\mathbf{V}(t)^{-\frac{1}{p}} \left( \int_{[0,t]} (Ph)^q w \right)^{\frac{1}{q}} \leq C \sup_{0<s<t} h(s),
\]

which obviously holds if and only if \( D_1 < \infty \). We use the same technique for the inequalities (11)-(13) to prove the rest of (iii).

In the case \( q = 1 \) we have

\[
C = \sup_{0 \leq g \leq 1} \int_{[0,\infty)} g \left( \int_{[0,\infty)} (Pg)^p v \right)^{\frac{1}{p}} = \int_{[0,\infty)} (Pg)^p v \left( \int_{[0,\infty)} (Pg)^p v \right)^{\frac{1}{p}},
\]

where \( dv(t) \) is defined by \( PQ^2 w(t) = \int_{[t,\infty)} dv(s) \). Then, according to [11, Theorem 5.1], we have

\[
C = \sup_{t>0} \left( \int_{[0,t]} s dv(s) + t \int_{[t,\infty)} dv(s) \right) \mathbf{V}(t)^{-\frac{1}{p}}
\]

when \( 0 < p \leq 1 \) and

\[
C \approx \left( \int_{[0,\infty)} \left( \int_{[0,t]} s dv(s) + t \int_{[t,\infty)} dv(s) \right)^{\frac{1}{q'}} V(t) dt \right)^{\frac{1}{p'}}
\]

when \( 1 < p < \infty \). We use the similar calculations as in the case \( q > 1 \) to obtain (iv) and (v).

\[\square\]

**Remark 2.** The similar criteria of boundedness for the Hilbert transform were obtained in [14, Theorem 3.4], but the function analogous to \( V(t) \) has the discrete form there.

**Remark 3.** The Riesz transforms \( R_j, 1 \leq j \leq n \), satisfy the same inequality as the inequality (6) for the Hilbert transform:

\[
(R_j f)^* (x) \leq C_1 [Pf^*(x) + Qf^*(x)] \leq C_2 \left( R_j \tilde{f} \right)^* (x), \ x > 0,
\]

where \( \tilde{f}(y) := f^* (|y|^{n}) \chi_{[0,\infty)}(y_j), A \) being the volume of \( S^{n-1} \) (see [4], [27]). Therefore, criteria for the boundedness of Riesz transforms from \( \Gamma_p(v) \) to \( \Gamma_q(w) \) are exactly the same as for the Hilbert transform.

### 4. The Mapping Properties of the Riesz Potentials

The result of this section concerns the boundedness of the Riesz potentials between weighted Lorentz \( \Gamma \)-spaces. It reads as follows:

**Theorem 4.** Let \( 0 < p < \infty, 1 \leq q < \infty \) and the weight \( v \) satisfy conditions

\[
\int_{[0,\infty)} \frac{v(s) ds}{s^p + v^p} < \infty, \quad \int_{[0,1]} \frac{v(s) ds}{s^p} = \int_{[1,\infty)} v(s) ds = \infty
\]
for every $t > 0$. Let

$$V(t) := \int_{[0,t]} v,$$

$$V(t) := V(t) + t^p \int_{t,\infty}^{t} \frac{v(s)}{s^p} ds,$$

$$\forall(t) := V(t)^{-p} V(t) \left( \int_{t,\infty}^{t} \frac{v(s)}{s^p} ds \right)^{p-1}.$$

Then the Riesz potential $I_\alpha$ is bounded from $\Gamma_p(v)$ to $\Gamma_q(w)$ if and only if

(i) $A := \max A_i < \infty$, $i = \overline{1,8}$, if $1 < p < q < \infty$, where

$$A_1 := \sup_{x>0} \left( \int_{[x,\infty)} (t-x)^p t^{p(\alpha/n-1)} \forall(t) dt \right)^{\frac{1}{p}} \left( \int_{[0,x]} w(t) dt \right)^{\frac{1}{q}},$$

$$A_2 := \sup_{x>0} \left( \int_{[x,\infty)} t^{p(\alpha/n-1)} \forall(t) dt \right)^{\frac{1}{p}} \left( \int_{[0,x]} (x-t)^q w(t) dt \right)^{\frac{1}{q}},$$

$$A_3 := \sup_{x>0} \left( \int_{[x,\infty)} \forall(t) dt \right)^{\frac{1}{p}} \left( \int_{[0,x]} t^{nq(q-1)/n} w(t) dt \right)^{\frac{1}{q}},$$

$$A_4 := \sup_{x>0} \left( \int_{[0,x]} t^{p(\alpha/n+1)} \forall(t) dt \right)^{\frac{1}{p}} \left( \int_{[x,\infty)} w(t) dt \right)^{\frac{1}{q}},$$

$$A_5 := \sup_{x>0} \left( \int_{[x,\infty)} t^{p\alpha/n} \forall(t) dt \right)^{\frac{1}{p}} \left( \int_{[0,x]} w(t) dt \right)^{\frac{1}{q}},$$

$$A_6 := \sup_{x>0} \left( \int_{[0,x]} t^{p'} \forall(t) dt \right)^{\frac{1}{p'}} \left( \int_{[x,\infty)} t^{q(\alpha/n-1)} w(t) dt \right)^{\frac{1}{q}},$$

$$A_7 := \sup_{x>0} \left( \int_{[0,x]} (x-t)^p t^{p(\alpha/n-1)} \forall(t) dt \right)^{\frac{1}{p}} \left( \int_{[x,\infty)} t^{q(\alpha/n-2)} w(t) dt \right)^{\frac{1}{q}},$$

$$A_8 := \sup_{x>0} \left( \int_{[0,x]} t^{p'} \forall(t) dt \right)^{\frac{1}{p'}} \left( \int_{[x,\infty)} (t-x)^q t^{q(\alpha/n-2)} w(t) dt \right)^{\frac{1}{q}}.$$

(ii) $B := \max B_i < \infty$, $i = \overline{1,8}$, if $1 < q < p < \infty$, where

$$B_1 := \left( \int_{[0,\infty)} \left( \int_{[x,\infty)} (t-x)^p t^{p(\alpha/n-1)} \forall(t) dt \right)^{\frac{1}{p}} \times \left( \int_{[0,x]} w(t) dt \right)^{\frac{1}{q}} w(x) dx \right)^{\frac{1}{r}},$$
\[ B_2 := \left( \int_{[0, \infty)} \left( \int_{[x, \infty]} t^{p'(\alpha/n-1)} \psi(t) dt \right)^{\frac{1}{p'}} \left( \int_{[0, x]} (x-t)^q w(t) dt \right)^{\frac{1}{q}} \right)^{\frac{1}{2}} \times x^{p'(\alpha/n-1)} \psi(x) dx \],

\[ B_3 := \left( \int_{[0, \infty)} \left( \int_{[x, \infty]} \psi(t) dt \right)^{\frac{1}{p'}} \left( \int_{[0, x]} t^{\alpha q/n-1} w(t) dt \right)^{\frac{1}{q}} \right)^{\frac{1}{2}} \times x^{\alpha q(n-1)/n} w(x) dx \],

\[ B_4 := \left( \int_{[0, \infty)} \left( \int_{[0, x]} t^{p'(\alpha/n+1)} \psi(t) dt \right)^{\frac{1}{p'}} \left( \int_{[x, \infty]} w(t) dt \right)^{\frac{1}{q}} \right)^{\frac{1}{2}} \times w(x) dx \],

\[ B_5 := \left( \int_{[0, \infty)} \left( \int_{[x, \infty]} t^{q'/n} \psi(t) dt \right)^{\frac{1}{p'}} \left( \int_{[0, x]} w(t) dt \right)^{\frac{1}{q}} \right)^{\frac{1}{2}} \times x^{q(n-1)/n} w(x) dx \],

\[ B_6 := \left( \int_{[0, \infty)} \left( \int_{[0, x]} t^{p'} \psi(t) dt \right)^{\frac{1}{p'}} \left( \int_{[x, \infty]} t^{q(n-1)} w(t) dt \right)^{\frac{1}{q}} \right)^{\frac{1}{2}} \times x^{q(n-1)/n} w(x) dx \],

\[ B_7 := \left( \int_{[0, \infty)} \left( \int_{[0, x]} (x-t)^{p'} \psi(t) dt \right)^{\frac{1}{p'}} \left( \int_{[x, \infty]} t^{q(n-2)/n} w(t) dt \right)^{\frac{1}{q}} \right)^{\frac{1}{2}} \times x^{q(n-2)/n} w(x) dx \],

\[ B_8 := \left( \int_{[0, \infty)} \left( \int_{[0, x]} t^{p'} \psi(t) dt \right)^{\frac{1}{p'}} \left( \int_{[x, \infty]} (t-x)^{q(n-2)} w(t) dt \right)^{\frac{1}{q}} \right)^{\frac{1}{2}} \times x^{p'} \psi(x) dx \].

(iii) \( D := \max D_i < \infty, \ i = 1, 6, \) if \( 0 < p \leq 1 < q < \infty, \) where

\[ D_1 := \sup_{x > 0} \left( \int_{[0, x]} (x-y)^q w(y) dy \right)^{\frac{1}{q}} x^{n/q-1} \psi(x)^{-\frac{1}{p}}, \]

\[ D_2 := \sup_{x > 0} \left( \int_{[0, x]} y^{q'/n} w(y) dy \right)^{\frac{1}{q'}} \psi(x)^{-\frac{1}{p}}, \]

\[ D_3 := \sup_{x > 0} \left( \int_{[x, \infty]} \frac{w(y)}{y^{q'/n}} dy \right)^{\frac{1}{q'}} x^{q(n-1)/n} \psi(x)^{-\frac{1}{p}}. \]
\[ D_4 := \sup_{x > 0} \left( \int_{[0,x]} w(y)dy \right)^{\frac{1}{q}} x^{\alpha/n} V(x)^{-\frac{1}{p}}, \]
\[ D_5 := \sup_{x > 0} \left( \int_{(x,\infty)} \frac{w(y)}{y^{p(1-\alpha/n)}}dy \right)^{\frac{1}{q}} x V(x)^{-\frac{1}{p}}, \]
\[ D_6 := \sup_{x > 0} \left( \int_{(x,\infty)} \left( \frac{y-x}{y^{2-\alpha/n}} \right)^q w(y)dy \right)^{\frac{1}{q}} x V(x)^{-\frac{1}{p}}; \]

(iv) \( F < \infty, \) if \( 0 < p + 1 = q, \) where
\[ F := \sup_{x > 0} \left( \int_{[0,x]} y^{\alpha/n} h(y)dy + x^{\alpha/n+1} \int_{[x,\infty)} \frac{h(y)}{y}dy \right. \]
\[ + x^{\alpha/n-1} \int_{[0,x]} (x-y) h(y)dy + x^{\alpha/n-1} \int_{[0,x]} h(y)dy \]
\[ + \left. \int_{(x,\infty)} \frac{h(y)}{y^{1-\alpha/n}}dy + \int_{[x,\infty)} \frac{h(y)}{y^{2-\alpha/n}}(y-x)dy \right) \left( x V(x) \right)^{-\frac{1}{p}}. \]

(v) \( G < \infty, \) if \( q = 1 < p < \infty, \) where
\[ G := \left( \int_{[0,\infty)} \left( \int_{[0,x]} y^{\alpha/n} h(y)dy + x^{\alpha/n+1} \int_{[x,\infty)} \frac{h(y)}{y}dy \right. \right. \]
\[ + x^{\alpha/n-1} \int_{[0,x]} (x-y) h(y)dy + x^{\alpha/n-1} \int_{[0,x]} h(y)dy \]
\[ + \left. \left. \int_{[x,\infty)} \frac{h(y)}{y^{1-\alpha/n}}dy + \int_{[x,\infty)} \frac{h(y)}{y^{2-\alpha/n}}(y-x)dy \right) \right) \left( x V(x) \right)^{\frac{1}{p}}. \]

**Proof.** It was shown in [5], [27] that the Riesz potentials satisfy the inequality
\[ (I_\alpha f)^*(x) \leq C_1 \left[ x^{\alpha/n-1} \int_{[0,x]} f^*(t)dt + \int_{[t,\infty)} t^{\alpha/n-1} f^*(t)dt \right] \]
\[ \leq C_2 \left( \int_{\mathbb{R}} (f^*)^*(s) \right)^\alpha, \quad x > 0, \]
where \( \tilde{f}(y) := f^*(A|y|^n), \) \( A \) being the volume of \( S^{n-1}. \) Let the operators \( P \) and \( Q \) be the same as in Theorem 3 and put
\[ P_\alpha f(x) := \frac{1}{x^{1-\alpha/n}} \int_{[0,x]} f(t)dt, \quad Q_\alpha f(x) := \int_{[x,\infty)} \frac{f(t)}{t^{1-\alpha/n}}dt. \]

Let
\[ C := \| I_\alpha \|_{\Gamma_p(v) \rightarrow \Gamma_q(w)}, \]
then \( C \) is the least possible constant in the inequality
\[ \left( \int_{[0,\infty)} \left( (I_\alpha f)^* \right)^q w(t)dt \right)^{\frac{1}{q}} \leq C \left( \int_{[0,\infty)} \left( f^* \right)^p v(t)dt \right)^{\frac{1}{p}}. \]
Since
\[(I_\alpha f)^* (x) \approx P (P_\alpha f^*) (x) + Q_\alpha f^* (x),\]
the Riesz potential \(I_\alpha\) is bounded from \(\Gamma_p(v)\) to \(\Gamma_q(w)\) if and only if the inequality
\[
\left( \int_{[0,\infty)} [P (P_\alpha g + Q_\alpha g)]^q w \right) \leq C \left( \int_{[0,\infty)} (P g)^p v \right)^{\frac{1}{p}}
\]
holds for all non-increasing functions \(g\).

Let \(q > 1\). Since the operator \(Q\) is adjoint of \(P\), by using the reverse Hölder inequality we deduce that (18) is equivalent to the inequality
\[
\sup_{0 \leq t \leq x} \int_{[0,\infty)} (P_\alpha g + Q_\alpha g) Q h \left( \int_{[0,\infty)} (P g)^p v \right)^{\frac{1}{p}} \leq C \left( \int_{[0,\infty)} h v^{2-q} \right)^{\frac{1}{q}}.
\]
which holds for all non-negative functions \(h\). We have
\[
P_\alpha g(x) + Q_\alpha g(x) = \frac{1}{x^{1-\alpha/n}} \int_{[0,x]} g(t) dt + \int_{[x,\infty)} \frac{g(t)}{t^{1-\alpha/n}} dt.
\]
Then
\[
\int_{[0,\infty)} (P_\alpha g + Q_\alpha g) Q h = \int_{[0,\infty)} t P g(t) \left( \int_{[0,\infty)} \frac{Q h(s) ds}{(s + t)^{2-\alpha/n}} \right) dt.
\]
Denote the left side of (19) by \(J\), then, according to [11, Theorem 5.1], we obtain
\[
J = \sup_{x > 0} \left( \int_{[0,x]} t \left( \int_{[0,\infty)} \frac{Q h(s) ds}{(s + t)^{2-\alpha/n}} \right) dt \right) + x \int_{[x,\infty)} \left( \int_{[0,\infty)} \frac{Q h(s) ds}{(s + t)^{2-\alpha/n}} \right) dt \right) V(x)^{-\frac{1}{p}}
\]
when \(0 < p \leq 1\) and
\[
J \approx \left( \int_{[0,\infty)} \left( \int_{[0,x]} t \left( \int_{[0,\infty)} \frac{Q h(s) ds}{(s + t)^{2-\alpha/n}} \right) dt \right) \right) \left( \int_{[0,\infty)} \frac{Q h(s) ds}{(s + t)^{2-\alpha/n}} \right) dt \right) \left( \int_{[0,\infty)} \frac{Q h(s) ds}{(s + t)^{2-\alpha/n}} \right) dt \right) V(x)^{\frac{1}{p}}
\]
when \(1 < p < \infty\).

First we calculate
\[
\int_{[0,\infty)} \frac{Q h(s) ds}{(s + t)^{2-\alpha/n}} = \int_{[0,\infty)} \left( \int_{[0,y]} ds \right) \frac{h(y)}{y} dy
\]
Therefore, we need to characterize the inequalities

\[
\left( \int_{0}^{\infty} \frac{h(y)}{t + y} \, dy \right)^{\frac{1}{p'}} \leq C \left( \int_{0}^{\infty} h^{q'} w^{1-q'} \, w \, dx \right)^{\frac{1}{p'}}.
\]

Similarly,

\[
\left( \int_{0}^{\infty} \frac{h(y)}{y} \, dy \right)^{\frac{1}{p'}} \leq C \left( \int_{0}^{\infty} h^{q'} w^{1-q'} \, dx \right)^{\frac{1}{p'}}.
\]

Therefore, we need to characterize the inequalities

\[
\sup_{x > 0} V(x)^{-\frac{1}{p'}} \int_{0}^{\infty} (x - y) h(y) \, dy \leq C \left( \int_{0}^{\infty} h^{q'} w^{1-q'} \, dx \right)^{\frac{1}{p'}}.
\]

\[
\sup_{x > 0} V(x)^{-\frac{1}{p'}} \int_{0}^{\infty} y^{\alpha/n} h(y) \, dy \leq C \left( \int_{0}^{\infty} h^{q'} w^{1-q'} \, dx \right)^{\frac{1}{p'}}.
\]

\[
\sup_{x > 0} V(x)^{-\frac{1}{p'}} \int_{0}^{\infty} \frac{h(y)}{y} \, dy \leq C \left( \int_{0}^{\infty} h^{q'} w^{1-q'} \, dx \right)^{\frac{1}{p'}}.
\]

\[
\sup_{x > 0} V(x)^{-\frac{1}{p'}} \int_{0}^{\infty} \frac{h(y)}{y^{1-\alpha/n}} \, dy \leq C \left( \int_{0}^{\infty} h^{q'} w^{1-q'} \, dx \right)^{\frac{1}{p'}}.
\]

\[
\sup_{x > 0} V(x)^{-\frac{1}{p'}} \int_{0}^{\infty} \frac{h(y)}{y^{2-\alpha/n}} \, dy \leq C \left( \int_{0}^{\infty} h^{q'} w^{1-q'} \, dx \right)^{\frac{1}{p'}}.
\]

for the case \(0 < p \leq 1\) and the inequalities

\[
\left( \int_{0}^{\infty} \left( \int_{0}^{\infty} (x - y) h(y) \, dy \right)^{\frac{1}{p'}} \, x^{\varepsilon'(\alpha/n-1)} V(x) \, dx \right)^{\frac{1}{p'}} \leq C \left( \int_{0}^{\infty} h^{q'} w^{1-q'} \, dx \right)^{\frac{1}{p'}}.
\]

\[
\left( \int_{0}^{\infty} \left( \int_{0}^{\infty} y^{\alpha/n} h(y) \, dy \right)^{\frac{1}{p'}} \, V(x) \, dx \right)^{\frac{1}{p'}} \leq C \left( \int_{0}^{\infty} h^{q'} w^{1-q'} \, dx \right)^{\frac{1}{p'}}.
\]
\[
\left( \int_{[0,\infty)} \left( \int_{[x,\infty)} \frac{h(y)}{y} \, dy \right)^{p'} x^{p'(\alpha/n+1)} V(x) \, dx \right)^{\frac{1}{p'}} \leq C \left( \int_{[0,\infty)} h^{q'} w^{1-q'} \right)^{\frac{1}{q'}} ,
\]
\[
\left( \int_{[0,\infty)} \left( \int_{[0,x]} \frac{h(y)}{y} \, dy \right)^{p'} x^{p'(1/\alpha)} V(x) \, dx \right)^{\frac{1}{p'}} \leq C \left( \int_{[0,\infty)} h^{q'} w^{1-q'} \right)^{\frac{1}{q'}} ,
\]
\[
\left( \int_{[0,\infty)} \left( \int_{[x,\infty)} \frac{h(y)}{y^{1-\alpha/n}} \, dy \right)^{p'} x^{p'} V(x) \, dx \right)^{\frac{1}{p'}} \leq C \left( \int_{[0,\infty)} h^{q'} w^{1-q'} \right)^{\frac{1}{q'}} ,
\]
\[
\left( \int_{[0,\infty)} \left( \int_{[x,\infty)} \frac{h(y)}{y^{2-\alpha/n}} \, dy \right)^{p'} x^{p'} V(x) \, dx \right)^{\frac{1}{p'}} \leq C \left( \int_{[0,\infty)} h^{q'} w^{1-q'} \right)^{\frac{1}{q'}}
\]
for the case \( 1 < p < \infty \).

The criteria for the mentioned inequalities are well known. For the case \( 1 < p, q < \infty \) see [23], [36] and [37]. In the case \( 0 < p \leq 1 \) we use the reverse Hölder’s inequality and finish the proof of (i)-(iii) in a similar way as the proof of Theorem 3.

In the case \( q = 1 \) we have
\[
C = \sup_{0 \leq g} \int_{[0,\infty)} P (P^{\alpha}g + Q^{\alpha}g) w \left( \int_{[0,\infty)} (Pg)^{p} v \right)^{\frac{1}{q'}}
\]
\[
= \int_{[0,\infty)} tPg(t) \left( \int_{[0,\infty)} Qw(s) ds \right) (s+t)^{2-\alpha/n} dt ,
\]
and by using again [11, Theorem 5.1] and the calculations for the case \( q > 1 \) we obtain also the proof of (iv) and (v). □

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