Studying the Views of Preservice Teachers on the Concept of Function

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Foreword

Because it is literary impossible to master higher mathematics in any intellectually honest way without a firm and deep understanding of functions, mathematics educators are trying to identify and understand the learning obstacles students encounter in mastering this notion. (Eisenberg, 1992, p. 158)

I became engaged in the learning of mathematics through my experiences as a teacher at upper secondary school and later as a lecturer involved in teacher education. When I received the opportunity to pursue further studies in mathematics education I wanted to study the views of preservice teachers on the central notion of function. In this thesis, the preservice teachers’ conceptions of function are frequently considered in relation to mathematical statements related to different concepts and topics on a variety of levels in mathematics. Moreover, preservice teachers’ opinions concerning the extent functions are of significance in mathematics and present in school mathematics are regarded as relevant aspects of the prospective teachers’ views of function. The final part of the study includes an intervention study where previous results of the study are considered in its design.

This thesis consists of an overview of the subject, where in particular the following five papers are put into a frame:


(The papers are ordered in a sequence that essentially reflects the implementation of the study. The papers contain minor corrections to language.)

The overview part consists of an introduction, a short description of some related research and theoretical framework, a summary of papers I-V and the concluding discussion.
1 Introduction

1.1 The function concept

The notion of function is fundamental in mathematics. According to the contemporary definition, a function is a correspondence between two nonempty sets that assigns to every element in the first set (the domain) exactly one element in the second set (the codomain). In the historical development of the concept, Peter Gustav Lejeune Dirichlet was one of the first to seriously consider this characterization of function during the first half of the 19th century. This was at a time when mathematicians “in practice … thought of functions as analytical expressions or curves” (Kleiner, 1989, p. 291). A common consensus of how to define a function was not established in the mathematics community until the first half of the 20th century. The definition of function we use nowadays was at that time more firmly established by Nicolas Bourbaki and is often called the Dirichlet-Bourbaki\(^1\) definition.

Ideas related to the concept of function can be argued to have been present in different contexts in the history of mathematics as dependencies between two quantities (e.g., Klein, 1972; Youschkevitch, 1976). The word “function” was introduced by Gottfried Wilhelm Leibniz at the end of the 17th century in very general terms to mean the dependence of a geometrical quantity (such as a subtangent or subnormal) related to a varying point of a curve where the curve was thought to be given by an equation. Influenced by Johann Bernoulli, who looked at problems in variational calculus where functions are sought as solutions, Leibniz came to use the term function as quantities that depend on a variable, and the two discussed how to designate functions by symbols. Toward the mid-18th century the concept of function was positioned at the center of analysis, mostly through the work of Leonhard Euler, who generally thought of functions as analytical expressions. The notion of an analytical expression was for Euler a broad description that, for instance, included power series and infinite products and a notion that was open insofar as newly defined operations could also appear (Jahnke, 2003b).

The concept of function has developed gradually by an evolution from vague and inexact notions. Discussions about what a function is or how it should

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\(^1\) The Bourbaki group defined in 1939 a function as a correspondence between two sets similar to what Dirichlet had done 1837 (Monna, 1972; Rüthing, 1984; Youschkevitch, 1976). However, Bourbaki’s view of function differs from Dirichlet’s view in that the domain and codomain no longer are restricted to sets of numbers. Bourbaki also formulated an equivalent definition of function as a set of ordered pairs, where a function from a set \(E\) to a set \(F\) is defined as a special subset of the Cartesian product \(E \times F\).

Dirichlet was the first one to take serious notion of functions characterized by the modern definition. He gave in 1829 at the end of a paper on Fourier series the Dirichlet function (a function with domain \([0, 1]\) and codomain \(\{0, 1\}\) that assigns rational numbers to 0 and irrational numbers to 1, Malik, 1980; Youschkevitch, 1976) as an example of a function consistent with his definition but neither possible to represent as an analytical expression nor as a curve. Dirichlet’s paper from 1837 (Dirichlet, 1837) with his description of function is an elaborate version of the paper from 1829 (Dirichlet, 1829), according to Hawkins (1975).
be defined has actualized on numerous occasions in the mathematics community (e.g., Eylon & Bruckheimer, 1986; Jahnke, 2003a; Klein, 1972; Kleiner, 1989; Markovits, Malik, 1980; Monna, 1972; Rüthing, 1984; Youschkevitch, 1976). Without going into details, the first extensive discussions on functions date back to the 18th century, when they are related to the solution of differential equation of the vibrating string. A second noticeable event to expand the concept of function was the development of Fourier series in the 19th century. The description of function suggested by Dirichlet was at that time by many mathematicians considered too general to characterize a function. Further polemics on the concept of function were brought to the fore around the year 1900, e.g. in the advancement of the theory of measure and integration and in discussions related to foundations of mathematics. Developments in mathematics successively expanded the classes of functions to consider and gave reasons to apply a more general notion of function. New problems and branches of mathematics also required a more inclusive characterization of function that was no longer related to numbers, but to general sets of domain and codomain.

The Dirichlet-Bourbaki definition of function simply utilizes the idea of uniqueness, i.e. for each element in the domain there is exactly one element in the codomain, with no other required properties of the correspondence. Uniqueness combined with domain and codomain as two arbitrary chosen nonempty sets makes the concept of function a highly general and abstract notion that proves to be demanding for students to assimilate (Akkoc & Tall, 2003; Eisenberg, 1991; Even, 1993; Sierpinska, 1992; Tall, 1992; Vinner & Dreyfus, 1989). Moreover, the concept of function has several synonyms associated with varied conceptions of function, such as mapping, operator, transform, etc. that are used in different contexts and in various forms of representation. Furthermore, it has an extensive set of sub-concepts and a large network of relations to other concepts (Blomhøj, 1997; Eisenberg, 1991, 1992; Selden & Selden, 1992; Tall, 1992, 1996) that make assimilation of the function concept and an understanding of its significance a long-term process for mathematics students.

1.2 A unifying concept, network of relations and school mathematics

The concept of function is part of most areas in mathematics and is frequently considered a unifying concept that provides a framework for the study of mathematics (e.g., Carlson, 1998; Cooney & Wilson, 1993; Selden & Selden, 1992). The significance of functions is also manifested through their large network of relations to other mathematical concepts (one aspect of this is illustrated in section 1.2.1). Developing an understanding of the function concept includes a comprehension of its network of relations. Gaining knowledge of the relationships and being able to use functions in different contexts is a learning process that requires a longer period of time. It is, therefore, appropriate to introduce the concept of function in school
mathematics, to gradually expand the students’ knowledge of functions, their applications, representations and relations to other concepts, and successively make the students able to handle functions in a more flexible way.

Historically, there have been initiatives to emphasize the concept of function in pre-tertiary education. According to Cooney and Wilson (1993) many mathematics educators during the early 20th century believed a greater emphasis on functional thinking in school mathematics was needed (referring to Breslich, 1928; Hamley, 1934; Hedrick, 1922; Schorling, 1936). The well-known mathematician Felix Klein became engaged in mathematics education and played a central role in a curriculum reform, the “Meran Programme”\(^1\), declared in 1905 (Cooney & Wilson, 1993; Fujita, Jones & Yamamoto, 2004; Sierpinska & Lerman, 1996). The concept of function as a dependency relation was a central part of the reform and considered as a unifying concept in mathematics. A similar development was seen in the curriculum in other countries, affected by the Meran Programme. But the effect of the reform was not noticeable (Cooney & Wilson, 1993; Sierpinska & Lerman, 1996), and Cooney and Wilson (1993) questioned if the emphasis on functions in reality reached as far as compulsory school, suggesting that one reason for this could be the teachers’ conceptions of functions. The concept of function was also advocated in the new mathematics movement of the 1960s, but usually in a more formalistic approach as a set of ordered pairs, which were later considered less appropriate to introduce the function concept in school (Cooney & Wilson, 1993; Eisenberg, 1991; Tall, 1992, 1996).

To become successful in dealing with the concept of function in their practice, it is important for mathematics teachers to have a well-developed conceptual knowledge of functions, including the concept’s significance in mathematics and relationships to other concepts (Cooney & Wilson, 1993; Eisenberg, 1992; Even, 1993; Thomas, 2003; Vollrath, 1994). The concept of function is currently a regular part of the school-mathematics curriculum. In Sweden, functions are introduced at compulsory school where some basic classes of functions are considered, and a dependency relation between two variables is typically stressed and the term “function” is less often used. At upper-secondary school the concept of function and its definition are more explicitly stated. The standard functions are part of courses in introductory calculus required for further studies in mathematics at a tertiary level (Skolverket, 2006).

1.2.1 The view of one preservice teacher on network of relations

The notion of function provides a framework rich of relations to other concepts. To illustrate one aspect of what this means, lets consider a concept map drawn by a preservice teacher. The map is derived from \(y=x+5\) and the student has

constructed the map according to his own thoughts related to $y=x+5$ (the map is further commented on in Paper II). The concept of function in the map is not a well-integrated concept, with very few links to other nodes that the student has chosen to include in his map.

Figure 1. A concept map derived from $y=x+5$ drawn by a preservice teacher.

The view of the illustrated map contrasts with an understanding that the notion of function provides a framework of reasoning incorporating a large network of relations. Developing an understanding of function as a concept rich of relations means the student has to realize that it is possible to relate the node “function” to many of the nodes in the map. The node “function” might connect to, for example, “in point (0, 5)” as it relates to the zero of the function, “table of values” gives values of the function, “slope 1” relates to an increasing function, “straight line” may be associated with the function graph, and “$y$ depends on $x$” relates to a dependent variable of a function.

1.3 Preservice teachers and aims of research

The aims of the research study concern the preservice teachers’ views on the concept of function at the end of their required courses in mathematics from a teacher preparation program in mathematics and science for the school grades 4 to 9. Preservice teachers’ conceptions of functions and their different properties are to a large extent examined in relation to mathematical statements, where the study has successively been expanded to include $y=x+5$, $y=\pi x^2$ and $xy=2$. The three mathematical statements can be related to the preservice teachers’ future teaching as well as concepts and topics at a tertiary level. Furthermore, questions concerning the significance of functions in mathematics and presence in school mathematics are considered as essential aspects of the preservice teachers’ views of the function concept as prospective teachers. The use of concept maps as a
research tool is also a question of interest in the study. The final part of the study includes an intervention study regarding the concept of function in one concluding course in mathematics from the educational programme. A more detailed account of the research questions is given in section four.

1.3.1 Mathematical statements considered in the study

The selection of the three statements\(^1\) \(y=x+5\), \(y=\pi x^2\) and \(xy=2\) will be briefly justified. The statements can be related to concepts and different topics on a variety of levels associated with the preservice teachers’ teaching as well as mathematics at a tertiary level. The statements may thus evoke numerous concepts and various chains of association in an individual’s reasoning, which span a network of relations between concepts in the context of the statements. These relations might include a range of mathematical topics, areas and applications, different representations of concepts, learning and teaching scenarios, etc. Although the concept of function is of primary interest in the study, the notation \(f(x)\) is avoided so that the preservice teachers will make their own interpretations of the three statements. Moreover, the statements are chosen to not represent prototypes, as for example \(y=x^2\) and \(y=1/x\) would have done. Additional reason for choosing \(y=x+5\) is given in Paper I.

The statements \(y=x+5\), \(y=\pi x^2\) and \(xy=2\) are relevant for the preservice teachers, as they all represent concepts and mathematical relations with applications that are common in school mathematics. For instance, the concept of formula, direct and indirect proportionality, function, area of a circle or rectangle, operation, equation, line, parabola, symmetry, hyperbola, to name a few concepts which also give preservice teachers opportunities to relate to future teaching scenarios. Of course, it is also possible for the preservice teachers to relate to topics primarily associated with tertiary education, such as vectors, conic sections, or a range of different properties of various classes of functions.

When \(y=x+5\), \(y=\pi x^2\) and \(xy=2\) are considered to represent functions – let us assume they represent real valued functions in one real variable, with maximal domains – it is then possible to link them to a range of concepts representing different properties of functions or classes of functions. For example, all represent standard classes of functions, like linear \((y=x+5)\), quadratic \((y=\pi x^2)\) and rational \((xy=2)\). As such, they have numerous properties in common, like continuous, differentiable, etc. However, they also have a number of different properties that set them apart from each other, such as even, odd, increasing, convex, asymptotic behaviors, injective, surjective, range, domain (in one case), concave (on an interval in one case), explicit versus implicit representation, extreme values, and existence of an inverse. The differences make the functions complement each other and together cover a range of properties of functions.

\(^1\) I chose to call \(y=x+5\), \(y=\pi x^2\) and \(xy=2\) statements and not expressions. This is because an expression according to Swedish terminology does not include an equality sign.
The concept of function has a large network of relations to other concepts in mathematics, and besides regarding \( y=x+5 \), \( y=\pi x^2 \) and \( xy=2 \) as functions with many properties, it is possible to further consider the unifying aspect of the function concept in mathematics, and thus observe relations to other concepts derived from \( y=x+5 \), \( y=\pi x^2 \) or \( xy=2 \). For instance, in this context it is possible to relate a root of an equation to the zero of a function, and a hyperbola to the graph of a function, the slope of a line to an increasing function, or symmetry about the y-axis to an even function, etc. Thus, the concept of function and its different properties span a considerable network of relations to a range of concepts in the context of \( y=x+5 \), \( y=\pi x^2 \) and \( xy=2 \). One part of the study aims to examine the preservice teachers’ view on functions in this context, with further details of the research questions in section four.
2 Related research

2.1 Studies related to the concept of function

The significance of the function concept in mathematics is reflected by substantial research literature regarding the notion of function in mathematics education. Conferences in mathematics education have been dedicated to the concept of function, which has generated books such as *The Concept of Function: Aspects of Epistemology and Pedagogy* (Harel & Dubinsky, 1992) or *Integrating Research on the Graphical Representation of Function* (Romberg, Fennema & Carpenter, 1993).

A number of studies have been conducted concerning conceptual knowledge of function of students at the tertiary level, confirming a frequent inconsistency in students’ conceptions of function and the definition of function (e.g., Breidenbach, Dubinsky, Hawks & Nichols, 1992; Carlson, 1998; Cuoco, 1994; Eisenberg & Dreyfus, 1994; Even, 1990, 1993, 1998; Romberg, Carpenter & Fennema, 1993; Slavit, 1997; Tall & Bakar, 1992; Thomas, 2003; Thompson, 1994; Vinner & Dreyfus, 1989; Williams, 1998). Vinner and Dreyfus (1989) conducted one such well-known study, showing that tertiary students during a course in calculus, even when the students were able to correctly account the definition of function, did not apply the definition of function successfully. Vinner (1983, 1992) describes a model using the notion of concept image consistent with these results, further delineated in section three. Analogous results have been reported in a range of studies (e.g., Tall & Bakar, 1992; Breidenbach et al., 1992; Even, 1993; Thomas; 2003), where Breidenbach et al. (1992) points out that “college students, even those who have taken a fair number of mathematics courses, do not have much of an understanding of the function concept” (p. 247). This confirms that the concept of function with its various sub-notions and contexts from which it can be approached is a complex concept for students to grasp. Conceptual development of function and the framework it provides is a long term process in which students are engaged in during their studies of mathematics from compulsory school to tertiary level.

A majority of researchers in the community of mathematics education (e.g., Confrey & Doerr, 1996; Eisenberg, 1991, 1992; Dubinsky & Harel, 1992; Freudenthal, 1983; Romberg, Carpenter et al., 1993; Selden & Selden, 1992; Sierpinska, 1992; Sfard, 1992; Tall, 1992, 1996; Yerushalmy & Chazan, 2002) seem to agree that the concept of function should be introduced in a dynamic form, such as a type of relation, correspondence or covariation – not favoring a static ordered pair version of the definition, as related to Bourbaki. Researchers then stress several different approaches representing different theoretical frameworks to develop students’ conceptual knowledge of function, such as modeling, programming, multiple representations, etc., to successively develop students conceptual understanding and first at the tertiary level utilize the definition of function in its full generality when it is required in the study of more advanced topics.
In students’ conceptual development of function, process-object models are frequently suggested (Dubinsky & Harel, 1992; Eisenberg, 1991; Selden & Selden, 1992; Sfard, 1992; Tall, 1992, 1996; Thompson, 1994). The concept of function is often used to illustrate conceptual development in different theoretical models, including the well-known theory of reification model by Sfard (1989, 1991, 1992) – further described in section three – and Dubinsky with colleagues APOS theory model (Asiala, Brown, DeVries, Dubinsky, Mathews & Thomas, 1996; Breidenbach et al., 1992, Dubinsky, 1991; Dubinsky & Harel, 1992), but also in the theory of procepts (Gray & Tall, 1994) underlining the role of symbols. The three theories are further discussed in, e.g. Mamona-Downs and Downs (2002) and Tall, Thomas, Davis, Gray and Simpson (2000).

While models of conceptual development using “object” as a central metaphor are frequent, they have also been criticized (e.g., Confrey & Costa, 1996; Dörfler, 1996). There have been requests to clarify the meaning of a mathematical object (Godino & Batanero, 1998) and alternative frameworks have been suggested. For example, prototypes (Akkoc & Tall, 2002; Schwarz & Hershkowitz, 1999; Tall & Bakar, 1992), multiple representations (Borba & Confrey, 1996; Kaput, 1992; Keller & Hirsch, 1998) or combinations of frameworks into broader perceptions referring to “horizontal growth” in different forms of representation and “vertical growth” in the development from process to object (Schwingendorf, Hawks & Beineke, 1992, or analogous models, such as DeMarois & Tall, 1996, using the terms “facet” and “layer”).

Aspects of students’ conceptions of function related to different representations of function and semiotics are considered in a range of studies (e.g., Eisenberg & Dreyfus, 1994; Even, 1998; Hitt, 1998; Keller & Hirsch, 1998; Leinhardt, Zaslavsky & Stein, 1990; Thomas, 2003; Yerushalmy, 1997). These studies essentially deal with the students’ abilities to do transformations of functions and their properties from one system of representation to another, e.g. from algebraic to graphic representation. This includes studies that apply technologies promoting the use of several systems of representation (Bloch, 2003; Borba & Confrey, 1996; Moschkovich, Schoenfeld & Arcavi, 1993; Schwarz & Dreyfus, 1995). Frameworks applied in relation to questions of semiotics frequently use theories not exclusively related to the function concept, including registers of semiotic representation (Duval, 1999), epistemological aspects (Steinbring, 2005) as well as cultural perspectives (Radford, 2003).

2.2 Teachers’ knowledge, preservice teachers and functions

Even if there is a considerable and growing body of research on the nature and learning of the function concept, most of this research has focused on the students’ conceptions of function. Only a minor part of the research has addressed the teachers’ or preservice teachers’ cognitions and appropriate knowledge of functions (e.g., Chinnappan & Thomas, 1999, 2001; Cooney & Wilson, 1993; Even, 1990, 1993, 1998; Even & Markovits, 1993; Grevholm,
2000, 2004; Leikin, Chazan & Yerushalmy, 2001; Haimes, 1996; Norman, 1992; Thomas, 2003). The idea that a teacher’s content knowledge base will influence the quality of the understanding students develop in an area of mathematics has received support from research findings (Even & Markovits, 1993; Even & Tirosh, 1995, 2002; Fennema & Franke, 1992). This is not particularly surprising, since one might expect both lesson goals and structures to be dependent on teacher’s understanding of the subject matter. Even so, teaching is a complex practice influenced by a range of factors with various aspects of interest, such as social and cultural aspects, further teacher competencies, etc. (e.g., Alrø & Skovsmose, 2002; Borko & Putman, 1996; Cooney & Wilson, 1993; Lehrer & Lesh, 2003; Shulman, 1986; Skovsmose & Valero, 2002; Whitcomb, 2003).

Shulman (1986) introduced the concept of pedagogical content knowledge in relation to teaching activities, suggesting the existence of links between explanations and representations generated during teaching and content knowledge. The distinction between being able to apply a relatively well determined set of instructions to a mathematical problem and being able to explain why doing so is crucial in this context. Skemp (1976, 1978) distinguishes between instrumental (knowing how) and relational (knowing how and why) understanding, and why teaching and learning in mathematics risks promoting instrumental instead of relational understanding.

The notions of knowledge and understanding are multidimensional and described in various forms in the mathematics education literature (Even & Tirosh, 2002; Sierpinska, 1994). Skemp’s instrumental and relational understanding is, for example, largely mirrored by the procedural and conceptual knowledge of Hiebert with colleagues (Hiebert & Carpenter, 1992; Hiebert & Lefevre, 1986). Where procedural knowledge is a form of sequential knowledge constructed in a succession of steps, and conceptual knowledge may be considered as a well-connected web of knowledge for flexibly accessing and selecting information. Both kinds of knowledge are required for mathematical expertise, according to Hiebert.

Studies regarding preservice teachers’ conceptual knowledge of function are limited, and even more so on the topic of its consequences for their learning and future teaching (e.g., Chinnappan & Thomas, 2001; Cooney & Wieg, 2003; Doerr & Bowers, 1999; Even, 1990, 1993, 1998; Grevholm, 2000, 2004; Sánchez & Llinares, 2003; Wilson, 1994). Furthermore, studies like Leikin, Chazan and Yerushalmy (2001) and Thomas (2003) show that inservice teachers’ conception of function is not consistent with contemporary characterizations of the concept. Even (1993) stresses the importance of (secondary) mathematics teachers having a concept image of function consistent with the contemporary definition of function. She emphasizes an understanding
of functions “arbitrary nature” \(^1\) (p. 96) and an understanding of the requirement of uniqueness, and preservice teachers’ ability to use various representations of functions (Even, 1998).

Even (1993) also points out that courses in mathematics for preservice teachers should be constructed to develop a better, more comprehensive and articulated understanding and knowledge of functions (and mathematics). Vollrath (1994) suggests concepts as starting points for didactical thinking in mathematics. He sees discussions about “central concepts” (p. 63) as an essential part in courses for preservice teachers, and considers knowledge about the importance of concepts, their use and relationships to other concepts as vital for teachers’ planning and teaching in mathematics.

Eisenberg (1992) describes what he calls “having a sense for functions” (p. 154) as a major goal in the curriculum, describing this notion as having insights about functions incorporating the integration of many skills. These skills are often taught in isolation where compartmentalization of knowledge risks occurring when a body of knowledge splits into a larger number of isolated bits (Eisenberg here uses parts of the theory of Chevallard’s didactical transposition, Chevallard, 1985, concerning the change knowledge undergoes as it is turned from scientific, academic knowledge to instructional knowledge as taught in school). More concerns regarding knowledge compartmentalization are considered in students not being able to assimilate different forms of representations of functions (Leinhardt et al., 1990; Mamona-Downs & Downs, 2002), with impact on understanding, facility in manipulation, mental imagery, etc.

2.3 Concept maps and the concept of function

Few studies have been conducted using concept maps in mathematics education – see Paper II for further details – with only some accessing the students’ conceptual knowledge of function (such as, Doerr & Bowers, 1999; Grevholm, 2000, 2004; Leikin et al., 2001; McGowen & Tall, 1999; Williams, 1998). The process of drawing concept maps might be valuable for preservice teachers to stimulate metacognitive activities and provide preservice teachers rich opportunities to reflect upon the function concept and its relations to other concepts. Another reason to draw concept maps is that concept mapping taps into teacher competencies required for organizing knowledge for presentation. It becomes necessary for preservice teachers to consider and describe concepts and relations between different concepts in the composition of their maps.

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\(^1\) Even (1993) points out two essential features that have evolved in the definition of function: arbitrariness and uniqueness (referring to Freudenthal, 1983). The arbitrary nature of function refers to both the relationship between the two sets on which the function is defined and the sets themselves.

Note: Even (1993) and Freudenthal (1983) use the term univalence to describe the criterion of uniqueness of function. However, this term has been avoided as not to mix the criterion of uniqueness of function with the criterion of univalence in relation to analytic functions (i.e. injective analytic functions).
Williams (1998) indicates how differentiated conceptual knowledge of functions are expressed in concept maps, comparing concept maps of professors with PhDs in mathematics to first-year university students taking a course in calculus. Williams found, for example, that the concept maps of many students contained trivial parts and parts of an algorithmic nature. In contrast, the professors’ concept maps reflected many properties, categorical groupings, function classes and common types of functions. None of the professors’ maps demonstrated the students’ inclination to think of a function as an equation. Instead, they defined it as a “correspondence, a mapping, a pairing, or a rule” (p. 420). Concept maps are also used in studies concerning the students’ conceptual development, like McGowen and Tall (1999) to document the process by which college students construct, organize and reconstruct their knowledge about functions, during a course in algebra. They conclude that high performing students build rich conceptual frameworks on anchoring concepts that develop in sophistication and power, whereas lower achievers reveal few stable concepts with conceptual frameworks that have few stable elements.

Concept maps have also been used to study the conceptual development of preservice mathematics teachers, such as Grevholm (2000, 2004) in a longitudinal study of preservice teachers’ conceptual development, including the concepts of equation and function. The presented results indicate that preservice teachers’ cognitive structures slowly develop to become a clearer and richer structure. Doerr and Bowers (1999) use concept maps as a tool to study preservice teachers’ conceptions of the function concept related to students learning. The concept maps show the concept of function to be largely disconnected from pedagogical strategies or learning paths students might encounter. However, after a course designed to challenge the preservice teachers’ knowledge about learning and the concept of function, such knowledge is integrated with their understanding of the function concept, thus calling for preservice teachers to conduct such activities.
3 Theoretical framework

The theoretical framework outlined in this section is an elaborated and extended version of that presented by Hansson (2004). The section starts with a broad overview of the framework, followed by further descriptions of some main notions regarding the learners’ conceptual development of the function concept in particular.

3.1 An overview of the framework

The theoretical framework relates to the field of constructivism. Knowledge is an individual construction built gradually, and understanding grows as an individual’s knowledge structures become larger and more organized, where existing knowledge influences constructed relationships. Understanding can be rather limited if only some mental representations of potentially related ideas are connected or if the connections are weak. To promote understanding includes critical dimensions in mental activities of the learner, such as constructing relationships, extending and applying mathematical knowledge, reflecting about experiences, articulating what one knows, etc. (Carpenter & Lehrer, 1999). These activities might be found in environments where students can identify and articulate their own views, exchange ideas and reflect on other students’ views, reflect critically on their own views and when necessary, reorganize their own views and negotiate shared meanings.

From this perspective learners build their own knowledge and understanding through personal experiences and learning encompasses several dimensions including, cognitive, motivational, collaborative, social and cultural (e.g., Cobb, 1994; Fennema & Romberg, 1999; Steffe & Gale, 1995). Learning is not just a passive absorption of information, but rather more interactive, involving the selection, processing, contemplation and assimilation of information of the learner. In environments that promote learning, students should be offered a broad range of teaching strategies, taking into account what students already know, presenting concepts and general ideas, and attending to appropriate material and activities, including the posing of situations encouraging reflection and interactive communication with peers and teachers.

A connection to Human constructivism (Novak, 1993, 1998) was established through application and interpretations of concept maps motivated by Novak and associates as described in Paper II. Human constructivism acknowledges that education takes place in a social context and emphasis on the role of concepts and conceptual frameworks in human learning in a constructivist framework where learning theories of Ausubel (2000) and Ausubel, Novak and Hanesian (1978) are incorporated. To obtain successful learning students must acquire knowledge actively and establish relations between what is to be learned and what the students know, which as well involves aspects of metacognition. Learning becomes meaningful when the learners are given opportunities to relate, and choose to relate, new knowledge.
to prior knowledge in a non-arbitrary and substantive way. Thus, meaningful learning is knowledge construction in which students also seek to “make sense” of their experiences.

Carpenter and Lehrer (1999) also express the idea that knowledge and understanding changes are produced through experiences interpreted in the light of prior understanding. They identify a variety of activities and components of relevant tasks to increase the opportunities for students to acquire knowledge with understanding in learning environments promoting understanding (Fennema & Romberg, 1999). Carpenter and Lehrer focus on defining and discussing understanding within a framework of connecting ideas and developing knowledge structures that is analogous to a general framework presented by Hiebert and Carpenter (1992). Hiebert and Carpenter also reflect on the learning theories of Ausubel, which they label as a bottom-up approach referring to the idea that the students’ prior knowledge is essential in establishing situations where meaningful learning is promoted. This is also stressed by David Ausubel: “If I had to reduce all of educational psychology to just one principle, I would say this: The most important single factor influencing learning is what the learner already knows” (as quoted in Hiebert & Carpenter, 1992, p. 80).

In the applied framework of the study, understanding is not an all-or-none phenomenon as practically all complex ideas or processes can be understood at a number of levels and in quite different ways (e.g., Sierpinska, 1994). Therefore, it is more appropriate to think of understanding as emerging or developing rather than assuming that someone does or does not understand a given topic, idea, or process. Moreover, it is not sufficient to think of the development of understanding simply as the appending of new concepts and processes to existing knowledge. Developing understanding involves the creation of rich, integrated knowledge structures. This structuring of knowledge is one of the features that makes learning with understanding generative (Carpenter & Lehrer, 1999), i.e. when students acquire knowledge with understanding, they can apply that knowledge to learn new topics and solve new and unfamiliar problems. When students do not understand, they perceive each topic as an isolated skill and cannot apply their skills to solve problems not explicitly covered by instruction or extend their learning to new topics.

More framework components are described below. Formal concepts like the concept of function are frequent and essential in mathematics and are considered in the framework by including the notion of concept image (Tall & Vinner, 1981; Vinner, 1991, 1992). This provides a means to describe relations between an individual’s conceptions of a mathematical concept and its definition. Moreover, in the conceptual development of the function concept, process and object conceptions are considered important, in common with various frameworks described in section two. The framework favors a property-oriented view in the development of object conceptions. Additional components are included in the framework and further described below.
3.2 Further notions of the framework

There is a considerable range of theoretical frameworks in practice within the field of mathematics education (Lerman & Tsatsaroni, 2004; Niss, 1999; Sierpinska, 2003; Steffe, Nesher, Cobb, Goldin & Greer, 1996). For the concept of function, a variety of frameworks are being used, as noticed in section two. I believe the observation made by Eisenberg (1991) to be still valid: when discussing learning associated with functions, there is no generally accepted theoretical framework as a basis for discussion. In the adopted framework of the study is human cognition based on a model of connected representations of knowledge. The processes of conceptualization of mathematical concepts apply the notion of concept image, which is further elaborated below with an emphasis on the concept of function.

3.2.1 Knowledge structures

Knowledge is represented internally, and understanding is described in terms of how an individual’s mental representations are structured. Internal representations can be linked, forming dynamic networks of knowledge with different structures, especially in forms of vertical hierarchies and webs. The network nodes can be thought of as knowledge entities of represented information and the threads between them as connections of relationships. Understanding grows as these cognitive structures become larger and more organized, where existing structures influence constructed relationships, thereby helping to shape the new, formed structures. The construction of new relationships may force a reconfiguration of affected structures. Ultimately, understanding increases as the reorganizations create more richly connected, cohesive knowledge structures.

Ausubel (e.g., 1968; 2000; Ausubel, Novak & Hanesian, 1978) describes learning in terms of “meaningful learning” as opposed to “rote learning” with consequences to linkages in a network model of an individual’s cognitive structure, where previous knowledge is essential in the learning of new knowledge, “only in rote learning does a simple arbitrary and nonsubstantive linkage occur with pre-existing cognitive structure” (Ausubel, 2000, p. 3). Hiebert and Carpenter (1992) outline an analogous approach using network metaphors in a general framework that is related to a variety of fields working with cognition. Hiebert and Carpenter refer to Ausubel’s theories as a bottom-up process in the way knowledge structures develop upon prior knowledge and describe a view of “learning with understanding” similar to Ausubel’s “meaningful learning” in the formation of internal dynamic knowledge.

1 The kind of network-based model of knowledge representation applied in the current framework is frequent in different fields of work with an interest in cognition and learning within psychology, cognitive science, neuroscience, linguistics, and others (e.g., Anderson, 2000; Baddeley, 1997; Gärdenfors, 2000; Hiebert & Carpenter, 1992). Furthermore, network models are used as support for learning in general frameworks in mathematics education, like “webbing” by Noss and Hoyles (1996).
structures, where structure and linkage are vital for understanding. Further contributions on these connections are presented in Papers I and II of this thesis.

The framework also includes the notions of conceptual knowledge and procedural knowledge (e.g., Hiebert & Carpenter, 1992; Hiebert & Lefevre, 1986). Procedural knowledge is primarily concerned with knowledge related to actions performed in a sequence of steps, whereas conceptual knowledge concerns knowledge rich in relationships like a web of knowledge for flexibly accessing and selecting information. Both types of knowledge are regarded as necessary for mathematical proficiency.

### 3.2.2 The notion of concept image

Vinner (1983, 1992) describes a model for the correspondence between the definition of the function concept and an individual’s understanding of the concept, which is applicable to formal concepts in general (Vinner, 1991). The key idea is the distinction between concept image and concept definition. Concept definition concerns a form of words used to describe a concept (it may be a personal concept definition different from a formal definition accepted by the mathematics community). The notion of concept image refers to the total cognitive structure in the mind of an individual that is associated with the concept, including “all the mental pictures and associated properties and processes” (Tall & Vinner, 1981, p. 152) and as such includes different forms of representation, etc. A concept image is built up during a longer period of time through an individual’s various experiences. The portion of a concept image activated at a particular time is called the evoked concept image. In thinking, the concept image will almost always be evoked, where the concept definition will remain inactive or even forgotten. When students meet an old concept in a new context, it is the concept image with all the implicit assumptions abstracted from earlier contexts that respond to the task.

In the process of learning, evoked concept images influence new constructions of relationships and reconfigurations of related cognitive structures during the interaction of new knowledge and relevant concepts in the existing knowledge structures. These reconfigurations may be local or widespread across numerous structures when different parts of a concept image are evoked or parts of several concept images are evoked, and thus one or several different concepts are called to mind. Reorganizations appear both as new insights, local or global, regarding one concept or relations between several concepts, as well as appearances of temporary confusion. Understanding increases when the reconfigurations provide more richly connected, cohesive knowledge structures.

A concept image might consist of less interconnected knowledge structures. This means for example that different forms or representations of a concept might correspond to segmented knowledge structures and thus develop in branches separate from each other – with consequences for properties related to the concept that can thus be linked to a specific form of concept representation.
or context. In the case of the function concept, this might for instance lead to situations where the graph of a function intersects the x-axis without a student realizing that the intersection points are related to zeros of the function, i.e. $f(x)=0$, or that a function graph symmetric about the y-axis represents an even function, i.e. $f(x)=f(-x)$. A concept image might also contain prototypical elements where a concept, e.g. the function concept, is greatly associated with a specific class of functions, properties of a particular function, or a certain representation of function, as well as representations of specific situations related to functions etc. Consequently, constructions of a concept image might raise obstacles in the conceptual development or cause evoked concept images with conflicting pieces of information.

3.2.3 Conceptual development and properties of functions

In students’ conceptual development of function, Sfard (1989, 1991, 1992) suggests a process-object model that may be applied to mathematical concepts in general, where the formation of an “operational” conception of function as a process precedes a later more mature phase in the formation of a “structural” conception regarding functions as objects. Both conceptions are essential and should coexist to form a dual view of the concept, according to Sfard. In the transition from operational to structural conception, a three-step pattern emerges: interiorization, condensation, and reification. Reification is the final step giving an individual the ability to conceptualize a concept as an object. Without reification an individual’s conceptual understanding will remain purely operational.

In the applied framework a structural conception is offered by a richly interconnected and structured concept image, gaining access to a view of the function concept as an object with a range of properties that are not specifically connected to a form of representation. In the process of developing a structural conception of function, Slavit (1997) suggests an emphasis on the functions’ properties to enhance the phase of reification, where the notion of invariance is significant (Bagni, 2003). He suggests a “property-oriented view” of functions to stimulate the development of a structural conception, basically through two types of experiences:

First, the property-oriented view involves an ability to realize the equivalence of procedures that are performed in different notational systems. Noting that the processes of symbolically solving $f(x)=0$ and graphically finding x-intercepts are equivalent (in the sense of finding zeroes) demonstrates this awareness. Second, students develop the ability to generalize procedures across different classes and types of functions. Here, students can relate procedures across notational systems, but they are also beginning to realize that some of these procedures have analogues in other types of functions. For example, one can find zeros of both linear and quadratic polynomials (as well as many other types of functions), and this invariance is what makes the property apparent. (Slavit, 1997, p. 266-267)
Slavit argues that experience with various examples of functions, and noticing their local and global properties (like, intercepts, points of inflection or periodicity, asymptotes, etc.) promotes reification, and that students would conceive functions as objects possessing or not possessing these properties. Considerations about properties of functions would thus be of significance in developing a more mature view of the concept of function. Moreover, it is possible that structural conceptions might develop within different forms of concept representation, in branches of the concept image, before a concept image is structured and interconnected to offer a view of the concept as an object with properties related to the concept rather than different representations of the concept. A similar course of action is possible for concepts other than functions in the development of structural conceptions and thus in the construction of a more cohesive concept image for the concept in question.

A property-oriented view of conceptual development emphasizes students to experience concepts in different contexts and notational systems to realize the invariance of properties. This will also bring a semiotic dimension in the process of conceptualization, as possibly described by Steinbring’s model (e.g., 1998, 2002, 2005) called “The epistemological triangle”.

The model illustrates connections between a concept, mathematical signs, a reference context, and the mediation between signs and reference contexts that is influenced by the epistemological conditions of mathematical knowledge. In the frame of mathematical knowledge it is possible to make different interpretations of the connections between the three corner points of the model.

The symbolic notation in Steinbring’s model, i.e. “sign/symbol”, is considered in a manifold of forms: verbal, tables, operations, diagrams, formulas, system of equations, graphs of functions, etc. The model may well be applied in a range of contexts, and specifically in the development of a property-oriented view of function advocated by Slavit (1997) and thus in the development of students’ concept image of function. This will be achieved if, for instance, “object/the reference context” is a specified function, “concept” is a particular property of the function, and “sign/symbol” denotes a notational system that, in accordance to Slavit, should vary in students’ experiences. These experiences would successively include a range of functions and their properties and none properties in different contexts. Further considerations related to
semiotics and properties of functions and the theoretical implication is a topic for future research as delineated in section 6.3.
4 Summary of the papers
The five papers Hansson (2004a, 2004b, 2005a, 2005b) and Hansson and Grevholm (2003) concern preservice teachers during a four and a half year long teacher preparation program in mathematics and science for grades 4 to 9. The papers are part of a larger study conducted until the sixth term, during the concluding mathematics courses of the program, with the exception of Hansson and Grevholm (2003) that also includes preservice teachers in their third term. The sixth term contains a calculus course where the concept of function is central; the preservice teachers’ view of function is primarily considered after the course. The five papers describe parts of a larger study that essentially was conducted during the spring terms of three consecutive school years. The papers comprise data collected from three groups of preservice teachers in their sixth term of the educational programme and data from one group of students in their third term. The papers of this thesis do not, however, sum up all parts of the conducted study and more details are given in section 4.2.6.

4.1 Aims of the study
The aims of the research are related to preservice teachers’ views on the concept of function. Preservice teachers’ conceptions of function are to a large part examined in relation to mathematical statements, where the study successively has been expanded to include \( y=x+5 \), \( y=\pi x^2 \) and \( xy=2 \). The statements can be linked to different concepts and topics on a variety of levels, as were more comprehensively discussed in section 1.3.1. In particular, they can be seen to represent real valued functions of a real variable in using a form of representation not uncommon for the concept of function (Eisenberg, 1991, 1992; Tall, 1996; Yerushalmy, 1997).

The three statements in question may evoke several concepts and various chains of associations in the preservice students’ reasoning that span a network of relations between concepts in the context of the statements. One aim of the study is to examine the preservice teachers’ views of the function concept in this setting, and how the preservice teachers relate to different properties and classes of functions and how they perceive relationships of various concepts to the concept of function. Moreover, the statements are selected so that the network of relations may include a range of concepts, applications, representations, teaching scenarios, etc., that are relevant to the preservice teachers, as argued in section 1.3.1.

In the study of relations between different concepts that the students understand a mathematical statement to represent, concept maps is obviously of interest. However, the application of concept maps concerning a mathematical statement is not common and may well be considered as an unorthodox utilization of concept maps, since a statement might be considered to not represent one, but a range of different concepts. Another purpose of the study is to investigate how preservice students respond to the task of constructing
concept maps derived from a mathematical statement and questions about the principles of how to interpret such maps.

The study also aims to examine preservice teachers’ conceptions of the function concept apart from mathematical statements, to compare with other groups of mathematics students. Further aims are to examine the preservice students’ opinions as to what extent functions are significant in mathematics as well as present in school mathematics, deemed to be relevant aspects of their view on the concept of function as prospective teachers. The final part of the study examines the effects of an intervention study where previous results of the study are considered in the design of the intervention.

The aims of the study, as per each paper, are included in the following section together with the research questions of the papers.

4.1.1 Aims and research questions of the papers

Below is a summary of the aims and research questions of the five papers:

Paper I: What conceptions do preservice teachers have and what is their concept of function in connection to \(y=x+5\)? What progression can be seen between two groups of students in their third and sixth terms from a teacher preparation program?

Paper II: The aim is to investigate the use of concept maps to reveal the knowledge and understanding of preservice teacher regarding the concept of function in relation to the mathematical statements \(y=x+5\) and \(y=\pi x^2\). More specifically: How do preservice teachers construct concept maps starting with the mathematical statements \(y=x+5\) and \(y=\pi x^2\)? How is the concept of function expressed in the maps? What knowledge is displayed and what qualities are desirable in such a map? What experiences of drawing the maps do the preservice teachers express?

Paper III (this paper describes a limited set of data presented in the previous paper with some overlap of the research questions): The purpose is to examine preservice teachers’ conceptual understanding of function in relation to \(y=x+5\) and \(y=\pi x^2\) through the utilization of concept maps. In particular: How are the concept of function and its network of relations to other concepts expressed in the maps? What properties of the function concept do the students choose to include in their maps? How do the students relate to teaching and learning in the context of the two statements? What opinions do the students express about the process of drawing maps derived from \(y=x+5\) and \(y=\pi x^2\)?

Paper IV: The purpose is to investigate how the function concept is expressed in relation to \(y=x+5\), \(y=\pi x^2\) and \(xy=2\) for preservice teachers at various stages of performance in their studies of mathematics. In particular: What conceptions of function do the preservice teachers have and what properties of function do they notice? How are relations between the concept of function and other concepts described in the context of the mathematical statements?
Paper V: The aim is to examine preservice teachers’ conceptions of the function concept, their conceptions of the significance of functions in mathematics and the presence of functions in school mathematics. A further aim is to study the effects of an intervention study concerning the concept of function. Based on the aims of the study the research questions are: What conceptions do preservice teachers have of the function concept? How do preservice teachers view the significance of functions in mathematics? How do they perceive the presence of functions in school mathematics? How does the intervention make a difference to the preservice teachers’ conceptions of the function concept?

4.2 Methods and design of the study

The teacher preparation program includes mathematics courses for 30 weeks of full-time study, where approximately one-third is related to mathematics education. The courses are distributed during the first, third and sixth terms of the educational programme, and correspond to a five-week introductory course in mathematics during the first term, a ten-week course about algebra and number theory during the third term, and courses in statistics (3 weeks full-time work), calculus (5 weeks) and geometry (7 weeks), during the sixth term. The groups participating in the study involved all preservice teachers in the teacher preparation program who specialized in mathematics and science, for each term the study takes place.

4.2.1 Paper I: Preservice teachers’ conceptions of y=x+5: Do they see a function?

The study includes one group of preservice teachers in their third and one group in their sixth term of the teaching training program. Grevholm (1998, 2002, 2004) began the study with a group of 38 preservice teachers studying algebra in their third term. One questions posed to the students in Grevholm’s study considering y=x+5 was inspired by Blomhøj (1997). Hansson (2003) repeated the study 3.5 years later with a group of 19 preservice students enrolled in a calculus course during their sixth term.

A survey including an open question about y=x+5 was used in the study. Grevholm (1998, 2002) categorized the answers to the question. The questionnaire was distributed before and after each course during a mathematics lecture and the answers were compiled for each group of preservice teachers. To guarantee that the groups’ answers were consistently divided into categories, the authors separated the answers from both groups of students into the six categories independent of each other. The results were compared and discussed and the procedure repeated until the division was such that the distribution of answers from the algebra group corresponded with the one from Grevholm (1998, 2002, 2004). Examples of how the answers were divided into categories are given in Paper I.
The preservice teachers’ survey answers were often brief. Eight students were thus interviewed after the algebra course to obtain a better understanding of their views on y=x+5. The interviews were recorded on tape. The students were selected based on their answers from the survey. A further seven students were interviewed after the calculus course, four were recorded on tape and notes were taken during the other three interviews. The interviews were conducted on the basis of the preservice teachers’ answers to the survey and each was given a chance to further expand their answers. Grevholm conducted the interviews and the transcription of the answers for the students in the third term and Hansson did the students in the sixth term (interview excerpts are disclosed in Grevholm, 1998, 2002, and Hansson, 2003).

To continue studying how preservice teachers view the statement y=x+5, Hansson used concept maps in an unorthodox manner by allowing the sixth-term students to draw concept maps derived from y=x+5 after the calculus course. A more detailed description of the procedure and how the maps were analyzed are presented in section 4.2.2.

4.2.2 Paper II: An unorthodox utilization of concept maps for mathematical statements: The responses of preservice teachers to a potential diagnostic tool

Two groups of preservice teachers participated in the study. The study was conducted over a period of two years, when each group was in the sixth term of the teaching program and had completed the course in calculus.

There are several studies in mathematics education that involve various types of maps, all referred to as “concept maps” (see Paper II). This agrees with the aims of the study to investigate how the preservice students construct different types of concept maps in relation to the mathematical statements, and how they describe the experience of drawing these maps. I have chosen to investigate how the students construct two common types of concept maps: concept maps with a freely chosen structure and with a hierarchical structure.

The first group consists of 19 students (the same group as in Paper I). The preservice teachers were introduced to concept maps during a lecture related to mathematics education; see Paper II for more detail. Each was then instructed to draw an individual concept map for y=x+5 in the manner of their choice. A week later, the preservice teachers drew a new map for y=x+5, though this time, they were instructed to construct the map in a hierarchical format. On the third occasion, they could comment on their maps by answering a set of questions that are further discussed in Paper II.

All of the concept maps were analyzed. Each map was analyzed as an integrated unity in which its contents and structure were noted. Furthermore, how the different sections of the map were related to each other was also studied. In particular, how the function concept was expressed on the maps and its relationship to other concepts and properties that were assigned to functions. The contents of the maps, the preservice teachers’ answers to the questions and
their comments on the maps were compiled in tables. A primarily quantitative analysis and a method of categorization based on the contents of the nodes and the number of links connecting them to other nodes was also tested. The quantitative analysis was abandoned in favor of a more qualitative analysis of the maps, since it seemed that valuable information about how different parts of the map were connected and how relations between different concepts were described was lost.

The study was repeated with a group of 25 preservice teachers the following year. Based on an analysis of the maps and comments from the previous year’s participants and the experienced gained, the hierarchical maps appeared to have greater potential for providing information about the preservice teachers’ perception of the function concept in relation to \( y=x+5 \). At the same time, this type of map was considered more demanding for the students to draw and was therefore often less detailed than the map that had been freely structured. This resulted in the students who participated in the study the following year drawing the maps on one occasion and were directed to begin with the freely formatted map before constructing the hierarchical map. Furthermore, the study was expanded to include \( y=\pi x^2 \). An analysis of the maps was conducted as in the previous year.

4.2.3 Paper III: Preservice teachers’ views of \( y=x+5 \) and \( y=\pi x^2 \) as expressed through the utilization of concept maps: A study of the concept of function

This study is part of the study described in Paper II and focuses on how the group of 25 students view the concept of function in relation to \( y=x+5 \) and \( y=\pi x^2 \) through the use of concept maps. In the current paper the preservice teachers’ account of the function concept and its properties and networks of relations in particular are considered.

After they finished the calculus course the preservice teachers were introduced to concept maps during a lecture in mathematics education as a part of the educational program. Examples of different types of concept maps were displayed during the introduction, such as non-hierarchical web-based maps and hierarchical maps where the nodes represented concepts and the links were labeled in each case. Concept maps derived from mathematical concepts were, however, largely avoided during the presentation, to not influence the contents of the maps drawn by the students in the subsequent assignment.

After the introduction, the preservice teachers were each directed to draw concept maps based on the statements \( y=x+5 \) and \( y=\pi x^2 \), for an hour or more, depending on when the students decided they had completed the task. In the process of drawing maps with a hierarchical structure the students started to draw maps with a freely formatted structure, as also described in 4.2.2, resulting in the preservice teachers constructing two maps for each mathematical statement – one with a freely formatted structure and one with a hierarchical...
structure. The preservice teachers were then asked to comment on their maps and their experiences of drawing maps through a set of questions.

Each of \( y=x+5 \) and \( y=\pi x^2 \) have the potential to represent a number of different concepts for an individual, as argued in 1.3.1. Both statements use a common form of representation for real valued functions of one real variable, and after the students had completed the calculus course the concept of function was thus expected to be included in their maps.

The construction of concept maps derived from a mathematical statement gives students an opportunity to illustrate what concepts they understand the statement to represent. Moreover, the maps allow the students to account for their perception of relations between the derived concepts, what properties the concepts have, how their different properties are related to each other, etc. The concept maps thus offer an option to study various chains of associations in the preservice students’ reasoning, their conceptions of different concepts in the context of \( y=x+5 \) and \( y=\pi x^2 \), and their perceptions of relationships between the derived concepts and accordingly how different parts of students’ knowledge are related.

All of the maps were analyzed in accordance to the analysis described in section 4.2.2, with further information in Paper III.

4.2.4 Paper IV: The views of preservice teachers on three mathematical statements: A case study regarding the concept of function

A group of 25 preservice teachers participated in the study while in the sixth term of the teaching program (the same group as in Papers II and III). A questionnaire\(^1\) was distributed before and after the calculus course, during a lecture in mathematics. The questionnaire contained questions asking the preservice teachers to describe a function, and answering open questions in relation to the statements \( y=x+5 \), \( y=\pi x^2 \) and \( xy=2 \). To study their views on the three statements and ask questions about their views, 20 students from the group were interviewed. The interviews were based on the answers the preservice teachers had given on their two questionnaires, where they had the opportunity to comment on and expand their answers, draw pictures, etc. (Further questions from the interviews are considered in Paper V, as described in 4.2.5). The preservice students also drew concept maps\(^2\) for \( y=x+5 \) and \( y=\pi x^2 \) after the calculus course, as described in Papers II and III.

Since the case study in Paper IV is based on the interviews, the method and the practical arrangements will be described in more detail. The students who were willing to be interviewed signed up for participation by writing their names

\(^1\) The current survey differs from the one in the study described in Paper I and includes, e.g., questions about \( y=\pi x^2 \), \( xy=2 \) but also questions related to Paper V; with the exception of the introductory questions and the question on \( y=x+5 \). The introductory questions were kept to create similar conditions for later groups of students in the study which attended the calculus course when they were to answer the open question on \( y=x+5 \), in case a comparison of the groups should be of interest.

\(^2\) To draw concept maps is a time-consuming process, and there was not an opportunity to draw concept maps for all three mathematical statements.
on a timetable. The interviews were conducted separately in a preparation room for mathematics during a three-week period. The length of the interviews varied, but often lasted for an hour or more and touched on all the questions on the questionnaire. If the student did not mention the function concept in relation to the three statements, I could ask them to comment on the statement in view of the previous questions in the survey, where the function concept is discussed. The interview was based on the preservice teachers’ answers to the survey, which they commented on. However, to obtain a better understanding of the preservice teachers’ thought processes I also asked questions tied to their answers. An interview normally took place during a predetermined time interval, which meant that I sometimes had to move on to the next question.

Brief notes were usually made after an interview to summarize the results. Each interview was recorded on tape and personally transcribed according to recommendations made by Kvale (1996) (this is described in more detail in Paper IV). Punctuation was inserted to make the transcription easier to understand. Sections raised in the interview were selected after the transcription of the interview had been read repeatedly and sections of the tape listened to. Recollections and notes from the interviews were also used. Sentence interpretation and focusing took place during the analysis of the interview, according to the method prescribed by Kvale (1996).

Of the 20 students interviewed, 3 were selected to participate in a case study. The selection was based on their performance in the mathematics courses during the sixth term. Their grades for the calculus course were “high pass”, “pass” and “fail”, respectively. Their performances on other mathematics courses confirm their distinctly different levels of competency in mathematics. In selecting the three preservice teachers, consideration was also given to their concept maps that were among the more detailed maps within the group with respect to the number of nodes.

4.2.5 Paper V: Preservice teachers’ conceptions of the function concept, its significance in mathematics and presence in school mathematics

The study is conducted for two consecutive spring terms regarding the calculus course, each representing the sixth term in the teacher education program for both groups of preservice students, i.e. one group for each term. Each group consists of 25 students and 17 students. The first group is the same group of preservice teachers that participated in the studies described in Papers II, III and IV.

The second group of preservice teachers took part in an intervention study regarding the concept of function that was partly designed with data from the previous group in the study. The intervention occurred during the first two weeks of the calculus course and is further described in Paper V. I implemented the intervention and the teacher who supervised the previous group in the study also supervised the remainder of the calculus course for the latter group.
In accordance with the method described in 4.2.4, a questionnaire was distributed to each group before and after the calculus course and handed in on each occasion. Students were asked voluntarily to sign up for an interview, implemented during the weeks after the calculus course. I conducted the interviews, which were based on the students’ answers to the questionnaire, as described in section 4.2.4. The preservice teachers were thus asked to study their answers to each question from before and after the calculus course, and comment upon them. Follow-up questions were asked to clarify the preservice students’ reasoning. The interview time varied, but often lasted for an hour or more. Each interview was recorded on tape and transcribed. Brief notes on the outcome were usually taken after an interview. Interviewed students totaled 20 from the first group and 9 from the second group. The interviewed students represent, for each group, students on a variety of levels of performance in their studies of mathematics.

Three of the questions on the questionnaire are considered in the current paper. In the questions, students are asked to account:

• their interpretations of the concept of function,
• their opinions about the extent to which functions are of significance in mathematics, and
• to what extent, in their opinion, functions are present in school mathematics.

The questions were open ended and frequently became a starting point for the preservice teachers’ further reasoning about the function concept in mathematics and school mathematics during the interviews.

The students’ written answers to the three questions from before and after the calculus course were compiled in tables for each group. The students’ interpretations of the function concept – their answers to the first of the three questions – were categorized in accordance to a categorization presented by Vinner and Dreyfus (1989). The choice of categorization is partly based on correlations from the theoretical frameworks in the use of the concept image, but also from the fact that the categorization was developed from a large number of participants, i.e. calculus students majoring in a variety of areas and a group of junior high school teachers in mathematics. The categorization thus gives an opportunity to compare the preservice teachers’ conceptions of function with different groups of students and in-service mathematics teachers. Vinner and Dreyfus both confirmed in a correspondence\(^1\) during the preparation of the current paper that they have not developed the categorization further – the categorization is an extended version of that created by Vinner (1983).

A categorization was constructed from the students’ answers to the significance of functions in mathematics – the second of the three questions – and is further described in Paper V. The categorization was derived from several distinctive themes in the students’ written answers. The themes were confirmed

\(^1\) The correspondence was conducted via e-mail the 29th and 30th of June 2005.
in the corresponding interviews where students were able to expand on their answers, if they were interviewed. The interviews gave valuable support in the selection of themes extracted from the students’ written answers and thus of importance in the creation of the categorization.

The students’ answers to the question of the presence of functions in school mathematics – the last of the three questions – were analyzed as with the question of the significance of functions in mathematics, and different themes were extracted from the answers. But no categorization was constructed because the students frequently diverged from their written answers during the interviews when they were given an opportunity to reflect upon the presence of functions in school mathematics. Hence, a categorization of answers did not seem to represent the preservice teachers’ conceptions of the presence of functions in school mathematics. This was not the case when the students reasoned the significance of functions in mathematics, when they were usually able to expand on their answers during the interviews, without making contradictions or obviously diverging from their previous opinions.

Finally, note that more data was collected than what is presented in the paper. The second week of the intervention was basically dedicated to properties of different classes of standard functions, and the first week was generally dedicated to the concept of function introduced as a special type of relation. The remaining data are basically related to questions about different properties of functions that students recognize and use in the context of problem solving – this was carried out in both groups of preservice students. The data consists mostly of solutions to problems that students solved individually and in the groups.

4.2.6 Methodological discussion

The study was basically conducted during a period of three years with three groups of preservice teachers at the end of their required courses in mathematics in an educational programme, during which a range of research questions was considered and variety of data collected. The overarching theme of interest was the preservice teachers’ conceptions of the function concept. The data described in the five papers is basically comprised of concept maps, questionnaires, and interviews. But the collected data also includes preservice teachers’ solutions to problems I gave them, video recorded sessions of preservice teachers discussing and solving problems in groups, problems constructed by preservice teachers with aspects they consider of importance in relation to functions, and more. Focus on the preservice teachers’ reasoning about functions in relation to problem solving was primarily conducted with the last two groups in the study, which is planned to be further described in future papers.

The five papers included in this thesis thus cover parts of a larger study. The papers neither summarize all the questions considered during the conducted study period nor include all collected data. Instead, the papers describe parts of the conducted study and focus more extensively on how the preservice teachers
reason the concept of function in relation to $y=x+5$, $y=\pi x^2$, and $xy=2$ in the Papers I-IV, as well as the use of concept maps in this context. Questions concerning how preservice teachers characterize the concept of function compared with other groups of students are also considered in Paper V, along with to what extent preservice teachers consider functions of significance in mathematics and how they consider functions to be present in school mathematics. Further questions relate to the intervention conducted in the third year of the study of how to facilitate rich experiences of the concept of function for preservice teachers. Section five further elaborates considerations on how teacher-training programs might support preservice teachers’ conceptual development of function as a result of the study.

Even if I personally hold a constructivist perspective that includes social and cultural perspectives – which I believe is more clearly visible when considering all parts of the conducted study – the data used in Papers I-V might not clearly illustrate this. The individual views of the preservice teachers in the five papers have been highly regarded, whereas collected data involving aspects of how preservice teachers working in groups articulate their own views, exchange ideas and reflect on other students’ views and negotiate shared meanings, etc., have not been used. In retrospect, this may well be a consequence of me as a teacher usually meeting the students in groups and that in the analysis of the collected data, my own curiosity has given more attention to how preservice teachers reason as individuals. The conducted study contains extensive data of preservice teachers working in groups during the calculus course, but which are primarily related to problem solving and not used in relation to the research questions in Papers I-V.

According to the theoretical framework, the knowledge of individuals is represented internally and the methods used in the study emphasize data collected individually to examine the preservice teachers’ different conceptions of function. The research design includes both quantitative and qualitative elements. Furthermore, different types of data are gathered during the study, such as questionnaires, interviews and concept maps, to enhance the validity of the findings and offer a more complete picture of the preservice teachers’ conceptions about the function concept. I was the instructor during the concept mapping and intervention and also conducted the interviews. This ensured the treatment was applied as designed. I was known as a teacher by the preservice teachers, which facilitated natural interactions in the interviews. I have frequently used codes to represent individuals rather than names in the compilation of data to prevent my previous knowledge about the preservice teachers influencing the data analysis. Furthermore, the data collected during the study includes information of the preservice teachers’ gender. However, an analysis of the data specifically related to gender has not been performed due to time restraints of the study.

An essential part of the collected data, concept maps are used to study conceptual relationships (Duit, Treagust & Mansfield, 1996; Novak, 1998;
Novak & Gowin, 1984; Williams, 1998) and investigate the preservice teachers’ conceptual understanding as the studies presented in section 2.3. But in contrast to those studies, the concept maps in the current study start with a mathematical statement (as elaborated in Paper II) to further examine the preservice teachers’ different conceptions related to the mathematical statements and the concept of function. Concept maps are mainly considered as a qualitative tool in the study, though their construction by the whole group of preservice teachers is described in the papers of this thesis. Other deciding aspects to use of concept maps in the study were, e.g. the fact that the construction of the concept maps tapped into the teachers’ knowledge in the requirement to organize and display knowledge, which are important skills for prospective teachers.

Even if the preservice teachers in the study were usually familiar with the notion of concept mapping from other subjects during teacher training, it takes practice to be accomplished in drawing concept maps, as is also noticed by Novak (1998). A lack of drawing concept map experience might have affected the preservice teachers construction of their concept maps in the negative with fewer links and concepts in the map. Moreover, even though the maps possess a wealth of information, it may be difficult to interpret concept maps and understand an individual’s thought process during the construction of these maps, and a degree of subjectivity is probably unavoidable when interpreting concept maps (Novak, 1998). Furthermore, in analogy to the open questions about $y=x+5$, $y=\pi x^2$, and $xy=2$ on the questionnaire, where the preservice teachers were free to make their own interpretation of the statements, the preservice teachers were free to include concepts in their maps according to their own ideas. Hence, possibly excluding concepts from the maps that could have been of interest to the research questions in the study. This might be a limitation in the application of concept maps in the study.

The interviews described in 4.2.4 and 4.2.5 permitted further study of the preservice teachers’ view on the concept of function. I informed the interviewee that I was interested in how she or he thought and reasoned. The interviewee studied the two questionnaires from before and after the calculus course (the questionnaires were collected at each occasion), and was asked to comment the answers. A preservice teacher was thus given the opportunity to reflect upon and comment on the answers for each question in the questionnaire and contemplate on how the answers had gradually changed during the calculus course in which the concept of function was a central concept. I asked further questions, probe and follow up questions (Kvale, 1996, 1997), in relation to the preservice teacher’s answers to understand how the reasoning of the preservice teacher.

The intervention, i.e. the last part of the study, was conducted with the intention to develop the preservice teachers’ view on the concept of function. Previous results (Hansson, 2004) were considered in the implementation of the intervention. Some important factors during the implementation of the intervention were to illustrate different properties of functions, to give experiences of functions in different contexts and representations, to stimulate
feedback from evoked concept images to the concept definition by problems and examples, and to provide the idea to picture the concept of function as a special type of relation. I supervised the two-week intervention, and wrote handouts to complement the course literature. Some lectures were also tape recorded during the intervention. I regularly took notes after lectures and associated problem solving sessions to collect students’ responses on topics and problems brought up for discussion, to be able to include some of these aspects in the description of the intervention in Paper V.

Further comments on the applied methodology are presented in section six.

4.3 Main results

4.3.1 Paper I: Preservice teachers’ conceptions of y=x+5: Do they see a function?

The written answers show a similar development in both groups. The preservice teachers use to a higher degree a numerical interpretation of y=x+5 before the course, which decreases after the course with a growth in linear and functional interpretation, with the existence of two variables as a large and rather stable category. For a majority of preservice teachers in both groups, the concept of function is not evoked in connection to y=x+5. The preservice teachers seem to have a tendency to use mathematical knowledge on a less advanced level than they have worked within their mathematics courses. More developed views of the function concept, as an object with many properties, are hardly visible. This became apparent in the written answers but it was also clearly visible in the concept maps. The concept maps contain more information than the written answers, and usually more developed in the area of a straight line. The preservice teachers’ concept of function is not seemingly represented by a rich cognitive structure in the context of y=x+5.

Examining data for the individual students confirm that concepts develop slowly. More than every second student gives answers in the same categories before and after the course. Other students just add an extra category. The group of preservice teachers who had progressed further in the teacher preparation program had a slightly more elaborated language (though not yet elaborate enough to become successful as an inservice teacher) and flexible way of looking at y=x+5, where, for instance, an interpretation related to the concept of equation was more common.

Finally, it is said in the paper that the preservice teachers did not make the mistake of describing x as being 5 greater than y, like the compulsory students did in the study by Blomhøj (1997). However, there is one analogous line of thought in the case of preservice teacher F7 (Paper I, p. 5), who gives a table of values with mixed values of x and y. Another is M6 who mix the x-axis and y-axis in his comment on the values of x and y when the line intersects the coordinate axis.”
4.3.2 Paper II: An unorthodox utilization of concept maps for mathematical statements: The responses of preservice teachers to a potential diagnostic tool

The preservice teachers construct their concept maps in a wide range of different styles when drawing them with a structure they find suitable. There is a tendency to let the mathematical statement become a hub surrounded by other parts of the map, where different parts have varying degrees of web structure with few tendencies to become hierarchical. Even if the preservice teachers’ concept maps have numerous links, they do not always seem to illustrate meaningful relationships between different concepts. However, the hierarchical maps became, for natural reasons by the given instructions, more homogenous in structure. When the preservice teachers in the first group constructed the hierarchical maps apart from their non-hierarchical maps, the hierarchical maps contained fewer links and the mathematical statement was very much linked to concepts it was deemed to represent. When the second group constructed their concept maps on the same occasion, there was a higher degree of overlap in both structure and content between the concept maps. The preservice teachers usually mix general with more specific concepts when they draw hierarchical maps.

The concept of function is not a well-integrated concept and is rarely well developed in the preservice teachers’ maps, indicating the preservice teachers’ concept of function to be not represented by a rich cognitive structure. When the preservice teachers add a concept to the concept map, it is less common that they notice its relations to the concept of function. An illustration of this is, for instance, a map showing $y=\pi x^2$ being understood to represent a parabola with a minimum point, where the map furthermore shows that $y=\pi x^2$ is recognized as a function. However, there are no connections between the concept of parabola and the concept of function, showing that the parabola is a function graph or that the minimum-point corresponds to a minimum-value of the function. The preservice teachers’ view of the function concept, as it appears in the concept maps, clearly diverges from the idea that functions play a central and unifying role in mathematics. Moreover, the concept maps usually do not express a more developed structural view of the concept of function as an object with a set of properties. The concept of function is often expressed as an operational conception in the form of a dependency relation between variables where some preservice teachers state the uniqueness requirement of an $x$ value giving one $y$ value, but no one mentions domain or codomain as parts of the function concept. The fact that the preservice teachers had just taken a course in calculus where the concept of function is a central concept, with a range of different properties and classes of functions, was rarely noticeable in the concept maps.

The concept maps indicate different concepts – including the concept of function – that the preservice teachers relate to the mathematical statement the concept map starts from, tend to be separated and thereby preventing preservice teachers from building a conceptual framework rich of meaningful connections. It is not unusual for the preservice teachers’ concept maps to contain trivial parts
not always related to mathematics, at the expense of relevant mathematical concepts and relations between concepts indicating rote learning and a less developed understanding of conceptual relationships in mathematics. There are also parts expressing procedural knowledge and knowledge of an “algorithmic nature”, e.g. if the concept of derivative is part of the map it is in the context of finding stationary points and not as a property of the function. Moreover, the concept maps show many preservice teachers use mathematical terminology infrequent or incorrect, which could be an obstacle for meaningful learning. Furthermore, even if the mathematical statements give the preservice teachers an opportunity to relate to future teaching, such themes are surprisingly rare in the concept maps. Using evaluation principles influenced by Ausubel’s assimilation theory, the utilization of concept maps seems to reveal important aspects of individuals’ knowledge and understanding in relation to the given mathematical statements.

The preservice teachers’ response to the activity of drawing concept maps indicates metacognitive activity and mediation along with an ability to evoke concept images with conflicting pieces of information, where hierarchical maps seem more mentally demanding to draw. Hierarchical maps also appear to have higher potential as diagnostic tools in the given context.

4.3.3 Paper III: Preservice teachers’ views of \( y = x + 5 \) and \( y = x^2 \) as expressed through the utilization of concept maps: A study of the concept of function

The function concept’s large network of relations to other concepts is frequently not part of the preservice teachers’ maps for \( y = x + 5 \) or \( y = x^2 \). This is usually due to the preservice teachers rarely observing and relating to different properties of functions or classes of functions, when \( y = x + 5 \) and \( y = x^2 \) are regarded as functions. When the preservice students write down a concept or some characteristic of a concept on the map, they do not usually observe relationships to the concept of function or make interpretations of how a concept is related to the concept of function. For instance, the preservice students do not notice a root of an equation being related to zero of a function in the current context, or the slope of a line is related to an increasing function, or a parabola symmetrical about the x-axis is related to an even function, etc.

Elements expressing procedural knowledge and skills of an algorithmic nature occur frequently in the maps. The maps may also contain completely trivial elements at the expense of important concepts and relations between them, indicating rote learning. Moreover, several misconceptions related to the concept of function can be identified on the maps. Furthermore, applications or links to real world situations are typically not present in the maps.

Preservice teachers usually express the function concept as a dependency between the variables \( x \) and \( y \), and thus describe a process conception. In some cases they give somewhat more elaborate explanations and state that an \( x \) gives one \( y \). But none of the students discuss domain or codomain in association with
the function concept, and a more developed conceptual understanding in the form of an object with a set of properties is less frequent in the maps.

The function concept is commonly developed with few relations to other parts of the maps. This might be an expression of compartmentalized knowledge structures that prevent the preservice teachers from building rich conceptual structures for the concept of function. The concept maps tend to branch off into substructures with often few cross-links. The frequent lack of connections between different parts of the map might have consequences for the preservice teachers’ ability to vary their reasoning and relate to concepts in different contexts with consequences for their future teaching.

The preservice teachers rarely relate to teaching and learning in their maps. This may be surprising, since their mathematics courses all contain parts related to mathematics education, particularly since the characteristics of the two statements $y=x+5$ and $y=\pi x^2$ make them suitable for connections to different teaching scenarios the preservice teachers will face as inservice teachers.

The concept maps seem to reveal a need for the preservice students to reflect upon the relevance of the function concept in mathematics, its different properties and its network of relations to other concepts. The preservice teachers’ comments about drawing the maps indicate that the process of drawing concept maps supports metacognitive activities and illustrate that such activities might promote the preservice teachers’ conceptual understanding of the function concept and its significance in mathematics.

Note: Because of an overlap in the research questions of Papers II and III, parts of the summary for the current paper are covered in the summary of the previous paper. The decision to give separate summaries for the two papers is based on the fact that there would be a loss of information in the summary for the current paper.

4.3.4 Paper IV: The views of preservice teachers on three mathematical statements: A case study regarding the concept of function

None of the preservice teachers in the case study describe the concept of function in a way consistent with the definition of function, i.e. a view of the function concept that includes the components of domain, codomain and a correspondence where each element in the domain gives one element in the codomain. For preservice teachers with lower performance in their mathematics studies, an operational understanding (Sfard, 1989, 1992) of the concept of function becomes more prominent. Knowledge structures for concepts they relate to the mathematical statements also seem to become more compartmentalized with fewer meaningful relations to other mathematical concepts, including the concept of function. The concept maps show that when the mathematical statement is recognized as a function, it is understood to have fewer properties and seems to be a less well-integrated concept as the preservice teachers’ level of performance in their mathematics studies decreases.
The preservice teachers do not often consider different classes of functions in relation to the mathematical statements. None of the preservice teachers in the case study mention, for example, that \( y=x+5 \) corresponds to a linear or \( xy=2 \) to a rational function, whereas the concept of quadratic function is evoked more frequently. Moreover, groupings or categorizations of functions regarding their different properties, e.g. properties encountered during the calculus course like continuous, monotone, differentiable, odd, etc., are hardly recognized in the case study. Concepts related to the calculus course are rarely described in the preservice teachers description of the mathematical statements, with a few exceptions like the concept of asymptote of a function.

The preservice teachers often make geometrical interpretations when describing the functions’ different properties, as well as numerical interpretations relating to the mathematical statements. In describing different properties of functions the preservice teachers less frequently use terminology related to the concept of function and mix terminology related to other mathematical concepts, indicating rote learning and less developed knowledge structures.

The form of the mathematical statements seems to influence which concepts the preservice teachers understand the mathematical statements to represent. For example, \( y-x=5 \) is clearly recognized as a diophantic equation, \( y-\pi x^2=0 \) as an equation, and \( y=2/x \) as a function. An explicit form of the mathematical statements seems more frequently recognized as a function.

In the preservice teachers’ evoked concept images related to the concept of function, there seem to be cognitive obstacles and prototypes. One cognitive obstacle seems to appear during the interview with the high achieving preservice teacher. Her understanding and experience of the relationships between the concept of an equation of a straight line and the concept of function, seems to interfere with her reasoning about the relations between the concept of equation and the concept of function. The preservice teachers in the case study seem to have been in contact with problems that to a lesser extent invite deep reflection upon the concept of function, functions’ different properties and relations to other concepts.

4.3.5 *Paper V: Preservice teachers’ conceptions of the function concept, its significance in mathematics and presence in school mathematics*

The preservice teachers in the two groups of the study clearly indicate different conceptions of the function concept after the calculus course. This contrasts the students’ views of the concept of function before the course, which is more similar in the two groups when applying the categorization by Vinner and Dreyfus (1989). Before the calculus course, more than half of the students in both groups expressed a function as a formula, an algebraic expression, an equation, or a representation possibly in a meaningless graphical or symbolic form. While the group that participated in the intervention after the course, considered a function much as an operation, i.e. one acts to get an image, and
usually without making any assumptions of a specific representation. This contrasts the first group, where the previously most frequent conceptions of function are strengthened after the course, i.e. students regard a function as a formula, an algebraic expression or an equation and frequently in combination with a dependence relation between variables.

Compared with the study by Vinner and Dreyfus (1989), the preservice teachers’ conceptions of function are similar to students majoring in less mathematical intense areas. Preservice teachers, who after the intervention in their survey answers, describe a function as an operation, fail to include the notions of domain and codomain as components of the function concept; hence, the students’ account of function in the second group would have been more comparable to students majoring in more mathematical intense fields in the study by Vinner and Dreyfus.

The preservice students in the first group are less aware of a definition of function and more frequently tend to mix terminology related to other concepts, e.g. formula or equation, in the context of functions – this became obvious during the interviews. The interviewed students from the second group were usually able describe a function more consistently with the definition of function – even if such conceptions did not appear to be spontaneously evoked in all cases. It was observed during the interviews that the preservice teachers were able to vary their reasoning to some extent and focus on different aspects of functions in different contexts, e.g. a dependency relation seemed to be of higher relevance for the students in the context of applications of functions.

Students in the first group almost exclusively associate functions with numbers, i.e. real valued functions in one real variable, and frequently in a symbolic or graphical form. However, this was not the case in the second group, where the students were able to reason about functions with a variety of representations and in a wider range of contexts. The concept of relation as an association between objects seems to be compatible with students’ previous conceptions regarding the notion of relation, and of importance for the students’ abilities to reason about functions in a range of contexts and representations – in the case a function is regarded as a special type of relation. The study indicates that the concept of relation has the potential to offer a conceptual infrastructure in the form of an anchoring idea for the function concept that facilitates meaningful learning.

Few students, including those in the second group, consider the notions of domain and codomain as two components of the function concept. The concept of set seems in this context to be an obstacle in the development of preservice students’ view on the concept of function. The preservice students seem to have limited experience in the concept of set – other than sets of numbers – which

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1 The concept of (binary) relation was during the intervention introduced as an association of elements in one set with elements in another set, and not as a subset of a Cartesian product of the two sets (which was looked upon rather as a representation of a relation). Relations were considered in different contexts related to mathematics as well as everyday situations.

2 A relation is here considered as an association between objects.
constitutes an obstacle in their ability to interpret the Dirichlet-Bourbaki definition of function.

The significance of functions in mathematics is, according to the preservice teachers, frequently motivated by the importance of functions in different applications of mathematics. This is a major theme in the first group both before and after the calculus course, with reference to science and especially physics. In the second group, applications of functions also became a major theme after the calculus course, with references to a variety of applications. For the students, the significance of functions in mathematics was highly connected with the ability to apply functions in different real world scenarios. The students’ conceptions of the significance of functions in mathematics indicate that applications of functions are relevant in the development of their concept image of function, and a factor of importance in meaningful learning and learning with understanding.

A majority of the preservice teachers consider functions as part of school mathematics at the end of compulsory school, and frequently a concept that mainly is relevant for high achieving compulsory students. The preservice teachers’ perception of the presence of functions in school mathematics is based on experiences of when the function concept is explicitly stated. It is, however, not uncommon during the interviews for the preservice teachers to get opportunities to further consider the presence of functions in school mathematics. This leads to a change in their view and they believe there is a high presence of functions in school mathematics, though usually “hidden” and not explicitly mentioned.

The idea that the teachers’ conceptions of the function concept will influence the quality of understanding the concept developed by students has received support from research findings in mathematics education. The current study illustrates that preservice teachers need opportunities to meet functions in a variety of contexts to develop their view on the function concept.
5 Concluding discussion

This section discusses the main results of the study. The discussion is primarily divided into the general conclusions of the study, topics related to teacher education, implications for teaching, and the study’s contribution of knowledge to mathematics education.

5.1 Discussion of results in the study

The preservice teachers’ view of functions contrasts with the idea that the concept of function is a unifying concept in mathematics. This is clearly illustrated in the preservice teachers’ concept maps, where the concept of function is rarely a well-integrated concept, as the preservice teachers less frequently notice relationships between the concept of function and other concepts included in their maps. A contributing factor to the function concept being less well-integrated in many of the maps is that the preservice teachers frequently omit different properties of functions and leave out interpretations of what these properties represent to other concepts. When preservice teachers do not notice different properties of functions or consider how the properties form a relationship to other concepts, they are less able to perceive the function concept’s network of relations in mathematics and demonstrate less advanced conceptual knowledge than one would expect.

The study indicates that the preservice teachers’ view of the function concept is represented by a less developed knowledge structure. When the preservice teachers pay attention to the concept of function, in relation to the mathematical statements in the study, they give few considerations of various functional properties or categories of functions and infrequently make interpretations of how such characteristics of functions are related to other concepts. They are thus representing a view that diverges from the reasoning in section 1.3.1. In case the preservice teachers do mention different properties of a function, they are often based on a geometrical and graphical view, with elements of numerical interpretations.

A more developed conceptual understanding of functions, as an object (Sfard, 1989, 1992) with a set of properties associated with the concept of function rather than with a specific representation of the function concept, is not prominent in the preservice teachers’ view of functions. Structural conceptions of the function concept as an object with properties can be traced within a representation, e.g. a graphical representation, rather than across different representations. Preservice teachers’ concept images for the function concept do not seem to be a construction of a richly connected and cohesive knowledge structures of meaningful relations. The concept maps give the overall impression that the preservice teachers’ knowledge structures tend to be compartmentalized, including the knowledge structures representing the concept of function, thus preventing preservice teachers from building a conceptual framework rich of meaningful connections. As a consequence, the preservice teachers may become
less flexible in their way of reasoning about mathematical concepts. This could in particular influence their reasoning of functions, their ability to interpret situations related to the function concept (in a variety of representations) and their perception of the function concept’s network of relations to other concepts. Also, they fail to realize that the concept of function is one of the central concepts underlying mathematics.

The preservice teachers often express an understanding of the function concept inconsistent with the contemporary definition of function, which became obvious in the case study and in the study related to the intervention (Papers IV and V of this thesis). Before the calculus course, more than half of the preservice teachers in both groups who participated in the study described in Paper V described a function as a formula, an algebraic expression, an equation, or a representation possibly in a meaningless graphical or symbolic form with respect to the categorization by Vinner and Dreyfus (1989), and frequently in combination with a dependence relationship between variables. Moreover, conceptions of functions as a formula, an algebraic expression or an equation were strengthened after the course for the group that did not participate in the intervention. The case study illustrates that the preservice teachers had to a lesser extent been in contact with tasks that invite to a deeper reflection upon the concept of function.

In the context of functions, preservice teachers less frequently use a terminology related to the concept of function than one could expect (e.g., Papers II and IV). They seem to lack a mathematical language to express themselves in and mix terminology related to concepts other than the function concept, which may be an expression of rote learning (Ausubel, 1968, 2000; Ausubel et al., 1978). One reason preservice teachers tend to mix terminology of different concepts in the context of function seems to be related to their conceptions of functions. When preservice teachers identify functions with formulas, algebraic expressions, equations, etc., it is not unexpected that they subsequently use terminology related to these concepts when they are reasoning about functions.

Few preservice teachers, including those who participated in the intervention, consider the notions of domain and codomain as two components of the function concept. The fact that the domain and codomain are sets seems in this context to be of significance, since the preservice teachers seem to have limited experience of the concept of set and may not previously been involved in meaningful learning situations related to the notion of set. The students’ conceptions of the notion of set seem to constitute an obstacle for their ability to interpret and apply the Dirichlet-Bourbaki definition of function. The preservice teachers’ limited experience of the notion of set may also increase their dependence on experiences of examples and applications of functions in their construction of a concept image for the function concept, as well as increase their demand for access to a conceptual infrastructure that provides support for their conceptual development of the function concept.
Preservice teachers in the study believe that a term for a mathematical concept does not always reveal or guide them in how to interpret the concept, as noticed in Paper V, and that their previous conceptions of a concept from everyday situations do not always support their understanding of a similarly named concept in mathematics. In this context the notion of function seems, from preservice teachers’ everyday use of the notion, to give limited support of how to consider the concept of function in mathematics – which might well be a consequence of the historical development of the function concept. One expression of these considerations might, for instance, be the high achieving preservice teacher’s comment on the definition of function as “the theoretical stuff that … has been drilled into us in school”, in the case study of Paper IV. During the interview, in her reasoning about functions, the preservice teacher initially relied on her concept image of function and regarded functions as a type of equation. But the student became aware of the necessity to use the definition of function in her attempts to clarify the concept of function to herself, thus creating a feedback from evoked concept images to the definition of function in a process of adjusting her concept image of function to be more consistent with the function concept.

The intervention indicates that the notion of relation may possibly offer a significant conceptual infrastructure for the preservice teachers’ conceptual development of the function concept – in case a function is regarded as a special type of relation, as described in Paper V. The concept of relation, as an association between objects, is commented upon by the preservice teachers as a notion they “recognize”, i.e. a notion largely consistent with their everyday experiences of the notion. As such, this seems to make the preservice teachers able to reason about relations between a variety of objects, and thus able to reason about functions as a special case of relation in a range of different contexts, representations and applications, and accordingly develop their concept image of function. These experiences might also render a possible feedback of evoked concept images of function to the definition of function, in a variety of contexts, representations, etc., and possibly stimulate the construction of a concept image more consistent with contemporary characterizations of the function concept. A further consequence of preservice teachers’ recognition of

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1 Originally, Leibniz “used the word ‘function’ to mean any quantity varying from point to point of a curve – for example, the length of the tangent, the normal, the subtangent, and the ordinate” (Klein, 1972, p. 340), where the curve itself was said to be given by an equation. From the late seventeenth century when Leibniz coined the term function to the first half of the twentieth century when Bourbaki established the contemporary definition of function, the term function became to represent a concept that evolved to become a highly general and abstract concept.

2 A (binary) relation from one set to another set is here regarded as an association between elements of the two sets. (This view represents a process conception of relation, and not the more advanced object conception of relation that a subset of a Cartesian product of the two sets would represent.) A function is considered as a special type of relation, such that each element in the first set is associated with exactly one element in the second set (i.e., a binary relation that is both total and functional). The descriptions of relation and function are thus compatible in that they both relate to process-conceptions in accordance to theories of conceptual development by Sfard (1989, 1991, 1992).
the notion of relation is that it may well offer a potential to create learning scenarios where they can relate new knowledge to their previous knowledge and experiences and thus promote personal meaningful learning (in the sense of Ausubel, 1968, 2000; Ausubel et al., 1978; Carpenter & Lehrer, 1999) for the function concept. The notion of relation can also give the preservice teachers a (mathematical) concept to relate to and a language to express themselves, both significant factors within the framework of meaningful learning in their reasoning about functions in general contexts that comprise the more inclusive notion of relation.

Preservice teachers who did not participate in the intervention almost exclusively associate functions with real valued functions in one real variable. Moreover, as commented earlier, after the calculus course more than half of these students regarded the concept of function as a formula, an algebraic expression, or were an equation when applying the categorization by Vinner and Dreyfus (1989). These conceptions of functions were strengthened during the calculus course, and accordingly, a view of the concept of function the preservice teachers frequently held at the end of their required mathematics courses in the teacher preparation program. These findings were considered when the intervention was implemented and led to discuss functions other than real valued functions in one real variable during the intervention. During the interviews, preservice teachers who participated in the intervention were typically able to reason about functions in a wider range of contexts and representations, in contrast to the previous students in the study, and did not exclusively consider functions to be real valued of one real variable. Their view of functions therefore did largely differ from the previous preservice teachers, who to a large part considered functions to be formulas, a type of equation, etc. The group of students that participated in the intervention more frequently began to consider the concept of function as an operation, as described in Paper V. The exclusion of domain and codomain in their descriptions of the function concept was a significant reason why the group that participated in the intervention did not become more comparable to the group of students majoring in more mathematic intensive fields or the group of inservice teachers, in the study by Vinner and Dreyfus (1989), when describing the concept of function.

The preservice teachers’ conceptions of function are similar to students majoring in less mathematic-intense areas compared with the study by Vinner and Dreyfus (1989). This result might be due to the educational system in Israel – where the study by Vinner and Dreyfus was conducted – probably being a more segregated educational system than that of Sweden. All applicants formally qualified for teacher education and specializing in mathematics are generally accepted in Sweden. This is most likely a contributing factor to the differences in conceptions of function between the preservice teachers in Paper V and the inservice teachers’ in the study by Vinner and Dreyfus, where most inservice teachers describe the function concept coherently with the Dirichlet-Bourbaki definition. Almost no preservice teachers in Paper V gave descriptions
of the function concept consistent with contemporary characterizations of functions. However, preservice teachers who participated in the intervention were, during the interviews, usually able to expand on their description of function and reason about a larger variety of functions more consistently with contemporary descriptions on the function concept.

5.2 Preparation of mathematics teachers

The preservice teachers who participated in the study were enrolled in the final mathematics courses of the educational program during the term the study was conducted, and completed their preparation in mathematics at the end of the term. Teaching is a complex process that is influenced by a range of factors, and different aspects of this process are of interest (e.g., Borko & Putman, 1996; Cooney & Wilson, 1993; Even & Tirosh, 2002; English, 2002; Shulman, 1986; Whitcomb, 2003). One aspect of importance is the role of the teachers’ subject matter knowledge and the process transforming subject matter for the purpose of teaching, particularly focused upon in the work of Shulman and colleagues (e.g., Shulman, 1986; Grossman, Wilson & Shulman, 1989). The opinion that teachers’ conceptions of a topic will influence the quality of the understanding developed by their students has received support from research findings (e.g., Even & Tirosh, 2002; Fennema & Loef, 1992; Lloyd & Wilson, 1998), though little is still know about the mechanisms by which teacher knowledge translates into student learning (e.g., Hill, Rowan & Ball, 2005; Wilson, Floden, & Ferrini-Mundy, 2002). However, based on a constructivist approach to learning, an important part of the teachers’ function is to establish mathematical environments where students are encouraged to explore and raise questions (e.g., Confrey, 1990). In these situations, teachers may well have to deal with instances of problems unfamiliar to them, which will possibly touch upon more advanced areas of mathematics where the teachers’ subject matter knowledge is important in deciding how to respond to students’ questions.

The subject matter preparation offered to preservice teachers should support and develop their prospects to become proficient mathematics teachers. However, when considering their conceptions of the concept of function and the study by Vinner and Dreyfus (1989) in Paper V, it appears that the prospective teachers’ view of the concept of function tends to be similar with that of students majoring in less mathematic-intensive areas. Teachers’ pedagogical decisions, e.g. questions they ask, examples they put forward, activities they design, ideas they consider of value and students’ suggestions they follow, are partly based on their understanding of the topics in question. Hence, it is important that teachers have a well-developed concept image of function in their reasoning about functions. From this perspective, is it desirable that preservice teachers have a better-developed concept image than what the findings in the study describe in this thesis. Preservice teachers should be better prepared to reason about functions in a variety of contexts and applications and from different points of view with their future students (e.g., Wilson, 1994; Cooney & Wiegel, 2003). In addition
to the pedagogical arguments where preservice teachers’ conceptions of functions are of importance, Even (1993) claims that there are cultural arguments for preservice teachers to have a well developed concept image of function and to be able to reason about a variety of functions, including functions characterized by the contemporary Dirichlet-Bourbaki definition. Even argues that the concept of function has historically evolved, not because someone arbitrarily decided to make changes to the function concept, but rather because new knowledge in mathematics created the need for those changes to take place, that the developments in mathematics created new branches of mathematics and caused changes to the function concept, and that this reality should be taken into account when considering preparation of mathematics teachers. Another argument is that mathematics teachers restricted by their limited concept image of function may be prevented in grasping the fundamentals of current mathematics that are based on contemporary characterizations of functions.

Still, having acquired a concept image of function that allows preservice teachers to reason about functions in various contexts is not sufficient. As prospective mathematics teachers they need to experience, for instance, why there is a need for uniqueness in the definition of function, though it seems as if the preservice teachers in the study have not yet developed this type of deeper understanding. A deeper understanding of the subject taught by teachers is regularly emphasized in mathematics education literature, as well as an emphasis on teachers having sensitivity about how mathematical ideas develop in learners (e.g., Ball, Lubienski & Mewborn, 2001; Cooney & Wiegel, 2003; Ma, 1999). Being familiar with the historical development of functions may possibly help preservice teachers, since it explains how the function concept came to be defined as it is today and thus gives meaning to the contemporary definition, as well as illustrates different views upon the concept of function that could be relevant for the preservice teachers’ ability to reason about functions from different points of view (e.g., Even, 1993; Vollrath, 1994). A meaningful understanding of the function concept might furthermore help teachers make knowledgeable decisions about the place of the concept of function in the curriculum. In particular, it could help preservice teachers create settings for their students to become aware of “powerful mathematical ideas”, as described by Skovsmose and Valero (2002), who promote four different perspectives on the topic of powerful mathematical ideas, in terms of: logical, psychological, cultural, and sociological perspectives, in the teaching of mathematics.

The findings of the study described in this thesis imply that prospective teachers require additional support to develop their view of functions and understanding of mathematics. Mathematics courses in teacher preparation programs ought to be constructed so that a better, more comprehensive and articulated understanding and knowledge of functions and mathematics is developed, as is also concluded by a number of studies involving preservice mathematics teachers (e.g., Cooney & Wiegel, 2003; Even, 1990, 1993;
Vollrath, 1994; Wilson, 1994; Norman, 1992, 1993). For example, Cooney, Wilson, Albright and Chauvot (as cited in Cooney, 2003) show that secondary preservice teachers display a limited view of mathematics, though they have completed a substantial part of their formal mathematical training. Their description of function often focused on equations or a formula involving computations – results consistent with those of the study described in this thesis. Even (1993) argues that the preservice teachers’ limited view of functions is due to most functions encountered by prospective secondary mathematics teachers during their mathematics courses being the kind that usually have a “nice” graph and can be described by an expression, and that brief experiences with other types of functions are not enough to affect the construction of the preservice teachers’ concept image of function. The circumstances described by Even are also applicable to the study described in this thesis, and most likely reasons that contribute to the results of the study, e.g. in the description of the function concept by preservice teachers that did not participate in the intervention, in Papers IV and V.

In constructing learning environments for preservice teachers that will possibly give the required additional support in their learning of mathematics, and thus develop a better, more comprehensive, and articulated understanding and knowledge of mathematics, Cooney and Wiegel (2003) identify three principles grounded in an overview of research literature in mathematics education. The first principle is that prospective teachers should experience mathematics as a pluralistic subject, and as such embraces human invention that basically involves seeing mathematics as an empirical science and envisioning connections within mathematics, between mathematics and the real world. This is not to exclude mathematical formalism, but rather to include other kinds of mathematical experiences. The second principle is that they should explicitly study and reflect on school mathematics, and generate ideas on how to teach that mathematics. The third principle is that they should experience mathematics in ways that support the development of process-oriented teaching styles, creating an active learning environment. The goal of process-oriented instruction is to teach preservice teachers about themselves as learners, and it involves teachers who model and think aloud as a way of uncovering for the learner the cognitive and metacognitive processes underlying learning. Students who possess these learning styles use deep processing and self-regulated learning strategies (Schunk & Zimmerman, 2003) and to them learning is seen as a personal construction of knowledge.

Tendencies of knowledge compartmentalization have been exposed in the study described in this thesis. Learning environments that foster powerful constructions of mathematical concepts may also have the goal to promote the development of what Eisenberg (1992) calls having a “sense for functions”. He describes this notion as linked to insights about functions that incorporate the integration of many skills, skills often taught in isolation, as a way to counteract compartmentalization of knowledge about functions. Such learning
environments would especially include the exploration of various properties of functions as an activity that integrates many skills that are also activities related to the development of structural conceptions of functions, where properties are recognized as belonging to a function rather than the outcome of isolated procedures applied to a function (Slavit, 1997; Sfard, 1989, 1991, 1992). Compartmentalization of knowledge is always an issue to consider in learning environments when a body of knowledge splits into a number of isolated pieces, which Eisenberg (1992) highlights regarding functions in referring to theory of didactical transposition of Chevallard (1985). This might be a further reason why preservice teachers need opportunities to encounter functions in a variety of contexts, applications and representations, to become aware that the concept of function is one of the central concepts underlying mathematics and to realize that it is an important concept and a powerful mathematical idea (Skovsmose & Valero, 2002) to present to future students.

The process of transforming subject-matter knowledge for the purpose of teaching is important to teachers, and teachers’ conception of concepts is an essential aspect for the outcome of this process. If one considers knowledge of concepts as central in the learning of mathematics (and believe that mathematicians are “jugglers of concepts rather than numbers”, in common with Lehrer & Lesh, 2003; Stewart, 1975), then it is natural to focus on fundamental concepts in mathematics, in the process of transformation of preservice teachers’ subject-matter knowledge to pedagogical content knowledge. Vollrath (1994) shares this view, suggesting that concepts should be regarded as starting points for didactical thinking in mathematics. From this point of view, considerations about what concepts should be taught in a mathematics course are central, as well as a range of topics related to a concept, e.g. the definition of a concept, different properties of a concept, examples and counterexamples, applications and the various roles of a concept, etc. Discussions about fundamental concepts in mathematics should consequently be an essential in the preparation of preservice teachers, which includes thinking about the significance of concepts, their use, and their network of relations to other concepts, etc., since such matters become vital for the teachers’ planning and organization of concepts in the teaching of mathematics.

However, the findings of the study described in this thesis show that significant relations between concepts are frequently excluded from preservice teachers’ reasoning, which becomes clear when considering their concepts maps. Preservice teachers in the study do not notice central relations between concepts and express low-level “networked learning” (Vollrath, 1994, p. 64), implying a lack of prerequisites to organize concepts and connections among ideas for teaching. Moreover, in the preservice teachers’ organization of concepts and their relationships, teaching perspectives are usually excluded, though the preservice teachers had taken most courses related to mathematics education in the teacher preparation program. Furthermore, when preservice teachers during the interviews were given opportunities to further reflect upon
the presence of functions in school mathematics, it regularly had the effect that preservice teachers came to consider the concept of function as present in school mathematics, but “hidden” in their experience and not made visible until the end of compulsory school, causing the students to wonder why this is the case. Reflection upon the significance and presence of different concepts in school mathematics seems to be unfamiliar to preservice teachers. The findings of this study imply that these kinds of reflections may possibly raise a series of questions to serve as a starting point for valuable didactical discussions with preservice teachers, which are related to subject-matter knowledge and other forms of knowledge in the tradition of Shulman and his colleagues, i.e. subject matter content knowledge, pedagogical content knowledge, and curricular content knowledge (e.g., Shulman, 1986; Wilson, Shulman & Richert, 1987).

5.3 Implications for teaching
The two previous sections included elements of implications for teaching mathematics to preservice teachers and further considerations of teaching implications are pursued in this section. If one begins by assuming that the preservice teachers’ view on the function concept is due to their experiences of functions and to a lesser extent of their knowledge of the concept definition (Vinner, 1983, 1991, 1992; Vinner & Dreyfus, 1989), then this should lead to attention being focused on those functions preservice teachers encounter during their mathematics courses. Highlighting the mutual and the distinct local and global properties of different functions could be one way of enhancing the development of structural conceptions with respect to the function concept (Sfard, 1989, 1991, 1992; Slavit, 1997; Bagni, 2003), as well as a means of contributing to the integration of various skills with the aim to restrain compartmentalization of knowledge (e.g., Eisenberg, 1992) and thus create more composite knowledge structures. In this context, the use of proper terms for different properties of functions might be significant in the preservice teachers’ development of cognitive structures (Ausubel, 2000) and in the support of preservice teachers’ ability to express themselves in the language of mathematics and in their reasoning about the function concept.

The preservice teachers appear to have little experience of problem situations that allow them to reflect on the definition of the function concept. Stimulating a feedback from evoked concept images of function to the definition of the concept in a range of different contexts, applications and representations may be one possible way to enhance the preservice teachers’ concept image of function and develop their understanding of the function concept. This form of re-association to the definition of a concept seems to be illustrated in the case study of the high achieving student in Paper IV, and occurs primarily with problems that are not of the standard variety according to Vinner (1991, 1992). The preservice teachers’ experiences of functions other than standard functions, might in this context also result in the development of a concept image of function that is not limited to specific forms of representation, e.g. formulas,
algebraic expressions, etc., as illustrated in Paper V. As well, this could contribute to preservice teachers’ reasoning of various relations the concept of function has to other concepts in mathematics, e.g. relations between the concept of function and the concept of equation as also shown in Paper IV.

For prospective mathematics teachers, knowing the definition of function as one part of their knowledge about functions is essential (e.g., Even, 1993, Vollrath, 1994). Without the knowledge of a definition to apply to their reasoning about functions, preservice students will certainly have difficulties of what to consider as a function in their thinking about functions, how the concept of function might be related to other concepts, and will probably contribute to statements like “… I am still uncertain about functions” (Paper V, p. 8). However, the preservice teachers in the study are frequently not able to give a description of function consistent with contemporary characterizations of the function concept. In particular, the notions of domain and codomain are excluded in preservice teachers’ descriptions of function (similar to studies of tertiary students by, e.g. Tall & Bakar, 1992; Williams, 1998). A contributing factor to preservice teachers’ omittance of domain and codomain in their descriptions of function is probably due to their experience of functions being largely based on real valued functions in one real variable, where the notion of domain and codomain might possibly tend to become tacit knowledge. One implication for teaching would be that it is necessary for preservice teachers to experience that the concepts of domain and codomain are part of the function concept. That changes to these two sets certainly give rise to different functions or, for instance, experience how changes to the domain and codomain of a binary relation give rise to a function, and so forth. Furthermore, the concept of set seems to be an obstacle in the preservice teachers’ interpretation of the Dirichlet-Bourbaki definition of function, possibly stemming from preservice teachers’ lack of experience of what the notion of set may represent. This suggests that the concept of set ought to be given more consideration in the teaching of functions to prevent preservice teachers from developing a limited view on functions, where they perhaps believe the notion of set to mean intervals of real numbers in the context of functions.

The current study indicates that students’ conceptions of the notion of relation, as an association between objects, is a notion that the preservice teachers “recognize” and allows them to reason about functions in different settings from their previous everyday experiences of relations between objects – when they consider a function to be a special case of relation – and thus able to reason about functions in more general context not always related to numbers or represented by formulas, expression, etc. An implication for teaching might be that the notion of relation can be considered to create meaningful situations for preservice teachers in which they can reason about the more inclusive concept of relation and experience how adjustments to the situation of a (binary) relation might give a function. Hence, the concept of relation would provide support, or anchorage, analogous to the notion of “cognitive root” by Tall (e.g., 1992), in
the conceptual development of function. This means that the notion of relation might present a cognitive infrastructure supporting the preservice teachers’ reasoning about functions between sets other than sets of numbers, and thus promote the development of a concept image enabling preservice teachers to reason about functions characterized by contemporary views on the function concept.

It appears that preservice teachers should be exposed more to problem situations involving reasoning of various relations between functions and other mathematical concepts, analogous to the description of “network-based” learning by Vollrath (1994), to realize that functions have large networks of relations in mathematics and the function concept is one of the fundamental concepts underlying mathematics (Eisenberg, 1991, 1992; Selden & Selden, 1992; Tall, 1992, 1996). These kinds of reflections may also promote the development of more flexible conceptual knowledge, and thus possibly create better conditions for learning with understanding and counteract the formation of segmented knowledge structures (Carpenter & Lehrer, 1999; Hiebert & Carpenter, 1992). The study indicates metacognitive activities in the preservice teachers’ responses to the process of drawing concept maps (as illustrated in, e.g. similar researches in Paper II and Novak, 1998), that might possibly stimulate the construction of more richly connected and coherent knowledge structures. The process of drawing concept maps might be a valuable tool, since it gives preservice teachers rich opportunities to reflect upon the function concept, its status and relations to other concepts, etc. Another reason for preservice teachers to regularly draw concept maps is that concept mapping taps into teacher competencies in the requirement of organizing knowledge for presentation, and makes it necessary for preservice teachers to consider and describe concepts and relations between concepts in their composition of the map. A possibility is to then also work with concept maps in the more unorthodox manner used in the study described in this thesis, and let preservice teachers reason about a variety of specific instances of functions and their properties and relations to other concepts.

When preservice teachers reason questions of the significance of functions in mathematics, they frequently touch upon applications of mathematics and often motivate the significance of functions with the use of functions in applications of mathematics. Even if the preservice teachers point out applications of functions as a theme of importance (in Paper V), and imply that applications of functions play an essential part of their construction of a concept image of the function concept, the students less frequently make interpretations of how to apply the mathematical relations represented by \( y = x + 5 \), \( y = \pi x^2 \) and \( xy = 2 \) in the context of the function concept. This might imply that the preservice teachers are less able to reason how to interpret what mathematical relations the statements represent and how to apply the relations. These results suggest that elements of mathematical modeling may be of importance in the preparation of preservice teachers to enhance their abilities to reason applications of
mathematical relations (considerations that are given support by, e.g. Blomhøj & Jensen, 2003; Brandell, 1997; Cooney & Wiegel, 2002). Moreover, elements of mathematical modeling might possibly enhance the preservice teachers ability to establish situations for meaningful learning of students in future teaching scenarios as well as opportunities to create situations of meaningful learning for preservice teachers themselves, as indicated in Paper V. Furthermore, the preservice teachers infrequent connections with school mathematics and teaching scenarios when reasoning y=x+5, y=x² and xy=2 might be a further indication of their limited experience with scenarios in which to apply basic mathematical relations, providing further motivation that elements of mathematical modeling should be considered in preservice teachers mathematics education.

5.4 Contribution to knowledge

As one of the fundamental concepts in mathematics, the concept of function has been the subject of substantial research in mathematics education. However, preservice teachers’ conception of function is not well researched, as described in 2.2, and very few studies in particular have been conducted in Sweden with this aim. The suggestion that mathematical statements, e.g. y=x+5, y=x², and xy=2, might evoke a network of concepts and relations in preservice teachers’ reasoning of the statements, where preservice teachers’ conceptions of function are examined, has been a theme of interest in Papers I-IV of this thesis.

The three statements y=x+5, y=x², and xy=2 are open to interpretation for preservice teachers and may well tap into teachers’ abilities to reason from a variety of perspectives with their students, and in these situations, are preservice teachers’ conception of function and its relations to other concepts certainly of relevance. This line of research contributes to knowledge about the characteristics of the preservice teachers’ conception of the concept of function (e.g., Chinnappan & Thomas, 2001; Cooney & Wiegel, 2003; Cooney & Wilson, 1993; Even, 1990, 1993, 1998; Grevholm, 2002; Sánchez & Llinares, 2003; Wilson, 1994). It also contributes to knowledge about the preservice teachers’ perception of relations between concepts (i.e., “networked learning”, Vollrath, 1994), as one part of teachers’ knowledge organization and connections among ideas (e.g., Doerr & Bowers, 1999; Grevholm, 2000, 2004; Leikin et al., 2001; Vollrath, 1994).

The findings of the study indicate that preservice teachers’ knowledge tends to be compartmentalized. This is an issue of concern in their development of conceptual frameworks rich in meaningful connections that might have consequences for their abilities to operate in a constructivist environment and reason with students from different points of view (e.g., Confrey, 1990). The function concepts’ network of relations to other concepts is often omitted in the preservice teachers’ reasoning, possibly indicating that they only are able to offer a limited range of teaching strategies in relation to functions. Properties
and different categories of functions are rarely identified by preservice teachers compared to more developed conceptual knowledge of function (e.g. professors with PhDs in mathematics, Williams, 1998). One can argue the importance of preservice teachers having a more developed understanding of the function concept, noticing conceptual relations, properties, terminology and the concepts significance in mathematics (e.g., Cooney & Wilson, 1993; Eisenberg, 1992; Even, 1990, 1993, 1998; Thomas, 2003; Vollrath, 1994).

It is not uncommon for preservice teachers to express beliefs that the function concept is a concept of value primarily to highly achieving students, and a concept of relevance first at the end of compulsory school, as described in Paper V. This may possibly imply that the preservice teachers will give issues related to the concept of function low priority in their teaching. However, the study indicates that preservice teachers’ reflect upon the significance of functions in mathematics (which they usually tie to applications of mathematics) and the presence of functions in school mathematics impacts the preservice teachers’ reasoning. They diverge from a line of thought in which the concept of function has low presence and relevance in compulsory school, and begin to notice situations where the concept of function is present, though not explicitly stated in school mathematics. This process of change in preservice teachers’ conceptions of the presence of functions in compulsory school may possibly contribute to emphasize the significance of the process of transformation of subject matter knowledge to pedagogical content knowledge (e.g., Shulman, 1986). Broadly speaking, this refers to the specialized knowledge that teachers have of how to represent content knowledge in multiple ways to learners, and even though a variety of concepts in mathematics are not explicitly stated, it represents powerful lines of thought (Skovsmose & Valero, 2002) which are issues for attention in the preparation of mathematics teachers.

A range of findings in the study has been discussed in sections 5.1-5.3. One finding is that preservice teachers’ frequently describe functions as a type of formula, algebraic expression, or equation, which are conceptions of function that risk strengthening during a calculus course, as described in Paper V. This result confirms assumptions from previous studies concerning the function concept (e.g., Vinner & Dreyfus, 1989; Even, 1993). Based on the findings of the study described in this thesis, an outline of measures is suggested in sections 5.2-5.3 of how one might possibly improve preservice teachers’ conception of function, and contributes to considerations about the preparation of mathematics teachers in teacher education by Cooney and Wiegel (2003). In the outline of measures is, e.g. the notion of relation, the notion of set, and properties of functions of significance.

Preservice teachers’ knowledge of properties of functions and interpretations of what these properties mean to other concepts has been found to be one reason why preservice teachers fail to perceive a network of relations from the concept of function to other concepts. Properties of functions are often perceived as results of isolated procedures linked to a form of representation,
rather than as properties of a function related to operational conceptions. These findings might possibly support the hypothesis by Slavit (1997), that properties of function play an essential part in the development of more advanced structural conceptions of the function concept, and that focusing on properties of functions, as an integration of many skills (Eisenberg, 1992), might play an essential part in the development of structural conceptions of functions.

Concept mapping is a method that has been used to less extent in mathematics education research. However, concept maps were applied in Papers I-IV with the main purpose to investigate preservice teachers’ conceptions about functions and their relations in the context of $y=x+5$ and $y=\pi x^2$. Concept maps have, therefore, not been derived from an explicitly stated concept, but used in a somewhat unorthodox manner in the study, further discussed in Paper II. Indications of metacognitive activities related to concept mapping have been confirmed, which might initiate processes of thought that stimulate the creation of conceptual frameworks of meaningful connections. Concept mapping can also be a tool that taps into teacher competencies in the organization of knowledge, and a tool with perhaps the potential to be of significance in the creation of learning environments for preservice teachers, as described in 5.2 and 5.3.

In the theoretical framework, knowledge is represented internally and understanding is described in terms of how an individual’s mental representation is structured. Network-based models are applied in the description of internal knowledge representation, as related to Hiebert and Carpenter (1992) and Ausubel (e.g., 2000; Ausubel, Novak & Hanesian, 1978). Moreover, the framework incorporates theoretical elements, such as the notion of concept image (Tall & Vinner, 1981; Vinner, 1983, 1991, 1992) and the development of structural conceptions (Sfard, 1989, 1991, 1992), making it possible to reason about a network-based concept image as well as the development of network-based structural conceptions. Properties of functions are considered to be of importance in the development of a richly interconnected and structured concept image that offers structural conceptions in agreement with Slavit (1997). The framework contributes to the field of mathematics education, and makes it possible to, e.g., consider the development of structural conceptions related to a representation of a concept, as a consequence of compartmentalization of a concept image.
6 Comments on the study and future research

6.1 Comments on limitations of the methods in the study and scientific work criteria for quality of research

This section is a continuation of the methodological discussion in 4.2.6. When considering the conducted study, it is possible to criticize it on several points. For example, the study could have involved a higher number of preservice teachers and been extended to teaching programs at several educational institutions. The study was conducted at a tertiary institution with relatively few preservice mathematics and sciences teachers. As a result, e.g., for ethical reasons (Sowder, 1998), it was not possible to reveal more detailed information about their backgrounds in connection to the case study in Paper IV. However, the study is part of a research education program and should thus be conducted during a limited period of time, ultimately affecting the scope of the study. Various practical arrangements were also of significance to conduct the study, such as having the opportunity to contact the preservice teachers, organizing the execution of the study, being in touch with their teachers and ensuring that the study could be conducted as planned. These would have influenced the choice of institution at which the study was to be conducted.

The preservice students answered questionnaires and drew concept maps during lecture time, which might have influenced how they answered the questions and designed the concept maps. Moreover, the preservice teachers were free to include concepts in their maps according to their own ideas, which possibly excluded concepts that could have been of interest to the research questions in the study. Arranging the preservice teachers to draw the concept maps and to answer the survey questions during lecture time ensured independent answers and maps and guaranteed the same conditions for each participant. This has significance in the comparison of their answers. Since I was the teacher during the lecture in which they drew the maps and the person who interviewed the preservice teachers, I could ensure that it proceeded as planned. The preservice students knew me as a teacher, which might have influenced their answers. I nevertheless found it was good for the interviewed students to know me, since it facilitated natural interactions and I sensed that it enabled them to express their opinions freely.

The interviews were based on the answers given to the preservice teachers during the survey. The students were asked to comment on their answers. Nevertheless, in order to gain a better understanding of their thought processes, they were asked questions that might have influenced how they answered. The questions in the survey were open, which meant that it was necessary to ask additional questions in order to investigate more closely the preservice teachers’ perception of the function concept in relation to the statements. The interviews were often conducted during a predetermined time interval. This meant that it was sometimes necessary to limit the subject and move on to the next question. The study is based on voluntary participation and the time allotted to an
interview (section 4.2.3) was adapted to increase the number of preservice teachers who participated in the study; this was of value to the study.

A number of criteria for quality of research have been promoted in mathematics education (e.g., Kilpatrick, 1993; Sierpinska, 1993; Lester & Lambdin, 1998), where Lester and Lambdin (1998) suggest criteria such as openness, credibility, worthwhileness, coherence and competence, which I would like to use as a starting point in a discussion of the scientific quality and relevance of the study. I believe the study favors openness in describing how the study was conducted, in describing relationships between researcher and subject, in appliance of methods, how data was collected and analyzed. The research findings are grounded in multiple sources of evidence in the form of questionnaires, interviews and concept maps, which should enhance credibility of the results in the study. The study is related to previous research in mathematics education and has gradually evolved over time where earlier results have made an impact on the theoretical framework, research questions and methods used in the study. The gradual development of the study favors issues related to worthwhileness, coherence and competence of the research.

Among their research criteria, Lester and Lambdin (1998) find the criterion of worthwhileness the most important. To further comment these criteria, related research (in section two) shows that teachers’ and preservice teachers’ conceptions of function is a less researched area internationally – in contrast to students’ conceptions of function. Moreover, few studies have used concept maps in the study of individuals’ conceptions of function (no previous study to my knowledge has used concept maps to investigate preservice teachers’ conceptions of function in relation to mathematical statements, as in the current study). Grevholm analyzes results from a longitudinal study of preservice teachers’ conceptual development (Grevholm, 2000, 2004), including the concepts of equation and function. Few other studies have been conducted on preservice mathematics teachers in Sweden (Bergsten & Grevholm, 2004; Björkqvist, 2003), and to my knowledge, no other study has been conducted with a focus on preservice teachers’ conceptions of functions as in the current study.

6.2 Ethical aspects on the study

The identities of the preservice teachers are protected. All names used in this thesis are fictitious. Furthermore, the data in the study are not presented in such manner as to enable the identification of the participants (Sowder, 1998). The preservice teachers have all been asked to participate willingly in the study. It is my impression that the preservice teachers who participated in the study were able to benefit from the experience. This was particularly evident during the interviews, when the preservice students commented on the interviews by revealing how the questions raised in the interview had helped them to improve their understanding of different concepts and relations. They also revealed that the interviews provided them with an opportunity to learn. This shows that their
learning is influenced by our observations, which in turn influences the information we measure; this is inevitable (Duit et al., 1996).

6.3 Future research

Preservice teachers reasoning about the concept of function show diverging lines of thought as to what extent functions are present in school mathematics. I believe this may be one reason to possibly motivate research concerning what different strategies teachers offer in their teaching and if they sufficiently promote functions and functional thinking. This research can be linked to questions about transformation of subject matter knowledge to pedagogical content knowledge (e.g., Shulman, 1986; Whitcomb, 2003), and questions related to the specialized knowledge that teachers have of how to represent subject knowledge in numerous ways to learners. As well, it relates to questions that a variety of concepts in mathematics represent powerful lines of thought (e.g., Skovsmose & Valero, 2002), regarded as issues for attention in the preparation of mathematics teachers. I also believe it would be valuable to investigate what aspects of function the school curriculum and literature used in school promote, and how teachers interpret this and it affects their conceptions of function.

Preservice teachers’ perceptions about properties of functions and interpretations of what these properties mean to other concepts have been found to be one reason why preservice teachers can be unsuccessful in noticing networks of relations from the concept of function to other concepts. Properties of functions seem frequently to be perceived as results of isolated procedures linked to a form of representation, rather than as properties of a function related to operational conceptions. I plan to conduct further studies about how attention to functions properties might support preservice teachers’ conceptual development of function and how to create incitements to promote development of richly connected conceptual frameworks for preservice students on the topic of function. This research may well be based on the outline of measures suggested in sections 5.2 and 5.3 of how one might possibly improve preservice teachers’ conceptions of function. I also intend to further study how semiotic interpretations might be linked to preservice teachers conceptions of mathematical expressions and their perception of what concepts and properties the expressions represent, as well as use technological tools in this research.

Preservice teachers seem to have limited conceptual frameworks of meaningful relations in their reasoning about fundamental concepts in mathematics. Indications of metacognitive activities related to concept mapping (as, e.g., described by Novak, 1998) have been confirmed, which might initiate processes of thought that stimulate the creation of conceptual frameworks of meaningful connections. I plan to study how to create incitements to promote preservice students’ development of conceptual frameworks and intend to use concept mapping as one tool in this research. Concept mapping might be a tool
that taps into possible teacher competencies of value in the creation of learning environments for preservice teachers as described in section five.
References


Chinnappan, M., & Thomas, M. (2001). Prospective teachers’ perspectives on function representations. In J. Bobis, B. Perry, & M. Mitchelmore (Eds.),
Proceedings of the 24th annual conference of the Mathematics Education Research Group of Australasia (pp. 155-162). Sydney: MERGA.


research in mathematics education (pp. 3-15). Mahwah, NJ: Lawrence Erlbaum Associates.


Hansson, Ö. (2005a). Preservice teachers’ views of y=x+5 and y=πx² as expressed through the utilization of concept maps: A study of the concept of function. In H. L. Chick & J. L. Vincent (Eds.), Proceedings of the 29th


Monna A. F. (1972). The concept of function in the 19th and 20th centuries, in particular with regard to discussions between Baire, Borel and Lebesgue. Archive for History of Exact Sciences, 9 (1), 57-84.


Abstract

This thesis consists of an overview of the subject, where in particular five papers are put into a frame. The study concerns preservice teachers during a four and a half year long teacher preparation program in mathematics and science for grades 4 to 9. The papers describe parts of a larger study essentially conducted until the sixth term, during the concluding mathematics courses of the program.

The aims of the research study concern the views of preservice teachers on the concept of function. Preservice teachers’ conceptions of functions are in particular examined in relation to mathematical statements that can be related to their future teaching as well as concepts and topics at a tertiary level. Furthermore, questions concerning the significance of functions in mathematics and presence in school mathematics are considered as significant aspects of the preservice teachers’ views of the function concept. The use of concept maps as a research tool is also a question of interest in the study. The final part of the study includes an intervention study regarding the concept of function located to a calculus course as one concluding course in mathematics from the educational programme.

The study is conducted during the spring terms of three consecutive school years. There is a range of findings from the study that is summarized and further discussed in the overview part of the thesis. The major results of the study and an outline of the implications based on the results are included in the concluding discussion.
PRESERVICE TEACHERS’ CONCEPTIONS OF Y=X+5: DO THEY SEE A FUNCTION?

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Kristianstad University/Luleå University of Technology, Sweden and Agder University, Norway

We study the conceptions, progression and especially the concept of function in connection to y=x+5 of two groups of preservice teachers when they take a course in algebra in their third term or a course in calculus in their sixth term of a teacher preparation program. There is a similar development in that they use a numerical interpretation before the course to a higher degree, which decreases after the course with a growth in linear and functional interpretation with the existence of the two variables as a large and rather stable category. The group in their sixth term have a slightly more elaborate language and way of looking at y=x+5 than the group in the third term. For a majority of preservice teachers in both groups, the concept of function is not evoked in connection to y=x+5.

INTRODUCTION

As parts of ongoing studies we have asked preservice teachers in mathematics and science for school years 4-9 to answer the following question “We write y=x+5. What does that mean?” (Grevholm, 1998, 2002; Hansson, 2001). One reason to study y=x+5 is the fact that Blomhøj (1997) reported on final year students in compulsory school, age 15-16 years, having an unsatisfactory (see below) way of handling a question concerning how x is related to y in y=x+5. Another reason is that linear relations are common subjects to be taught by the preservice teachers in different teaching situations as inservice teachers. Linear relations are also often used in introducing the function concept during the later years of compulsory school.

The questions of the study are: What conceptions do preservice teachers have and what is their concept of function in connection to y=x+5? What progression can be seen between two groups in their third and sixth terms in a teacher preparation program?

THEORETICAL FRAMEWORK

Hiebert and Carpenter (1992) present a framework to examine issues of learning and teaching with understanding. The framework assumes that the knowledge of individuals is represented internally; that internal representations are structured and can be related or connected to one another to produce dynamic networks1 of knowledge. They suggest that we think about these networks basically in terms of two metaphors, vertical hierarchy and web:

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1 The idea is supported by the fact that human memory, conceived as a network of entities, is a central and well founded theoretical construct in psychology and neuroscience (Anderson, 2000).
When networks are structured like hierarchies, some representations subsume other representations; representations fit as details underneath or within more general representations. Generalizations are examples of overarching or umbrella representations, whereas special cases are examples of details. … a network may be structured like spider’s web. The junctures, or nodes, can be thought of as the pieces of represented information, and the threads between them as the connections or relationships. … The webs may be very simple, resembling linear chains, or they may be extremely complex, with many connections emanating from each node. (Hiebert & Carpenter, 1992, p. 67)

The two metaphors can also be mixed, resulting in additional forms of networks. Mathematics is understood if its mental representation is part of a network of representations. Understanding grows as the networks become larger and more organized; “a mathematical idea, procedure, or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections” (p. 67). Existing networks influence the constructed relationship, thereby helping to shape the new networks that are formed.

Some parts of the network are so tightly structured that they are accessed and applied as a whole or a single chunk: “accessing any part of the chunk means accessing the entire network” (p. 75). Other parts, called schemata, are relatively stable internal networks that serve as templates to interpret specific events, i.e. abstract representations to which specific situations are connected as special cases.

Ausubel (2000) presents a similar hierarchical cognitive structure to the network model of Hiebert and Carpenter. He also presents his theory for learning in an institutionalized setting and talks about meaningful learning and rote learning, which has consequences for the students’ cognitive structures. For students to accomplish meaningful learning teachers have to activate relevant “anchoring” ideas in the learners’ cognitive structures, necessarily building upon the learners’ prior knowledge, termed by Hiebert and Carpenter as the bottom-up approach (p. 81). When meaningful learning is accomplished, then:

… eventually they [emergence of new meanings in semantic\textsuperscript{2} memory] become, sequentially and hierarchically, part of an organized system, related to other similar, topical organizations of ideas (knowledge) in cognitive structure. It is the eventual coalescence of many of these sub-systems that constitutes or gives rise to a subject-matter discipline or a field of knowledge. Rote learning, on the other hand, obviously do not add to the substance or fabric of knowledge inasmuch as their relation to existing knowledge in cognitive structure is arbitrary, non-substantive, verbatim, peripheral, and generally of transient duration, utility, and significance. (Ausubel, 2000, p. x)

We consider what Ausubel calls meaningful learning to be similar to the learning with understanding of Hiebert and Carpenter, where the dynamic network becomes larger and more organized with a growing understanding. The similar phenomenon occurring in Ausubel’s model is the following:

\textsuperscript{2} Ausubel describes semantic memory as “Semantic memory is the ideational outcome of a meaningful (not rote) learning process as a result of which new meaning(s) emerge.” (p. x).
It is important to recognize that meaningful learning does not imply that new information forms a kind of simple bond with pre-existing elements of cognitive structure. On the contrary, only in rote learning does a simple arbitrary and nonsubstantive linkage occur with pre-existing cognitive structure. In meaningful learning the very process of acquiring information results in a modification of both the newly acquired information and of the specifically relevant aspect of cognitive structure to a specific relevant concept or proposition. (Ausubel, 2000, p. 3)

Tall and Vinner (1981) introduced the notion of concept image as “the concept image consists of all the cognitive structure in the individual’s mind that is associated with a given concept” (p. 151). Different parts of the concept image are evoked in different contexts, as stated by Tall and Vinner, “we shall call the portion of the concept image which is activated at a particular time the evoked concept image” (p. 152). In this paper, we see a concept image as a chunk of the knowledge structure described above, and an evoked concept image as a portion of the concept image as per Tall and Vinner.

**BLOMHOJ’S STUDY**

Preservice teachers supervised by Blomhøj (1997) studied the concept of function in a group of 22 final year compulsory school pupils (the 9th year). The pupils were asked to write their answers to the question “y= x+5, what can you say about x in relation to y?” and followed up the answers with interviews. In his report Blomhøj distributes the answers in four categories: a) x is 5 less than y, b) which interpret the equation without answering the question, c) x is 5 more than y, and finally d) which neither interpret the equation nor answer the question.

The distribution of answers was as follows: a) 6, b) 4, c) 7 and d) 5. Hence, category c), which is a wrong answer, received the most answers. Moreover, the pupils’ answers often contained contradictions and more than half of the students could not give an acceptable interpretation.

**METHOD AND RESULTS**

The preservice teachers at Kristianstad University study mathematics in their first, third and sixth terms and take a total of 30 weeks of full-time study courses where approximately one-third relates to educational studies in mathematics. We studied two separate groups of preservice teachers in their third and sixth terms of a four and a half-year teacher preparation program. The first study occurred during the third term when Grevholm (1998, 2002) asked a group of 38 preservice teachers to answer a questionnaire containing the question of interest before and after a five-week course in algebra and also interviewed some of the preservice teachers. The second study took place during the sixth term when Hansson (2001) replicated the first study with a group of 19 preservice teachers in connection to a five-week calculus course. Hansson also asked them to draw a map that represented their way of thinking about y=x+5 after the course.
Grevholm created a categorization based on the preservice teachers’ written answers to the question “We write $y=x+5$. What does that mean?” The categorization was based on the gathered data. The categories separate answers that:

1) describe how x and y are related numerically, here called category N
2) state that there are two variables, V
3) give a table of values for $y=x+5$, T
4) describe the relation as a straight line, L
5) describe the relation as a function, F
6) give other specific descriptions, O

Table 1 shows the distribution of answers. Hansson used the same categories and Table 2 shows the distribution\(^3\) of his answers. The tables are based on the total number of categories included in the preservice teachers’ answers.

<table>
<thead>
<tr>
<th>Category</th>
<th>N</th>
<th>V</th>
<th>T</th>
<th>L</th>
<th>F</th>
<th>O</th>
</tr>
</thead>
<tbody>
<tr>
<td>Before</td>
<td>19</td>
<td>11</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>2</td>
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<td></td>
<td>(46%)</td>
<td>(27%)</td>
<td>(5%)</td>
<td>(7%)</td>
<td>(10%)</td>
<td>(5%)</td>
</tr>
<tr>
<td>After</td>
<td>12</td>
<td>12</td>
<td>2</td>
<td>8</td>
<td>9</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>(27%)</td>
<td>(27%)</td>
<td>(5%)</td>
<td>(18%)</td>
<td>(20%)</td>
<td>(2%)</td>
</tr>
</tbody>
</table>

Table 1. 28 preservice teachers answered the question before and after the course in algebra, where 36 answered the questionnaire before and 31 after the course. Table from Grevholm (1998).

<table>
<thead>
<tr>
<th>Category</th>
<th>N</th>
<th>V</th>
<th>T</th>
<th>L</th>
<th>F</th>
<th>O</th>
</tr>
</thead>
<tbody>
<tr>
<td>Before</td>
<td>12</td>
<td>14</td>
<td>0</td>
<td>4</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>(38%)</td>
<td>(44%)</td>
<td>(0%)</td>
<td>(12%)</td>
<td>(6%)</td>
<td>(0%)</td>
</tr>
<tr>
<td>After</td>
<td>8</td>
<td>17</td>
<td>0</td>
<td>13</td>
<td>8</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>(16%)</td>
<td>(34%)</td>
<td>(0%)</td>
<td>(26%)</td>
<td>(16%)</td>
<td>(8%)</td>
</tr>
</tbody>
</table>

Table 2. 18 preservice teachers answered the question before and after the calculus course, where 17 answered the questionnaire both before and after the course.

The following illustrates how the categorization was assessed (F\(n\) belonging to the first study and n\(F\) the second study):

N) F1: $y=(\text{value of } x)+5$, 8M: *y is a number that is 5 units larger than the number x*, F4: *that y is the sum of 5 and the number you decide x to be*.

V) F3: *Two unknowns, x and y are variables*, 14M: *y depends on x*, M10: *Different for different people. For me, it means that one x-value represents one y-value*.

T) F7: *A table of values with x-values on the first line “x 5 4 3 2” and y-values on the second line “y 0 1 2 3”*

L) M6: *y=x+5 is a line that intersects the y-axis when x=-5 and intersects the x-axis when y=5*, 12F: *it can also be a straight line*, 4F: *You can also see it as an equation for a straight line that uniquely determines what the line looks like*.

F) M5: *Function $y=\text{variable } x+\text{number } 5$, 3F: *y is a function of x*, 4F: *y is a function of $x+5$.*

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\(^3\) The distribution of answers is adjusted compared to Hansson (2001) to become more uniform with the categorization of answers in Grevholm’s studies. In Hansson (2001) the categories F and V were broader and narrower respectively. Functional thinking (Vollrath, 1986) like “y depends on x” was graded F and more specific statements like “… variables x, y …” graded V.

\(^4\) In this case the preservice teacher puts x and y on the wrong values.
Eight preservice teachers were also interviewed and tape-recorded in the first study, and seven were interviewed and four were tape-recorded in the second study. The interviews reveal that the students have more to say than they express in the questionnaire answers. In conversation, they usually interpret “y=x+5” to cover more of the categories N-O than in the questionnaire.

The use of concept maps

Students give only a few knowledge propositions to the question in the questionnaires, when only one and at most four propositions are normally given. In earlier research Grevholm (2000a, 2000b) demonstrated the use of concept maps as one way to get students to reveal more about their mental representations. It is intellectually more demanding to draw a concept map than to answer a question. In the concept maps students activate more concepts and more links between them than in a verbal written proposition. Inspired by this experience Hansson also decided to use concept maps in his study.

Map made by 5M. A concept map about y=x+5.

From previous experience, the preservice teachers in the second study were familiar with drawing mind maps and concept maps in pedagogy and biology. Hansson (2001) gave a lecture of how to use concept maps in mathematics education (as introduced by Novak & Gowin, 1984) and discussed how different maps can help visualise knowledge and understanding and also be used as Ausubel’s advance organizers. At the end of the lecture, he asked the preservice teachers to draw a map

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5 An advance organizer is a pedagogic device that helps … bridging the gap between what the learner already knows and what he needs to know if he is to learn new material most actively and expeditiously. (Ausubel, 2000, p. 11)
of \( y = x + 5 \), which they did for almost 30 minutes. One of the preservice teacher’s maps, made by 5M, is shown above.

**DISCUSSION**

Both groups of the studied preservice teachers had a similar development of answers to the question before and after the courses they took through a reduction in category N and growth in categories L and F as shown in Tables 1 and 2. Category V is large and quite stable in the algebra course, as well as being the largest category and rather stable in the calculus course. Surprisingly, the preservice teachers in their sixth term before the calculus course came up numerous answers in category N, a category deemed less advanced than categories L and F. The ordering of categories N, V, T, L and F reflects our view of order of more advanced levels of thinking, and hence demands more developed cognitive structures with the function concept having the most complex structure with connections to numerous sub concepts.

The answers in the second questionnaire were more detailed and covered more categories, and were more explicit from the group who took the calculus course. The authors perceived a slightly more mature language in the answers of the second questionnaire, and especially from the group who had advanced further in the teacher preparation program, though a more elaborate language is desired to become successful as an inservice teacher. None of the groups used more advanced mathematics than that from the curriculum of upper secondary level. The interviews indicate, however, a somewhat more mature understanding of the students than what seems to be the case in the written answers.

Examining data for individual students confirms the slow development of concepts. More than every second student gives answers in the same categories before and after the course and other students just add an extra category. For those who keep the same categories it is notable that after the course, they express themselves in a more advanced professional language considered appropriate for teachers. (The students were not aware of the categories used here. They were only asked to answer as honestly and openly as they could to show us their knowledge.)

The number of answers in category F grows when comparing the first and second questionnaires. However, the majority of preservice teachers do not mention \( y = x + 5 \) as being a function. Only one preservice teacher (M13) who took the algebra course gave \( y = x + 5 \) any properties as a function, writing it is a linear function in the second questionnaire. Only one preservice teacher (4F) before taking the calculus course wrote that \( y = x + 5 \) becomes a line. More answers mentioned that the function becomes a line (no one referred to the line as a function graph) after the calculus course and no other property was mentioned.

It is notable that so few preservice teachers write \( y = x + 5 \) as ‘an equation’ in the questionnaires. Only one preservice teacher mentions it in the algebra course and four (in the second questionnaire) in the calculus course, compared to the concept maps where ten preservice teachers mention it. The concept of equation is one the
students have worked with for many years, much longer than the concept of function. It is also obvious that none of the groups actively use the term linear. However, the concept of line is used in both groups and more frequently in connection to functions in the group studying in their sixth term.

The maps contain more information than the written answers. They are clearly more developed in the area of a straight line and mention things like slope, intersection with the axis in a coordinate system and the equation of a straight line \( y=mx+b \), which was also in the written answers. Eight preservice teachers have function as a part of their map, though with a few links\(^6\) connected to it (as in the map drawn by 5M). One exception is 17M who writes \( f(x)=x+5 \) and gives the derivative and primitive function. Moreover, links between function and straight line are not common and only one map (14M) mentions graph and makes links between function, graph and then straight line. Other properties of functions like monotony and continuity are not mentioned. The concept of equation is more explicit in the maps than in the written answers and some maps are also connected to applications and learning and teaching.

The fact that the group just took a course in calculus where the function concept is central was generally not visible in the written answers or the drawn maps, indicating that the function concept is less meaningful in the context of \( y=x+5 \). The students seem to connect with mathematical knowledge on a less advanced level than what they have worked with in their later mathematics courses. A premature concept of function (Vollrath, 1986) is also visible in an answer like “\( y \) depends on \( x \)” (14M) in category V. Even category F has answers with a less developed concept of function like “\( y \) is a function of \( x+5 \)” (4F).

CONCLUSIONS

There was no indication that incorrect answers like those shown in the study by Blomhøj were frequent among the preservice teachers. A similar development exists in both groups of preservice teachers, whose tendencies of a numerical \( y=x+5 \) interpretation before the course lessened after the courses with a growth of linear and functional interpretation, comprising two variables as a large and rather stable category. The group of preservice teachers that had progressed further in the teacher preparation program had a slightly more developed language and flexible way of looking at \( y=x+5 \), where, for example, the concept of equation was more common. The maps gave valuable information about how the students look upon \( y=x+5 \), and the connections between different parts of knowledge became more explicit. The function concept in connection to \( y=x+5 \) was not well developed; if mentioned, it was done so without any properties except as a line in a few cases. Views upon the function concept, as an object with many properties, were hardly visible. This became apparent in the written answers as well as the maps.

\(^6\) We assume the number of links is positively correlated to the concept's importance in relation to each other in the context of \( y=x+5 \).
This study indicates that preservice teachers’ concept of function do not correspond to cognitive structures that evoke rich concept images of function in the context of $y=x+5$. It could mean that, as inservice teachers, they give less attention (Chinnappan & Thomas, 1999; Even, 1993; Fennema & Loef, 1992; Vollrath, 1994) to the function concept in linear relations. The fact that teachers do not give enough explicit attention to the functional aspects of linear relations can be one explanation for the results from 9th pupils year in Blomhøj’s study. Niss (2001, p 43) concludes that “If it is something we want our pupils to know, understand or manage, we must make this part of an explicit and carefully designed teaching” (authors’ translation from Swedish). If we want students (pupils) to be able to interpret a given expression as a function this aspect must be part of the teaching offered to students.

Acknowledgement. We thank Professor Barbara Jaworski for her valuable comments on an earlier draft of this paper.

References


Paper II
An unorthodox utilization of concept maps for mathematical statements: The responses of preservice teachers to a potential diagnostic tool

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This paper considers the construction of concept maps of two groups of preservice teachers in an unorthodox setting, starting with a mathematical statement. The purpose of drawing these maps is to investigate the knowledge and understanding about the statement of individuals; what concepts the statement represents, conceptions about these concepts and how they relate to each other. I describe how the preservice teachers respond to the activity of drawing such maps, starting with the mathematical statements $y=x+5$ and $y=\pi x^2$, with a focus on the concept of function. The preservice teachers’ responses have signs of metacognitive activity and mediation in drawing the maps, as well as indications of an ability to evoke conflicting cognitive pieces of information where drawing hierarchical maps seem more demanding. In a given context, the hierarchical maps also appear to have a higher potential as a diagnostic tool. Moreover, the maps indicate that different concepts – including the concept of function – related to the mathematical statements tend to be separated, preventing preservice teachers from building a conceptual framework rich in meaningful connections. Using evaluation principles influenced by Ausubel’s assimilation theory, the utilization of concept maps seems to reveal important aspects of the knowledge and understanding of individuals in relation to the given mathematical statements.

1 Background

A concept map is a graphical representation of various connections between numerous concepts. The map consists of a network of nodes and links, where the nodes depict concepts and the links represent relations between the concepts. Developed by Joseph Novak together with colleagues (Novak, 1990a, 1998; Novak & Gowin, 1984), the maps have been widely used in sciences education (Al-Kunifeed & Wandersee, 1990; Novak, 1998; Novak & Wandersee, 1990). Similar maps were later used within a range of different subject areas with varying theoretical frameworks (Ruiz-Primo & Shavelson, 1996), e.g., semantic networks (Fisher, 1990).

Concept maps have more recently been used in mathematics education (Doerr & Bowers, 1999; Grevholm, 2000, 2004; Haseman & Mansfield, 1995; Leikin, Chazan & Yerushalmy, 2001; McGowen & Tall, 1999; Ryve, 2003; Williams, 1998, 2003), though to a limited extent and have thus been formulated
in many different ways. Concept maps, as presented by Novak, are based on Ausubel’s assimilation theory (Ausubel, 1968, 2000; Ausubel, Novak & Hanesian, 1978). This theory stipulates that the concept map be derived from a concept placed uppermost in a map with a hierarchical structure, with general and inclusive concepts placed above more specific concepts.

Part of the data in Hansson (2003) and Hansson and Grevholm (2003) is derived from an unorthodox use of concept maps, where the maps are drawn from a mathematical statement rather than a specific concept. The concept maps were used to investigate preservice teachers’ comprehension of the mathematical statement \( y=x+5 \) and, in particular, how the function concept is derived. The investigations indicate that the maps contain a wealth of information on preservice teachers’ knowledge and understanding of the statement and the concepts it represents.

A mathematical statement (e.g. \( y=x^2 \)) can be assumed to represent a number of different mathematical concepts (Dreyfus, 1991; Goldin, 2002; Goldin & Kaput, 1996), where \( y=x^2 \) may be, for example, an equation, a statement, function or a parabola. Concept maps derived from a mathematical statement differ from those explicitly stating a mathematical concept, since it could be interpreted as if the map was derived from the different concepts thought to represent the statement. The maps illustrate the concepts that an individual connects to the statement, the meaning and characteristics of the concepts, and the relations between them regarding the statement.

This paper is organized as follows: aims and the theoretical framework of the study are followed by a section of methods and results from some of the few related studies in mathematics education. Thereafter, the method and procedure for the present study are described as well as a framework for the analysis of concept maps. Subsequent sections present the results of the study, including case studies of preservice students. The final section contains a concluding discussion.

2 Aims of the research

The aim of this paper is to investigate and describe the use of concept maps to reveal the knowledge and understanding of preservice teachers regarding the

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1 Novak defines concepts as “perceived regularities in events or objects, or records of events or objects, designated by a label” (Novak, 1998, p. 21) and Ausubel defines them as “objects, events, situations, or properties that possess common criterial attributes and are designed by the same sign or symbol” (Ausubel, 2000, p. 2).

2 Let us assume that the statement is perceived to represent a real-valued function of a real variable.

3 Here, a given mathematical statement (e.g. \( y=x^2 \)) is not a concept according to the definitions given by Novak (1998) or Ausubel (2000). On the contrary, \( y=x^2 \) can be perceived to represent a number of different mathematical concepts. Moreover, if we use Novak’s hierarchical concept maps and place a mathematical statement uppermost in the concept hierarchy then the map immediately assumes an unorthodox structure, since the hierarchy is lost. The statement is placed above the more general concepts that it is deemed to represent, i.e. where the statement constitutes a specific example for each of the concepts.
concept of function in relation to the mathematical statements \( y = x + 5 \) and \( y = \pi x^2 \).

More specifically, the study aims to answer the following questions: How do preservice teachers construct concept maps starting with the mathematical statements \( y = x + 5 \) and \( y = \pi x^2 \)? How is the concept of function expressed in the maps? What knowledge is displayed and what qualities are desirable in such a map? What experiences of drawing the maps do the preservice teachers express?

3 Theoretical framework

The framework is based on the assumption that knowledge is represented internally and an understanding is described in terms of the way an individual’s mental representation is structured. Internal representations can be linked metaphorically to form dynamic networks\(^4\) of knowledge with different structures, especially in the form of webs and vertical hierarchies (Hiebert & Carpenter, 1992). How an individual deals with or produces an external representation reveals something of how the individual has represented that information internally. Conversely, an external representation (a picture etc.) with which an individual interacts makes a difference in the way individuals represent the quantity or relationship internally.

Networks of knowledge are gradually built as new information is connected to existing networks or as new relationships are constructed between previously disconnected information. According to Hiebert and Carpenter (1992), “a mathematical idea, procedure, or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections” (p. 67). Understanding grows as the networks become larger and more organized, where existing networks influence relationships that are constructed, thereby helping to shape the new formed networks. The construction of new relationships may force a reconfiguration of affected networks. Ultimately, understanding increases as the reorganizations yield more richly connected, cohesive networks. Understanding can be rather limited if only some of the mental representations of potentially related ideas are connected or if the connections are weak. An individual’s ability to structurally handle abstract notions, e.g. function (Sfard, 1991, 1992), as an object is in my view a reflection of more highly interconnected parts of the network.

Meaningful learning is a central idea in Ausubel’s assimilation theory (Ausubel, 1968, 2000; Ausubel et al., 1978). It is accomplished by using relevant anchoring ideas in the individual’s cognitive structure to obtain new meanings that “become, sequentially and hierarchically, part of an organized system, related to other similar, topical organizations of ideas (knowledge) in cognitive structure.” (Ausubel, 2000, p. x). Contrary to meaningful learning, rote learning

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\(4\) The network model is a common model of semantic memory in cognitive psychology (Anderson, 2000) where some models, like Ausubel’s assimilation theory (Ausubel, 2000), make further assumptions about the network, such as it being hierarchal, etc.
has a different impact on the cognitive structure in that “only in rote learning does a simple arbitrary and non-substantive linkage occur with pre-existing cognitive structure” (Ausubel, 2000, p. 3). New concepts are assimilated to the existing cognitive structure in the process of subsumption where the anchoring concepts are subsumers. Subsumers can be more elaborate and specific through integration with related concepts and new linkages can be established where the network of concepts will thus be modified in the process of progressive differentiation. When new concepts are actually introduced they can have a superordinate relationship to concepts that already exist in the cognitive structure, i.e. superordinate learning – when subordinate concepts acquire new meanings. When new ideas are integrated, the existing concepts can recombine themselves and new meanings can be added to them in what Ausubel calls integrative reconciliation. Assimilation theory applies what Hiebert and Carpenter (1992) describe as a bottom-up approach in the way knowledge structures develop, building upon the individual’s prior knowledge. I consider Ausubel’s meaningful learning similar to what Hiebert and Carpenter call learning with understanding.

All parts of an individual’s cognitive structure associated with a given concept are called a concept image (Tall & Vinner, 1981). Different parts of the concept image are evoked at different times. The portion of the concept image evoked at a particular time is called an evoked concept image.

4 The use, definition and evaluation of concept maps in studies related to the concept of function

There exists a relatively small number of studies using concept maps in mathematics education, and very few about the concept of function (e.g., Doerr & Bowers, 1999; Grevholm, 2000, 2004; Leikin et al., 2001; McGowen & Tall, 1999; Williams, 1998). This section details studies related to the concept of function, their use of concept maps and the results.

4.1 The view of calculus students on the concept of function compared to professors with a Ph.D. in mathematics

Williams (1998) examines concept maps as an instrument for the assessment of conceptual understanding. The maps are used to compare the knowledge of function held by students enrolled in university calculus classes. The students who participated in the study had different backgrounds, with half coming from non-traditional sections and the other half from traditional sections of a first-year calculus course. Several professors with PhDs in mathematics made similar maps called experts’ maps.

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5 A reform section emphasising modeling and technology.
The study participants manually drew their concept maps starting with the concept of “function”. They were free to draw the maps as they pleased after an introduction to concept maps by Williams, where she showed numerous examples of concept maps with different structures, such as hierarchical, non-hierarchical and web based concept maps, all with labeled links.

Williams states that the maps drawn by the participants proved to be widely divergent and did not lend themselves to a numerical scoring scheme. She takes a holistic approach to the evaluation process and describes her analysis of the maps, “I looked at the maps as integrated wholes and searched for differences between the two student groups and between the experts and the students.” (p. 416). Williams emphasis two observations in her analysis of the students’ maps. The first observation is that the concepts and propositions in many maps are trivial or irrelevant regarding both the reform and the traditional students. The second observation is that the students’ maps have an “algorithmic nature”, particularly among the traditional students, who reflect upon the steps in a procedure. Another characteristic of the students’ maps is the almost nonexistent integration of concepts, by linking a concept to a concept in another branch (cross-links). This contrasted to the experts’ maps where cross-links were more frequent.

The experts’ maps compared by Williams with the students’ maps were a restricted task where the expert was asked to draw a concept map of function representing what he or she would expect students completing a first-year calculus sequence to know. Williams also asked the experts to draw an unrestricted concept map, but did not include them in the analysis. The expert maps were more homogeneous than the students’ maps. Moreover, unlike many of the students’ maps, the experts’ maps showed no hint of algorithmic nature. Instead, they reflected properties, categorical groupings and classes of functions.

Williams concludes that the concept maps indicate differences in conceptual understanding. The general homogeneity of the experts’ maps and their distinct variance from the students’ maps give credibility to her conclusion’s opinion that “concept maps do capture a representative sample of conceptual knowledge and can differentiate well among fairly disparate levels of understanding.” (p. 420). Moreover, in Williams’ experience the analysis of the maps also provided information about the students’ understanding not readily gained from a traditional pen-and-paper test, providing important information about students’ conceptual understanding that makes concept maps useful in the mathematics education researchers’ collection of tools.
4.2 The cognitive development of high achieving versus low achieving algebra students in relation to the function concept

McGowen and Tall (1999) use concept maps\(^6\) to document the processes by which students construct, organize and reconstruct their knowledge about functions during a course in mathematics. During a 16-week algebra course at a community college, the students were asked to construct concept maps starting with the concept of “function” on three occasions at five-week intervals. The students were advised to use Post-it notes to move items around before they drew their final maps, and at each occasion were given an opportunity to review their map and redraw it one week later, without any given preference about the structure of the map.

Two groups of students comprising four of the most successful and four of the least successful student concept maps were analyzed (the selection of students was based on a pre- and post-test, and the final exam). The principal focus of the study was to trace the cognitive development of students throughout a mathematics course and to seek the qualitative differences between those of different levels of achievement. In addition to a qualitative analysis of the concept maps, McGowen and Tall also used a pictorial technique to document the changes in the maps in what they call “schematic diagrams”. A student’s schematic diagram illustrates how the student built the first concept map and the subsequent two maps by keeping some old elements, reorganizing and introducing new elements.

The most successful students who built their two successive concept maps added new elements to old elements in a structure that gradually increased in complexity and richness. In contrast, the least successful students showed little constructive growth, building new maps on each occasion and, according to the authors:

Analysis of the … selected students reveals striking similarities among the schematic diagrams for each group. Each student in the more successful group produced a sequence of concept diagrams [concept maps] whose schematic diagrams retained the basic structure of the first within a growing cognitive collage\(^7\). Each set of schematic diagrams for the least successful also exhibited a common characteristic: a new structure replaced the previous structure in each subsequent map, with few, if any elements of the previous map retained in the new structure. No basic structure was retained throughout. (McGowen & Tall, 1999, p. 287, emphasis in original)

The concept maps of the low achieving students also reveal procedural undertones by concentrating on routines (find slope, find constant, solve, etc.).

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\(^6\) Of note, McGowen and Tall (1999) define a concept map as “A concept map is a diagram representing the conceptual structure of a subject discipline as a graph in which nodes represent concepts” (p. 281), and do not include labeled links in the concept maps in the study.

\(^7\) The term “cognitive collage” is used to describe the notion of an individual’s conceptual framework in a given context.
evaluate, etc.) with few or no links to other concept branches. A quantitative analysis of the maps triangulated with the students’ written work and interview data confirms that their knowledge is compartmentalized, preventing them from building a conceptual framework with meaningful connections.

McGowen and Tall conclude that there is a wide divergence in the quality of thinking processes, with high achievers showing a level of flexible thinking by using various representational forms and building rich conceptual frameworks on anchoring concepts that develop in sophistication and power. The lower achievers, however, reveal few stable concepts with conceptual frameworks containing few stable elements, leaving the student with much work to use learned routines inflexibly and often inappropriately.

4.3 The views of preservice teachers on the function concept related to teaching and learning

Doerr and Bowers (1999) see concept maps as a tool for expressing the ideas of preservice secondary mathematics teachers about the concept of function. Concept maps were drawn individually before and after a course with several instructional sequences designed to challenge their existing knowledge about the concept of function and the mathematics of change, and to evoke models of how others might learn these ideas. The task was presented in an open-ended way and the preservice teachers were free to structure their concept maps in a way that seemed reasonable to them. They were also asked to write accompanying interpretative essays about the concept of function.

The data analysis by Doerr and Bowers involves three phases, all qualitatively. In the first phase, the overall structure and most central features were identified in the pre-course maps, followed by a second phase comparing these to the preservice teachers’ essays and finally, a third phase focusing explicitly on the differences between individuals’ pre- and post-course concept maps.

In their analysis, Dorr and Bowers notice structural differences between the maps. The pre-course concept maps were structurally of two types, the first characterized by a central idea, usually “function” with a relatively large number of sub-concepts with few details directly related to function. These maps were often structurally similar to the central hub of a wheel, though with limited interconnectivity among the sub-concepts (few cross-links). The majority was of the second type of map and more of an unstructured web with several clusters of sub-concepts linked to the concept of function. No preservice teacher had constructed a hierarchically organized map that flowed from top to bottom. According to the authors, this could be partly due to the web-based illustrations of concept maps given in class. The authors do not further describe how they define a concept map, e.g. if they use labeled links, etc. The authors continue that most of the preservice teachers had prior experience with concept maps and consider their preference for web-based maps a partial reflection of their thinking.
about the concept of function as non-hierarchical and non-linear. The preservice teachers’ post-course concept maps showed shifts from their initial maps to more web-like structures with a higher degree of interconnectivity, a larger number of sub-concepts and more structured relationships among the sub-concepts.

Besides the maps structures, Doerr and Bowers make some observations about the contents of the maps. On their initial maps, all but one of the eleven preservice teachers who participated in the study included some ideas about multiple representations (table, graphs, symbols, words) of functions. For a large majority of the preservice teachers, these representations were not linked to each other. On the final map, half of them now expressed a linked relationship between various representations of a function. A major shift from the initial to the final maps was related to ideas about teaching and learning about functions. In his initial map, only one preservice teacher had two single nodes related to pedagogical strategies or student understandings. However, on the final maps, nine of the eleven preservice teachers had added multiple nodes explicitly related to teaching about functions and learning issues for students. This suggests that prior to this particular course, the views of preservice teachers about the concept of function were largely disconnected from any pedagogical strategies or learning paths or obstacles that students might encounter. The preservice teachers’ final concept maps indicated a link between teaching and learning issues and their mathematical understanding of the function concept, according to Doerr and Bowers.

4.4 High school teachers’ understanding of the mathematics they teach

Leikin, Chazan and Yerushalmy (2001) asked high school mathematics teachers to draw concept maps for the concept “equation”. The purpose was to identify the teachers’ conceptions of equation and how those conceptions fit inside a larger approach to teach algebra, and especially if the teachers have a functions-based or an equations-based approach to teach algebra. Moreover, they also wanted to examine the strengths and weakness of concept maps as a tool to thus examine the teachers’ understanding of the mathematics they teach.

The teachers drew concept maps as a part of interviews at the beginning and end of a school year, when they taught an algebra course that adopted a district-developed curriculum built around the assumption of technological support. The teachers where given a table of concepts and asked to draw a concept map, but were also allowed to add other concepts to the map. An example of a concept map for the notion of quadrilateral was shown if the interviewee was unfamiliar with a concept map. The authors do not specify how they define a concept map, other than saying, “In science education, concept maps – defined as a two-dimensional representation of relationships between selected concepts – have been used extensively…”, p. 289.)

At the end of the school year the teachers were asked to compare their new concept map with their map drawn at the beginning of the school year. The
authors do not generally attempt to interpret or assess the maps other than to make some qualitative comments about the maps’ structure, which contained web-like and hierarchical components. Instead, they saw the interviews as central to the evaluation of the maps and commented whether the maps exposed the teachers’ understanding:

As a result, we would certainly advocate the importance of discussion of maps with interviewees, as well as the combination of concept maps with other interview tasks. Without such supports, it seems very difficult to assess what one sees and to determine whether repeated use of concept maps over time reveals changes in perspective or representations of different aspects of a teacher’s thinking. (Leikin et al., 2001, p. 295)

This point of view is somewhat different from other studies using concept maps in mathematics education that highlight the difficulty in assessing concept maps. They also conclude that “One strength of concept maps is that they can indicate a conception of a concept as well as how this concept fits into a larger web, potentially revealing tensions in a teachers’ thinking.” (p. 295).

Another outcome of the study was illustrated in two case studies using concept maps in interviews, showing the difficulties of teachers in understanding the relationships between the concept of equation and the concept of function.

4.5 A longitudinal study of preservice students’ conceptual development

In a longitudinal study of preservice teachers’ conceptual development in mathematics, Grevholm (2000, 2004) uses concept maps in a number of different ways:

Concept maps are used as a tool both for analyzing the content of the teacher education to find the fundamental concepts, to investigate students’ answers in questionnaires and interviews and for the students to express their current concept structure. (Grevholm, 2000, p. 1)

In her description of concept maps Grevholm refers to Novak (1998), who defines a concept map in the same manner. Grevholm presents findings of the students’ conceptual development that illustrates her kind of analysis. In the longitudinal study, preservice teachers drew concept maps of the concept of equation and the concept of function at three occasions (Grevholm, 2004) with intervals of about six and nine months; the preservice teachers were not taking any mathematics courses during that period. The first concept map was often constructed in collaboration with other students whereas the second and third maps were drawn individually. The preservice students participating in Grevholm’s study are in the same teacher preparation program described in this paper.

The concept maps were analyzed qualitatively to describe how the maps change over time. Grevholm (2004) concludes “The knowledge about students’ conceptual structures given to me from the maps is much richer than the one I get
from interview questions” (p. 16). Moreover, the results show concept maps occasionally produced by the students to be similar to each other: “The slow development shown by the maps is towards clearer structure, richer maps and better verbal propositions. Maps from different students can be quite different but the knowledge a student expresses seems to be lasting with minor changes”. Grevholm also express her opinion that concept maps give valuable information of the students’ conceptual development “With the concept maps it is possible to see where changes in structures take place, how the individual concept images develop and where the learner still has some conceptual structure to explore and assimilate.” (p. 16). With the preservice teachers’ concept maps, Grevholm also observes that the maps are useful in revealing when the students over-generalize and when they not are able to group concepts belonging together.

5 Method and procedure
The present study, conducted during two consecutive spring terms, is part of a larger study concerning the understanding of the function concept by preservice teachers. It comprises two groups of preservice teachers at a small university in Sweden and takes place at the end of each spring term when the students have completed a course in calculus. At the end of each term the preservice teachers had completed all courses in mathematics in the teacher preparation program for mathematics and science, school grades 4 - 9.

Several studies are conducted in mathematics education with various types of maps, called “concept maps”. In agreement with the aims of the study, I have chosen to investigate how preservice teachers construct concept maps in a freely structured form and in a hierarchical form, both derived from a mathematical statement.

The first group of preservice teachers began by drawing a concept map for the statement y=x+5, structured in a way that seemed reasonable to them. One week later, they drew a hierarchical map for the same statement (the maps were collected at the end of both sessions). As a result, there was a lower degree of mutual influence between the contents of the maps and the student’s work methods. The preservice teachers’ maps were distributed shortly afterwards; they were able to comment on the contents of the maps and how they experienced the drawing process.

Each concept map was analyzed as part of an integrated entity and elements such as structure, central features and included concepts were studied. The section of the map containing the function concept (in case it was included on the map), its structure and scope, its relations to other concepts on the map and the highlighted characteristics regarding the map were especially considered. The different concepts thought to be represented by the statement and their relations to other concepts were also studied. The comments from the first group were

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8 The university has approximately 9,000 students, half of whom are enrolled full-time.
compiled with the contents of the concept maps. The same procedure was later repeated during analysis of the concept maps drawn by the second group of preservice teachers.

After analyzing the concept maps prepared by the first group and their accounted experiences during the drawing process, the work method for the second group was modified. Both drew a freely formatted map and a hierarchical map the following term, but gave written comments on the maps at the same occasion. Like the preservice teachers in the previous group, this group members drew maps for the statement $y=x+5$; the study was then widened to include the statement $y=\pi x^2$.

The study was scheduled for when the preservice students’ mathematics course concerned mathematics education. This resulted in lower absenteeism and ensured the same preconditions for all preservice teachers when drawing the concept maps, but also limited the amount of time allotted to the process of drawing the maps. The students were informed that the maps were intended to be part of a research project in which participation was voluntary.

5.1 The groups of studied preservice teachers

The preservice teachers who participated in the study specialized in mathematics and sciences for school grades 4 to 9. The program’s duration was four and a half years. Applicants to the teacher training program should have completed a program in high school level mathematics and sciences or be equally qualified. The two groups were comprised of 19 and 25 students, of which women were in the majority. On each occasion, the preservice teachers made up all of the students specializing in mathematics and sciences during the sixth term of the program. The preservice teachers studied 15 weeks of full-time mathematics during the sixth term. After the sixth term the students had completed their required mathematics courses on the educational programme and a total of 30 weeks full-time mathematics studies, one-third was dedicated to mathematics education.

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9 The maps produced by the first group suggested that the freely formatted maps were often more extensive, but with a higher incidence of free associations (bringing mind maps to mind, Buzan, 1995) than the hierarchical maps. Among the nodes related to the statement, the number of concepts that the statement was understood to represent were generally lower in the freely formatted maps. Furthermore, relationships between the concepts deemed to be represented by the statement – among them the function concept – tended to receive less attention in the freely formatted maps than in the hierarchical maps.

The evaluation of the first group indicated that the hierarchical maps had a greater potential to provide information about the preservice teachers’ function concepts and the relation between these and other concepts than the freely structured maps had. The written comments and the author’s own observations of the students as they drew the maps revealed that the preservice teachers in the first group felt that the drawing of hierarchical maps was a more arduous task. As a result, the work method for the second group of preservice teachers was changed. The preservice teachers were now able to draw both maps at the same time and could thus manage the time as they chose.

10 Gymnasieskolans naturvetarprogram.
5.2 Concept mapping

Concept maps were presented to each student group, along with examples of different types of concept maps, such as non-hierarchical, web-based maps and hierarchical maps where the nodes represented concepts and the links were labeled in each case (the hierarchical maps were constructed according to Novak & Gowin, 1984, and Novak, 1998). The displayed concept maps were often related to science education. Concept maps derived from mathematical concepts were largely avoided during the presentation to not influence the contents of the maps that the students were to draw in the subsequent assignment.

After the introduction, the preservice teachers in the first group were each directed to draw a concept map based on the statement \( y = x + 5 \). They were thus able to construct a concept map with a structure they themselves considered suitable. One week later during the second occasion, they were asked to draw a hierarchical concept map for the statement \( y = x + 5 \). The maps were distributed shortly thereafter and the students were instructed to examine their maps and provide written comments about them. The instructions were as follows: “1. What are your thoughts on the two concept maps?” In particular, the preservice teachers were asked to comment on the different sections of the maps by answering the questions: “2. Is there any feature that you believe you have described successfully/unsuccessfully? Is there any feature about which you are uncertain?” and finally, against the background of their experiences of drawing the maps, “3. In your opinion, what are the advantages and disadvantages of concept maps?” The preservice teachers were also instructed to select, check and comment on a map drawn by another student within their group.

The second group of preservice teachers received a similar introduction to concept maps the following spring term. They subsequently each drew two maps derived from the statement \( y = x + 5 \). Both maps were constructed according to their own ideas were of one of the maps should be hierarchically constructed. The procedure was repeated for the statement \( y = \pi x^2 \). Like the participants of the first group, the students were also given the opportunity to comment on their maps and their experiences in drawing the maps.

The first group of preservice teachers were given 30 minutes to draw their maps on each occasion, and slightly more time to check and comment on their own maps and those of a fellow student. The other group received 60 - 80 minutes to draw and comment on their maps, depending on when the students decided they had completed the assigned task. The author took about 30 minutes to introduce the maps to each group. In total, 17 students from the first group and 24 from the second submitted all of their maps.
6 Framework for analysis

The maps in this study are viewed as a qualitative tool to study the individual’s knowledge and grasp of concepts with respect to a mathematical statement. Several studies on concept maps support the theory that they provide valid information about an individual’s knowledge of a particular subject area, though most studies have been conducted on areas other than mathematics (Baralos, 2002; Laturno, 1994; Markham, Mintzes & Jones, 1994; Mintzes, Wandersee & Novak, 1999; Novak, 1998). The studies indicate concept maps derived from a mathematical statement to be perhaps used as a diagnostic tool to qualitatively study an individual’s knowledge and interpretation of concepts (as indicated by Hansson, 2003; Hansson & Grevholm, 2003).

The nodes in the actual maps may be explicitly specified concepts or mathematical symbols, statements and expressions that are connected by links with linking words to explain their meaning. The mathematical statements, symbols and expressions complement the map with a symbolic language that enables associations with many different mathematical concepts (Dreyfus, 1991; Goldin, 2002; Goldin & Kaput, 1996).

6.1 Evaluation based on Ausubel’s assimilation theory

A hierarchical structure opens the map up to an evaluation that is partially analogous with how Novak and his colleagues utilize Ausubel’s assimilation theory in their evaluation of concept maps (Ausubel, 1968, 2000; Ausubel et al., 1978). In using Ausubel’s theory, Novak et al. concluded that concept maps should be labeled with appropriate linking words and be hierarchical, starting with broad inclusive concepts that lead to more specific, less inclusive concepts. Moreover:

Concept maps . . . are a representation of meaning . . . specific to a domain of knowledge, for a given context. We define concept as a perceived regularity in events or objects . . . designated by a label. . . . Two or more concepts can be linked together with words to form propositions and we see propositions as the units of psychological meaning. The meaning of any concept for a person would be represented by all the propositional linkages that the person could construct that include that concept. (Novak, 1990b, p. 29)

Concept maps were intended to “tap into a learner’s cognitive structure and to externalize, for both the learner and the teacher to see, what the learner already knows” (Novak & Gowin, 1984, p. 40). Novak and Gowin recognized that any representation would be incomplete – where not all concepts or propositions would be represented. Nevertheless, such maps would provide a “workable representation” (p. 40).

Ausubel’s assimilation theory suggests a hierarchical cognitive structure where new information is often relatable to and subsumable under more general, inclusive concepts. According to Ausubel’s principle of progressive
differentiation, the structure expands where new concepts and new links are added to the cognitive structure. Meaning increases for individuals as they recognize new links between different parts of the cognitive structure.

Using Ausubel’s assimilation theory Novak and Gowin (1984) suggest how to score their concept maps based on the following three main reasons:

1) More inclusive concepts and propositions are superordinate to less inclusive concepts and propositions. It takes a more active integration of concepts to construct a hierarchical concept. Individuals will have to grasp new meanings and actively integrate them into their existing conceptual framework. Representations of a hierarchical concept map thus show that an individual is able to differentiate a more inclusive concept from a less inclusive concept.

2) Concepts in a cognitive structure continuously undergo changes of progressive differentiation, i.e. greater inclusiveness of concepts and propositions are acquired through linkages with other related concepts. As learning proceeds, more and more related concepts are progressively differentiated. These progressive differentiations of concepts are enhanced when concept maps of one topic are cross-linked to concept maps for other related topics.

3) As more meanings are acquired, concepts are recognized and interconnected to other relevant concepts (in the process of progressive differentiation). Some newly acquired meanings can conflict available yet relatable concepts. Integrative reconciliation occurs as new knowledge is learned, existing knowledge structures recombine themselves, and new and different meanings arise, as when conflicting meanings of concepts are resolved or when misconceptions are uncovered. Concept maps showing valid cross-links between numerous concepts that are otherwise viewed as independent or conflicting can reveal the learners’ integrative reconciliation of concepts.

6.2 Maps starting with a mathematical statement

The maps in this study differ from concept maps because they are derived from a mathematical statement and not an explicitly expressed concept. An individual may believe that the mathematical statement represents several different concepts and as such, the map is derived from not one, but several concepts. If the statement is thought to represent several different concepts, then they are then naturally and directly linked to the statement (this is supported by the empirical results of the study). Thus, like the maps recommended by Novak et al. (see the figure below), ordinary concept maps can be drawn for each and every one of the concepts with most of their evaluation principles still applied. Studies in

11 A scoring procedure of concept maps is described in Novak and Gowin (1984), giving points for each meaningful, valid proposition (1 point), valid level in the hierarchy (5 points), cross-links (valid and significant 10 points, valid but do not illustrate a synthesis between sets of related concepts or propositions 2 points), examples (1 p). In addition, a criterion concept map may be constructed, and scored for the material to be mapped and the student scores divided by the criterion map score to give a percentage for comparison.
mathematics education, as outlined in Williams (1998) and McGowen and Tall (1999), clearly indicate how differentiated, conceptual interpretations can arise in the maps. The various concepts represented by the mathematical statement can be analyzed individually, but the map also provides information on the individual’s interpretation of the relationship between the different concepts in relation to the statement.

![Diagram 1](image1.png)

Figure 1. This is an example of a map, derived from the statement $y=x^2$, which represents the concepts of equation, formula, parabolas, … , function to an individual. It is possible to further extend the construction of the map by drawing ordinary concept maps for each of the concepts.

![Diagram 2](image2.png)

Figure 2. Here is another example of a map where the various concepts are arranged into a somewhat more integrated structure than in the previous map. The map contains information of the concepts that an individual associates with the statement and the meaning assigned to the concepts that can be further developed to illustrate their relationship to each other.

The current maps can also differ from hierarchical concept maps where the mathematical statement violates the principle that general, more inclusive concepts are placed above specific concepts when it is linked to the various concepts it is thought to represent. The links from the statement violate the hierarchy, since the statement constitutes a specific example of each different concept. Nevertheless, the other parts of the map can be constructed hierarchically.

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12 Let us assume that the statement represents a real-valued function of one real variable.
7 Results

7.1 The first group

The structures of the two maps drawn by the preservice teachers differ greatly from each other. The students were allowed to draw their first concept map in whichever manner they chose and thus designed their maps in various ways. Furthermore, they did not always write down to explain the meaning of the links on the maps. On the second occasion the students were instructed to draw the maps hierarchically with the statement placed at the highest point on the map and asked to explain the links. It transpired in their comments on the maps and during their construction of the maps that the students found it more mentally demanding to draw the maps in the latter manner. There were fewer concepts and fewer links drawn in the second map than in the first, possibly due to the same amount of time being allotted for both sessions and more time being required for the students to draw hierarchical maps.

A number of concepts were arranged around the statement on most of the freely structured maps, such as the central part of a wheel (e.g. 4F, 5M, 12F). Other map sections consisted of various web-like structures (including cross-links, even if they were in the minority) that could also resemble clusters (e.g. 3F, 7F, 13M). None of the freely formatted maps were hierarchical in structure; only a few of the maps had anything resembling a hierarchical structure (e.g. 1F and 17M). Although the maps had numerous links, they did not give the impression that the preservice teachers had attempted to capture meaningful relationships between the concepts. As a result, clusters could be found in one section of the map, whereas other sections primarily consisted of chains of nodes and links (e.g. 10F and 7F).

The hierarchical maps drawn by the students generally consisted of fewer links and nodes, with the exception of cross-links, where a distinct trend could not be identified. The structure of the maps became naturally more homogenous. The number of nodes connected to the statement $y=x+5$ was commonly reduced to fewer than half the number of links compared to the freely structured map (e.g. 3F, 8M, 11M) where the statement in the extreme case had just one link (4F, 10F, 14M). As the number of links to the statement was reduced in the hierarchical maps, the proportion of concepts that the statement was deemed to represent increased and were thus allotted greater space on the maps.

The students tend to mix inclusive and specific concepts in the map. Less successful students at constructing maps seem to focus on formulating sentences that are written down in the form of a chain, consisting of concepts and links (e.g., 6F, 7F, 13M), instead of focusing on constructing meaningful relationships between the different concepts in the map (this would result in fewer cross-links).

The maps also show signs of a less developed language among the preservice teachers, who do not always use appropriate mathematical terminology. For example, 7F states that an “equation” contains an “unknown
variable”, 8M labels \( y=x+5 \) a “mathematical expression”\(^{13}\) and says the line “slopes upwards”. 11M states the solution to an “equation” may be “ununiquely”\(^{14}\) defined (as opposed to “uniquely”\(^{15}\) defined), 12F writes \( y=x+5 \) results in a “straight stroke”\(^{16}\) and “the letters” (meaning the variables x and y) may be “different numbers”. The maps also contain traces of an algorithmic nature (similar to results in Williams, 1998 and McGowen & Tall, 1999), which 5M raises in “the point-slope equation \( y-y_1=k(x-x_1) \)” in his map and 16F describes how the slope can be calculated “\( k=(y_2-y_1)/(x_2-x_1) \)”.

Of the various concepts the preservice teachers connect to \( y=x+5 \), a graphical or geometric perspective in the form of a straight line occurs more frequently; in this case, the students grasp concepts such as sloping and intersection through the use of coordinate axes (e.g. 3F, 13M, 16F). In general, a more developed view of the function concept in the form of an object (Sfard, 1991, 1992) possessing various properties (Slavit, 1997) and a set of sub-concepts (Eisenberg, 1991) will not usually be expressed in the maps. Eight students raise the function concept in both versions of their maps. While the students raise the function concept, none of them from the first group mentions that the statement represents a linear function. This is contrary to those on the expert maps in Williams (1998), which represent various different classes of functions to a great extent. On the other hand, some students (14M and 17M) are increasingly able to reconnect to the function concept, even though this does not always take place when connecting to the node “function”. Instead, they connect to other nodes on the map (often to “\( y=x+5 \)”), where 17M raises the “derivative”, “the anti-derivative” and “the inverse”. Incidentally, 17M is the only student in the group who uses the symbol \( f(x)=x+5 \) in connection to the function concept. None of the students mentioned domains or codomains in the maps containing the function concept.

One characteristic of the preservice teachers’ maps was that when the function concept was actually present, it was not a well-integrated node with connections to several nodes on the map. The preservice teachers do not observe the influence of the other concepts on the function concept. For example, only one student (14M) who had the terms “straight line” and “function” in his map states that the line is a graph of the function. Furthermore, when the preservice teachers included the concepts “function” and “increasing” (e.g. 2M, who uses the term “increasing” to refer to the slope of the line), they did not link “function” and “increasing” and seemed to not grasp the fact that “increasing” is also a property of the function. In another example, 7F states in her (freely structured) map that the x-value of -5 results in a “point of intersection” on the x-axis, but does not state -5 as being zero of the “function” shown on her map. The examples all represent an absence of cross-links regarding the function concept.

\(^{13}\) According to Swedish terminology, a mathematical expression does not contain an equality sign.
\(^{14}\) “Mångtydig” (swe.).
\(^{15}\) “Ensyntig” (swe.).
\(^{16}\) “Rakt streck” (swe.).
Although the two maps of the preservice teachers have completely different structures, since the second map has a hierarchical arrangement whereas the first does not, and the second map has fewer nodes and links, common “themes” such as straight lines, equations and functions are continuously repeated as a rule. Connections to teaching and learning rarely appear in the maps. Only a few preservice teachers (such as 4F and 11M) make the connection to student learning, though this does not constitute a well-developed area, but consists of one or a few nodes on the map.

7.1.1 Preservice teachers’ response on drawing the maps

The students in the group constructed their hierarchical maps with varying degrees of success. Their comments about the maps reveal a preference by several of them to use the maps as a tool during brainstorming sessions when they can write down their thoughts about the statement and connect them with links as they did when they drew the first map. Several students indicated that they found this work method to be more natural, such as 9F who wrote the following comment: “I think that it is more difficult to make a hierarchical concept map than the first type of map we did. When I constructed the first map I could follow my own ideas and work in accordance with my own sense of logic. In the hierarchical map I have to “rearrange” my thoughts.” Similarly, 12F wrote: “I could consider using this during a brainstorming session with other students when they write down the ideas they come up with and what they think about. However, this would result in a map much more like the first. To draw the second map one has to know what one is doing to be able to proceed”.

The lack of experience of drawing maps is also reflected in the students’ comments. Many feel that the maps are “confusing”, as can be seen from 6F’s comment, “they can quickly become confusing”, or 2M’s comment, “it can be difficult to arrange the concepts in order of hierarchy, but it becomes apparent that the hierarchical order is easier to understand after attempts are made to do so. The latter [hierarchal map] has an order and a structure, while the former [non-hierarchal map] has concepts all over the place.” It can be mentioned that Novak & Gowin, 1984, and Novak, 1998, state that students must practice drawing hierarchical maps so as to be able to utilize them more successfully. Although their comments refer to (orthodox) concept maps, their experiences may be relevant even if the maps are used in a more unorthodox manner.

The students’ comments concerning their colleagues’ maps were often brief. The hierarchical map were generally thought to be easier to follow and understand, as evidenced by 4F’s comments on 2M’s map: “the second (map) is clearer and easier to follow from start to finish.” Or 12F’s comments on the map drawn by 15F: “the first map is more confusing than the second, but one cannot say whether it is good or bad, since concept maps are so individualistic. It is as if one were to say that notes are also individualistic, thus the person who wrote them may understand them, but another reader may not need to understand.” At
this point, 12F refers to a point also raised in the comments, in which the students
did not appreciate the maps as a tool for the assessment of knowledge and
understanding, but rather as very individualistic constructions of value to the
creators themselves and not others. Their comments reveal a perception of
concept maps as a metacognitive tool (Novak, 1990b).

In general, the students are more expansive when they comment on their
own maps. They also go as far as to establishing connections to the maps’
structure and the suggested work methods. The first map was considered to be
“confusing” to interpret, but arose from the work method where participants were
invited to write down their thoughts and associations, which was preferred by
many students. The second map was considered easier to follow and interpret, but
more difficult and demanding to draw.

The answers to the questions connected to the maps’ layout vary, but are
nevertheless often mainly connected to the first task, where the structure of the
maps is affected and includes “confusing” maps as well as a lack of experience in
drawing maps. Among the common answers to the question of how the students
felt from their experiences about working with maps is that while the maps can
be useful in obtaining a clear image or a comprehensive impression of how an
individual interprets the mathematical statement, they can also quickly become
unclear and “confusing”. There are also answers concerning metacognitive
aspects (Novak, 1985, 1990b) of drawing concept maps, as 2M writes “good tool
to help a person to gather his/her thoughts and focus his/her skills”, or 9F writes
“It helps a person to structure his/her thoughts.”.

7.1.2 Two preservice teachers’ maps

The layouts of maps drawn by two students, Ted (5M) and Ann (7F), and their
accompanying comments are presented below. Both preservice teachers raise the
function concept in their maps. The maps are considered examples of how the
various students in the group drew their maps and illustrate their very
individualistic nature.

In their freely structured maps, both Ted and Ann placed the statement at the
center, as did most students in the group. Nevertheless, they are closer to the
group’s extremes regarding the maps’ web-like structure and trend towards
clusters, which are more predominant in Ann’s map than in Ted’s. Ted’s
hierarchical map is also more representative of how the students designed their
own hierarchical maps, whereas Ann’s map is an extreme case with respect to the
absence of cross-links.
7.1.2.1 Ted’s maps and comments

Figure 3. The first map that Ted drew.

Figure 4. The second map that Ted drew.

<table>
<thead>
<tr>
<th>Found only in the first map</th>
<th>Found in both maps</th>
<th>Found only in the second map</th>
</tr>
</thead>
<tbody>
<tr>
<td>motion, distance, acceleration can describe a physical event, graphical solution, variable, infinite, calculator, intersects line 5</td>
<td>straight line, coordinate system, origin, y=kx+m, y depends on x, a slope of 1, function, table of values.</td>
<td>y-y1=k(x-x1), y-axis 5 steps above the origin, in point (0,5), mathematical expression, an x-value returns one y-value only. (Links: point-slope equation,</td>
</tr>
</tbody>
</table>
Table 1. Overview of the contents of the maps (the description of the links’ meaning has also been included in the other map).

**Ted’s answers:**

1) More order in the second map. More “objects” in the first. Would nevertheless use the second map in a teaching scenario. 2) I have managed to give a good description of the straight line [this is in reference to the second map]. The first map is confusing. 3) Overview. Some people may find it confusing.

The first map resembles a mind map (Buzan, 1995), where the statement \( y = x + 5 \) is located at the center of the map, from where links (12) are drawn to the various concepts and assertions made by Ted in connection to the statement. Hardly any attempt is made to place the concepts on the map in order of hierarchy, or to arrange them in a more systematic manner. The structure is similar to one of the types of pre-course maps, “the central hub of a wheel”, described in Doerr and Bowers (1999). The meanings of the various links are not explained in the map. The map simply branches off at the statement “can describe a physical event”, indicating that the notion of physical event represents a more progressively differentiated structure of knowledge (Ausubel, 1968, 2000; Ausubel et al., 1978). Other links do not branch off, with the exception of “straight line,” and different sections of the map are not linked, i.e. the map has no cross-links. It is observed that a fundamental concept such as “function” is not connected to many sections of the map. This possibly indicates that the statement does not evoke a rich concept image of function (Vinner, 1983, 1992; Vinner & Dreyfus, 1989), and that the concept of function is thus represented by a less developed knowledge structure with few relations to other concepts (Hiebert & Carpenter, 1992).

The second map is, to a great extent, constructed hierarchically and contains more inclusive concepts placed above specific concepts, with no descriptions of certain links. The structure of the map is completely different from that of the previous map, where even the contents have changed (see the above table). Ted uses three terms when the concepts are to be arranged in hierarchical order: “straight line”, “function” and “mathematical expression”. Links to these concepts are drawn from the statement. The three expressions are the most representative of the concepts in relation to the statement. The placement of the first two concepts close to the statement seems to indicate that they are more prominent. “Straight line” is the most developed of the three concepts, since it has the most links and thus gives rise to the majority of the detailed concepts and statements on the map. No links exist between the three concepts or their underlying sections on the map, indicating a lower degree of progressive differentiation that is particularly apparent for the concepts “function” and
“mathematical expression”. Also, there is a potential need of integrative reconciliation (Ausubel, 1968, 2000; Ausubel et al., 1978), e.g. in the absence of links between “y depends on x” and the region related to the function concept in the map. A less developed view of the concept “mathematical expression” is confirmed in this case due to the concept not having been further developed on the map and because y=x+5 does not represent a mathematical expression (according to Swedish terminology, since it contains an equality sign). “Coordinate system” is the term with the most links, indicating it to be one of the more conceptually developed notions and (in Ted’s view) one of the most meaningful concepts in the map (Hiebert & Carpenter, 1992).

If the function concept is studied in detail as with the previous map, the link labeled “definition” from “function” to “an x value returns one y value only” then indicates an understanding of function as a process (Eisenberg, 1991; Sfard, 1991, 1992). The link labeled “definition” indicate that the node is to be viewed as a definition of the function concept. Ted notices the uniqueness criteria “an x value returns one y value only” of function, but other components of the function concept like domain and codomain are not raised. Although the function concept is (via a link from y=x+5) considered to represent or “describe” the statement and a more prominent concept (located on the row closest to the statement) on the map, it does not result in a more developed sub-structure on the map. It is not linked to a representational structure such as a graph, table of values or any characteristic of y=x+5 as a function (similar observations have been made by, e.g., Chinnappan & Thomas, 2001; Even, 1998; Thomas, 2003). The evoked concept images that Ted has of a function do not lead him to connect the concept to other nodes on the map. Together, they indicate a less developed knowledge structure of the function concept, particularly for linear functions. A comparison with the previous map reveals more information on Ted’s conceptions of the function concept in his second map than his first. The previous map may give the impression that Ted did not view function as a more representative concept than the other 11 concepts linked to the statement y=x+5. This seems to not be the case for the second map, and for Ted a highly representative concept of y=x+5. Nevertheless, the function concept is less developed in the map and has few properties, since it lacks an underlying section of the map able to accommodate concepts such as “linear”, “increasing”, etc.

Ted has constructed the two maps in completely different ways. As per his comments, he identifies the structure of the maps and thus describes the second map as having “more structure”. Ted states that of the two maps, he would be more likely to use the second in a teaching scenario; he also feels that he has succeeded in describing the straight line well. It can be noted that Ted uses one of the three concepts which are directly linked to the statement y=x+5 to conclude that the lower section of the map includes the concept “straight line”. Moreover, Ted describes the first map as “confusing”, but states that it contains more “objects”. Although Ted here uses the term “object”, this conception is not
reflected in all sections of the maps, including those sections where the definition of “function” suggests a process conception (Eisenberg, 1991; Sfard, 1992; Tall, 1992, 1996). This view is rather related to procedural understanding (Hiebert & Carpenter, 1992; Hiebert & Lefevre, 1986), which is prominent in studies by Williams (1998), and McGowen and Tall (1999). He also raises the term “overall impression” as an advantage of the type of maps the preservice teachers have drawn. After comparing the maps, he states that a “confusing” map, i.e. an unstructured map, is not to his preference.

7.1.2.2 Ann’s maps and comments

Figure 5. The first map that Ann drew.

Figure 6. The second map that Ann drew.
Table 2. Overview of map contents.

<table>
<thead>
<tr>
<th>Found only in the first map</th>
<th>Found in both maps</th>
<th>Found only in the second map</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 y-axis, -5 x-axis, point of intersection, slope, unknown</td>
<td>Coordinate system, x-axis, y-axis, y=lx+m, the equation</td>
<td>variables, slope of a line,</td>
</tr>
<tr>
<td>variable, equation, the slope of the line, line, mathematical</td>
<td>of the line, dependence, function</td>
<td>describe the intersection on</td>
</tr>
<tr>
<td>expression</td>
<td></td>
<td>the y axis, coordinates, a</td>
</tr>
<tr>
<td></td>
<td></td>
<td>curve, straight line, special</td>
</tr>
<tr>
<td></td>
<td></td>
<td>case curves</td>
</tr>
</tbody>
</table>

Ann’s answers:

1) I think that they are confusing and difficult to comprehend. The non-hierarchical map is nevertheless slightly better, since it has more features that are connected in various ways. 2) I think that I have managed to give a relatively good description of the line’s equation. The structure of the map is poor and confusing. 3) I think that they are confusing and difficult to visualize; however, they provide students with a new way of learning by using an image instead of simply text. In other words, a more aesthetic approach to learning.

In the first map drawn by Ann, the statement $y=x+5$ is surrounded by six different nodes representing concepts and a statement to which directed links are drawn from the main statement, i.e. $y=x+5$. The map has a somewhat web-like structure and is more developed around the nodes “y=kx+m”, “coordinate system” and “point of intersection”, where the first node has six connections and the latter two have five connections to other nodes on the map (a two-way link is counted as one connection). The higher number of links indicates that these are more developed and meaningful concepts to Ann.

It might be natural inserting more links between the nodes on the map, e.g. between “line” and “slope” or “line” and “the equation of the line”, which could indicate a limited progressive differentiation (Ausubel, 1968, 2000; Ausubel et al., 1978) of the related concepts. Furthermore, an expression such as “function” may be considered peripheral in relation to the statement, since it is only connected to one other node on the map. Similarly, the term “line” has only one link and its contrast with the statement “y=kx+m”, for example, is remarkable. Moreover, the expression “unknown variable” is found on the map (with a link to “equation”), clearly indicating a less developed variable concept (which is also confirmed by the fact that the concept has just a single link on the map). No clear attempt to structure the contents of the map has been identified.

Ann uses four expressions when drawing a hierarchically structured map: “function”, “the equation of a line”, “coordinates” and “a curve”. They are all connected to the statement “$y=x+5$” located at the same level and can be viewed as the most prominently represented concepts with reference to the statement. As in the maps produced by most students in the group, the number of links has been reduced in the hierarchical map compared to Ann’s previous map, including the number of links from the statement $y=x+5$.

The map’s structure is dominated by the four most prominently represented concepts, each linked to underlying concepts and statements through separate
chains\textsuperscript{17} that are not connected to each other. The task of arranging concepts in a hierarchical structure obviously leads Ann to focus on four separately developed expressions. There are significant contrasts between this and the first map. Many concepts are now connected in a different sequence than that of the previous map. In the second map, Ann has thoroughly linked the nodes in chains and has even gone so far as to construct a chain (regarding the statement “\(y=kx+m\)”) where the links “where \(k\)” and “and \(m\)” follow each other instead of allowing both links to derive from “\(y=kx+m\)”.

A comparison of the two maps drawn by Ann reveals many more links between the nodes on the first map than the second. The second map has mainly developed from concepts that have resulted in separate chains of associations. This indicates that Ann probably has devoted too short a time to reflect upon the included concepts and their relations when drawing the map to use it successfully as a diagnostic tool.

When studying the function concept in detail, it recurs in the hierarchical map as a concept directly linked to \(y=x+5\) and is thus one of the statement’s most prominently represented concepts. Upon comparing this to the first map, Ann changes the order between “function” and “dependence”. The function concept on the hierarchical map is thus viewed as a central concept in relation to the statement \(y=x+5\), in contrast to the freely structured map, but is still a less developed concept.

From Ann’s comments, the maps are “confusing and difficult to interpret”. She states that the first map drawn was the better of the two, since it contained “more information” and “is interconnected in a different way”. “The equation of the line” is implied by Ann as something she has “been relatively successful at describing”. As well, it can be noticed that the first map is more developed in the equation of a straight line environment, and “the equation of the line” constitutes the longest chain on the second map. The comments reveal that Ann perceives the maps to be of minor importance. This may even bring some bearing on the layout of the second map, since it reflects few meaningful relationships and a limited conceptual understanding (Hiebert & Carpenter, 1992). One can observe that Ann also makes connections to learning in her comments when she states that the maps give rise to “a different style of learning” and discusses the term “image” in relation to “text”.

7.2 The second group

After having evaluated the maps drawn by the first group, it was observed that while the freely structured maps were often more comprehensive, they

\textsuperscript{17} The students in the group tend to include separate chains on their hierarchical maps, but to a significantly lower degree than that drawn by Ann. Nonetheless, it is more common to find just a few cross-links on the maps; this trend is also found on the maps drawn by students in studies conducted by Williams (1998) and Doerr and Bowers (1999).
nevertheless had a higher component of free associations (bringing “mind maps” to mind) than the hierarchical maps. The proportion of nodes the statement was perceived to represent was generally lower, and any meaningful relations between them tended to receive less attention than those in the hierarchical maps. The students also perceived the hierarchical maps to be more difficult to draw, resulting in a modified work method in the second group.

The preservice teachers in the second group drew their maps during a longer, continuous period; they were asked to draw and comment on two maps for the statements $y=x+5$ and $y=\pi x^2$. One map was to be of a freely formatted design, whereas the other was to have a hierarchical format. It was suggested that the students begin by drawing the free-format map, followed by the hierarchical map. A consequence was their hierarchical map often being based on their free-format maps. As a result, the hierarchical maps generally became more comprehensive than the freely structured maps. Most students supplemented the second map with additional concepts and statements, since they frequently were drawing from their freely formatted map.

7.2.1 One preservice teacher’s maps

Eva’s (F16) map, below, can illustrate how the maps drawn by the students were constructed, as well as trace how she designed the layout and commented on her maps. Her work method corresponds with the general format adopted by the students in the group.

7.2.1.1 Eva’s maps and comments

![Eva's map](image)

Figure 7. The first map that Eva drew for $y=x+5$. 

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Figure 8. The first map that Eva drew for $y=\pi x^2$.

Eva’s comments on the map for $y=x+5$:
Thoughts on this map: Difficult to get started. A little bit “fixed”. Happiest with the function section. The inverse section is rather poor.

Eva’s comments on the map for $y=\pi x^2$:
Would have been able to write much more on this one. Easy to form associations to it. Personally, I rarely use concept maps, but I think it is an excellent idea to gather and present my thoughts (to myself). I think the “function” and “$\pi$” sections are ok. This is something that needs practice, just like other work methods. Good alternative to supporting terminology and posts, e.g. at a presentation. Maps are useful for presenting an overall view of a given area.

Figure 9. The second map that Eva drew for $y=x+5$.
Eva’s comments on the two hierarchical maps:
Thoughts on these: Clarify by using a hierarchical order. Tried to make it clearer with color\textsuperscript{18}. Difficult to make it look good. Perhaps it should be constructed more like a family tree. Reworking the origin is a positive move. Need to think about it one more time.

Eva clearly made use of information from her first freely formatted maps when she drew the hierarchical maps. If one considers the two maps for $y=x+5$, it then becomes evident that while she has extended the hierarchical map with “y depends on x”, she has also kept the nodes and links from the first map. There are, however, some small deviations in her choice of words, the biggest of which is “function of x”, which was later changed to “function” when she added the node “y depends on x”.

On the hierarchical map, the four concepts “mathematical formula”, “equation”, “inverse” and “function” were directly linked to the statement $y=x+5$ and might thus be seen as strongly represented concepts with respect to the statement. In addition, “function” has the most links, indicating it to be in this context the most meaningful concept of the four and possibly confirmed by Eva who writes that she is “happiest with the function section” in her comments. The four concepts and their underlying structures are not linked on the map, though there is opportunity for several cross-links. This implies a need for progressive differentiation and integrative reconciliation (Novak, 1998; Novak & Gowin, 1984).

By specifically studying the function concept in relation to $y=x+5$, we realize that Eva ties the variable concept and a dependent variable to “function” and

\textsuperscript{18} The student colored the nodes to highlight the level they belonged to in the hierarchy.
applies a geometric interpretation concerning the function concept. However, she does not use specific terminology such as “graph” instead writing “give graphically”. Neither does she describe any properties of function (Slavit, 1997; Williams, 1998) such as being “linear”, “increasing” and “continuous”, which apparently do not exist in her evoked concept image of “function” in the given context (Vinner, 1983, 1992; Vinner & Dreyfus, 1989). She also does not mention domain and codomain. Furthermore, she has directly linked “inverse” to the statement $y=x+5$ without making any connection to the function concept. Her comment “the inverse section is rather poor” indicates that the map invites contemplation of the term “inverse”, which may benefit initiation of integrative reconciliation in her case (Novak, 1998; Novak & Gowin, 1984).

Several differences exist between the nodes and links in the two maps drawn by Eva for the statement $y=\pi x^2$. In the second map, the terms “equation” and “unknown” have disappeared, whereas “inverse” and “$x = \frac{y}{\pi}$” have appeared. Furthermore, “$\pi$” has been expanded to become a “Greek letter”, an “infinite number of decimals” and “3.14”. The link to “min” (previously labeled “min value”) has also been changed. “$\pi$” has the largest number of links on both maps, followed by “function”, and apparently belongs to one of her more progressively differentiated knowledge structures relating to the content on the maps (Ausubel, 1968, 2000; Ausubel et al., 1978; Novak, 1998).

From her first map for $y=\pi x^2$ drawn in freely-chosen format, Eva states that it is “easy to associate (things) to it”, thus indicating her view of the construction of the map as a process with associations to various concepts, expressions and so on. The majority of the students (e.g. M8 and F12) were of the same opinion when constructing their freely formatted maps. The other map was slightly changed; from Eva’s comments, she believes that the hierarchical format makes the map “clearer”, though it is not adapted to a hierarchical structure considering the concepts. The three terms “$\pi$”, “inverse” and “function” are directly linked to the statement and may be perceived as the most strongly associated concepts in relation to $y=\pi x^2$. “$\pi$” and “function” have the highest number of links of the three terms, indicating their representation by more progressively developed knowledge structures. Eva believes these concepts to be “ok”, as she writes in her comments. She also writes in the hierarchical map that she can develop the two further. There is a large contrast between “inverse” and the two other concepts connected to $y=\pi x^2$, since these both lead to more comprehensive underlying structures, and “inverse” is only linked to one underlying node. The number of links connected to “inverse” indicates a low degree of progressive differentiation (Ausbel, 1968, 2000; Ausubel et al., 1978) of the related cognitive structure and what seems to be a low degree of comprehension (Hiebert & Carpenter, 1992). This is verified by the fact that $y=\pi x^2$ has no inverse (assuming that the statement represents a real-valued function of a real variable), without any link between
“inverse” and “function”. Of note concerning the function concept, Eva draws a parallel to a function class “2nd degree function” of polynomial functions – as opposed to y=x+5, where no function class was specified – and highlights characteristics such as “max” and “min”, even though they are tied to “parabola” and there are no links to “2nd degree function”.

Parts of the two hierarchical maps are similar in structure for the two mathematical expressions (primarily the sections surrounding “inverse” and “function”), since the maps were constructed simultaneously and subject to mutual influences during construction. Similar tendencies are found among other students (e.g. F1, F2, M8) in the group.

The written comments indicate that Eva believes the maps provide a good overview or “an overall view of a given area” and appears to view them primarily as a metacognitive tool (Novak, 1985, 1990b), commenting that it is “an excellent way to assemble and present my ideas (to myself)”. She also comments on the work method applied during the construction of her first map, “reworking the original” describing it as “a positive move” and stating that she needed to think one more time. This reflects the general comments given by many students in the group (e.g. F2, F5, F8) on the work method, though some students preferred to draw the hierarchical map after having first written a list of concepts (e.g. M7).

7.2.2 Preservice teachers’ response to the drawing of the maps

In addition to Eva’s comments, there were other comments from the group related to metacognition (e.g. F9, F11, F14) and comments indicating that the student perceived the maps to have a mediating role, e.g. M6 stating, “you see relations that you have not considered”, or F3 writing, “one really has to consider the meaning of what the different things mean and where they lead”, or F8 saying, one “learns from the map, facts are entered once again.” The student’s comments also indicated that the concept maps evoke concept images containing conflicting pieces of information, e.g. M5: “It only causes confusion… does not bring order, to me, it is just confusing.”

The preservice teachers first mentioned the hierarchical maps when commenting on their maps as the hierarchical maps often was an end product of their work method. The students rarely mentioned the concepts and statements on their maps; instead, they tended to describe their experience of drawing the maps, which they often perceived to be a demanding task. F5 puts this view forward: “The used concepts are extremely abstract. It is difficult to see what can be included and what can’t”. M4 writes: “it is difficult to get started”. F6 comments: “Tiresome is my first thought. Unfamiliar is the next thought.” F1 states: “It is hard to know what to include and where to set the limit.” F4 comments: “At first it is difficult, otherwise it is fun.”

Through their comments, most students thought the hierarchical maps gave a clearer overview of the map contents, even if they often mixed general and
specific concepts. This view is held by F2, for example: “It was good to subsequently arrange the concepts hierarchically ... since I am thus able to see everything in an organized manner.” M6 agrees: “... it became clearer when it was redrawn. Additional relations were identified.” F16 writes: “The hierarchical format is clearer.” But there are also comments pointing to the more taxing effort to draw a hierarchical map, as indicated by F17’s comments: “It was difficult to draw hierarchical maps because I did not really know in which direction the arrows should be drawn.” Indicating difficulties for the student to determine the inclusiveness of various concepts and the relations of the different concepts she included in her map.

The students hierarchical maps were often based on the freely formatted map, but the contents were thus further processed, as indicated by F8: “It was good that we were able to redo them, firstly because it is easier (naturally) to understand the new redrawn version, but also because you remember and add several things you did not in the previous one.” M8: “I tried to arrange 3 and 4 [referring to the two hierarchical maps] according to general concepts and then according to sub-concepts in decreasing order. At the same time I corrected (added, subtracted or re-formulated some things).” F5: “When I drew the maps in the hierarchical format I got a clearer image and thus changed some sections.”

As in the previous group, the students clearly find it easier to interpret each other’s hierarchical maps. This is observed in F9’s comments about the maps drawn by F3: “When I look at it [the freely formatted map for y=x+5] I become confused and can no longer follow her train of thought. The same applies to [the freely formatted map for y=πx²]; one is forced to ask her to explain it. It is easier to follow her train of thought in [the hierarchical map for y=x+5] and it appears to be more structured. It is also easier to interpret [the hierarchical map for y=πx²]; one gets the impression that she knows what she is doing.” From the F15’s comments on F2’s maps: “… It is very easy to get a clear perspective of the hierarchical maps. The maps simplify the process of detecting the connections. The first and second maps [freely formatted] may appear to be rather confusing. The structures of maps three and four [hierarchical maps] are better.”

The students in the second group had a longer, continuous time to draw their maps than the students in the first group; they were thus able to draw more detailed maps. Still, there were practical problems when the maps were to be constructed in the hierarchical format. As F11 writes: “it is difficult to organize space and surface. One identifies several things that are related and draws a line over half of the paper”; or as F9 puts it: “New thoughts crop up now that we are to begin from scratch. It means that we make good use of the eraser and the scrap paper and thus quickly may abandon the attempt to draw it again on a new sheet of paper.” This is also illustrated with the map F11 submitted as her hierarchical map:
y = \frac{x^2}{2} is a function second degree consists of among others variables a same square \(x \cdot x\) \(y = \frac{x^2}{2}\) "square of" gives "2 answers" (as described in McGowen & Tall, 1999; Williams, 1998). F11 seems to make free associations without defining a clear structure of meaningful concepts and relations on the map, as with the node "\(x^2\)", which seems to have the highest number of links on the map and is surrounded by many trivial nodes and links, e.g. "\(xx\)".

**7.2.3 Some observations about the preservice teachers’ maps**

The statements, like with the first group, are placed at the center of the freely formatted maps; the map has a web-like structure, but very few cross-links (e.g. those drawn by F3, F5 and F18). The maps are less detailed than the hierarchical maps, which often served as a draft of the information to be later included in the hierarchical maps. The hierarchical maps tend to branch off into substructures that often have few cross-links (as in those drawn by F2, F8 and F16), implying that the concepts perceived to be represented by the statement often develop...
separately from each other. Although the maps have a hierarchical form, the conceptual structure is less hierarchical; thus, more general and overlying concepts tend to be mixed with more specific concepts, though a completely hierarchical conceptual structure is not possible, since the two maps are derived from specific examples. Furthermore, it is clear that when the preservice teachers simultaneously construct maps for \( y = x + 5 \) and \( y = px^2 \), the maps contents and structure were influenced (as indicated by the maps and comments of F1, M8 and F16).

When studying the students’ maps, it becomes evident that they do not always use or master mathematical terminology (this is also described by Grevholm, 2000, 2004). For example, F1 links \( y = px^2 \) to “algebraic expression”\(^{19}\), F2 refers to \( x \) and \( y \) as “two unknown numbers” (in relation to \( y = px^2 \)), F4 links “\( y = x + 5 \)” to “algebraic expression” and F8 describes the exponent in \( px^2 \) as “an elevated number”\(^{20}\). The preservice teachers also raise rather trivial matters, to say the least, when commenting on their maps (as also described by Williams, 1998). Hence, M1 states, \( y = x + 5 \) is “easy to draw”; F2 states, \( x^2 \) is \( x \) “multiplied by itself”; F8 writes the Greek letters “\( \alpha, \beta \)” in relation to \( \pi \); and F18 states, \( \pi \) may denote a plane in space. In addition, there are elements of an “algorithmic nature” (e.g. McGowen & Tall, 1999; Williams, 1998) where F3 states \( y = px^2 \) has two roots \( x = \pm \sqrt{\frac{y}{\pi}} \), F8 establishes a relation to the area of a rectangle \( A = ab \) and a triangle \( a = bh/2 \) (in relation to \( y = px^2 \)) and F5 states “the derivative” of \( y = px^2 \) gives a “minimum point”.

A few students establish links between teaching and learning in their maps (similar to Doerr & Bowers, 1999), e.g. F5 states that “\( y = px^2 \)” is a “difficult” “second degree function” that “primary school students cannot manage”. This is in contrast to “\( y = x + 5 \)”, which F5 labels a “simple function” “used in primary schools.” The preservice teachers also expressed similar opinions on the degree of difficulty without establishing any connection to learning: F3 states, \( y = x + 5 \) is a “simple function” and M6 states, “\( y = x + 5 \)” is “easier” than “\( y = px^2 \)” (on his map for \( y = px^2 \)).

The maps also illustrate the incorrect impressions of the students related to the function concept. M2 links “inverse” to “equation”, F16 states that \( x = \sqrt{\frac{y}{\pi}} \) is the inverse\(^{21}\) of \( y = px^2 \), M17 links “equation” to “has derivative” and F13 links “\( y = px^2 \)” to “exponential function”. The preservice teachers also experience difficulties in distinguishing between the function concept and the equation concept (similar to Leikin et al., 2001; Vinner & Dreyfus, 1989; Williams, 1998).

\(^{19}\) Giving the impression that she understand \( y = px^2 \) to be an algebraic expression.
\(^{20}\) “Ett upphöjt tal” (swe).
\(^{21}\) Without any reference to domain or codomain of the function.
to name a few). This is revealed on the maps drawn by F1 and F14, for example, where they have double links between “equation” and “function”. M8 places “function” as a sub-concept to “equation” in his hierarchical maps (or as F3 notes in her comments on her maps, “I am not sure of the difference between equation and function. They are basically the same thing to me”).

If studying how the function concept is expressed on the maps, it is observed that all of the preservice teachers in the group have included the concept of function in their maps. There are clear indications that the preservice teachers have compared their maps for the two statements y=x+5 and y=πx² before adding more concepts to each map (e.g. F2, M5, F16). Concerning the classes of functions, about half of the preservice teachers did not tie the function concept to any special function class (e.g. F1, M6, F18), while nearly one-third give “second degree function” for y=πx² and “function” for y=x+5 (M5, F14, F16). Other preservice teachers state that y=πx² is a “second degree function” and y=x+5 is a “first degree function” (M4 and M8) or “linear function” (M2, F15, F17). The preservice teachers also associated y=πx² with other function classes, of which F13 names “exponential function”. Otherwise, the preservice teachers do not mention any properties of functions, implying that they view the statements y=x+5 and y=πx² as belonging to different categories of functions. For example, when discussing the derivative (only a few students mention derivatives in their maps, e.g. M1, M4 and F18), it is not viewed as a property (Slavit, 1997; Williams, 1998) leading to a class of functions, but rather a procedural skill (Hiebert & Carpenter, 1992) in the form of a calculation process. For example, the derivative determines the slope (stated by M2), the derivative determines the minimum point (by F5), or derivatives are related to the slope of a curve (by F18). Derivatives are, however, more common in the maps derived from y=πx² (6 maps) than maps derived from y=x+5 (4 maps). Furthermore, the preservice teachers often express the function concept as a “relationship” (e.g. F1, F12, F16) or a “dependence” (e.g. F3, F10, F11) between the variables x and y, reflecting a process conception (Eisenberg, 1991; Sfard, 1991, 1992). No preservice teachers discuss domain or codomain in relation to the function concept.

The node containing the function concept is usually connected to the statement deriving the map and often leads to an underlying structure with very few links to other sections of the map (e.g. F2, F9, F10) – even if there are students who integrate the function concept in their maps to a great extent (e.g. 22 M8 links “y=f(x)” to “function”; the concept map implies that M8 interprets the notation “y=f(x)” as an equation and thus views “function” as a sub-concept to “equation”.

23 When the preservice teachers in the second group had a longer continuous period to work with the concept maps, there may have been some conversations going on between some of the preservice teachers that possibly could have had an influenced on the contents on their maps.

24 “Samband” (swe.)
25 “Beroende” (swe.)
F1, M4, F14). Maps containing fewer cross-links highlight to a lesser degree the manner in which the different concepts – represented by the statement – are related to each other. For example, F2 has the nodes “straight line” and “function” but does not state that the straight line is a graph of the function. Similarly, M7 links “curve” to “minimum value”, but does not link “minimum value” and “function”. Whereas preservice teachers who largely link the function concept to other nodes on the map also link it to concepts that give the impression that they do not fully understand the function concept. F14, for example, has double links between “function” and “equation” with link words “have a” (for function) and “determine” (for equation), while F1 links “function” to “algebraic expression” with the link word “contains”.

8 Evaluation – a subjective activity

The concept maps in the study are intended to be used as an open diagnostic tool in the qualitative study of people’s knowledge and understanding of concepts in relation to a mathematical statement. Studies in mathematics education using different types of concept maps support the idea of a concept maps possessing a wealth of information about an individual’s conceptual understanding (Baralos, 2002; Doerr & Bowers, 1999; Grevholm, 2000, 2004; Haseman & Mansfield, 1995; Heerer & Kommers, 1992; Laturno, 1994; Leikin et al., 2001; McGowen & Tall, 1999; Williams, 1998, 2003). Furthermore, they also highlight how a diverse conceptual interpretation is reflected in the maps (Baralos, 2002; Haseman & Mansfield, 1995; Laturno, 1994; McGowen & Tall, 1999; Williams, 1998, 2003).

Although the maps possess a wealth of information, it may be difficult to interpret them and understand an individual’s thought process during the construction of these maps. In this regard, Leikin et al. (2001) believe it is necessary to supplement the maps with interviews to interpret the information they contain. I believe that it may be desirable to supplement the maps with additional information as to how they should be interpreted. However, the need to supplement the maps with more information is reduced when the maps are constructed more homogenously, i.e. a hierarchical structure with labeled links.

Novak reflects the following considering the evaluation of concept maps:

Although there is some subjectivity in scoring the maps, the great freedom given to individuals to demonstrate their idiosyncratic meanings for the subject matter removes an important source of bias and subjectivity that is present when the test writer chooses the specific content and form in which answers must be selected. (Novak, 1998, p. 194)

A degree of subjectivity is unavoidable when interpreting concept maps. I have mainly noted certain evaluation principles developed by Novak and his colleagues (Novak, 1998; Novak & Gowin, 1984) based on Ausubel’s theory of learning. The validity of the evaluation principles is primarily studied in science
education (Markham et al., 1994; Mintzes et al., 1999; Novak, 1998), but also in mathematics education (Laturno, 1994).

9 Discussion and conclusions

When the preservice teachers in the first group were instructed to draw a freely formatted map starting from the mathematical statement $y = x + 5$, they constructed their maps in many different ways, varying the structure of the maps. Nevertheless, there was a tendency to surround the statement with nodes much like the central hub of a wheel. While sections of the map often had a web-like structure of a varying density, the structure varied from one with nodes lacking many connecting links to one linked with several nodes resembling clusters, such as those described in Doerr and Bowers (1999). There was rarely any hint of a hierarchical structure in the maps and none of the students drew a completely hierarchical map starting with more general and inclusive concepts successively leading to more specific concepts.

A comparison of the freely structured maps and the hierarchical maps revealed that the students in the first group raise different concepts and connect them differently on both of their maps. This is not surprising, since none of the students had drawn their first maps with a hierarchical structure, but were subsequently directed to draw the second maps with a hierarchical format without access to their first maps. Although the number of links and nodes is fewer in the hierarchical maps (in the first group) and in particular the number of links to the statement $y = x + 5$, similar “themes” tend to be found on the two maps, i.e. if a connection to a line, equation or function exists on one map, then it will usually exist on the other map in a somewhat modified form. Furthermore, even if the number of links is reduced in the hierarchical maps – compared to the freely formatted maps – there is no clear trend regarding the number of cross-links on the maps, which increase or decrease from one individual to another. There were generally very few cross-links on the preservice teachers’ maps, in agreement with William (1998) and Doerr and Bowers (1999).

Concerning the hierarchically formulated maps drawn by the students for the mathematical statements $y = x + 5$ and $y = x^2$, the students often mixed general (i.e. more inclusive) and specific concepts or used them interchangeably. Some students even tended to try forming sentences with nodes and links without considering a conceptual hierarchy, possibly due to their inexperience in drawing hierarchical maps. Prior experience in using hierarchically constructed maps in science education shows the maps to be more successfully utilized as the student’s (the subject’s) experience in drawing that type of map grew (Novak, 1998; Novak & Gowin, 1984).

The students found the task of drawing a hierarchical map to be more challenging than that of drawing a map according to a structure of their own choosing. One reason for the students to mix inclusive and specific concepts
could be that the task of drawing a map based on a mathematical statement places greater demands on a conceptual understanding of the various concepts the statement is perceived to represent and their relationship to each other, rather than what a map only derived from a specifically stated concept does. The various concepts thought to represent the statement evoke concept images of different concepts (Tall & Vinner, 1981) that when joined together become more comprehensive and thus more difficult to process than would have been the case if the map had been derived from just one specific concept.

Opinions on the working methods adopted by the students when they drew their maps were more prominent in the written comments submitted by the first group. The students generally preferred the method used during the construction of the freely structured map. Their comments reveal that they often viewed the maps as tools to illustrate ideas in a less structured format (e.g. a sort of mind map, Buzan, 1995) in which the maps were primarily of value to the individual. There were clear indications that the students themselves often preferred their own freely formatted maps, while preferring the hierarchical maps of their colleagues, since they perceived these to be easier to follow and understand. The working method used in the second group received more positive comments from many students. The differences between the working methods applied to the construction of freely structured and hierarchical maps were not raised. Instead, the entire project was viewed as a working process in which the goal was to draw a map with a hierarchical structure, perceived to be clearer by the students. This structure subsequently became more comprehensive than the freely formatted map they had initially drawn. This could imply that the working method promoted progressive differentiation and integrated reconciliation (Ausubel, 1968, 2000; Ausubel et al., 1978). Many of the students' written comments revealed clear perceptions that construction of concept maps promote metacognition and have a mediating role, but there are also signs that the maps evoke concept images with conflicting segments of information.

It can also be mentioned that preparing hierarchical maps by hand is fraught with practical problems. The more time the students were allotted to draw the maps, the more detailed the maps became and the greater the problems faced by the students (who often limited a map to one sheet of paper) when developing the maps with a prescribed hierarchical structure.

The maps drawn by the students reveal that they link the statement to a number of nodes. These nodes seemed to be the most strongly represented concepts in relation to the statement (Goldin, 2002; Goldin & Kaput, 1996; Hiebert & Carpenter, 1992), and thus mainly in the hierarchical maps. When the students drew maps with hierarchical structures compared to the freely formatted, the number of links from the mathematical statement was often largely reduced, whereas the concepts connected to the statement, which the statement was understood to represent, increased and were thus generally allotted more space on the map. Nonetheless, this was more evident in the first group of students who
had not had access to their freely formatted maps. Moreover, one can observe that the nodes on the maps do not always represent the statement the map is derived from, but may possibly represent a concept to which the preservice teachers establish a strong association in relation to the statement (e.g., \(\pi\) in relation to \(y=\pi x^2\)).

Concepts, particularly those representing the statement, were obviously often developed independent of each other, i.e., they had few cross-links, thus indicating a less developed ability of the preservice teachers to relate different concepts to each other in a meaningful manner (similar tendencies were identified in Doerr & Bowers, 1999; McGowen & Tall, 1999; Williams, 1998). This may prevent the preservice teachers from building rich conceptual structures that form a basis for meaningful learning (Ausubel, 1968, 2000; Ausubel et al., 1978) and learning with understanding (Hiebert & Carpenter, 1992).

The maps further indicate that many students did not use any mathematical terminology or used them incorrectly (Grevholm, 2000, 2004), indicating a less developed language amongst the preservice teachers that can constitute an obstacle to meaningful learning (Ausubel, 2000). Elements that express procedural knowledge and skills of an “algorithmic nature” occur frequently, in agreement with results presented by Williams (1998) and McGowen and Tall (1999). The maps may also contain completely trivial elements at the expense of important concepts and relationships between them, which indicates rote learning with a lower degree of conceptual understanding (Ausubel, 1968, 2000; Ausubel et al., 1978; Hiebert & Carpenter, 1992).

The preservice teachers rarely relate to teaching and learning in their maps. This might be surprising, since their mathematics courses all contain parts related to mathematics education. In particular because the characteristics of the two statements, \(y=x+5\) and \(y=\pi x^2\), make them suitable to connect with different teaching scenarios the preservice teachers will face as inservice teachers.

Barely half of the preservice teachers in the first group included the function concept in their maps, where the function concept is not usually a well-integrated concept. Instead, it often takes the shape of a node with a few links that are not connected to other sections of the map. Different properties of functions, where the preservice teachers mention, for example, whether it is differentiable or has an inverse, are only affected in exceptional cases and even then, mainly in the form of procedural skills. No student states that \(y=x+5\) is a linear function, while connections to straight lines were rather common.

All preservice teachers in the second group included the function concept in both maps, where some categorical groupings of function were also present in the maps as opposed to the first group. The maps drawn by the preservice teachers in the second group for the expressions \(y=x+5\) and \(y=\pi x^2\) clearly affected the content and structure of their maps when constructed at the same occasion. Concepts raised in the map for one statement occur quite frequently on the map drawn for the other statement. The result suggests that the statement \(y=\pi x^2\) leads
to mental images containing the function concept to a greater degree than \( y = x + 5 \). This view is supported by Tall and Bakar (1992), who show that first-year mathematics students largely perceive \( y = x^2 \) to represent a function. Several misconceptions can be identified when the function concept is found on all maps, e.g. inverse and derivative of a function are linked to “equation” and the preservice teachers are uncertain as to the relations between “function” and “equation”. The function concept is also not properly integrated with other concepts on the maps drawn by the preservice teachers, often due to them not observing the functions’ different properties and their relations to other concepts. When the students give a concept, or some characteristic of a concept, they do not usually observe the relationships to the function concept. Furthermore, the preservice teachers often express the function concept as a dependency between the variables \( x \) and \( y \). In some cases, they give somewhat more elaborate explanations and state that an \( x \) gives one \( y \), though none of the students discuss domain or codomain in association with the function concept.

The perception of the function concept illustrated in the concept maps drawn by the preservice teachers contrasts highly with the idea of functions playing a central and unifying role in mathematics (Carlson, 1998; Cooney & Wilson, 1993; Selden & Selden, 1992). The concept maps seem to reveal a need for the students to reflect upon relations between the concept of function and other concepts in mathematics, in an attempt to facilitate more integrated knowledge structures and promote the preservice teachers’ understanding of the concept of function.

References


concept of function: Aspects of epistemology and pedagogy (pp. 59-84). Washington, DC: Mathematical Association of America.


Paper III
PRESERVICE TEACHERS’ VIEWS OF Y=X+5 AND Y=πX² AS EXPRESSED THROUGH THE UTILIZATION OF CONCEPT MAPS: A STUDY OF THE CONCEPT OF FUNCTION

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This paper considers the construction of concept maps derived from y=x+5 and y=πx² of a group of preservice teachers, with an emphasis on their conceptual understanding of function. The two statements are thought to represent a number of different concepts with indications of compartmentalized knowledge structures that might prevent the preservice teachers from building rich conceptual structures. Their view of the concept of function contrasts with the function concept viewed as a unifying concept in mathematics with a large network of relations to other concepts. Different properties and categorizations of functions are less frequently recognized. The preservice teachers’ responses of drawing concept maps indicate signs of metacognitive activity.

INTRODUCTION

A limited number of studies have been conducted using concept maps in mathematics education, with only a few accessing the conceptual understanding of function of students (e.g. Doerr & Bowers, 1999; Grevholm, 2000a, 2000b; Hansson, 2004; Hansson & Grevholm, 2003; McGowen & Tall, 1999; Williams, 1998). Williams (1998) concludes that concept maps assess the conceptual knowledge of studying maps drawn by calculus students and professors with PhDs in mathematics. McGowen and Tall (1999) give further support to this conclusion in studying concept maps drawn by students at different levels of achievement in their studies. Other studies in mathematics education (e.g. Laturno, 1994) give further credibility to concept maps as an assessment technique.

Functions are present within all areas of mathematics, and for preservice teachers to become successful dealing with the concept of function and its framework of reasoning it is important to have a well-developed conceptual understanding of function, including the concept’s relations to other concepts and its significance in mathematics (Cooney & Wilson, 1993; Eisenberg, 1992; Even & Tirosh, 2002; Vollrath, 1994). Mathematical statements like y=x+5 and y=πx² can be perceived to represent a number of concepts on a variety of levels related to the future teaching of preservice teachers as well as more advanced mathematical concepts. It is possible to let the statements represent concepts such as a straight line, parabola, equation, formula, proportionality and function (if we assume that domain and codomain are also considered). The two statements, y=x+5 and y=πx², can in particular be recognized as real functions of a real variable by using a representation not uncommon for the concept of function (Eisenberg, 1991, 1992; Tall, 1996). The
function concept can in this context be related to different properties and classes of functions, and recognized as a concept with a large network of relations to other concepts (Cooney & Wilson, 1993; Eisenberg, 1991, 1992).

The purpose of the current study is to examine preservice teachers’ conceptual understanding of function in relation to $y=x+5$ and $y=\pi x^2$ through the utilization of concept maps.

**THEORETICAL FRAMEWORK**

In the chosen theoretical framework, knowledge is considered as a gradually built individual construction. Knowledge is represented internally and described in terms of the way an individual’s mental representation is structured (Goldin, 2002; Hiebert & Carpenter, 1992). Internal representations can be metaphorically linked, forming dynamic networks of knowledge with different structures, and especially webs and vertical hierarchies. Understanding grows as an individual’s knowledge structures become larger and more organized, where existing knowledge influences constructed relationships. Understanding can also be rather limited if only some of the mental representations of potentially related ideas are connected or if the connections are weak. Hiebert and Carpenter (1992) describe the construction of larger and more organized networks of knowledge as learning with understanding.

Ausubel (2000) describes a similar notion in terms of meaningful learning, as opposed to rote learning, “only in rote learning does a simple arbitrary and nonsubstantive linkage occur with preexisting cognitive structure.” (Ausubel, 2000, p. 3).

A central part of Ausubel’s assimilation theory of meaningful learning is the idea that new meanings are acquired through the interaction of new, potentially meaningful ideas with what is previously learned. This interactional process results in a modification of both the potential meaning of new information and the meaning of the knowledge structure to which it becomes anchored. The process of assimilation results in progressive differentiation in the consequent refinement of meanings, and in an enhanced potential to provide anchorage for further learning.

Tall and Vinner (1981) view the total cognitive structure associated with a concept in the mind of an individual as a concept image, including “all the mental pictures and associated properties and processes” (p. 152). When an individual encounters an old concept in a new context, it is the evoked concept image at that time – a portion of the concept image – with all the implicit assumptions abstracted from earlier contexts that respond to the task.

In an individual’s conceptual development, Sfard (1991, 1992) suggests a process-object model. The formation of an operational conception, i.e. a process conception, precedes a more mature phase in the formation of a structural conception with a focus on objects.
METHOD AND PROCEDURE

The current study is part of an ongoing study on the view of preservice teachers regarding the concept of function (Hansson, 2004). Preservice teachers participating in the study are in their third year of a four and half-year teacher preparation program and specialize in mathematics and science, grades 4 to 9. During the sixth term they are enrolled in the final mathematics courses of the program, and the study is conducted after a course in calculus where the concept of function is a central concept.

During a lecture, a group of 25 preservice teachers was presented examples of different types of concept maps, such as non-hierarchical web-based maps and hierarchical maps, where the nodes represented concepts and the links were labeled in each case. The hierarchical maps were constructed according to Novak and Gowin (1984) and Novak (1998). The displayed maps were often related to science education. Concept maps derived from mathematical concepts were largely avoided during the presentation to not influence the contents of the maps that the students were to draw in the subsequent assignment.

After the introduction, the preservice students were each directed to draw concept maps based on the statements $y=x+5$ and $y=\pi x^2$. In the process of drawing maps with a hierarchical structure, they began with a freely formatted structure resulting in the construction of two maps for each mathematical statement – one freely formatted map and one hierarchical map. The preservice students were also asked to comment on their maps and their experiences of drawing the maps.

Drawing concept maps based on $y=x+5$ and $y=\pi x^2$ differs from the use of concept maps derived from “function” (e.g. Doerr & Bowers, 1999; Grevholm, 2000b; McGowen & Tall, 1999; Williams, 1998), since a concept is not explicitly stated. The statements can thus be perceived to represent a number of different concepts. The derived concept maps permit the subject to illustrate what concepts the statements are perceived to represent, and their different properties and relations.

All of the concept maps were analyzed, each as an integrated unity in which its contents and structure were noted. Furthermore, how the different sections of the map were related to each other was also studied. In particular, how the function concept was expressed on the maps was noted, and its relationships to other concepts and properties that were assigned to functions were considered. The contents of the maps and the comments of the preservice students about the maps were also compiled in tables.

The presentation took about 30 minutes and the group received 60 to 80 minutes, depending on when the students decided they had completed the assigned task. From the group, 24 students submitted all of their maps.
RESULTS

The statement \( y = x + 5 \) or \( y = \pi x^2 \) is usually placed at the center on freely formatted maps. These maps have a web-like structure, but very few cross-links, i.e. links connecting different parts of a map (e.g. those drawn by F3, F5 and F18). The maps are less detailed than hierarchical maps, and often served as a draft of the information to be later included in the hierarchical maps. The hierarchical maps derived from \( y = x + 5 \) or \( y = \pi x^2 \) tend to branch off into substructures that often have few cross-links (as in those drawn by F2, F8 and F16), implying that the concepts perceived to be represented by the statement often develop separately from each other. Although the maps have a hierarchical form, their conceptual structure is less hierarchical. Therefore, more general and inclusive concepts tend to be mixed with more specific concepts (a completely hierarchical conceptual structure is, however, not possible, since the maps are derived from specific examples). Furthermore, it is clear that when the preservice teachers simultaneously construct maps for \( y = x + 5 \) and \( y = \pi x^2 \), the maps' contents and structure are influenced (as indicated by the maps and comments of, e.g. F1, M8, F16).

All of the preservice teachers in the group included the function concept in their maps, though none discuss domain or codomain in relation to the function concept. They often express the function concept as a “relationship” (e.g. F1, F12, F16) or a “dependence” (e.g. F3, F10, F11) between the variables \( x \) and \( y \), reflecting a process conception (Eisenberg, 1991; Sfard, 1992). The node containing the function concept is usually connected to the statement from which the map is derived, and often leads to an underlying structure with very few links to other sections of the map (e.g. F2, F9, F10) – even if there are students who greatly integrate the function concept in their maps (e.g. F1, M4, F14).

Maps containing fewer cross-links highlight to a lesser degree how the different concepts – represented by the statement – are related to each other. For example, F2 has the nodes “straight line” and “function”, but does not state that the straight line is a graph of the function. Similarly, M7 links “curve” to “minimum value”, but does not link “minimum value” and “function”. However, preservice teachers who mainly link the function concept to other nodes on the map, also link it to concepts that give the impression that they do not fully understand the function concept. The preservice teachers are observed to experience difficulties in distinguishing relations between the function concept and the equation concept (similar to e.g. Grevholm, 2000b; Leikin, Chazan & Yerushalmy, 2001; Williams, 1998), where, for example, M8 places “function” as a sub-concept\(^1\) to “equation” in his hierarchical maps, M2 links “inverse” to “equation” and M17 links “equation” to “has derivative”.

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\(^1\) M8 links “\( y = f(x) \)” to “function” and links “function” to “equation”. The concept map implies that M8 interprets the notation “\( y = f(x) \)” as an equation and thus views “function” as a sub-concept to “equation”.
Concerning classes of functions, about half of the preservice teachers did not link the function concept to any special function class (e.g. F1, M6, F18). Nearly one-third gave “second degree function” as a polynomial function for \( y=\pi x^2 \) and “function” for \( y=x+5 \), and thus link \( y=\pi x^2 \) but not \( y=x+5 \) to a function class (e.g. M5, F14, F16). Other preservice teachers state that \( y=\pi x^2 \) is a “second degree function”, and that \( y=x+5 \) is a “first degree function” (M4 and M8) or a “linear function” (M2, F15, F17). Preservice teachers who incorrectly associated \( y=\pi x^2 \) with other function classes were also noted, e.g. F13 who link it to “exponential function”. Otherwise, the preservice teachers do not mention any properties of functions, implying they view \( y=x+5 \) and \( y=\pi x^2 \) to belong to categories of functions that they met during their calculus course, such as even, continuous, increasing, differentiable, etc. In those cases where, for example, the derivative is discussed (only a few of the students mention derivatives in their maps, e.g. M1, M4, F18), it is not viewed as a property that gives rise to a class of functions, but rather as a procedural skill in the form of a calculation process, e.g. just as M2 states that the derivative determines the slope or F5 states that the derivative determines the minimum point. Derivatives are, however, more common in the maps derived from \( y=\pi x^2 \) (6 maps) than maps derived from \( y=x+5 \) (4 maps).

Only a few preservice teachers establish links between teaching and learning in their maps (similar to Doerr & Bowers, 1999). F5, for example, states in her map that \( y=\pi x^2 \) is a “difficult” “second degree function” and that “primary school students cannot handle it”. This is in contrast to \( y=x+5 \), which she labels a “simple function” “used in primary schools”. The preservice teachers also expressed similar opinions on the degree of difficulty without establishing any connection to learning: F3 states that \( y=x+5 \) is a “simple function” and M6 states that \( y=x+5 \) is “easier” than \( y=\pi x^2 \) (on his map for \( y=\pi x^2 \)).

There are elements of an “algorithmic nature” in the maps (in common with McGowen & Tall, 1999; Williams, 1998), where, for example, F3 states that \( y=\pi x^2 \) has two roots \( x=\pm\sqrt{\frac{y}{\pi}} \), F8 establishes a relationship to the area of a rectangle “A=ab” and a triangle “a=bh/2” (in relation to \( y=\pi x^2 \)), and F5 states that “the derivative” of \( y=\pi x^2 \) gives a “minimum point”. Moreover, in their maps, the preservice teachers also raise certain trivial matters, e.g. M1 states that \( y=x+5 \) is “easy to draw”, F2 states that \( x^2 \) is \( x \) “multiplied by itself” or F8 who writes the Greek letters “\( \epsilon, \alpha, \beta \)” in relation to \( \pi \).

The preservice students’ written comments revealed clear perceptions that the maps promote metacognition and have a mediating role (Novak, 1998), e.g. M6 who states that “you see relations that you have not considered”, F3 writes that “one really has to consider the meaning of what the different things mean and where they lead”, or F8 who says that one “learns from the map, facts are entered once again”. But there are also signs that the maps evoke concept images with conflicting pieces of information, e.g. M5 with the opinion that “It only causes confusion...”.

5
One preservice teacher’s maps

The two concept maps below are examples of how the preservice teachers might draw their maps and illustrate a tendency for the maps in the study to have few cross-links – or in this case, a lack of cross-links – where different parts of a map are developed separately. Moreover, the two maps illustrate a common feature of the preservice teachers’ maps, namely graphical interpretations – though this feature is not always in relation to the concept of function in contrast to the maps below. They also show how preservice teachers less frequently recognize relationships to different properties of functions. For instance, “k=1” (in Figure 1) might be related to an increasing function. Furthermore, “min” and “min max” (in Figure 2) are related to the concept of “parabola” rather than “function” (for “min”) or “2nd degree function” (for “min max”).

![Figure 1. The hierarchical map of y=x+5 drawn by F16.]

![Figure 2. The hierarchical map of y=πx² drawn by F16.]

Both maps also show individual characteristics, e.g. in the case of “inverse” where the function concept seems to be less meaningful. This may possibly indicate rote learning (Ausubel, 2000), since y=πx² in particular does not have an inverse (when considered as a real function of a real variable of maximal domain).
DISCUSSION AND CONCLUSIONS

There are clear indications that the function concept is often developed independently with few relations to other parts of the maps. This may be an expression of compartmentalized knowledge structures that prevent the preservice teachers from building rich conceptual structures that form a basis for meaningful learning (Ausubel, 2000) and learning with understanding (Hiebert & Carpenter, 1992), with consequences for the preservice teachers’ future teaching (Even & Tirosh, 2002; Vollrath, 1994). Moreover, the preservice teachers rarely relate to teaching and learning in their maps. This might be surprising, since their mathematics courses all contain parts related to mathematics education. Particularly as the characteristics of the two statements $y=x+5$ and $y=\pi x^2$ make them suitable for connection to different teaching scenarios which the preservice teachers will face as inservice teachers.

The preservice teachers often express the function concept as a dependency between the variables $x$ and $y$ – describing a process conception (Eisenberg, 1991; Sfard, 1991, 1992). In some cases, they give somewhat more elaborate explanations and state that an $x$ gives one $y$. However, none of the students discuss domain or codomain in association with the function concept, and a more developed conceptual understanding in the form of an object with a set of properties is less frequent in the maps.

Elements that express procedural knowledge and skills of an algorithmic nature occur frequently (in agreement with results presented by, e.g. Grevholm, 2000b; McGowen & Tall, 1999; Williams, 1998). The maps may also contain completely trivial elements at the expense of important concepts and relationships between them, indicating rote learning (Ausubel, 2000). Moreover, several misconceptions related to the concept of function can be identified on the maps.

It is found that the function concept’s large network of relations to other concepts is frequently not part of the concept maps. This is usually a consequence of the preservice teachers not observing and relating to different properties of $y=x+5$ and $y=\pi x^2$ when regarded as functions. In those cases, the preservice teachers give a concept, or some characteristic of a concept, they do not usually observe the relations to the concept of function. The perception of the function concept illustrated in the concept maps contrasts highly with the idea of functions playing a central and unifying role in mathematics (Cooney & Wilson, 1993; Eisenberg, 1991, 1992).

The concept maps seem to reveal a need for the preservice students to reflect upon the function concept’s relevance in mathematics, its different properties and its network of relations to other concepts. The preservice teachers’ comments on drawing the maps indicate that the process of drawing concept maps supports metacognitive activities (Novak, 1998). Such activities might promote the preservice
teachers’ conceptual understanding of the function concept and its significance in mathematics and thus require further research.

References


The views of preservice teachers on three mathematical statements: A case study regarding the concept of function

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Three preservice teachers participate in a case study during the final mathematics courses of a teaching program. The purpose of the study is to investigate how preservice teachers at various stages of competence in mathematics view the statements $y=x+5$, $y=\frac{x}{2}$ and $xy=2$, with particular emphasis on how the function concept is expressed. None of the preservice teachers in the case study describe the concept of function as consistent with the definition of function. As the performance of the preservice teachers in their studies of mathematics decreases, a process-based conception of the function concept becomes more prominent. The preservice teachers’ knowledge structure of different concepts, which the statements are perceived to represent, also seems to become more compartmentalized. The number of meaningful relations to other mathematical concepts becomes less, as their level of performance in their studies of mathematics decreases. Preservice teachers often use geometrical interpretations when describing various properties of a function in relation to the statements. The form of the statements influences the concepts that they are deemed to represent. Cognitive obstacles and prototypes seem to exist in relation to the function concept. Preservice teachers often seem to lack a mathematical language to describe the properties of functions, and use function classes and categorizations to a low degree when discussing the concept of function.

1 Background

Functions are applied in most areas of mathematics and the concept of function has a comprehensive network of relations to other concepts (Blomhøj, 1997; Eisenberg, 1991; Selden & Selden, 1992). It is imperative that preservice teachers have a well developed understanding of the function concept, partly to become successful in their mathematics studies (Carlson, 1998; Eisenberg, 1991, 1992; Tall, 1992, 1996; Thompson, 1994), partly to be better equipped to manage the concept as a teacher (Chinnappan & Thomas, 2001; Cooney & Wilson, 1993; Even, 1990, 1993; Even & Tirosh, 1995, 2002; Fennema & Loe, 1992; Thomas, 2003; Vollrath, 1994). In this regard, it is relevant to study the views of preservice teachers on mathematical statements that can be connected to different topics and levels of mathematics, at the tertiary level as well as to their future teaching, to examine how the function concept is expressed.
The results from previous studies indicate the views of preservice teachers to be less developed on the concept of function than was desirable in a teacher (Hansson, 2004, 2005; Hansson & Grevholm, 2003), and motivate further studies of preservice teachers’ conceptions of function.

2 Aims of the study

The mathematical statements \( y = x + 5 \), \( y = \pi x^2 \) and \( xy = 2 \) may be perceived to represent the numerous concepts related to different areas of mathematics the preservice teachers have come in contact with before and during the teaching program. In particular, the three statements may be thought to represent real valued functions of a real variable. In this regard, the preservice teachers could get the opportunity to link them to different classes of functions and identify their various properties and relations to other concepts.

The purpose of this study is to investigate, by means of a case study, how the function concept is expressed in relation to \( y = x + 5 \), \( y = \pi x^2 \) and \( xy = 2 \) for preservice teachers at various stages of performance in their studies of mathematics. What properties the preservice teachers identify and how relations between the function concept and other concepts are presented will in particular be examined.

3 Theoretical framework

In the chosen theoretical framework, knowledge is represented internally and described in terms of how an individual’s mental representation is structured. Internal representations can be metaphorically linked, forming dynamic networks of knowledge with different structures, especially in forms of webs and vertical hierarchies (Hiebert & Carpenter, 1992). Understanding grows as the networks, i.e. the cognitive structure, become larger and more organized where existing networks influence the constructed relationships, thereby helping to shape the new networks that are formed. Hiebert and Carpenter (1992) describe the construction of larger and more organized networks of knowledge as learning with understanding. Understanding can be somewhat limited if only some of mental representations of the potentially related ideas are connected or if the connections are weak. Ausubel (2000) describes a similar notion in the terms of meaningful learning, as opposed to rote learning, “only in rote learning does a simple arbitrary and nonsubstantive linkage occur with pre-existing cognitive structure.” (p. 3).

A central part of Ausubel’s theory of meaningful learning (Ausubel, 2000; Ausubel et al., 1978) is the idea that new meanings are acquired through the interaction of new, potentially meaningful ideas (knowledge) with what is previously learned. This interactional process results in a modification of the potential meaning of new information and the meaning of the knowledge.
structure to which it becomes anchored. The process results in a progressive
differentiation in the consequent refinement of meanings, and an enhanced
potential for providing anchorage for further knowledge. When new knowledge
is learned the already existing knowledge structures can recombine themselves,
new and different meanings may develop, and conflicting meanings may
possibly be resolved through a process of integrative reconciliation.

The total cognitive structure associated with a concept in the mind of an
individual is called a concept image (Tall & Vinner, 1981), including “all the
mental pictures and associated properties and processes” (p. 152). A concept
image is built up through an individual’s various experiences of the concept.
When an individual encounters an old concept in a new context, as in the case of
function (Vinner, 1983, 1992), it is the concept image that responds to the task
with all the implicit assumptions abstracted from earlier contexts. The portion of
a concept image activated at a particular time is called the evoked concept
image.

In an individual’s conceptual development, Sfard (1991) suggests a
process-object model that is especially applicable to the concept of function
(Sfard, 1989, 1992). The formation of an operational conception, as a process,
precedes a later more mature phase in the formation of a “structural” conception
regarding functions as objects. According to Sfard, both conceptions are
essential and should coexist in a dual view of the function concept. In the
transition from operational to structural conception, Slavit (1997) suggests
emphasising the functions’ properties to enhance the development of a structural
conception.

4 Method and procedure

This study is part of a larger study on preservice teachers’ understanding of the
function concept in the sixth term of a teaching program, and specializing in
mathematics and science, grades 4 to 9. They are enrolled in the final
mathematics courses of the program during the term. Data collection occurs
primarily at the end of the term, after a calculus course when the function
concept has been a central concept.

The choice of the statements \( y=x+5, \ y=\pi x^2 \) and \( xy=2 \) enables the preservice
teachers to relate to previously encountered concepts and areas in mathematics,
as well as others encountered during the teaching program. The statements can
possibly represent concepts such as a straight line, parabola, hyperbola,
equation, formula, proportionality, function\(^1\) and others. In connection to the
function concept, the preservice teachers can identify various classes of
functions related to \( y=x+5, \ y=\pi x^2 \) and \( xy=2 \), e.g. linear, polynomial or rational,
along with continuous, differentiable, even or odd. The preservice teachers may

\(^1\) If we assume that a domain and a codomain are given.
also indicate if the functions have inverse, asymptote, extremes or other properties dealt with in a calculus course. The statements also give the preservice teachers an opportunity to relate to future teaching situations on the function concept.

4.1 Method

To investigate the preservice teachers’ views on the statements, they were given a questionnaire before and after the calculus course. The questions were open and read: “We write y=x+5. What does that mean?”, “What can you say about y=\pi x^2? Please give as detailed an answer as possible” and “What can you say about xy=2? Please give as detailed an answer as possible.” The questionnaire also contained a question asking the preservice teachers to describe a function: “Describe, in your own words, your interpretation of the concept of function.” They further drew concept maps for the statements y=x+5 and y=\pi x^2 to describe their knowledge and understanding of the statements, the concepts they are considered to represent, the conceptions about these concepts and their relations. Finally, the preservice teachers were interviewed to allow them to expand their answers.

The preservice teachers answered the questionnaire and drew the concept maps individually to facilitate a comparison of their differences in perception concerning the function concept and the mathematical statements. This was done in conjunction with their mathematics class, enabling them to answer the questionnaires and draw the concept maps under similar conditions (meaning that the students did not get the opportunity to also draw concept maps for xy=2, since it is a time consuming activity).

The interview was formulated so that the preservice teachers could comment on their answers from the two questionnaires. They were then allowed to supplement their answers and clarify their views on the function concept and how it is expressed in relation to the three statements. To gain a better understanding of their thought processes, they were also frequently asked follow-up questions based on their previous answers (Kvale, 1996). For practical reasons, the interviews were conducted during predetermined time intervals. This meant that it was sometimes necessary to limit the time somewhat for a topic and move onto the next question. The preservice teachers were previously acquainted with me, as I was one of their teachers. This contributed to a natural interaction between interviewer and interviewee.

Three preservice teachers were selected to participate in a case study. They were drawn from a large batch of data that was compiled on all preservice teachers in the sixth term of the mathematics and science program, and based on their performance in the calculus course using the three-grade scale: “high pass”, “pass” and “fail”. Consideration was also given to the performance of the three students in other mathematics courses during the term, which clearly
showed that the preservice teachers had distinctly different levels of competency in mathematics (note that preservice teachers with consistently high grades in mathematics constitute the smallest of the three categories). Consideration was also given to the drawn concept maps, which were among the more extensive maps regarding the number of nodes.

The study was limited to the investigation of how preservice teachers view the function concept concerning the three statements when enrolled in the last mathematics courses of the program. The nature of the statements is such that they may also be applied to different teaching situations the preservice teachers may be exposed to as inservice teachers. This study does not treat how the preservice teachers handle the function concept in this case.

4.2 Procedure of the study

A questionnaire consisting of eight questions was distributed before and after the calculus course. The questionnaire is appended to this paper. Question 3 concerned the statement \( y = x + 5 \), question 4: \( y = \pi x^2 \), question 5: \( xy = 2 \) and question 2c a description of the function concept. Moreover, the first two questions on the survey\(^2\) concerned variables, equations, function and algebraic expressions, and the last three questions addressed the function concept. The preservice teachers were able to answer the questionnaires during the lecture and spent 40 minutes on the task, with somewhat less time on the first questionnaire. A few days after the students had answered the second questionnaire, they drew concept maps for the statements \( y = x + 5 \) and \( y = \pi x^2 \) during a 90-minute lecture, which began with an introduction to concept maps (a more detailed explanation of the process can be found in Hansson, 2004, 2005). Students who volunteered to be interviewed signed up for participation by writing their names on a timetable. Twenty students from the group were interviewed during a three-week period at the end of the term. The length of each interview varied, but often lasted one hour. The interviews were conducted in a preparation room for mathematics and recorded on tape. An interview included all questions in the survey. The recordings were usually supplemented with handwritten observations. In total, 24 preservice teachers drew concept maps, with 22 answering the questionnaire before taking the calculus course and 24 afterwards. The answers and the contents of the maps were compiled. All interviews were transcribed in their entirety. Three preservice teachers were subsequently selected to participate in this study as outlined above.

\(^2\) The three introductory questions on the questionnaire are similar to the two introductory questions on the questionnaire used by Hansson and Grevholm (2003). Otherwise, the contents of the two surveys are different. This was done to create similar conditions for the groups of students that attended calculus courses when answering the question on \( y = x + 5 \), in case a comparison should become necessary. The survey used in Hansson and Grevholm (2003) was originally formulated by Grevholm in conjunction with an algebra course for preservice students.
During the transcription process, sounds such as “hmm”, “ee”, etc., were removed according to recommendations from Kvale (1996, 1997). Furthermore, periods, commas, question marks, etc., were inserted to make the transcription easier to understand. The chosen sections raised in the interview took place after repeatedly reading the transcription of the interview and listening to sections of the tape. Recollections from the interviews and notes from the interview were also used. Sentence interpretation and focusing took place during the interview analysis according to the method prescribed by Kvale (1996, 1997).

4.3 The group of preservice teachers
The preservice teachers who participated in the study were enrolled in a four and a half-year teacher-training program in mathematics and science, grades 4 to 9. The group consisted of 25 students, the majority being women, and included all 3 preservice teachers who specialized in these subjects during the sixth term of the program. At the end of the term, they would have completed the required 30 weeks of full-time studies in mathematics, of which one-third would have been dedicated to mathematics education.

Before reaching the sixth term, the preservice teachers had studied mathematics during the first and third terms of the program. In the first term, they enrolled in the introductory course in mathematics consisting of five weeks of full-time study. In the third term, the preservice teachers took a course in algebra and number theory, together comprising 10 weeks of full-time studies. They then took the final mathematics courses in the sixth term of the program, which included statistics (3 weeks of full-time work), calculus (5 weeks full-time) and geometry (7 weeks full-time). Applicants to the teaching program in mathematics and science, grades 4-9, should have successfully completed a program 4 in mathematics and science at upper secondary school or the equivalent.

5 Results
Three preservice teachers, Emma (F14), Nils (M6) and Vera (F8), participated in the case study. Their level of performance in mathematics during the sixth term can be arranged as follows in descending order: Emma, Nils, Vera. Their answers to the two questionnaires and the corresponding interview sections are compiled and presented below. For each preservice teacher who participated in the case study, there are 2 answers for questions 3 to 5 and an account of the

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3 The teacher preparation program had relatively few students specialized in mathematics and science. The study was conducted at a tertiary institution with 9,000 students, half enrolled full-time.

4 Naturvetarprogrammet.
interview in conjunction with the questions. The preservice teachers’ concept maps\(^5\) are appended to the paper.

### 5.1 A case study with Emma

#### 5.1.1 Answers to the questionnaires

3. We write \(y = x + 5\). What does that mean?

<table>
<thead>
<tr>
<th>First questionnaire</th>
<th>Second questionnaire</th>
</tr>
</thead>
<tbody>
<tr>
<td>Any value can be assigned to (x), and (y) will “automatically” be obtained. (y) is always greater than (x) by 5.</td>
<td>(y) is greater than (x) by 5. Here, one must personally decide (x) in order to know which (y)-value corresponds with this (x)-value.</td>
</tr>
</tbody>
</table>

4. What can you say about \(y = \pi x^2\)? Please give as detailed an answer as possible.

<table>
<thead>
<tr>
<th>First questionnaire</th>
<th>Second questionnaire</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y) has the same value when one chooses a positive or negative (x). It is the absolute value that matters.</td>
<td>(y) is greater than (x), how much greater depends on (x). The greater (x) is, the greater the difference between the two variables. (\pi) is a constant of proportionality. (y) is proportional to the square of (x). If (x) is the radius of a circle then (y = \text{the area of the circle}).</td>
</tr>
</tbody>
</table>

5. What can you say about \(xy = 2\)? Please give as detailed an answer as possible.

<table>
<thead>
<tr>
<th>First questionnaire</th>
<th>Second questionnaire</th>
</tr>
</thead>
<tbody>
<tr>
<td>A person should choose 2 numbers that give a product of 2.</td>
<td>If a person multiplies two independent variables, the product will always then be 2. One must determine the value of one of the variables, e.g. (x). The person then has to solve for (y) to obtain a value for (y). (y = \frac{2}{x}). You get pairs of numbers: an (x)-value corresponds to a (y)-value.</td>
</tr>
</tbody>
</table>

The answers are generally more detailed in the second questionnaire and also differ in character. The answers to question 3 are more or less similar, whereas the answers to question 4 differ and discuss completely different aspects (e.g. absolute value and area calculation). The answer to question 5 is much more detailed in the second questionnaire and includes much of the previous answer.

Emma discusses the variables \(x\) and \(y\) and their mainly numerical relationships in the first questionnaire (with the exception of question 5) and, to a higher extent, in the second. It can be noted that Emma mentions the concept of proportionality in her answer to question 4 in the second questionnaire, but

\(^5\) A more comprehensive analysis of the concept maps will be conducted in a planned further developed version of the.
not in question 5. She also describes the difference between variables x and y in questions 3 and 4, as opposed to question 5. Furthermore, pairs of numbers are mentioned in question 5 but do not lead to a geometric interpretation of the statement (geometric interpretations are generally not mentioned in any of the answers, other than in the form of a circle in question 4).

Question 5 in the second questionnaire begins with a contradiction: the first sentence contradicts the second and the third sentences. Question 4 also begins with an incorrect statement (since y is not greater than x if $0 \leq x \leq 1/\pi$).

5.1.2 The interview coupled to the questionnaires

Emma read her answers to question 3 and realized that they were similar. She thought about whether they should be supplemented with anything “graphical”, before saying that $y = x + 5$ was the equation for a straight line. She also mentioned that “pairs of numbers” arise because “if you choose x then you get y”, but still added that $y = x + 5$ is a function, referring to a previous question on the function concept. Emma did not know more to say, but stated that she had now discussed the issues she had wanted to in connection to question 3.

Emma moved on to question 4 and read out the answers in the questionnaire (without correcting the statement that y is greater than x), adding that $y = x^2$ is “of course also a function. A quadratic function…” At this point, she mentioned a function class in relation to the function concept, contrary to the previous statement. On her own initiative, she subsequently tried to determine the appearance of the graph. Here are some excerpts of the interview:

E: Let me see here, what does $\pi$ do? Yes, what does $\pi$ do? … I just asked a question to which I don’t know the answer. [Laugh]

\[8\]

Emma went back to what was previously discussed about functions in the interview in connection to question 2c, where she should have described a function in her own words. On the first questionnaire, she answered, “An expression with two variables (e.g. x and y), for each value of x, there is just one value of y” and on the second questionnaire, “An expression that contains two different variables. You can draw the graph of the function in a coordinate system that shows the different solution pairs.”

In the interview, Emma said that the condition “for each value of x, there is just one value of y” is “the theoretical stuff that one has learnt, has been drilled into us in school,” which clearly indicates rote learning (Ausubel, 2000; Ausubel et al., 1978). Emma stated that “both [answers] felt good” and that “solution pairs” in the second questionnaire reflected “the same idea” as the answer on the first questionnaire, which according to Emma implied “that there are then numbers, which belong together in pairs”. (Later, in connection to question 3, she said “pairs of numbers” instead of “solution pairs”.) What Emma meant by “expression” in the questionnaire was never clarified, though many preservice teachers in the group, according to Swedish terminology, used the term “expression” incorrectly when they actually meant “statement” in questions 3-5. When Emma subsequently raises the issue of “solution pairs”, it indicates that she (with “expression” signifies an equation with two variables and) apparently makes a connection between equation and function. This is also evident in the concept map that she drew for $y = x + 5$, which had double links between “function” and “equation”, with labels “have a” for function and “determine” for equation. Otherwise, Emma neither mentions domain or codomain in her answers on the questionnaires, nor does she do so in the interview, in conjunction to question 2c (which is similar to the results from Tall & Bakar, 1992; Even, 1993).
I: What happens when you have π there, in front of \(x^2\)?
E: Well, then … in that case one assumes … one cannot begin from … well the parabola then? And … is it that which rises and thus falls along the y-axis?
I: You need to clarify. Rises and falls?

It is apparent that Emma does not realize how the factor π affects the shape of the parabola. In the following extract she begins with a parabola that is defined by \(y = x^2\) to explain how it changes if multiplied by π.

E: So if you have the parabola, i.e. when you begin with the parabola, then it will be at the origin. Then if you put a constant in front of it, it will be at another point on the y-axis.

Emma interpreted π to cause a vertical translation of the parabola. I informed her that this was not the case. She thus tried to explain the effect on the graph of \(y = x^2\) multiplied by π, considering whether the parabola’s “gap” would be changed – referring to the distance between points on the parabola with the same y-value. She rejected her line of reason, stating that the exponent determines “the gap”. Emma used \(x^4\) as an example: “I mean, if you have \(x^4\) instead of \(x^2\), then it would look different. They do not have the same width” (thus making a correct observation regarding the \(x^2\) and \(x^4\) graphs). Emma could not decide how the factor π would affect the graph before she had plotted the coordinates and sketched the graph.

Before Emma left question 4, she commented on her answer in the second questionnaire by saying that her first statement (i.e., “\(y\) is greater than \(x\), how much greater depends on \(x\)) relates to functions and explains, “Assign any value to \(x\) and you get a corresponding value for \(y\)”.” Furthermore, she said that by mentioning proportionality in the second questionnaire, she also relates to functions (a viewpoint shared by Vollrath, 1986, and Leinhardt, Zaslavsky & Stein, 1990, as a preliminary stage of the function concept) and points out that it can be visualized in a system of coordinates.

Emma then continued to read her answers to question 5 (without noticing her contradictions to the answer in the other questionnaire). Concerning the answers in the questionnaire, she spontaneously raised the relationship between function and equation, as follows:

E: (…) You get pairs of numbers: an x-value corresponds to a y-value [completed the last sentence in the answer of the second questionnaire]. So it is also a function. Although it is written in a different form than I am used to. But it is… well … yes, exactly, it is an equation. But one says … yes, what does one say? A function’s … no, an equation of a graph … they… they belong together. Was this

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7 Emma also mentioned “the gap” on her concept map and linked it to π. Nevertheless, she rejects the connection in the interview.
what we had problems with, when we were supposed to draw those thought maps \(^8\) the other day …

I: I see.
E: … to relate it to that business of equation and function … That it is [pause]… If you have an expression, then you can of course call it an equation. But you can also say that it is a function and that it ties the two together. Because we say the equation of the graph, or … people talk about the equation of a straight line and … so I didn’t quite express this on the thought maps. [Laugh]

Emma completed her reply by stating that \(xy=2\) is a function. In this regard, she felt more familiar with the explicit form \(y=2/x\), which is also reflected in the answers to the second questionnaire. The implicit form \(xy=2\) is what Emma claims to be a “form” she is “unfamiliar with”. It appears that \(xy=2\) and \(y=2/x\) (as opposed to the two previous statements) evoked concept images containing conflicting information for equation and function (Tall & Vinner, 1981; Vinner, 1983, 1992). Emma could obviously not explain how the function and equation concepts related to each other, stating that “they correspond to each other”. Similar observations of students’ difficulties to explain relations between function and equation are made in various studies (e.g. Even, 1993; Grevholm, 2002; Leikin, Chazan, Yerushalmy, 2001; Tall, 1996; Thomson, 1994; Vinner & Dreyfus, 1989). The conception that function and equation correspond to each other was also used when she single-handedly made up incorrect expressions such as “the equation of the graph” \(^9\), illustrating a need for integrative reconciliation by related cognitive structures (Ausubel, 2000; Ausubel et al., 1978). Emma seemed to make generalizations based on the equation of a straight line, since she states that “people talk about the equation of a straight line”, clearly indicating she believed the solution set for an equation generally produces a graph of a function. The straight-line equation appears to be a cognitive obstacle (Goldin, 2002; Thompson, 1994) when Emma tries to describe relations between the concepts of function and equation.

I mentioned to Emma what she had discussed about equation and function and asked her to explain the difference:

E: Well, earlier I thought there was a huge difference, but I don’t think that anymore. It is more likely that it can be demonstrated; if there is an equation with two unknowns, then it can be shown in some way with the aid of a graph. That way you can extract … after drawing it, you can extract pairs of numbers. These contain all solutions, so to speak … If you choose an x-value, then you get a y-value.

---

\(^8\) Emma referred here to the concept maps that the preservice teachers drew for \(y=x+5\) and \(y=\pi x^2\) in the days leading up to the interview. Emma’s map for \(y=x+5\) had double links between function and equation, showing that she interprets the two concepts to be closely related.

\(^9\) Such an expression is not correct according to Swedish terminology.
It appears that Emma likens an equation with two variables to a function. She makes a geometric interpretation of the solution set (which she calls a “graph”) and seems to believe that a pair of numbers (x, y) in the set of solutions to the equation has the property that the x-value results in just one y-value, without realizing that the solution set may contain several different pairs of numbers with the same first component. Emma apparently believes that the two concepts approach each other, and she should thus refer to the period when she took the calculus course and the subsequent period, since she had previously stated that she saw “an enormous difference” between the concepts.

When she was once again asked to attempt to further explain her views of function and equation, it then became apparent that her understanding of equations had been influenced by equations with one variable (though she did mention the straight line equation – an equation with two variables – in her interview, which is something that she ought to have been familiar with for a long time). She said that she had now “changed her opinion” of what an equation was and that “it can also have two unknowns”. She then combined equations with two variables with the function concept, saying, “they [two concepts] begin to approach each other”. Her very language indicated that Emma could not distinguish between the two terms “function” and “equation”, since she sometimes said, “there are roots of the equation where the function intersects the x-axis” (similar results are presented by, e.g. Even, 1993). Emma seemed to rely on the fact that it is possible to illustrate solution sets for equations with two variables in a system of coordinates, when she combined the terms “equation” and “function”. She said that she did not have an answer that she was personally very happy with, but added after a short period of contemplation the following:

E: But a function is such that there is just one y-value for each x-value. Isn’t this what defines a function? Still, there may be many x-values connected to a y-value … I have not really thought about what that may look like if the equations were drawn in a system of coordinates … properly. It may not be valid, that theory of there is just one y-value for each x-value. Although … I don’t know … It may be valid, as far as I can see at the moment. I have never seen anything else, but that does not mean that it doesn’t exist. I have never encountered anything like that before.

Emma now considered the condition “there is just one y-value for each x-value” and began to see a difference between the two concepts (though other components of the function concept, such as domain and codomain, are not mentioned). The interview revealed that she had not worked, or had any memory of having worked with questions and problems that clarify the relations between the two concepts.

To help Emma further explain the function concept, I asked, “If one were to write $y^2=x$, what could you say about that?” Emma replied that one (positive) x-
value then gives two y-values. She looked at her sketch of the graph of \( y = \pi x^2 \) and noted that \( y^2 = x \) also results in a parabola, before saying, “though it is unsymmetrical\(^{10} \) around the x-axis”. I then asked, “Is it a function, is \( y \) a function of \( x \)?” To which she replied:

E: No, it is not, because of the definition of a function. It is perhaps just that there is a … Now I put it together. One, there is only one … y-value to each x-value … and this is not the case in \( y^2 = x \). There are two y-values for each value of \( x \).

Emma seems to now try to use the definition of function in the first place and her concept image of function (Vinner, 1983, 1992) in the second place.

Emma also seems to obtain greater clarity regarding the relationships between “equation” and “function”, indicating that integrative reconciliation (Ausubel, 2000; Ausubel et al., 1978) was taking place. The statement “Now I put it together” may be also be a sign of that. Emma also illustrated that a positive y-value gives two x-values in a correct sketch of \( y^2 = x \).

I asked Emma to decide if \( x \) was a function of \( y \) in the questions questions 3, 4, or 5, when she immediately answered that this was the case for \( y = x + 5 \) and \( xy = 2 \). This, she said, was because \( y = x + 5 \) gives a straight line and then a corresponding value of \( x \) for each \( y \). She explained \( xy = 2 \) by saying that one could solve for \( x \) and get \( x = 2/y \) (Emma based her answer on a geometric argument \( y = x + 5 \), but not for \( xy = 2 \), which is based on an explicit form and a symbolic argument. It appeared later that she did not know what the \( xy = 2 \) graph looked like before she had taken the time to draw it). Emma stated that she was able to solve for the variable \( x \) in both \( y = x + 5 \) and \( xy = 2 \) and hence, obtain one value of \( x \) for each value of \( y \), but that this did not apply for \( y = \pi x^2 \), and \( x \) is no function of \( y \). She subsequently used the term “inverse function”, stating that \( y = x + 5 \) and \( xy = 2 \) have inverse functions, but \( y = \pi x^2 \) does not. She also stated that \( y = \pi x^2 \) may have an inverse function (and thus introduce a restriction of the function) for an interval not including zero; she mentioned the intervals \( x > 0 \) and \( x < 0 \), among others.

After being asked, Emma acknowledged the notion of “domain” and “range” and explained what they were. She explained that she had not thought about it in relation to the statements in the three questions (corresponding with results from Tall & Bakar, 1992). Emma explained “domain” and “range” for \( y = 2/x \), but did not completely understand how the graph would look and thus plotted some coordinates and did a sketch of the graph. She subsequently stated that the x and y axes are asymptotes, observing, “There was a lot that could have been said about it \( y = 2/x \), that I had not said”. However, to further discuss the statement she said that she wanted to get “some kind of lead” to build on. Emma

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\(^{10}\) Emma understood that \( y^2 = x \) describes a parabola which is symmetrical about the x-axis (and unsymmetrical about the y-axis), since she sketched a correct parabola for \( y^2 = x \) during the interview. I assume that it was a slip of the tongue when Emma said “unsymmetrical about the x-axis”. 

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did not consider that the statement represented concepts such as reverse proportionality and hyperbola, though she had mentioned proportionality and parabola in connection to the statement $y=\pi x^2$. Of note, Emma did not discuss any function class, e.g. rational functions, or other properties in connection to the function concept, such as odd functions, or state that the function is the inverse of itself, etc.

During the interview, Emma believed she was now “better at illustrating problems using images and not just words”. She also declared that she was presently more ready to “explain it with a picture” and thought it was a process that probably began during the calculus course and was now continuing. She said an example of this was when she discussed “absolute value” in her answer to question 4 in the first questionnaire, while proportionality was discussed in the other. She associated proportionality with the function concept because it can be illustrated in a system of coordinates.

5.2 A case study with Nils

5.2.1 Answers to the questionnaires

<table>
<thead>
<tr>
<th>Question</th>
<th>First questionnaire</th>
<th>Second questionnaire</th>
</tr>
</thead>
<tbody>
<tr>
<td>3. We write $y=x+5$. What does that mean?</td>
<td>If we have a number $x$, then $y$ is larger by 5, i.e. $x+5$.</td>
<td>That $y$ changes depending on the chosen value for $x$. It is also a line.</td>
</tr>
</tbody>
</table>

4. What can you say about $y=\pi x^2$? Please give as detailed an answer as possible.

<table>
<thead>
<tr>
<th>Question</th>
<th>First questionnaire</th>
<th>Second questionnaire</th>
</tr>
</thead>
<tbody>
<tr>
<td>First questionnaire</td>
<td>Second questionnaire</td>
<td></td>
</tr>
<tr>
<td>You insert a value for $x$, do the calculation and get a new value for $y$.</td>
<td>$y$ is dependent on $x$. $y\neq 0$ if $x\in\mathbb{R}$. A curve which roughly looks like this</td>
<td></td>
</tr>
</tbody>
</table>

5. What can you say about $xy=2$? Please give as detailed an answer as possible

<table>
<thead>
<tr>
<th>Question</th>
<th>First questionnaire</th>
<th>Second questionnaire</th>
</tr>
</thead>
<tbody>
<tr>
<td>That the product of two numbers will be 2. Any numbers, so long as the product is 2.</td>
<td>$y$ is dependent on $x$ or vice versa. $y=\frac{2}{x}$, $y$ can be any number apart from 0.</td>
<td></td>
</tr>
</tbody>
</table>

Both questionnaire answers have different characteristics. In the first survey, Nils considers the $x$ and $y$ variables (except in question 5) and primarily discusses the numerical aspects before shifting to a more geometric interpretation in the second survey (again, with the exception of question 5), where he stressed the dependence between the variables in each statement.

Of note, Nils brought up the “dependence” between variables in question 4 of the second survey, drew a figure, specified the intervals of the variables and
used the term “curve”. This outlook is strongly associated to functions, without explicitly naming it. He also makes explicit the domain in saying \(x \in \mathbb{R}\), and range in \(y \geq 0\).

5.2.2 The interview coupled to the questionnaires

Nils read out his answers to question 3, stated that both answers are correct and continued:

N: One could even … well, carry over the x.
I: To give \(y-x=5\).
N: Which results in a diophantine equation.
I: I see.
N: So there will be more of an equation there, but in its current state, it is then more of a function. And it is a line, a straight line, in my opinion.

Nils clearly distinguishes between \(y-x=5\) and \(y=x+5\) because he viewed the first as an equation and the second as a function\(^{11}\) in which the function concept is connected to an explicit form. He also viewed \(y=x+5\) as a straight line and not as the equation of a straight line, which he nevertheless reflected upon in his concept map for \(y=x+5\). Nils used different concept images to interpret equivalent statements and said that he “saw” different concepts, as illustrated in the following excerpt from his interview:

N: Yes, well, if you see it as a function then you can say that if you make \(x\) equal to 0 then it will intersect the y-axis at 5. Thus \(y\) would be equal to 5… and the slope of the line would be 1.

In conjunction with his understood \(y=x+5\) to represent a function, he applied a geometric interpretation and brought up an intersection by referring to “the y-axis at 5” in a system of coordinates when \(x\) is equal to 0. Of note in this context, he also viewed \(y=x+5\) as a straight line, which became apparent when he specifies the slope of the line.

Nils continued to state values that the \(x\) and \(y\) variables could assume, mentioning that when \(y\) becomes 5 when \(x\) is 0, and \(y\) becomes negative when \(x < -5\). He said, “one can set \(x\) to absolutely any value”, including “infinity” (which gives “infinity”, and would then represent \(y\)). Nils noted that the function was not bounded. In relation to this, he applied another geometric interpretation, stating that \(y=x+5\) “rises completely and sinks below…” in the form of a straight line. He tried to describe the properties of the function by combining both numerical and geometric interpretations. Although Nils claimed to view

\(^{11}\) The function concept was previously discussed during question 2 of the interview, when Nils was asked to describe a function in his own words. He answered the question on the first questionnaire by giving two examples: “E.g. \(y=3x+1\) or \(f(x)=4x^2+3x+1\)”, whereas on the second questionnaire he wrote “A variable is dependent on another variable”. During the interview, he commented on his first questionnaire answers by saying that he had given “a good example” of functions and that they illustrate “a variable which is dependent on another variable”, thus making a link to the answers in the first survey.
y=x+5 as a function, he used no terminology related to the function concept, such as domain, graph, increase and decrease. He also did not highlight the fact that y=x+5 is a linear function or appeared to master the terminology associated with the function concept when he attempted to describe the different properties of the function.

He proceeded to question 4 and read what he had written on the questionnaires:

N: You enter a value of x, run the calculation and get a new value of y [reading the answer from the first questionnaire aloud]. Yes, but it is just as if they are dependent, like a function. And it is an x²-curve, a variant of an x²-curve.
I: Like the one you drew in the second questionnaire?
N: Yes, that can be checked if we insert … let us say \( \pi x^2 \). Set \( x = 1 \) so that \( y \) becomes equal to \( \pi \) … then we need to raise …
I: But what happens if you set \( x \) to 0?
N: Then \( y \) becomes 0 … Yes, \( x \) and \( y \), well… We say that \( y \) is 0, then I should move down, by the way [Nils realized that the graph was incorrectly drawn on the second questionnaire]. Yes, it should go through the origin, in that case.

Nils supplemented his answers to the questionnaires by stating that \( y=\pi x^2 \) is a function, since the variables are “dependent” on each other. He also applied a geometric interpretation, perceiving \( y=\pi x^2 \) as “a variant of an x²-curve” and using \( y=x^2 \) as the basis to describe \( y=\pi x^2 \). During the interview and by the figure drawn in the answer in the second questionnaire, Nils (like Emma) believed that the factor \( \pi \) means the intersection of the graph and the y-axis above the origin. It was not until he was asked to consider an x-value of zero when he realized that the origin is a part of the graph. Nils was then asked to draw the graph, upon which he stated:

N: (…) yes, it will be much narrower [Nils drew the graph] … than a normal x²-curve. So it is the actual width, if one could say so, which is changed when the constant is changed …

Nils obviously looked at \( x^2 \) at the right hand side and applied a geometric interpretation in the form of an “\( x^2 \)-curve”, viewing \( y=x^2 \) more as a prototype in that context. This is also illustrated in the concept map drawn by Nils for \( y=\pi x^2 \), in which he also stated that \( y=\pi x^2 \) is “narrower” than an “\( x^2 \)-curve”, referring to “\( y=x^{2/3} \)”. In his concept map, Nils appeared to realize how the shape of the graph of \( y=\pi x^2 \) changed if it is multiplied by \( \pi \) while he was drawing it. However, it is not apparent if he also believed that \( \pi \) results in a translation of the graph.

Nils outlined the values the variables could have, saying “one can set \( x \) to absolutely any value” and \( y \) is always positive “since it is the square of \( x \)”. He returned to his sketch of \( y=\pi x^2 \) to comment on the effect of \( x^2 \) on the graph: “it turns at the zero point and then rises at both ends, if one could say so… so \( y \) can never be negative” and described which, like the previous statement, was based
on a geometrical interpretation and which values could possibly be assigned to
the variables without referring to either domain or range.

Nils continued to comment on $y=\pi x^2$:

N: Well, even that [meaning $y=\pi x^2$] may be … you may write it as an equation.
Although it is extremely tricky, that is of course my first impression. But I don’t
know …
I: What …
N: Well, you can also write it as an equation. But I guess it is better as a function.

In Nils’ case, “write it as an equation” meant gathering the variables on one side
of the equal sign (as with $y=x+5$, in writing it as $y-x=5$). He was now of the
impression that $y=\pi x^2$ represented a function:

I: You see it as a function?
N: Yes, automatically when it is presented as $y$ equals something with $x$. It is
easier.
I: Did you also do this with the third question [$y=x+5$]?
N: Yes, as a function. You do this automatically. But when you think about it,
what you can change … play around with, etc… But this [meaning $y=x+5$, in
question 3] is significantly easier than that [meaning $y=\pi x^2$, in question 4].
I: The third one is easier?
N: Yes.
I: In what way?
N: Well, because it [$y=x+5$] is a straight line, so … it is easier to consider than a
quadratic curve [$y=\pi x^2$] … But a person would automatically think that it would
be more difficult when they see $\pi$ as well. I mean, it really doesn’t matter whether
there is $\pi$ or 5. But once the person sees $\pi$, the automatic thought is that it is much
more difficult.

Nils said he viewed $y=x+5$ and $y=\pi x^2$ as functions, seemingly basing this on the
explicit form in which they are written when he said, “yes, automatically, when
it is presented as $y$ equals something with $x$”, describing an expected form of
representation of the function concept. Nils, as was the case of Emma, associates
different concepts to different forms of the statements, describing a semiotic
function of the forms of a statement (Steinbring, 2005). Nils also evaluated the
degree of difficulty of the statements in questions 3 and 4, stating that $y=x+5$ “is
significantly easier” than $y=\pi x^2$, based on the appearance of the graph, and
saying that a “straight line” is “easier to consider” than a “quadratic curve”. He
thus appeared to base the difficulty of the statements on a geometric
interpretation. Also of note, Nils used a different terminology, saying a
“quadratic curve” instead of an “$x^2$-curve”. He did not explicitly imply a “graph
of a function” when he used the term “curve”, but his concept map for $y=\pi x^2$
nevertheless indicated this as his intention, since it contains a horizontal
sequence of nodes and links from the term “functions” to “$x^2$-curve”, via “table”
and “graph”. Furthermore, Nils found the presence of $\pi$ in the statement to be
“much more difficult”, without realizing that $y = \pi x^2$ also gives the area of a circle with radius $x$.

His conception of the function concept – which he seemed to tie to an explicit form and dependence between the variables – was subsequently investigated. In relation to $y = \pi x^2$, he was thus asked, “… which of the variables is a function of the other, or what is your view?” Nils replied, “$y$ is a function of $x$”. To incite him to expand his view of functions I asked, “can it be said that $x$ is a function of $y$?” He then stated, “… there will be two values of $x$ for each value of $y$” and was uncertain of whether it was a function. Nevertheless, he pointed to a dependence between the variables and did not rule out the possibility that $x$ could be a function of $y$:

N: When you set a value for $y$, you get one or two corresponding values for $x$. And then you may even set a value for $x$ and get a corresponding value for $y$ … thus they could … they are thus dependent on each other …

Nils further contemlplated the function concept and stated that $y = x + 5$ in question 3 can be written in the form $x = y - 5$, which in his opinion means that “$x$ is dependent on $y$” and thus is a function of $y$. In the resulting discussion, Nils stated that $x$ should to be a function of $y$ when $y = \pi x^2$, since a person is able to write $x$ “freely”, i.e. solved for $x$. However, he was uncertain as to whether it is a function because it produces two values of $y$. Furthermore, regarding the function concept, he stated that “a function is automatically perceived to be expressed as $y$ followed by the equal sign and then an expression containing $x$” and found it strange to let $x$ become a function of $y$. I therefore suggested that he study $x = \pi y^2$ and determine whether $y$ is a function of $x$. Nils solved for $y$ and obtained $y = \pm \frac{x}{\pi}$, but could not decide if $y$ is a function of $x$:

N: I don’t know if one can do that, so I can’t really … If $y$ is then dependent because we get two answers or whether that is it. Because if this is the case, i.e. if this is allowed, then it is dependent on $x$.

Nils could not apparently remember a definition of function. I asked, “What would such a function graph look like?” Nils then plotted some values and drew a correct parabola for $x = \pi y^2$, though this did not enable him to decide whether $y$ is a function of $x$. Drawing the parabola apparently did not evoke any mental images of function graphs and hence provide any insight into the fact that an $x$ value corresponds to only one value on $y$ (as opposed to Emma’s case).

The interview continued with Nils reading his answers to question 5, demonstrating that $x$ and $y$ depend on each other by letting the variables assume values whose product was 2. He also said that one could “rewrite it [xy=2] as a function $y = 2/x$”, stating that neither $x$ nor $y$ could be zero. In contrast to the previous statements, Nils did not apply a geometric interpretation and was asked
to describe the function graph. He replied by plotting the coordinates and made a sketch of the graph, pointing out that x and y axes form asymptotes.

Nils was asked again to try and explain whether x is a function of y in questions 3 to 5. Nils replied, “it can be reversed”, explaining that one could solve for the variable x in y=x+5 and xy=2, continuing with:

N: (...) actually, it also should be, in question 4 \([y=\pi x^2]\). Although it is a square … It’s just that you get two answers. So yes. It should be, since you can make both [variables] independent, but then I don’t really know if it is allowed, in the case where there are two answers. It is easier to see in that case [referring to xy=2], since it is just x and y.

It appeared that Nils’ view of functions was mainly based on a dependence between variables expressed in an explicit form. He did not take note of a contemporary definition of function, which was apparently unknown. Questions on whether or not x is a function of y did not seem to lead to any recognition of the concept of inverse function (in contrast to Emma).

5.3 A case study with Vera

5.3.1 Answers to the questionnaires

3. We write \(y=x+5\). What does that mean?

<table>
<thead>
<tr>
<th>First questionnaire</th>
<th>Second questionnaire</th>
</tr>
</thead>
<tbody>
<tr>
<td>That something added with 5 becomes the sum y.</td>
<td>The number “hidden” behind x added with five gives the sum y.</td>
</tr>
<tr>
<td></td>
<td>It could also be said that y depends on x.</td>
</tr>
</tbody>
</table>

4. What can you say about \(y=\pi x^2\)? Please give as detailed an answer as possible.

<table>
<thead>
<tr>
<th>First questionnaire</th>
<th>Second questionnaire</th>
</tr>
</thead>
<tbody>
<tr>
<td>That a number raised to the power of 2 and multiplied by (\pi) will become y.</td>
<td>y depends on (\pi) and (x^2).</td>
</tr>
<tr>
<td></td>
<td>(y=\pi x^2) explains the area of a circle.</td>
</tr>
<tr>
<td></td>
<td>(a=\pi r^2).</td>
</tr>
</tbody>
</table>

5. What can you say about xy=2? Please give as detailed an answer as possible.

<table>
<thead>
<tr>
<th>First questionnaire</th>
<th>Second questionnaire</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two unknown numbers will become the sum 2.</td>
<td>If you multiply the variables x and y the sum becomes 2.</td>
</tr>
</tbody>
</table>

In the first questionnaire, Vera interprets the statements as mainly numerical calculations. She divides each statement into two expressions and begins each answer by calculating an expression, before stating that the result “becomes” the other expression. In the second questionnaire, she to a higher degree takes note of the variables in the statements, and a dependency relationship in using the term “depends” (with the exception of question 5). Vera also includes new
variables in conjunction with her interpretation of the statement in question 4 as the calculation of an area.

5.3.2 The interview coupled to the questionnaires

Vera read her answers for question 3 of the questionnaire. She said the answer to the first questionnaire and the first sentence in the second questionnaire are “exactly the same” and proceeded to explain the dependency of \( y \) on \( x \) by determining \( y \) for different integers of \( x \). In relation to her explanation, she was asked, “Does \( x \) depend on \( y \)?” This prompted her to consider whether it could possibly be so:

\[ V: \ldots \text{Yes, it does. Yes, because we had set ... Well it must of course do that? Because if we had brought } y \text{ over ... and } x \text{ had become free [explicit] ... If we had done that then it would have been the same thing. It would have depended on ... Yes, it does. Or, am I wrong?} \]

Vera obviously tied the “dependence” of the variables to an explicit form, but was uncertain whether this was the case, i.e. if \( x \) was dependent on \( y \). She was asked if anything else could be said about \( y=x+5 \), to which she replied:

\[ V: \text{That ... do you mean like a formula or something ... or not? [Pause.] Oh! That it is a straight line!} \]

Vera also associated \( y=x+5 \) with a straight line and then had nothing more to add.

The interview with Vera differs from those with Emma and Nils since she was often unable to expand on her answers as much as the others. Thus, she was asked to examine her previous answers to the questionnaires, which included the function concept to allow her to give an additional response in relation to the statement \( y=x+5 \). The questions concerned variable, equation, function and algebraic expression. The interview continued:

\[ V: \text{Yes, I think I see it as an equation ... I don’t think it is a function}. \]

I: No ...

\[ V: \text{[Laugh]} ... \text{But it is, actually, if you think back to what it was. When I look at it I don’t think: “Oh! A function!”} \]

I: But ...

\[ V: \text{I can’t say why I don’t, but I don’t think about it. I would much rather see it as an equation.} \]

\[ ^{12} \text{In question 2c of the first questionnaire, Vera was asked to describe a function in her own words and wrote: “curves that can be calculated.” In the second questionnaire, she wrote “something that depends on each other”. During the interview, Vera did not understand what she meant by her first answer, but settled on her answer from the second questionnaire. To further explain this answer she gave } y=\pi x^2 \text{ in question 4 as an example, saying that “} y \text{ is dependent on } \pi x^2 \text{”. She obviously did not perceive this to be a dependency, which was first based on the } x \text{ and } y \text{ variables.} \]
Vera now chose to interpret the statement as an equation, which is something she did not mention in her questionnaire. She continued:

V: Wait … One could say that y is dependent on x. So it should be a function as well. So let us say both function and equation.

By “y is dependent on x”, Vera clearly means that \( y=x+5 \) is a function. She also expressed function on her concept map for \( y=x+5 \), but then stated that y was dependent on \( x+5 \), indicating a lower dependence of variables. Vera had nothing more to add regarding to the statement afterwards and was able to move on to question 4.

Vera read her answers to question 4. She said the answer in the first questionnaire only states how to read \( y=\pi x^2 \), and continued:

V: They are equal in weight, y is the one on that side [refers to the right hand side of \( y=\pi x^2 \)].

I: Hmm … on the right hand side, you mean …

V: Yes, so y is the answer. So, y is exactly the same as the result of the right hand side … And it says here that y is dependent on \( \pi \) and x. It does, because if I change x then I change y. And I did not realize it then [in the first questionnaire], that it was the area of a circle. But it is.

Vera’s comments indicate that she viewed the statement as an equation, since she used a weight scale as a metaphor to show the two sides “are equal in weight”. She also used a viewpoint, which could be interpreted as being analogous to numerical calculations, when she said, “y is the one on that side” (referring to \( \pi x^2 \)) as well as “y is the answer”, thus expressing procedural knowledge (Hiebert & Carpenter, 1992; Hiebert & Lefevre, 1986). An element of numerical interpretation also existed when she said, “y is dependent on \( \pi \) and x”, and that the y-value was dependent on the constant \( \pi \). Nevertheless, Vera notes the significance of the variables by stating, “if I change x then I change y”.

Vera did not apply any geometric interpretation to \( y=\pi x^2 \) in the form of a graph (in contrast to Emma and Nils). However, like Emma, she interpreted the statement as the calculation of an area and attempted to clarify it by writing \( a=\pi r^2 \) and stating that \( a \) is dependent on \( \pi \) and \( r^2 \) during the interview, i.e. a viewpoint analogous with a calculation is once again applied, emphasizing an operation rather than a relationship between the variables \( a \) and \( r \).

Vera did not have any further comments on her answers to the questionnaires and thus again studied the previous questions on the first page of the questionnaire (which concerning variable, equation, function and algebraic expression) before saying:

V: It is a function. And an equation as well, I would like to say. It has several … It consists of two variables, y and x, as well as the constant \( \pi \).

Not until Vera read the previous questions did she state that the statement represented an equation and a function. She had previously used expressions
such as “weigh” and “dependent”\(^1\), which could be seen as metaphors indicating a simplified view of the two concepts. The metaphors appeared to be at the centre of her concept images for equation (i.e. weight) and function (i.e. dependent), something thought to represent less developed knowledge structures. A less developed knowledge structure seems to also be reflected in the concept maps drawn by Vera. The concept maps do not have any cross links to connect different sections of the map, giving the impression that they have few meaningful relations and are thus represented by segmented knowledge structures that imply a less developed understanding (Ausubel, 2000; Ausubel et al., 1978; Hiebert & Carpenter, 1992; Novak, 1998).

Vera read out her answers to question 5. She wrote “sum” in the two questionnaires instead of product. I attempted to focus her attention to this issue:

I: The sum of two?
V: Hmm ... But actually [laugh]. If one could consider what we have previously discussed ... yes, the sum of two ... here it is. Yes, because they should actually be equal in weight, where we can, e.g. have x is 2 y gives 1, for them to be equal.
I: ... x. What did you say ...
V: So we can actually just set x to 2 and y to 1, which results in an answer of 2.
I: Yes.
V: And that ... I don’t know if one can say that x and y are dependent on each other here? Perhaps not?
I: Why not?
V: But it must be possible to say that, if that is the basis on which I ... Because if x is free then we have 2/y ... and when we change y then x is also changed.

Vera did not realize that she had written “sum” instead of “product”. When she read her answers to question 5, she seemed to notice the equality sign in the statement. She again used a “weight scale” metaphor in which the left hand and right hand sides should “weigh the same”, giving the impression of the statement representing an equation. She suggested values for x and y that fulfilled the statement, but was uncertain if it meant that “x and y were dependent on each other”. To confirm this she extracted x (to give x=2/y) and established that x changed for different values of y.

During the interview, Vera was uncertain how to comment on the statement in question 5 (with the shortest answer of all questions in the second questionnaire) and considered whether she was on the “wrong track”. She once again explored the previous questions before saying that x and y are variables, and xy=2 is an equation, since it has “several unknowns”. She ended with the following:

\(^1\) A dependency that is analogous to a numerical calculation is often observed, where the result is dependent on the formulation of a mathematical expression rather than on a variable dependent on another variable. This leads to a pronounced operational conception (Sfard, 1991, 1992) of function.
V: And then I began to figure out … I remember that it is a function, since they are dependent on each other.

Vera was then asked what she meant by writing “sum” in the questionnaire. She said, “The sum of x multiplied by y is two” to interrupt herself and wondered why she wrote, “sum” in both questionnaires. She apparently then realized that what she had written was incorrect, but did not recall the word “product” until I suggested it. A discussion on terminology for addition and multiplication followed. Vera stated that this uncertainty had likely existed “since I learned to count” early in school. Helldén (2004) presented the theory that knowledge gained during the early years of school is of significance during subsequent years.

Vera then applied a graphical interpretation to the statement in question 3, regarding the shape of a line, but not to the statements in questions 4 and 5:

I: Considering questions 3, 4 and 5, do you see any images before you, or… for the first question, I have got the impression that you saw a line.

V: Yes.

I: How was it for questions 4 and 5?

V: In question 4, I saw the area of a circle. When I see it in front of me, I see a round shape, is it a circle? Yes, that’s what it looks like. No, I see the area of a circle … and the radius and so on … though it doesn’t say “radius”. But I have explained that here [referring to a=\pi r^2 in the answer in the questionnaire]. What does a radius looked like? Yes, I can quite simply see the area in front of me… of a circle … I don’t see it as a function14.

I: Hmm, no.

V: Although I thought that it was. But I don’t see it like that, not at first glance.

I: No.

V: Concerning y … I don’t see anything in particular at all [referring to question 5].

Vera obviously could evoke mental images for \(y=x+5\) in the form of a line and for \(y=\pi r^2\) in the form of a circle, but not for \(xy=2\) (as with Emma and Nils). I noted that Vera had previously mentioned \(xy=2\) as being written in another form \((x=2/y)\), to which she replied that one could either write \(xy=2\), so that “x or y are free”, as well as \(y=2/x\). She also considered whether \(y=2/x\) resulted in a “line”, saying that she thought about “the equation of the straight line”. Vera made no attempt to draw the graph, but instead followed up her assessment of straight lines by noting that the statement \(y=x+5\) in question 3 results in a straight line, since it has the form \(y=kx+m\,\), and stated that \(k\) is 1 and \(m\) is 5. She realized that the statement in question 5 could not be written in the form \(y=kx+m\) and was therefore not a line.

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14 Vera’s answers are not completely consistent with those given earlier in the interview regarding question 2c, when she gave \(y=\pi x^2\) as an example of a function (or a “dependency”).
Vera said “I don’t like the [statement in] number 5” and requested assistance with interpreting $xy=2$. She was then asked if she thought the statement was more familiar in the form $y=2/x$. She did not think so, but wondered if it should. She did not make any other associations. Upon hearing the concept of proportionality as an example of what the statement could represent, she became more enthusiastic, though still uncertain as to how it should be linked to $xy=2$. During the interview, Vera associated reverse proportionality with “$x^2$ and such things…” and appeared to relate to mathematical models in science.

The concept of inverse function was mentioned, which Vera recognized but did not understand. Further, I asked whether $x$ was a function of $y$ amongst questions 3 to 5. Vera stated that $x$ is a function of $y$ in $y=x+5$ and $xy=2$ after she had solved for the variable $x$. Vera did not succeed in solving for $x$ in $y=\pi x^2$ and was uncertain as to whether $x$ was a function of $y$. She subsequently realized that $\pi x^2$ is the area of a circle, where $x$ is the radius; she did not think it was a function, but could not explain why.

Vera mentioned a dependency when she said that the statements represented functions, but did not always refer to a dependency between the $x$ and $y$ variables; instead, she said $y$ is a function of $\pi x^2$ or $x^2$. It was also revealed in the interview that Vera associated $x^2$ with quadratic functions, which she most closely identified with $y=x^2$; she could not decide if $y=\pi x^2$ was a quadratic function, since it included $\pi$.

6 Discussion and conclusions

6.1 The function concept

Although the function concept is a central concept in the calculus course, none of the preservice teachers in the case study mentioned that the statements represented functions in their second questionnaire. However, formulations highlighting the dependencies of variables were common (as with Emma and Nils, who emphasized this by explicitly writing $xy=2$ as $y=2/x$).

The preservice teachers did not describe a function consistent with contemporary characterizations of function. Nonetheless, Emma noticed an important component of the function concept, the uniqueness criterion, when she wrote, “for each value of $x$, there is just one value of $y$” in the first questionnaire. At the same time, she deviated from a contemporary definition of function by assuming that $x$ and $y$ are part of “an expression with two variables”. She commented on her answer in the questionnaire by stating, “… the theoretical stuff that one has learn, has been drilled into us…” and does not mention the uniqueness criterion in the second questionnaire, thereby showing signs of rote learning (Ausubel, 2000; Ausubel et al., 1978). In addition, Nils viewed the function concept as dependence between variables. Like Nils, Vera based her conceptual interpretation of function on a dependency relation. But in
contrast to Nils, she specified a dependency that did not exclusively consist of variables, but also included constants analogous to numerical calculations.

One reason the preservice teachers are unsuccessful in describing the function concept in a manner consistent with a contemporary definition of function may be their apparent lack of experience of working with problem formulations requiring them to contemplate the definition of function (Even, 1993; Tall, 1996; Tall & Bakar, 1992; Vinner, 1992; Vinner & Dreyfus, 1989). The preservice teachers give the impression of only having been exposed to problems that encourage little reflection of the function concept. Emma revealed this during the interview when she stated that she had not “come across” such situations. This was confirmed in the interview with Nils, who clearly could not recall the definition of function because he could not determine whether \( y \) was a function of \( x \), though he stated that a positive value of \( x \) results in two \( y \)-values for \( x = \pi y^2 \).

The function concept to a higher degree is tied to a structural conception (Sfard, 1991, 1992) in the form of ordered pairs in Emma’s case. The preservice teachers increasing use an operational conception (Sfard, 1991, 1992) of function, identifying dependencies between the variables in the statements, as their level of performance in their studies of mathematics become lower, i.e. Vera’s case often highlights a process that is analogous to numerical calculations.

6.2 Function classes, properties and language

Mathematicians with PhDs often take note of various function classes and properties of functions when they draw concept maps for the concept of function (Williams, 1998). Slavit (1997) states that focusing on the properties of functions may lead to a more developed structural conception in the form of objects. A comparison of preservice teachers in the case study reveals that they did not stress \( y = x + 5 \), \( y = \pi x^2 \) and \( xy = 2 \) to represent different function classes. Nevertheless, \( y = \pi x^2 \) was a quadratic function for Emma, both in her interview and on her concept map. In this respect, Nils referred to classes of curves (\( x^2 \)-curves and quadratic curves) and did not explicitly mention that he was referring to function graphs. However, the concept map drawn by Nils for \( y = \pi x^2 \) contains a horizontal sequence of nodes and links from the term “function” to “\( x^2 \)-curve” via “table” and “graph, thus indicating his intention. Vera seemed to associate quadratic function with \( y = x^2 \), but was uncertain if \( y = \pi x^2 \) was also a quadratic function. Furthermore, all stated that \( y = x + 5 \) was a straight line (Emma stated that it was the equation of a straight line) and a function, though none mentioned that it was a linear function. During their interviews, they also stated that \( xy = 2 \)

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15 With the condition that the contents should be tied to a basic course in calculus.
was a function (which they often preferred to write in the form $y=2/x$), but without connecting it to a function class.

Emma and Nils leaned more towards a geometric (holistic) approach for $y=x+5$ and $y=\pi x^2$ during their interviews and on their concept maps, and to a greater extent than in their answers to the questionnaires. This is also true of Vera in the case of $y=x+5$, since she largely viewed $y=\pi x^2$ as a formula to calculate the area, as opposed to $xy=2$ where none of the students gave a geometrical description without first plotting the values to sketch the graph. When the students discussed properties in conjunction with the function concept, they primarily did so from a geometric perspective (in Nils’ case this was often combined with a numerical approach where he refers to the values of the variables). A geometric approach is also reflected in the concept map drawn by Emma for $y=\pi x^2$, where she places “parabola” (referring to the function graph) in an underlying structure to “function”. To this, she tied “min. point” and “symmetrical”, among others. Emma further claimed that she now preferred to explain and illustrate problems with the aid of “images”.

When the preservice teachers describe the properties of functions, they often lack the mathematical language (Grevholm, 2004a) related to the function concept. This was apparent when Nils said “rises completely” instead of “increase” or when Emma said “roots of the equation where the function intersects the x-axis” instead of “zero of the function”. It was nevertheless shown that the preservice teachers occasionally used a mathematical language, e.g. Emma mentioned the term “inverse function” in relation to the questions of whether $x$ was a function of $y$ and both Emma and Nils stated that $xy=2$ has asymptotes. The absence of language reduces the possibilities for a concept to become well integrated in the cognitive structure (Ausubel, 2000). The language deficiencies of the preservice teachers regarding the different properties of functions may prevent the construction of well-developed knowledge structures related to the function concept. This could create poor conditions for meaningful learning (Ausubel, 2000; Ausubel et al., 1978) and learning with understanding (Hiebert & Carpenter, 1992).

6.3 Knowledge structures and relations between concepts

The concept maps illustrate the preservice teachers’ views of how the different concepts deemed to represent the statements are related to each other (Hansson, 2004). It can be said that the maps drawn by the three preservice teachers are different in character. Grevholm (2004a) shows the concept maps drawn by the preservice students to also preserve some individual characteristics. Emma and

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16 The concept map Emma drew for $y=\pi x^2$ differs from that which she drew for $y=x+5$ because it contained no cross-links. This is because (according to her) she first drew the concept map for $y=x+5$ and thought that it was too time-consuming. The map of $y=\pi x^2$ subsequently became a first draft that she did not develop during the remaining time allotted to the preservice teachers to draw their maps.
Nils drew maps that largely connected different sections of the map. They contained more17 cross-links (Novak, 1998; Novak & Gowin, 1984) compared to Vera’s two maps, neither of which contained any cross-links. This could imply that the knowledge structures representing the concepts are more compartmentalized for Vera than Emma and Nils. Furthermore, the maps drawn by Nils and Vera tended to contain more trivial elements (e.g. “draw”, “Greek letter”, “car romeo”), whereas the maps drawn by Emma were more subject-specific. This implies that Emma has knowledge structures with more meaningful relations to mathematical concepts.

The function concept is a more integrated part of the concept maps drawn by Emma and Nils than those drawn by Vera (with the exception of the map Emma drew for $y=\pi x^2$). In the map Emma drew for $y=x+5$, the function concept is placed in close relation to the equation concept. Of note, both function and equation concepts exist in all of the maps drawn by the preservice students for $y=x+5$ (all three also include $y=kx+m$, thus referring to the equation of a straight line). This is in contrast to the concept maps for $y=\pi x^2$; Vera is the only one who discusses the equation concept (by the link “remind me of” to the node “quadratic equation”). The preservice teachers appeared to see it as if the relationship between the concepts “function” and “equation” changed in relation to the two statements $y=x+5$ and $y=\pi x^2$.

The form of a statement appears to determine which concepts the preservice teachers decide it represents, illustrating a semiotic function of the statements (Steinbring, 2005). During his interview, Nils says he “automatically” sees a “$y = x$-expression” as a function, and both he and Emma write $xy=2$ in the form $y=2/x$ in relation to the function concept. Nils evidently did not view $y=\pi x^2$ to represent an equation, since he stated in his interview that “you can write it as an equation” and considered gathering the variables on one side of the equal sign. He also said it would then become “extremely tricky” and did not appear to realize that the solution set of the equation coincided with the graph of the function. Furthermore, Emma did not state that $y=\pi x^2$ represented an equation, not mentioning the equation concept in the questionnaires, the concept map or the interview, as opposed to $y=x+5$ and $xy=2$. The relationships between the concepts represented in the concept maps also appear to have been influenced, e.g. function was strongly related to equation in the concept map drawn by Emma for $y=x+5$, whereas the concepts had no relations in the map she drew for $y=\pi x^2$, which had no equation concept.

The concept map drawn by Emma for $y=x+5$ contains both the function and equation concepts. She also mentioned these during the interview, saying that she had had “problems with them” when she drew the maps, indicating that the concepts evoked conflicting concept images during construction that contributed to the time-consuming experience for Emma.

17 With the exception of the map which Emma drew for $y=\pi x^2$. 

26
The preservice teachers hardly make use of subject matter knowledge gained from a tertiary level in relation to the statements, with concepts and relations such as those from the recently concluded calculus course. Concepts mentioned to which they have been exposed to in the teacher-training program, but possibly not prior, were diophantine equations (Nils), asymptotes (Nils, Emma) and inverse functions (Emma). Furthermore, the preservice teachers did not make any connections to future teaching scenarios or their learning in connection to the statements.

6.4 Dominant elements in concept images

During the interviews, Emma and Nils both seem to use \( y = x^2 \) as a prototype when they draw the graph of \( y = \pi x^2 \) (Akkoc & Tall, 2002; Hershkowitz, Schwarz, 1997; Tall & Bakar, 1992). The concept map drawn by Nils for \( y = \pi x^2 \) confirms his view of \( y = x^2 \) as a prototype in relation to \( y = \pi x^2 \), whereas Emma states on her map that \( \pi \) influences the shape of the graph, indicating her assumptions based on \( y = x^2 \). The concept images presented by Emma and Nils \( y = x^2 \) appear to dominate for a quadratic function and parabola, and an “\( x^2 \)-curve”. However, it does not immediately lead them to draw a correct function graph of \( y = \pi x^2 \), as both Emma and Nils mistake making a vertical translation of the graph during the interview. Hershkowitz and Schwarz (1997) present somewhat different results showing that prototype elements may have a beneficial effect on problem solving. Furthermore, the statement \( y = x^2 \) appears to have a strong influence on Vera’s concept image for quadratic functions. She identifies \( y = x^2 \) as a quadratic function and is uncertain whether \( y = \pi x^2 \) is a quadratic function, since the factor \( \pi \) is included.

In relation to \( y = x + 5 \) and a straight line, the interviews and concept maps reveal \( y = kx + m \) to constitute a dominant feature in the preservice teachers’ related concept images, and they state that \( m \) and \( k \) are the coordinates’ \( y \)-value for the point of intersection with the \( y \)-axis and the slope of the line, respectively. Vera, for example, related to \( y = kx + m \) when investigating whether the graph of \( y = 2/x \) is a straight line. It can be observed that in relation to \( y = kx + m \), the preservice teachers seem to apply a general model in their reasoning without a specific example.

Another situation arises during the interview, when Emma spontaneously raises the relationship between equation and function in connection with \( xy = 2 \) (“equation” is thought to be represented by \( xy = 2 \) and “function” by \( y = 2/x \). Emma seems to find it difficult to distinguish between the two concepts and invents expressions, such as “the equation of the graph”. The concept map for \( y = x + 5 \) (with double links between the two concepts) and her reply to the question 2c of the second questionnaire (containing words such as “solution pair”) also show that the cognitive structures related to the concepts seem to require integrated reconciliation. During the interview, her understanding of the
relations between “the equation of a straight line” and “function” appeared to be a cognitive obstacle (Goldin, 2002; Thompson, 1994), making it difficult for her to distinguish the concept of equation and the concept of function. Nevertheless, Emma came to a better understanding of the function concept in the interview when she considered \( y^2 = x \) and then tried to determine if \( y \) is a function of \( x \). It appears as if the situation stimulated a feedback of the concept image to the concept definition that in Emma’s case lead to a better understanding with signs of progressive differentiation and integrative reconciliation (Ausubel, 2000; Ausubel et al., 1978).

6.5 Closing comments

It is well known that mathematics students in the first year of university often do not have a conceptual understanding of functions considered consistent with contemporary characterizations of function, as with the preservice teachers in this study (e.g. Breidenbach, Dubinsky, Hawks & Nichols, 1992; Carlson, 1998; Tall & Bakar, 1992; Thompson, 1994; Vinner & Dreyfus, 1989; Williams, 1998, to name a few). However, Even (1993) argues that preservice teachers ought to have a deeper understanding of the function concept, whereas Vollrath (1994) states that the understanding of mathematical concepts including their definition, properties and relationships to other concepts are important aspects of teachers’ knowledge skills. This is closely related to what Schulman (1986) refers to as pedagogical content knowledge. The preservice teachers in the case study were found to have a limited understanding of the function concept and still lack the skills an inservice teacher should possess. This is also expressed in the way they use mathematical terminology regarding the concept of function; similar results relating to preservice students’ language can be found in Grevholm (2004a, 2004b).

The preservice teachers in the study seemingly need to be exposed to problem formulations concerning the relationships between mathematical concepts, including problems inviting reflection upon the definition of function. The preservice teachers do not appear to be experienced in working with such problems. A way to develop their understanding of the function concept could then be to stimulate a feedback of evoked concept image to the concept definition, as possibly when Emma comes to a better understanding during the interview. According to Vinner (1992), this type of reconnection is primarily possible with problems that are not of the standard variety. Moreover, if the preservice teachers’ concept images of function were characterized by those examples of function with which they come in contact, and to a lesser degree by the formal definition of the concept (a premise supported by, e.g., Vinner, 1983, 1992; Vinner & Dreyfus, 1989; Tall, 1996; Tall & Bakar, 1992), this should result in paying closer attention to those examples of functions preservice teachers encounter during mathematics courses. Furthermore, highlighting
different properties of functions and their relations to other concepts could be one way of stimulating a structural conception of function (Slavit, 1997; Sfard, 1991, 1992), and help the preservice teachers to develop more composite knowledge structures in relation to the function concept, which the preservice teachers in the study seem to be in need of.

References


Note: In the translation of the interviews from Swedish to English, preservice teachers’ reasoning concerning 2nd degree polynomial functions and equations has been translated as concerning quadratic functions and equations. This has however not been done in the translation of the preservice teachers’ concept maps.
Nils
Vera

The sum of \((x+5)\) depends on \(y = f(x+5)\), which is the equation of a straight line with the equation \(y = kx + m\) that shows where the \(y\)-axis is intersected. Two values of \(x\) move vertical, and that \(x\)-axis moves horizontal.

In this case, the equation of a straight line is \(y \in kx + m\) is an alphabet letter that has an own axis on which moves horizontal. That is a variable that has its own axis on which moves vertical. In this case, \(x\) is a number that can be any number.

The \(y\)-axis is intersected in the equation \(y = x + 5\), which is a function of a circle. That is also a variable of a circle. The area of a circle is \(\pi r^2\). That has its own axis on which moves horizontal. That moves horizontal such as circle circumference.

Some others are square and triangle. The area of a square is \(a^2\) and that of a triangle is \(\frac{1}{2}bh\). That has its own axis on which moves horizontal.
Questionnaire for students on the teacher education program specializing in mathematics for the grades 4 to 9

Name:..............................................................................................

Answer as clearly as you can. Try to express an idea even if you are not fully satisfied with your answer. Give examples if you feel it makes your answers more comprehensible.

1) Put a mark in one or more squares according to your opinion.

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<th>Is an equation</th>
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2) Describe in your own words your interpretation of the concept of:

a) Variable

b) Equation

c) Function

d) Algebraic expression
3) We write $y = x + 5$. What does that mean?

4) What can you say about $y = x^2$? Please give as detailed an answer as possible.

5) What can you say about $xy = 2$? Please give as detailed an answer as possible.
6) We write $y = f(x)$. What can you say about it?

7) Give your opinion about the extent to which functions are of significance in mathematics. Provide reasons for your opinion.

8) To what extent are, in your opinion, functions present in school mathematics? Provide reasons for your opinion.
The purpose of this paper is to examine preservice teachers’ conceptions of the function concept as well as their conceptions of the significance of functions in mathematics and the presence of functions in school mathematics. A further purpose is to study the effects of an intervention regarding the function concept. Two groups of preservice teachers who are specializing in mathematics and science are participating in the study. The findings indicate changes, firstly in the preservice teachers’ view on the function concept related to the intervention, and secondly with respect to different themes of relevance in their reasoning on the significance and presence of functions in various contexts.

INTRODUCTION

The concept of function is one of the central concepts underlying mathematics. It is important that preservice mathematics teachers acquire a well-developed understanding of the function concept, partly to become successful in their studies in mathematics but also critical in their planning and teaching of mathematics (e.g., Even, 1993; Even & Tirosh, 2002; Lloyd & Wilson, 1998; Vollrath, 1994). Having a deeper understanding of the function concept can help teachers to make knowledgeable decisions about the place of functions in the curriculum and create settings for their students to become aware of the function concept as a powerful mathematical idea. However, previous studies indicate that preservice teachers’ conceptions of function might possibly not be as well-developed as one may expect of proficient mathematics teachers (e.g., Even, 1993; Hansson, 2004; Wilson, 1994). The aim of the current study is to examine preservice teachers’ conceptions of the function concept, their conceptions of the significance of functions in mathematics as well the presence of functions in school mathematics. A further aim is to study the effects of an intervention concerning the concept of function.

RESEARCH QUESTIONS

Based on the aims of the study the research questions are: What conceptions do preservice teachers have of the function concept? How do preservice teachers view the significance of functions in mathematics? How do they perceive the presence of functions in school mathematics? How does the intervention make a difference to the preservice teachers’ conceptions of the function concept?
THEORETICAL FRAMEWORK

The framework relates to principles of constructivist learning where knowledge is an individual construction built gradually (seen as, e.g., Novak, 1993; Steffe & Gale, 1995). Understanding is described in terms of building mental structures where previous built structures affect subsequent constructions. To promote understanding includes important dimensions in mental activities of the learner such as constructing relationships, articulating what one knows, extending and applying mathematical knowledge, reflecting about experiences etc. A central part of the theoretical framework is the notion of concept image (Tall & Vinner, 1981; Vinner, 1983, 1992; Vinner & Dreyfus, 1989) as well as theory related to the notions of learning with understanding (Carpenter & Lehrer, 1999; Hiebert & Carpenter, 1992) and meaningful learning (Ausubel, 2000; Ausubel, Novak & Hanesian, 1978).

Ausubel (2000; Ausubel et al., 1978) describes learning in terms of meaningful learning as opposed to rote learning, where learning becomes meaningful when the learners are able to relate new knowledge to prior knowledge. In order to obtain successful learning students must acquire knowledge actively and establish relationships between concepts to be learned and the ones the students know. Hiebert and Carpenter (1992) refer to Ausubel’s theories as a “bottom-up approach” in the way knowledge develops building upon prior knowledge, and describe a view of learning with understanding similar to meaningful learning. That is, understanding grows as an individual’s cognitive structures become larger, more organized and richly connected networks of knowledge. This progressive process involves reconfigurations of cognitive structures where existing structures influence the relationships that are constructed. Understanding can be rather limited if only some of the mental representations of potentially related ideas are connected. These ideas are consistent with a more concrete approach by Carpenter and Lehrer (1999) considering how students construct meaning for mathematical concepts and processes and how classrooms support that kind of learning.

An individual’s understanding of the function concept and its correspondence to the definition of function is by Vinner (1983, 1992; Vinner & Dreyfus, 1989) described by utilizing the notion of concept image, which is applicable to formal concepts in general (Vinner, 1991). A concept image is in the current framework considered to consist of all parts of cognitive structure associated with a concept in the mind of an individual, “including all the mental pictures and associated properties and processes” (Tall & Vinner, 1981, p. 152). In thinking, different parts of the concept image are evoked at different times. During an individual’s reasoning, the concept image will almost always be evoked, whereas the concept definition will remain inactive or even be forgotten. When students meet an old concept in a new context, it is the concept image, with all the implicit assumptions abstracted from earlier contexts, that responds to the task.

The framework is further discussed below in relation to the intervention in the study.
METHOD AND PROCEDURE

The current study is part of a larger study related to the views of preservice teachers on the concept of function. It is conducted during two consecutive spring terms with two groups of preservice teachers, one group for each term, in relation to a course in calculus in which the function concept is a central concept. The calculus course is part of the concluding courses in mathematics during the sixth term of a four and a half year teacher preparation program. Both groups of preservice teachers are specializing in mathematics and science for grades four to nine. They comprise twenty-five and sixteen students, respectively. The study is conducted at a smaller university in Sweden and the two groups of preservice teachers make up all the third-year students specializing in mathematics and science on the education programme.

A questionnaire was completed on two occasions, before and after the calculus course. Twenty-two students in the first group and fifteen students in the second group answered the questionnaire on both occasions. The questionnaire comprises eight questions that the students typically spent half an hour or more to answer. Shortly after the second questionnaire was distributed volunteer preservice teachers were interviewed individually. The interviews were based on the questionnaire, and the preservice teachers were asked to study their responses from before and after the calculus course, for each question, and to comment on them. Follow-up questions were asked to clarify the preservice teachers’ reasoning. The time for each interview varied but often continued for an hour or more. Each interview was recorded on tape and transcribed. Twenty students in the first group and nine students in the second group were interviewed. The interviewed students represent, in each group, students on a variety of levels in their studies of mathematics.

Three of the questions in the questionnaire are considered in this paper. The first question is “Describe in your own words your interpretation of the concept of function”, the second is “Give your opinion about the extent to which functions are of significance in mathematics. Provide reasons for your opinion”, and the third question is “To what extent are, in your opinion, functions present in school mathematics? Provide reasons for your opinion”. The questions are open ended, and they frequently became a starting point for further reasoning about the function concept in mathematics and school mathematics by the preservice teachers during the interviews.

The second group of preservice teachers also participated in an intervention that was partly designed with respect to data from the first group in the study. The intervention is further described in a section below. (Because the study described in this paper is part of a larger study, more data was collected from both groups of

1 In this paper the preservice teachers of the study are sometimes called students, since they are students on the teacher preparation program.
2 There are approximately 10000 students enrolled at the university with about half of the students in full-time study.
preservice teachers during the period the study was conducted. For instance, concept maps, preservice teachers’ solutions to mathematical problems, video recordings of preservice teachers working in groups etc. These data are however not used in the current paper.)

**Categorizations used in the study**

The students’ conceptions of the function concept were categorized following a categorization scheme presented by Vinner and Dreyfus (1989). Furthermore, a categorization of the preservice teachers’ accounts of the significance of functions in mathematics was derived from a number of distinctive themes that appear in their answers to the questionnaire. The categorization of the significance of functions is further described in the section of results from the study.

Different groups of college students, in addition to inservice teachers in mathematics, participated in the study conducted by Vinner and Dreyfus (1989). A categorization based on their conceptions of the function concept was developed. The categorization consists of the following seven categories:

I. A function is any correspondence between two sets that for each element in the first set assigns exactly one element in the second set (the Dirichlet-Bourbaki definition).

II. A function is a dependence relation between two variables.

III. A function is a rule, which is expected to have some regularity (whereas a correspondence may be “arbitrary”).

IV. A function is an operation or a manipulation (one acts, e.g. on a given number, in order to get its image).

V. A function is a formula, an algebraic expression, or an equation.

VI. A function is identified with a representation, possibly in a meaningless graphical or symbolic form.

VII. Others.

The categories are derived from the answers to the questionnaire where the participants were asked to give their opinion on what a function is. The current study applies the categorization in the same manner as Vinner and Dreyfus, where an answer only is allotted to one of the seven categories, and thus makes it possible to compare results from the two studies. The categories are, however, not disjoint, and from the samples of answers that are presented by Vinner and Dreyfus, one may conclude that among the four categories II, III, IV, V a higher numbered category is generally given higher precedence in deciding to which category an answer will be assigned. Also from the given sample of answers, the four categories can often be seen as a sequence of decreasing subsets; this was also observed in the current study. Moreover, if an answer contains a description of function equivalent to the definition, then it is always allotted to category I. The categories VI and VII will be

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During the preparation of the current paper, Vinner and Dreyfus confirmed in a correspondence that they have not made any changes to the categorization or later developed it further.
considered if an answer is not applicable to any of the categories I-V.

THE STUDY BY VINNER AND DREYFUS

The participants in the study by Vinner and Dreyfus (1989) were drawn from several groups of first-year college students as well as a group of junior high school mathematics teachers in inservice training. The study makes it possible to compare a range of different groups of students and the preservice teachers in the current study. The college students majored in different areas and were divided into four groups by the level of mathematics courses required for their majors: Low level, 33 students majoring in industrial design; intermediate level, 67 students majoring in economics or agriculture; high level, 113 students majoring in chemistry, biology, or technological education; and mathematics level, 58 students majoring in mathematics or physics. Moreover, a fifth group consisted of 36 junior high school mathematics teachers.

<table>
<thead>
<tr>
<th>Category</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
<th>VII</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>2 (6)</td>
<td>12 (36)</td>
<td>4 (12)</td>
<td>2 (6)</td>
<td>6 (18)</td>
<td>4 (12)</td>
<td>3 (9)</td>
</tr>
<tr>
<td>Intermediate</td>
<td>12 (18)</td>
<td>18 (27)</td>
<td>9 (13)</td>
<td>1 (1)</td>
<td>13 (19)</td>
<td>6 (9)</td>
<td>8 (12)</td>
</tr>
<tr>
<td>High</td>
<td>17 (15)</td>
<td>36 (32)</td>
<td>9 (8)</td>
<td>7 (6)</td>
<td>8 (7)</td>
<td>11 (10)</td>
<td>25 (22)</td>
</tr>
<tr>
<td>Mathematics</td>
<td>26 (45)</td>
<td>12 (21)</td>
<td>7 (12)</td>
<td>3 (5)</td>
<td>3 (5)</td>
<td>3 (5)</td>
<td>4 (7)</td>
</tr>
<tr>
<td>Teachers</td>
<td>25 (69)</td>
<td>3 (8)</td>
<td>3 (8)</td>
<td>1 (3)</td>
<td>0</td>
<td>1 (3)</td>
<td>3 (8)</td>
</tr>
</tbody>
</table>

Table 1: Distribution of conceptions of the function concept for the participants in the study by Vinner and Dreyfus (1989), number and percentage x (y).

The study is conducted before the concept of function was introduced in the courses and focus on participants who gave an account of function that is consistent with the definition of function, that is category I. Due to limited space, the reader is directed to Vinner and Dreyfus (1989) for further details.

THE INTERVENTION

The intervention was conducted during the first two weeks of a five-week calculus course. I implemented the intervention, and the same teacher who had supervised the first group in the study supervised the remaining part of the course. The preservice teachers were informed about the intervention but not about the research questions of the project. The intervention also became apparent for the students for instance as a result of handouts that complemented or replaced the course literature during the execution of the intervention. Some ideas and purposes of the intervention that are related to the research questions in this paper are described below.

The concept of function was introduced as a special case of the concept of relation4. One reason for this approach was to give the students a concept to relate to with consequences for meaningful learning (Ausubel, 2000; Ausubel et al., 1978), and

4 A (binary) relation was given as an association of elements between two non-empty sets, rather than a subset of a Cartesian product. A function was thereafter given as a special case of relation; which for each element in the first set (the domain) gives exactly one corresponding element in the second set (the codomain), and thus satisfying the criterion of uniqueness.
learning with understanding (Carpenter & Lehrer, 1999; Hiebert & Carpenter, 1992), and appropriate language in which they could express themselves in situations of a more general context related to the concept of function. Based on experiences from the first group in the study (where the students’ conceptions of function frequently were limited to a formula, an algebraic expression, or an equation), the inclusive notion of relation was intended as an anchoring idea for the concept of function (Ausubel, 2000; Ausubel et al., 1978). The concept of relation was considered as an association between objects, in everyday situations as well as in mathematics, and as such thought to create opportunities for meaningful learning and render possible for the students to further develop their view on the concept of function.

An individual’s concept image of the function concept is constructed during a longer period of time and is based on experiences of all kinds of the concept. Different portions of the concept image, that is evoked concept images, are used at different times in an individual's reasoning (Tall & Vinner, 1981; Vinner, 1983, 1992; Vinner & Dreyfus, 1989). An individual’s concept image may be based on experiences that do not necessarily form a concept image that is consistent with the definition of the concept. One intention of the intervention was to stimulate feedback from evoked concept images to the concept definition. This would give the preservice teachers an opportunity to further develop their concept image of function in different contexts, and construct a concept image that is more consistent with the Dirichlet-Bourbaki definition of function. For this reason, the concept of function was given in a range of examples and problems with varying domain, codomain and representation, in applied as well as in non-applied contexts (which is coherent with results by, e.g., Vinner & Dreyfus, 1989; Schwarz & Dreyfus, 1995).

Feedback from an evoked concept image to the concept definition is usually not established with standard types of problems, according to Vinner (1992). This justifies the inclusion of examples and problems that traditionally are not given in a calculus course. For instance, functions between finite sets, the number of such functions, and the use of sets other than sets of numbers (this was mainly part of the first week of the intervention). The second\(^5\) week of the intervention was generally dedicated to real valued functions in one real variable and their different properties. Furthermore, the preservice teachers were given opportunities to reflect upon the concept of function and the presence of functions in mathematics and school mathematics, for instance in assignments with the task to formulate problems where the function concept was considered to be present to a varying degree. These are examples of activities consistent with principles of constructivist learning were preservice teachers are invited to reflect upon the concept of function and articulate what one knows about functions and their relations to other concepts etc. (Carpenter & Lehrer, 1999).

\(^5\) Further elaboration on the subject of preservice students’ view on properties of different classes of functions is planned to be presented in a future paper.
RESULTS

The concept of function

Before the calculus course, category V is the dominant category in the first group, while the students’ conceptions of the function concept are somewhat more evenly distributed among the categories in the second group with V and VI as the two largest categories, see Table 2. In both groups of students, the two categories V and VI together comprise half of the students’ answers before the course, that is answers reflecting a view of the function concept as a formula, an algebraic expression, an equation, or a representation possibly in a meaningless graphical or symbolic form. None of the students gives an account of a view that is consistent with the definition of function. The students’ conceptions of function are distributed over five categories in each group, where no answers are allotted to category III in the first group or category IV in the second group, that is answers describing a function as a rule or an operation, respectively. (However, this does not exclude the case where an answer allotted to a higher numbered category includes conceptions related to a lower numbered category, for the categories II-V). In comparison with the study by Vinner and Dreyfus (1989), the preservice teachers’ conceptions of function are in both groups, before the calculus course, more similar to students in the lower level groups than to students majoring in mathematics and physics or the inservice teachers.

After the calculus course, conceptions related to category V are yet more dominant in the first group than before the course, that is answers that give account of a function as a formula, an algebraic expression, or an equation. (However, these answers may also include parts related to categories II-IV, for example a dependency relation between two variables). There are still five categories represented in the first group, where category V now is more than twice as large as any other category. Moreover, answers related to category VI are no longer represented, that is conceptions of a meaningless graphical or symbolic representation of function. The students’ answers now also represent category III with conceptions of function as a rule. As before the course, none of the students’ answers in the first group describe the function concept in a way that is consistent with the contemporary definition of function.

While category V is the dominant category in the first group, both before and after the calculus course, there are more significant changes in the students’ conceptions of the function concept in the second group. The previously two largest categories V and VI are no longer part of the students’ answers after the course, while category IV now is the largest category in the second group, reflecting a view of function as an operation. (This is sometimes a less dramatic change in a student’s conception of function, where parts of a previous answer in category V may include conceptions related to category IV). After the course, four categories are covered by the students’
answers in the second group where categories V, VI, and VII are no longer represented. Only one student’s answer is consistent with the contemporary definition of function, that is a view of the concept which includes a domain, a codomain, and a correspondence that satisfies the uniqueness criterion. However, answers in category IV in the second group often contain conceptions of the function concept which involve an operation where the uniqueness criterion is made explicit but the answers lack the notion of domain and codomain, as well as the notion of set. The lack of domain and codomain is a significant reason for the large differences in the distribution of answers after the calculus course when the second group is compared to the group of inservice teachers in the study by Vinner and Dreyfus (1989).

Table 2 below displays the distribution of students’ answers for the 21 preservice teachers in the first group and 15 preservice teachers in the second group who gave their interpretation of function both before and after the calculus course. The table displays number and percentage x (y), respectively. (The number of students who gave their opinion on the concept of function both before and after the calculus course was reduced. This was due to students’ absence from lectures at the time the survey was carried out and due students who before the calculus course did not answer the question in the survey.)

<table>
<thead>
<tr>
<th>Category</th>
<th>I (x)</th>
<th>II (y)</th>
<th>III (z)</th>
<th>IV (a)</th>
<th>V (b)</th>
<th>VI (c)</th>
<th>VII (d)</th>
</tr>
</thead>
<tbody>
<tr>
<td>The first group Before</td>
<td>0</td>
<td>5 (24)</td>
<td>0</td>
<td>3 (14)</td>
<td>8 (38)</td>
<td>4 (19)</td>
<td>1 (5)</td>
</tr>
<tr>
<td>After</td>
<td>0</td>
<td>4 (19)</td>
<td>4 (19)</td>
<td>1 (5)</td>
<td>11 (52)</td>
<td>0</td>
<td>1 (5)</td>
</tr>
<tr>
<td>The second group Before</td>
<td>0</td>
<td>2 (13)</td>
<td>3 (20)</td>
<td>0</td>
<td>4 (27)</td>
<td>4 (27)</td>
<td>2 (13)</td>
</tr>
<tr>
<td>After</td>
<td>1 (7)</td>
<td>2 (13)</td>
<td>5 (33)</td>
<td>7 (47)</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 2: Distribution of students’ answers.

During the interviews it was common that the preservice teachers express an uncertainty in their interpretation of the function concept, and to a higher degree in the first group. It was not unusual with responses like “… I am still uncertain about functions. After five weeks of calculus, I am still uncertain about functions. It feels very silly, but that’s the way it is.” (F107, in the first group). The interviewed students in the second group could usually give a description of function that was more consistent with the contemporary definition when given an opportunity to reflect upon their two answers. However, this was not a conception of function that seems to be spontaneously evoked. Students in the first group gave the impression to be less aware of a definition of function, and during the interviews they were more likely to mix terminology related to the concept of function with concepts associated with category V, like the concept of equation, formula, or a mathematical expression. In their reasoning about functions, students in the first group also more frequently seem to think of functions in terms of symbolic manipulations and procedural techniques.

\[7 \text{ Code (F-female) used during data analysis}\]
Few students included the notions of domain and codomain in their description of the function concept, in the questionnaire or during the interviews, but usually related to numbers in the context of function. It was also typical for students from the first group to limit their view on different forms of representation of function to a symbolic or visual representation. Students in the second group may during the interviews however refer to examples of functions from the intervention with domain and codomain other than sets of numbers as well as a larger variety of representations. The notion of set was almost absent in the students’ answers to the questionnaire and seems to be a notion that forms an obstacle in the preservice teachers’ interpretation of domain and codomain as two components of the function concept. The students would during the interviews sometimes use terms like “bubbles” or “groupings” and so on, in the context of set, which made it obvious that they were not really accustomed to the concept of set.

During the interviews, the students, primarily in the second group, referred to the language used in mathematics – this was also a theme brought up by the students in relation to lectures during the intervention. The preservice teachers noticed that a chosen name for a concept did not always seem to have any meaning for them and did not help them to clarify the meaning of the concept. In this context the preservice students in the second group referred to the concept of relation as a concept they believed to be more true to their everyday experiences of the notion, than the notion of function. This was for instance the case for the student (M2) in the second group, who in response to the questionnaire after the calculus course described the notion of function in a way that is consistent with current characterizations of function. For him the concept of relation felt natural and like “a word that you recognize”, and he applied it in a variety of contexts during the interview, and then looked upon the concept of function as a special type of relation.

The significance of functions in mathematics

A number of distinctive themes appears in the preservice teachers’ answers to the question of the significance of functions in mathematics, which gives rise to the following categorization together with samples of answers:

1. Functions in different applications of mathematics: “Functions are ‘real’ math where relations can be described that are taken from reality”, “To theoretically be able to describe a practical example”, “Good examples are everyday situations as fuel consumption (the cost of this) for a certain distance [for a vehicle]”.

2. Presence of functions in mathematics: “Functions are part of quite a lot of mathematics, but you really don’t think about it”, “Most of mathematics is built by functions, so that’s why they are of importance”, “I guess that harder mathematics uses functions”.

3. Use of functions in mathematics: “Functions are useful to ‘see’ math!”, “They are used to solve problems in mathematics which can be solved both graphically and by
“calculations”, “To describe relations”.

4. Functions in the context of teaching and learning mathematics: “Pupils can learn to illustrate relations graphically”, “If the functions are related to the pupils’ interests, then I think its good”, “In ninth grade you (the pupils) don’t think it is relevant, I suppose”.

5. No opinion: “I don’t know!”, “?”, “I have forgotten how, when and why you use functions”.

6. Other: “... I think it is fun ... (Maybe more [people] do so, then it is significant!!)”, “Important to know what a function is to be able to consider yourself mathematical”, “You should be able to calculate with unknown numbers, variables”.

More than a third of the preservice teachers’ answers in both groups is allotted to category 5 before the calculus course. They are students who did not know how to answer a question on the significance of functions in mathematics. After the course all students gave an opinion on functions’ significance in mathematics, see Table 3 below. The table displays the distribution of students’ answers on functions’ significance in mathematics, number and percentage x (y). Participants are the 22 preservice students in the first group and 15 preservice students in the second group, who took part in the survey at both occasions before and after the calculus course. (In the survey, 6 students in the first group and 4 students in the second group did not give any answer to the question on the first questionnaire. They are assumed to have no opinion and have been allotted to category 5.)

<table>
<thead>
<tr>
<th>Category</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>The first group</td>
<td>Before</td>
<td>10 (45)</td>
<td>3 (14)</td>
<td>3 (14)</td>
<td>2 (9)</td>
<td>8 (36)</td>
</tr>
<tr>
<td></td>
<td>After</td>
<td>12 (55)</td>
<td>2 (9)</td>
<td>9 (41)</td>
<td>5 (23)</td>
<td>0</td>
</tr>
<tr>
<td>The second group</td>
<td>Before</td>
<td>3 (20)</td>
<td>0</td>
<td>3 (20)</td>
<td>1 (7)</td>
<td>6 (40)</td>
</tr>
<tr>
<td></td>
<td>After</td>
<td>8 (53)</td>
<td>6 (40)</td>
<td>3 (20)</td>
<td>2 (13)</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 3: Distribution of students’ answers on the significance of functions in mathematics.

In the first group, a major theme in the students’ answers to functions’ significance in mathematics is related to the use of functions in applications of mathematics, represented by category 1. Almost half of the students’ answers relate to category 1 before the calculus course; that is more than two-thirds of the students who actually expressed an opinion on the significance of functions in mathematics. The significance of functions in applications of mathematics is still a major theme after the calculus course. It is a theme that is part of more than half of the students’ answers in the first group; this represents however a smaller part of those students who actually gave an opinion on the significance of functions than before the course. The applications of functions that the students give are frequently related to the areas of science, usually physics. Furthermore, there is a threefold increase of answers related to the use of functions in mathematics, that is category 3, which often is connected to the use of functions in standard problems from the calculus course, and this is the second largest category after the course. Moreover, answers
that relate to teaching mathematics, category 4, have more than doubled after the course and represent almost one-quarter of the students’ answers, while answers in category 6 – answers that not always seem to make sense – have decreased.

In the second group, before the course, answers with no opinion on the significance of functions in mathematics, that is category 5, are twice as frequent as for any other category. After the course all students gave an opinion on the significance of functions in mathematics, where the two major themes are: functions in applications of mathematics, and the presence of functions within mathematics, that is categories 1 and 2. Half of the students’ answers relate to category 1 after the course, which is almost a threefold increase. Furthermore, more than one-third of the students’ answers relate to category 2, which is a category that did not exist before the course. Answers in category 2 are basically of two types of reasoning: the first type of reasoning is based on a quantitative view from the students’ experience of functions in different contexts, and the second type of reasoning is to a higher degree based on the ability to perceive the concept of function in different situations. The numbers of answers that concerns categories 3 and 6 have not changed in the second group, while answers related to teaching in mathematics, that is category 4, have increased although this is still a small category.

The preservice teachers perceived the significance of functions in mathematics motivated from their importance in different applications of mathematics and often in the area of science. Applications in the area of physics are more dominant in the first group both before and after the course. Whereas in the second group, applications of functions are less frequent before the course while the majority uses it as an argument in their answers after the course, but with fewer references to science than in the first group. Moreover, connections to teaching were only made by a few students in the two groups in relation to the significance of functions in mathematics, while more opinions on this subject were given in the subsequent question on the presence of functions in school mathematics.

During the interviews, students in the first group did to, a higher degree, relate to their experiences of the function concept in physics, as a formula etc., in the context of the significance of functions in mathematics. Moreover, students in the first group usually restricted their view of functions to real valued functions in one real variable. In the interviews with students from the second group, the use of functions in science was also a central part in their reasoning of the significance of functions in mathematics, but they did, in addition, relate to functions with domain and codomain other than sets of real numbers. From the interviews with students it became obvious, in both groups, that applications of functions in real world situations were an important factor in their reasoning about the significance of functions in mathematics.

In the context of applications of functions during the interviews students often seemed to relate to functions as a dependency relation between two quantities. After further consideration students frequently came to the understanding that such
relations are “all around us” and believed that this is something you usually do not talk about. Many students thought the question of functions’ significance in mathematics was a hard question to answer, and some students felt that they did not really have the knowledge to answer the question.

The presence of functions in school mathematics

In the preservice teachers’ accounts of the presence of functions in school mathematics, they often refer to: their experiences of teaching in compulsory school during teacher training, the literature used in compulsory school, mathematical problems they associate with compulsory school, and in some cases experiences of having their own children in school suggest. Leave with no explicit reference to the syllabus in their answers to the questionnaire. Almost half of the students who answered the questionnaire both before and after the calculus course did, before the course, not know how to answer or gave no opinion to the question of functions’ presence in school mathematic (that is nine students in the first group and seven students in the second group). All of the students gave an opinion to the question after the course.

In the first group of preservice teachers a large majority expressed the opinion in their responses to the questionnaire, that the concept of function is only part of school mathematics at the end of compulsory school, usually not until the last year. Some of the preservice teachers thought that functions are not present at all in compulsory school. Moreover, it is common for the preservice teachers in the first group to believe, that the function concept is a hard concept for compulsory students to learn, and quite frequently a concept that primarily is relevant for high achieving students. A minority of the preservice teachers, in the first group, perceives a high presence of functions in school mathematics, but “hidden” and not mentioned explicitly. In the preservice teachers’ perceptions of the presence of functions in compulsory school, they use different types of reasoning. A high perception of functions’ presence in school mathematics is based on interpretations of situations where the preservice teachers rely on their conceptions of function and make associations to the function concept. While a low perception of functions’ presence in compulsory school is based on the preservice teachers’ experiences of a context in which functions are explicitly stated.

In the second group in the study the majority of preservice teachers expressed in their responses to the questionnaire, in consistency with the first group, the opinion that functions are to a higher degree present in the later years of compulsory school. None of the preservice teachers expressed the opinion that functions are not present in school mathematics. In their reasoning on functions’ presence in school, preservice teachers in the second group more frequently seemed to distinguish between contexts where the concept of function is explicitly stated, and contexts in which they perceived the concept to be present but not openly mentioned.

During the interviews it was confirmed that preservice teachers who argue that the
presence of functions is low in school mathematics, base their reasoning on contexts where the function concept is explicitly stated. While preservice teachers who argue that there is high presence of functions in school mathematics, base their reasoning on their conceptions of function and situations which they relate to the concept of function like graphs, formulas, the dependence of two quantities etc. During the interviews, it was not uncommon that students, who gave the opinion that the function concept has low presence in school mathematics, changed their opinion when they were given an opportunity to reflect upon the presence of functions in school mathematics. Some students even became upset, and questioned why the concept was “hidden away” and not explicitly stated earlier in school. In their experience, the concept was introduced first at the end of compulsory school. Other students who reflected upon the presence of functions in school mathematics, became aware of more situations they thought the concept of function to be present than they previously had realized. In some cases, quite early in school, in relation to operations like addition and multiplication, and observed that this was something they never really had thought about or discussed in the context of school mathematics.

**CONCLUDING DISCUSSION**

The study implies that a more traditional calculus course may strengthen preservice teachers’ conceptions of the function concept that are inconsistent with current characterizations of function. While their conceptions of function are more similar before the course there are clear indications of different conceptions of the function concept between the two groups of preservice teachers after the calculus course. One significant difference between the groups is that a function is, after the calculus course, to a higher degree considered as an operation in the second group. This is in contrast to the first group, where the previously most frequent conceptions of function are strengthened after the course, that is conceptions of function as a formula, an algebraic expression, or an equation. Students in the first group are also less aware of a definition of function and more frequently tend to mix terminology related to other concepts in the context of functions. It was however observed during the interviews that the preservice teachers could vary their reasoning to some extent and focus on different aspects of functions in different contexts. For example a dependency relation seemed to be of higher relevance for the students in the context of applications of functions.

Few students, including students in the second group, gave an account of the notions of domain and codomain as two components of the function concept. The notion of set seemed in this context to be an obstacle in the preservice teachers’ interpretations of the definition of function. Students in the first group almost exclusively associated numbers with functions, that is real valued functions in one real variable, usually in a symbolic or graphical form. This is not the case in the second group in which students during the interviews were usually able to reason about functions with different representations and in contexts not always related to numbers.
Preservice teachers in the second group said they “recognized” the notion of relation from their everyday experiences and that the use of relation in mathematics felt in accord with their view of the concept as an association between objects. The students’ previous conceptions of the notion of relation seem to be important for their ability to reason about functions with domain and codomain other than sets of numbers and about functions with a variety of representations, in the case a function is known as a special type of relation. Students’ preexisting ideas of the notion of relation appear to be an essential conceptual infrastructure to which the notion of function can be anchored – in the sense of Ausubel (2000), analogous to the notion of cognitive root by Tall (1992) – and seem to have the potential to play an important role in students’ conceptual development of the function concept. The study indicates that with the notion of relation as an anchoring idea, the function concept is less likely to be learned by rote or limited to real valued functions or linked to a specific representation.

The significance of functions in mathematics is by the preservice teachers frequently justified by the importance of functions in different applications of mathematics. This is a major theme in the first group both before and after the calculus course, with reference to science and especially physics which is most likely a consequence of their experiences of functions in physics. In the second group, applications of functions also became a major theme after the calculus course, with reference to a variety of applications. The significance of functions in mathematics was for the students to a high extent connected to the ability to apply functions in different real world contexts. The students’ conceptions of functions’ significance in mathematics indicate that applications of functions are relevant in the development of their concept image of function, and a factor of importance to create opportunities for meaningful learning (Ausubel, 2000; Ausubel et al., 1978) and learning with understanding (Carpenter & Lehrer, 1999; Hiebert & Carpenter, 1992).

A majority of the preservice teachers considers functions to be part of school mathematics at the end of compulsory school. Their perception of functions’ presence in school mathematics is based on experiences of contexts in which the function concept is explicitly stated. It is however not uncommon that, during the interviews, further consideration of the presence of functions in school mathematics leads to a change in the preservice teachers’ view. This involves that they believe there is a high presence of functions in school mathematics, with reactions on why the concept is not made explicit earlier in school. This illustrates that reflections on the presence of functions in school mathematics are relevant for preservice teachers as a starting point for further reasoning on mathematics teaching, and might tap into teacher competencies that would possibly be suitable for further investigations.

The idea that teachers’ conceptions of function will influence the quality of the understanding developed by their students has received support from research findings (e.g., Even & Tirosh, 2002; Lloyd & Wilson, 1998). A teacher’s view on functions (for instance, as a formula, an algebraic expression, or an equation) will
most likely have consequences for the students’ conceptions of the function concept. One might expect both lesson goals and structures to be consistent with teachers’ views of functions, and thus expressed by examples they put forward, activities they design, questions they ask, and ideas they consider of value and so forth.

The current study illustrates that preservice teachers need support to develop their view of the function concept. In comparison with the study by Vinner and Dreyfus (1989), the preservice teachers’ views on the concept of function tend to be similar to students majoring in less mathematics intense fields. This might further indicate that preservice teachers need opportunities to further develop their conceptions of function and meet functions in a variety of contexts. Contexts that provide opportunities for preservice teachers to realize that the concept of function is a central concept in mathematics and an important concept to introduce to their future students.

References


