Path-Planning with Obstacle-Avoiding Minimum Curvature Variation B-splines

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Abstract

We study the general problem of computing an obstacle-avoiding path that, for a prescribed weight, minimizes the weighted sum of a smoothness measure and a safety measure of the path. We consider planar curvature-continuous paths, that are functions on an interval of a room axis, for a point-size vehicle amidst obstacles. The obstacles are two disjoint continuous functions on the same interval. A path is found as a minimizer of the weighted sum of two costs, namely 1) the integral of the square of arc-length derivative of curvature along the path (smoothness), and 2) the distance in $L^2$ norm between the path and the point-wise arithmetic mean of the obstacles (safety).

We formulate a variant of this problem in which we restrict the path to be a B-spline function and the obstacles to be piece-wise linear functions. Through implementations, we demonstrate that it is possible to compute paths, for different choices of weights, and use them in practical industrial applications, in our case for use by the ore transport vehicles operated by the Swedish mining company Luossavaara-Kiirunavaara AB (LKAB). Assuming that the constraint set is non-empty, we show that, if only safety is considered, this problem is trivially solved. We also show that properties of the problem, for an arbitrary weight, can be studied by investigating the problem when only smoothness is considered. The uniqueness of the solution is studied by the convexity properties of the problem. We prove that the convexity properties of the problem are preserved due to a scaling and translation of the knot sequence defining the B-spline. Furthermore, we prove that a convexity investigation of the problem amounts to investigating the convexity properties of an unconstrained variant of the problem. An empirical investigation of the problem indicates that it has one unique solution. When only smoothness is considered, the approximation properties of a B-spline solution are investigated. We prove that, if there exists a sequence of B-spline minimizers that converge to a path as the number of B-spline basis functions tends to infinity, then this path is a solution to the general problem. We provide an example of such a converging sequence.
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To Charlotte
Part I

Introduction and Summary
Chapter 1

Introduction

This licentiate thesis\textsuperscript{1} consists of four parts of which the last three are research papers\textsuperscript{2}. This first part contains an introduction, a summary of contributions, and future work. This chapter gives an introduction to our field of research and describes in detail the problems studied. It is divided into four sections. First, Section 1.1 contains, in more general terms, a motivation for, and a background to, the problem of planning a path. In Section 1.2, we discuss some preliminaries and restrict the scope of the licentiate thesis. Then, in Section 1.3, we review some related work. Finally, in Section 1.4, we introduce the general path-planning problem that we would like to solve, together with some of its properties and its relation to previous work. We also present a discretized version of the problem.

The results put forward in this licentiate thesis are summarized in the form of a list in Chapter 2. The possible future extension of them is treated in Chapter 3 which ends this introductory part of the licentiate thesis.

1.1 Background

Path-planning is a problem area that grows more and more important as the level of automation increases [20, 22, 29, 45]. Today, there are many industrial applications that require the pre-computation of paths for autonomous vehicles and robot arms to follow and move along. In this licentiate thesis, we address the problem of finding an obstacle-avoiding path that combines the properties of being both smooth and safe.

The work space of autonomous vehicles and robot arms are often cluttered with obstacles that these machines must avoid. An example is the

\textsuperscript{1}This thesis is submitted for the degree of licentiate, which is a Swedish degree between a Master of Science and a PhD.

\textsuperscript{2}The papers are reproduced in their original form. However, a common format has been applied to all of the papers causing some cosmetic changes in their layout in order to fit the format.
path-planning of the autonomous vehicles used for transporting iron ore in the underground mines of the Swedish mining company Luossavaara-Kirunavaara Aktiebolag (LKAB) [36]. It is crucial that these autonomous vehicles avoid the obstacles constituted by the mine walls in order for the production to go on. Another example is a robot arm holding a cutting tool that follows a pre-computed path. It should perform its task while staying away from other machines and it should not damage itself.

Much work has been devoted to planning paths in the presence of obstacles and there are many results yielding piecewise linear paths [20, 21]. However, practical applications involving physical machines also require that the paths are smooth [2, 46]. Consider the autonomous ore transport vehicle of LKAB. It is an articulated vehicle, which means that the whole vehicle is involved in the steering. Its steering gear is worn out by even small jerks in its path. Moreover, such jerks easily cause the machine to drop ore on the road which eventually must be cleared before the transports may continue. The turning speed of the transport vehicle is limited by its speed. Its speed is therefore adjusted such that the vehicle is capable of following the path. The smoother the path, the higher speed of the vehicle, which directly increases the production rate. Another example of the importance of smoothness comes from the offset path of a cutting tool. Its path must be curvature-continuous to avoid jumps in the acceleration that damage the driver motors.

There are many definitions of smoothness. Depending on context, a path or curve is referred to as being smooth if it is tangent continuous, has a continuous curvature, or even has a continuous derivative of curvature. [37]. Smooth curves are of importance not only to achieve paths that do not cause unnecessary wear. They have also found application in designing curves that look aesthetically pleasing. Such applications are found in the car, airplane, and film industry [16, 39].

Still, obstacle-avoidance and smoothness is not enough for practical purposes. The path should also be safe. The smoother the path, the larger the radius of the curve. This means that a path will tend to go close to obstacles in order for it to have as high smoothness as possible. Once again, consider the ore transportation vehicle that should avoid the walls of the mine and follow an as smooth path as possible. A smooth path makes it go close to the walls of the mine when turning. As touching the walls is the main danger for the vehicle, it should stay close to the walls as short time as possible. In this case, the path is most safe if it is centered between the mine walls. Centering of a path or a tool between obstacles has been considered in the literature [38].
1.2 Preliminaries

This study is on the fundamental properties of the problem of planning a path that combines the qualities of being smooth and safe while avoiding a set of obstacles. An exact definition of what this means is given in Section 1.4.2. The goal of our study is to gain a deep theoretical understanding about the problem in order to solve it and to enable others to make use of the solutions in practice. Many of the definitions and much of the notation used here can be found in standard textbooks [19, 29, 42].

A vehicle – autonomous or not – has bounds on its ability to steer and the steering is closely related to its traversed path. Assuming that the vehicle has a steering-wheel, the angle and the angular velocity of the steering-wheel directly corresponds to the curvature and the derivative of curvature of the path, respectively. Although important to apply our results in practice, we leave out the many problems there are of making a real vehicle or robot operate. We do not treat hardware issues, like sensors and actuators. Nor do we treat software issues on how to make a robot navigate and how to control its movement. We rather focus on making the path suitable for an ideal machine capable of following any smooth and safe path.

We consider planar paths for point-size vehicles moving at unit-speed with prescribed initial and terminal endpoint constraints. By a unit-speed path is meant a curve that is parameterized in its arc-length [42]. Taking the real sweep area of a vehicle into account when planning the path yields a more realistic but also a much more complicated problem. The sweep area can be taken into account by extending the borders of the obstacles before the actual path-planning is begun. In Part II we present a method of doing so, but from here on, this issue is left aside. Furthermore, from here on, we make no difference between a path and a curve. Depending on context, endpoint constraints are position, slope angle or tangent, curvature, and arc-length derivative of curvature of the curve. Constraints like these are also referred to as endpoint conditions, postures, or endpoint configurations.

Due to the constraints imposed on the path, an optimal path is found as a minimizer of a certain cost function or functional. We formulate our problem by defining a general cost function over smooth curves which is then approximated by a discretized cost function defined over a subset of these smooth curves. This approximation is explained in more detail in Section 1.4.4. Through the implementation of algorithms that solve the discretized problem, we have, which is shown in Part II, successfully computed paths based on industrial data from LKAB. These paths have been tested with good results in a simulator used by LKAB, which means that the paths can be used in the control of the real mining vehicles. The implementation is written in \textsc{Matlab}\textsuperscript{3} and it involves, e.g., constrained nonlinear

\textsuperscript{3}\textsc{Matlab} is a trademark of The MathWorks, Inc.
programming routines, quadrature routines, and the computation of curves.

Many problems and questions concerning to the implementation and the use of it, has to do with optimization theory, which constitutes a research area of its own [19]. Such questions concern, for example, the choice of initial value, convergence rate and termination of the solver. Even though these are interesting issues, except from in the discussion on future work in Chapter 3, we do not treat these topics here. The implementation shows that our paths can actually be computed based upon our ideas of practical applications. We use it to solve instances of our discretized problem in order to study the problem and its properties.

1.3 Previous Work

Path-planning is an area that has been widely studied [20, 22, 29, 45]. Early research on path-planning with a smoothness aspect considered the primary cost as being the length of the path. The goal was to minimize the length of the path subject to constraints on various curvature properties. This research was boosted in 1957 when Dubins [15] presented an algorithm for computing a shortest path between two postures in the plane for a vehicle having a limited curvature, i.e., a maximum angle of its steering-wheel. He showed that such a path consists of the concatenation of line segments and arcs of circles. Dubins’ seminal work has since then been extended for other more capable vehicles but still confined to line segments and arcs of circles [5, 6, 9, 10, 17, 23, 33, 43, 47].

The drawback of a such a path is that its curvature is not continuous at the joints connecting the lines and arcs of the circles. It changes abruptly from one (constant) value to another. This means that the angle of the steering-wheel has to change momentarily whenever a joint is passed. Assuming that the vehicle is moving at a certain speed, this is neither physically possible nor something to wish for. Sudden jerkily changes in curvature causes heavy wear on the steering-gear of the vehicle.

Algorithms for computing shortest paths with continuous curvature\textsuperscript{4} were published during the second half of the 1980:s [30, 32]. If there are limitations in both the curvature and the derivative of curvature, the shortest path between two postures in the plane consists of line segments, arcs of circles, and clothoids [7, 27]. During the 1990:s algorithms for computing such paths were developed [28]. A clothoid is a curve with the curvature being a linear function of its arc-length [6, 7]. Thus, inserting a proper part of a clothoid between an arc of a circle and a straight line segment maintains continuity of curvature. Clothoids have been successfully applied for control purposes and they are used in practice to create bends in roads [2, 46].

\textsuperscript{4}Curvature-continuity is required for regulating a vehicle using a feedback controller [35].
1.3. PREVIOUS WORK

However, a path that is a concatenation of line segments, arcs of circles, and clothoids, cause both a maximum angle and discontinuities in the angular velocity of the steering-wheel of a vehicle traversing the paths. Even though it is continuous in curvature, its derivative of curvature jumps momentarily between its limit values. Such sudden changes in the derivative of curvature, or the angular velocity of the steering wheel, also cause jerks that contribute to the wear of the steering-gear of the vehicle [25].

Since the end of the 1980:s research has been made in the field of planning paths with a higher degree of smoothness than that of the paths mentioned above. This was done by considering other curves like B-spline functions [26], quintic polynomials [49], and polar splines [41]. It was also done by changing the cost function of the problem. Instead of primarily focussing on the minimization of curve length, the focus was put on the smoothness itself. In 1989, Kanayama and Hartman [24] proposed two related cost functions over paths between two postures in the plane. The first cost was the integral over the square of curvature and the other cost was the integral over the square of derivative of curvature. They showed that the minimizers of these cost functions are the concatenations of clothoids and the concatenations of the so-called cubic spirals, respectively. A cubic spiral or a cubic is a curve with a curvature that is a quadratic function of its arc-length. The last decade, further research was made concerning the properties and the applications of the cubic and curves related to it [13, 40].

None of the problems mentioned above consider the avoidance of obstacles. In the general case, taking obstacles into account yields a more complex problem. Unless there are some simplifications, it is a difficult problem to know whether and where a curve intersects another curve or not. In the general case, there are infinitely many positions along the curve itself – and also along the obstacles – that has to be checked, in order to know if the path is feasible or not. However, there are previous work done when generating paths among obstacles. Some of this work is done in a manner where there is a brute initial guess, which is later on iteratively refined in some of its critical parts where the path is not feasible [3, 18, 31, 48]. Another way is to constrain the representation of the obstacles. This is done in for example the problem of computing shortest path with bounded curvature. The obstacles are constrained to be convex and as smooth as the path ("moderate obstacles") [8]. Then the problem of planning a path is transformed into a graph problem. When a higher degree of smoothness is required, the approach of moderating the obstacles is more difficult and has, to our knowledge, not been studied further. Due to the difficulties in handling constraints required for obstacle-avoidance, little is done in the field of path-planning with obstacle-avoiding curvature-continuous curves. The difficulties persists with or without bounds on curvature or derivative of curvature. However, the issue is discussed and remains of interest [34].

Moving on to the safety aspects addressed in Section 1.1, we consider
yet another cost function, namely a cost that strives to center a path between obstacles. If centering is the only concern, then the (safe) path, with respect to the Euclidian metric in the plane, is found as a part of a Voronoi diagram \[29\]. However, in the general case, the Voronoi diagram is not curvature-continuous and is therefore unsuitable for representing a smooth curve. In 1999, Lutterkort and Peters \[38\] proposed a method to compute curvature-continuous curves centered between an upper and a lower obstacle. They considered a subset of smooth curves, namely the B-spline functions \[12\], and the obstacles were restricted to being polygonal chains.

1.4 Problem formulation

As we have learned, there are many different ways to approach the problem of planning a path. In this licentiate thesis we address variants of the problem of planning an obstacle-avoiding curvature-continuous path that combines measures of smoothness and safety. We refer to a path as smooth when it is designed to have a small curvature variation along the path. In detail, this means that the integral over the square of derivative of curvature along the path is small. We refer to a path that has been deliberately designed to avoid a set of obstacles as safe. The level of safety is considered as being high if the path is far away from its nearest obstacle. In our problem formulation, the notion of safety is explained in more detail.

This section is divided into four subsections. First, we point out the restrictions made when addressing our problem together with the distinctions of our problem in relation to previous work. Second, we pose the general path-planning problem addressed in this licentiate thesis and present some of its properties. Third, we describe why a B-spline function is a good and suitable approximation to the general curve sought for. Fourth, we pose the discretized version of the general problem. The discretization is done by restricting the curves to being B-spline functions and restricting the obstacles to being polygonal chains. The idea with addressing the discretized problem is to indirectly gain knowledge about the general problem. Furthermore, the discretization allows a computation of smooth paths. In Parts II, III, and IV, we study the discretized problem.

1.4.1 Distinctions and restrictions

Here, we present important distinctions and restrictions made for the problem addressed in this licentiate thesis to put it in comparison with the problems described in previous work. The meaning of these distinctions become clearer with the formulation of the problem.

Curvature-continuous curves over an interval of a room axis. The subset of paths considered here are curvature-continuous that can be
parameterized not only in their arc-length, but also as functions over an interval of a room axis in a Euclidian coordinate system, i.e., as functions $y = f(x)$. These paths are consistent with our smoothness measure.

**Upper and lower obstacles.** We consider obstacles as being two non-intersecting curves that are continuous functions of $x$. These curves need not be curvature continuous. The path is intended to lie in between these two curves.

**Smoothness.** We consider the same smoothness measure as Kanayama and Hartman [24], namely the integral over the square of arc-length derivative of curvature along the path.

For a function $f(x)$ with curvature $K(x)$, $x \in [x_0, x_1]$, there is the one-to-one relation $ds = \sqrt{1 + f'(x)^2} \, dx$, between $x$ and arc-length $s$. Therefore, letting $\dot{\nu} = d\nu/ds$ and $\nu' = d\nu/dx$, this measure can be written as

$$
\int_{s(x_0)}^{s(x_1)} \dot{K}(s)^2 \, ds = \int_{x_0}^{x_1} \frac{K'(x)^2}{\sqrt{1 + f'(x)^2}} \, dx.
$$

**Safety.** There is a curve that is exactly centered in between the two obstacles. We call it the center curve and measure safety through the distance, in $L^2$ norm, between the path and the center curve.

**Other distinctions and restrictions.** We do not consider bounds of curvature or bounds of derivative of curvature of the path. Neither do we introduce the length of the curve as a cost.

Our idea is to take the whole path into account in our planning process instead of concatenating path segments, that are locally optimal, into a suboptimal path. In practice, for example in the case with mining vehicles, these operate in considerably straight connected tunnels, where the paths can be written as functions over one room axes. If it is not possible to fulfill all of our requirements between two postures in the mine, it is possible to produce a suboptimal path by inserting intermediate postures. In the mine, the length of the path can not vary to a great extent. It is instead the smoothness of turns that allows the vehicle to maintain a high speed. While smoothness requirements makes a curve come close to the obstacles, safety requirements makes it stay away from them. In practice, bounds on curvature are important and they are discussed as future work in Chapter 3. Our aim is to compute a path with as high smoothness and safety as possible.

**1.4.2 A general problem formulation**

In its general form the situation we study consists of two flanking obstacles stretched out next to each other. We model these using two disjoint contin-
uous functions \( l(x) \) and \( u(x) \) (the lower and upper obstacles), respectively, defined on a real interval \( I = [x_0, x_1] \) such that \( l(x) < u(x) \) for all \( x \in I \). Given certain endpoint constraints on \( f(x) \) and its derivatives, the problem is to compute a curvature-continuous function \( f(x) \) defined on \( I \) such that

a) The path \( f(x) \) satisfies given endpoint constraints.

b) The path \( f(x) \) lies in between the two obstacles, i.e., \( l(x) \leq f(x) \leq u(x) \).

c) The path \( f(x) \) minimizes the (weighted) cost function

\[
\lambda \int_I \frac{K'(x)^2}{\sqrt{1 + f'(x)^2}} \, dx + (1 - \lambda) \int_I (f(x) - \Theta(x))^2 \, dx,
\]

where \( \Theta(x) = (l(x) + u(x))/2 \) is the center function between \( l(x) \) and \( u(x) \), 
\( \nu'(x) = d\nu/dx \), \( K(x) = f''(x)/(1 + f'(x)^2)^{3/2} \) is the curvature of \( f(x) \), and \( \lambda \in [0, 1] \) is a given constant (the weight). We refer to this problem as the general problem.

The cost function (1.1) has two terms. Smoothness is expressed through the first term and safety is expressed through the second term. The path \( f(x) \), that minimizes the cost function, can, through different choices of \( \lambda \), be balanced between high and low levels of safety and smoothness. The value of \( \lambda \) is a design parameter that is considered to be given before solving for \( f(x) \) and it can be chosen as desired depending on the context.

We consider the following questions regarding the general problem:

1. **Existence.** Is there a function \( f(x) \) that minimizes the object function?

2. **Uniqueness.** Is the function \( f(x) \) unique?

3. **Efficiency.** Can \( f(x) \) be computed efficiently?

We directly distinguish two special cases of the design parameter \( \lambda \) that makes our problem similar to problems considered in previous work, namely

\( \lambda = 0 \). Omitting the restriction of curvature-continuity, the path is trivially found to be the center function, i.e., \( f(x) = \Theta(x) = (l(x) + u(x))/2 \). But, as we require curvature-continuity of the path, there has to be additional restrictions on the curvature of the obstacles for the existence of a solution. Otherwise, we have a degenerate case, c.f., the discussion by Dubins [15].

\( \lambda = 1 \). Omitting the obstacles, the path is a cubic [25]. Considering obstacle-avoidance makes the problem more complicated. In general, to our knowledge, the minimizer in this case is unknown.
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We have not found any previous work on how to solve the general problem in the case when $\lambda \in (0, 1)$.

Since we do not know how to find an exact solution, we turn our attention to approximations of the minimizer through a simplification of the general problem. The idea is to indirectly gain knowledge about the general problem and to being able to answer the questions posed above. Furthermore, it allows us to compute a path.

We want this approximation to have all of the properties listed in below:

1. It should be able to satisfy the endpoint constraints.
2. It should in an easy manner be able to avoid obstacles.
3. It should have enough an intrinsic smoothness, i.e. at least be curvature-continuous.
4. It should be efficiently computed.
5. It should be flexible in such a way that accuracy of the approximation can be increased or decreased.

There are numerous curve representations and we consider the B-spline function.

1.4.3 B-splines for path-planning

B-spline functions [11, 12, 14] meet up with the demands on path approximations that we stated in the previous section. A B-spline function or B-spline $B(x)$ is a piecewise polynomial of a certain degree. Here, we consider quartic, i.e., degree 4, uniform B-splines that are functions of $x$. Quartic B-splines are sufficiently smooth for our purposes as they are continuous even in their derivative of curvature.

By means of its envelope [38, 44], a B-spline function can avoid obstacles $l(x) \leq u(x)$ that are piecewise linear functions. The envelope of $B(x)$ is a pair of piecewise linear functions $\varepsilon(x)$ and $\tau(x)$, given by $B(x)$, such that $\varepsilon(x) \leq B(x) \leq \tau(x)$. In order for the B-spline to avoid the obstacles it suffices to impose the constraints $l(x) \leq \varepsilon(x)$ and $\tau(x) \leq u(x)$. This, in turn, is the key that opens the door to obstacle-avoidance of the approximation $B(x)$. The constraints $l(x) \leq \varepsilon(x)$ and $\tau(x) \leq u(x)$ can be formulated as a finite number of linear constraints as they amount to comparing the piecewise linear functions, representing the obstacles, at their respective vertices. Then, instead of having to compare the function itself, at an infinite number of points, against the obstacles, we have finitely many constraints depending only on the number of vertices of the envelope and the vertices of the piecewise linear obstacles.

B-splines are written in closed form, and they are also efficiently computed [12]. Another thing that makes B-splines attractive is the ease by
which the shape of the resulting curve can be controlled. Their approximation ability and the properties of their shape depend on the number of B-spline basis functions used to define \( B(x) \). The larger number of basis functions \( B(x) \) is built from, the more flexible its representation. For these reasons, B-splines are also widely used in a variety of contexts such as data fitting, computer aided design (CAD), automated manufacturing (CAM), and computer graphics [16].

### 1.4.4 A discretized problem formulation

The discretized version of the general problem of Section 1.4.2 is stated as follows.

Given as input are two flanking obstacles that are piecewise linear functions \( L(x) \leq U(x) \), defined on a real interval \( I = [x_0, x_1] \), certain endpoint constraints, and a certain number, \( n \), of B-spline basis functions. The problem is to compute a quartic uniform B-spline \( B(x) \) defined on \( I \) such that

a) The B-spline \( B(x) \) satisfies the given endpoint constraints.

b) \( L(x) \leq e(x) \) and \( e(x) \leq U(x) \).

c) The B-spline \( B(x) \) minimizes the (weighted) cost function

\[
\lambda \int_I \frac{K'(x)^2}{\sqrt{1 + B'(x)^2}} \, dx + (1 - \lambda) \sum_{i=1}^{n} (\Theta(t_i^*) - b_i)^2,
\]

where \( K(x) = B''(x)/(1+B'(x)^2)^{3/2} \) is the curvature of \( B(x) \), \( b_i, i = 1, \ldots, n \), are called B-spline coefficients of \( B(x) \), and \( t_i^* \), \( i = 1, \ldots, n \), are called the Greville abscissae [12]. The value \( \lambda \in [0, 1] \) is a given constant (the weight).

The sum to the right of (1.2) is a discretization of the integral for centering in the general problem [38]. It is a measure of the distance between the path \( B(x) \) and the center function \( \Theta(x) = (U(x) + L(x))/2 \). The choice of \( \lambda \) is, also in this case, a given design parameter balancing the smoothness and the safety of the resulting path. We refer to this problem as the discretized problem. A variant of this problem is stated in more detail in Part II.

The B-spline is uniquely determined by the B-spline coefficients so solving the problem for \( B(x) \) amounts to finding the optimal B-spline coefficients. Furthermore, all constraints of the discretized problem are written as linear constraints in terms of the B-spline coefficients [38]. This yields a convex constraint set, i.e., a convex set of B-splines over which we seek a minimizer.

There are two differences between the general problem and the discretized problem. First, the curvature-continuous path is restricted to being a quartic uniform B-spline. Second, the obstacles are polygonal chains instead of general continuous functions. Polygons can approximate every
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general parameterized curve arbitrary well and they have a countable and finite representation [1].

We have not yet shown that a solution of the discretized problem approximates a solution of the general problem, but a study of the approximation properties of the discretized problem is made in Part IV. However, if the polygonal obstacles have infinitely short edges and the number of B-spline basis functions tends to infinity, the discretized problem resembles the general problem stated in Section 1.4.2.

We know already that the B-spline is effectively computed and its properties are well-known, so the questions stated at this point are:

1. **Existence.** Is there a B-spline \( B(x) \) that minimizes the object function?

2. **Uniqueness.** Is the B-spline \( B(x) \) unique?

3. **Approximation properties.** What are the differences between an optimal B-spline \( B(x) \) of the discretized problem compared to an optimal solution \( f(x) \) of the general problem?

In order to seek the answers to these questions, we consider two special cases of \( \lambda \), namely:

\( \lambda = 0 \). If we have a nonempty constraint set, the B-spline is trivially found by \( b_i = \Theta(t_i^*) \), \( i = 1, \ldots, n \). This results in a B-spline close to the center function \( \Theta(x) \). Note that the cost function is in this case convex with respect to the B-spline coefficients. As the constraint set is also convex, the whole problem is convex, and the therefore there is only one unique solution to this problem [4].

\( \lambda = 1 \). In this case we only consider the smoothness of the B-spline. We call this problem the minimum curvature variation B-spline problem and we pay it special attention in the next section. A solution to this problem is referred to as a minimum variation B-spline (MVB). In practice, using sufficiently many basis functions to have a non-empty constraint set, we are able to solve instances of this problem.

We are also able to compute solutions to instances of the discretized problem when \( \lambda \in (0, 1) \). This is useful in order to investigate properties of the problem.

**The minimum curvature variation B-spline problem**

An investigation of the properties of the minimum curvature variation B-spline problem, i.e., the discretized problem of the previous section, where \( \lambda = 1 \), is of great interest in the study of the whole discretized problem, i.e.,
where $\lambda \in [0, 1]$. We already know that the centering problem, i.e., when $\lambda = 0$, is convex and therefore has a unique solution.

When $\lambda = 1$ we have the so-called minimum curvature variation B-spline cost function. If this cost function is convex (or quasi-convex), then together with the convexity of the centering cost function, i.e., $\lambda = 0$, the whole discretized cost function is convex (or quasi-convex). If this is the case, together with the convex constraint set (which is assumed to be non-empty), the whole discretized problem has one unique solution. This is an important reason for investigating the convexity properties of the minimum curvature variation cost function for B-splines. Such an investigation is performed in Part III.

This minimum curvature variation B-spline problem is also interesting from another point of view. If we omit the obstacles, i.e., assume that $L(x) = -\infty$ and $U(x) = +\infty$, then we have exactly the same cost function and setting as the one yielding the cubics [25]. This means that we have a setting in which we already know the optimal solution $f(x)$ of the general problem of Section 1.4.2, i.e. a cubic. In turn, this means that we can compare a B-spline minimizer $B(x)$ with an optimal solution $f(x)$. This can be done for a sequence of problems with an increasing number of B-spline coefficients in order to investigate the approximation properties of the B-spline solutions. If the approximation properties of B-splines in this special case are not satisfactory, then they are probably not satisfactory when it comes to solutions of the general problem in the general case. A study of the approximation properties of B-splines in a part of this case is treated in Part IV.
Chapter 2

Summary of contributions

In this chapter, we list our contributions in the study of the problems described in the previous chapter. The contributions are ordered by the papers that constitutes the last three parts of this licentiate thesis.

Part II: Automatic Generation of Smooth Paths Bounded by Polygonal Chains

- A first formulation of the discretized problem, cf., Section 1.4.4, of computing an obstacle-avoiding path, that combines the properties of smoothness and safety, is given. The paths are restricted to being quartic B-splines and the obstacles are restricted to being polygonal chains. Smoothness is defined as small magnitude of the integral over the square of derivative of curvature along the path. Safety is defined as the degree of centering of the path between the polygonal chains.

- An implementation of an algorithm for solving the discretized problem is made. The implementation is made in MATLAB and uses a standard nonlinear programming solver that handles constraints.

- A test on application data provided by the Swedish mining company LKAB is made. The test demonstrates that the path-planning method works in practice.

- A method for generating a safety margin is proposed and implemented. It preprocesses the polygonal chains in order to remove excess parts and it introduces safety margins for a vehicle.

Part III: An Obstacle-Avoiding Minimum Variation B-spline Problem

- The minimum variation B-spline problem is introduced. This is the problem of computing a uniform quartic B-spline function that, for
given endpoint constraints and due to the constraint of lying between two polygonal chains, minimizes the integral of the square of arc-length derivative of curvature along the curve. The solution is called a minimum variation B-spline. In this licentiate thesis we refer to this problem as the minimum curvature variation B-spline problem, which is a special case, i.e., when \( \lambda = 1 \), of the discretized problem of Section 1.4.4.

- The uniqueness of the minimum variation B-spline is studied through an investigation of the convexity of the minimum variation B-spline problem. If a problem is convex, then it has one unique solution. A proof is given, showing that, if a particular unconstrained minimum variation B-spline problem is convex, then so is also the general minimum variation B-spline problem. Therefore, it suffices to study the convexity properties of the particular problem in order to study the convexity properties of the general minimum variation B-spline problem.

- A proof is given, showing that, for any B-spline function, the convexity properties of the problem are preserved subject to a scaling and translation of the knot sequence defining the B-spline.

- An empirical investigation is made, indicating that the minimum variation B-spline problem has one unique solution.

**Part IV: Epi-Convergence of Minimum Curvature Variation B-splines**

- A proof, showing that, if there exists a sequence of quartic uniform B-spline minimizers of the curvature variation functional that converge to a curve as the number of B-spline basis functions tends to infinity, then this curve is a minimizer of the curvature variation functional. Therefore, if such a sequence exists, then the B-spline can be found as a good approximation to a general smooth curve that minimizes the curvature variation functional. The curvature variation functional is the integral over the square of arc-length derivative of curvature along a planar curve. It is also the smoothness measure of the problems in Section 1.4. The idea is to compare a B-spline minimizer with a general smooth curve, that is a curve from a superset of the B-spline curves, that minimizes the curvature variation functional.

- An example of a sequence of computed B-spline minimizers that converge to a cubic spiral is shown.
Chapter 3

Future Work

The formulation of the general problem of Section 1.4.2 is new. It relates to on-going research in diverse fields like robotics, computer aided design, numerical analysis, theoretical computer science, and computer graphics. We have successfully managed to compute smooth and obstacle-avoiding paths. We have done so using real data supplied by LKAB. Furthermore, we have studied the uniqueness of these paths and their approximation properties.

In this chapter, we pose questions, suggest, and discuss suitable future work. We begin with a large scope and end by zooming in on the discretized problem.

A reformulation of the general problem

We use functions \( f(x) \) over an interval of a room axis to model paths. This has the drawback that we can only compute paths constrained by obstacles that are also functions on the same interval. It would be interesting to generalize our general problem and consider paths and obstacles parameterized in their arc-length. This means that our measure of safety or centering needs to be changed. Would it be possible to discretize such a problem with B-splines and polygonal chains?

The general problem

For the general problem posed in Section 1.4.2 there are still the open problems:

- Under which circumstances does there exist a solution? Always?
- Is there but one solution to each instance of the problem?
- What class of functions does the solutions belong to? What about the special cases when \( \lambda = 0 \) or \( \lambda = 1 \)?
B-splines and polygonal chains

In the licentiate thesis, we escape from the hardness of the general problem by formulating a discretized problem, cf., Section 1.4.4, limiting the solution to being a B-spline and by restricting the obstacles to being piecewise linear functions. A natural question is then:

- Is there some other way than to employ B-splines, envelopes and polygonal obstacles to find an approximate solution to the general problem?

While the discretized problem enable us to compute solutions it also introduce interesting problems. Here we propose future work for the discretized problem but many of the issues brought up in this section are also of interest in the general problem.

Improvements of the model

In practice, there are limitations on the steering capabilities of a vehicle and it does not necessarily move at constant speed. It would be of interest to model:

- Limited curvature (angle of the steering-wheel).
- Limited derivative of curvature (angular velocity of the steering wheel).
- Variable speed (discrete/continuous).
- Geometry of the vehicle (sweep area).

How are these changes of the model performed? What is the impact of changing the model? How about existence and uniqueness of a solution? Are we able to compute a solution?

Existence and uniqueness

As was shown in Section 1.4.4, properties of the entire discretized problem can be found by studying the curvature variation functional over B-splines. Two main questions are still unanswered, namely

- Given a nonempty constraint set, is there a B-spline function that minimizes the curvature variation functional?
- If there exists a B-spline minimizer of the curvature variation functional, is it unique?

We have begun to study these issues indirectly by an investigation of the convexity and the approximation properties of the discretized problem. Are there other ways of answering these questions?
Approximation properties

As we show in Part IV, B-splines have some good approximation properties in a symmetric endpoint setting without other constraints. The questions raised due to this result are:

- Is there a converging sequence of B-spline minimizers in the symmetric case?
- In that case, is it unique? This question is strongly connected to the question on the uniqueness of a B-spline minimizer mentioned in the previous section.
- What if we consider general endpoint constraints and pose the same questions as above?
- What happens if we also consider polygonal obstacles? How much will the envelope contribute to an error in the approximation?
- How much better is the approximation if we add one B-spline basis function?
- How many B-spline basis functions should be used to obtain a non-empty constraint set?
- Given the polygonal chains and a number of B-spline functions; what is a good knot placement? We have mostly considered uniform knot sequences.
- Given a certain error limit between the B-spline approximation and the optimal solution; what is the lowest number of knots needed and how should these knots be placed?

Computation issues

In practice, when implementing the algorithms for solving the discretized problem, a number of questions are raised, e.g.,:

- What is the rate of convergence of our algorithms?
- Is it reasonable to expect an answer that is (numerically) correct?
- Is it possible to bound the number of iterations used by the solver in terms of the initial value or other properties of the instance of the problem?
- What quadrature routine is best fitted to use in the solver with respect to time and accuracy? We have tested a few, but there are certainly others. There is maybe some approximations that can be made as we are dealing with B-splines?
• How should the initial value (the seed of the solver) be chosen? The convergence of the solver is faster if the initial value is chosen close to the solution.

• In what ways is it possible to preprocess the obstacles such that we have a simpler problem to solve, while the solutions approximate each other well?

More extensive simulations should be performed and the algorithms should be studied more thoroughly.
Bibliography


Part II

Automatic Generation of Smooth Paths Bounded by Polygonal Chains
Automatic Generation of Smooth Paths Bounded by Polygonal Chains

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Abstract

We consider the problem of planning smooth paths for a vehicle in a region bounded by polygonal chains. The paths are represented as B-spline functions. A path is found by solving an optimization problem using a cost function designed to care for both the smoothness of the path and the safety of the vehicle. Smoothness is defined as small magnitude of the derivative of curvature and safety is defined as the degree of centering of the path between the polygonal chains. The polygonal chains are preprocessed in order to remove excess parts and introduce safety margins for the vehicle. The method has been implemented for use with a standard solver and tests have been made on application data provided by the Swedish mining company LKAB.

1 Introduction

We study the problem of planning a smooth path between two points in a planar map described by polygonal chains. Here, a smooth path is a curve, with continuous derivative of curvature, that is a solution to a certain
optimization problem. Today, it is not known how to find a closed form of an optimal path. Instead, we give a good approximation using B-spline functions. Even though the emphasis of this paper lies on path-planning with mine maps as input, our solution is generally applicable to any map described by polygonal chains.

1.1 Background

The Swedish mining company LKAB is using unmanned autonomous vehicles for underground ore transportation. An on-board control system guides each vehicle along precomputed paths. The paths are described in a planar map of the mine that consists of polygonal chains.

A fully loaded vehicle has a weight of about 120 tons and its maximum speed is about 20 km/h. The larger the vehicle and the faster its speed, the greater is the strain put upon its construction and the smaller is the margin for error. This puts a high demand on the smoothness of precomputed paths as the steering gear of a vehicle is worn out more quickly if there are fast changes in the curvature of its path (high speed of the turning wheel). Today, the paths are handmade and, according to LKAB, the existing path-planning is time-consuming and not always satisfactory. Several successive refinements are needed to save the vehicle and the road.

Generating smooth paths is one step towards the company’s goal of lowering the ore production costs. Smooth paths allow a high speed, but more important, give low maintenance costs both for the vehicles and the road.

1.2 Related work

The path-planning problem involves planning a collision-free path for a vehicle or a robot moving amid obstacles. It is one of the main problems in robotics and has been widely studied [1, 2]. Dubins [3] was the first to study curvature-constrained shortest paths. His paths are concatenations of straight lines and arcs of circles and his theories have been extended for various problems [4, 5, 6, 7].

Dubins’ paths have discontinuous curvature yielding excessive wear of the steering gear of a vehicle following them (with nonzero speed). Path-planning with bounded derivative of curvature has been studied by, for example, Boissonnat et al. [8] and Kostov and Degtiariova-Kostova [9]. These authors work with paths formed by a concatenation of straight line segments and arcs of clothoids. Such paths have a higher degree of smoothness than Dubins’ paths, but their explicit computation tend to be difficult. Lutterkort and Peters [10, 11] present a method for computing smooth paths in polygonal channels that depends on their bound on the envelope of a B-spline function. This bound was recently extended by Reif [12].
1.3 Our contribution

Restricting the path to being a B-spline function [13], we apply the result of Lutterkort and Peters. Their envelope is suitable for our purpose, but their technique demands that the mine map is decomposed into channels. A channel is a pair of polygonal chains that are strictly monotone with respect to the same parametrization used for their corresponding B-spline function. We assume that the pairs of polygonal chains are given and solve our path-planning problem for each pair separately. Different algorithms for decomposing a simple polygon into pairs of monotone polygonal chains are presented by, for example, Keil [14] and Liu and Ntafos [15].

In order to account for a predefined safety margin and the size of the vehicle, the commonly used Minkowski sum [16] is applied to fatten the original polygonal chains. Optionally, polygon approximation techniques can be used with the objective of lowering the number of vertices of the fattened polygonal chains. A lower number of vertices decrease the time complexity of our path-planning. The generation of a safety margin is treated in Section 2.

In Section 3, we formulate our path-planning problem as an optimization problem that can be solved using standard nonlinear programming solvers. The cost function, based on experience at LKAB, has been designed to give smooth paths as well as concern about vehicle safety by a centering term. An example of our path-planning technique is shown in Section 4, followed by concluding remarks in Section 5.

2 Generating a safety margin

We use the Minkowski sum [16] on a polygonal chain to account for the size of a vehicle and to impose a safety margin. The result, in turn, needs to be a polygonal chain for use in our optimization problem. In order to decrease the time complexity of the path-planning, the number of vertices in the new polygonal chain should be minimized.

Our problem can be formulated as; Given a polygonal chain \( C \) construct another polygonal chain \( C' \), containing no point closer to \( C \) than \( \tau \) and no point farther from \( C \) than \( \tau + \epsilon \), having as few vertices as possible. A sketch of the problem is shown in Figure 1. We solve a restricted version of this problem where the resulting polygonal chain has its vertices at distance \( \tau + \epsilon/2 \) from \( C \). Our resulting polygonal chain is the solution to a polygon approximation problem on \( \beta \), that is a polygonal chain built from sufficiently frequent sample points at distance \( \tau + \epsilon/2 \) from \( C \).

Let \( d(p, p') \) be the Euclidean distance between the points \( p \) and \( p' \). Furthermore, let the distance between the two polygonal chains \( B \) and \( B' \) be defined by \( D(B, B') = \max_{p \in B} \min_{p' \in B'} d(p, p') \). The polygon approximation problem is: Given a polygonal chain \( B \) with \( n \) vertices and an error
3 Generating smooth paths

Our paths are generated by solving a nonlinear optimization problem. The input is given as a pair of monotone polygonal chains, $c < c'$, that correspond to the permitted region for a point-sized vehicle. We are also given start and endpoint configurations of the sought path, $S$ and $T$ respectively, defined by position and velocity. We look for a path combining smoothness and closeness to a center function between $c$ and $c'$.

Our resulting path is a B-spline function $b(z) = \sum_{j=0}^{m} b_j N_j^d(z)$ [13]. The B-spline coefficients are found by solving our optimization problem in which $x = [b_0, \ldots, b_m]^T$ is the vector of unknowns. The B-spline basis functions $N_j^d$ of degree $d$ are defined by a recursion formula and a knot sequence of $z$-values, namely $t_0 \leq t_1 \leq \cdots \leq t_{m+d+1}$. We consider the knot sequence fixed. The Greville abscissae $t_i^* = \sum_{k=i+1}^{i+d+1} t_k/d$, $i = 0, \ldots, m$, are the abscissae of

Figure 1: The polygonal chain $C'$ approximates $C$. The minimal distance between a point on $C'$ and a point on $C$ lies between $\tau$ and $\tau + \epsilon$. 

bound $\delta$, find a polygonal chain $B'$, consisting of a minimal length subsequence of the vertices of $B$, such that $D(B, B') \leq \delta$. A solution to this problem is presented by Iri and Imai [17]. They build a graph $G$ by extending $B$ with edges for all valid shortcuts from one vertex $v_i \in B$ to another vertex $v_j \in B$. A shortcut is said to be valid if all vertices $v_k \in B$, $k = i, \ldots, j$, are at distance less than $\delta$ from the straight line connecting $v_i$ and $v_j$. Finding $B'$ with minimum number of vertices is equivalent to finding the minimum number of edges in $G$ that connect $v_1$ and $v_n$. Since $G$ is a directed and acyclic graph, this can be done in a straightforward manner using, for example, dynamic programming techniques.

We apply the polygon approximation solution by Iri and Imai [17] to our restricted problem by letting their original polygonal chain $B$ equal our sampled polygonal chain $\beta$ and their error bound $\delta$ equal $\epsilon/2$, see Figure 1. Note that there is no guarantee for our resulting polygonal chain to have less vertices than the original chain. A special case where a high number of vertices is needed is seen in Figure 2.
Figure 2: The polygon approximation algorithm does not guarantee a reduction in the number of vertices of the resulting polygonal chain (dashed) when $\epsilon$ is small compared to $\tau$.

the vertices of the control polygon $l$ for which $l(t_i^*) = b_i$. We denote the (signed) curvature of $b$ by $K_s = b''/(1 + (b')^2)^{3/2}$ and its derivative with respect to $z$ by $K'_s$.

Using B-splines and the Lutterkort and Peters’ envelope [11, 10], $\underline{c} \leq b \leq \overline{c}$, it is possible to ensure the continuous inequality $\underline{c} \leq b \leq \overline{c}$. The B-spline function $b$ lies between the pair of polygonal chains $\underline{c} \leq \overline{c}$ if both $\underline{c} \leq \underline{e}$ and $\overline{e} \leq \overline{c}$ hold for the finitely many vertices of the envelope and the finitely many vertices of the polygonal chains. This is illustrated in Figure 3. We work with B-splines of degree $d = 4$ as this is the highest degree, corresponding to the most smooth B-spline function, for which the envelope is shown according to Lutterkort and Peters.

We formulate an optimization cost function combining both smoothness and centering of the path. Smoothness of a path is measured by its derivative of curvature, $K'_s$. The difference between the control polygon $l$ and a center function $c = (c + \overline{c})/2$ at discrete values of the parameter $z$ is a measure of the centering of the path. These two measures are combined by the given weight $\lambda$, $0 \leq \lambda \leq 1$, to form our nonlinear cost function

$$F(x) = \lambda \int_{z_S}^{z_T} (K'_s)^2 \, dz + (1 - \lambda) \sum_{i=0}^{m} (c(t_i^*) - b_i)^2,$$  \hspace{1cm} (1)

where $z_S$ and $z_T$ are the values of the parameter $z$ at configurations $S$ and $T$.

In order to compute (1) we have made the discretization

$$F_d(x) = \lambda \sum_{j=1}^{m} (K'_s(z_j))^2 (z_j - z_{j-1}) + (1 - \lambda) \sum_{i=0}^{m} (c(t_i^*) - b_i)^2,$$  \hspace{1cm} (2)
where the evaluation points $z_S = z_0 < z_1 < \cdots < z_n = z_T$ are chosen to give a good approximation of the integral.

Our optimization problem is $\min_x F_d(x)$ s.t. $x \in \Omega$. We use a constraint set $\Omega$ in which the strict nonlinear definitions of $\Delta_1^+$ and $\Delta_1^-$, posed by Lutterkort and Peters, hold, see Ref. [10] for details. This yields an $\Omega$ which is a nonlinear counterpart of the linear constraint set proposed by Lutterkort and Peters. Furthermore, $\Omega$ contains constraints imposed by the start and target configurations, $S$ and $T$. We have experienced trouble solving the problem resulting from the combination of Lutterkort and Peters’ linear constraint set and (2), but our nonlinear problem can be solved with standard solvers. We use the center function $c$ to produce an initial value $x = x_0$ for the solver by $b_j = c(t_j^*)$, $j = 0, \ldots, m$.

4 Path generation example

Selected parts of our proposed method to automatically design paths for vehicles have been implemented and tested on application data from mining industry. The technique is visualized through an example of a path generation.

In Figure 4(a) a part of a mine production area (dotted), a start configuration $S$, and a target configuration $T$, are shown. The shaded area in the figure is bounded by a pair of monotone polygonal chains connecting $S$ and $T$. 

Figure 3: Sketch of a corridor $c \leq \bar{c}$ and a B-spline function $b$ with control polygon $l$ and envelope $e \leq \bar{e}$. The B-spline function is contained in the corridor, i.e. $c \leq e \leq b \leq e \leq c$. 

Selected parts of our proposed method to automatically design paths for vehicles have been implemented and tested on application data from mining industry. The technique is visualized through an example of a path generation.
A safety margin is added to each of the two monotone chains. The result is simplified and shown with dashed lines in Figure 4(b).

An optimal smooth B-spline path (solid) of degree \(d = 4\) having its envelope inside the simplified polygonal chains is computed and shown in Figures 4(c) and 4(a). We have used a uniform knot sequence and a design parameter \(\lambda\) yielding a path both smooth and to some extent centered between the walls of the mine.

5 Concluding remarks

Our main result is a proposal and an implementation of an automatic method for the generation of smooth paths. The method depends on the minimization of a specific cost function. Our cost function is a combined measure of both smoothness and safety. We define smoothness as the derivative of curvature of the path and safety as the degree of centering of the path between two polygonal chains. The resulting path is a B-spline function minimizing the cost function while still having its envelope inside a permitted region. The permitted region is generated from some original polygonal chains and accounts for the size of a vehicle and a safety margin.

The method has been tested on maps produced from sample data. Our tests show that smooth paths can be obtained in a straightforward manner by applying a standard nonlinear optimization solver on our cost function together with our constraints.

Future work include a comparison between the value of our cost function for paths produced with our method and the corresponding value for handmade paths. It would also be interesting to compare the optimality of our paths with paths that need not be described by B-spline functions only. For better understanding of the properties of the produced optimal path, further investigation of the number and placement of B-spline knot points as well as the definition of the center function is needed. Today, our proposed safety margin assumes a circular shape of the vehicle. This is a crude model as an LKAB vehicle has an extension strongly depending on the curvature of its path. Incorporating the sweep area of the vehicle in the optimization would give a more accurate modeling of the safety margin. The safety margin generation, with its proposed polygon approximation algorithm, could be improved by canceling the restriction of having vertices of the resulting polygonal chain at a specific distance from the original polygonal chain.

We conclude that, even though there is considerable work still to be done, our proposed technique is useful for automatic generation of smooth paths.
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References


(a) Mine production area (dashed) with a smooth path between start and target configurations, S and T.

(b) A pair of polygonal chains (dashed) as the result of a generation of two safety margins.

(c) Smooth path generated between the pair of polygonal chains.

Figure 4: An example of a path generation.
Part III

An Obstacle-Avoiding Minimum Variation B-spline Problem
An Obstacle-Avoiding Minimum Variation B-spline Problem

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Abstract

We study the problem of computing a planar curve, restricted to lie between two given polygonal chains, such that the integral of the square of arc-length derivative of curvature along the curve is minimized. We introduce the Minimum Variation B-spline problem which is a linearly constrained optimization problem over curves defined by B-spline functions only.

An empirical investigation indicates that this problem has one unique solution among all uniform quartic B-spline functions. Furthermore, we prove that, for any B-spline function, the convexity properties of the problem are preserved subject to a scaling and translation of the knot sequence defining the B-spline.

1 Introduction

There are at least two areas dealing with the construction of smooth curves. One area concerns path-planning for autonomous vehicles and robots, see e.g. [9, 5, 13, 2]. A smooth path is easy for a vehicle to follow and yield low wear on for example the vehicle steering gear [9]. The other area deals with curve and surface design and reconstruction [12]. In this case, the smoothness is, for example, used to make surfaces resemble real objects having a smooth shape or to create an appealing form.

The literature contains different ways to define smoothness. One of the more natural definitions comes from minimizing the square of the arc-length derivative of curvature along the curve. This results in a curve that is known as a minimum variation curve (MVC) [12].

Consider any curve in the plane and let \( K(s) \) denote the curvature at arc-length \( s \) along it. Then the curve is an MVC if it minimizes the cost
function
\[
\int \left( \frac{d}{ds} K(s) \right)^2 ds,
\]
subject to constraints on curve position, and optionally, curve tangent and curvature, at certain points along the curve. We refer to the problem of computing an MVC as the \( \text{MVC-problem} \) \( \left( \text{P}_{\text{MVC}} \right) \).

Even though an MVC has pleasing properties it also possesses some negative ones. Solely imposing constraints on the position, direction, and curvature of the curve at its endpoints, the MVC is described by an \textit{intrinsic spline} of degree 2 [5]. The position of an intrinsic spline can not be written as a closed form expression and it is costly to compute. More importantly, to our knowledge, it is not known how to effectively bound it in order for the curve to avoid, i.e. not intersect, obstacles such as other curves.

Another class of curves that are used for describing smooth curves are the so called \textit{B-splines} [4, 6], which are defined in terms of \textit{B-spline bases}. One thing that makes B-splines attractive is the ease by which the shape of the resulting curve can be controlled. For this reason, B-splines are widely used in a variety of different contexts such as data fitting, computer aided design (CAD), automated manufacturing (CAM), and computer graphics [7, 15]. Another advantage of B-splines compared to intrinsic splines is that the former can be given in closed form. It is also possible to bound them in the plane by piecewise linear envelopes in terms of the parameters describing the splines, as shown by Lutterkort and Peters [10].

\textbf{Contribution}

In contrast to the general MVC-problem, mentioned above, we introduce the \textit{minimum variation B-spline problem} and some variants of it. These problems are linearly constrained optimization problems minimizing the same cost function as in (1) but for curves restricted to being B-spline functions only.

We study the problem of how to compute a uniform B-spline of degree 4 (a quartic B-spline) in the plane restricted to lie between two given polygonal chains such that the integral of the square of arc-length derivative of curvature along the curve is minimized. In particular, we are interested in the convexity properties of the problem since we want to solve it in practice [2]. If the problem is convex, there is only one local minimum and a solution computed by a numerical solver is expected to be the global minimum [1].

We investigate if there is a unique solution to this problem by studying the convexity properties of the related problems. We undertake an empirical investigation that indicates that it has one unique solution in cases involving 1 up to 20 B-spline bases. Furthermore, we prove that the convexity properties of all the problems are preserved subject to a scaling and translation of the knot sequence defining the B-spline.
2 The problem of computing an obstacle-avoiding uniform quartic B-spline

In this section we define the main problem that we study. We formulate it as an optimization problem with a cost function $\hat{f}_B(b)$ being minimized over the vector $b$ subject to a set of linear constraints (the complete formulation can be found on page 41).

2.1 B-splines and their minimum variation cost function

The cost function involves a measure of smoothness, which is based on the arc-length derivative of curvature along the curve. The curve is defined as a B-spline function. Let $B(b, x) = \sum_{i=1}^{m} b_i N_{i,d}(x) \in R$ be a B-spline function (B-spline) of degree $d$. It is defined by the B-spline basis functions $N_{i,d}(x)$ [4, 6], the B-spline coefficient vector $b = [b_1, \ldots, b_m]^T \in R^m$, and a non-decreasing knot sequence $\tau = \{\tau_i\}_{i=1,...,m+d+1}$. We refer to the number of B-spline bases $m$ as being the dimension of the problems that we study.

In particular, we consider two different quartic B-splines, i.e. B-splines of degree 4. These are $\hat{B}(b, x)$ and $\tilde{B}(b, x)$ defined by the two uniform knot sequences $\hat{\tau} = \{\hat{\tau}_i\}_{i=1,...,m+5}$ and $\tilde{\tau} = \{\tilde{\tau}_i\}_{i=1,...,m+5}$ respectively. These sequences are defined by

$$\hat{\tau}_i = \hat{\tau}_1 + (i-1)\Delta, \quad i = 2, \ldots, m + 5. \quad (2)$$

$$\tilde{\tau}_i = \begin{cases} \hat{\tau}_1, & i < 5, \\ \hat{\tau}_i, & i = 5, \ldots, m + 1 \\ \hat{\tau}_{m+1}, & i > m + 1. \end{cases} \quad (3)$$

where $\hat{\tau}_1, \Delta \in R$. Note that $\hat{\tau}$ has 5 multiple knots at each end.

The curvature $K_B(b, x)$ (see O'Neill [14]) of $B(b, x)$ is given by

$$K(s) = K_B(b, x) = \frac{\frac{\partial^2}{\partial x^2} B(b, x)}{(1 + (\frac{\partial}{\partial x} B(b, x))^2)^{\frac{3}{2}}}. \quad (4)$$

Our measure of smoothness of a curve is based on the arc-length derivative of curvature $d/ds(K(s))$. Using $ds = \sqrt{1 + (\partial/\partial x(B(b, x)))^2} dx$ we find that

$$\frac{d}{ds} K(s) = \frac{\frac{\partial}{\partial x} K_B(b, x)}{\sqrt{1 + (\frac{\partial}{\partial x} B(b, x))^2}}. \quad (5)$$

It can be derived, using the recursive definition of $B(b, x)$ [4], that 4 is the least degree of $B(b, x)$ for which $\partial^3/\partial x^3(B(b, x))$, and thereby the derivative of curvature $d/ds(K(s(x)))$, is continuous with respect to $x$. Thus, for a curve defined by a general B-spline $B(b, x)$ with knot sequence $\tau$, the minimum
variation B-spline cost function $f_B(b)$ is given by

$$f_B(b) = \int_{\tilde{t}_1}^{\tilde{t}_{m+1}} \frac{\left( \frac{\partial}{\partial x} K_B(b, x) \right)^2}{\sqrt{1 + \left( \frac{\partial}{\partial x} B(b, x) \right)^2}} \, dx.$$  

(6)

Note that this is the same cost function as (1) but for a B-spline.

2.2 Linear constraints

We now consider a uniform quartic B-spline $\tilde{B}(b, x)$, defined by $b$ and $\tilde{t}$ from (3). The set of constraints consists of two parts. The first part stipulates boundary conditions by requiring that the value of $\tilde{B}(b, x)$ and its first three derivatives are equal to given constants at the end points, $\tilde{t}_1$ and $\tilde{t}_{m+5}$, of the curve. These constraints are:

$$\begin{cases}
\frac{\partial}{\partial x} \tilde{B}(b, \tilde{t}_1) = \alpha_k, & k = 0, \ldots, 3, \\
\frac{\partial}{\partial x} \tilde{B}(b, \tilde{t}_{m+5}) = \beta_k, & k = 0, \ldots, 3,
\end{cases}$$

where $\alpha_k$ and $\beta_k$, $k = 0, \ldots, 3$, are given constants.

The second part makes sure the curve does not intersect two given obstacles defined as non-intersecting polygonal chains which in this case are piecewise linear functions of $x$ with domain $[\tilde{t}_1, \tilde{t}_{m+5}]$. These chains are denoted $\tau$ and $\sigma$, where $\tau(x) \leq \sigma(x)$, $x \in [\tilde{t}_1, \tilde{t}_{m+5}]$. The region between the chains constitutes the region in the plane which contains the curve. To ensure that the curve does not intersect the obstacles we use an envelope to bound it and require that the envelope does not intersect the obstacles. The envelope also consists of two piecewise linear functions $\epsilon(b, x)$ and $\bar{\epsilon}(b, x)$ that are uniquely defined by $\tilde{B}(b, x)$ [10]. For every $x \in [\tilde{t}_1, \tilde{t}_{m+5}]$, they satisfy $\epsilon(b, x) \leq \tilde{B}(b, x) \leq \bar{\epsilon}(b, x)$. This is used for the second part of the constraints which enforces

$$\begin{cases}
\epsilon(x) \leq \epsilon(b, x), & x \in [\tilde{t}_1, \tilde{t}_{m+5}], \\
\bar{\epsilon}(b, x) \leq \bar{\epsilon}(x), & x \in [\tilde{t}_1, \tilde{t}_{m+5}].
\end{cases}$$

(8)

2.3 The optimization problem

We summarize the outline in the preceding sections as follows:

**Problem P_O** (The obstacle-avoiding uniform quartic minimum variation B-spline problem):

Given a knot sequence $\tilde{t}$ (according to (3)), two polygonal chains $\tau(x) \leq \sigma(x)$, and constants $\alpha_k$ and $\beta_k$, $k = 0, \ldots, 3$, compute

$$\min_{b \in \mathbb{R}^n} \int_{\tilde{t}_1}^{\tilde{t}_{m+5}} \frac{\left( \frac{\partial}{\partial x} K_{\tilde{B}}(b, x) \right)^2}{\sqrt{1 + \left( \frac{\partial}{\partial x} B(b, x) \right)^2}} \, dx$$
Figure 1: A B-spline function that is computed to solve an instance of $P_O$. The B-spline is built from 10 B-spline bases and its envelope and the obstacles of the problem are shown in the figure.

such that

$$
\frac{\partial^k}{\partial x^k} \tilde{B}(b, \tilde{t}_1) = \alpha_k, k = 0, \ldots, 3,
\frac{\partial^k}{\partial x^k} \tilde{B}(b, \tilde{t}_{m+5}) = \beta_k, k = 0, \ldots, 3,
\epsilon(x) \leq \epsilon(b, x), x \in [\tilde{t}_1, \tilde{t}_{m+5}],
\bar{\tau}(b, x) \leq \bar{\tau}(x), x \in [\tilde{t}_1, \tilde{t}_{m+5}],
$$

where $\tilde{B}(b, x)$ is the uniform quartic B-spline defined by $b$ and $\tilde{t}$, $\epsilon(b, x) \leq \bar{\tau}(b, x)$ is the envelope and $K_{\tilde{B}}(b, x)$ is the curvature of $\tilde{B}(b, x)$.

The last two constraints can be expressed by constraints linear in $b$ [10]. The number of such constraints is no more than $6m - 2$ plus the number of vertices in the obstacles. These constraints express the fact that since the computed curve is contained in its envelope it is enough to require that the envelope lies between the obstacles in order to ensure that the computed curve does not intersect the obstacles. Note that, as the derivatives of $\tilde{B}(b, x)$ are also linear in $b$, all constraints are linear in $b$. Figure 1 shows a computed solution to an instance of $P_O$, in which $\tilde{B}(b, x)$ is built from 10 B-spline bases. It also visualizes how $\tilde{B}(b, x)$ is constrained to lie between the obstacles, i.e. the two polygonal chains, by means of its envelope.

3 Convexity properties of $P_O$

We are interested in if there is a unique solution to $P_O$. The problem has a unique solution if it is convex. In this section we show that a convexity
investigation of $P_O$ can be reduced to a convexity investigation of a simpler problem defined using one particular knot sequence.

3.1 Related problems

Below we define three problems that relate to $P_O$. Except from the first problem they are used to investigate if there is a unique solution to $P_O$.

**Problem $P_{MVB}$:** The *minimum variation B-spline problem* is defined as $P_{MVC}$ but the curve is restricted to being a B-spline function $B(b, x)$ with knot sequence $\tau$.

**Problem $P$:** The *uniform quartic minimum variation B-spline problem* is a subproblem to $P_{MVB}$. It is defined in the same way as $P_O$ (see Section 2.3), but the constraints on the envelope of $\tilde{B}(b, x)$ are omitted. Therefore, $P_O$ itself is in fact a subproblem of $P$. Since $P_O$ is defined for $\tilde{B}(b, x)$ on the knot sequence $\tilde{t}$, so is also $P$.

**Problem $P_T$:** The *trivial uniform quartic minimum variation B-spline problem* is stated in the same way as $P_{MVB}$ with the restriction that it only considers quartic B-splines $\hat{B}(b, x)$ on the knot sequence $\hat{t}$ according to (2). This knot sequence implies that the value of the B-spline and its three first derivatives vanish at the endpoints $\hat{t}_1$ and $\hat{t}_{m+5}$. In turn, there is always the trivial solution to $P_T$, namely $b = [0, \ldots, 0]^T$ yielding $\hat{B}(b, x) \equiv 0$.

The relations between the problems of this paper are outlined in Figure 2. Each subproblem is presented with its additional restrictions or constraints compared to the problem on its left hand side.
3.2 Problem $P_O$ is convex if $P_T$ is convex

We show that investigating the convexity of $P_O$ can be done by investigating the convexity of the simpler problem $P_T$, for which the global minimum (the zero function) is known.

**Lemma 1** Assume that $P_T$ is a convex problem. Then $P_O$ is convex.

**Proof:** First we introduce a new problem, $P_{\hat{T}}$, which we get from $P_T$ by adding the same constraints as those in $P$ at the points $\hat{t}_5 = \check{t}_5$ and $\hat{t}_{m+1} = \check{t}_{m+1}$ (see (7)). This implies that the values of the B-spline and its first three derivatives are specified at those points. Imposing linear constraints on a convex problem yields a problem that is convex. Hence, by the assumption that $P_T$ is convex, it follows that $P_{\hat{T}}$ is convex.

A B-spline function $\hat{B}(\hat{b}, x)$ defined on knot sequence $\check{t}$ (as in $P_{\hat{T}}$) is closely related to a B-spline function $\hat{B}(\check{b}, x)$ defined on the knot sequence $\check{t}$ (as in $P$). It can be verified, by applying the recursive definition of the

---

**Figure 3:** An example of when $\check{B}(\check{b}, x) \equiv \hat{B}(\check{b}, x)$ over $x \in [\check{t}_5, \check{t}_{m+1})$. 

---
B-spline, that, \( \hat{B}(\tilde{b}, x) \equiv \hat{B}(b, x), \ x \in [\hat{t}_5, \hat{t}_{m+1}] \), if and only if

\[
\begin{bmatrix}
\tilde{b}_1 \\
\tilde{b}_2 \\
\tilde{b}_3 \\
\tilde{b}_4 \\
\end{bmatrix}
= 
\begin{bmatrix}
\frac{1}{7} & \frac{11}{7} & \frac{11}{7} & \frac{1}{7} \\
0 & \frac{1}{5} & \frac{7}{12} & \frac{7}{12} \\
0 & 0 & \frac{4}{5} & \frac{4}{5} \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4 \\
\end{bmatrix},
\]

\( \tilde{b}_j = b_j, \ j = 5, \ldots, m - 4, \)

The cost function of \( \tilde{P} \) can be split into three terms

\[
\int_{\hat{t}_5}^{\hat{t}_1} g(b, x)dx + \int_{\hat{t}_5}^{\hat{t}_{m+1}} g(b, x)dx + \int_{\hat{t}_{m+1}}^{\hat{t}_{m+5}} g(b, x)dx
\]

where

\[
g(b, x) = \frac{\left( \frac{\partial}{\partial x} K_B(b, x) \right)^2}{\sqrt{1 + \left( \frac{\partial}{\partial x} \hat{B}(b, x) \right)^2}}
\]

Since the constraints completely determine the coefficients \( b_1, \ldots, b_4 \) and \( b_{m-3}, \ldots, b_m \), the first and third integral is fixed and only the second integral is affected by the minimization. Hence, the convex problem \( \tilde{P} \) is the same as finding \( b_5, \ldots, b_{m-4} \) such that this second integral is minimized. The linear one-to-one relationship between \( b \) and \( \tilde{b} \) for \( x \in [\hat{t}_5, \hat{t}_{m+1}] \) implies that this is problem equivalent to \( P \). Hence, \( \tilde{P} \) and \( P \) have the same convexity properties. Since \( P_O \) is constructed from \( P \) by adding linear constraints, i.e. the ones considering the envelope, it follows that \( P_O \) is convex.

3.3 The convexity of \( P_{MVB} \) is preserved due to scaling and translation.

We show that the convexity of \( P_{MVB} \) is preserved due to a scaling and translation of the B-spline \( B(b, x) \) and its defining knot sequence \( \tau \). The result is valid for an arbitrary knot sequence \( \tau \), dimension \( m \), and B-spline degree \( d \). In turn, as we are only dealing with linear constraints, this property holds for any of the subproblems of \( P_{MVB} \). In particular, it holds for \( P_T \) and implies that it is sufficient to investigate the convexity of \( P_T \) for a specific uniform knot sequence (for example \( \hat{t} = \{0, 1, \ldots, m + d\} \)) in order to investigate the convexity of \( P_T \) for a general uniform knot sequence \( \hat{t} \).
3.3 The convexity of $P_{MV B}$

Problem $P_{MV B}$ is convex on $\Omega \subset R^m$ if the cost function $f_B(b)$ defined by (6) is convex with respect to $b \in \Omega$. Therefore, we turn our attention to the convexity of $f_B(b)$.

**Lemma 2** Let the cost function $f_B(b)$ be defined by (6) using a B-spline function $B(b,x)$ of degree $d$ defined on a knot sequence $\tau = \{\tau_1,\tau_2,\ldots, \tau_{m+d+1}\}$. Let the cost function $f_{\tilde{B}}(b)$ be defined by (6) using the B-spline function $\tilde{B}(b,x) = y_0 + wB(b,(x-x_0)/v)$ defined on the knot sequence $\tilde{\tau} = \{x_0 + v\tau_1, x_0 + v\tau_2, \ldots, x_0 + v\tau_{m+d+1}\}$, where $v,w > 0$ and $x_0, y_0 \in R$.

If $f_B(b)$ is convex on $\Omega \subset R^m$, then $f_{\tilde{B}}(b)$ is convex on $\tilde{\Omega} = \{\tilde{b} | \tilde{b} = \nu b/w, b \in \Omega\}$.

**Proof:** The cost function for $B(b,x)$ is 

$$f_B(b) = \int_{\tau_1}^{\tau_{m+d+1}} \frac{\left(\frac{\partial}{\partial x} K_B(b,x)\right)^2}{\sqrt{1 + \left(\frac{\partial}{\partial x} B(b,x)\right)^2}} \, dx$$

and the cost function for $\tilde{B}(x)$ is 

$$f_{\tilde{B}}(b) = \int_{x_0 + v\tau_1}^{x_0 + v\tau_{m+d+1}} \frac{\left(\frac{\partial}{\partial x} \tilde{K}(b,x)\right)^2}{\sqrt{1 + \left(\frac{\partial}{\partial x} \tilde{B}(b,x)\right)^2}} \, dx.$$ 

Letting $z = (x-x_0)/v$ implies that $dx = vdz$ and for $k = 1,2,\ldots$, 

$$\frac{\partial^k}{\partial x^k}(\tilde{B}(b,x)) = w\frac{\partial^k}{\partial z^k}(B(b,z))/v^k.$$ 

We use the relation 

$$w\frac{\partial^k}{\partial z^k}(B(b,z))/v = \frac{\partial^k}{\partial z^k}(B(\nu b/w,z)),$$

for $k = 0,1,\ldots$, which comes from the linearity of B-splines, and derive 

$$f_{\tilde{B}}(b) = \frac{1}{v^3} f_B \left(\frac{w}{\nu} b\right). \quad (9)$$

If $f_B(b)$ is (strictly) convex on $\Omega \subset R^m$, then, for every $b^{(1)}, b^{(2)} \in \Omega$ and $\lambda \in [0,1]$, 

$$f_B(\lambda b^{(1)} + (1-\lambda) b^{(2)}) < \lambda f_B(b^{(1)}) + (1-\lambda) f_B(b^{(2)}).$$

Relation (9) states that $f_B(\nu b/w) = f_B(b)/v^3$. This in turn yields that 

$$f_B \left(\lambda \left(\frac{w}{\nu} b^{(1)}\right) + (1-\lambda) \left(\frac{w}{\nu} b^{(2)}\right)\right)$$

$$= \frac{1}{v^3} f_B \left(\lambda b^{(1)} + (1-\lambda) b^{(2)}\right)$$

$$< \frac{1}{v^3} \left(\lambda f_B \left(b^{(1)}\right) + (1-\lambda) f_B \left(b^{(2)}\right)\right)$$

$$= \lambda f_B \left(\frac{w}{\nu} b^{(1)}\right) + (1-\lambda) f_B \left(\frac{w}{\nu} b^{(2)}\right).$$

Thus, if $f_B(b)$ is convex on $\Omega \subset R^m$, then $f_B(b)$ is convex on $\tilde{\Omega} = \{\tilde{b} | \tilde{b} = \nu b/w, b \in \Omega\}$. \qed
4 AN EMPIRICAL INVESTIGATION

In order to investigate the convexity of \( P_O \) we investigate the convexity of \( P_T \) (see Lemma 1). We have made some attempts to investigate the convexity of the cost function \( f_B(b) \) analytically by for example studying if the Hessian of \( f_B(b) \) is positive definite. But expressions involved in the analysis grow rapidly in size, and yet we have no significant analytical results. Therefore, we rely on an empirical investigation.

We use two different empirical methods for studying the convexity of \( f_B(b) \). Both methods use the fact that \( f_B(b) \), where \( b \in \mathbb{R}^m \), has at least one minimum, namely the origin \( b = [0, \ldots, 0]^T \). In each method, we study a set \( \Omega \subset \mathbb{R}^m \), of B-spline coefficient vectors \( b \), that contains the origin. We perform our investigations using integer knot sequences only since, by Lemma 2, the results carry over to investigations performed using any uniform knot sequence. First, we plot \( f_B(b) \) in one and two dimensions, i.e. for \( b \in \Omega \subset \mathbb{R} \) and \( b \in \Omega \subset \mathbb{R}^2 \), and rely on a visual inspection of the plots to draw conclusions about the convexity of \( f_B(b) \). Second, we search for other minima of \( f_B(b) \) than the one at the origin by solving instances of \( P_T \) with a numerical solver.

The plots are based on numerical computations performed in MAPLE. In one dimension we have successfully plotted a positive second derivative – indicating convexity – for sizes of \( \Omega \) up to \([-20000, 20000]\) before having numerical problems. Figure 4 shows the second derivative of \( f_B(b) \) for \( b = b_1 \in \Omega = [-6, 6] \). In the two-dimensional case we plotted \( f_B(b) \) for \( [b_1, b_2]^T \in \Omega = [-k, k] \times [-k, k] \subset \mathbb{R}^2 \) for \( k \leq 5000 \) without problems. For \( k > 5000 \) we get numerical difficulties as the value of \( f_B(b) \) grows rapidly with the size of \( b \). Plots for the two-dimensional case also indicate convexity but are omitted here due to page limitations. They are included in the full report [3].

Figure 4: Plot of the second derivative of the cost function \( f_B(b) \) for \( b = b_1 \in \Omega = [-6, 6] \).
Our search for other minima is performed using a numerical solver with many randomly chosen initial values. Given an initial value \( b^{(0)} \in \mathbb{R}^m \), the solver provides a local minimum by iteratively converging to a solution \( b^* \) [8, 11]. For dimensions 1 to 20 \((m = 1, \ldots, 20)\), we try to find out if \( f_{\hat{B}}(b) \) has any other local minima than the global minimum \( b = [0, \ldots, 0]^T \). The space over which the search is performed is \( \Omega = [-10^q, 10^q] \times \cdots \times [-10^q, 10^q] \subset \mathbb{R}^m \), where \( q = 0, 1, 2 \). If the cost function \( f_{\hat{B}}(b) \) is convex over \( b \in \Omega \) the only minimum of \( f_{\hat{B}}(b) \) is at the origin. Thus, if the search is successful, \( f_{\hat{B}}(b) \) is not convex and if the search is unsuccessful, \( f_{\hat{B}}(b) \) might be convex. To minimize \( f_{\hat{B}}(b) \) starting at \( b = b^{(0)} \), we use \texttt{fminunc}, which is a solver for unconstrained optimization problems provided by MATLAB.

Test results show that, for all \( m = 1, \ldots, 20 \) and \( q = 0, 1, 2 \), the solver converges to the origin. Thus, we have not found any local minima other than the origin. Even if this does not necessarily mean that \( f_{\hat{B}}(b) \) is convex, the test results together with the plots, indicate that \( f_{\hat{B}}(b) \) (and hence \( P_T \) and thereby \( P_O \)) is convex.

5 Conclusions and future work

In this paper we have studied the problem of computing a smooth planar curve where the smoothness of the curve was defined as the integral of the square of arc-length derivative of curvature along the curve. We introduced the minimum variation B-spline problem which is a linearly constrained optimization problem over curves defined by B-spline functions only. Our focus lay on properties of the variant of the problem asking for a curve restricted to lie between two given polygonal chains. This problem finds application in path planning among obstacles [2].

An empirical investigation, based on plots and numerous tests using numerical methods, indicates that each instance of this problem has one unique solution among all uniform quartic B-spline functions. We conjecture that there is but one unique solution of the problem. The problem might in fact be convex. The practical implication of this is that a solution computed by a numerical solver can be trusted to be the global minimum. Furthermore, we prove that, for any B-spline function, the convexity properties of the problem are preserved subject to a scaling and translation of the knot sequence defining the B-spline.

Our use of envelopes makes it possible to compute curves that go free of obstacles but it also has a drawback. There is, by construction, some space to accommodate an envelope between the obstacles and a curve [10]. Therefore, the curve is probably not the smoothest possible. It would be interesting to determine how much worse than optimal our curves actually are. It is likely that the difference eventually vanishes as the number of knots is increased. However, this is at the expense of having to spend more
time computing the curve. What is the relation between time and improved smoothness?

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References


Part IV

Epi-Convergence of Minimum Curvature Variation B-splines
Epi-Convergence of Minimum Curvature Variation B-splines

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Abstract

We study the curvature variation functional, i.e., the integral over the square of arc-length derivative of curvature, along a planar curve. With no other constraints than prescribed position, slope angle, and curvature at the endpoints of the curve, the minimizer of this functional is known as a cubic spiral. It remains a challenge to effectively compute minimizers or approximations to minimizers of this functional subject to additional constraints such as, for example, for the curve to avoid obstacles such as other curves.

In this paper, we consider the set of smooth curves that can be written as graphs of three times continuously differentiable functions on an interval, and, in particular, we consider approximations using quartic uniform B-spline functions. We show that if quartic uniform B-spline minimizers of the curvature variation functional converge to a curve, as the number of B-spline basis functions tends to infinity, then this curve is in fact a minimizer of the curvature variation functional. In order to illustrate this result, we present an example of sequences of B-spline minimizers that converge to a cubic spiral.

1 Introduction

Let \( \varphi = \varphi(s) \) be a curve in the plane, parameterized by its arc-length \( s \), and let \( \kappa_\varphi \) and \( L_\varphi \) be the curvature and the length of \( \varphi \), respectively. We study
the curvature variation functional
\[ F(\varphi) = \int_0^{L_\varphi} \kappa_\varphi^2 \, ds, \] (1)

where \( \dot{\nu} = d\nu/ds \).

The literature contains a number of studies of this functional and the problem of computing a curve that minimizes (1), subject to various constraints [13, 18, 14, 3]. Following Moreton [18], we call such a minimizer a minimum variation curve (MVC). The computation of these curves is of great interest in path-planning [13, 8, 19, 3] as well as curve and surface design and reconstruction [18].

In the general case, when arbitrary constraints are imposed on the curve, to the best of the authors’ knowledge, the representation of the MVC is not known. An example of such a case is when the curve is constrained not to intersect obstacles such as other curves [4]. Although, in the special case where the only constraints are second order constraints, i.e., prescribed position, slope angle, and curvature, at the endpoints of the curve, the representation of the MVC is known as a cubic spiral. It is also known as a cubic [13] and an intrinsic spline of degree 2 [8].

One drawback with the cubic is that its position cannot be written as a closed form expression and that it is costly to compute. Therefore, both in the special and the general case, efficiently computed approximations, that can be constrained in various ways, of a cubic and a general MVC, is of great interest.

We consider approximations that belong to a certain class of smooth curves, namely the so called B-spline functions, which are also referred to as B-splines [6, 9]. One thing that makes B-splines attractive is the ease by which the shape of the resulting curve can be controlled. For this reason, B-splines are widely used in a variety of contexts such as data fitting, computer aided design (CAD), automated manufacturing (CAM), and computer graphics [11]. An advantage of B-splines compared to cubics is that they can be given in closed form and they are efficiently computed. It is also possible to bound them in the plane by piecewise linear envelopes in terms of the parameters describing the splines [15].

In this paper, we investigate whether B-splines can serve as good approximations to cubics in the special case where we only have second order constraints at the endpoints of the curve. We obtain convergence results for the curvature variation functional of (1) over a class of smooth curves containing B-spline functions. To this end, we use epi-convergence, or \( \Gamma \)-convergence, theory [12, 2]. Our approach follows the lines of Bruckstein
et al. [5], who obtained convergence results for polygons with respect to the functional $\int_0^L \kappa^\alpha ds$, $1 \leq \alpha < \infty$.

In Section 2, we give some mathematical preliminaries regarding spaces, metrics, constraints, functionals, and $\Gamma$-convergence. In Section 3 we prove our convergence results. This is followed by an example of a converging sequence of B-spline minimizers in Section 4. Finally, in Section 5 we conclude the paper.

2 Preliminaries

The canonical problem of finding a curve $\varphi$ that minimizes $F(\varphi)$ of (1) subject to second order endpoint constraints, i.e., specified position, tangent, and curvature at the endpoints of the curve, is treated thoroughly by Kanayama and Hartman [13]. They show that there is one unique curve of finite length, having curvatures equal to zero at its endpoints, that minimizes $F(\varphi)$. This curve is either a so called symmetric cubic spiral or a pair of connected symmetric cubic spirals, and it is referred to as a cubic spiral, or cubic for short.

We do not address the problem of finding minimizers of $F(\varphi)$ as their representation is already given by the cubic. Instead, we are interested in the convergence and approximation properties of B-spline minimizers of $F(\varphi)$ for the case when we have symmetric endpoint constraints. These are the constraints that yield a symmetric cubic, which all cubics are concatenations of.

In this section, we first present the space and metrics of the smooth curves considered in this paper together with a short description of quartic uniform B-splines and the notion of symmetry. Second, we briefly present the method, relying on $\Gamma$-convergence theory, that we use to obtain convergence results.

2.1 Smooth curves, B-splines, symmetry, and metrics

We let $x$ and $y$ be Euclidian coordinates in the plane, and we consider a particular space of smooth curves, namely functions in $C^3[x_0, x_1]$, or more briefly $C^3$. Here, a curve $\varphi$ is a vector $\varphi(x) = [x, f_\varphi(x)]^T$, where $f_\varphi \in C^3$. The curve $\varphi$ can also be parameterized by its arc-length $s$, so as to be a unit-speed curve, through the one-to-one relation $ds = \sqrt{1 + f_\varphi'(x)^2} dx$, where $\nu' = dv/dx$. Throughout this paper, where appropriate, we use both of these parameterizations. We refer to $\|\varphi_1(s) - \varphi_2(s)\|$ as the Euclidian distance.
between the vectors \( \varphi_1(s) = [x_{\varphi_1}(s), y_{\varphi_1}(s)]^T \) and \( \varphi_2(s) = [x_{\varphi_2}(s), y_{\varphi_2}(s)]^T \). The curvature \( \kappa_\varphi \) of \( \varphi \) is defined as \( \kappa_\varphi(s) = \| \varphi'(s) \| \), where \( \varphi' = dv/ds \), and the length of \( \varphi \) is denoted \( L_\varphi \). We consider the following symmetric second order endpoint constraints for \( \varphi = \varphi(s) \):

\[
\begin{align*}
\varphi(0) &= [x_0, y_0(0)]^T, \\
\varphi(L_\varphi) &= [x_1, y_\varphi(L_\varphi)]^T, \\
\| \varphi(L_\varphi) - \varphi(0) \| &= \| \varphi(L_\varphi) - \varphi(0) \|', \\
\dot{\varphi}(0) &= 0, \\
\varphi(L_\varphi) &= 0.
\end{align*}
\]

For the approximation, we consider subsets \( S_{1,n} \subset C^3 \), \( n \geq 6 \), that are sets of uniform quartic B-spline functions built from \( n \) basis functions that are able to match the symmetric endpoint constraints. The reason for using quartic B-splines is that they are the B-splines of the lowest degree for which the derivative of curvature is continuous. A uniform quartic B-spline \( y = B_n(b, x) = \sum_{i=1}^n b_i N_{i,4}(x) \) is a piecewise polynomial of degree 4 [6, 9]. It is defined by the B-spline basis functions \( N_{i,4}(x) \), \( i = 1, \ldots, n \), the B-spline coefficient vector \( b = [b_1, \ldots, b_n]^T \), and a non-decreasing real number knot sequence. To handle the endpoint constraints, we use a uniform knot sequence with 5 multiple knots at the end points.

Our convergence results are shown with respect to the norm in \( C^2 \). To simplify the notation in this paper, for the curves \( \varphi_1 = [x, f_{\varphi_1}(x)]^T \) and \( \varphi_2 = [x, f_{\varphi_2}(x)]^T \), where \( f_{\varphi_1}, f_{\varphi_2} \in C^3 \) and \( x \in [x_0, x_1] \), the \( C^2 \) norm is written as

\[
D(\varphi_1, \varphi_2) = \| f_{\varphi_2} - f_{\varphi_1} \|_{C^2} = \max_{x \in [x_0, x_1]} | f_{\varphi_2} - f_{\varphi_1} | + \max_{x \in [x_0, x_1]} | f_{\varphi_2}' - f_{\varphi_1}' | + \max_{x \in [x_0, x_1]} | f_{\varphi_2}'' - f_{\varphi_1}'' |.
\]

In order to obtain results under the \( C^2 \) norm \( D(\cdot, \cdot) \) for smooth curves. These curves can be normalized and parameterized over \( t \in [0, 1] \) (instead of in arc-length \( s \)). For the curves \( \varphi_1 \) and \( \varphi_2 \) that are parameterized as \( \varphi_1(t) = [x_{\varphi_1}(t), y_{\varphi_1}(t)]^T \) and \( \varphi_2(t) = [x_{\varphi_2}(t), y_{\varphi_2}(t)]^T \), this metric is

\[
d(\varphi_1, \varphi_2) = \inf_{\psi: [0,1] \rightarrow [0,1]} \sup_{t \in [0,1]} \| \varphi_2(t) - \varphi_1(\psi(t)) \|.
\]
where \( \psi \) is a homeomorphism (reparametrization). Informally, the metric 
\[ d(\varphi_1, \varphi_2) \]
can be seen as a measure of how far apart two pencils would have to 
separate if the curves \( \varphi_1 \) and \( \varphi_2 \) were drawn simultaneously. Furthermore, 
from the definition of distance between curves given by Alexandrov and 
Reshetnyak [1], it follows that the \( C^2 \) norm and \( d(\cdot, \cdot) \) are related as 
\[ d(\varphi_1, \varphi_2) \leq D(\varphi_1, \varphi_2). \]  

\section*{2.2 Epi-convergence and curvature variation functionals}

Corresponding to the general curvature variation functional \( F \), cf. (1), de-
defined by 
\[ F(\gamma) = \int_0^L \kappa_\gamma^2 \, ds, \quad \gamma \in C^3, \]  
we consider functionals of the form 
\[ F_n(\gamma) = \begin{cases} 
F(\gamma), & \gamma \in S_{4,n} \subset C^3 \\
+\infty, & \gamma \in C^3 \setminus S_{4,n}, 
\end{cases} \]  
where \( S_{4,n} \subset C^3 \), \( n = 6, 7, \ldots \), are sets of uniform quartic B-spline functions 
\( B_n \) that are built from \( n \) basis functions. We prove that such functionals 
as in (6) approximate the functional in (5), with respect to the \( C^2 \) norm. 
In order to do this, we use epi-convergence theory, also referred to as \( \Gamma \)-
convergence theory [12, 2, 16]. The main implication of this approximation 
is that minimizers of \( F_n \) converge to minimizers of \( F \) as \( n \to \infty \).

The general structure of \( \Gamma \)-convergence is the following [16]. Let \( \Omega \) be a 
separable metric space, where \( F_n, n = 1, 2, \ldots \), and \( F \) are functionals defined 
over \( \Omega \). We say that \( F_n \) \( \Gamma \)-converges to \( F \), which we denote \( F_n \Gamma \longrightarrow F \), if 
\[ \begin{align*}
(L) \quad \forall \omega \in \Omega, \quad \omega_n \rightarrow \omega : \quad & \liminf_{n \to \infty} F_n(\omega_n) \geq F(\omega), \\
(U) \quad \forall \omega \in \Omega, \quad \exists \tilde{\omega}_n \rightarrow \omega : \quad & \limsup_{n \to \infty} F_n(\tilde{\omega}_n) \leq F(\omega),
\end{align*} \]  
where \( \omega_n, \tilde{\omega}_n \in \Omega \). We think of \( (L) \) and \( (U) \) as a lower and upper limit 
respectively. Proving \( \Gamma \)-convergence amounts to proving \( (L) \) and \( (U) \).

The theory of \( \Gamma \)-convergence provides the following key result [16]:

\textbf{Theorem 1} Let \( F_n \Gamma \longrightarrow F \) and let \( \omega_n \) minimize \( F_n \). If \( \omega \) is a cluster point 
of \( \{ \omega_n \} \) then \( \omega \) minimizes \( F \).

This means that all limit points of minimizers of \( F_n \) are minimizers of \( F \) 
which in turn implies that we can obtain approximations to minimizers of \( F \) 
without having explicit representations of them. In this paper, the separable 
metric space \( \Omega \) is \( C^3 \), and the convergence is shown for \( n = 6, 7, \ldots \), with 
respect to the \( C^2 \) norm.
3 Γ-convergence of curvature variation B-splines

In order to prove Γ-convergence of $F_n$ to $F$, we are interested in showing that $F(\gamma)$ is lower semicontinuous, cf. (L) of (7). Then we first need the following property of a curve $\gamma \in C^3$ which comes from the fact that $\gamma$ is an element of a much larger class of curves, namely the class of rectifiable curves with finite total absolute curvature (RFT-curves) [1]. This is a class of curves that admit an arc-length parametrization and that are endowed with the metric $d(\cdot, \cdot)$ as defined in (3). Such curves have the following property [1]:

**Theorem 2** Let $c_i, i = 1, 2, \ldots$, be a sequence of RFT curves converging to an RFT curve $c$ with respect to $d(\cdot, \cdot)$. Then

$$L_c \leq \liminf_{i \to \infty} L_{c_i}$$

We use this in order to state the following lemma regarding semicontinuity of $F$ with respect to the $C^2$ norm.

**Lemma 1** If, for $\gamma_n, \gamma \in C^3$, $\lim_{n \to \infty} D(\gamma_n, \gamma) = 0$, then

$$F(\gamma) \leq \liminf_{n \to \infty} F(\gamma_n).$$

**Proof:** We define the functional $H$ as follows

$$H(\gamma) = \int_0^{L_\gamma} \| \ddot{\gamma} \|^2 \, ds \quad \gamma \in C^3.$$ 

Now, we use the Frenet formulas for a planar unit-speed curve [20]. By differentiating $\ddot{\gamma}$ we relate the curvature $\kappa_\gamma = \| \dddot{\gamma} \|$ by

$$\ddot{\gamma} = \kappa_\gamma N_\gamma - \kappa_\gamma^2 T_\gamma,$$

where $N_\gamma$ and $T_\gamma$ are the principal normal and the tangent unit-vector fields on $\gamma$ respectively. As $N_\gamma$ and $T_\gamma$ are orthonormal vectors, we conclude that

$$\| \dddot{\gamma} \|^2 = \kappa_\gamma^2 + \kappa_\gamma^4.$$

This in turn means that

$$H(\gamma) = F(\gamma) + G(\gamma),$$

(8)
where

\[ F(\gamma) = \int_0^{L(\gamma)} \kappa_\gamma^2 \, ds \]

is the same functional as in (5) and

\[ G(\gamma) = \int_0^{L(\gamma)} \kappa_\gamma^4 \, ds = \int_0^{L(\gamma)} \|\dddot{\gamma}\|^4 \, ds. \]

As \( d(\gamma_n, \gamma) \leq D(\gamma_n, \gamma) \), by (4), then \( L_\gamma \leq \liminf_{n \to \infty} L_{\gamma_n} \), by Theorem 2. It follows easily, from standard lower semicontinuity of \( L^p \) norms, that

\[ H(\gamma) \leq \liminf_{n \to \infty} H(\gamma_n). \]

This means that \( H \) is lower semicontinuous with respect to the \( C^2 \) norm. According to (8), \( F(\gamma) = H(\gamma) - G(\gamma) \). It is evident that \( G(\gamma) = \int_0^{L(\gamma)} \|\dddot{\gamma}\|^4 \, ds \) is continuous with respect to the \( C^2 \) norm. We treat \( G \) as a continuous perturbation of the lower semicontinuous functional \( H \) and conclude that \( F \) is also lower semicontinuous.

Now, we are in position to prove the main result of this paper, namely the \( \Gamma \)-convergence of \( F_n \) to \( F \) with respect to the \( C^2 \) norm.

**Theorem 3** Let \( F_n \) and \( F \) be functionals defined over \( C^3 \) according to (5) and (6), respectively. Then, with respect to the \( C^2 \) norm,

\[ F_n \xrightarrow{\Gamma} F. \]

**Proof:** We prove the \( \Gamma \)-convergence by proving the lower and upper limits \((L)\) and \((U)\) as described in (7).

The lower limit \((L)\) follows directly from the definition of \( F_n \), cf. (6), together with the lower semicontinuity of \( F \) with respect to the \( C^2 \) norm, cf. (5) and Lemma 1. We have, for \( \gamma_n, \gamma \in C^3 \), \( \lim_{n \to \infty} D(\gamma_n, \gamma) = 0 \), that

\[ \liminf_{n \to \infty} F_n(\gamma_n) \geq \liminf_{n \to \infty} F(\gamma_n) \geq F(\gamma), \]

which is also true for \( \gamma_n = B_n \).

To prove the upper limit \((U)\) we use an approximation scheme, applicable to uniform B-splines, introduced by de Boor and Fix [7], which is also mentioned by de Boor [6]. Using the scheme, it is possible to produce a uniform quartic B-spline function \( B_n \), built from \( n \) basis functions, that is able to approximate curves with respect to the \( C^4 \) norm at the most. As we are dealing with curves in \( C^3 \), we can obtain approximations
with respect to the $C^3$ norm. For $\gamma_1, \gamma_2 \in C^3$ the $C^3$ norm is given by $\tilde{D}(\gamma_1, \gamma_2) = D(\gamma_1, \gamma_2) + \max_{x \in [x_0, x_1]} |f''_{\gamma_2} - f''_{\gamma_1}|$. The scheme implies that, given a curve $\gamma \in C^3$, $\lim_{n \to \infty} \tilde{D}(B_n, \gamma) = 0$. As our functionals $F$ and $F_n$ are continuous with respect to the $C^3$ norm, $\lim_{n \to \infty} F(B_n) = F(\gamma)$, and as $0 \leq D(\cdot, \cdot) \leq \tilde{D}(\cdot, \cdot)$, we also have that $\lim_{n \to \infty} D(B_n, \gamma) = 0$, i.e. convergence in the $C^2$ norm. Altogether, we know that, given a curve $\gamma \in C^3$ there exists a sequence of quartic uniform B-splines $B_n \in C^3$ for which $\lim_{n \to \infty} D(B_n, \gamma) = 0$, such that $\lim_{n \to \infty} F(B_n) = F(\gamma)$.

This concludes the proof of $(U)$ and also the proof of $\Gamma$-convergence.

4 An example of convergent B-spline minimizers

In this section, we give an indication of the existence of a sequence of B-spline minimizers $B_n$ of $F_n$, that converge to a minimizer $\gamma$ of the curvature variation functional $F$. Here, we do this by a numerical study of one example of symmetric second order endpoint constraints. In this case, we know already that the unique minimizer $\gamma \in C^3$ of $F$, is a symmetric cubic spiral. Using standard nonlinear constrained programming software, we compute B-spline minimizers to the curvature variation functional over sets of B-spline functions. We investigate the convergence of the B-spline minimizers by comparing them with the already known cubic spiral solution.

In order to compute the B-spline minimizers, we use a B-spline implementation and a solver for constrained optimization problems $fmincon$, that are both provided by MATLAB [17]. Involved integrals are computed by the routine $\text{coteglob}$, which is a globally doubly adaptive quadrature based on Newton-Cotes 5 and 9 points rules over a finite interval [10]. As an initial value for the solver, we use the straight line segment between the endpoints.

For B-splines built from a larger number of basis functions than 10, we get problems with the convergence when using the solver. The reason might be errors in the implementation of the solver itself or the accuracy of the quadrature routine. A feasible initial value, that is closer to the cubic than the one used here, would probably yield faster and more reliable convergence. Due to these problems mentioned, we study the convergence of B-spline minimizers for low numbers of basis functions only and take this as an indication of what happens for a larger number.

In our example, we consider the symmetric endpoint constraints $y(0) = 0$, $y'(0) = \tan(1)$, $y''(0) = 0$, and $y(1) = 0$, $y'(1) = \tan(-1)$, $y''(1) = 0$. The
Table 1: Difference between cost functions and the distance between the symmetric cubic spiral $\gamma$ and the B-spline minimizers $B_n$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$F_n(B_n) - F(\gamma)$</th>
<th>$D(\gamma, B_n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>83.0496</td>
<td>3.4807</td>
</tr>
<tr>
<td>7</td>
<td>10.4786</td>
<td>2.3571</td>
</tr>
<tr>
<td>8</td>
<td>3.4175</td>
<td>1.7753</td>
</tr>
<tr>
<td>9</td>
<td>0.8430</td>
<td>1.2122</td>
</tr>
<tr>
<td>10</td>
<td>0.0811</td>
<td>0.8129</td>
</tr>
</tbody>
</table>

cubic spiral $\gamma$ that minimizes $F$, and that can easily be derived [13], yield $F(\gamma) = 22.0615$.

In Table 4, for an increasing number, $n$, of basis functions, we present the relation of the B-spline minimizers $B_n$ to $\gamma$, with respect to difference in cost function and $C^2$ norm. Figure 1 shows a plot of the cubic spiral solution together with the B-spline minimizers. Both numerical and visual inspection indicates that there is a sequence of B-spline minimizers that converge to the optimal curve, which is a symmetric cubic.

5 Conclusions and future work

In this paper we have studied the problem of computing a smooth planar curve $\gamma \in C^3$ that, with symmetric second order endpoint constraints, minimizes the curvature variation functional $F$ over $C^3$, i.e., the integral over the square of arc-length derivative of curvature, along the curve. The curve $\gamma$ that solves this problem is already known as a symmetric cubic spiral.

We have considered approximations in $C^3$ represented by efficiently computed uniform quartic B-splines $B_n \in S_{4,n} \subset C^3$ built from $n$ B-spline basis functions. The functional $F_n$, which is the same as $F$ over the space $S_{4,n}$ of B-splines and $+\infty$ otherwise, has been introduced. We have proved that $F_n$ $\Gamma$-converges, or epi-converges, to $F$ with respect to the $C^2$ norm. The main implication of this is that, if there is a converging sequence of minimizers $B_n$ of $F_n$, then $B_n$ converges, in $C^2$ norm, to a minimizer $\gamma$ of $F$ as $n \rightarrow \infty$.

Through an example with symmetric endpoint constraints, for which we already know the cubic spiral minimizer $\gamma$ of $F$, we gave an indication of the existence of such a converging sequence of B-spline minimizers $B_n$ of $F_n$. It is in fact possible to prove that such a sequence exists, using the Sobolev embedding theorem and assuming that curve lengths are uniformly
Figure 1: Plots of B-spline minimizers $B_n$ of $F_n$, $n = 6, \ldots, 10$, together with the symmetric cubic spiral $\gamma$ (dashed) that minimizes $F$.

bounded. We will treat this issue in forthcoming work.

Other future work includes the convergence of quartic uniform B-spline approximations for general second order endpoint constraints as well as for various other constraints, e.g., for the curve to not intersect other prescribed curves [4]. Is the minimizer $B_n$ of $F_n$ unique even for these more general constraints? Another interesting issue is the convergence rate, i.e., the decrease in $F_n$ with respect to $n$. How many B-spline basis functions are needed in order to obtain a given accuracy of the approximation?

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