LICENTIATE THESIS

Coercive Estimates for the Solutions of some Singular Differential Equations and their Applications

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Abstract

This Licentiate thesis deals with the study of existence and uniqueness together with coercive estimates for solutions of certain differential equations.

The thesis consists of four papers (papers A, B, C and D) and an introduction, which put these papers into a more general frame and which also serves as an overview of this interesting field of mathematics.

In the text below the functions $r(x)$, $q(x)$, $m(x)$ etc. are functions on $(-\infty, +\infty)$, which are different but well defined in each paper.

In paper A we study the separation and approximation properties for the differential operator

$$ly = -y'' + r(x)y' + q(x)y$$

in the Hilbert space $L_2 := L_2(R),\ R = (-\infty, +\infty)$, as well as the existence problem for a second order nonlinear differential equation in L_2.

Paper B deals with the study of separation and approximation properties for the differential operator

$$ly = -y'' + r(x)y' + s(x)\bar{y}'$$

in the Hilbert space $L_2 := L_2(R),\ R = (-\infty, +\infty)$, (here $\bar{y}$ is the complex conjugate of y). A coercive estimate for the solution of the second order differential equation $ly = f$ is obtained and its applications to spectral problems for the corresponding differential operator l is demonstrated. Some sufficient conditions for the existence of the solutions of a class of nonlinear second order differential equations on the real axis are obtained.

In paper C we study questions of the existence and uniqueness of solutions of the third order differential equation

$$(L + \lambda E)\, y := -m(x)\,(m(x)y')'' + [q(x) + ir(x) + \lambda]y = f(x), \qquad (0.1)$$

and conditions, which provide the following estimate:

$$\|m(x)(m(x)y')''\|_p^p + \|(q(x) + ir(x) + \lambda)y\|_p^p \le c\,\|f(x)\|_p^p$$

for a solution y of (0.1).

Paper D is devoted to the study of the existence and uniqueness for the solutions of the following more general third order differential equation with unbounded coefficients:

$$-\mu_1(x)\left(\mu_2(x)\left(\mu_1(x)y'\right)'\right)' + \left(q(x) + ir(x) + \lambda\right) y = f(x).$$

Some new existence and uniqueness results are proved and some norm-estimates of the solutions are given.

Preface

This Licentiate thesis consists of four papers (papers A, B, C and D) and an introduction, which puts these papers into a more general frame.

[**A**] K. N. Ospanov and R. D. Akhmetkaliyeva, On separation of a degenerate differential operator in Hilbert space, *CRM-1080*, Barcelona, (2011), 12 pp.

[**B**] K. N. Ospanov and R. D. Akhmetkaliyeva, Separation and the existence theorem for second order nonlinear differential equation, *Electron. J. Qual. Theory Differ. Equ.* (2012), No. 66, 12 pp.

[**C**] R. D. Akhmetkaliyeva, K. N. Ospanov, L.-E. Persson and P. Wall, Some new results concerning a class of third order differential equations, *Research Report 2012 - 05, Department of Engineering Sciences and Mathematics, Luleå University of Technology, ISSN: 1400 - 4003*, (submitted), 25 pp.

[**D**] R. D. Akhmetkaliyeva, About conditions for the solvability of a class of third-order differential equations, *Research Report 2012 - 07, Department of Engineering Sciences and Mathematics, Luleå University of Technology, ISSN: 1400 - 4003*, (submitted), 18 pp.

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Introduction

This Licentiate thesis deals with the smoothness and approximation properties of solutions of differential equations defined in the Lebesgue space and having real and sometimes complex coefficients.

The main questions in the investigation of differential equations can be classified into the following three categories: existence, uniqueness and qualitative behavior of the solutions. The first two questions are responsible for the compliance of the equations as a mathematical model of the real process, and the third question is necessary to investigate in order to know more about the nature of the process. In the study of the qualitative behavior of solutions of linear and nonlinear differential equations we are interested in the following questions:

1) the smoothness of the solutions;

2) estimates of solutions in different weighted norms;

3) approximation properties of the solutions.

The problem of smoothness for solutions of elliptic equations and estimates of solutions in various norms are well studied in the case when the domain is bounded and the coefficients are reasonable "regular". In this case we have methods which are nowadays developed to the classical perfection and presented in detail in well-known monographs. Complete bibliography of works in this field can be found e.g. in the books of O. A. Ladyzhenskaya and N. N. Ural'tseva [35], Zh.-L. Lions and E. Madzhenes [36].

Unfortunately, these methods are not applicable for differential equations given in an unbounded domain and with increasing (not integrable) coefficients. Studies of problems of this type were first made by W. N. Everitt and M. Giertz [25-30] as singular Sturm-Liouville problems. In particular, the formulation of the fundamental problem of separability for a differential operator belong to them. Moreover, in [25-30] the same authors basically elucidated the conditions on the potential function $q(x)$, providing the separability of the Sturm-Liouville operator

$$Ly(x) = -y''(x) + q(x)y(x), \quad x \in R.$$

In these papers W. N. Everitt and M. Giertz called the indicated operator *separable* in the space $L_2 = L_2(-\infty, +\infty)$, if from $y \in D(L)$, $Ly \in L_2$ it follows that $q(x)y,\ y'' \in L_2$.

It is well-known that the separability of the operator L is equivalent to the existence of the estimate

$$\|y''\|_{L_2(R)} + \|qy\|_{L_2(R)} \le c\left(\|Ly\|_{L_2(R)} + \|y\|_{L_2(R)}\right), \quad y \in D(L), \qquad (0.2)$$

where $D(L)$ is the domain of L. In [25-30] it was shown that if $\inf q(x) > -\infty$ and $\left(q^{-\frac{1}{4}}(x)\right)'' q^{\frac{1}{4}}(x) \in L_1$, then the operator L is separable. Moreover, an example of a non-separable operator L with non-smooth potential q was given. Independently of each other F. V. Atkinson [7], K. H. Boimatov [14], [17], M. Otelbaev [48] and D. Z. Raimbekov [56] weakened the condition used by W. N. Everitt and M. Giertz. In particular, in [14], [48] and [56], the condition $\left(q^{-\frac{1}{4}}(x)\right)'' q^{\frac{1}{4}}(x) \in L_1$ was replaced by weaker conditions (different by different authors), which are similar to the known conditions of Levitan - Titchmarsh, which commonly are used in investigations of the resolvent (concerning these conditions, see e.g. [16], [47] and [60]). In [48] the problem of separability was considered not only in the Hilbert space L_2, but also in non-Hilbert weighted spaces $L_{p,l}$ (where l is a continuous weight function). Here $L_{p,l}$ is defined by the norm

$$\|f\|_{p,l} := \left(\int_{-\infty}^{+\infty} |f(x)l(x)|^p dx\right)^{\frac{1}{p}}, \ (1 \le p < +\infty).$$

In particular, it was shown that the separability of the Sturm-Liouville operator holds for an extensive class of rapidly oscillating potentials (for example, $q(x) = e^{|x|}\sin^2 e^{|x|^5}$). Later on M. Otelbaev proposed a special method with local representation of the resolvent to solve the problem concerning the smoothness of solutions of some differential equations, which he called variational. Multivariate equations were considered in [15], where K. H. Boimatov essentially verbatim transferred results from [14] to a class of elliptic operators.

The existence and smoothness of solutions of nonlinear differential equations (with a singular potential) for unbounded domains equipped with the Sturm-Liouville equation was considered by M. B. Muratbekov and M. Otelbaev [44]. Later on this problem was investigated in the works T. T. Amanova [4] and M. B. Muratbekov [42].

In [32] the authors investigated the separability of the nonlinear Sturm-Liouville operator

$$Ly = -y'' + q(x, y)y$$

in the space $L_1(-\infty,+\infty)$. Moreover, in [1], [11], [20], [39], [63] and [64] the differential expression

$$Ly = -\left(P(x)y'\right)' + Q(x)y, \quad x \in (-\infty,+\infty),$$

with operator coefficients was considered.

We have thus motivated the fact that in the case when the differential equation is given in an infinite domain and has unbounded coefficients, the problem of determining the estimates of separability of the type (0.2) for the corresponding differential operator is meaningful. The presence of estimates of separability allows us to accurately describe the class of functions, where the generalized solution of the singular boundary value problem for the differential equation belongs. At the same time the estimate of separability provides a precise description of the domain generated by the indicated singular boundary value problem for the differential operator. This domain is usually a weighted Sobolev space. Thus, if we have estimates of separability, then we can use the modern theory of function spaces to study qualitative properties of the solutions of singular differential equations. We recall that the famous scientist I. M. Gelfand considered that finding estimates of separability is one of the most central problems in the study of elliptic equations in the general theory of linear operators (see e.g. preface of the book [35, p.8]).

Separability of a wide class of linear elliptic differential operators was investigated in [6], [7], [14-30], [37-56] and [62-64], where, in particular, important smoothness and approximation properties of solutions of these equations and the spectral properties of the associated singular differential and integral operators were investigated. The methods of proofs in the indicated works are based on deep facts of the theory of embedding between function spaces, of spectral theory of operators, and also widely used advances in the theory of integral operators in function spaces and non-local apriori estimates of generalized solutions. These studies had an enormous influence on the development of the theory of singular differential equations, spectral theory of operators, the theory of weighted function spaces and integral operators in them.

However, the results in all these papers concern only those linear differential operators whose first order terms can be estimated in norms with the other terms involved. However, many practical problems lead us to study elliptic equations, whose properties depend strongly on the behavior of the components with intermediate derivatives of the solution involved and where we have no such norm estimate. Such equations are in the literature called degenerate differential equations. These include for example equations of

Schrödinger type with intermediate members with unlimited potential from below, the Korteweg - de Vries type equation, where the coefficient of the constant term depends on the derivative of the unknown function, as well as the differential equation of oscillations in environments with resistance proportional to the velocity or acceleration (see e.g. [59]). Despite of this, the study of degenerate differential equations was carried out only in the symmetric case for the corresponding differential operators in [31], [33], [34] and [58], where, in particular, the problem of self-adjoint operators assessing their eigenvalues and determination of the structure of the spectrum was solved.

In papers A and B of this thesis we study the more general case concerning a degenerate differential equation having non-symmetric form. In the same papers we consider the question of solvability (apparently for the first time) for a quasilinear degenerate differential equation.

The approaches developed in the above studies also allows us to study some classes of non semibounded differential operators, i.e., such energy spaces that are not enclosed in a Sobolev space. The non semibounded operators include all differential operators of odd order. Linear and nonlinear differential operators of odd order were investigated e.g. in [2], [3], [5], [8-10], [12], [13], [43], [57] and [61]. However, all of them except Zh. Zh. Aytkozha and M. B. Muratbekov [9], A. Birgebaev and M. Otelbaev [13] and M. B. Muratbekov, M. M. Muratbekov and K. N. Ospanov [43] was devoted to the case of a real potential and in [9] and [13] the case of a Hilbert space was considered. Odd order differential equations with singular complex coefficients in Banach space have not been studied systematically. Such equations constantly arise in the application of the projection methods, in particular in the Fourier's method of separation of variables for solving partial differential equations.

In Papers C and D we investigate some more general third order equations than those above. Usually, the previous mentioned authors only consider equations of the type

$$Ly = -y''' + q(x)y = f(x),$$

where $f \in L_p(R),\ R = (-\infty, +\infty)$. However, we consider the more general case, when the coefficients are not constant in the leading term.

Before starting the presentation of the results obtained in the papers A, B, C and D we present a number of well-known definitions and necessary notations.

R^n is a n-dimensional real Euclidean space; in particular when $n = 2$ we obtain a two-dimensional Euclidean space of points $z = (x, y)$, where $-\infty < x < \infty, \quad -\infty < y < \infty$.

Ω denotes an open domain in R^n and we denote by $\overline{\Omega}$ the closure of Ω.

Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, where $\alpha_j \geq 0$ $(j = 1, 2, ...n)$ are integers. We also use the notation $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$.

$C^l(\overline{\Omega})$, $l = 0, 1, 2, ...$, is the set of continuous functions with continuous partial derivatives of order up to l inclusive in $\overline{\Omega}$, which can be written as

$$D^\alpha(u) := \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}}, \quad \text{where} \quad |\alpha| \leq l.$$

$C^\infty(\overline{\Omega})$ is a set of infinitely differentiable functions in $\overline{\Omega}$.

Definition 0.1. *The set $\{x \in \Omega : u(x) \neq 0\}$ is called the support of the function u defined on the set $\overline{\Omega}$ and it is denoted by $\operatorname{supp} u$.*

$C_0^\infty(\overline{\Omega})$ denotes the set of infinitely differentiable and compactly supported functions in $\overline{\Omega}$.

$L_2 = L_2(\Omega)$ is the Hilbert space of Lebesgue measurable functions on Ω with a finite norm

$$\|u\|_{2,\Omega} := \left[\int_\Omega |u|^2 d\Omega \right]^{\frac{1}{2}}.$$

$W_2^k(\Omega)$ denotes the space of functions from $L_2(\Omega)$ having all the generalized Sobolev derivatives up to order $k \geq 1$ also belonging to $L_2(\Omega)$ with the norm

$$\|u\|_{W_2^k(\Omega)} := \left[\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^2 d\Omega \right]^{\frac{1}{2}}.$$

The domain of the operator A is denoted by $D(A)$ and the range of A is denoted by $R(A)$.

Definition 0.2. *An operator A is called a bijection if, for any x_1 and x_2 belonging to $D(A)$, such that $Ax_1 = Ax_2$, it follows that $x_1 = x_2$.*

If A maps $D(A)$ onto $R(A)$ bijective, then there exists an inverse mapping or inverse A^{-1} which maps $R(A)$ onto $D(A)$.

Definition 0.3. *The operator A is said to be closed if, for every sequence $\{x_n\} \subset D(A)$, the fact that $x_n \to x_0$ and $Ax_n \to y_0$ implies that $x_0 \in D(A)$ and $y_0 = Ax_0$.*

If the operator A is not closed, then sometimes it can be extended to be closed. This operation is called the closure of the operator A and the operator is called closable.

A criterion to guarantee that an operator has a closed extension: an operator A has a closed extension if and only if the properties $\{x_n\} \subset D(A)$, $x_n \to 0$ and $Ax_n \to y$ follows that $y = 0$.

Definition 0.4. *An operator A is said to be completely continuous if it maps every bounded set into a compact set or, for every bounded sequence $\{x_n\}$ of elements of $D(A)$, the sequence $\{Ax_n\}$ contains a convergent subsequence.*

Let X and Y be normed spaces and let A be a bounded operator from X to Y. We define a functional φ by

$$\varphi(x) = (x, \varphi) = (Ax, f), \quad x \in X, \quad f \in Y^*, \tag{0.3}$$

where Y^* denotes the conjugate space of the space Y.

It is easy to see that φ is linear and $D(\varphi) = X$. Hence, according to (0.3), for each $f \in Y^*$ there exists an element $\varphi \in X^*$, where X^* is the conjugate space to X. Thus a linear continuous operator $\varphi = A^* f$ is given. This operator A^* is called the adjoint of A.

Definition 0.5. *An operator A acting in the Hilbert space $L_2(\Omega)$ is said to be self-adjoint if it is symmetric, i.e., if the scalar product $\langle Au, v\rangle = \langle u, Av\rangle$ for any $u,\ v \in D(A)$ and from the identity*

$$\langle Au, v\rangle = \langle u, w\rangle \,,$$

where v and w are fixed, u is any element from $D(A)$, it follows that $v \in D(A)$ and $w = Av$.

Now we give the definition of Kolmogorov's k-widths and their properties.

Let M be a centrally symmetric subset of H (H is a Hilbert space), i.e., $M = -M$.

The value

$$d_k = \inf_{\{G_k\}} \sup_{u \in M} \inf_{v \in G_k} \|u - v\| \,, \quad k = 0, 1, 2, ...$$

is called Kolmogorov's k-width of the set M, where G_k is a subset with dimension k.

The k-widths d_k ($k = 1, 2, ...$) have the following properties:

1) $d_0 \le d_1 \le d_2 \le ...$;

2) $d_k(\widetilde{M}) \leq d_k(M), \quad \widetilde{M} \subset M, \quad k = 1, 2, 3, ...;$
3) $d_k(nM) = nd_k(M), \quad n > 0, \quad nM = \{x' = nx, \ x \in M\}.$

Let $\overset{\circ}{L}{}^l_p(\Omega, q)$ be the completion of $C_0^\infty(\Omega)$, defined by the norm

$$\left\| (-\Delta)^{\frac{l}{2}} u \right\|^p_{L_p(\Omega)} + \int\limits_\Omega q(t)|u(t)|^p dt,$$

where $q(t)$ is a nonnegative function, Ω is an open (bounded or unbounded) set in R^n, $\quad l > 0, \quad 1 \leq p < \infty$. We pronounce that the space $\overset{\circ}{L}{}^l_p(\Omega, q)$ uniquely arises in many situations in the study of differential equations.

We also define the following function $q^*(x)$ introduced by M. Otelbaev (see e.g. [55]):

$$q^*(x) = \inf_{Q_d(x) \subseteq \Omega} \left(d^{-1} : \ d^{-pl+n} \geq \int\limits_{Q_d(x)} q(t) dt \right), \tag{0.4}$$

where $Q_d(x)$ is a cube with sides equal to d and with center $x \in \Omega, \quad pl > n$.

Definition 0.6. *Let B_1 and B_2 be Banach spaces. B_1 is said to be embedded in B_2 if B_1 is a subspace B_2 and there is a constant $c > 0$ such that*

$$\|x\|_{B_2} \leq c \|x\|_{B_1} \quad \textit{for all} \quad x \in B_1.$$

In this case we write $B_1 \hookrightarrow B_2$.

Definition 0.7. *Let B_1 and B_2 be Banach spaces. Then a transformation E mapping each element x from B_1 to the same element in B_2 is called the embedding operator and denoted by $E : B_1 \to B_2$.*

Theorem 0.1 ([52]). *The embedding operator $E : \overset{\circ}{L}{}^l_p(\Omega, q) \hookrightarrow L_p$ is compact if and only if*

$$q^*(x) \to \infty \quad \textit{when} \quad |x| \to \infty.$$

Let B_1 and B_2 be Banach spaces and $B_1 \hookrightarrow B_2$.

Definition 0.8. *The Kolmogorov k-width of the unit ball of the space B_1 in B_2 is called the k-width of the embedding $B_1 \hookrightarrow B_2$.*

We introduce a function $N(\lambda) = \sum_{d_k>\lambda} 1$ as the number of k-widths of the embeddings $B_1 \hookrightarrow B_2$ greater than $\lambda > 0$. $N(\lambda)$ is also called the distribution function of the k-widths d_k.

We also observe that the k-widths d_k can be recovered from their distribution function using the formula

$$d_k = \inf\{\lambda > 0 : \; N(\lambda) \le k\}, \quad \text{for any} \quad k > 0.$$

Let $N(\lambda)$ be a distribution function of the k-widths $\{d_k\}$ related to the embedding

$$\overset{\circ}{L}{}^l_p(\Omega, q) \hookrightarrow L_p.$$

Then the following theorem holds:

Theorem 0.2 ([49], [53]). *Let $pl > n$. Then the following estimates*

$$c^{-1}\lambda^{-\frac{n}{l}}\mu\left(x \in \Omega : \; q^*(x) \le \lambda^{-\frac{1}{l}}\right) \le N(\lambda) \le c\lambda^{-\frac{n}{l}}\mu\left(x \in \Omega : \; q^*(x) \le \lambda^{-\frac{1}{l}}\right)$$

hold, where $\mu(\cdot)$ is the Lebesgue measure and c depends only on p, l and n.

It is easy to see that if $d = 1$ and the condition

$$\sup_{\substack{|x-y|\le 1 \\ x,y\in R^n}} \frac{q(x)}{q(y)} \le C \tag{0.5}$$

holds, then $c_0^{-1}q^{pl-n}(x) \le q^*(x) \le c_0 q^{pl-n}(x)$, where $c_0 > 1$ and $q^*(x)$ is defined by (0.4). In this case Theorems 0.1 and 0.2 can be restated in terms of the function $q(x)$ in the following way:

Theorem 0.3. *Let $pl > n$ and for a positive function $q(x)$ the condition (0.5) holds. Then the embedding operator $\overset{\circ}{L}{}^l_p(\Omega, q) \hookrightarrow L_p$ is compact if and only if*

$$q(x) \to \infty \quad \textit{when} \quad |x| \to \infty.$$

Theorem 0.4. *Let $pl > n$ and the condition (0.5) holds. Then the following estimates*

$$c^{-1}\lambda^{-\frac{n}{l}}\mu\left(x \in \Omega : \; q(x) \le \lambda^{-\frac{1}{l(pl-n)}}\right) \le N(\lambda) \le c\lambda^{-\frac{n}{l}}\mu\left(x \in \Omega : \; q(x) \le \lambda^{-\frac{1}{l(pl-n)}}\right)$$

hold, where $\mu(\cdot)$ is the Lebesgue measure and c depends only on p, l and n.

Theorem 0.5 (Schauder). *Let D be a nonempty closed bounded convex subset of a Banach space X and let the operator $A : X \to X$ be compact and map D into itself. Then A has a fixed point in D.*

Now let us briefly present the most important results of papers A, B, C and D. In the sequel the functions $r(x)$, $q(x)$, $m(x)$ etc. are functions on $(-\infty, +\infty)$, which are different but well defined in each paper.

In paper A of this Licentiate thesis we consider a problem of separation and approximate properties for the differential operator

$$ly := -y'' + r(x)y' + q(x)y \tag{0.6}$$

in the Hilbert space $L_2 := L_2(R),\ R = (-\infty, +\infty)$, as well as the existence problem for the following nonlinear differential equation in L_2

$$Ly = -y'' + [r(x, y)]y' = f(x), \tag{0.7}$$

where $x \in R$, r is real-valued function and $f \in L_2$.

Definition 0.9. *A function $y \in L_2$ is called a solution of (0.7) if there is a sequence of twice continuously differentiable functions $\{y_n\}_{n=1}^{\infty}$ such that $\|\theta(y_n - y)\|_2 \to 0,\quad \|\theta(Ly_n - f)\|_2 \to 0$ as $n \to \infty$ for any $\theta \in C_0^\infty(R)$.*

The operator l is said to be separable in the space L_2 if the following estimate holds:

$$\|y''\|_2 + \|ry'\|_2 + \|qy\|_2 \le c\left(\|ly\|_2 + \|y\|_2\right), \quad y \in D(l),$$

where $\|\cdot\|_2$ is the norm in L_2.

We assume that the function r is positive and increases at infinity faster than $|q|$.

We denote

$$\alpha_{g,h}(t) = \|g\|_{L_2(0,t)} \left\|h^{-1}\right\|_{L_2(t,+\infty)} \quad (t > 0),$$

$$\beta_{g,h}(\tau) = \|g\|_{L_2(\tau,0)} \left\|h^{-1}\right\|_{L_2(-\infty,\tau)} \quad (\tau < 0),$$

$$\gamma_{g,h} = max\left(\sup_{t>0} \alpha_{g,h}(t), \sup_{\tau<0} \beta_{g,h}(\tau)\right),$$

where g and h are given functions.

By $C_{loc}^{(1)}(R)$ we denote the set of functions f such that $\psi f \in C^{(1)}(R)$ for all $\psi \in C_0^{\infty}(R)$.

The main results of paper A read as follows:

Theorem 0.6. *Let the function r satisfy the conditions*

$$r \in C_{loc}^{(1)}(R), \quad r \geq \delta > 0, \quad \gamma_{1,r} < \infty, \tag{0.8}$$

$$c^{-1} \leq \frac{r(x)}{r(\eta)} \leq c \ \ at \ \ |x-\eta| \leq 1, \quad c > 1, \tag{0.9}$$

and let the function q be such that

$$\gamma_{q,r} < +\infty. \tag{0.10}$$

Then for $y \in D(l)$ the estimate

$$\|y''\|_2 + \|ry'\|_2 + \|qy\|_2 \leq c_l \|ly\|_2$$

holds, i.e., in particular, the operator l is separable in L_2.

Theorem 0.7. *Let the functions q, r satisfy the conditions (0.8)-(0.10) and the equalities $\lim\limits_{t \to +\infty} \alpha_{q,r}(t) = 0$, $\lim\limits_{\tau \to -\infty} \beta_{q,r}(\tau) = 0$ hold. If l is defined by (0.6), then an inverse operator l^{-1} exists and it is completely continuous in L_2.*

We assume that the conditions of Theorem 0.7 hold and consider the set

$$M := \{y \in L_2 : \|ly\|_2 \leq 1\}.$$

Let

$$d_k = \inf_{\Sigma_k \subset \{\Sigma_k\}} \sup_{y \in M} \inf_{w \in \Sigma_k} \|y - w\|_2 \, (k = 0, 1, 2, ...)$$

be the Kolmogorov's k-widths of the set M in L_2. Here $\{\Sigma_k\}$ denotes the set of all subspaces Σ_k of L_2 whose dimensions are not more than k. By $N_2(\lambda)$ we denote the number of k-widths d_k which are not smaller than a given positive number λ. Estimates of the k-width's distribution function $N_2(\lambda)$ are important in the approximating problem of solutions of the equation $ly = f$. The following statement holds:

Theorem 0.8. *Let the conditions of Theorem 0.7 be fulfilled. Then the following estimates hold:*

$$c_1 \lambda^{-2} \mu \left\{x : |q(x)| \leq c_2^{-1} \lambda^{-1}\right\} \leq N_2(\lambda) \leq c_3 \lambda^{-2} \mu \left\{x : |q(x)| \leq c_2 \lambda^{-1}\right\}.$$

Theorem 0.9. *Let the function r be continuously differentiable with respect to both arguments and satisfy the following conditions:*

$$r \geq \delta_0(1+x^2) \quad (\delta_0 > 0), \qquad \sup_{|x-y|\leq 1} \ \sup_{|C_1|\leq A, |C_2|\leq A, |C_1-C_2|\leq A} \frac{r(x,C_1)}{r(\eta,C_2)} < \infty.$$

Then there is a solution y of the equation (0.7) and

$$\|y''\|_2 + \|[r(\cdot,y)]y'\|_2 < \infty.$$

In paper B we study a degenerate second order differential operator with complex coefficients.

Let l be the closure in $L_2 := L_2(\mathbb{R})$, $\mathbb{R} = (-\infty, +\infty)$ of the expression $l_0 y = -y'' + r(x)y' + s(x)\bar{y}'$ defined in the set $C_0^\infty(\mathbb{R})$ of all infinitely differentiable and compactly supported functions. Here r and s are complex-valued functions and $\bar{y}$ is the complex conjugate to y.

The operator l is said to be separable in L_2 if the following estimate holds:

$$\|y''\|_2 + \|ry'\|_2 + \|s\bar{y}'\|_2 \leq c\left(\|ly\|_2 + \|y\|_2\right), \ y \in D(l),$$

where $\|\cdot\|_2$ is the L_2- norm.

The main results of this paper are the following:

Theorem 0.10. *Let the functions r and s satisfy the conditions*

$$r, s \in C_{loc}^{(1)}(\mathbb{R}), \quad Re\ r - |s| \geq \delta > 0, \quad \gamma_{1,Re\ r} < \infty.$$

Then l is invertible and l^{-1} is defined in all L_2.

Theorem 0.11. *Assume that the functions r and s satisfy the conditions*

$$\begin{cases} r, s \in C_{loc}^{(1)}(\mathbb{R}), \ Re\ r - \rho[|Im\ r| + |s|] \geq \delta > 0, \ \gamma_{1,Re\ r} < \infty, \ 1 < \rho < 2, \\ c^{-1} \leq \frac{Re\ r(x)}{Re\ r(\eta)} \leq c \quad at \quad |x-\eta| \leq 1, \ c > 1. \end{cases} \tag{0.11}$$

Then, for $y \in D(l)$, the estimate

$$\|y''\|_2 + \|ry'\|_2 + \|s\bar{y}'\|_2 \leq c_l \|ly\|_2$$

holds, i.e., in particular, the operator l is separable in L_2.

Theorem 0.12. *Assume that the functions r and s satisfy (0.11) and let $\lim\limits_{t\to+\infty}\alpha_{1,Re\,r}(t)=0$, $\lim\limits_{\tau\to-\infty}\beta_{1,Re\,r}(\tau)=0$. Then l^{-1} is completely continuous in L_2.*

Theorem 0.13. *Assume that the conditions of Theorem 0.12 are fulfilled and let the function q satisfy that $\gamma_{q,Re\,r}<\infty$. Then the following estimates hold:*

$$c_1\lambda^{-2}\mu\left\{x:|q(x)|\le c_2^{-1}\lambda^{-1}\right\}\le N_2(\lambda)\le c_3\lambda^{-2}\mu\left\{x:|q(x)|\le c_2\lambda^{-1}\right\},$$

where μ is the Lebesgue measure.

Thus, in paper B sufficient conditions for the invertibility and separability of the differential operator l are obtained. Moreover, spectral and approximation results for the inverse operator l^{-1} are achieved. Using a separation theorem, which is obtained for the linear case, the solvability of the degenerate nonlinear second order differential equation $-y''+r(x,y)y'=F$ $(x\in\mathbb{R})$ is proved.

In paper C we investigate the problem of existence and uniqueness of solutions of the third order differential equations

$$(L+\lambda E)y:=-m(x)\left(m(x)y'\right)''+[q(x)+ir(x)+\lambda]y=f(x), \qquad (0.12)$$

where $f\in L_p$, $(1<p<+\infty)$, $\lambda\ge 0$ and $r(x)$, $q(x)$ and $m(x)$ are given functions. We also derive conditions so that for a solution y of (0.12) the following estimate holds:

$$\|m(x)(m(x)y')''\|_p^p+\|(q(x)+ir(x)+\lambda)y\|_p^p\le c\,\|f(x)\|_p^p. \qquad (0.13)$$

In the case when $m(x)=1$ sufficient conditions for unique solvability of the equation (0.12) and the estimate of the form (0.13) for its solution in spaces $L_{p,l}$ were obtained by Zh. Zh. Aytkozha [8] and Zh. Zh. Aytkozha and M. B. Muratbekov [9].

In the case when $m(x)=1$ and $r(x)=0$ the existence and uniqueness questions for the solutions of (0.12) and also non-local estimates of the solutions and its derivatives have been studied in [2], [3], [57].

Definition 0.10. *A function $y(x)\in L_p(R)$, is called a solution of (0.12), if there exists a sequence $\{y_n\}_{n=1}^{\infty}$ of continuously differentiable functions with compact support such that $\|y_n-y\|_p\to 0$ and $\|(L+\lambda E)y_n-f\|_p\to 0$ as*

$n \to \infty$.

By $C^{(k)}(R)$ $(k = 1, 2, ...)$ we denote the set of all k times continuously differentiable functions $\varphi(x)$ for which the value $\sum_{j=0}^{k} \sup_{x \in R} |\varphi^{(j)}(x)|$ is finite. Let $W_\lambda(x) := \dfrac{|q(x) + \lambda + ir(x)|}{m^2(x)}$.

Our main results in this paper are formulated in the following two theorems:

Theorem 0.14. *Assume that the functions $q = q(x)$ and $r = r(x)$ are continuous on R, $m = m(x) \in C^{(2)}_{loc}(R)$ and that the following conditions hold:*

$$m(x) \geq 1, \ \frac{q(x)}{m^4(x)} \geq 1, \ r(x) \geq 1, \tag{0.14}$$

$$c^{-1} \leq \frac{m(x)}{m(\eta)}, \frac{q(x)}{q(\eta)}, \frac{r(x)}{r(\eta)} \leq c, \ x, \eta \in R, \quad |x - \eta| \leq 1, \quad \textit{for some } c > 0, \tag{0.15}$$

$$|m^{(j)}(x)| \leq c_j m(x), \ x \in R, \quad \textit{for some } c_j > 0, \quad j = 1, 2, \tag{0.16}$$

$$\sup_{|x-\eta| \leq 1} \frac{|W_\lambda(x) - W_\lambda(\eta)|}{|W_\lambda(x)|^\nu |x - \eta|^\mu} < +\infty, \quad 0 < \nu < \frac{\mu}{3} + 1, \quad \mu \in (0, 1], \ \lambda \geq 0. \tag{0.17}$$

Then there exists a number $\lambda_0 \geq 0$, such that the equation (0.12) has a solution y for all $\lambda \geq \lambda_0$.

Theorem 0.15. *Let the functions $q = q(x)$ and $r = r(x)$ be continuous on R, $m = m(x) \in C^{(3)}_{loc}(R)$ and satisfy the conditions (0.14) - (0.17) and*

$$|m^{(3)}(x)| \leq c_3 m(x), \quad x \in R.$$

Then the solution of the equation (0.12) is unique and the estimate (0.13) holds.

Finally, in paper D the results from paper C are generalized to the situation when the equation (0.12) is replaced by the following more general equation:

$$(l + \lambda E)\, y := -\mu_1(x) \left(\mu_2(x) \left(\mu_1(x) y'\right)'\right)' + [q(x) + ir(x) + \lambda] y = f(x).$$

Here, $\lambda \geq 0$ is a constant, and $\mu_1(x)$, $\mu_2(x)$, $q(x)$ and $r(x)$ are given functions and $f \in L_p$.

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Paper A

ON SEPARATION OF A DEGENERATE DIFFERENTIAL OPERATOR IN HILBERT SPACE

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ABSTRACT. A coercive estimate for a solution of a degenerate second order differential equation is installed and its applications to spectral problems for the corresponding differential operator is demonstrated. The sufficient conditions for existence of the solutions of one class of the nonlinear second order differential equations on the real axis are obtained.

Key words and phrases: Hilbert space, separability of the operator, completely continuous resolvent.

Mathematics Subject Classification. 35J70

1. Introduction and main results

The concept of a separability was introduced in the fundamental paper [1]. The Sturm-Liouville's operator

$$Jy = -y'' + q(x)y, \quad x \in (a, +\infty),$$

is called separable [1] in space $L_2(a, +\infty)$, if $y, -y'' + qy \in L_2(a, +\infty)$ imply $-y'', qy \in L_2(a, +\infty)$. The separability of the operator J is equivalent to the following inequality

$$\|y''\|_{L_2(a,+\infty)} + \|qy\|_{L_2(a,+\infty)} \leq c\left(\|Jy\|_{L_2(a,+\infty)} + \|y\|_{L_2(a,+\infty)}\right), \quad y \in D(J). \tag{1.1}$$

In [1] (see also [2, 3]) for J some criteria of the separability depended on the behavior q and its derivatives are received, and an examples of not separable J with non-smooth potential q is shown. When q isn't necessarily differentiable function the sufficient separabilities conditions of J is obtained in [4, 5]. In [6, 7] it was developed so-called "the localization principle" of proof of the separability of higher order binomial elliptic operators in Hilbert space. In [8,9] it was shown that the local integrability and the semi-boundedness from below of q are sufficient for separability of J in space $L_1(-\infty, +\infty)$. The valuation method of Green's functions [1-3, 8, 9] (see also [10]), a parametrix method [4, 5], as well as a method of local estimates of the resolvents of some regular operators [6, 7] have been used in these works.

The sufficient conditions of the separability for the Sturm-Liouville's operator

$$y'' + Q(x)y$$

are obtained in [11-15] where Q is an operator. There are a number of works where a separation of the general elliptic, hyperbolic and mixed-type operators is discussed.

The separability estimate (1.1) is used in the spectral theory of J [15-18] and it allows us to prove an existence and a smoothness of solutions of one class of nonlinear differential equations in unbounded domains [11, 17-20]. Brown [21] has shown that certain properties of positive solutions of disconjugate second order differential expressions imply the separation. The connection of separation with the concrete physical problems is noted in [22].

The main aim of this paper is to study the separation, and approximate properties for the differential operator

$$ly := -y'' + r(x)y' + q(x)y$$

in Hilbert space $L_2 := L_2(R),\ R = (-\infty, +\infty)$, as well as the existence problem for certain nonlinear differential equation in L_2. The operator l is said to be separable in space L_2, if the following estimate holds:

$$\|y''\|_2 + \|ry'\|_2 + \|qy\|_2 \le c\left(\|ly\|_2 + \|y\|_2\right), \quad y \in D(l), \tag{1.2}$$

where $\|\cdot\|_2$ is the norm in L_2.

We assume that the function r is positive and increases at infinity faster than $|q|$. The operator l occurs in the oscillatory processes in a medium with a resistance that depends on velocity [23] (page 111-116). The operator J same as the operator l when $r = 0$. Nevertheless, note that the sufficient conditions for the invertibility, respectively of l and of J are principally different from each other. The separability estimate for l can not be obtained by applying of results of the works [1-15].

We denote

$$\alpha_{g,h}(t) = \|g\|_{L_2(0,t)} \left\|h^{-1}\right\|_{L_2(t,+\infty)} (t > 0), \quad \beta_{g,h}(\tau) = \|g\|_{L_2(\tau,0)} \left\|h^{-1}\right\|_{L_2(-\infty,\tau)} (\tau < 0),$$

$$\gamma_{g,h} = max\left(\sup_{t>0} \alpha_{g,h}(t), \sup_{\tau<0} \beta_{g,h}(\tau)\right),$$

where g and h are given functions. By $C^{(1)}_{loc}(R)$ we denote the set of functions f such that $\psi f \in C^{(1)}(R)$ for all $\psi \in C_0^\infty(R)$.

Theorem 1. *Let the function r satisfy the conditions*

$$r \in C^{(1)}_{loc}(R), \quad r \ge \delta > 0, \quad \gamma_{1,r} < \infty, \tag{1.3}$$

$$c^{-1} \le \frac{r(x)}{r(\eta)} \le c \ \ at \ \ |x - \eta| \le 1, \quad c > 1, \tag{1.4}$$

and the function q such that

$$\gamma_{q,r} < +\infty. \tag{1.5}$$

Then for $y \in D(l)$ the estimate

$$\|y''\|_2 + \|ry'\|_2 + \|qy\|_2 \le c_l \|ly\|_2 \tag{1.6}$$

holds, in particular, the operator l is separable in L_2.

Following Theorems 2-4 are applications of Theorem 1.

Theorem 2. *Let functions q and r satisfy the conditions (1.3)-(1.5) and the equalities $\lim_{t\to+\infty}\alpha_{q,r}(t)=0$, $\lim_{\tau\to-\infty}\beta_{q,r}(\tau)=0$ hold. Then an inverse operator l^{-1} is completely continuous in L_2.*

We assume that the conditions of Theorem 2 hold and consider a set

$$M=\left\{y\in L_2:\|ly\|_2\le 1\right\}.$$

Let

$$d_k=\inf_{\Sigma_k\subset\{\Sigma_k\}}\sup_{y\in M}\inf_{w\in\Sigma_k}\|y-w\|_2\,(k=0,1,2,...)$$

be the Kolmogorov's widths of the set M in L_2. Here $\{\Sigma_k\}$ is a set of all subspaces Σ_k of L_2 whose dimensions are not more than k. Through $N_2(\lambda)$ denote the number of widths d_k which are not smaller than a given positive number λ. Estimates of the width's distribution function $N_2(\lambda)$ are important in the approximating problem of solutions of the equation $ly=f$. The following statement holds.

Theorem 3. *Let the conditions of Theorem 2 be fulfilled. Then the following estimates hold:*

$$c_1\lambda^{-2}\mu\left\{x:|q(x)|\le c_2^{-1}\lambda^{-1}\right\}\le N_2(\lambda)\le c_3\lambda^{-2}\mu\left\{x:|q(x)|\le c_2\lambda^{-1}\right\}.$$

Example. Let $q=-x^{\alpha}$, $(\alpha\ge 0)$ and $r=(1+x^2)^{\beta}$, $(\beta>0)$. Then the conditions of Theorem 1 are satisfied if $\beta\ge\frac{1+\alpha}{2}$. If $\beta>\frac{1+\alpha}{2}$ then the conditions of Theorem 3 are satisfied and for some $\epsilon>0$ the following estimates hold:

$$c_0\lambda^{\frac{-7-2\beta+\epsilon}{4}}\le N_2(\lambda)\le c_1\lambda^{\frac{-7-2\beta+\epsilon}{4}}.$$

Consider the following nonlinear equation

$$Ly=-y''+[r(x,y)]y'=f(x), \tag{1.7}$$

where $x\in R$, r is real-valued function and $f\in L_2$.

Definition 1. *A function $y\in L_2$ is called a solution of (1.7), if there is a sequence of twice continuously differentiable functions $\{y_n\}_{n=1}^{\infty}$ such that $\|\theta(y_n-y)\|_2\to 0$, $\|\theta(Ly_n-f)\|_2\to 0$ as $n\to\infty$ for any $\theta\in C_0^{\infty}(R)$.*

Theorem 4. *Let the function r be continuously differentiable with respect to both arguments and satisfy the following conditions*

$$r\ge\delta_0(1+x^2)\quad(\delta_0>0),\quad\sup_{|x-\eta|\le 1}\quad\sup_{|C_1|\le A,|C_2|\le A,|C_1-C_2|\le A}\frac{r(x,C_1)}{r(\eta,C_2)}<\infty. \tag{1.8}$$

Then there is a solution y of the equation (1.7) and

$$\|y''\|_2+\|[r(\cdot,y)]y'\|_2<\infty. \tag{1.9}$$

2. Auxiliary statements

The next statement is a corollary of the well known Muckenhoupt's inequality [24].

Lemma 2.1. *Let the functions g and h such that $\gamma_{g,h} < \infty$. Then for $y \in C_0^\infty(R)$ the following inequality holds:*

$$\int_{-\infty}^{\infty} |g(x)y(x)|^2 dx \leq C \int_{-\infty}^{\infty} |h(x)y'(x)|^2 dx. \tag{2.1}$$

Moreover, if C is a smallest constant for which the estimate (2.1) holds then $\gamma_{g,h} \leq C \leq 2\gamma_{g,h}$.

The following lemma is a special case of Theorem 2.2 [25].

Lemma 2.2. *Let the given function r satisfy conditions*

$$\lim_{x\to+\infty} \sqrt{x}\left\|r^{-1}\right\|_{L_2(x,+\infty)} = \lim_{x\to+\infty} \sqrt{x}\left(\int_x^\infty r^{-2}(t)dt\right)^{\frac{1}{2}} = 0,$$

$$\lim_{x\to-\infty} \sqrt{|x|}\left\|r^{-1}\right\|_{L_2(-\infty,x)} = \lim_{x\to-\infty} \sqrt{|x|}\left(\int_{-\infty}^x r^{-2}(t)dt\right)^{\frac{1}{2}} = 0. \tag{2.2}$$

Then the set

$$F_k = \left\{ y : y \in C_0^\infty(R),\ \int_{-\infty}^{+\infty} |r(t)y'(t)|^2 dt \leq K \right\}, \quad K > 0,$$

is a relatively compact in $L_2(R)$.

Denote by $\mathscr{L}$ a closure in L_2-norm of the differential expression

$$\mathscr{L}_0 z = -z' + rz \tag{2.3}$$

defined on the set $C_0^\infty(R)$.

Lemma 2.3. *Let the function r satisfy conditions (1.3) and (1.4). Then the operator $\mathscr{L}$ is boundedly invertible and separable in L_2. Moreover, for $z \in D(\mathscr{L})$ the following estimate holds:*

$$\|z'\|_2 + \|rz\|_2 \leq c\,\|\mathscr{L}z\|_2\,. \tag{2.4}$$

Proof. Let $\mathscr{L}_\lambda = \mathscr{L} + \lambda E, \quad \lambda \geq 0$. Note that the operators $\mathscr{L}$ and $\mathscr{L}_\lambda = \mathscr{L} + \lambda E$ are separated to one and the same time. If z is a continuously differentiable function with the compact support, then

$$(\mathscr{L}_\lambda z, z) = -\int_R z'\bar{z}dx + \int_R [r(x) + \lambda]|z|^2 dx. \tag{2.5}$$

But

$$T := -\int_R z'\bar{z}dx = \int_R z\bar{z}'dx = -\bar{T}.$$

Therefore $ReT = 0$ and it follows from (2.5)

$$Re(\mathscr{L}_\lambda z, z) = \int\limits_R [r(x) + \lambda]|z|^2 dx. \tag{2.6}$$

We estimate the left-hand side of the equality (2.6) by using the Hölder's inequality. Then we have

$$\left\| \sqrt{r(\cdot) + \lambda} z \right\|_2 \le \left\| \frac{1}{\sqrt{r(\cdot) + \lambda}} \mathscr{L}_\lambda z \right\|_2 . \tag{2.7}$$

It is easy to show that (2.7) holds for any solution of (2.3).

Let $\Delta_j = (j-1, j+1)$ $(j \in Z)$, $\{\varphi_j\}_{j=-\infty}^{+\infty}$ be a sequence of such functions from $C_0^\infty(\Delta_j)$, that

$$0 \le \varphi_j \le 1, \quad \sum_{j=-\infty}^{+\infty} \varphi_j^2(x) = 1.$$

We continue $r(x)$ from Δ_j to R so that its continuation $r_j(x)$ was a bounded and periodic function with period 2. Denote by $\mathscr{L}_{\lambda,j}$ the closure in $L_2(R)$ of the differential operator $-z' + [r_j(x) + \lambda]z$ defined on the set $C_0^\infty(R)$. Similarly to the derivation of (2.7) one can proof the inequality

$$\left\| (r_j + \lambda)^{\frac{1}{2}} z \right\|_2 \le \left\| (r_j + \lambda)^{-\frac{1}{2}} \mathscr{L}_{\lambda,j} z \right\|_2, \quad z \in D(\mathscr{L}_{\lambda,j}). \tag{2.8}$$

It follows from the estimates (2.7), (2.8) and from general theory of linear differential equations that the operators $\mathscr{L}_\lambda, \mathscr{L}_{\lambda,j}$ $(j \in Z)$ are invertible and their inverses $\mathscr{L}_\lambda^{-1}$ and $\mathscr{L}_{\lambda,j}^{-1}$ are defined in all L_2. From the estimate (2.8) by (1.4) follows

$$\|\mathscr{L}_{\lambda,j} z\|_2 \ge c \sup_{x \in \Delta_j} [r_j(x) + \lambda] \, \|z\|_2 , \quad z \in D(\mathscr{L}_{\lambda,j}). \tag{2.9}$$

Let us introduce the operators B_λ and M_λ:

$$B_\lambda f = \sum_{j=-\infty}^{+\infty} \varphi_j'(x) \mathscr{L}_{\lambda,j}^{-1} \varphi_j f, \quad M_\lambda f = \sum_{j=-\infty}^{+\infty} \varphi_j(x) \mathscr{L}_{\lambda,j}^{-1} \varphi_j f.$$

At any point $x \in R$ the sums of the right-hand side in these terms contain no more than two summands, so B_λ and M_λ is defined on all L_2. It is easy to show that

$$\mathscr{L}_\lambda M_\lambda = E + B_\lambda. \tag{2.10}$$

Using (2.9) and properties of the functions φ_j $(j \in Z)$ we find that $\lim\limits_{\lambda \to +\infty} \|B_\lambda\| = 0$, hence there exists a number λ_0, such that $\|B_\lambda\| \le \frac{1}{2}$ for all $\lambda \ge \lambda_0$. Then it follows from (2.10)

$$\mathscr{L}_\lambda^{-1} = M_\lambda (E + B_\lambda)^{-1}, \quad \lambda \ge \lambda_0. \tag{2.11}$$

By (2.11) and using properties of the functions φ_j $(j \in Z)$ again, we have

$$\left\| (r + \lambda) \mathscr{L}_\lambda^{-1} f \right\|_2 \le c_1 \sup_{j \in Z} \left\| (r + \lambda) \mathscr{L}_{\lambda,j}^{-1} \right\|_{L_2(\Delta_j)} \|f\|_2 . \tag{2.12}$$

From (2.9) by conditions (1.4) follows

$$\sup_{j\in Z}\left\|(r+\lambda)\mathscr{L}_{\lambda,j}^{-1}F\right\|_{L_2(\Delta_j)}\le\frac{\sup\limits_{x\in\Delta_j}[r(x)+\lambda]}{\inf\limits_{x\in\Delta_j}[r(x)+\lambda]}\left\|F\right\|_{L_2(\Delta_j)}\le$$

$$\le\sup_{|x-z|\le 2}\frac{r(x)+\lambda}{r(z)+\lambda}\left\|F\right\|_{L_2(\Delta_j)}\le c_2\left\|F\right\|_{L_2(\Delta_j)}.$$

From the last inequalities and (2.12) we obtain $\|(r+\lambda)z\|_2\le c_3\|\mathscr{L}_\lambda z\|_2, \quad z\in D(\mathscr{L}_\lambda)$, therefore

$$\|z'\|_2+\|(r+\lambda)z\|_2\le(1+2c_3)\|\mathscr{L}_\lambda z\|_2.$$

From this taking into account (2.7) we have the estimate (2.4). The lemma is proved. □

Denote by L a closure in the L_2 -norm of the differential expression

$$L_0y=-y''+r(x)y'$$

defined on the set $C_0^\infty(R)$.

Lemma 2.4. *Assume that the function r satisfies the condition (1.3). Then for $y\in D(L)$ the estimate*

$$\left\|\sqrt{r}y'\right\|_2+\|y\|_2\le c\|Ly\|_2 \tag{2.13}$$

holds.

Proof. Let $y\in C_0^\infty(R)$. Integrating by parts, we have

$$(Ly,y')=-\int\limits_R y''\bar{y}'dx+\int\limits_R r(x)|y'|^2dx. \tag{2.14}$$

Since

$$A:=-\int\limits_R y''\bar{y}'dx=\int\limits_R y'\bar{y}''dx=-\bar{A},$$

we see $ReA=0$.
Therefore, it follows from (2.14)

$$Re\,(Ly,y')=\int\limits_R r(x)|y'|^2dx.$$

Hence, applying the Hölder's inequality and using the condition (1.3) we obtain the following estimate

$$c_0\left\|\sqrt{r}y'\right\|_2\le\|Ly\|_2. \tag{2.15}$$

The inequality (2.15) and Lemma 2.1 imply the estimate (2.13) for $y\in C_0^\infty(R)$. If y is an arbitrary element of $D(L)$, then there is a sequence of functions $\{y_n\}_{n=1}^\infty\subset C_0^\infty(R)$ such that $\|y_n-y\|_2\to 0, \quad \|Ly_n-Ly\|_2\to 0$ as $n\to\infty$. For y_n the estimate (2.13) holds. From (2.13) taking the limit as $n\to\infty$ we obtain the same estimate for y. The lemma is proved. □

Remark 2.1. The statement of Lemma 2.1 is valid, if $r(x)$ is a complex-valued function and instead of (1.3) the conditions

$$Re\ r \geq \delta > 0, \quad \gamma_{1,Re\ r} < \infty, \tag{2.16}$$

hold. It follows from Lemma 2.1 that the conditions related to the function r in Lemma 2.4 are natural.

We consider the equation

$$Ly \equiv -y'' + r(x)y' = f, \quad f \in L_2. \tag{2.17}$$

By a solution of (2.17) we mean a function $y \in L_2$ for which there exists a sequence $\{y_n\}_{n=1}^{\infty} \subset C_0^{\infty}(R)$ such that $\|y_n - y\|_2 \to 0$ and $\|Ly_n - f\|_2 \to 0$, $n \to \infty$.

Lemma 2.5. *If the function r satisfies the condition (1.3) then the equation (2.17) has a unique solution. If, in addition, the function r satisfies the condition (1.4) then for a solution y of the equation (2.17) the following estimate*

$$\|y''\|_2 + \|ry'\|_2 \leq c_L \|Ly\|_2$$

holds i.e. the operator L is separated in the space L_2.

Proof. It follows from the estimate (2.13) that a solution y of the equation (2.17) is unique and belongs to $W_2^1(R)$. Let us prove that the equation (2.17) is solved. Assume the contrary. Then $R(L) \neq L_2$, and there exists a non-zero element $z_0 \in L_2$ such that $z_0 \perp R(L)$. According to operator's theory z_0 is a generalized solution of the equation

$$L^* y \equiv -y' + [r(x)y]' = 0,$$

where L^* is an adjoint operator. Then

$$-z_0' + r(x)z_0 = C.$$

Without loss of generality, we set $C = 1$. Then

$$z_0 = c_0 exp\left[-\int_a^x r(t)dt\right] + \int_a^x exp\left[-\int_a^t r(\tau)d\tau\right] dt := z_1 + z_2. \tag{2.18}$$

In (2.18) if $c_0 > 0$, then $z_0 \geq c_0$ when $x > a$. If in (2.18) $c_0 \leq 0$, then $z_1 \to 0$ when $x \to -\infty$ and $|z_2(x)| \geq c_1 exp[-\delta_0 x]$ $(0 < \delta_0 < \delta)$ when $x << a$. So $z_0 \notin L_2$. We obtained a contradiction, which shows that the solution of the equation (2.17) exists.

Further, it follows from Lemma 2.3 that the operator $\mathscr{L}$ is separated in L_2. Then by construction the operator L is also separated in L_2. The proof is complete. $\square$

Lemma 2.6. *Let the function r satisfy conditions (1.3), (1.4), $\gamma_{1,r} < \infty$ and*

$$\lim_{t\to+\infty} \sqrt{t}\left\|r^{-1}\right\|_{L_2(t,+\infty)} = 0, \quad \lim_{t\to-\infty} \sqrt{|t|}\left\|r^{-1}\right\|_{L_2(-\infty,t)} = 0. \tag{2.19}$$

Then the inverse operator L^{-1} is completely continuous in L_2.

Proof. From Lemma 2.5 follows that the operator L^{-1} exists and translates L_2 into space $W_{2,r}^2(R)$ with the norm $\|y''\|_2 + \|ry'\|_2 + \|y\|_2$. By Lemma 2.2 and (2.19) space $W_{2,r}^2(R)$ is compactly embedded into L_2. The proof is complete. $\square$

3. Proofs of Theorems 1-4

Proof of Theorem 1. It follows from Lemma 2.5 that the operator $Ly \equiv -y''+r(x)y'$ is separated in L_2. From (1.5) and (2.1) we get the estimates

$$\|qy\|_2 \leq 2\gamma_{q,r}\|ry'\|_2 \leq \frac{2}{\sqrt{\delta}}\gamma_{q,r}c\,\|Ly\|_2\,, \quad y \in D(L).$$

This means that the operator $l = L + qE$ is also separated in L_2. The theorem is proved. □

Theorem 2 is a consequence of Lemma 2.2, Lemma 2.5 and Theorem 1.

Statement of Theorem 3 follows from Theorem 2 and Theorem 1 [26].

Proof of Theorem 4. Let ϵ and A be positive numbers. We denote

$$S_A = \left\{ z \in W_2^1(R) : \|z\|_{W_2^1(R)} \leq A \right\}.$$

Let ν be an arbitrary element of S_A. Consider the following linear "perturbed" equation

$$l_{0,\nu,\epsilon}y \equiv -y'' + \left[r(x,\nu(x)) + \epsilon(1+x^2)^2\right]y' = f(x). \tag{3.1}$$

Denote by $l_{\nu,\epsilon}$ the minimal closed in L_2 operator generated by expression $l_{0,\nu,\epsilon}y$. Since

$$r_\epsilon(x) := r(x,\nu(x)) + \epsilon(1+x^2)^2 \geq 1 + \epsilon(1+x^2)^2,$$

the function $r_\epsilon(x)$ satisfies the condition (1.3). Further, when $|x-\eta| \leq 1$ for $\nu \in S_A$ we have

$$|\nu(x) - \nu(\eta)| \leq |x-\eta|\,\|\nu'\|_p \leq |x-\eta|\,\|\nu\|_{W_2^1} \leq A. \tag{3.2}$$

It is easy to verify that

$$\sup_{|x-\eta|\leq 1}\frac{(1+x^2)^2}{(1+\eta^2)^2} \leq 3.$$

Then, assuming $\nu(x) = C_1, \quad \nu(\eta) = C_2$, by (1.8) and the inequality (3.2) we obtain

$$\sup_{|x-\eta|\leq 1}\frac{r_\epsilon(x)}{r_\epsilon(\eta)} \leq \sup_{|x-\eta|\leq 1}\ \sup_{|C_1|\leq A,|C_2|\leq A,|C_1-C_2|\leq A}\frac{r(x,C_1)}{r(\eta,C_2)} + 3 < \infty.$$

Thus the coefficient $r_\epsilon(x)$ in (3.1) satisfies the conditions of Lemma 2.5. Therefore, the equation (3.1) has unique solution y and for y the estimate

$$\|y''\|_2 + \left\|[r(\cdot,\nu(\cdot)) + \epsilon(1+x^2)^2]y'\right\|_2 \leq C_3\,\|f\|_2 \tag{3.3}$$

holds (an operator $l_{\nu,\epsilon}$ is separated). By (1.8) and (2.1)

$$\|y\|_2 \leq C_0\,\|ry'\|_2\,, \quad \left\|(1+x^2)y\right\|_2 \leq C_4\left\|(1+x^2)^2y'\right\|_2. \tag{3.4}$$

Taking them into account from (3.3) we have

$$\|y''\|_2 + \frac{1}{2}\left\|(1+x^2)y'\right\|_2 + \frac{1}{2C_0}\|y\|_2 + \frac{\epsilon}{C_4}\left\|(1+x^2)y\right\|_2 \leq C_3\,\|f\|_2\,.$$

Then for some $C_5 > 0$ the following estimate

$$\|y\|_W := \|y''\|_2 + \left\|(1+x^2)y'\right\|_2 + \left\|[1+\epsilon(1+x^2)]y\right\|_2 \leq C_5\,\|f\|_2 \tag{3.5}$$

holds. We choose $A = C_5 \|f\|_2$ and denote $P(\nu,\epsilon) := L_{\nu,\epsilon}^{-1} f$. From the estimate (3.5) follows that the operator $P(\nu,\epsilon)$ translates the ball $S_A \subset W_2^1(R)$ to itself. Moreover, the operator $P(\nu,\epsilon)$ translates the ball S_A into a set

$$Q_A = \left\{ y : \|y''\|_2 + \left\|(1+x^2)y'\right\|_2 + \left\|[1+\epsilon(1+x^2)]y'\right\|_2 \le C_5 \|f\|_2 \right\}.$$

The set Q_A is the compact in Sobolev's space $W_2^1(R)$. Indeed, if $y \in Q_A$, $h \neq 0$ and $N > 0$ then the following relations (3.6), (3.7) hold:

$$\|y(\cdot+h) - y(\cdot)\|_{W_2^1(R)}^2 = \int_{-\infty}^{+\infty} \left[|y'(t+h) - y'(t)|^2 + |y(t+h) - y(t)|^2 \right] dt =$$

$$= \int_{-\infty}^{+\infty} \left[\left| \int_t^{t+h} y''(\eta) d\eta \right|^2 + \left| \int_t^{t+h} y'(\eta) d\eta \right|^2 \right] dt \le$$

$$\le |h| \int_{-\infty}^{+\infty} \left[\left| \int_t^{t+h} y''(\eta) d\eta \right| + \left| \int_t^{t+h} y'(\eta) d\eta \right| \right] dt =$$

$$= |h|^2 \int_{-\infty}^{+\infty} \left[|y''(\eta)|^2 + |y'(\eta)|^2 \right] d\eta \le C_5 \|f\|_2 |h|^2, \tag{3.6}$$

$$\|y\|_{W_2^1(R \setminus [-N,N])}^2 = \int_{|\eta| \ge N} \left[|y'(\eta)|^2 + |y(\eta)|^2 \right] d\eta \le$$

$$\le \int_{|\eta| \ge N} (1+\eta^2)^{-2} \left[|y''(\eta)|^2 + (1+\eta^2)^2 |y'(\eta)|^2 + (1+\eta^2)^2 |y(\eta)|^2 \right] d\eta \le$$

$$\le C_5^2 \|f\|_2^2 (1+N^2)^{-2}. \tag{3.7}$$

The expressions in the right-hand side of (3.6) and (3.7), respectively, tend to zero as $h \to 0$ and as $N \to +\infty$. Then by Frechét-Kolmogorov criterion the set Q_A is compact in space $W_2^1(R)$. Hence $P(\nu,\epsilon)$ is a compact operator.

Let us show that the operator $P(\nu,\epsilon)$ is continuous with respect to ν in S_A. Let $\{\nu_n\} \subset S_A$ be a sequence such that $\|\nu_n - \nu\|_{W_2^1} \to 0$ as $n \to \infty$, and y_n and y such that $L_{\nu,\epsilon}^{-1} y = f$, $L_{\nu_n,\epsilon}^{-1} y_n = f$. Then it is sufficient to show that the sequence $\{y_n\}$ converges to y in $W_2^1(R)$ - norm as $n \to \infty$. We have

$$P(\nu_n,\epsilon) - P(\nu,\epsilon) = y_n - y = L_{\nu_n,\epsilon}^{-1} [r(x,\nu_n(x)) - r(x,\nu(x))] y_n'.$$

The functions $\nu(x)$ and $\nu_n(x)$ $(n = 1, 2, ...)$ are continuous, then by conditions of the theorem the difference $r(x, \nu_n(x)) - r(x, \nu(x))$ is also continuous with respect to x, so that for each finite interval $[-a, a]$, $\quad a > 0$, we have

$$\|y_n - y\|_{W_2^1(-a,a)} \le c \max_{x\in[-a,a]} |r(x, \nu_n(x)) - r(x, \nu)| \cdot \|y_n'\|_{L_2(-a,a)} \to 0 \tag{3.8}$$

as $n \to \infty$. On the other hand, it follows from Lemma 2.4 that $\{y_n\} \in Q_A$, $\quad \|y_n\|_W \le A$, $\quad y \in Q_A$, $\quad \|y\|_W \le A$. Since the set Q_A is compact in $W_2^1(R)$, then $\{y_n\}$ converges in the norm of $W_2^1(R)$. Let z be a limit. By properties of $W_2^1(R)$

$$\lim_{|x|\to\infty} y(x) = 0, \quad \lim_{|x|\to\infty} z(x) = 0. \tag{3.9}$$

Since $L_{\nu,\epsilon}^{-1}$ is a closed operator, from (3.8) and (3.9) we obtain $y = z$. So $\|P(\nu_n, \epsilon) - P(\nu, \epsilon)\|_{W_2^1(R)} \to 0, \quad n \to \infty$.

Hence $P(\nu, \epsilon)$ is the completely continuous operator in space $W_2^1(R)$ and translates the ball S_A to itself. Then, by Schauder's theorem the operator $P(\nu, \epsilon)$ has in S_A a fixed point y $\ (P(y, \epsilon) = y)$ and y is a solution of the equation

$$L_\epsilon y := -y'' + \left[r(x, y) + \epsilon(1 + x^2)^2\right] y' = f(x).$$

By (3.3) for y the estimate

$$\|y''\|_2 + \left\| \left[r(\cdot, y) + \epsilon(1 + x^2)^2\right] y' \right\|_2 \le C_3 \|f\|_2$$

holds.

Now, suppose that $\{\epsilon_j\}_{j=1}^\infty$ is a sequence of the positive numbers converged to zero. The fixed point $y_j \in S_A$ of the operator $P(\nu, \epsilon_j)$ is a solution of the equation

$$L_{\epsilon_j} y_j := -y_j'' + \left[r(x, y_j) + \epsilon_j(1 + x^2)^2\right] y_j' = f(x).$$

For y_j the estimate

$$\left\|y_j''\right\|_2 + \left\| \left[r(\cdot, y_j(\cdot)) + \epsilon(1 + x^2)^2\right] y_j' \right\|_2 \le C_3 \|f\|_2 \tag{3.10}$$

holds.

Suppose (a, b) is an arbitrary finite interval. By (3.10) from the sequence $\{y_j\}_{j=1}^\infty \subset W_2^2(a, b)$ one can select a subsequence $\left\{y_{\epsilon_j}\right\}_{j=1}^\infty$ such that $\left\|y_{\epsilon_j} - y\right\|_{L_2[a,b]} \to 0$ as $j \to \infty$. A direct verification shows that y is a solution of the equation (1.7). In (3.10) passing to the limit as $j \to \infty$ we obtain (1.9). The theorem is proved. $\square$

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Paper B

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Separation and the existence theorem for second order nonlinear differential equation[1]

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Abstract. Sufficient conditions for the invertibility and separability in $L_2(-\infty, +\infty)$ of the degenerate second order differential operator with complex-valued coefficients are obtained, and its applications to the spectral and approximate problems are demonstrated. Using a separability theorem, which is obtained for the linear case, the solvability of nonlinear second order differential equation is proved on the real axis.

Keywords: separability of the operator, complex-valued coefficients, completely continuous resolvent.

Mathematics subject classifications: 34B40

1. Introduction and main results

A concept of the separability was introduced in the fundamental paper [1]. The Sturm-Liouville's operator

$$Jy = -y'' + q(x)y, \quad x \in (a, +\infty)$$

is called separable [1] in $L_2(a, +\infty)$, if $y,\ -y'' + qy \in L_2(a, +\infty)$ imply $-y'', qy \in L_2(a, +\infty)$. From this it follows that the separability of J is equivalent to the existence of the estimate

$$\|y''\|_{L_2(a,+\infty)} + \|qy\|_{L_2(a,+\infty)} \leq c \left(\|Jy\|_{L_2(a,+\infty)} + \|y\|_{L_2(a,+\infty)} \right), \quad y \in D(J), \qquad (1.1)$$

where $D(J)$ is the domain of J. In [1] (see also [2, 3]) some criteria of the separability depended on a behavior q and its derivatives has been obtained for J. Moreover, an example of non-separable operator J with non-smooth potential q was shown in this papers. Without differentiability condition on function q the sufficient conditions for the separability of J has been obtained in [4, 5]. In [6,7] so-called Localization Principle of the proof for the separability of higher order binomial elliptic operators was developed in Hilbert space. In [8,9] it was shown that local integrability and semiboundedness from below of q are enough for separability of J in $L_1(-\infty, +\infty)$. Valuation method of Green's functions [1-3,8,9] (see also [10]), parametrix method [4,5], as well as method of local estimates for the resolvents of some regular operators [6, 7] have been used in these works.

Sufficient conditions of the separability for the Sturm-Liouville's operator

$$y'' + Q(x)y$$

have been obtained in [11-15], where Q is an operator. A number of works were devoted to the separation problem for the general elliptic, hyperbolic and mixed-type operators.

An application of the separability estimate (1.1) in the spectral theory of J has been shown in [15-18], and it allows us to prove an existence and a smoothness of solutions of nonlinear differential equations in unbounded domains [11, 17-20]. Brown [21] has shown that certain properties of positive solutions of disconjugate second

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order differential expressions imply the separation. The connection of separation with concrete physical problems has been noted in [22].

We denote $L_2 := L_2(\mathbb{R})$, $\mathbb{R} = (-\infty, +\infty)$, the space of square integrable functions. Let l is a closure in L_2 of the expression $l_0 y = -y'' + r(x)y' + s(x)\bar{y}'$ defined in the set $C_0^\infty(\mathbb{R})$ of all infinitely differentiable and compactly sapported functions. Here r and s are complex - valued functions and $\bar{y}$ is the complex conjugate to y.

In this report we investigate some problems for the operator l. Although the operator l, similarly to the Sturm-Liouville operator J, is a singular differential operator of second order, their properties are different. The theory of the Sturm-Liouville operator J, in contrast to the operator l, developing a long time, while the idea of research is often based on the positivity of the potential $q(x)$ (see e.g. [1-20]). Because of the coefficients r and s, are the methods developed for the Sturm-Liouville problems are often not applicable to the study of the operator l. The spectral properties for self-adjoint singular differential operators of second order have been to a certain extent investigated, without the free term; a review of literature can be found in [23, 24]. Note that the differential operator l is used, in particular, in the oscillatory processes in the medium with resistance depended on velocity [25, pp. 111-116].

The operator l is said to be separable in L_2 if the following estimate holds:

$$\|y''\|_2 + \|ry'\|_2 + \|s\bar{y}'\|_2 \le c\left(\|ly\|_2 + \|y\|_2\right), \ y \in D(l),$$

where $\|\cdot\|_2$ is the L_2- norm. In the present communication the sufficient conditions for the invertibility and separability of the differential operator l are obtained. Moreover, spectral and approximate results for the inverse operator l^{-1} are achieved. Using a separation theorem, which is obtained for the linear case, the solvability of the degenerate nonlinear second order differential equation $-y'' + r(x,y)y' = F$ $(x \in \mathbb{R})$ is proved.

Let's consider the degenerate differential equation

$$ly = -y'' + r(x)y' + s(x)\bar{y}' = f. \tag{1.2}$$

The function $y \in L_2$ is called a solution of (1.2) if there exists a sequence $\{y_n\}_{n=1}^{+\infty}$ such that $\|y_n - y\|_2 \to 0$, $\|ly_n - f\|_2 \to 0$ as $n \to +\infty$. If the operator l is separable, then the solution y of (1.2) belongs to the weighted Sobolev space $W_2^2(\mathbb{R}, |r| + |s|)$ with the norm $\|y''\|_2 + \|(|r| + |s|)y'\|_2$. So, the study of the qualitative behavior of solutions of (1.2) and spectral and approximative properties of l can be reduced to the investigation of embedding $W_2^2(\mathbb{R}, |r| + |s|) \hookrightarrow L_2$.

We denote

$$\alpha_{g,h}(t) = \|g\|_{L_2(0,t)} \|1/h\|_{L_2(t,+\infty)} \, (t > 0), \quad \beta_{g,h}(\tau) = \|g\|_{L_2(\tau,0)} \|1/h\|_{L_2(-\infty,\tau)} \, (\tau < 0),$$

$$\gamma_{g,h} = \max\left(\sup_{t>0} \alpha_{g,h}(t), \sup_{\tau<0} \beta_{g,h}(\tau)\right),$$

where g and h are given functions. By $C_{loc}^{(1)}(\mathbb{R})$ we denote the set of functions f such that $\psi f \in C^{(1)}(\mathbb{R})$ for all $\psi \in C_0^\infty(\mathbb{R})$.

Theorem 1. *Let functions r and s satisfy the conditions*

$$r, s \in C_{loc}^{(1)}(\mathbb{R}), \quad Re\ r - |s| \ge \delta > 0, \quad \gamma_{1,Re\ r} < \infty. \tag{1.3}$$

Then l is invertible and l^{-1} is defined in all L_2.

Theorem 2. *Assume that functions r and s satisfy the conditions*

$$\begin{cases} r, s \in C^{(1)}_{loc}(\mathbb{R}),\ Re\ r - \rho[|Im\ r| + |s|] \geq \delta > 0,\ \gamma_{1,Re\ r} < \infty,\ 1 < \rho < 2, \\ c^{-1} \leq \frac{Re\ r(x)}{Re\ r(\eta)} \leq c \ \text{ at } \ |x - \eta| \leq 1,\ c > 1. \end{cases} \tag{1.4}$$

Then for $y \in D(l)$ the estimate

$$\|y''\|_2 + \|ry'\|_2 + \|s\bar{y}'\|_2 \leq c_l \|ly\|_2 \tag{1.5}$$

holds, i.e. the operator l is separable in L_2.

We use the statement of Theorem 2 for proof of the following Theorems 3-5.

Theorem 3. *Assume that functions r and s satisfy (1.4) and let $\lim\limits_{t\to+\infty} \alpha_{1,Re\,r}(t) = 0$, $\lim\limits_{\tau\to-\infty} \beta_{1,Re\,r}(\tau) = 0$. Then l^{-1} is completely continuous in L_2.*

We assume that the conditions of Theorem 3 hold and consider a set

$$M = \{y \in L_2 : \|ly\|_2 \leq 1\}.$$

Let

$$d_k = \inf_{\Sigma_k \subset \{\Sigma_k\}} \sup_{y\in M} \inf_{w\in\Sigma_k} \|y - w\|_2 \quad (k = 0, 1, 2, ...)$$

be the Kolmogorov's widths of the set M in L_2. Here $\{\Sigma_k\}$ is a set of all subspaces Σ_k of L_2 whose dimensions are not greater than k. Through $N_2(\lambda)$ denote the number of widths d_k which are not smaller than a given positive number λ. Estimates of the width's distribution function $N_2(\lambda)$ are important in the approximation problems of solutions of the equation $ly = f$. The following statement holds.

Theorem 4. *Assume that the conditions of Theorem 3 be fulfilled and let a function q satisfy $\gamma_{q,Re\ r} < \infty$. Then the following estimates hold:*

$$c_1\lambda^{-2}\mu\left\{x : |q(x)| \leq c_2^{-1}\lambda^{-1}\right\} \leq N_2(\lambda) \leq c_3\lambda^{-2}\mu\left\{x : |q(x)| \leq c_2\lambda^{-1}\right\},$$

where μ is the Lebesgue measure.

Example. Assume that $r = (1 + x^2)^\beta \ \ (\beta > 0)$ and let $s = 0$. Then the conditions of Theorem 2 are satisfied if $\beta \geq 1/2$. If $\beta > 1/2$, then the conditions of Theorem 4 are satisfied and the following estimates hold:

$$c_4\lambda^{\frac{-2\beta+3}{2(2\beta-1)}} \leq N_2(\lambda) \leq c_5\lambda^{\frac{-2\beta+3}{2(2\beta-1)}}.$$

We consider the following nonlinear equation

$$Ly = -y'' + [r(x, y)]y' = f(x), \tag{1.6}$$

where $x \in \mathbb{R}$, r is a real-valued function and $f \in L_2$.

A function $y \in L_2$ is called a solution of equation (1.6), if there exists a sequence of twice continuously differentiable functions $\{y_n\}_{n=1}^\infty$ such that $\|\theta(y_n - y)\|_2 \to 0$, $\|\theta(Ly_n - f)\|_2 \to 0$ as $n \to \infty$ for any $\theta \in C_0^\infty(\mathbb{R})$.

Theorem 5. *Let the function r be continuously differentiable with respect to both arguments and satisfy the following conditions*

$$r \geq \delta_0\sqrt{1+x^2} \quad (\delta_0 > 0), \qquad \sup_{x,\,\eta\in R:\,|x-y|\leq 1} \quad \sup_{A>0} \quad \sup_{|C_1|\leq A, |C_2|\leq A, |C_1-C_2|\leq A} \frac{r(x,C_1)}{r(\eta,C_2)} < \infty. \tag{1.7}$$

Then there exists a solution y of (1.6), and

$$\|y''\|_2 + \|[r(\cdot,y)]y'\|_2 < \infty. \tag{1.8}$$

2. Auxiliary statements

The next statement is a corollary of the well known Muckenhoupt's inequality [26].

Lemma 2.1. *Let functions g and h such that $\gamma_{g,h} < \infty$. Then for all $y \in C_0^\infty(\mathbb{R})$ the following inequality holds:*

$$\int_{-\infty}^{\infty} |g(x)y(x)|^2 dx \leq C \int_{-\infty}^{\infty} |h(x)y'(x)|^2 dx. \tag{2.1}$$

Moreover, if C is a smallest constant for which estimate (2.1) holds, then $\gamma_{g,h} \leq C \leq 2\gamma_{g,h}$.

The following lemma is a particular case of Theorem 2.2 [23].

Lemma 2.2. *Let the given function h satisfy conditions*

$$\lim_{x\to+\infty} \sqrt{x}\left(\int_x^{\infty} h^{-2}(t)dt\right)^{\frac{1}{2}} = 0,$$

$$\lim_{x\to-\infty} \sqrt{|x|}\left(\int_{-\infty}^{x} h^{-2}(t)dt\right)^{\frac{1}{2}} = 0.$$

Then the set

$$F_K = \left\{ y : y \in C_0^\infty(\mathbb{R}), \int_{-\infty}^{+\infty} |h(t)y'(t)|^2 dt \leq K \right\}, \quad K > 0$$

is a relatively compact in $L_2(\mathbb{R})$.

Denote by L a closure in L_2-norm of the differential expression

$$L_0 z = -z' + rz + s\bar{z} \tag{2.2}$$

defined on the set $C_0^\infty(\mathbb{R})$.

Lemma 2.3. *Assume that functions r and s satisfy condition (1.3). Then the operator L is boundedly invertible in L_2.*

Proof. Let $L_\lambda = L + \lambda E$, where $\lambda \geq 0$, and E be the identity map of L_2 to itself. Note that L is separable if and only if $L_\lambda = L + \lambda E$ is separable for some λ. If z is a continuously differentiate function with a compact support, then

$$(L_\lambda z, z) = -\int_{\mathbb{R}} z'\bar{z}dx + \int_{\mathbb{R}} [(r+\lambda)|z|^2 + s\bar{z}^2]dx. \tag{2.3}$$

But

$$T := -\int_{\mathbb{R}} z'\bar{z}dx = \int_{\mathbb{R}} z\bar{z}'dx = -\bar{T}.$$

Therefore $Re\, T = 0$ and from (2.3) it follows that

$$Re(L_\lambda z, z) \geq c\int_{\mathbb{R}} [Re\ r + \lambda - |s|]|z|^2 dx. \tag{2.4}$$

We estimate the left-hand side of inequality (2.4) by using the Hölder's inequality. Then by (1.3) we have $\|L_\lambda z\|_2 \geq \delta \|z\|_2$. This estimate implies that L_λ is invertible. Let us proof that L_λ^{-1} is defined in all L_2. Assume the contrary. Let $R(L_\lambda) \neq L_2$. Then there exists a non-zero element $z_0 \in L_2$ such that $z_0 \perp R(L_\lambda)$. According to operator's theory z_0 satisfies the equality

$$L_\lambda^* z_0 := z_0' + (\bar{r}+\lambda)z_0 + s\bar{z}_0 = 0, \tag{2.5}$$

where L_λ^* is an adjoint operator.

Let $\theta \in C_0^\infty(\mathbb{R})$ be a real function. Denote $\psi = \theta z_0$. From (2.5) it follows that $z_0 \in W_{2,loc}^1(\mathbb{R})$, then $\psi \in D(L_\lambda^*)$. Using (2.5), we get $L_\lambda^*\psi = \theta' z_0$. Hence

$$(L_\lambda^*\psi, \psi) = \int_{\mathbb{R}} \theta'\theta|z_0|^2 dx. \tag{2.6}$$

On the other hand using the expression $L_\lambda^*\psi$ we have

$$Re(L_\lambda^*\psi, \psi) = \int_{\mathbb{R}} \theta^2[Re(\bar{r}+\lambda)|z_0|^2 + Re(s\bar{z}_0^2)]dx \geq$$
$$\geq \int_{\mathbb{R}} \theta^2[Re\bar{r} + \lambda - |s|]|z_0|^2 dx.$$

Hence by (2.6) the following estimate

$$\delta\int_{\mathbb{R}} \theta^2|z_0|^2 dx \leq \int_{\mathbb{R}} \theta'\theta|z_0|^2 dx \tag{2.7}$$

holds. Choose the function θ such that

$$\theta(x) = \begin{cases} 1, & |x| \leq \xi \\ 0, & |x| \geq \xi + 1, \end{cases}$$

$0 \le \theta \le 1$, $|\theta'| \le C$. Here $\xi > 0$. Then it follows from (2.7)

$$\delta \int\limits_{-\xi-1}^{\xi+1} \theta^2 |z_0|^2 dx \le C \left[\int\limits_{-\xi-1}^{-\xi} |z_0|^2 dx + \int\limits_{\xi}^{\xi+1} |z_0|^2 dx \right].$$

Since $z_0 \in L_2$, passing to the limit as $\xi \to +\infty$ in the last inequality, we have $\|z_0\|_2 = 0$. Then $z_0 = 0$. We obtain the contradiction, which gives that $R(L_\lambda) = L_2$. The lemma is proved. □

Lemma 2.4. *Assume that functions r and s satisfy condition (1.4). Then L is separable in L_2 and for $z \in D(L)$ the following estimate holds:*

$$\|z'\|_2 + \|rz\|_2 + \|s\bar{z}\|_2 \le c\,\|Lz\|_2 . \tag{2.8}$$

Proof. From inequality (2.4) it follows that

$$\left\| \sqrt{Re\ r(\cdot) + \lambda} z \right\|_2 \le c_1 \left\| \frac{1}{\sqrt{Re\ r(\cdot) + \lambda}} L_\lambda z \right\|_2 . \tag{2.9}$$

It is easy to show that (2.9) holds for all z from $D(L_\lambda)$.

Let $\Delta_j = (j-1, j+1)$ $(j \in \mathbb{Z})$ and let $\{\varphi_j\}_{j=-\infty}^{+\infty}$ be a sequence of functions from $C_0^\infty(\Delta_j)$ such that

$$0 \le \varphi_j \le 1, \quad \sum_{j=-\infty}^{+\infty} \varphi_j^2(x) = 1.$$

We continue $r(x)$ and $s(x)$ from Δ_j to $\mathbb{R}$ so that its continuations $r_j(x)$ and $s_j(x)$ are bounded and periodic functions with period 2. Denote by $L_{\lambda,j}$ the closure in $L_2(\mathbb{R})$ of the differential operator $-z' + [r_j(x) + \lambda]z + s_j(x)\bar{z}$ defined on $C_0^\infty(R)$. Using the method which was applied for L_λ one can proof that $L_{\lambda,j}$ are invertible and $L_{\lambda,j}^{-1}$ are defined in all L_2. In addition, the following inequality

$$\left\| (Re\ r_j + \lambda)^{\frac{1}{2}} z \right\|_2 \le c_2 \left\| (Re\ r_j + \lambda)^{-\frac{1}{2}} L_{\lambda,j} z \right\|_2, \quad z \in D(L_{\lambda,j}), \tag{2.10}$$

holds. From estimate (2.10) by (1.4) it follows

$$\|L_{\lambda,j} z\|_2 \ge c_3 \sup_{x \in \Delta_j} [Re\ r_j(x) + \lambda]\, \|z\|_2, \quad z \in D(L_{\lambda,j}). \tag{2.11}$$

Let us introduce the operators B_λ and M_λ:

$$B_\lambda f = \sum_{j=-\infty}^{+\infty} \varphi_j'(x) L_{\lambda,j}^{-1} \varphi_j f, \quad M_\lambda f = \sum_{j=-\infty}^{+\infty} \varphi_j(x) L_{\lambda,j}^{-1} \varphi_j f.$$

At any point $x \in \mathbb{R}$ the sums of the right-hand side in these terms contain no more than two summands, therefore B_λ and M_λ is defined on all L_2. It is easy to show that

$$L_\lambda M_\lambda = E + B_\lambda. \tag{2.12}$$

Using (2.11) and properties of φ_j $(j \in \mathbb{Z})$ we find that $\lim\limits_{\lambda \to +\infty} \|B_\lambda\| = 0$, hence there exists a number λ_0 such that $\|B_\lambda\| \le 0.5$ for all $\lambda \ge \lambda_0$. Then it follows from (2.12)

$$L_\lambda^{-1} = M_\lambda (E + B_\lambda)^{-1}, \quad \lambda \ge \lambda_0. \tag{2.13}$$

Using (2.13) and properties of φ_j $(j \in \mathbb{Z})$ we have

$$\left\|(Re\ r+\lambda)L_\lambda^{-1}f\right\|_2 \le c_4 \sup_{j\in\mathbb{Z}} \left\|(Re\ r_j+\lambda)L_{\lambda,j}^{-1}\right\|_{L_2\to L_2} \|f\|_2 . \tag{2.14}$$

Further, (1.4) and (2.11) imply that

$$\sup_{j\in\mathbb{Z}} \left\|(Re\ r_j+\lambda)L_{\lambda,j}^{-1}F\right\|_{L_2(\mathbb{R})} \le c_5 \frac{\sup\limits_{x\in\Delta_j}[Re\ r(x)+\lambda]}{\inf\limits_{t\in\Delta_j}[Re\ r(t)+\lambda]} \|F\|_{L_2(\mathbb{R})} \le$$

$$\le c_5 \sup_{|x-z|\le 2} \frac{Re\ r(x)+\lambda}{Re\ r(z)+\lambda} \|F\|_{L_2(\mathbb{R})} \le c_6 \|F\|_{L_2(\mathbb{R})} .$$

From the last inequalities and (2.14) we obtain $\|(Re\ r+\lambda)z\|_2 \le c_7 \|L_\lambda z\|_2$, $z \in D(L_\lambda)$, therefore it follows from condition (1.4)

$$\|z'\|_2 + \|(r+\lambda)z\|_2 + \|s\bar{z}\|_2 \le c_8 \|L_\lambda z\|_2 .$$

When $\lambda = 0$ from this inequality we have estimate (2.8). The lemma is proved. □

Lemma 2.5. *Assume that functions r and s satisfy condition (1.3). Then for $y \in D(l)$ the estimate*

$$\|y'\|_2 + \|y\|_2 \le c \|ly\|_2 \tag{2.15}$$

holds.

Proof. Let $y \in C_0^\infty(\mathbb{R})$. Integrating by parts, we have

$$(ly, y') = -\int_{\mathbb{R}} y''\bar{y}'dx + \int_{\mathbb{R}} [r(x)|y'|^2 + s(x)(\bar{y}')^2]dx. \tag{2.16}$$

Since

$$A := -\int_{\mathbb{R}} y''\bar{y}'dx = \int_{\mathbb{R}} y'\bar{y}''dx = -\bar{A},$$

we see $Re\ A = 0$. Therefore, it follows from (2.16)

$$Re\ (ly, y') \ge \int_{\mathbb{R}} [Re\ r - |s|]|y'|^2 dx \ge \delta \|y'\|_2 .$$

Hence, using the Hölder's inequality, the condition $\gamma_{1,Re\ r} < \infty$ in (1.3) and Lemma 2.1 we obtain (2.15) for any $y \in C_0^\infty(\mathbb{R})$. If y is an arbitrary element of $D(l)$, then there exists a sequence $\{y_n\}_{n=1}^\infty \subset C_0^\infty(\mathbb{R})$ such that $\|y_n - y\|_2 \to 0$, $\|ly_n - ly\|_2 \to 0$ as $n \to \infty$. The estimate (2.15) holds for y_n. From (2.15) passing to the limit as $n \to \infty$ we obtain the same estimate for y. The lemma is proved. □

A function $y \in L_2$ is called a solution of the equation

$$ly \equiv -y'' + r(x)y' + s(x)\bar{y}' = f, \quad f \in L_2, \tag{2.17}$$

if there exists a sequence $\{y_n\}_{n=1}^\infty \subset C_0^\infty(\mathbb{R})$ such that $\|y_n - y\|_2 \to 0$, $\|ly_n - f\|_2 \to 0$, $n \to \infty$.

Lemma 2.6. *If functions r and s satisfy condition (1.3) then the equation (2.17) has a unique solution.*

Proof. From (2.15) it follows that the solution y of (2.17) is unique and belongs to $W_2^1(\mathbb{R})$. Lemma 2.3 shows that L^{-1} is defined in all L_2. Then by the construction (2.17) is solvable. The proof is complete. $\square$

3. Proofs of Theorems 1-4

Proof of Theorem 1. From (1.3) and Lemma 2.6 we obtain that l is invertible and l^{-1} is defined in all L_2. $\square$

Proof of Theorem 2. From Lemma 2.4 it follows that L is separated in L_2 under condition (1.4). And consequently, by construction $ly \equiv -y'' + r(x)y' + s(x)\bar{y}'$ is separated in L_2 and the estimate (1.5) holds. The theorem is proved. $\square$

Proof of Theorem 3. The estimate (1.5) shows that l^{-1} maps L_2 into space $\tilde{W}_2^2(\mathbb{R})$ with the norm $\|y''\|_2 + \|ry'\|_2 + \|s\bar{y}'\|_2 + \|y\|_2$. By condition of the theorem Lemma 2.2 implies that $\tilde{W}_2^2(\mathbb{R})$ is compactly embedded into L_2. The proof is complete. $\square$

Proof of Theorem 4. By Lemma 2.1 Theorem 2 implies that $\|y''\|_2 + \|qy\|_2 \leq c\,\|ly\|_2\,,\ \ y \in D(l)$. Then Theorem 1 [27] gives the estimates in Theorem 4. $\square$

Proof of Theorem 5. Let ϵ and A be positive numbers. We denote

$$S_A = \left\{ z \in W_2^1(\mathbb{R}) :\ \|z\|_{W_2^1(\mathbb{R})} \leq A \right\}.$$

Let ν be an arbitrary element of S_A. Consider the following linear "perturbed" equation

$$l_{0,\nu,\epsilon}y \equiv -y'' + [r(x,\nu(x)) + \epsilon(1+x^2)^2]y' = f(x). \tag{3.1}$$

Denote by $l_{\nu,\epsilon}$ the minimal closed operator in L_2 generated by expression $l_{0,\nu,\epsilon}y$. Since

$$r_\epsilon(x) := r(x,\nu(x)) + \epsilon(1+x^2)^2 \geq 1 + \epsilon(1+x^2)^2,$$

the function $r_\epsilon(x)$ satisfies condition (1.3). Further, if $|x-\eta| \leq 1$, then for $\nu \in S_A$ we have

$$|\nu(x) - \nu(\eta)| \leq |x-\eta|\,\|\nu'\|_p \leq |x-\eta|\,\|\nu\|_{W_2^1} \leq A. \tag{3.2}$$

It is easy to verify that

$$\sup_{|x-\eta|\leq 1} \frac{(1+x^2)^2}{(1+\eta^2)^2} \leq 9.$$

Now we assume that $\nu(x) = C_1,\ \ \nu(\eta) = C_2$. Then by (1.7) and (3.2) we obtain

$$\sup_{|x-\eta|\leq 1} \frac{r_\epsilon(x)}{r_\epsilon(\eta)} \leq \sup_{|x-\eta|\leq 1}\ \sup_{A>0}\ \sup_{|C_1|\leq A,\ |C_2|\leq A, |C_1-C_2|\leq A} \frac{r(x,C_1)}{r(\eta,C_2)} + 9\varepsilon < \infty.$$

Thus the coefficient $r_\epsilon(x)$ in (3.1) satisfies the conditions of Theorem 2. Therefore, (3.1) has a unique solution y and for y the estimate

$$\|y''\|_2 + \left\|[r(\cdot,\nu(\cdot)) + \epsilon(1+x^2)^2]y'\right\|_2 \leq C_3\,\|f\|_2 \tag{3.3}$$

holds (i.e. an operator $l_{\nu,\epsilon}$ is separated). By (1.7) and (2.1)

$$\|y\|_2 \leq C_0\,\|ry'\|_2\,, \quad \left\|(1+x^2)y\right\|_2 \leq C_4 \left\|(1+x^2)^2 y'\right\|_2. \tag{3.4}$$

Taking into account (3.4) from (3.3) we have

$$\|y''\|_2 + \frac{1}{2}\left\|(1+x^2)^2 y'\right\|_2 + \frac{1}{2C_0}\|y\|_2 + \frac{\epsilon}{C_4}\left\|(1+x^2)y\right\|_2 \le C_3\|f\|_2 .$$

Then for some $C_5 > 0$ the following estimate

$$\|y\|_W := \|y''\|_2 + \left\|(1+x^2)^2 y'\right\|_2 + \left\|[1+\epsilon(1+x^2)]y\right\|_2 \le C_5\|f\|_2 \tag{3.5}$$

holds. We choose $A = C_5\|f\|_2$ and denote $P(\nu,\epsilon) := L_{\nu,\epsilon}^{-1} f$. From estimate (3.5) it follows that the operator $P(\nu,\epsilon)$ maps $S_A \subset W_2^1(\mathbb{R})$ to itself. Moreover, $P(\nu,\epsilon)$ maps S_A into the set

$$Q_A = \{y:\ \|y''\|_2 + \left\|(1+x^2)^2 y'\right\|_2 + \left\|[1+\epsilon(1+x^2)]y\right\|_2 \le C_5\|f\|_2\}.$$

Q_A is the compact in Sobolev's space $W_2^1(\mathbb{R})$. Indeed, if $y \in Q_A$, $h \neq 0$ and $N > 0$, then the following relations hold:

$$\|y(\cdot+h) - y(\cdot)\|^2_{W_2^1(\mathbb{R})} = \int_{-\infty}^{+\infty} [|y'(t+h) - y'(t)|^2 + |y(t+h) - y(t)|^2]dt =$$

$$= \int_{-\infty}^{+\infty} \left[\left|\int_t^{t+h} y''(\eta)d\eta\right|^2 + \left|\int_t^{t+h} y'(\eta)d\eta\right|^2\right] dt \le$$

$$\le |h| \int_{-\infty}^{+\infty} \left[\int_t^{t+h} |y''(\eta)|^2 d\eta + \int_t^{t+h} |y'(\eta)|^2 d\eta\right] dt =$$

$$= |h|^2 \int_{-\infty}^{+\infty} \left[|y''(\eta)|^2 + |y'(\eta)|^2\right] d\eta \le C_6\|f\|_2^2 |h|^2, \tag{3.6}$$

and

$$\|y\|^2_{W_2^1(\mathbb{R}\setminus[-N,N])} = \int_{|\eta|\ge N} \left[|y'(\eta)|^2 + |y(\eta)|^2\right] d\eta \le$$

$$\le \int_{|\eta|\ge N} (1+\eta^2)^{-1}\left[|y''(\eta)|^2 + (1+\eta^2)^2|y'(\eta)|^2 + (1+\eta^2)|y(\eta)|^2\right] d\eta \le$$

$$\le C_7\|f\|_2^2 (1+N^2)^{-1}. \tag{3.7}$$

Expressions in the right-hand side of (3.6) and (3.7) tend to zero as $h \to 0$ and as $N \to +\infty$, respectively. Then by Frechét-Kolmogorov criterion the set Q_A is compact in $W_2^1(\mathbb{R})$. Hence $P(\nu,\epsilon)$ is a compact operator.

Let us show that $P(\nu, \epsilon)$ is continuous with respect to ν in S_A. Let $\{\nu_n\} \subset S_A$ be a sequence such that $\|\nu_n - \nu\|_{W_2^1} \to 0$ as $n \to \infty$, and y_n and y such that $L_{\nu,\epsilon} y = f, \;\; L_{\nu_n,\epsilon} y_n = f$. Then it is enough to show that the sequence $\{y_n\}$ converges to y in $W_2^1(\mathbb{R})$ - norm as $n \to \infty$. We have

$$P(\nu_n, \epsilon) - P(\nu, \epsilon) = y_n - y = L_{\nu_n,\epsilon}^{-1}[r(x, \nu_n(x)) - r(x, \nu(x))]y_n'.$$

The functions $\nu(x)$ and $\nu_n(x)$ $(n = 1, 2, ...)$ are continuous. Then by conditions of the theorem the difference $r(x, \nu_n(x)) - r(x, \nu(x))$ is also continuous with respect to x. Hence for each finite interval $[-a, a], \;\; a > 0$, we have

$$\|y_n - y\|_{W_2^1(-a,a)} \leq c \max_{x \in [-a,a]} |r(x, \nu_n(x)) - r(x, \nu)| \cdot \|y_n'\|_{L_2(-a,a)} \to 0 \tag{3.8}$$

as $n \to \infty$. On the other hand, from Theorem 2 it follows that $\{y_n\} \in Q_A, \;\; \|y_n\|_W \leq A, \;\; y \in Q_A, \;\; \|y\|_W \leq A$. Since the set Q_A is compact in $W_2^1(\mathbb{R})$, $\{y_n\}$ converges in the $W_2^1(\mathbb{R})$ - norm. Let z be the limit of $\{y_n\}$. By properties of $W_2^1(\mathbb{R})$

$$\lim_{|x| \to \infty} y(x) = 0, \quad \lim_{|x| \to \infty} z(x) = 0. \tag{3.9}$$

Since $L_{\nu,\epsilon}^{-1}$ is the closed operator, from (3.8) and (3.9) we obtain $y = z$. Then $\|P(\nu_n, \epsilon) - P(\nu, \epsilon)\|_{W_2^1(\mathbb{R})} \to 0$, as $n \to \infty$.

Summing up, we have that $P(\nu, \epsilon)$ is the completely continuous operator in $W_2^1(\mathbb{R})$ and maps S_A to itself. Then by Schauder's theorem $P(\nu, \epsilon)$ has a fixed point $y \;\; (P(y, \epsilon) = y)$ in S_A. And consequently, y is a solution of the equation

$$L_\epsilon y := -y'' + \left[r(x, y) + \epsilon(1 + x^2)^2\right] y' = f(x).$$

By (3.3) for y the estimate

$$\|y''\|_2 + \left\| \left[r(\cdot, y) + \epsilon(1 + x^2)^2\right] y' \right\|_2 \leq C_3 \|f\|_2$$

holds.

Now, suppose that $\{\epsilon_j\}_{j=1}^\infty$ is a sequence of positive numbers converged to zero. The fixed point $y_j \in S_A$ of $P(\nu, \epsilon_j)$ is a solution of the equation

$$L_{\epsilon_j} y_j := -y_j'' + \left[r(x, y_j) + \epsilon_j(1 + x^2)^2\right] y_j' = f(x).$$

For y_j the estimate

$$\left\|y_j''\right\|_2 + \left\| \left[r(\cdot, y_j(\cdot)) + \epsilon(1 + x^2)^2\right] y_j' \right\|_2 \leq C_3 \|f\|_2 \tag{3.10}$$

holds.

Suppose (a, b) is an arbitrary finite interval. From $\{y_j\}_{j=1}^\infty \subset W_2^2(a, b)$ one can select a subsequence $\left\{y_{\epsilon_j}\right\}_{j=1}^\infty$ such that $\left\|y_{\epsilon_j} - y\right\|_{L_2[a,b]} \to 0$ as $j \to \infty$. A direct verification shows that y is a solution of (1.6). In (3.10) passing to the limit as $j \to \infty$ we obtain (1.8). The theorem is proved. $\square$

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Paper C

Some new results concerning a class of third order differential equations

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Abstract: We consider the following third order differential equation with unbounded coefficients:

$$-m(x)\left(m(x)y'\right)'' + [q(x) + ir(x) + \lambda]y = f(x).$$

Some new existence and uniqueness results are proved, and precise estimates of the norms of the solutions are given. The obtained results may be regarded as a unification and extension of all other results of this type.

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1 Introduction and statement of the main results

Linear and nonlinear equations of odd order are sometimes called non-classical equations of mathematical physics. The study of boundary value problems and qualitative properties of solutions of such equations begun fairly late and is reflected e.g. in the works of A.I. Kozhanov, N.A. Larkin, N.N. Yanenko [18] and some others. As important representatives of such equations we mention the Korteweg-de Vries equation and its modifications arising in the theory of distribution of long waves of small finite amplitudes, as well as the composite type equations arising in problems of hydrodynamics.

Questions of smoothness of solutions of differential equations are of great interest due to their importance for applications (e.g. for many problems of gas dynamics, hydrodynamics, hydromechanics, etc.). The case with bounded domains and smooth scalar coefficients are well understood and sufficiently well described in the known literature. In the case with unbounded domains the problem of separability was started in 1970 by W.N. Everitt and M. Geertz [14-15]. They proved some interesting separation results for the Sturm-Liouville differential operator

$$Ly(x) = -y''(x) + q(x)y(x), \quad x \in R,$$

in the space $L_2(R)$. They studied the following question: what are the conditions on $q(x)$ such that if $y(x) \in L_2(R)$, $R = (-\infty, +\infty)$, and $Ly(x) \in L_2(R)$ imply both of $y''(x)$ and $q(x)y(x) \in L_2(R)$. More fundamental results of separation of differential operator were obtained later by the same authors [16,17]. A number of results concerning the property referred to as the separation of differential operators was investigated e.g. by K. Kh. Boimatov [9], M. Otelbaev [26], A. Zettle [31] and A.S. Mohamed et al. [19-24]. The separation for differential operators with matrix potentials was first studied by A. Birgebaev [8]. R.C. Brown [12] proved that certain properties of positive solutions of disconjugate second-order differential expressions imply the separation property. Some separation criteria and inequalities associated with linear second-order differential operators were studied by R.C. Brown et al. [10-11]. A.S.Mohamed et al. [22] investigated the separation property of the Sturm-Liouville differential operator

$$Ly(x) = -\left(m(x)y'\right)' + V(x)y(x), \quad x \in R,$$

in the Hilbert space $H_p(R)$ $(p = 1, 2)$, where $V \in L(l_p)$ is an operator potential which is a bounded linear operator on l_p and $m(x) \in C^1(R)$ is a positive continuous function on R.

In [20] the same authors studied the separation property for the linear differential operator

$$Ly(x) = (-1)^m D^{2m} y(x) + V(x)y(x), \quad x \in R,$$

in the Banach space $L_p(R)^l$, where $V(x)$ is an $l \times l$ positive Hermitian matrix and $D^{2m} = \dfrac{d^{2m}}{dx^{2m}}$ is the classical differentiation of order $2m$.

Moreover, in [23] again the same authors investigated the separation of the Schrödinger operator

$$Ly(x) = -\Delta y(x) + V(x)y(x), \quad x \in R^n,$$

with the operator potential $V(x) \in C^1(R^n, L(H_1))$ in the Hilbert space $L_2(R^n, H_1)$ and $\Delta = \sum\limits_{i=1}^{n} \dfrac{\partial^2}{\partial x_i^2}$ is the Laplace operator in R^n. Finally, in [24] they studied the separation for the general second-order differential operator

$$Ly(x) = -\sum_{i,j=1}^{n} a_{ij}(x) D_i^j y(x) + V(x)y(x), \quad x \in R^n,$$

with the operator potential $V(x)$, in the weighted Hilbert space $L_{2,k}(R^n, H_1)$, where $a_{ij} \in C^2(R^n)$ and $D_i^j = \dfrac{\partial^2}{\partial x_i \partial x_j}$.

Recently, in [28] E.M.E. Zayed et al. derived some new results on the separation of linear and nonlinear Schrödinger-type operators with operator potentials in Banach spaces. Furthermore, in [29] the same authors studied the separation of the elliptic differential operator

$$Ly(x) = -\sum_{i,j=1}^{n} \left[D_i \left(P_{ij}(x) D_j y(x)\right) - P_{ij}(x) b_i(x) b_j(x) y(x)\right] + V(x)y(x),$$

with the operator potential $V(x) \in C^1(R^n, L(H_1))$, in the weighted Hilbert space $L_{2,k}(R^n, H_1)$, where $P_{ij}(x)$ and $b_i(x)$ are real-valued continuous functions while $D_i = \dfrac{\partial}{\partial x_i}$. Finally, in [30] E.M.E. Zayed et al. studied the separation for the Laplace-Beltrami differential operator in Hilbert spaces.

In the Hilbert case, the differential operator corresponding to the linear equation of odd-order is not semibounded; thus, estimating intermediate derivatives in the solution, not to mention the highest derivative, is a nontrivial problem.

We also note that B.I. Aliev [1] and T.T. Amanova [2] proved the separation of the differential operator L in $L_p(R)$ $(1 \leq p < \infty)$, generated by the expression $-y''' + q(x)y$, under some natural assumptions. In particular, they proved that for all sufficiently large $\lambda > 0$, the operator $L + \lambda E$ has a bounded inverse operator in $L_p(R)$.

It is also important to mention that sufficient conditions for the separability of the nonlinear operator

$$Ly = -y'' + q(x, y(x))y + \lambda y, \quad \lambda > 0, \quad x \in R,$$

have been established by M. Otelbaev and A. Birgebaev[7].

In [27] M. Sapenov and L.A. Shuster studied the problem if the solutions of the differential equation

$$-y^{(2n+1)}(t) + q(t)y(t) = f(t) \in L_p(R), \quad t \in R$$

belongs to a weighted Lebesgue space under certain conditions imposed on $q(t)$.

Moreover, very recently in [25] K.N. Ospanov et al. proved some interesting results concerning the smoothness and approximative properties of generalized solutions of the nonlinear equation

$$(-1)^k y^{2k+1} + [q(x, y) + \lambda + ir(x, y)]y = f$$

on R. Here $\lambda \geq 0$ is a constant, and $q = q(x, y)$ and $r = r(x, y)$ are given functions, which may increase near infinity.

Let $1 < p < +\infty$. By $L_p \equiv L_p(R)$, $R = (-\infty, +\infty)$, we denote the space of functions with finite norm

$$\|\varphi\|_p := \left(\int_R |\varphi(x)|^p dx \right)^{\frac{1}{p}}.$$

We consider the third order differential equation with unbounded coefficients:

$$(L + \lambda E)\, y := -m(x)\, (m(x) y')'' + [q(x) + ir(x) + \lambda] y = f(x), \qquad (1)$$

where $f \in L_p$, $\lambda \geq 0$, and where $m(x)$, $q(x)$ and $r(x)$ are functions to be defined later on.

In this paper we study questions of the existence and uniqueness of the solutions of (1) and conditions, which for a solution y of (1) the following estimate holds:

$$\|m(x)(m(x)y')''\|_p^p + \|(q(x) + ir(x) + \lambda)y\|_p^p \leq c\,\|f(x)\|_p^p. \qquad (2)$$

In the case of $m(x) = 1$ sufficient conditions for unique solvability of the equation (1) and the estimate of the form (2) for its solution when $r(x) \equiv 0$ can be found in [3,7,25], and when $r(x) \geq 1$ in [4-6].

Hence, the results in this paper can be regarded as a unification and extension of the results in the papers [3-7,25].

Definition 1. *A function $y(x) \in L_p(R)$, is called a solution of (1), if there is a sequence $\{y_n\}_{n=1}^{\infty}$ of continuously differentiable functions with compact support, such that $\|y_n - y\|_p \to 0$ and $\|(L + \lambda E)y_n - f\|_p \to 0$ as $n \to \infty$.*

By $C^{(k)}(R)$ $(k = 1, 2, ...)$ we denote the set of all k times continuously differentiable functions $\varphi(x)$ for which the value $\sum_{j=0}^{k} \sup_{x\in R} |\varphi^{(j)}(x)|$ is finite. Let $W_\lambda(x) := \dfrac{|q(x) + \lambda + ir(x)|}{m^2(x)}$.

Our main results in this paper read:

Theorem 1. *Assume that the functions $q = q(x)$ and $r = r(x)$ are continuous on R, $m = m(x) \in C^{(2)}_{loc}(R)$ and that the following conditions hold:*

$$m(x) \geq 1,\ \frac{q(x)}{m^4(x)} \geq 1,\ r(x) \geq 1, \qquad (3)$$

$$c^{-1} \leq \frac{m(x)}{m(\eta)}, \frac{q(x)}{q(\eta)}, \frac{r(x)}{r(\eta)} \leq c,\ x, \eta \in R, \quad |x - \eta| \leq 1, \quad \textit{for some } c > 0 \qquad (4)$$

$$|m^{(j)}(x)| \leq c_j m(x),\ x \in R, \quad \textit{for some } c_j > 0, \quad j = 1, 2. \qquad (5)$$

$$\sup_{|x-\eta|\leq 1} \frac{|W_\lambda(x) - W_\lambda(\eta)|}{|W_\lambda(x)|^\nu |x - \eta|^\mu} < +\infty, \quad 0 < \nu < \frac{\mu}{3} + 1, \quad \mu \in (0, 1],\ \lambda \geq 0. \qquad (6)$$

Then there exists a number $\lambda_0 \geq 0$, such that the equation (1) has a solution y for all $\lambda \geq \lambda_0$.

Theorem 2. *Let the functions $q = q(x)$ and $r = r(x)$ be continuous on R, $m = m(x) \in C^{(3)}_{loc}(R)$ and satisfy the conditions (3) - (6) and*

$$|m^{(3)}(x)| \le c_3 m(x), \quad x \in R. \tag{7}$$

Then the solution of the equation (1) is unique and the estimate (2) holds.

In order to make the proofs of Theorems 1 and 2 easy to follow some auxiliary Lemmas and other preliminaries are given in Section 2. Some identities and inequalities in these Lemmas are of independent interest. The detailed proofs can be found in Section 3.

2 Preliminaries

Let $\xi_s = \xi_s(x),\ s = 1,2,3,$ be the roots of equation $m^2(x)\xi^3 - r(x) + i(q(x) + \lambda) = 0$, where $m(x)$, $r(x)$ and $q(x)$ satisfy the properties of Theorem 1. From the conditions of Theorem 1 it follows that $0 < arg\xi_1 < \pi,\ \pi < arg\xi_j < 2\pi,\ j = 2,3.$ We introduce the function

$$M_0(x,\eta,\lambda) = \begin{cases} -\dfrac{1}{3m^2(x)} \dfrac{e^{i(x-\eta)\xi_1}}{\xi_1^2}, & -\infty < \eta < x, \\ \\ \dfrac{1}{3m^2(x)} \displaystyle\sum_{j=2}^{3} \dfrac{e^{i(x-\eta)\xi_j}}{\xi_j^2}, & x < \eta < +\infty. \end{cases} \tag{8}$$

By a direct computation we get the following equalities:

$$\left.\frac{\partial^j M_0(x,\eta,\lambda)}{\partial \eta^j}\right|_{x=\eta-0} = \left.\frac{\partial^j M_0(x,\eta,\lambda)}{\partial \eta^j}\right|_{x=\eta+0},\ j = 0,1, \tag{9}$$

$$\left.\frac{\partial^2 M_0(x,\eta,\lambda)}{\partial \eta^2}\right|_{x=\eta-0} - \left.\frac{\partial^2 M_0(x,\eta,\lambda)}{\partial \eta^2}\right|_{x=\eta+0} = -\frac{1}{m^2(x)}, \tag{10}$$

and

$$-m(x)\left(m(x)\frac{\partial M_0(x,\eta,\lambda)}{\partial \eta}\right)''_{\eta\eta} + [q(x) + ir(x) + \lambda]M_0(x,\eta,\lambda) = 0. \tag{11}$$

Let the function $d(\eta) \in C_0^\infty(-1,1)$ be such that

$$d(\eta) = \begin{cases} 1, & |\eta| \le \frac{1}{2}, \\ 0, & |\eta| \ge 1. \end{cases}$$

We denote

$$M_1(x,\eta,\lambda) = \left[(q(\eta)+ir(\eta)) - \frac{m^2(\eta)}{m^2(x)}(q(x)+ir(x))\right] M_0(x,\eta,\lambda)d(\eta-x),$$

$$M_2(x,\eta,\lambda) = -\left[2m'(\eta)m(\eta)d(\eta-x)+3m^2(\eta)d'(\eta-x)\right]\frac{\partial^2 M_0(x,\eta,\lambda)}{\partial\eta^2} -$$

$$-\left[m''(\eta)m(\eta)d(\eta-x)+4m'(\eta)m(\eta)d'(\eta-x)+3m^2(\eta)d''(\eta-x)\right]\frac{\partial M_0(x,\eta,\lambda)}{\partial\eta} -$$

$$-\left[m''(\eta)m(\eta)d'(\eta-x)+2m'(\eta)m(\eta)d''(\eta-x)+m^2(\eta)d'''(\eta-x)\right]M_0(x,\eta,\lambda),$$

and

$$M_3(x,\eta,\lambda) = M_0(x,\eta,\lambda)d(\eta-x).$$

We introduce the following integral operators:

$$\left(M_j(\lambda)f\right)(\eta) = \int\limits_R M_j(x,\eta,\lambda)f(x)dx, \quad (j=1,2,3).$$

The following statement is known ([13, p. 902]):

Lemma 1. *Let $1 < p < +\infty$, $k(x,\eta)$ be a continuous function and*

$$(K\nu)(\eta) = \int\limits_R k(x,\eta)\nu(x)dx.$$

Then

$$\|K\|_{L_p\to L_p} \le \sup_{\eta\in R}\int\limits_R \left[|k(x,\eta)|+|k(\eta,x)|\right]dx.$$

Our next auxiliary result reads:

Lemma 2. *Let all the conditions of Theorem 1 be satisfied. Then the operators $M_j(\lambda)$, $j=1,2,3$, are continuous in the space L_p, and the following estimates hold ($\lambda \ge 0$):*

$$\|M_1(\lambda)\|_{L_p\to L_p} \le \frac{c_1}{b_\lambda^{\mu+3-3\nu}(\eta)}, \quad \mu\in(0,1], \quad 0<\nu<\frac{\mu}{3}+1, \tag{12}$$

$$\|M_2(\lambda)\|_{L_p\to L_p} \le \frac{c_2}{b_\lambda(\eta)}, \tag{13}$$

and

$$\|M_3(\lambda)\|_{L_p\to L_p} \le \frac{c_3}{m^2(\eta)b_\lambda^3(\eta)}. \tag{14}$$

Here $b_\lambda(x) = \sqrt[3]{\dfrac{|r(x) - i(q(x)+\lambda)|}{m^2(x)}}$.

Remark 1. The statement of Lemma 2 remains true if the condition $r(x) \ge 1$ in (3) is replaced by the condition $r(x) \le -1$, but we do not need this fact in our further investigations.

Denote by $L+\lambda E$ $(\lambda \ge 0)$ the closure in L_p of the differential expression

$$(l+\lambda E)y \equiv -m(x)\,(m(x)y')'' + [q(x) + ir(x) + \lambda]y,$$

defined on the set $C_0^\infty(R)$ of infinitely differentiable and compactly supported functions. From definition 1 it is easy to see that the function $y \in L_p$ is a solution of the equation (1) if it belongs to $D(L+\lambda E)$ and the equality $(L+\lambda E)y = f$ holds.

Also the following identify of independent interest is crucial for the proof of Theorem 1:

Lemma 3. Let the conditions of Theorem 1 be satisfied. Then the following equality holds:

$$(L+\lambda E)\,[M_3(\lambda)f]\,(\eta) = f(\eta) + [M_1(\lambda)f]\,(\eta) + [M_2(\lambda)f]\,(\eta). \tag{15}$$

Let the functions $m(x)$, $q(x)$, $r(x)$ satisfy the conditions of Theorem 2, and let p' denote the conjugate number of p i.e. $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. We denote by $(L+\lambda E)'$ an operator acting in the space $L_{p'}(R)$ such that

$$((L+\lambda E)\,y, z) = \big(y, (L+\lambda E)'\,z\big)\,, \quad y \in D(L+\lambda E), \quad z \in D\left((L+\lambda E)'\right).$$

Obviously, it yields that

$$(L+\lambda E)'z \equiv \big(m(x)\,(m(x)z)''\big)' + (q(x) + \lambda - ir(x))\,z.$$

We consider the differential equation

$$(L+\lambda E)'z \equiv \big(m(x)\,(m(x)z)''\big)' + (q(x) + \lambda - ir(x))\,z = g(x), \tag{16}$$

where the function $m(x) \ge 1$ is continuous together with its derivatives up to third order, $q(x)$ and $r(x)$ are continuous real-valued functions, $\lambda \ge 0$ and $g(x) \in L_{p'}(R)$.

The next Lemma is crucial for the proof of Theorem 2.

Lemma 4. *Let the continuous functions $q(x)$, $r(x)$ and the function $m \in C^{(3)}_{loc}(R)$ satisfy the conditions (3) - (6) and (7). Then there exists a number $\lambda_1 \geq 0$, such that the equation (16) for all $\lambda \geq \lambda_1$ has a solution.*

Before we can prove Lemma 4 we introduce some notations and give some further auxiliary statements.

Let, $\zeta_l = \zeta_l(x)$, $l = 1, 2, 3$, be the roots of the equation $m^2(x)\zeta^3 - r(x) - i(q(x) + \lambda) = 0$. From the assumptions of Lemma 4 it follows that $0 < arg\zeta_j < \pi$, $j = 1, 2$ and $\pi < arg\zeta_3 < 2\pi$. We introduce the function

$$N_0(x,\eta,\lambda) = \begin{cases} -\dfrac{1}{3m^2(x)} \displaystyle\sum_{j=1}^{2} \dfrac{e^{i(x-\eta)\zeta_j}}{\zeta_j^2}, & -\infty < \eta < x, \\ \\ \dfrac{1}{3m^2(x)} \dfrac{e^{i(x-\eta)\zeta_3}}{\zeta_3^2}, & x < \eta < +\infty. \end{cases} \tag{17}$$

By direct computations we easily obtain the following:

$$\left.\frac{\partial^j N_0(x,\eta,\lambda)}{\partial\eta^j}\right|_{x=\eta-0} = \left.\frac{\partial^j N_0(x,\eta,\lambda)}{\partial\eta^j}\right|_{x=\eta+0}, \quad j = 0, 1, \tag{18}$$

$$\left.\frac{\partial^2 N_0(x,\eta,\lambda)}{\partial\eta^2}\right|_{x=\eta-0} - \left.\frac{\partial^2 N_0(x,\eta,\lambda)}{\partial\eta^2}\right|_{x=\eta+0} = -\frac{1}{m^2(x)}, \tag{19}$$

$$\Big(m(x)\left(m(x)N_0(x,\eta,\lambda)\right)''_{\eta\eta}\Big)'_{\eta} + [q(x) - ir(x) + \lambda]N_0(x,\eta,\lambda) = 0. \tag{20}$$

Moreover, we denote

$$N_1(x,\eta,\lambda) = \left[(q(\eta) - ir(\eta)) - \frac{m^2(\eta)}{m^2(x)}\left(q(x) - ir(x)\right)\right] N_0(x,\eta,\lambda)d(\eta - x),$$

$$N_2(x,\eta,\lambda) = \left[4m'(\eta)m(\eta)d(\eta - x) + 3m^2(\eta)d'(\eta - x)\right] \frac{\partial^2 N_0(x,\eta,\lambda)}{\partial\eta^2} +$$

$$+ \left[3m''(\eta)m(\eta)d(\eta - x) + 2(m'(\eta))^2 d(\eta - x) + 8m(\eta)m'(\eta)d'_{\eta}(\eta - x) +\right.$$

$$+3m^2(\eta)d''_{\eta\eta}(\eta-x)\big]\,\frac{\partial N_0(x,\eta,\lambda)}{\partial\eta}+$$

$$+\Big[m''(\eta)m'(\eta)d(\eta-x)+m'''(\eta)m(\eta)d(\eta-x)+3m''(\eta)m(\eta)d'_{\eta}(\eta-x)+$$

$$+2\left(m'(\eta)\right)^2 d'_{\eta}(\eta-x)+4m'(\eta)m(\eta)d''_{\eta\eta}(\eta-x)+m^2(\eta)d'''_{\eta\eta\eta}(\eta-x)\Big]\,N_0(x,\eta,\lambda),$$

and

$$N_3(x,\eta,\lambda)=N_0(x,\eta,\lambda)d(\eta-x).$$

We also introduce the following integral operators

$$\left(N_j(\lambda)f\right)(\eta)=\int\limits_R N_j(x,\eta,\lambda)f(x)dx,\quad j=1,2,3.$$

Lemma 5. *Let all the conditions of Lemma 4 be satisfied. Then the operators $N_j(\lambda)$ are continuous in the space $L_{p'}$, and the following estimates hold ($\lambda\geq 0$):*

$$\|N_1(\lambda)\|_{L_{p'}\to L_{p'}}\leq\frac{c_1}{b_\lambda^{\beta+3-3\alpha}(\eta)},\quad\beta\in(0,1],\quad 0<\alpha<\frac{\beta}{3}+1, \tag{21}$$

$$\|N_2(\lambda)\|_{L_{p'}\to L_{p'}}\leq\frac{c_2}{b_\lambda(\eta)}, \tag{22}$$

and

$$\|N_3(\lambda)\|_{L_{p'}\to L_{p'}}\leq\frac{c_3}{m^2(\eta)b_\lambda^3(\eta)}. \tag{23}$$

We also need the following identity of independent interest:

Lemma 6. Let the conditions of Lemma 4 be fulfilled. Then the following identity holds:

$$(L+\lambda E)'\left[N_3(\lambda)g\right](\eta)=g(\eta)+\left[N_1(\lambda)g\right](\eta)+\left[N_2(\lambda)g\right](\eta). \tag{24}$$

3 Proofs

Proof of Lemma 2. Under the assumptions of Theorem 1 for the functions $q(x), r(x)$ and $m(x)$, there exists a constant $\sigma > 0$ such that $Im\xi_1 \geq \sigma$ and $Im\xi_1 \leq -\sigma$ $(j = 2, 3)$. Then, for the function M_0 defined by (8), we have the following estimates:

$$|M_0(x, \eta, \lambda)| \leq \begin{cases} \dfrac{1}{3m^2(x)} \dfrac{e^{-\sigma(x-\eta)b_\lambda(x)}}{b_\lambda^2(x)}, & -\infty < \eta < x, \\ \\ \dfrac{2}{3m^2(x)} \dfrac{e^{\sigma(x-\eta)b_\lambda(x)}}{b_\lambda^2(x)}, & x < \eta < +\infty, \end{cases} \tag{25}$$

and

$$\left| \frac{\partial^j M_0(x, \eta, \lambda)}{\partial \eta^j} \right| \leq \begin{cases} \dfrac{1}{3m^2(x)} \dfrac{e^{-\sigma(x-\eta)b_\lambda(x)}}{b_\lambda^{2-j}(x)}, & -\infty < \eta < x, \\ \\ \dfrac{2}{3m^2(x)} \dfrac{e^{\sigma(x-\eta)b_\lambda(x)}}{b_\lambda^{2-j}(x)}, & x < \eta < +\infty, \quad j = 1, 2. \end{cases} \tag{26}$$

According to our choice $M_j(x, \eta, \lambda) = 0$ for $|x - \eta| > 1$. Taking into account the conditions (3) - (6) of Theorem 1 and (25) - (26) for the functions $M_j(x, \eta, \lambda)$, $j = 0, 1, 2$, at $|x - \eta| \leq 1$ we obtain the following estimates:

$$|M_1(x, \eta, \lambda)| \leq \begin{cases} \bar{c}_1 m^2(\eta)|x - \eta|^\mu b_\lambda^{3\nu-2}(x) \dfrac{e^{-\sigma(x-\eta)b_\lambda(x)}}{m^2(x)}, & -\infty < \eta < x, \\ \\ \bar{c}_2 m^2(\eta)|x - \eta|^\mu b_\lambda^{3\nu-2}(x) \dfrac{e^{\sigma(x-\eta)b_\lambda(x)}}{m^2(x)}, & x < \eta < +\infty. \end{cases} \tag{27}$$

$$|M_2(x, \eta, \lambda)| \leq \begin{cases} \dfrac{m^2(\eta)}{m^2(x)} \sum\limits_{k=0}^{2} \tilde{c}_k \dfrac{e^{-\sigma(x-\eta)b_\lambda(x)}}{b_\lambda^k(x)}, & -\infty < \eta < x, \\ \\ \dfrac{m^2(\eta)}{m^2(x)} \sum\limits_{k=0}^{2} \tilde{c}_k \dfrac{e^{\sigma(x-\eta)b_\lambda(x)}}{b_\lambda^k(x)}, & x < \eta < +\infty, \end{cases} \tag{28}$$

and

$$|M_3(x,\eta,\lambda)| \le \begin{cases} \dfrac{1}{3m^2(x)} \dfrac{e^{-\sigma(x-\eta)b_\lambda(x)}}{b_\lambda^2(x)}, & -\infty < \eta < x, \\ \\ \dfrac{2}{3m^2(x)} \dfrac{e^{\sigma(x-\eta)b_\lambda(x)}}{b_\lambda^2(x)}, & x < \eta < +\infty. \end{cases} \tag{29}$$

We shall estimate the norms $\|M_j(\lambda)\|_{L_p \to L_p}$ of the operators $M_j(\lambda)$, $j = 1, 2, 3$, by using Lemma 1 and the inequalities (27) - (29). We have that

$$\|M_1(\lambda)\|_{L_p \to L_p} \le \sup_{\eta \in R} \int\limits_R [|M_1(x,\eta,\lambda)| + |M_1(\eta,x,\lambda)|]\, dx \le$$

$$\le c_1 \sup_{\eta\in R} \int\limits_{\eta-1}^{\eta+1} \left(\frac{m^2(\eta)}{m^2(x)} \frac{\exp[-\sigma(x-\eta)b_\lambda(x)]}{b_\lambda^2(x)} + \frac{m^2(x)}{m^2(\eta)} \frac{\exp[-\sigma(x-\eta)b_\lambda(\eta)]}{b_\lambda^2(\eta)} \right) \times$$

$$\times \left| \frac{q(\eta)+\lambda+ir(\eta)}{m^2(\eta)} - \frac{q(x)+\lambda+ir(x)}{m^2(x)} \right| dx.$$

By using (3), (4), (6) we obtain that

$$\|M_1(\lambda)\|_{L_p \to L_p} \le$$

$$\le c_2 \sup_{\eta\in R} \int\limits_{\eta-1}^{\eta+1} \left(\frac{\exp[-\sigma(x-\eta)\tilde{c}b_\lambda(\eta)]}{\tilde{c}b_\lambda^2(\eta)} + \frac{\exp[-\sigma(x-\eta)b_\lambda(\eta)]}{b_\lambda^2(\eta)} \right) \times$$

$$\times \left| \frac{q(\eta)+\lambda+ir(\eta)}{m^2(\eta)} \right|^\nu |\eta - x|^\mu dx.$$

Hence, by making the change of variable $\eta - x = \dfrac{1}{\sigma b_\lambda(\eta)} z$, we find that

$$\|M_1(\lambda)\|_{L_p \to L_p} \le$$

$$\le \frac{c_3 \left|\frac{q(\eta)+\lambda+ir(\eta)}{m^2(\eta)}\right|^\nu}{(\tilde{c}b_\lambda(\eta))^{\mu+3}} + \frac{c_4 \left|\frac{q(\eta)+\lambda+ir(\eta)}{m^2(\eta)}\right|^\nu}{(b_\lambda(\eta))^{\mu+3}} = \frac{c_5}{\left(\frac{|q(\eta)+\lambda+ir(\eta)|}{m^2(\eta)}\right)^{\frac{\mu}{3}+1-\nu}}.$$

Moreover, according to the condition (3), we have that

$$\frac{|q(\eta)+\lambda+ir(\eta)|}{m^2(\eta)} \geq \sqrt{1+\lambda}\,.$$

Therefore, from the previous inequality we obtain (12). Furthermore, in view of the conditions (3) - (5) of Theorem 1, we can deduce that

$$\|M_2(\lambda)\|_{L_p\to L_p} \leq \sup_{\eta\in R}\int\limits_R \left[|M_2(x,\eta,\lambda)|+|M_2(\eta,x,\lambda)|\right]dx \leq$$

$$\leq c_2 \sup_{\eta\in R}\int\limits_{\eta-1}^{\eta+1}\left(\frac{m^2(\eta)}{m^2(x)}e^{-\sigma(x-\eta)b_\lambda(x)}+\frac{m^2(\eta)}{m^2(x)}\frac{e^{-\sigma(x-\eta)b_\lambda(x)}}{b_\lambda(x)}+\frac{m^2(\eta)}{m^2(x)}\frac{e^{-\sigma(x-\eta)b_\lambda(x)}}{b_\lambda^2(x)}+\right.$$

$$\left.+\frac{m^2(x)}{m^2(\eta)}e^{-\sigma(x-\eta)b_\lambda(\eta)}+\frac{m^2(x)}{m^2(\eta)}\frac{e^{-\sigma(x-\eta)b_\lambda(\eta)}}{b_\lambda(\eta)}+\frac{m^2(x)}{m^2(\eta)}\frac{e^{-\sigma(x-\eta)b_\lambda(\eta)}}{b_\lambda^2(\eta)}\right)dx.$$

Hence, by calculating the integrals, we obtain that

$$\|M_2(\lambda)\|_{L_p\to L_p} \leq \bar{c_3}\sup_{\eta\in R}\int\limits_{\eta-1}^{\eta+1}\left[e^{-\sigma(x-\eta)b_\lambda(\eta)}+\frac{e^{-\sigma(x-\eta)b_\lambda(\eta)}}{b_\lambda(\eta)}+\frac{e^{-\sigma(x-\eta)b_\lambda(\eta)}}{b_\lambda^2(\eta)}\right]dx =$$

$$=\bar{c_3}\sup_{\eta\in R}\left[\frac{1}{\sigma b_\lambda(\eta)}\left(1-e^{-\sigma b_\lambda(\eta)}\right)+\frac{1}{\sigma b_\lambda^2(\eta)}\left(1-e^{-\sigma b_\lambda(\eta)}\right)+\frac{1}{\sigma b_\lambda^3(\eta)}\left(1-e^{-\sigma b_\lambda(\eta)}\right)\right]\leq$$

$$\leq \frac{c_2}{b_\lambda(\eta)}.$$

Finally, we shall prove the inequality (14). According to the estimate (29) it yields that

$$\|M_3(\lambda)\|_{L_p\to L_p} \leq \sup_{\eta\in R}\int\limits_R \left[|M_3(x,\eta,\lambda)|+|M_3(\eta,x,\lambda)|\right]dx \leq$$

$$\leq \sup_{\eta\in R}\int\limits_{\eta-1}^{\eta+1}\left(\frac{2}{3m^2(x)}\frac{\exp[-\sigma(x-\eta)b_\lambda(x)]}{b_\lambda^2(x)}+\frac{2}{3m^2(\eta)}\frac{\exp[-\sigma(x-\eta)b_\lambda(\eta)]}{b_\lambda^2(\eta)}\right)dx.$$

Thus, by taking the condition (4) into account, we find that

$$\|M_3(\lambda)\|_{L_p\to L_p} \le \bar{c}_3 \sup_{\eta\in R} \int\limits_{\eta-1}^{\eta+1} \left(\frac{2}{3(c_1^{-1})^2 m^2(\eta)} \frac{\exp[-\sigma(x-\eta)\tilde{c}b_\lambda(\eta)]}{\tilde{c}b_\lambda^2(\eta)} + \right.$$

$$\left. + \frac{2}{3m^2(\eta)} \frac{\exp[-\sigma(x-\eta)b_\lambda(\eta)]}{b_\lambda^2(\eta)} \right) dx \le$$

$$\le 2\bar{c}_3 \sup_{\eta\in R} \left(\frac{c_{41}}{(c_1^{-1})^2 m^2(\eta) b_\lambda^3(\eta)} + \frac{c_{42}}{m^2(\eta) b_\lambda^3(\eta)} \right) \le \frac{\bar{c}_4}{m^2(\eta) b_\lambda^3(\eta)}.$$

The proof is complete. □

Proof of Lemma 3. Obviously, it yields that

$(L+\lambda E)\,[M_3(\lambda)f]\,(\eta) =$

$$-m(\eta)\left(m(\eta)\left(\int\limits_{-\infty}^{\eta} M_0(x,\eta,\lambda)d(\eta-x)f(x)dx + \int\limits_{\eta}^{+\infty} M_0(x,\eta,\lambda)d(\eta-x)f(x)dx \right)'_{\eta} \right)''_{\eta\eta} +$$

$$(q(\eta)+ir(\eta)+\lambda)\left(\int\limits_{-\infty}^{\eta} M_0(x,\eta,\lambda)d(\eta-x)f(x)dx + \int\limits_{\eta}^{+\infty} M_0(x,\eta,\lambda)d(\eta-x)f(x)dx \right)(\eta).$$

Moreover,

$$\frac{d}{d\eta}\left(M_3(\lambda)f\right)(\eta) =$$

$$= \left[\int\limits_{-\infty}^{\eta} M_0(x,\eta,\lambda)d(\eta-x)f(x)dx + \int\limits_{\eta}^{+\infty} M_0(x,\eta,\lambda)d(\eta-x)f(x)dx \right]'_{\eta} =$$

$$= M_0(x,\eta,\lambda)d(\eta-x)f(x)|_{x=\eta-0} - M_0(x,\eta,\lambda)d(\eta-x)f(x)|_{x=\eta+0} +$$

$$+ \int\limits_{-\infty}^{\eta} M'_{0\eta}(x,\eta,\lambda)d(\eta-x)f(x)dx + \int\limits_{-\infty}^{\eta} M_0(x,\eta,\lambda)d'_{\eta}(\eta-x)f(x)dx +$$

$$+ \int\limits_{\eta}^{+\infty} M'_{0\eta}(x,\eta,\lambda)d(\eta-x)f(x)dx + \int\limits_{\eta}^{+\infty} M_0(x,\eta,\lambda)d'_{\eta}(\eta-x)f(x)dx.$$

According to (9) we have that

$$M_0(x,\eta,\lambda)d(\eta-x)f(x)|_{x=\eta-0}-M_0(x,\eta,\lambda)d(\eta-x)f(x)|_{x=\eta+0}=0.$$

By using this fact we find that

$$\frac{d}{d\eta}\left(m(\eta)M'_{3\eta}(\lambda)f\right)(\eta)=m'(\eta)\int\limits_R M'_{0\eta}(x,\eta,\lambda)d(\eta-x)f(x)dx+$$

$$+m'(\eta)\int\limits_R M_0(x,\eta,\lambda)d'_\eta(\eta-x)f(x)dx+m(\eta)\int\limits_R M''_{0\eta\eta}(x,\eta,\lambda)d(\eta-x)f(x)dx+$$

$$+2m(\eta)\int\limits_R M'_{0\eta}(x,\eta,\lambda)d'_\eta(\eta-x)f(x)dx+m(\eta)\int\limits_R M_0(x,\eta,\lambda)d''_{\eta\eta}(\eta-x)f(x)dx.$$

Differentiating this expression with respect to η, we obtain that

$$\frac{d^2}{d\eta^2}\left(m(\eta)M'_{3\eta}(\lambda)f\right)(\eta)=m''(\eta)\int\limits_R M'_{0\eta}(x,\eta,\lambda)d(\eta-x)f(x)dx+$$

$$+m''(\eta)\int\limits_R M_0(x,\eta,\lambda)d'_\eta(\eta-x)f(x)dx+2m'(\eta)\int\limits_R M_0(x,\eta,\lambda)d''_{\eta\eta}(\eta-x)f(x)dx+$$

$$+2m'(\eta)\int\limits_R M''_{0\eta\eta}(x,\eta,\lambda)d(\eta-x)f(x)dx+4m'(\eta)\int\limits_R M'_{0\eta}(x,\eta,\lambda)d'_\eta(\eta-x)f(x)dx+$$

$$+\,m(\eta)M''_{0\eta\eta}(x,\eta,\lambda)d(\eta-x)f(x)\Big|_{x=\eta-0}-m(\eta)M''_{0\eta\eta}(x,\eta,\lambda)d(\eta-x)f(x)\Big|_{x=\eta+0}+$$

$$+m(\eta)\int\limits_R M'''_{0\eta\eta\eta}(x,\eta,\lambda)d(\eta-x)f(x)dx+3m(\eta)\int\limits_R M''_{0\eta\eta}(x,\eta,\lambda)d'_\eta(\eta-x)f(x)dx+$$

$$+3m(\eta)\int\limits_R M'_{0\eta}(x,\eta,\lambda)d''_{\eta\eta}(\eta-x)f(x)dx+m(\eta)\int\limits_R M_0(x,\eta,\lambda)d'''_{\eta\eta\eta}(\eta-x)f(x)dx.$$

Hence, by using (10), (11), and the notations $M_j(x,\eta,\lambda)$, $j=1,2,3$, we obtain the equality (15). The proof is complate. $\square$

We are now ready to present the

Proof of Theorem 1. By using the estimates (12) and (13) in Lemma 2, we conclude that there exists a number $\lambda_0>0$ such that the inequality

$\|M_1(\lambda)\|_{L_p\to L_p} + \|M_2(\lambda)\|_{L_p\to L_p} \le \dfrac{1}{2}$ holds if $\lambda \ge \lambda_0$. Therefore the operator $G(\lambda) := E + M_1(\lambda) + M_2(\lambda)$ has a bounded inverse $G^{-1}(\lambda)$ in L_p. Hence, by letting $h = [E + M_1(\lambda) + M_2(\lambda)]\, f$, in view of the relation (15) in Lemma 3, we obtain that $(L + \lambda E)\,[M_3(\lambda)G^{-1}(\lambda)h](\eta) = h$. Thus, for all λ, $\lambda \ge \lambda_0$ it yields that the function $y = M_3(\lambda)G^{-1}(\lambda)f$ is a solution of the equation (1). The proof is complete. $\square$

Proof of Lemma 5. Under the assumptions of Lemma 4 for the functions $q(x), r(x)$ and $m(x)$, there exists a constant $\delta > 0$ such that $Im\zeta_j \ge \delta$ $(j = 1, 2)$ and $Im\zeta_3 \le -\delta$. Therefore, in view of the definition (17) of N_0, we have the following estimates:

$$|N_0(x,\eta,\lambda)| \le \begin{cases} \dfrac{2}{3m^2(x)} \dfrac{e^{-\delta(x-\eta)b_\lambda(x)}}{b_\lambda^2(x)}, & -\infty < \eta < x, \\ \\ \dfrac{1}{3m^2(x)} \dfrac{e^{\delta(x-\eta)b_\lambda(x)}}{b_\lambda^2(x)}, & x < \eta < +\infty. \end{cases} \tag{30}$$

and

$$\left|\frac{\partial^j N_0(x,\eta,\lambda)}{\partial\eta^j}\right| \le \begin{cases} \dfrac{2}{3m^2(x)} \dfrac{e^{-\delta(x-\eta)b_\lambda(x)}}{b_\lambda^{2-j}(x)}, & -\infty < \eta < x, \\ \\ \dfrac{1}{3m^2(x)} \dfrac{e^{\delta(x-\eta)b_\lambda(x)}}{b_\lambda^{2-j}(x)}, & x < \eta < +\infty, \quad j = 1, 2. \end{cases} \tag{31}$$

Moreover, it yields that $N_j(x, \eta, \lambda) = 0$ for $|x - \eta| > 1$. Furthermore, by taking into account the conditions of Lemma 4 for the functions $m(x)$, $q(x)$ and $r(x)$ and (30), (31) at $|x - \eta| \le 1$, we obtain the following estimates:

$$|N_1(x,\eta,\lambda)| \le \begin{cases} \bar{c}_1 m^2(\eta)|x-\eta|^\beta b_\lambda^{3\alpha-2}(x) \dfrac{e^{-\delta(x-\eta)b_\lambda(x)}}{m^2(x)}, & -\infty < \eta < x, \\ \\ \bar{c}_2 m^2(\eta)|x-\eta|^\beta b_\lambda^{3\alpha-2}(x) \dfrac{e^{\delta(x-\eta)b_\lambda(x)}}{m^2(x)}, & x < \eta < +\infty, \end{cases} \tag{32}$$

$$|N_2(x,\eta,\lambda)| \le \begin{cases} \frac{m^2(\eta)}{m^2(x)} \sum_{k=0}^{2} \tilde{c}_k \frac{e^{-\delta(x-\eta)b_\lambda(x)}}{b_\lambda^k(x)}, & -\infty < \eta < x, \\ \\ \frac{m^2(\eta)}{m^2(x)} \sum_{k=0}^{2} \tilde{c}_k \frac{e^{\delta(x-\eta)b_\lambda(x)}}{b_\lambda^k(x)}, & x < \eta < +\infty, \end{cases} \tag{33}$$

and

$$|N_3(x,\eta,\lambda)| \le \begin{cases} \frac{2}{3m^2(x)} \frac{e^{-\delta(x-\eta)b_\lambda(x)}}{b_\lambda^2(x)}, & -\infty < \eta < x, \\ \\ \frac{1}{3m^2(x)} \frac{e^{\delta(x-\eta)b_\lambda(x)}}{b_\lambda^2(x)}, & x < \eta < +\infty. \end{cases} \tag{34}$$

We shall now estimate the norms $\|N_j(\lambda)\|_{L_{p'}\to L_{p'}}$ of the operators $N_j(\lambda)$, $j = 1,2,3$, by using Lemma 1 and the inequalities (32) - (34). We have that

$$\|N_1(\lambda)\|_{L_{p'}\to L_{p'}} \le \sup_{\eta\in R} \int_R \left[|N_1(x,\eta,\lambda)| + |N_1(\eta,x,\lambda)|\right] dx \le$$

$$\le c_1 \sup_{\eta\in R} \int_{\eta-1}^{\eta+1} \left(\frac{m^2(\eta)}{m^2(x)} \frac{\exp[-\delta(x-\eta)b_\lambda(x)]}{b_\lambda^2(x)} + \frac{m^2(x)}{m^2(\eta)} \frac{\exp[-\delta(x-\eta)b_\lambda(\eta)]}{b_\lambda^2(\eta)} \right) \times$$

$$\times \left| \frac{q(\eta)+\lambda-ir(\eta)}{m^2(\eta)} - \frac{q(x)+\lambda-ir(x)}{m^2(x)} \right| dx.$$

Moreover, by using the conditions (3), (4) and (6), we obtain that

$$\|N_1(\lambda)\|_{L_{p'}\to L_{p'}} \le$$

$$\le c_2 \sup_{\eta\in R} \int_{\eta-1}^{\eta+1} \left(\frac{\exp[-\delta(x-\eta)\tilde{c}b_\lambda(\eta)]}{\tilde{c}b_\lambda^2(\eta)} + \frac{\exp[-\delta(x-\eta)b_\lambda(\eta)]}{b_\lambda^2(\eta)} \right) \times$$

$$\times \left| \frac{q(\eta)+\lambda+ir(\eta)}{m^2(\eta)} \right|^\alpha |\eta-x|^\beta dx.$$

Hence, by making the change of variable $\eta - x = \dfrac{1}{\delta b_\lambda(\eta)} z$, we find that

$$\|N_1(\lambda)\|_{L_{p'}\to L_{p'}} \le$$

$$\leq \frac{c_3 \left|\frac{q(\eta)+\lambda+ir(\eta)}{m^2(\eta)}\right|^{\alpha}}{\left(\tilde{c}b_\lambda(\eta)\right)^{\beta+3}} + \frac{c_4 \left|\frac{q(\eta)+\lambda+ir(\eta)}{m^2(\eta)}\right|^{\alpha}}{\left(b_\lambda(\eta)\right)^{\beta+3}} = \frac{c_5}{\left(\frac{|q(\eta)+\lambda+ir(\eta)|}{m^2(\eta)}\right)^{\frac{\beta}{3}+1-\alpha}}.$$

Moreover, according to the condition (3), we have that

$$\frac{|q(\eta)+\lambda+ir(\eta)|}{m^2(\eta)} \geq \sqrt{1+\lambda}\,.$$

Therefore, from the previous inequality we obtain (21). Furthermore, in view of the conditions (3) - (5) and (7), we can deduce that

$$\|N_2(\lambda)\|_{L_{p'}\to L_{p'}} \leq \sup_{\eta\in R}\int_R \left[|N_2(x,\eta,\lambda)| + |N_2(\eta,x,\lambda)|\right]dx \leq$$

$$\leq c_2 \sup_{\eta\in R}\int_{\eta-1}^{\eta+1}\left(\frac{m^2(\eta)}{m^2(x)}e^{-\delta(x-\eta)b_\lambda(x)} + \frac{m^2(\eta)}{m^2(x)}\frac{e^{-\delta(x-\eta)b_\lambda(x)}}{b_\lambda(x)} + \frac{m^2(\eta)}{m^2(x)}\frac{e^{-\delta(x-\eta)b_\lambda(x)}}{b_\lambda^2(x)} + \right.$$

$$\left. + \frac{m^2(x)}{m^2(\eta)}e^{-\delta(x-\eta)b_\lambda(\eta)} + \frac{m^2(x)}{m^2(\eta)}\frac{e^{-\delta(x-\eta)b_\lambda(\eta)}}{b_\lambda(\eta)} + \frac{m^2(x)}{m^2(\eta)}\frac{e^{-\delta(x-\eta)b_\lambda(\eta)}}{b_\lambda^2(\eta)} + \right)dx.$$

Hence, by calculating the integrals, we obtain that

$$\|N_2(\lambda)\|_{L_{p'}\to L_{p'}} \leq \bar{c_3}\sup_{\eta\in R}\int_{\eta-1}^{\eta+1}\left[e^{-\delta(x-\eta)b_\lambda(\eta)} + \frac{e^{-\delta(x-\eta)b_\lambda(\eta)}}{b_\lambda(\eta)} + \frac{e^{-\delta(x-\eta)b_\lambda(\eta)}}{b_\lambda^2(\eta)}\right]dx =$$

$$\bar{c_3}\sup_{\eta\in R}\left[\frac{1}{\delta b_\lambda(\eta)}\left(1-e^{-\delta b_\lambda(\eta)}\right) + \frac{1}{\delta b_\lambda^2(\eta)}\left(1-e^{-\delta b_\lambda(\eta)}\right) + \frac{1}{\delta b_\lambda^3(\eta)}\left(1-e^{-\delta b_\lambda(\eta)}\right)\right] \leq$$

$$\leq \frac{c_2}{b_\lambda(\eta)},$$

and also (22) is proved. Moreover, the inequality (23) is proved in a similar way as the proof of (14) so we omit the details. The proof is complete. □

Proof of Lemma 6. Obviously, it yields that

$$(L+\lambda E)'\left[N_3(\lambda)g\right](\eta) =$$

$$= \left[m(\eta)\left(m(\eta)\left(\int_{-\infty}^{\eta} N_0(x,\eta,\lambda)d(\eta-x)f(x)dx + \int_{\eta}^{+\infty} N_0(x,\eta,\lambda)d(\eta-x)f(x)dx\right)\right)''_{\eta\eta}\right]'_{\eta} +$$

$$+(q(\eta)-ir(\eta)+\lambda)\left(\int\limits_{-\infty}^{\eta} N_0(x,\eta,\lambda)d(\eta-x)g(x)dx+\int\limits_{\eta}^{+\infty} N_0(x,\eta,\lambda)d(\eta-x)g(x)dx\right)(\eta).$$

Moreover, we have that

$$\frac{d}{d\eta}\left(m(x)N_3(\lambda)g\right)(\eta)=m'(x)\int\limits_R N_0(x,\eta,\lambda)d(\eta-x)g(x)dx+$$

$$+m(\eta)\int\limits_R N'_{0\eta}(x,\eta,\lambda)d(\eta-x)g(x)dx+m(\eta)\int\limits_R N_0(x,\eta,\lambda)d'_\eta(\eta-x)g(x)dx$$

and

$$\frac{d^2}{d\eta^2}\left(m(\eta)N_3(\lambda)g\right)(\eta)=$$

$$=m''(\eta)\int\limits_R N_0(x,\eta,\lambda)d(\eta-x)g(x)dx+2m'(\eta)\int\limits_R N'_{0\eta}(x,\eta,\lambda)d(\eta-x)g(x)dx+$$

$$+2m'(\eta)\int\limits_R N_0(x,\eta,\lambda)d'_\eta(\eta-x)g(x)dx+m(\eta)\int\limits_R N''_{0\eta\eta}(x,\eta,\lambda)d(\eta-x)g(x)dx+$$

$$+2m(\eta)\int\limits_R N'_{0\eta}(x,\eta,\lambda)d'_\eta(\eta-x)g(x)dx+m(\eta)\int\limits_R N_0(x,\eta,\lambda)d''_{\eta\eta}(\eta-x)g(x)dx.$$

We also note that

$$\frac{d}{d\eta}\left(m(\eta)\frac{d^2[m(\eta)N_3(\lambda)g]}{d\eta^2}\right)(\eta)=$$

$$=(m''(\eta)m'(\eta)+m'''(\eta)m(\eta))\int\limits_R N_0(x,\eta,\lambda)d(\eta-x)g(x)dx+$$

$$+\left(3m(\eta)m''(\eta)+2(m'(\eta))^2\right)\int\limits_R N'_{0\eta}(x,\eta,\lambda)d(\eta-x)g(x)dx+$$

$$+\left(3m(\eta)m''(\eta)+2(m'(\eta))^2\right)\int\limits_R N_0(x,\eta,\lambda)d'_\eta(\eta-x)g(x)dx+$$

$$+4m(\eta)m'(\eta)\int\limits_R N''_{0\eta\eta}(x,\eta,\lambda)d(\eta-x)g(x)dx+3m^2(\eta)\int\limits_R N'_{0\eta}(x,\eta,\lambda)d''_{\eta\eta}(\eta-x)g(x)dx+$$

$$+2m(\eta)m'(\eta)\int\limits_R N_0(x,\eta,\lambda)d''_{\eta\eta}(\eta-x)g(x)dx+m^2(\eta)\int\limits_R N'''_{0\eta\eta\eta}(x,\eta,\lambda)d(\eta-x)g(x)dx+$$

$$+m^2(\eta)m'(\eta)\int\limits_R N''_{0\eta\eta}(x,\eta,\lambda)d'_\eta(\eta-x)g(x)dx+2m^2(\eta)\int\limits_R N''_{0\eta\eta}(x,\eta,\lambda)d'_\eta(\eta-x)g(x)dx+$$

$$-m^2(\eta)N''_{0\eta\eta}(x,\eta,\lambda)d(\eta-x)g(x)\big|_{x=\eta+0}-m^2(\eta)N''_{0\eta\eta}(x,\eta,\lambda)d(\eta-x)g(x)\big|_{x=\eta-0}+$$

$$+8m(\eta)m'(\eta)\int\limits_R N'_{0\eta}(x,\eta,\lambda)d'_\eta(\eta-x)g(x)dx+m^2(\eta)\int\limits_R N_0(x,\eta,\lambda)d'''_{\eta\eta\eta}(\eta-x)g(x)dx.$$

Now, by using (18)-(20) and the notations $N_j(x,\eta,\lambda)$, $j=1,2,3$, we obtain the equality (24). The proof is complete. □

We are now ready to prove Lemma 4 and, finally, Theorem 2.

Proof of Lemma 4. By using the estimates (21) and (22) we conclude that there exists a number $\lambda_1>0$ such that when $\lambda\geq\lambda_1$ the inequality $\|N_1(\lambda)\|_{L_{p'}\to L_{p'}}+\|N_2(\lambda)\|_{L_{p'}\to L_{p'}}\leq\dfrac{1}{2}$ holds. Then the operator $\Phi(\lambda):=E+N_1(\lambda)+N_2(\lambda)$ has bounded inverse $\Phi^{-1}(\lambda)$ in $L_{p'}$. Therefore, defining $h:=[E+N_1(\lambda)+N_2(\lambda)]\,g$, from the relation (24) we obtain that
$(L+\lambda E)'\,[N_3(\lambda)\Phi^{-1}(\lambda)h](\eta)=h$. Hence, for all λ, $\lambda\geq\lambda_1$ we find that the function $y=N_3(\lambda)\Phi^{-1}(\lambda)g$ is a solution of the equation (16). The proof is complete. □

Proof of Theorem 2. By using Lemma 4 we conclude that the operator $(L+\lambda E)'$, acting in the space $L_{p'}(R)$ at $\lambda\geq\lambda_1$, has a right inverse, which is defined on $L_{p'}(R)$. Thus, $ker\,((L+\lambda E)')^*=\{0\}$, where $((L+\lambda E)')^*$ is the adjoint operator of $(L+\lambda E)'$. From here since $((L+\lambda E)')^*$ is an extension of the operator $L+\lambda E$, we have $ker(L+\lambda E)=\{0\},\ \ \lambda\geq\tilde\lambda=\max(\lambda_0,\lambda_1)$. Thus, the operator $L+\lambda E$ is a boundedly invertible operator in the space $L_{p'}(R)$ and, in fact, we have that

$$(L+\lambda E)^{-1}=M_3(\lambda)G^{-1}(\lambda),\quad \lambda\geq\tilde\lambda=\max(\lambda_0,\lambda_1). \tag{35}$$

Let y be a solution of the equation (1), where $\lambda\geq\tilde\lambda=\max(\lambda_0,\lambda_1)$. We shall prove the estimate (2). By using (35), Lemma 1 and the conditions (3) - (6) we have that

$$\left\|(q+\lambda+ir)(L+\lambda E)^{-1}\right\|_{L_p\to L_p}=\left\|(q+\lambda+ir)M_3(\lambda)G^{-1}(\lambda)\right\|_{L_p\to L_p}\leq$$

$$\leq c \sup_{\eta \in R} \int_{\eta-1}^{\eta+1} b_\lambda^3(\eta) b_\lambda^{-2}(x) \exp[-\sigma|x-\eta| b_\lambda(x)]dx \leq$$

$$\leq c_1 \sup_{\eta \in R} b_\lambda(\eta) \int_{\eta-1}^{\eta+1} \exp[-\sigma|x-\eta| b_\lambda(x)]dx < \infty.$$

From this and (1) we conclude that $\left\| m(x)\left(m(x)y'\right)'' \right\|_p \leq c\left(\|f\|_p + \|y\|_p \right)$. Finally, by combining the last two estimates we obtain (2). The proof is complete. $\square$

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Paper D

About conditions for the solvability of a class of third-order differential equations

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R.D.Akhmetkaliyeva. About conditions for the solvability of a class of third-order differential equations, Luleå University of Technology, Department of Engineering Sciences and Mathematics, Research Report 7 (2012).

Abstract: This paper is a generalization of some results recently obtained in [1]. The following third order differential equation with unbounded coefficients is considered:

$$-\mu_1(x)\left(\mu_2(x)\left(\mu_1(x)y'\right)'\right)' + \left(q(x) + ir(x) + \lambda\right)y = f(x).$$

Some new existence and uniqueness results are proved and some norm-estimates of the solutions are given.

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Keywords and Phrases: Separability, coercive estimate, third-order differential equations.

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1 Introduction

The separation of differential expressions was early studied W.N. Everitt and M. Giertz [4-7], and they proved some fundamental results.

Later on a number of results concerning the property referred to as separation of differential expressions have been obtained e.g. by K.Kh. Boimatov [2,3], M. Otelbaev [12], A. Zettl [13], A.S. Mohamed [8,9], and A.S. Mohamed et al. [10,11].

Some a very recents results in this direction was presented and proved by R.D. Akhmetkaliyeva, K.N. Ospanov, L.-E.Persson and P.Wall (see [1]).

In this paper we consider the following more general situation when the equation

$$(l+\lambda E)\,y := -m(x)\,(m(x)y')'' + [q(x)+ir(x)+\lambda]y = f(x),$$

investigated in [1] is replaced by a more general equation (see (2.1) below). The history and complementary literature connected to this equation can be found in [1].

This paper is organized as follows: In section 2 we state and discuss the main results together with some necessary Lemmas, necessary for the proof but also sometimes of independent interest. The proofs can be found in Section 3.

2 Main results and auxiliary statements

Let $1 < p < +\infty$. By $L_p \equiv L_p(R)$, $R = (-\infty, +\infty)$, we denote the space of functions with finite norm

$$\|\varphi\|_p := \left(\int\limits_R |\varphi(x)|^p dx\right)^{\frac{1}{p}}.$$

We study questions of the existence and uniqueness of solutions of the equation

$$(l+\lambda E)\,y := -\mu_1(x)\left(\mu_2(x)\left(\mu_1(x)y'\right)'\right)' + [q(x)+ir(x)+\lambda]y = f(x), \quad (2.1)$$

and conditions, so that, when a solution y of (2.1) exists, the estimate

$$\left\|\mu_1(x)\left(\mu_2(x)\left(\mu_1(x)y'\right)'\right)'\right\|_p^p + \|(q(x)+ir(x)+\lambda)y\|_p^p \le c\,\|f(x)\|_p^p \quad (2.2)$$

holds. Here, $\lambda \geq 0$ is a constant, and $\mu_1(x)$, $\mu_2(x)$, $q(x)$ and $r(x)$ are given functions, $f \in L_p$.

Definition 1. *A function $y(x) \in L_p(R)$, is called a solution of (2.1), if there is a sequence of three times continuously differentiable functions with compact support $\{y_n\}_{n=1}^{\infty}$ such that $\|y_n - y\|_p \to 0$ and $\|(L+\lambda E)y_n - f\|_p \to 0$ as $n \to \infty$.*

By $C^{(k)}(R)$ $(k = 1, 2, ...)$ we denote the set of all k times continuously differentiable functions $\varphi(x)$, for which the value $\sum_{j=0}^{k} \sup_{x\in R} |\varphi^{(j)}(x)|$ is finite.

Let $W_\lambda(x) := \dfrac{|q(x) + \lambda + ir(x)|}{\mu_1^2(x)\mu_2(x)}$.

Our main results in this paper are formulated in the following two theorems:

Theorem 1. *Assume that the functions $q(x)$, $r(x)$ are continuous on R, $\mu_1 \in C_{loc}^{(2)}(R)$, $\mu_2 \in C_{loc}^{(1)}(R)$ and satisfy the following conditions:*

$$\mu_1(x) \geq 1, \quad \mu_2(x) \geq 1 \quad \frac{q(x)}{\mu_1^4(x)\mu_2^2(x)} \geq 1, \ r(x) \geq 1, \tag{2.3}$$

$$c^{-1} \leq \frac{\mu_i(x)}{\mu_i(\eta)}, \frac{q(x)}{q(\eta)}, \frac{r(x)}{r(\eta)} \leq c, \quad (i = 1, 2), \ x, \eta \in R, \quad |x - \eta| \leq 1, \tag{2.4}$$

$$|\mu_1^{(j)}(x)| \leq c\mu_1(x), \quad |\mu_2'(x)| \leq c\mu_2(x), \ j = 1, 2, \ x \in R, \tag{2.5}$$

$$\sup_{|x-\eta|\leq 1} \frac{|W_\lambda(x) - W_\lambda(\eta)|}{|W_\lambda(x)|^\alpha |x-\eta|^\beta} < +\infty, \quad 0 < \alpha < \frac{\beta}{3} + 1, \quad \beta \in (0, 1], \ \lambda \geq 0. \tag{2.6}$$

Then there exists a number $\lambda_0 \geq 0$, such that the equation (2.1) for all $\lambda \geq \lambda_0$ has a solution y.

In conditions (2.4), (2.5), and elsewhere, c denotes a fixed constant which may, in general, be different in the various places it is used.

Theorem 2. *Let the functions $q(x)$, $r(x)$ be continuous, $\mu_1 \in C_{loc}^{(3)}(R)$, $\mu_2 \in C_{loc}^{(2)}(R)$ and satisfy the conditions (2.3), (2.4), (2.6) and*

$$|\mu_1^{(j)}(x)| \leq c\mu_1(x), \quad |\mu_2^{(i)}(x)| \leq c\mu_2(x), \ j = \overline{1,3}, \quad i = 1, 2, \ x \in R. \tag{2.7}$$

Then the solution of the equation (2.1) is unique and estimate (2.2) holds.

Remark 1: *For the case $\mu_2(x) \equiv 1$ these results coincides with those discussed and proved in [1].*

We now present the necessary Lemmas for the proofs of Theorems 1 and 2.

Let, $\xi_l = \xi_l(x)$ $(l = 0, 1, 2)$ be the roots of the equation $\mu_1^2(x)\mu_2(x)\xi^3 - r(x) + i(q(x) + \lambda) = 0$. From the conditions of Theorem 1 it follows that $0 < arg\xi_0 < \pi,\ \pi < arg\xi_j < 2\pi,\ j = 1, 2.$

We introduce the kernels

$$M_0(x,\eta,\lambda) = \begin{cases} -\dfrac{1}{3\mu_1^2(x)\mu_2(x)} \dfrac{e^{i(x-\eta)\xi_0}}{\xi_0^2}, & -\infty < \eta < x \\ \dfrac{1}{3\mu_1^2(x)\mu_2(x)} \sum\limits_{l=1}^{2} \dfrac{e^{i(x-\eta)\xi_l}}{\xi_l^2}, & x < \eta < +\infty, \end{cases} \tag{2.8}$$

$$M_1(x,\eta,\lambda) = \mu_1^2(\eta)\mu_2(\eta)\times$$

$$\times \left[\frac{q(\eta) + ir(\eta) + \lambda}{\mu_1^2(\eta)\mu_2(\eta)} - \frac{q(x) + ir(x) + \lambda}{\mu_1^2(x)\mu_2(x)}\right] M_0(x,\eta,\lambda)\omega(\eta - x),$$

$$\begin{aligned} M_2(x,\eta,\lambda) = &- \left[2\mu_1'(\eta)\mu_1(\eta)\mu_2(\eta)\omega(\eta - x) + \mu_1^2(\eta)\mu_2'(\eta)\omega(\eta - x) + \right.\\ &\left. + 3\mu_1^2(\eta)\mu_2(\eta)\omega_\eta'(\eta - x)\right] M_{0\eta\eta}''(x,\eta,\lambda) - \\ &- \left[\mu_1'(\eta)\mu_1(\eta)\mu_2'(\eta)\omega(\eta - x) + \mu_1''(\eta)\mu_1(\eta)\mu_2(\eta)\omega(\eta - x) + \right.\\ &+ 4\mu_1'(\eta)\mu_1(\eta)\mu_2(\eta)\omega_\eta'(\eta - x) + 2\mu_1^2(\eta)\mu_2'(\eta)\omega_\eta'(\eta - x) + \\ &\left. + 3\mu_1^2(\eta)\mu_2(\eta)\omega_{\eta\eta}''(\eta - x)\right] M_{0\eta}'(x,\eta,\lambda) - \\ &- \left[\mu_1'(\eta)\mu_1(\eta)\mu_2'(\eta)\omega_\eta'(\eta - x) + \mu_1''(\eta)\mu_1(\eta)\mu_2(\eta)\omega_\eta'(\eta - x) + \right.\\ &+ 2\mu_1'(\eta)\mu_1(\eta)\mu_2(\eta)\omega_{\eta\eta}''(\eta - x) + \\ &\left. + \mu_1^2(\eta)\mu_2'(\eta)\omega_{\eta\eta}''(\eta - x) + \mu_1^2(\eta)\mu_2(\eta)\omega_{\eta\eta\eta}'''(\eta - x)\right] M_0(x,\eta,\lambda), \end{aligned}$$

and

$$M_3(x,\eta,\lambda) = M_0(x,\eta,\lambda)\omega(\eta - x),$$

where the function $\omega(\eta) \in C_0^\infty(-1, 1)$ is such that

$$\omega(\eta) = \begin{cases} 1, & |\eta| \le \frac{1}{2} \\ 0, & |\eta| \ge 1. \end{cases}$$

We get the following equalities:

$$\left.\frac{\partial^j M_0(x,\eta,\lambda)}{\partial\eta^j}\right|_{x=\eta-0}=\left.\frac{\partial^j M_0(x,\eta,\lambda)}{\partial\eta^j}\right|_{x=\eta+0},\ j=0,1, \tag{2.9}$$

$$\left.\frac{\partial^2 M_0(x,\eta,\lambda)}{\partial\eta^2}\right|_{x=\eta-0}-\left.\frac{\partial^2 M_0(x,\eta,\lambda)}{\partial\eta^2}\right|_{x=\eta+0}=-\frac{1}{\mu_1^2(x)\mu_2(x)}, \tag{2.10}$$

and

$$-\mu_1(x)\left(\mu_2(x)\left(\mu_1(x)\frac{\partial M_0(x,\eta,\lambda)}{\partial\eta}\right)'_\eta\right)'_\eta+$$
$$+[q(x)+ir(x)+\lambda]M_0(x,\eta,\lambda)=0. \tag{2.11}$$

Moreover, we define the operators $M_j(\lambda)$, $(j=\overline{1,3})$ by means of the equation

$$\left(M_j(\lambda)f\right)(\eta)=\int\limits_R M_j(x,\eta,\lambda)f(x)dx, \quad (j=\overline{1,3}).$$

The following statment is well-known:

Lemma 1. *Let $1<p<+\infty$, $k(x,\eta)$ be continuous function and*

$$\left(K\nu\right)(\eta)=\int\limits_R k(x,\eta)\nu(x)dx.$$

Then

$$\|K\|_{L_p\to L_p}\le\sup_{\eta\in R}\int\limits_R\left[|k(x,\eta)|+|k(\eta,x)|\right]dx.$$

Our next lemma reads:

Lemma 2. *Let all the conditions of Theorem 1 be satisfied. Then the operators $M_j(\lambda)$, $j=\overline{1,3}$, are continuous in the space L_p, and the following estimates hold $(\lambda\ge 0)$:*

$$\|M_1(\lambda)\|_{L_p\to L_p}\le\frac{c}{b_\lambda^{\beta+3-3\alpha}(\eta)},\quad \beta\in(0,1],\quad 0<\alpha<\frac{\beta}{3}+1, \tag{2.12}$$

$$\|M_2(\lambda)\|_{L_p\to L_p}\le\frac{c}{b_\lambda(\eta)}, \tag{2.13}$$

and
$$\|M_3(\lambda)\|_{L_p\to L_p} \le \frac{c}{\mu_1^2(\eta)\mu_2(\eta)b_\lambda^3(\eta)}. \tag{2.14}$$

Here $b_\lambda(x) = \sqrt[3]{\dfrac{|r(x) - i(q(x)+\lambda)|}{\mu_1^2(x)\mu_2(x)}}$.

Remark 2. *The statement of Lemma 2 remains true if the condition* $r(x) \ge 1$ *in (3) is replaced by the condition* $r(x) \le -1$*, but we do not need this fact in our further investigations.*

Denote by $L + \lambda E$ $(\lambda \ge 0)$ the closure in L_p of the differential expression

$$(l+\lambda E)y \equiv -\mu_1(x)\left(\mu_2(x)\left(\mu_1(x)y'\right)'\right)' + [q(x) + ir(x) + \lambda]y,$$

defined on the set $C_0^\infty(R)$ of infinitely differentiable and compactly supported functions. From definition 1 it is easy to see that the function $y \in L_p$ is a solution of the equation (1) if it belongs to $D(L+\lambda E)$ and the equality $(L+\lambda E)y = f$ holds.

The next identity is crucial for the proof of Theorem 1 and also of independent interest.

Lemma 3. *Let the conditions of Theorem 1 be satisfied. Then the following equality holds:*

$$(L+\lambda E)\left[M_3(\lambda)f\right](\eta) = f(\eta) + \left[M_1(\lambda)f\right](\eta) + \left[M_2(\lambda)f\right](\eta). \tag{2.15}$$

Let the functions $\mu_1(x)$, $\mu_2(x)$, $q(x)$ and $r(x)$ satisfy the conditions of Theorem 2, and let p' denote the conjugate number of p i.e. $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. We denote by $(L+\lambda E)'$ an operator acting in the space $L_{p'}(R)$ and such that

$$\left((L+\lambda E)\,y, z\right) = \left(y, (L+\lambda E)'\,z\right), \quad y \in D(L+\lambda E), \quad z \in D\left((L+\lambda E)'\right).$$

Obviously, we have that

$$(L+\lambda E)'z := \Big(\mu_1(x)\left(\mu_2(x)\left(\mu_1(x)z\right)'\right)'\Big)' + \left(q(x) + \lambda - ir(x)\right)z.$$

We consider the differential equation

$$(L+\lambda E)'z :=$$

$$:= \left(\mu_1(x)\left(\mu_2(x)\left(\mu_1(x)z\right)'\right)'\right)' + \left(q(x)+\lambda-ir(x)\right)z = g(x), \qquad (2.16)$$

where the function $\mu_1(x) \geq 1$ is continuous together with derivatives up to third order and $\mu_2(x) \geq 1$ is continuous together with derivatives up to second order, $q(x)$ and $r(x)$ are continuous real-valued functions, $\lambda \geq 0$ and $g(x) \in L_{p'}(R)$.

The next Lemma is crucial for the proof of Theorem 2.

Lemma 4. *Let the continuous functions $q(x)$, $r(x)$ and the functions $\mu_1 \in C^{(3)}_{loc}(R)$, $\mu_2 \in C^{(2)}_{loc}(R)$ satisfy the conditions (2.3), (2.4), (2.6) and (2.7). Then there exists a number $\lambda_1 \geq 0$, such that the equation (2.16) for all $\lambda \geq \lambda_1$ has a solution.*

To prove Lemma 4 we need the following auxiliary assertions. Let $\zeta_l = \zeta_l(x)$ $(l = 1, 2, 3)$ be the roots of the equation

$$\mu_1^2(x)\mu_2(x)\zeta^3 - (r(x) + i(q(x) + \lambda)) = 0.$$

From the condition of Lemma 4 follows that $0 < arg\zeta_j < \pi, \quad (j = 1, 2)$ and $\pi < arg\zeta_3 < 2\pi$. We introduce the function

$$\widetilde{M}_0(x,\eta,\lambda) = \begin{cases} -\dfrac{1}{3\mu_1^2(x)\mu_2(x)} \displaystyle\sum_{j=1}^{2} \dfrac{e^{i(x-\eta)\zeta_j}}{\zeta_j^2}, & -\infty < \eta < x \\ \\ \dfrac{1}{3\mu_1^2(x)\mu_2(x)} \dfrac{e^{i(x-\eta)\zeta_3}}{\zeta_3^2}, & x < \eta < +\infty. \end{cases} \qquad (2.17)$$

It satisfies the following equalities:

$$\left.\frac{\partial^j \widetilde{M}_0(x,\eta,\lambda)}{\partial \eta^j}\right|_{x=\eta-0} = \left.\frac{\partial^j \widetilde{M}_0(x,\eta,\lambda)}{\partial \eta^j}\right|_{x=\eta+0}, \ j = 0, 1, \qquad (2.18)$$

$$\left.\frac{\partial^2 \widetilde{M}_0(x,\eta,\lambda)}{\partial \eta^2}\right|_{x=\eta-0} - \left.\frac{\partial^2 \widetilde{M}_0(x,\eta,\lambda)}{\partial \eta^2}\right|_{x=\eta+0} = -\frac{1}{\mu_1^2(x)\mu_2(x)}, \qquad (2.19)$$

and

$$\left(\mu_1(x)\left(\mu_2(x)\left(\mu_1(x)\widetilde{M}_0(x,\eta,\lambda)\right)'_\eta\right)'_\eta\right)'_\eta +$$

$$+[q(x) - ir(x) + \lambda]\widetilde{M}_0(x,\eta,\lambda) = 0. \qquad (2.20)$$

We denote:

$$\widetilde{M_1}(x,\eta,\lambda) =$$

$$= \mu_1^2(\eta)\mu_2(\eta)\left[\frac{q(\eta)+\lambda-ir(\eta)}{\mu_1^2(\eta)\mu_2(\eta)} - \frac{q(x)+\lambda-ir(x)}{\mu_1^2(x)\mu_2(x)}\right]\widetilde{M_0}(x,\eta,\lambda)\omega(\eta-x),$$

$$\widetilde{M_2}(x,\eta,\lambda) = [4\mu_1(\eta)\mu_2(\eta)\mu_1'(\eta)\omega(\eta-x) + 2\mu_1^2(\eta)\mu_2'(\eta)\omega(\eta-x) +$$

$$+3\mu_1^2(\eta)\mu_2(\eta)\omega'(\eta-x)]\,\frac{\partial^2\widetilde{M_0}(x,\eta,\lambda)}{\partial\eta^2}+$$

$$+[5\mu_1(\eta)\mu_2'(\eta)\mu_1'(\eta)\omega(\eta-x) + 3\mu_1(\eta)\mu_2(\eta)\mu_1''(\eta)\omega(\eta-x)+$$

$$+2(\mu_1'(\eta))^2\mu_2(\eta)\omega(\eta-x) + \mu_1^2(\eta)\mu_2''(\eta)\omega(\eta-x)+$$

$$+8\mu_1(\eta)\mu_2(\eta)\mu_1'(\eta)\omega_\eta'(\eta-x) + 4\mu_1^2(\eta)\mu_2'(\eta)\omega_\eta'(\eta-x)+$$

$$+3\mu_1^2(\eta)\mu_2(\eta)\omega_{\eta\eta}''(\eta-x)]\,\frac{\partial\widetilde{M_0}(x,\eta,\lambda)}{\partial\eta}+$$

$$+\left[3(\mu_1'(\eta))^2\mu_2'(\eta)\omega(\eta-x) + \mu_1(\eta)\mu_2''(\eta)\mu_1'(\eta)\omega(\eta-x)+\right.$$

$$+2\mu_1(\eta)\mu_2'(\eta)\mu_1''(\eta)\omega(\eta-x) + \mu_1'(\eta)\mu_2(\eta)\mu_1''(\eta)\omega(\eta-x)+$$

$$+\mu_1(\eta)\mu_2(\eta)\mu_1'''(\eta)\omega(\eta-x) + 5\mu_1(\eta)\mu_2'(\eta)\mu_1'(\eta)\omega_\eta'(\eta-x)+$$

$$+3\mu_1(\eta)\mu_2(\eta)\mu_1''(\eta)\omega_\eta'(\eta-x) + 2(\mu_1'(\eta))^2\mu_2(\eta)\omega_\eta'(\eta-x)+$$

$$+4\mu_1(\eta)\mu_2(\eta)\mu_1'(\eta)\omega_{\eta\eta}''(\eta-x) + \mu_1^2(\eta)\mu_2''(\eta)\omega_\eta'(\eta-x)+$$

$$+2\mu_1^2(\eta)\mu_2'(\eta)\omega_{\eta\eta}''(\eta-x) + \mu_1^2(\eta)\mu_2(\eta)\omega_{\eta\eta\eta}'''(\eta-x)]\,\widetilde{M_0}(x,\eta,\lambda),$$

and

$$\widetilde{M_3}(x,\eta,\lambda) = \widetilde{M_0}(x,\eta,\lambda)\omega(\eta-x).$$

We introduce following integral operators:

$$\left(\widetilde{M_j}(\lambda)f\right)(\eta) = \int\limits_R \widetilde{M_j}(x,\eta,\lambda)f(x)dx, \quad (j=\overline{1,3}).$$

The following estimates hold for these operators:

Lemma 5. *Let all the conditions of Lemma 4 are satisfied. Then the operators* $\widetilde{M}_j(\lambda)$ *are continuous in the space* $L_{p'}$*, and the following estimates hold* $(\lambda \geq 0)$

$$\left\| \widetilde{M}_1(\lambda) \right\|_{L_{p'} \to L_{p'}} \leq \frac{c}{b_\lambda^{\beta'+3-3\alpha'}(\eta)}, \quad \beta' \in (0,1], \quad 0 < \alpha' < \frac{\beta'}{3} + 1, \tag{2.21}$$

$$\left\| \widetilde{M}_2(\lambda) \right\|_{L_{p'} \to L_{p'}} \leq \frac{c}{b_\lambda(\eta)}, \tag{2.22}$$

and

$$\left\| \widetilde{M}_3(\lambda) \right\|_{L_{p'} \to L_{p'}} \leq \frac{c}{\mu_1^2(\eta)\mu_2(\eta) b_\lambda^3(\eta)}. \tag{2.23}$$

Finally, for the proof of Theorem 2 we need the following identity also this of independent interest:

Lemma 6. Let the conditions of Lemma 4 be fulfilled. Then the following identity holds:

$$(L+\lambda E)' \left[\widetilde{M}_3(\lambda) g \right](\eta) = g(\eta) + \left[\widetilde{M}_1(\lambda) g \right](\eta) + \left[\widetilde{M}_2(\lambda) g \right](\eta) \tag{2.24}$$

3 Proofs

For completeness we include a proof of the well-known Lemma 1.

Proof of Lemma 1. With $\frac{1}{p} + \frac{1}{p'} = 1$ we have the inequalities

$$\|K\|_{L_p \to L_p} = \sup_{\|f\|_{L_p=1}} \sup_{\|g\|_{L_{p'}=1}} |\langle kf, g \rangle| \leq$$

$$\leq \sup_{\|f\|_{L_p=1}} \sup_{\|g\|_{L_{p'}=1}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |k(x,\eta) f(x) g(\eta)| \, dx d\eta.$$

To the last inequality we apply the well known inequality of Young, namely, $|ab| \leq \frac{|a|^p}{p} + \frac{|b|^{p'}}{p'}$, and obtain that

$$\|K\|_{L_p \to L_p} = \sup_{\|f\|_{L_p=1}} \sup_{\|g\|_{L_{p'}=1}} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(|k(x,\eta)| \left(\frac{|f(x)|^p}{p} + \frac{|g(\eta)|^{p'}}{p'} \right) dx d\eta \right) \right) \leq$$

$$\le \sup_{\eta\in R} \int\limits_{-\infty}^{\infty} (|k(x,\eta)| + |k(\eta,x)|)\, dx.$$

The lemma is proved. $\square$

Proof of Lemma 2. Under the assumptions of Theorem 1 for the functions $q(x), r(x)$, $\mu_1(x)$ and $\mu_2(x)$, there exists a constant $\sigma > 0$ such that $Im\xi_1 \ge \sigma$ and $Im\xi_l \le -\sigma$ $(l = 2, 3)$. Then from the definition (2.8) we can derive that

$$|M_0(x,\eta,\lambda)| \le \begin{cases} \dfrac{1}{3\mu_1^2(x)\mu_2(x)} \dfrac{e^{-\sigma(x-\eta)b_\lambda(x)}}{b_\lambda^2(x)}, & -\infty < \eta < x, \\ \\ \dfrac{2}{3\mu_1^2(x)\mu_2(x)} \dfrac{e^{\sigma(x-\eta)b_\lambda(x)}}{b_\lambda^2(x)}, & x < \eta < +\infty, \end{cases} \tag{3.1}$$

and

$$\left|\frac{\partial^j M_0(x,\eta,\lambda)}{\partial \eta^j}\right| \le \begin{cases} \dfrac{1}{3\mu_1^2(x)\mu_2(x)} \dfrac{e^{-\sigma(x-\eta)b_\lambda(x)}}{b_\lambda^{2-j}(x)}, & -\infty < \eta < x, \\ \\ \dfrac{2}{3\mu_1^2(x)\mu_2(x)} \dfrac{e^{\sigma(x-\eta)b_\lambda(x)}}{b_\lambda^{2-j}(x)}, & x < \eta < +\infty. \end{cases} \tag{3.2}$$

According to our choice $M_j(x,\eta,\lambda) = 0$ at $|x-\eta| > 1$. Taking the conditions (2.3) - (2.6) of Theorem 1 and (3.1) - (3.2) for the functions $M_j(x,\eta,\lambda)$ $(j = 0, 1, 2)$ at $|x-\eta| \le 1$ into account we obtain the following estimates:

$$|M_1(x,\eta,\lambda)| \le \begin{cases} c\mu_1^2(\eta)\mu_2(\eta)|x-\eta|^\beta b_\lambda^{3\alpha-2}(x) \dfrac{e^{-\sigma(x-\eta)b_\lambda(x)}}{\mu_1^2(x)\mu_2(x)}, & -\infty < \eta < x, \\ \\ c\mu_1^2(\eta)\mu_2(\eta)|x-\eta|^\beta b_\lambda^{3\alpha-2}(x) \dfrac{e^{\sigma(x-\eta)b_\lambda(x)}}{\mu_1^2(x)\mu_2(x)}, & x < \eta < +\infty. \end{cases} \tag{3.3}$$

$$|M_2(x,\eta,\lambda)| \le \begin{cases} \dfrac{\mu_1^2(\eta)\mu_2(\eta)}{\mu_1^2(x)\mu_2(x)} \displaystyle\sum_{k=0}^{2} \tilde{c}_k \dfrac{e^{-\sigma(x-\eta)b_\lambda(x)}}{b_\lambda^k(x)}, & -\infty < \eta < x, \\ \\ \dfrac{\mu_1^2(\eta)\mu_2(\eta)}{\mu_1^2(x)\mu_2(x)} \displaystyle\sum_{k=0}^{2} \tilde{c}_k \dfrac{e^{\sigma(x-\eta)b_\lambda(x)}}{b_\lambda^k(x)}, & x < \eta < +\infty. \end{cases} \tag{3.4}$$

and

$$|M_3(x,\eta,\lambda)| \leq \begin{cases} \dfrac{1}{3\mu_1^2(x)\mu_2(x)} \dfrac{e^{-\sigma(x-\eta)b_\lambda(x)}}{b_\lambda^2(x)}, & -\infty < \eta < x, \\ \\ \dfrac{2}{3\mu_1^2(x)\mu_2(x)} \dfrac{e^{\sigma(x-\eta)b_\lambda(x)}}{b_\lambda^2(x)}, & x < \eta < +\infty. \end{cases} \tag{3.5}$$

We shall estimate the norms $\|M_j(\lambda)\|_{L_p \to L_p}$ of the operators $M_j(\lambda)$ $(j = \overline{1,3})$ using the Lemma 1 and the inequalities (3.3) - (3.5). We have that

$$\|M_1(\lambda)\|_{L_p \to L_p} \leq \sup_{\eta \in R} \int\limits_R [|M_1(x,\eta,\lambda)| + |M_1(\eta,x,\lambda)|]\, dx \leq$$

$$\leq c \sup_{\eta \in R} \int\limits_{\eta-1}^{\eta+1} \left(\frac{\mu_1^2(\eta)\mu_2(\eta)}{\mu_1^2(x)\mu_2(x)} \frac{\exp[-\sigma(x-\eta)b_\lambda(x)]}{b_\lambda^2(x)} + \frac{\mu_1^2(x)\mu_2(x)}{\mu_1^2(\eta)\mu_2(\eta)} \frac{\exp[-\sigma(x-\eta)b_\lambda(\eta)]}{b_\lambda^2(\eta)} \right) \times$$

$$\times \left| \frac{q(\eta)+\lambda+ir(\eta)}{\mu_1^2(\eta)\mu_2(\eta)} - \frac{q(x)+\lambda+ir(x)}{\mu_1^2(x)\mu_2(x)} \right| dx.$$

By using (2.3), (2.4) and (2.6) we obtain that
$\|M_1(\lambda)\|_{L_p \to L_p} \leq$

$$\leq c \sup_{\eta \in R} \int\limits_{\eta-1}^{\eta+1} \left(\frac{\exp[-\sigma(x-\eta)cb_\lambda(\eta)]}{cb_\lambda^2(\eta)} + \frac{\exp[-\sigma(x-\eta)b_\lambda(\eta)]}{b_\lambda^2(\eta)} \right) \times$$

$$\times \left| \frac{q(\eta)+\lambda+ir(\eta)}{\mu_1^2(\eta)\mu_2(\eta)} \right|^\alpha |\eta - x|^\beta dx.$$

Hence, by making the change of variable $\eta - x = \dfrac{1}{\sigma b_\lambda(\eta)} z$, we receive that
$\|M_1(\lambda)\|_{L_p \to L_p} \leq$

$$\leq \frac{c \left| \dfrac{q(\eta)+\lambda+ir(\eta)}{\mu_1^2(\eta)\mu_2(\eta)} \right|^\alpha}{(cb_\lambda(\eta))^{\beta+3}} + \frac{c \left| \dfrac{q(\eta)+\lambda+ir(\eta)}{\mu_1^2(\eta)\mu_2(\eta)} \right|^\alpha}{(b_\lambda(\eta))^{\beta+3}} =$$

$$= \frac{c}{\left(\frac{|q(\eta)+\lambda+ir(\eta)|}{\mu_1^2(\eta)\mu_2(\eta)}\right)^{\frac{\beta}{3}+1-\alpha}}.$$

Moreover, according to condition (2.3), we find that

$$\frac{|q(\eta)+\lambda+ir(\eta)|}{\mu_1^2(\eta)\mu_2(\eta)} \geq \sqrt{1+\lambda}.$$

Therefore, from the previous inequality we obtain (2.12). According to the conditions (2.3) - (2.5) of Theorem 1 we have that

$$\|M_2(\lambda)\|_{L_p\to L_p} \leq \sup_{\eta\in R} \int_R \left[|M_2(x,\eta,\lambda)| + |M_2(\eta,x,\lambda)|\right] dx \leq$$

$$\leq c \sup_{\eta\in R} \int_{\eta-1}^{\eta+1} \left(\frac{\mu_1^2(\eta)\mu_2(\eta)}{\mu_1^2(x)\mu_2(x)} e^{-\sigma(x-\eta)b_\lambda(x)} + \frac{\mu_1^2(\eta)\mu_2(\eta)}{\mu_1^2(x)\mu_2(x)} \frac{e^{-\sigma(x-\eta)b_\lambda(x)}}{b_\lambda(x)} + \right.$$

$$+ \frac{\mu_1^2(\eta)\mu_2(\eta)}{\mu_1^2(x)\mu_2(x)} \frac{e^{-\sigma(x-\eta)b_\lambda(x)}}{b_\lambda^2(x)} + \frac{\mu_1^2(x)\mu_2(x)}{\mu_1^2(\eta)\mu_2(\eta)} e^{-\sigma(x-\eta)b_\lambda(\eta)} +$$

$$\left. + \frac{\mu_1^2(x)\mu_2(x)}{\mu_1^2(\eta)\mu_2(\eta)} \frac{e^{-\sigma(x-\eta)b_\lambda(\eta)}}{b_\lambda(\eta)} + \frac{\mu_1^2(x)\mu_2(x)}{\mu_1^2(\eta)\mu_2(\eta)} \frac{e^{-\sigma(x-\eta)b_\lambda(\eta)}}{b_\lambda^2(\eta)} \right) dx.$$

Hence, calculating the integrals, we obtain that

$$\|M_2(\lambda)\|_{L_p\to L_p} \leq c \sup_{\eta\in R} \int_{\eta-1}^{\eta+1} \left[e^{-\sigma(x-\eta)b_\lambda(\eta)} + \frac{e^{-\sigma(x-\eta)b_\lambda(\eta)}}{b_\lambda(\eta)} + \frac{e^{-\sigma(x-\eta)b_\lambda(\eta)}}{b_\lambda^2(\eta)} \right] dx =$$

$$= c \sup_{\eta\in R} \left[\frac{\left(1-e^{-\sigma b_\lambda(\eta)}\right)}{\sigma b_\lambda(\eta)} + \frac{\left(1-e^{-\sigma b_\lambda(\eta)}\right)}{\sigma b_\lambda^2(\eta)} + \frac{\left(1-e^{-\sigma b_\lambda(\eta)}\right)}{\sigma b_\lambda^3(\eta)} \right] \leq \frac{c}{b_\lambda(\eta)}.$$

We shall prove the inequality (2.14). According to the estimate (3.5) we find that

$$\|M_3(\lambda)\|_{L_p\to L_p} \leq \sup_{\eta\in R} \int_R \left[|M_3(x,\eta,\lambda)| + |M_3(\eta,x,\lambda)|\right] dx \leq$$

$$\leq \sup_{\eta\in R}\int\limits_{\eta-1}^{\eta+1}\left(\frac{2}{3\mu_1^2(x)\mu_2(x)}\frac{e^{-\sigma(x-\eta)b_\lambda(x)}}{b_\lambda^2(x)}+\frac{2}{3\mu_1^2(\eta)\mu_2(\eta)}\frac{e^{-\sigma(x-\eta)b_\lambda(\eta)}}{b_\lambda^2(\eta)}\right)dx.$$

Now, take in account the condition (2.4) and we get that

$$\|M_3(\lambda)\|_{L_p\to L_p}\leq c\sup_{\eta\in R}\int\limits_{\eta-1}^{\eta+1}\left(\frac{2}{3(c^{-1})^2\mu_1^2(\eta)\mu_2(\eta)}\frac{\exp[-\sigma(x-\eta)cb_\lambda(\eta)]}{cb_\lambda^2(\eta)}+\right.$$

$$\left.+\frac{2}{3\mu_1^2(\eta)\mu_2(\eta)}\frac{\exp[-\sigma(x-\eta)b_\lambda(\eta)]}{b_\lambda^2(\eta)}\right)dx\leq$$

$$\leq 2c\sup_{\eta\in R}\left(\frac{c}{(c^{-1})^2\mu_1^2(\eta)\mu_2(\eta)b_\lambda^3(\eta)}+\frac{c}{\mu_1^2(\eta)\mu_2(\eta)b_\lambda^3(\eta)}\right)\leq\frac{c}{\mu_1^2(\eta)\mu_2(\eta)b_\lambda^3(\eta)}.$$

The lemma is proved. □

Proof of Lemma 3. Obviously, it yields that
$(L+\lambda E)\left[M_3(\lambda)f\right](\eta)=$

$$=-\mu_1(\eta)\left(\mu_2(\eta)\left(\mu_1(\eta)\left(\int\limits_R M_0(x,\eta,\lambda)\omega(\eta-x)f(x)dx\right)_\eta'\right)_\eta'\right)_\eta'+$$

$$+(q(\eta)+ir(\eta)+\lambda)\int\limits_R M_0(x,\eta,\lambda)\omega(\eta-x)f(x)dx.$$

By calculating the derivatives and taking equality (2.9) into account we have that
$\frac{d}{d\eta}\left(M_3(\lambda)f\right)(\eta)=$

$$=\left(\int\limits_{-\infty}^{\eta}M_0(x,\eta,\lambda)\omega(\eta-x)f(x)dx+\int\limits_{\eta}^{+\infty}M_0(x,\eta,\lambda)\omega(\eta-x)f(x)dx\right)_\eta'=$$

$$=\int\limits_R M_{0\eta}'(x,\eta,\lambda)\omega(\eta-x)f(x)dx+\int\limits_R M_0(x,\eta,\lambda)\omega_\eta'(\eta-x)f(x)dx.$$

Furthermore, by calculating all derivatives, we obtain that

$$\frac{d}{d\eta}\left(\mu_2(\eta)\left(\mu_1(\eta)M'_{3\eta}(\lambda)f\right)\right)(\eta)=$$

$$=\left[\mu_2'(\eta)\mu_1'(\eta)+\mu_2(\eta)\mu_1''(\eta)\right]\int\limits_R M'_{0\eta}\omega(\eta-x)f(x)dx+$$

$$+\mu_2(\eta)\mu_1(\eta)\int\limits_R M'''_{0\eta\eta\eta}\omega(\eta-x)f(x)dx+3\mu_2(\eta)\mu_1(\eta)\int\limits_R M''_{0\eta\eta}\omega'_\eta(\eta-x)f(x)dx+$$

$$+\mu_2(\eta)\mu_1(\eta)M''_{0\eta\eta}\omega(\eta-x)f(x)\big|_{x=\eta-0}-\mu_2(\eta)\mu_1(\eta)M''_{0\eta\eta}\omega(\eta-x)f(x)\big|_{x=\eta+0}+$$

$$+3\mu_2(\eta)\mu_1(\eta)\int\limits_R M'_{0\eta}\omega''_{\eta\eta}(\eta-x)f(x)dx+\mu_2(\eta)\mu_1(\eta)\int\limits_R M_0\omega'''_{\eta\eta\eta}(\eta-x)f(x)dx+$$

$$+\left[2\mu_2(\eta)\mu_1'(\eta)+\mu_2'(\eta)\mu_1(\eta)\right]\int\limits_R M''_{0\eta\eta}\omega(\eta-x)f(x)dx+$$

$$+\left[4\mu_2(\eta)\mu_1'(\eta)+2\mu_2'(\eta)\mu_1(\eta)\right]\int\limits_R M'_{0\eta}\omega'_\eta(\eta-x)f(x)dx+$$

$$+\left[\mu_2'(\eta)\mu_1'(\eta)+\mu_2(\eta)\mu_1''(\eta)\right]\int\limits_R M_0\omega'_\eta(\eta-x)f(x)dx+$$

$$+\left[2\mu_2(\eta)\mu_1'(\eta)+\mu_2'(\eta)\mu_1(\eta)\right]\int\limits_R M_0\omega''_{\eta\eta}(\eta-x)f(x)dx.$$

Now, by using (2.10), (2.11), and the notations $M_j(x,\eta,\lambda)$ $(j=\overline{1,3})$ we obtain the equality (2.15). The lemma is proved. □

Proof of Lemma 5. The Lemma can be proved similarly as the proof of Lemma 2. So we leave out the details.

Proof of Lemma 6. The Lemma can be proved similarly as the proof of Lemma 3. So we leave out the details.

Proof of Lemma 4. By the estimates (2.21) and (2.22) there exists a number $\lambda_1>0$ such that when $\lambda\geq\lambda_1$ the inequality $\left\|\widetilde{M}_1(\lambda)\right\|_{L_{p'}\to L_{p'}}+\left\|\widetilde{M}_2(\lambda)\right\|_{L_{p'}\to L_{p'}}\leq\frac{1}{2}$ holds. Then the operator $\Phi(\lambda)=E+\widetilde{M}_1(\lambda)+\widetilde{M}_2(\lambda)$ has bounded inverse $\Phi^{-1}(\lambda)$ in $L_{p'}$. Therefore, assuming $h = \left[E + \widetilde{M}_1(\lambda) + \widetilde{M}_2(\lambda)\right] g$, from the relation (2.24) we obtain that $(L+\lambda E)\,[\widetilde{M}_3(\lambda)\Phi^{-1}(\lambda)h](\eta) = h$. Hence, for all λ, $\lambda \geq \lambda_1$ we see that the function $y = \widetilde{M}_3(\lambda)\Phi^{-1}(\lambda)g$ is a solution of the equation (2.16). The theorem is proved. □

Proof of Theorem 1. By using the estimates (2.12) and (2.13) in Lemma 2 we conclude that there exists a number $\lambda_0 > 0$ such that when $\lambda \geq \lambda_0$ the inequality $\|M_1(\lambda)\|_{L_p \to L_p} + \|M_2(\lambda)\|_{L_p \to L_p} \leq \dfrac{1}{2}$ holds. Then the operator $G(\lambda) := E + M_1(\lambda) + M_2(\lambda)$ has a bounded inverse $G^{-1}(\lambda)$ in L_p . Therefore, by letting $h = [E + M_1(\lambda) + M_2(\lambda)]\, f$, according to the relation (2.15) in Lemma 3 we obtain that $(L+\lambda E)\,[M_3(\lambda)G^{-1}(\lambda)h](\eta) = h$. Hence, for all λ, $\lambda \geq \lambda_0$, we conclude that the function $y = M_3(\lambda)G^{-1}(\lambda)f$ is a solution to equation (2.1). The proof is complete. □

Proof of Theorem 2. Lemma 4 implies that the operator $(L+\lambda E)'$ acting in the space $L_{p'}(R)$, at $\lambda \geq \lambda_1$ has a right inverse, which is defined on $L_{p'}(R)$. Therefore $ker\,((L+\lambda E)')^* = \{0\}$, where $((L+\lambda E)')^*$ is the operator adjoint to $(L+\lambda E)'$. From here, since $((L+\lambda E)')^*$ is an extension of the operator $L+\lambda E$, we have that $ker(L+\lambda E) = \{0\}$, $\lambda \geq \tilde{\lambda} = \max(\lambda_0, \lambda_1)$. Thus, the operator $L+\lambda E$ is a boundedly invertible operator in the space $L_{p'}(R)$ and, in fact, it yields that

$$(L+\lambda E)^{-1} = M_3(\lambda)G^{-1}(\lambda), \quad \lambda \geq \tilde{\lambda} = \max(\lambda_0, \lambda_1) \tag{3.6}$$

Let y be a solution of equation (2.1), where $\lambda \geq \tilde{\lambda} = \max(\lambda_0, \lambda_1)$. We shall prove the estimate (2.2). By using the relation (3.6), Lemma 1 and the conditions (2.3) - (2.6) we have that

$$\left\|(q+\lambda+ir)(L+\lambda E)^{-1}\right\|_{L_p\to L_p} = \left\|(q+\lambda+ir)M_3(\lambda)G^{-1}(\lambda)\right\|_{L_p\to L_p} \leq$$

$$\leq c \sup_{\eta\in R} \int_{\eta-1}^{\eta+1} b_\lambda^3(\eta) b_\lambda^{-2}(x)\exp[-\sigma|x-\eta| b_\lambda(x)]dx \leq$$

$$\leq c_1 \sup_{\eta\in R} b_\lambda(\eta) \int_{\eta-1}^{\eta+1} \exp[-\sigma|x-\eta| b_\lambda(x)]dx < \infty.$$

From this and (2.1) we find that $\left\|\mu_1(x)\left(\mu_2(x)\left(\mu_1(x)y'\right)'\right)'\right\|_p \leq c\left(\|f\|_p + \|y\|_p\right)$. By combining the last two estimates we obtain (2.2). The theorem is proved. □

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