Dynamics of some Vibro-impacting Systems with Amplitude Constraints

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ABSTRACT

This thesis concerns the dynamics of some vibro-impacting systems with fixed or moving amplitude constraints. It is based on and includes five papers, marked A to E. Simple models of three different vibro-impacting systems with applications in the fields of impact hammers, granular flow and disk brakes in vehicles are analysed.

A 2-DOF (two-degree-of-freedom) model of a threshold-limited impact hammer is studied (Paper A). The stability of a class of periodic motions is analysed. For some parameter values these periodic motions are found to be qualitatively similar to the ones observed for a corresponding 1-DOF system. At other parameter combinations, however, new kinds of periodic or chaotic motions can be observed. For low damping, phenomena resembling antiresonance for linear systems can also be observed.

Granular shear flows show a transitional behaviour in the rapid flow regime as the shear speed or the concentration of the grains is varied. The motion can, for example, change from smooth and orderly to erratic and turbulent. Some aspects of this transitional behaviour in granular shear flow are studied numerically, analytically and experimentally (Papers B, C and D). Simple vibro-impacting models are suggested to get some analytical insight into the dynamics of shear layers. Results from a 1-DOF model show that for high forcing frequencies, which correspond to high shear speeds, periodic as well as chaotic motions can exist, whereas, for low forcing frequencies the vibrations are completely damped out to a stationary state (Paper B). Stability of this stationary state is studied analytically (Paper C), and experimentally (Paper D), where the motions of granular particles in a transparent shear cell are followed by using video techniques. For low shear speeds a single shear layer adjacent to the bottom boundary of the shear cell is observed. As the shear speed is increased, a transition to a random like state involving many layers is found to occur.

In order to understand the phenomenon of squeal in disk brakes, a 3-DOF model is suggested to simulate the dynamics of a brake pad. The region of contact between the brake pad and the disk is described by using a coefficient of friction and distributed stiffness. The brake pad is allowed to have adjustable support locations and possibilities of impacts with its surroundings. The equilibrium state of the pad is determined by using a static analysis. The assumption is that the instability of this stationary state is a possible explanation of squeal, therefore, the stability is analysed in detail. Examples of different kinds of pad motions are presented. A rich variety of motions are found to exist including periodic, seemingly chaotic, stationary behaviour in slip, vibrations with full contact with the disk, stick-slip and impacts.
THESIS

This thesis concerns the dynamics of some vibro-impacting systems. It contains a survey and the following five papers:


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**ACKNOWLEDGEMENTS**  

REFERENCES  

**APPENDED PAPERS:**  

A: Periodic and chaotic behaviour of a threshold-limited two-degree-of-freedom system  

B: One-Dimensional model for the transition from periodic to chaotic motions in granular shear flows  

C: Stability and bifurcations of a stationary state for an impact oscillator  

D: Experimental and numerical studies of shear layers in a granular shear cell  

E: Stability of a vibro-impacting model of a brake pad: A possible explanation of squeal in disk brakes
1. INTRODUCTION

Vibrating systems with fixed or moving amplitude constraints arise in many engineering applications. Some examples are impact hammers, flow of granular materials, vehicle suspensions, linkages, gear-transmissions, joints and braking systems. See, e. g., [1-5]. Such systems are inherently nonlinear due to the possibility of impacts. A detailed knowledge of the dynamics of such systems is essential in order to exploit the desirable effects of the amplitude limitations, as in applications like percussive drilling, or to avoid the undesirable effects like noise and squeal in disk brakes. In granular materials, however, the qualitative nature of the flow strongly depends on the presence or absence of inter-particle collisions.

It is now well known that even with the simplest conceivable equations of motion, almost any nonlinear system can exhibit a chaotic behaviour. See, e. g., [6]. Given infinitesimally different starting points or parameters, one can end up with widely different solutions [7-9]. Also the qualitative behaviour of nonlinear systems may show sensitive dependence on one or more parameters. In systems with amplitude constraints one can observe periodic response, sub harmonic response or chaotic response [10-12]. So careful parameter studies are needed to avoid the undesirable instabilities.

The above mentioned features of nonlinear systems have been known for a long time [9,13,14]. Poincaré (1892) knew that, given the initial conditions one can not always determine what a deterministic system will do far into the future. Yet detailed analyses of even the simplest kinds of nonlinear systems has only been possible during the last decade, due to the availability of relatively fast and cheap computing power. As regards impact oscillators, a large number of papers have appeared where different aspects of the mechanics of such systems are studied using a discrete one-degree-of-freedom model. It is of interest to find out if any new phenomena occur when an additional mass is included in such models. In this thesis a discrete model of an impact hammer with two degrees of freedom is studied (Paper A). See [15-18] for other examples of nonlinear systems with more than one degrees of freedom. The influence of system parameters like damping, coefficient of restitution, distribution of masses and clearance etc. is studied for some extreme values of these parameters. The stability of a class of periodic motions is analysed. The results are presented in a convenient form so that an impact hammer designer can identify the appropriate parameter combinations which give, e. g., a high impact velocity. The results are compared with those from a corresponding one-degree-of-freedom model. For some parameter combinations the results from the two models show qualitative and quantitative agreement as expected. For some other combinations, however, the two-degree-of-freedom model yields motions which lack a counterpart in the simpler model.

Granular shear flow occurs when an aggregate of discrete solid particles undergo shear. Some examples related to geophysical phenomena are rock slides, debris flows, snow avalanches and motions of Arctic ice pack. Granular shear flow
can also be found in industrial processes involving bulk transportation of coal, grain or powder. Particle inertia, stiffness, damping and inter-particle collisions play an important role in the rapid flow regime [19-21]. Recently, the dynamics of granular flows in this regime has been studied by several authors using discrete two-dimensional models. The particles are modelled as circular disks. The motion of each disk in a two-dimensional assembly of a limited number of disks is determined by integrating the equations of motion numerically [22-23]. This approach has resulted in some useful observations like the occurrence of shear layers etc. [24] but it suffers from the disadvantage that no general conclusions can be drawn. Therefore, in order to understand the qualitative features of the dynamics of a typical particle in granular flow, it is desirable to establish even simpler models, i.e., models of models. A one-degree-of-freedom model proposed in Paper B is a step in this direction. A typical particle is modelled as an assembly of mass, spring and dashpot. The influence of the surrounding particles is described by using moving amplitude constraints. Due to the simplicity of the model, it is possible to derive analytical solutions for the particle motions and closed form analytical expressions for the stability of stationary motions. See Paper C where detailed parameter studies and stability diagrams are presented.

As mentioned above, some studies of two-dimensional discrete models of granular flow have revealed the existence of the so called shear layer(s). Roughly, it means that as a two-dimensional assembly of disks is gradually sheared, one or more layers are developed which separate the whole assembly into two zones. Each of these zones moves as a block relative to the other. The locations and extents of the shear layers depend on parameters like shear velocity, friction and concentration, defined as the fraction of the area covered by the disks. A transparent shear cell was built to study such behaviour experimentally. Motions of several layers of different granular materials in the form of identical spheres were followed by using video techniques. The results, reported in Paper D, show a transitional behaviour as the shear velocity is increased. A one-dimensional model consisting of a series of masses, springs and dashpots is proposed to gain some analytical insight into this transitional behaviour. The model captures some aspects of the dynamics but, due to its simplicity, fails to give detailed quantitative agreement with the experimental results. The model is a candidate for further investigation.

Another engineering problem of some importance is the occurrence of squeal in vehicle braking systems. A number of attempts have been made to identify the cause(s) of this noise [25-28]. One difficulty is that such systems contain several different sources of nonlinearities. In disk brakes, for example, we have friction, geometric nonlinearities as well as the possibilities of impacts. It is therefore no surprise that the motion of a brake pad, and hence the occurrence of squeal, often shows a rather unpredictable behaviour. Brakes that did not show any squeal under test conditions started to squeal during operation where the parameters were apparently identical. See, e.g., [29]. The above mentioned nonlinearities cause a variety of motions like, e.g., stationary behaviour, stick slip, possibly chaotic motions and motions with impacts. See Paper E, where the dynamics of a typical brake pad has been studied by using a three-degree-of-freedom model. Geometric nonlinearities, possibilities of impacts and frictional effects are taken into account simultaneously.
2. NONLINEAR DYNAMICS

For linear systems the resulting motion is composed by superposition of a transient and a steady state solution. The transient part is due to the free oscillations and in the presence of damping it is damped out after some time, leaving the steady state solution with oscillations of the frequency of the forcing. Thus, for damped linear systems there is no long term dependence on initial conditions. Once the system and the forcing is chosen, there exists only one unique solution. For nonlinear systems, however, the principle of superposition is not valid and a rich variety of motions is possible. Sometimes even for the same parameter combinations, different initial conditions can lead to qualitatively different kinds of motions.

Basically most dynamical problems are nonlinear from the outset. Nevertheless, linearised approximations are commonly used and are satisfactory for most purposes. The kinds of motions generally predicted by the linear theory may be classified as (i) equilibrium, points (ii) periodic motions or limit cycles, and (iii) quasiperiodic motions. These limit sets are called attractors as the presence of damping causes the long term behaviour of the system to be attracted to one of these states. Linear systems subjected to a random input of course show a random response. Nonlinear systems can lead to essentially new phenomena which cannot be predicted by the corresponding linearised models. Response of nonlinear systems can be quite rich including all of the linear responses mentioned earlier, but also more complex motions like e. g., subharmonic motions, superharmonic motions, self excited motions and jump phenomena. For a nonlinear system different attractors can also coexist for the same parameter combinations. But more interesting is that new attractors can be found which are none of the above, namely the strange attractors. The classical attractors are associated with a geometrical object in the phase space, e. g., point, closed curve or a surface. The strange attractors are associated with a more complex geometrical object of a fractal nature. This is the new class of motions called chaotic motions. For a chaotic system a periodic input can cause randomlike output with wide frequency spectra. There is, however, some amount of regularity in this seemingly randomlike behaviour. The system is still attracting towards a strange attractor and in the strobed subspace of the phase plane (Poincaré section) a fractal limit set can be observed.

Another interesting feature of chaotic systems is their sensitivity to initial conditions and parameters. Two points initially close to each other can after some time be separated very far from each other. Small differences in initial conditions can also result in attraction to different attractors. In order to quantify these new behaviours a rich bifurcational theory is under development [30].

2.1 Dynamical systems

Dynamics deals with the evolution of systems, i.e., the change of state with time. Such systems can be described by ordinary differential equations (ODE), partial
differential equations (PDE) or by difference equations (DE). This thesis deals with systems that can be characterized by a set of first order ODE of the form

$$\dot{x} = f(x, t)$$  \hspace{1cm} (1)

where $x \in \mathbb{R}^n$ is a state vector, $t \in \mathbb{R}$ is time and $f$ is a vector field in $n$-dimensional euclidean space ($\mathbb{R}^n$). The overdot in equation (1) represents differentiation with respect to time. In particle dynamics, the state vector $x$ contains both displacements and velocities. The solution of these equations is called the flow $\phi(x, t)$

$$x = \phi(x, t)$$  \hspace{1cm} (2)

Given the initial conditions $(x_0, t_0)$, the set of points defined by equation (2) in the interval $-\infty < t < \infty$ defines a trajectory in state space through $x_0$ at $t_0$. The state of the system at a given time is determined by a point $x$ in the state space $\mathbb{R}^n$ (also called state space). The flow $\phi(x, t)$ has the geometrical interpretation of a curve in $\mathbb{R}^n$. At each point of this curve equation (1) gives the tangent vector. Hence the reason for calling equation (1) a vector field. If there exists a $\tau > 0$ such that $\phi(x, t) = \phi(x, t+\tau)$ then the system is said to be time periodic with period $\tau$.

In some problems of this thesis the ODE have been simplified to DE. In these cases, a periodic motion have been located and the dynamics of the system is analysed by studying the map over a period $\tau$. This discrete time (time step $\tau$) evolution can be described by a DE

$$x_{k+1} = g(x_k)$$  \hspace{1cm} (3)

where the index $k$ denotes the state at a discrete time.

### 2.2 Poincaré maps

An important tool for analysing dynamical systems is the Poincaré map due to Poincaré who first introduced its use. It replaces the flow of a (n)th-order continuous time system with a (n-1)th-order discrete time system called the Poincaré map. The usefulness of the method lies in the reduction of the order and the fact that it bridges the gap between continuous (ODE) and discrete-time systems (DE). Below the method is described for a non-autonomous case that is similar to the typical systems treated in this thesis.

Given a forced oscillator with the frequency of the forcing $\omega$. There are (n+1) dimensions in the state space: $n$ displacements and velocities, $x$, and the phase of the forcing $\theta = \omega t$. Note that in a forced system the time (or phase) becomes another dimension. Since the motion of the forcing is periodic of period $\tau = 2\pi/\omega$ the phase can be restricted to the interval $[0, 2\pi]$ and is described by the circle $S^1$ with a period $\tau$. Thus points in the (n+1) dimensional state space are given by $(x, \theta) \in \mathbb{R}^n \times S^1$. For a constant value of the phase $\theta_p$, the Poincaré section $\Sigma$ is chosen as the cross section
of this state space.

\[ \Sigma = \{ x, \theta \mid \theta = \theta_p \} \]  

(4)

Points in this n dimensional Poincaré section are the intersections of trajectories with the plane \( \Sigma \) positioned at constant phase \( \theta_p \). The Poincaré map, \( P: \Sigma \to \Sigma (\mathbb{R}^n \to \mathbb{R}^n) \), is the map for an intersection of a trajectory with \( \Sigma \) to the next intersection with \( \Sigma \). In this example the Poincaré map can also be simply described as a stroboscopic image of phase space taken at time intervals equal to the period \( \tau \).

For impacting systems, the Poincaré map is often taken as the map between consecutive impacts. Then the section is defined at the impacting surface. This map is sometimes called the impact map. An advantage of using the impact map is that it gives direct information on the impact velocities. See, e.g., Paper A. One disadvantage is that this map will lead to discontinuities in the map for zero impact velocity. This grazing impact problem has been deeply analysed in [31].

2.3 Impact oscillators

Systems with impacts can often be described by equations of motion which are valid only up to a certain time, namely the instant of impact. At this instant, impact conditions are used to determine the state of the system just after impact. The choice of impact conditions is crucial and depends on the problem studied. If the duration of impacts is short compared with characteristic times of oscillation a simple coefficient of restitution approach, described below, can be used. If a body moving with a velocity of approach \( V_a \) is impacting with a rigid (no change in momentum) boundary moving with a velocity \( V_b \leq V_a \) the coefficient of restitution approach gives a change in momentum according to

\[ \alpha = \frac{V_b - V_r}{V_a - V_b} \]  

(5)

where \( V_r \) is the rebound velocity and \( \alpha \) the coefficient of restitution, \( 0 \leq \alpha \leq 1 \). Due to its simplicity, this model is frequently used and many useful results have been obtained [32-34].

More complicated models are needed when the duration of the impacts cannot be neglected. Commonly, the contact is modeled with springs and dashpots for these cases. But, it is important to consider the choice between kinematic and kinetic contact entrance and exit. In the kinetic case the force in the contact is always in compression and the time of contact entrance and exit is given when the force is zero. In the kinematic case the contact entrance and exit are normally given by the relative distance between the body and the boundary. This model will result in traction forces at contact exit. From physical point of view it seems unrealistic to have traction
forces. But, the model is normally much simpler and may be justified because of the possibility of deeper analysis. For the case of low damping, the difference between these cases is small. By adjusting the damping coefficient the coefficient of restitution can be made equal for both models. For more information about kinetic and kinematic contact conditions see, e.g., [35].

A simple oscillator with amplitude constraints is taken as an example. Consider a simple one-degree-of-freedom model, with a mass attached to a linear spring and dashpot. The impact is described by a coefficient of restitution, \( \alpha \). Scaling out the mass and stiffness the resulting equation of motion can be written

\[
\begin{align*}
\ddot{y} + 2\beta \dot{y} + y &= \gamma \cos(\omega t) ; \quad y < b \\
\dot{y} &\rightarrow -\alpha \dot{y} \quad ; \quad y = b
\end{align*}
\]

The model has a damping proportional to \( \beta \) and is excited by a force proportional to \( \gamma \). Given a set of initial conditions the first equation completely defines the evolution of the system until the stop is encountered (i.e. \( y = b \)). When the amplitude equals \( b \) the coefficient of restitution abruptly changes the direction of the velocity according to the second equation. A new set of initial conditions is obtained, and thereafter the equation of motion together with the new initial conditions determine the evolution of the system until the next impact occurs. The existence of analytical solutions which are valid between impacts, allows some particular long term solutions of such systems to be found analytically. See, e.g., [36-38] where harmonic and subharmonic solutions undergoing a small number of impacts are analysed. Such methods of analyses, however, become quickly impractical for large number of impacts or when additional masses are included, see Paper A. The existence of partial analytical solutions facilitates a numerical approach where some root finding routine (to find the time when impact occurs) can be used to determine the motion of the system over several impacts quickly and accurately.

For cases when the duration of impact may not be neglected, the second equation, in (6), is replaced by a similar equation as the first one. For this case a root finding routine is also used to find the condition when contact is lost. Two possibilities exist for the condition of contact exit and entrance. The first one is kinetic contact condition when times for zero contact force are located. The second method uses kinematic condition (as in equation (6)) to locate times for contact entrance and exit.

3. STABILITY

Stable limit sets are of interest since they are the only ones that can be observed naturally. In the classical linear theories stabilities have been analysed and quantified. It has been proven that locally a linearised map of a nonlinear system can be used which facilitates the stability analysis of nonlinear systems. Even for
complicated motions with impacts [1-3] the stability of periodic motions can be
analysed. In the following sections, some examples of stability problems relevant to
the work in this thesis are described.

3.1 Equilibrium point

Consider a general autonomous vector field on a n-dimensional state space \( \mathbb{R}^n \)
\[
\dot{x} = f(x)
\]  (7)

An equilibrium point of equation (7) is a point \( x_{eq} \in \mathbb{R}^n \) such that
\[
f(x_{eq}) = 0
\]  (8)
i.e., a solution which does not change with time. The point \( x_{eq} \) is usually denoted as
the equilibrium, singular or stationary point. The stability is solved by linearisation
of the equations for solutions close to the equilibrium point. Let
\[
x = x_{eq} + z
\]  (9)
Substituting equation (9) into equation (7) and Taylor expanding about the
equilibrium point gives
\[
\dot{x} = \dot{x}_{eq} + \dot{z} = f(x_{eq}) + Df(x_{eq})z + O(|z|^2)
\]  (10)
The behaviour in the neighbourhood of \( x_{eq} \) is linearised to
\[
\dot{z} = Df(x_{eq})z
\]  (11)
of which the solution through \( z_0 \in \mathbb{R}^n \) at \( t = 0 \) is
\[
z = e^{Df(x_{eq})t}z_0
\]  (12)
Thus, the stability can be determined by the eigenvalues \( \lambda \) of \( Df(x_{eq}) \) which gives:
- asymptotically stable solutions if \( \text{Re}(\lambda)<0 \) for all \( \lambda \),
- and unstable solutions if \( \text{Re}(\lambda)>0 \) for some \( \lambda \).
In addition complex eigenvalues imply oscillatory motions and real aperiodic motion.

3.2 Periodic motion

Again let \( x \in \mathbb{R}^n \) represent the state vector as in the previous section. The
stability of periodic motion can be studied by defining a map \( g \) over a period of
oscillation as shown in equation (3).

\[ x_{k+1} = g(x_k) \]  \hspace{1cm} (13)

A periodic solution is a fixed point \( x_{fix} \) of the map \( g \). Let

\[ x_k = x_{fix} + z_k \]  \hspace{1cm} (14)

Using similar steps as in the case of vector fields, the linearised map is given by

\[ z_{k+1} = Dg(x_{fix})z_k \]  \hspace{1cm} (15)

Assuming a solution on the form \( z_k = m^k z_0 \), \( z_{0}, m \in \mathbb{R}^n \) equation (16) gives

\[ m = Dg(x_{fix}) \]  \hspace{1cm} (16)

The eigenvalues \( m \) are called characteristic multipliers of the periodic solution. Like the eigenvalues at an equilibrium point, the characteristic multipliers position in the complex plane determine the stability of the fixed point. Thus, the stability can be determined from the eigenvalues \( m \) of \( Dg(x_{fix}) \) which gives
- asymptotically stable solutions if \( |m| < 1 \) for all \( m \),
- and unstable solutions if \( |m| > 1 \) for some \( m \).

Again complex eigenvalues imply oscillatory motions in the map \( g \).

### 3.3 Impacting motion

Stability analyses of impacting motions require more effort than the cases described above. Even for two degrees of freedom the analysis becomes complicated and analytical solutions are hard to derive. See, e. g., Paper A. In here the method is described for the simplest case of one-degree-of-freedom system with a coefficient of restitution description of the impact. Consider a forced system with a mass impacting against a rigid surface when the amplitude is equal to \( b \). The velocity of rebound is given by the coefficient of restitution \( \alpha \) times the velocity of approach. Let \( \omega \) be the frequency of the forcing and \( n \) the number of forcing cycles between two consecutive impacts. Then periodic motion (period \( T = \frac{n2\pi}{\omega} \)) with equal state of the system at each impact corresponds to the conditions

\[ x(0) = b \]
\[ x(0) = x\left(\frac{n2\pi}{\omega}\right) \]
\[ \dot{x}(0) = -\alpha \dot{x}\left(\frac{n2\pi}{\omega}\right) \]  \hspace{1cm} (17)

Where the time is put to zero at each impact. The initial condition \( x(0) \) and the phase of the forcing \( \theta \) can then be determined at impact, \( t = 0 \), for this \( n \)-periodic motion.
The stability is however more complicated to determine. Introducing a small perturbation to the periodic motion the stability is determined by following the perturbed solution. See Figure 1. At point $S$ the periodic solution $x$ (solid line) is disturbed which results in the dashed path $y$. The resulting perturbed solutions are termed $y^{(1)}, y^{(2)}, y^{(3)}, \ldots$ and the corresponding phase shifts $\Delta \theta^{(1)}, \Delta \theta^{(2)}, \Delta \theta^{(3)}, \ldots$. For each cycle the origin of time is chosen as the beginning of each cycle (time between impacts). The perturbed solution, $y$, and its derivative, $\dot{y}$, in the $k$:th interval after the perturbation are

$$
y^{(k)}(t) = x(t) + \Delta x^{(k)}(t)
$$

$$
\dot{y}^{(k)}(t) = \dot{x}(t) + \Delta \dot{x}^{(k)}(t)
$$

where

$$
\Delta x^{(k)}(t) = \frac{\partial x(t)}{\partial x(0)} \Delta x^{(k)}(0) + \frac{\partial x(t)}{\partial \dot{x}(0)} \Delta \dot{x}^{(k)}(0) + \frac{\partial x(t)}{\partial \theta} \Delta \theta^{(k)}
$$

$$
\Delta \dot{x}^{(k)}(t) = \frac{\partial \dot{x}(t)}{\partial x(0)} \Delta x^{(k)}(0) + \frac{\partial \dot{x}(t)}{\partial \dot{x}(0)} \Delta \dot{x}^{(k)}(0) + \frac{\partial \dot{x}(t)}{\partial \theta} \Delta \theta^{(k)}
$$

have been used. The variables $x(0), \dot{x}(0)$ and $\theta$ are initial conditions for each impact. Applying the conditions at impact (17) to the perturbed solution gives
Using equations (18), (20) and (17) and only taking the first order differentials into account, the following equations are obtained

\[
\begin{align*}
\Delta x^{(k)}(0) &= 0 \\
\Delta x^{(k)}(\frac{n2\pi}{\omega}) + \dot{x}(\frac{n2\pi}{\omega})(\Delta \theta^{(k)} - \Delta \theta^{(0)}) &= 0 \\
\Delta \dot{x}^{(k+1)}(0) &= -\alpha \Delta x^{(k)}(\frac{n2\pi}{\omega}) + \dot{x}(\frac{n2\pi}{\omega})(\Delta \theta^{(k+1)} - \Delta \theta^{(0)})
\end{align*}
\]

This equation can be written as function of the deviations from the initial conditions \((\Delta \theta^{(k)}(0), \Delta x^{(k)}(0), \Delta \dot{x}^{(k)}(0))\) by inserting equation (19). At the times of impact \((t=0\text{ and } n2\pi/\omega)\) the displacement is always zero and hence \(\Delta x^{(k)}= 0\). After some algebra the equations can be written in the form

\[
\begin{bmatrix}
\Delta x(0) \\
\Delta \dot{x}(0) \\
\Delta \theta(0)
\end{bmatrix}^{(k+1)} = [C] \begin{bmatrix}
\Delta x(0) \\
\Delta \dot{x}(0) \\
\Delta \theta(0)
\end{bmatrix}^{(k)}
\]

The stability is then determined as in section 3.2. This method becomes impractical for systems with more than one degrees of freedom. For such cases Kobrinskii [39] has described a method to determine the eigenvalues directly from the difference equations. Even this method becomes impractical for more than two degrees of freedom as can be seen in paper A.

4. GRANULAR FLOW

A granular material is an aggregate of discrete solid particles. Flows of such materials, are widely met in nature and in various industrial processes. These flows occur in many technological processes such as pneumatic transport, slurry pipelines, powder processing, material handling in bins and hoppers, as well as in many geophysical phenomena such as sediment transport in rivers, ice jams and drift of packed ice, landslides, snow avalanches, rock falls, debris flows, etc. For a broad overview of granular flows see [40-43].
4.1 Models of the motion in granular flows

Motion of dry, cohesionless granular media may occur in several regimes which can be subdivided into grain-inertia (rapid flow, fluid like), transitional and quasi-static (slow flow, solid like). See, e.g., [44]. The latter flows are characterized by particles constantly contacting each other during their motion. In these regimes bulk properties of moving granular media are controlled by the Coulomb interparticle friction forces. On the other hand, in grain-inertia regime particles interact by fast impacts, occurring during their collisions. Most of the time particles spend freely flying between successive collisions, during which interactions the particles' kinetic energy and momentum are transferred. Particle impacts within granular media are accompanied by kinetic energy losses, associated with the nonconservative nature of these interactions. The effect of these impacts is to increase internal energy of the particle. This internal energy is sometimes called granular temperature. Due to the energy losses at each impact, a constant source of mechanical energy is needed to sustain the collisional motion regime of granular material. The key difference between the classical theories of dense fluids and those for granular flows is the need to account for the energy dissipation which occur during collisions and sliding contacts between particles.

Strict solid or fluid phases are not the only ways that granular materials can behave. In many situations both fluid-like and solid-like behaviour may occur simultaneously, such as a funnel flow in hoppers. When transition from fluid-like to solid-like behaviour occurs, undesired phenomena such as clogging are not unusual.

A complete granular flow theory should be able to include all of the above transport mechanisms, each of which would become dominant in appropriate regime. However, most of the research has been concentrated on these limiting regimes only, and a complete theory is not available yet. In particular, the problem of transition between limiting regimes stands out as very poorly understood and unsolved. Granular material are discrete systems that can be viewed as a continuum only upon suitable statistical averaging. Macroscopic behaviour of the system is then described by differential form of mass, momentum and energy conserving laws. The central problem is the determination of constitutive laws for fluxes of momentum (stresses), mechanical energy (energy diffusion) and energy dissipation over the full range of concentrations and flow conditions. In search for these constitutive equations current research efforts include theoretical, experimental and numerical studies. Some examples of analytical techniques are thermodynamic [45] or hydrodynamic [46], approaches and some examples of numerical techniques are cellular automata [47], Monte-Carlo simulation [48] and molecular dynamics [49].

Discrete particle numerical simulations of granular motion have become a valuable tool for investigation of particulate media behaviour. These models determine the behaviour of idealized granular material by calculating the motion of individual particles as they interact with each other and the boundaries. In a recent numerical study [24] a shear cell was simulated in a two-dimensional model consisting of a number of viscoelastic disks. The contacts was modelled by a linear
springs and dashpots which are activated when the distance between neighboring disk centers becomes less than the diameter of a disk. It was found that concentration plays a role in the flow regime. For fixed shear-rate and material properties, at a very high solid concentration, the shearing motion only occurred in one shear layer, sandwiched between two blocks formed by adjacent materials. In Figure 2 this shear layer is demonstrated. In the sequence of figures one can observe one shear layer between the blocks. In Figure 2 (a) one can observe hexagonal packing of all particles. To increase the visibility, particles in the shear layer are shaded and lines are drawn between some disc centers to show the shearing action. At a small time step later, Figure 2 (b) the configuration is changed to cubic packing in the shear layer. Other particles remain hexagonal. Finally, in Figure 2 (c) all particles are in hexagonal packing again. One can conclude that blocks are moving with constant configuration and between these blocks all shearing occur in one shear layer. This was frequently observed for high concentrations, $c$, defined as the fraction of the area covered by the disks. By decreasing the concentration, random motion and collisions among grains resulted in uniform shearing of the whole granular mass.

4.2 Transitional behaviour

Because of the co-existence of the fluid-like and the solid-like behaviour, knowing how to describe the two extreme cases of granular flows is not sufficient. For the purpose of prediction and control of many industrial and geological processes, we need to know the criterion that determines when a granular material will switch from the slow state to the rapid state. Bagnold (1954) [50] was the first to formulate a theory for granular materials that can switch from one constitutive law to the other. His study was for a granular mixture consisting of a viscous fluid and neutrally buoyant solids. The corresponding flows are either in the "macro-viscous" regime, where the stress depends on the first power of the strain-rate, or the rapid flow regime, where a square dependence is observed. The criterion for the transition from one to the other is defined by the so-called Bagnold number, which, like Reynold's number, represents a ratio of particle inertial to fluid viscous effects. As this number increases, the granular material switches from the macro-viscous regime

![Figure 2. The formation of shear layers according to reference [24].](image-url)
to the rapid flow regime. For dry granular flows, the fluid viscosity is much less dominant than the interparticle forces. Yet rate-independent constitutive law exists in slow shear. Therefore for dry granular materials, there must exist other criterion for the transition from one regime to the other. Furthermore, in the dry case, the range of power law dependence between the stress and the strain-rate is much greater than that for a liquid/solid mixture. Several other attempts have been made to combine all transport mechanisms [51-52]. These theories are based on heuristic argument that stresses due to different mechanisms can simply be added together. Total stresses are then equated to the sum of kinetic and collisional stresses (as obtained by kinetic theory) and "plastic", or "frictional" contribution (typically assumed with contact forces, multiple contacts, sliding, rolling and other mechanisms not considered in kinetic theory).

Evidence of the transition from one regime to other in a dry granular flow and the co-existence of both regimes in the same granular flow has been found in a number of recent studies. Zhang and Campbell [53] investigated a shear flow of uniform disks in a gravity field by a computer simulation. They found that the flowing material consisted of a rapid shear zone at the top, adjacent to the moving wall, and a solid non-moving zone at the bottom, adjacent to the fixed wall. In a laboratory experiment of a shear cell, Craig [54] observed a sudden expansion of the granular material as the shear rate increases. This expansion is accompanied by a sudden change of the stress level. Although the shear cell is opaque thus prevents direct observation of the internal motion, this expansion strongly suggests a change of the flow characteristics.

The fact that granular materials can have various constitutive laws makes it important to determine which parameters control the transition, and how these parameters are inter-related in controlling the transition. In this thesis some simple models are introduced to study transitional behaviours in dry cohesionless granular shear flows. Due to the simplicity of the models, deep analysis is possible and some new insight about the mechanism of the transitional behaviour can be obtained.

5. SUMMARY OF THE PAPERS

5.1 Paper A

Paper A deals with the dynamics of a two-degree-of-freedom system with amplitude constraints. The system consists of two masses in series, coupled through linear springs and dashpots where the maximum displacement of one of the masses is limited to a threshold value by a rigid wall. Both masses are allowed to have harmonic excitations with possibly different amplitudes but with the same frequency and phase. The duration of each impact between the (impacting) mass and the wall is assumed to be short in comparison with the forcing period. The impacts are described by a coefficient of restitution kind of impact law. Dynamics of this model
is amenable to analytical treatment for the case of proportional damping. This work is an extension of the undamped case treated in [16]. Periodic as well as chaotic motions are studied with a special emphasis on the stability of a class of impacting periodic motions.

The method used for the analysis of the above mentioned system is as follows. The motion of the system until the occurrence of the first impact is determined analytically by solving the linear equations of motion together with the given initial conditions. The state of the system just before the first impact is noted. Impact law together with the continuity requirements on the state of the non impacting mass are then used to determine the state at the end of this impact. This state forms new initial conditions, which together with the same linear equations of motion determine the system behaviour until the occurrence of the second impact. Noting the state just before the second impact and applying the impact law again, the procedure is repeated for the desired number of impacts. The stability of impacting motions is analysed by using a perturbation technique due to Kobrinski [39]

The motion of the system is analysed for various parameter combinations and the results are presented in the form of time histories, Poincaré plots, bifurcation diagrams and stability diagrams. As a bifurcation parameter is varied, in most cases one observes the typical behaviour with periodic windows separated by other periodic or chaotic regions as also observed in the corresponding one-degree-of-freedom systems. For some extreme values of parameters, however, there are some significant differences. The 2-DOF model yields motions which lack a counterpart in the corresponding degenerate case, e.g., occurrence of new periodic motions and antiresonance phenomena. Another observation of interest is the co-existence of different kinds of periodic motions in the stability diagrams. This implies that for a given system stable periodic motions with different kinds of periodicities can be obtained by changing the initial conditions.

In designing impact tools it is of interest to achieve the desired (high) impact velocity. Diagrams are presented which can be conveniently used for picking the appropriate parameter combinations. Increased damping generally leads to lower impact velocities.

5.2 Paper B

In a recent numerical study [24] two-dimensional granular shear flow was modelled by using a discrete model consisting of a packing of identical circular disks to which linear springs and dashpots were attached. These springs and dashpots become active (in compression) as the distance between two neighbouring disk centres becomes less than a disk diameter. An interesting observation was that as such an assembly of disks is sheared, a transitional phenomenon occurs for some parameter combinations. A block of disks moves w.r.t another block through an intervening layer of disks, called the shear layer. During the motion, the two blocks maintain their internal configuration of disks.
In Paper B a one dimensional model was suggested to simulate the dynamics of one typical disk in such a shear layer. In a co-ordinate system which follows the horizontal motion of the shear layer, the disk experiences periodic excitations through the blocks above and below it. This feature was simulated by using a one-degree-of-freedom model consisting of a mass-spring-dashpot aggregate which bounces between two moving walls representing the two blocks. The model is an assembly consisting of linear springs and dashpots attached to both sides of the mass, which can perform oscillations in the vertical direction. The walls are considered to be rigid and they are prescribed to have harmonic motions in opposite phases. The amplitude and frequency of this excitation is related to the shear velocity and the concentration of the grains, which is defined as the fraction of the total area covered by the grains, i.e., the disks.

The system considered above is one-dimensional but it has two natural frequencies. One corresponding to the case when there is contact with only one of the walls and another corresponding to the case when there is contact with both of the walls. Moreover, at times the assembly may be in free flight with no contact with either of the walls. The dynamics of this system is analysed by studying these different phases of motions and combining them appropriately. The main result is that for high forcing frequencies, corresponding to high shear speeds, the motion of the disk can become chaotic. Periodic motions are also possible at intermediate values of the parameters. For low forcing frequencies, however, the motion is damped out and it approaches a stationary state with zero displacement and velocity. The stability of this state is a crucial factor in determining the system behaviour at low frequencies.

5.3 Paper C

Paper C concerns the stability analysis of the stationary state mentioned above in Paper B. Methods of the theory of dynamical system are used to perform a detailed stability analysis. As mentioned earlier, the motion of the mass can undergo four different phases or modes, namely (i) free flight, (ii) contact with the upper wall, (iii) contact with the lower wall, and (iv) contact with both walls. The sequence of these phases and the duration in each phase is not known a priori, and must be determined through analysis. This fact poses significant difficulties in an analytical treatment of the stability aspects.

A careful analysis of the trajectories close to the stationary state shows, however, that it should be a good approximation to assume that the stability of this state depends mainly on the two modes, namely free flight and contact with both walls. Moreover, the duration for each mode can be calculated from a knowledge of the state of the system at the instants of establishing contact and losing contact with both walls. The stability analysis is strongly based on these observations. Closed form expressions for the eigenvalues of the Poincaré map are derived.

Some of the important parameters influencing the system behaviour are
damping, concentration of grains and the forcing frequency. The results provide a
detailed understanding of the stability regions in the parameter space. In general, the
sizes of the regions of stability in the parameter space decrease as the forcing
frequency is increased. Stable solutions are mainly concentrated around high
damping and high concentrations as expected. It is, however, possible to obtain some
stable solutions even for low damping and low concentrations. Such solutions are
concentrated along thin strips in the parameter space, which means that the
parameters have to be precisely tuned to observe these solutions.

Regions of possible chaos are studied with the help of Poincaré sections. For
chaotic systems the necessary condition of stretching and folding is often proven by
confirming the existence of transversal homoclinic points. The existence of Smale
horseshoe kind of maps is shown for some forcing frequencies by detecting such
points. This simple one-dimensional model shows a variety of motions, including
periodic and chaotic, even for frequencies as low as 1% of the lowest natural
frequency of the system.

5.4 Paper D

In Paper D the stability of a shear layer inside a granular material in a gravity
field is studied experimentally and numerically. A shear cell is built and the motion
of the granular material inside the cell is followed by using video techniques. The
granular material, consisting of identical spheres, is located between two fixed,
concentric cylinders, which are made of a transparent acrylic material (Plexiglas).
Four different granular materials were studied, namely polystyrene, polypropylene,
acrylic and cellulose acetate.

The mean diameter between the cylinders is 0.081 m and the gap between the
cylinders is 0.0045 m, which is slightly larger than one sphere diameter (0.0040 m).
The spheres of are packed at the highest concentration inside the gap between the
cylinders. The number of layers of spheres varies from 10 to 30 in steps of 5. The
cylinders are placed vertically in the gravity field and the shearing motion is
performed with the help of a rotating bottom which is driven by a variable speed
motor. The rotating bottom is roughened by gluing spheres identical to those in the
cell. The experiments are recorded with a video camera and the speed of rotation is
measured with a tachometer. In each experiment the granular material is initially
packed densely without any voids. The speed is slowly increased from 100 rpm in
steps of 50 rpm.

A typical scenario of one such experiment can be described as follows. At low
rpm, a single layer of the spheres just above the bottom moves at half of the speed
of the bottom boundary. All mass above this single shear layer remains structured
and nearly stationary. Visible vibration of small amplitude is present in all layers.
Increasing the rpm, vertical vibration in all layers increases. Further increase of the
rpm, leads to a stick-slip motion in the second layer just above the boundary. From
there, a small increase of the rpm causes a chaotic state, which involves many layers.
adjacent to the rotating boundary. In this chaotic state, the layering structure next to the rotating boundary disappears. The evolution from a single layer shearing to a chaotic, multiple layer shearing occurs in a narrow range of rpm values. Occurrence of such a qualitative change is here in after termed as a transition. An important parameter influencing the transitional behaviour is the number of layers overlain the bottom boundary. In the case of 10 layers, all of the spheres are mixing in an irregular fashion. When the number of layers is increased to 20 and above, only a few layers of spheres adjacent to the rotating boundary participate in the mixing. All other spheres on the top remain layered. As the number of layers increases, the rotational velocity for transition to initiate also increases. Furthermore, the chaotic mixing region becomes smaller.

When the number of layers goes beyond 40, the rotational velocity required to cause the transition suddenly jumps to a much higher value. In some cases no transition could be observed in the test range. A series of experiments were performed to determine the critical rotational velocities, i.e., the velocities required to initiate the transition. In each case the wall velocity at which the layer above the shearing layer starts to slip is noted.

An one-dimensional model was used to simulate the mechanics of the shear cell. In the numerical experiments a qualitative agreement was found but the uncertainties in the friction force made it impossible to get any good quantitative agreement. The model is interesting but the friction needs to be understood better before any improvements can be made.

5.5 Paper E

A vibro-impacting model is suggested to study some aspects of the dynamics of a brake pad in a disk brake. The purpose is to gain some understanding about the phenomenon of squeal in such brakes. The hypothesis is that the transversal vibrations of the disk are excited by instabilities in the pad motion. So the focus is on the motion of the pad only. The disk is assumed to be rigid and the brake pad is modelled as rectangular mass performing vibrations in a plane orthogonal to the plane of contact between the pad and the disk. Thus, the model has three degrees of freedom; two components of translation and a rotation around the mass center. The brake pad is supported by springs and dampers at variable positions and it is pressed against the disk by a brake force. The interface between the pad and the disk is modelled by using springs with distributed stiffness and friction is also allowed at this interface.

The equilibrium state for the pad is determined first by using a static analysis. As the stability of this state is assumed to be of crucial importance, a detailed stability analysis is performed next. A large region of the parameter space is found to be unstable for the case when the three natural frequencies of the (linearised) system are close to each other. The shape and size of this region are strongly influenced by the location of the pad supports, damping and the coefficient of friction but not so much
by the brake force. An observation of some interest is that there exist some unstable regions even for parameter combinations which correspond to the three natural frequencies being far apart from each other.

The system considered here has more than one sources of nonlinearities and shows a variety of motions including, periodic, chaotic, stick-slip and impacts. For high disk velocities, generally an increase in the brake force results in an increased amplitude of vibrations, the two being almost directly proportional. For low disk velocities, however, if the brake force is increased, the system passes through a transition to stick-slip kind of behaviour, beyond which the proportionality between the force and vibration amplitude is no longer valid.

The phenomena of squeal may be caused by a number of different factors including geometric nonlinearities, possibilities of impacts, velocity dependent frictional forces etc. It is quite likely that more than one of these factors come into play simultaneously for different parameter combinations. The model studied here confirms that having equal natural frequencies for different modes of vibrations is obviously undesirable from the point of view of squeal. But it is not the only possible explanation as relatively large regions of instability are also found for the case of unequal frequencies.

6. CONCLUDING REMARKS

6.1 Conclusions

This thesis comprises five papers which deal with the mechanics of some vibro-impacting systems with fixed or moving amplitude constraints. A two-degree-of-freedom model of an impact hammer has been analysed. It is shown that the dynamic behaviour of such systems can show features which lack a counterpart in the corresponding single-degree-of-freedom case. New kinds of motions may become possible for some parameter combinations.

It is also shown that simple vibro-impacting models can be used to understand some features of the dynamics of granular shear flow as revealed by models using colliding disks. These simpler models have the advantage of being amenable to analytical studies. Thus, they can provide some insight into the mechanics of such transitional phenomena as occurrence and stability of shear layers in two-dimensional shear flow. Shear cells of the kind described in this thesis can also be used to study the transitional behaviour experimentally.

The phenomena of squeal in disk brakes is an unresolved problem as the exact cause or source of excitation remains unidentified. Analysis of a three-degree-of-freedom vibro-impacting model studied in this thesis indicates that instabilities in the brake pad motion may be one possible explanation. Several
nonlinearities like friction and geometric nonlinearities including impacts may be present separately or simultaneously, giving rise to wider areas of instability in the parameter space.

All of the models shows that complicated dynamics should be expected when impacts are involved. One can expect features like, e.g., periodic doublings, jump phenomena, sub-harmonic motion, super-harmonic motion and chaos. However, there are often parameter combinations which gives more predictable behaviour. Two such behaviours are stationary positions or periodic motions with equal state of the system at each impact. For these motions stability analysis can be performed and therefore deeper analysis and fast investigation of such motions in a large parameter space.

6.2 Future Work

The work reported in this thesis can be extended in several directions. The two-degree-of-freedom model treated in Paper A has employed proportional damping. This model should be investigated further for the case of general damping in order to explore the possible benefits from distributed damping in an impact hammer. As indicated by Paper A, analytical methods for the stability analysis of systems with more than two degrees of freedom are likely to become rather cumbersome. So a numerical approach should be considered for multi-degree-of-freedom systems.

The idea of applying low dimensional vibro-impacting models to simulate the transitional behaviour in granular flows should be pursued further. The model of Paper B should be supplemented with a random component added to the wall motion, in order to simulate the inevitable variations in particle sizes etc. The shape of the horse shoe map (S-like curve) in Paper C strongly suggests that there may exist an analytical expression for this one-dimensional mapping. This possibility should be explored further to get a clear understanding of the dynamics in the chaotic region. The numerical model suggested in Paper D fails to accurately predict the behaviour observed in the experiments. The main reason for this seems to be that friction has not been modelled sufficiently accurately. Further attempts towards improved modelling of inter particle friction as well as friction between the granular material and the shear cell walls should be continued.

In the model of Paper E, aimed at understanding the phenomenon of squeal in disk brakes, the key assumption is that the pad motions excite the disk and not the opposite. The validity of this assumption should be looked into further. Also the model should be extended to include coupling between the pad and the disk motions.
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PERIODIC AND CHAOTIC BEHAVIOUR OF A THRESHOLD-LIMITED TWO-DEGREE-OF-FREEDOM SYSTEM

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A two-degree-of-freedom impact oscillator with proportional damping is considered. The maximum displacement of one of the masses is limited to a threshold value by a rigid wall, which gives rise to a non-linearity in the system. Impacts between the mass and the wall are described by a coefficient of restitution. The behaviour of the system is rich and includes features like period doublings, period halvings, jumps, chaos, etc. Periodic motions of the system are studied by analytical methods. The influence of system parameters such as damping, coefficient of restitution, distribution of masses and clearance, etc., is studied for some extreme values of these parameters. The stability of a class of periodic motions is investigated. Parameter ranges which result in stable periodic multiple impacts are identified. Application of the results to the design of impact tools is discussed.

1. INTRODUCTION

Vibrating systems with clearances between the moving parts are frequently encountered in technical applications. Impacts occur when the amplitudes of vibration of some parts of the system exceed critical values. Some examples are impacting hammers [1], linkage mechanisms [2-4] and gear transmissions [5]. Such systems have been the subject of several investigations in recent years.

A vibro-impacting system is usually modelled as a spring-mass system with amplitude constraint. In a number of studies one-degree-of-freedom models have been used. It is well known that such systems are inherently non-linear and that they can show periodic or chaotic behaviour even when the input is harmonic. It has been possible to study (the various aspects of) the dynamics of such systems in great detail, mainly due to the small number of parameters involved [6-10]. For the opposite reason few results are available on corresponding multi-degree-of-freedom systems [11-14]. For the design of real systems, however, it may sometimes be crucial to consider the influence of more than one degree of freedom.

In this paper a damped two-degree-of-freedom model of a vibro-impacting system is studied. This example is amenable to analytical treatment for the case of proportional damping. The amplitude of one of the masses is limited by a stop. The impacts are described by a coefficient of restitution. Excitation on both masses is harmonic with different amplitudes but with the same forcing frequency. The dynamics of the system are studied analytically with special attention to periodic motions and to their stability. This work is an extension of the undamped case treated in reference [15]. Results are presented
in the form of time histories, phase portraits, Poincaré plots, bifurcation diagrams and stability diagrams. It is shown that the additional mass strongly influences the dynamics in some parameter ranges. For the degenerate case the dynamics agrees with that of a single-degree-of-freedom model, as expected. Application of the results to the design of impact tools is discussed.

2. DESCRIPTION OF THE SYSTEM

2.1. MODEL

The model for a two-degree-of-freedom vibrator with masses $M_1$ and $M_2$ is shown in Figure 1. The masses are connected to linear springs with stiffnesses $K_1$ and $K_2$, and to linear viscous dashpots with damping constants $C_1$ and $C_2$. Proportional damping of the Rayleigh type is assumed, which in this case implies $K_1/K_2 = C_1/C_2$. The excitations on both masses are harmonic with amplitudes $F_1$ and $F_2$. The excitation frequency $\Omega$ and the phase $\delta$ is the same for both masses. The phase angle is used only to make a suitable choice for the origin of time $T$ in the calculations. The lower mass impacts against a rigid surface when its displacement $X_1$ equals the gap $B$. The impact is described by a coefficient of restitution $\alpha$, and it is assumed that the duration of impact is negligible compared to the period of the force. Between impacts, for $X_1 < B$, the equations of motion are

\[
\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \frac{d^2}{dT^2} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} C_1 & -C_1 \\ -C_1 & C_1 + C_2 \end{bmatrix} \frac{d}{dT} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} K_1 & -K_1 \\ -K_1 & K_1 + K_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \cos(\Omega T + \delta),
\]

or, in non-dimensional form, for $x_1 < b$,

\[
\begin{bmatrix} 1 & 0 \\ 0 & m_2 \frac{1}{1-m_2} \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2\zeta & -2\zeta \\ -2\zeta & 2\zeta \frac{1}{1-c_2} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \frac{1}{1-k_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1-f_2 \\ f_2 \end{bmatrix} \cos(\omega t + \delta),
\]

Figure 1. Schematic of the two-degree-of-freedom impact oscillator.
where the non-dimensional quantities
\[ m_2 = \frac{M_2}{M_1 + M_2}, \quad k_2 = \frac{K_2}{K_1 + K_2}, \quad c_2 = \frac{C_2}{C_1 + C_2}, \quad f_2 = \frac{F_2}{|F_1| + |F_2|}, \quad \omega = \frac{\Omega}{\sqrt{K_1/M_1}}, \]
\[ \zeta = \frac{C_1}{2\sqrt{K_1 M_1}}, \quad t = \sqrt{K_1/M_1} T, \quad b = \frac{B K_1}{|F_1| + |F_2|}, \quad x_i = \frac{X_i K_1}{|F_1| + |F_2|}, \quad i = 1, 2, \] (3)
have been introduced. In equation (2) a dot denotes differentiation with respect to the non-dimensional time \( t \).

2.2. SOLUTION PROCEDURE

The equations of motion (2) are solved by using the normal coordinates and the modal matrix approach; see, e.g., reference [16]. Rayleigh damping implies \( c_2 = k_2 \), and therefore the general solution takes the form
\[ x_1(t) = e^{-\zeta_1 \omega_1 t} [A_1 \sin (\omega_1 t) + A_2 \cos (\omega_1 t)] + e^{-\zeta_2 \omega_2 t} [A_3 \sin (\omega_2 t) + A_4 \cos (\omega_2 t)] + (D_{11} + D_{12}) \cos (\omega t) + (D_{21} + D_{22}) \sin (\omega t), \] (4)
\[ x_2(t) = (1 - \omega_1^2) e^{-\zeta_1 \omega_1 t} [A_1 \sin (\omega_1 t) + A_2 \cos (\omega_1 t)] + e^{-\zeta_2 \omega_2 t} [A_3 \sin (\omega_2 t) + A_4 \cos (\omega_2 t)] + ((1 - \omega_1^2) D_{11} + (1 - \omega_2^2) D_{12}) \cos (\omega t) + ((1 - \omega_1^2) D_{21} + (1 - \omega_2^2) D_{22}) \sin (\omega t), \] (5)
where \( \omega_1 \) and \( \omega_2 \) are the eigenfrequencies of the undamped system, given by
\[ \{\omega_i^2\} = \frac{1}{2} \left[ 1 + \frac{1-m_2}{(1-k_2)m_2} - \left( \frac{1-m_2}{(1-k_2)m_2} \right)^2 + 4 \frac{1-m_2}{m_2} \right]^{1/2}. \] (6)

For \( i = 1 \) and \( 2 \),
\[ \zeta_i = [\zeta (1 - m_2) \{ k_2 + \omega_i^2 (\omega_i^2 - 2k_2) \}] / \{ \omega_i (1 - k_2) \{ 1 + m_2 \omega_i^2 (\omega_i^2 - 2) \} \} \] (7)
are the modal damping constants, and
\[ \omega_{di} = \omega_i \sqrt{-\zeta_i^2} \] (8)
are the frequencies of damped oscillations.

The constants of integration \( A_1 - A_4 \) are given by
\[ A_1 = \left[ \frac{(1 - \omega_1^2) \dot{x}_1(0) - \dot{x}_2(0)}{\omega_1^2 - \omega_2^2} + \zeta_1 \omega_1 \left( \frac{(1 - \omega_2^2) x_1(0) - x_2(0)}{\omega_1^2 - \omega_2^2} - D_{11} \right) - \omega D_{21} \right] / \omega_{di}, \] (9)
\[ A_2 = \left[ \frac{(1 - \omega_2^2) x_1(0) - x_2(0)}{\omega_1^2 - \omega_2^2} - D_{11}, \right] \] (10)
\[ A_3 = \left[ \frac{-(1 - \omega_1^2) \dot{x}_1(0) + \dot{x}_2(0)}{\omega_1^2 - \omega_2^2} + \zeta_2 \omega_2 \left( \frac{-(1 - \omega_2^2) x_1(0) + x_2(0)}{\omega_1^2 - \omega_2^2} - D_{12} \right) - \omega D_{22} \right] / \omega_{d2}, \] (11)
\[ A_4 = \left[ \frac{-(1 - \omega_2^2) x_1(0) + x_2(0)}{\omega_1^2 - \omega_2^2} - D_{12}. \right] \] (12)
The amplitude parameters $D_{11}$, $D_{12}$, $D_{21}$ and $D_{22}$ are given by

$$D_{ii} = \frac{1 - \omega_i^2 f_i^2 (1 - m_2)}{1 + m_2 \omega_i^2 (\omega_i^2 - 2)} \left[ \frac{2 \zeta_i \omega_j \omega \sin (\delta) + (\omega_j^2 - \omega^2) \cos (\delta)}{(\omega_j^2 - \omega^2)^2 + (2 \zeta_i \omega_j \omega)^2} \right],$$

(13)

$$D_{2i} = \frac{1 - \omega_i^2 f_i^2 (1 - m_2)}{1 + m_2 \omega_i^2 (\omega_i^2 - 2)} \left[ \frac{2 \zeta_i \omega_j \omega \cos (\delta) - (\omega_j^2 - \omega^2) \sin (\delta)}{(\omega_j^2 - \omega^2)^2 + (2 \zeta_i \omega_j \omega)^2} \right],$$

(14)

where $i = 1$ and 2.

If the state of the system $(x_1, \dot{x}_1, x_2, \dot{x}_2, \delta)$, is known at some instant, then by setting $t = 0$ at this instant, equations (4)-(14) can be used to determine the motion of the system until the next impact. When impact occurs, for $x_1 = b$, the velocity of the impacting mass is changed according to the impact law, $[\dot{x}_1]_{after \ impact} = -a[\dot{x}_1]_{before \ impact}$.

After impact, by resetting the time $t$ to zero and shifting the phase according to $[\delta]_{after \ impact} = [\omega t + \delta]_{before \ impact}$, new values of the eight constants $A_1-A_4$ and $D_{11}-D_{22}$ can be derived. Then, by using equations (4)-(14) again the motion of the system can be determined until the next impact. By repeated use of this procedure, the motion of the system can be followed over an arbitrary number of impacts.

3. CONDITIONS FOR PERIODIC IMPACTS

Under suitable conditions, the system of Figure 1 exhibits periodic behaviour. In order to study a class of periodic motions, let $n$ stand for the number of forcing cycles between two consecutive impacts. This means that if the dimensionless time $\tau$ is set to zero directly after an impact, it becomes $n \ln/a$ just before the next impact. Periodic motion corresponds to the conditions

$$x_1(0) = b, \quad x_2(n2\pi/\omega) = b, \quad \dot{x}_1(0) = -a \dot{x}_1(n2\pi/\omega),$$

(15-17)

$$x_2(0) = x_2(n2\pi/\omega), \quad \dot{x}_2(0) = \dot{x}_2(n2\pi/\omega),$$

(18, 19)

where $\dot{x}_1(n2\pi/\omega)$ is the impact velocity. Equations (15) and (16) express the instantaneous nature of each impact, (17) is the impact law, and (18) and (19) express the continuity of position and velocity of the mass $M_2$ at the instant of impact. Using the five conditions (15)-(19) in the general solution of equation (2) makes it possible to solve for the state of the system immediately after impact: i.e., for the five unknowns $x_1(0)$, $x_2(0)$, $\dot{x}_1(0)$, $\dot{x}_2(0)$ and $\delta$. One of these quantities, $x_1(0)$, is already determined by equation (15). In order to solve for the remaining four quantities, the formal solution of equation (2) is written in the form

$$x_1(t) = \bar{\Gamma}_1(t)x_1(0) + \bar{\Gamma}_2(t)x_2(0) + \bar{\Gamma}_3(t)\dot{x}_1(0) + \bar{\Gamma}_4(t)\dot{x}_2(0) + \bar{\Gamma}_5(t) \sin (\delta) + \bar{\Gamma}_6(t) \cos (\delta),$$

$$x_2(t) = \bar{\Gamma}_7(t)x_1(0) + \bar{\Gamma}_8(t)x_2(0) + \bar{\Gamma}_9(t)\dot{x}_1(0) + \bar{\Gamma}_{10}(t)\dot{x}_2(0) + \bar{\Gamma}_{11}(t) \sin (\delta) + \bar{\Gamma}_{12}(t) \cos (\delta),$$

$$\dot{x}_1(t) = \bar{\Gamma}_{13}(t)x_1(0) + \bar{\Gamma}_{14}(t)x_2(0) + \bar{\Gamma}_{15}(t)\dot{x}_1(0)$$

$$+ \bar{\Gamma}_{16}(t)\dot{x}_2(0) + \bar{\Gamma}_{17}(t) \sin (\delta) + \bar{\Gamma}_{18}(t) \cos (\delta),$$

$$\dot{x}_2(t) = \bar{\Gamma}_{19}(t)x_1(0) + \bar{\Gamma}_{20}(t)x_2(0) + \bar{\Gamma}_{21}(t)\dot{x}_1(0)$$

$$+ \bar{\Gamma}_{22}(t)\dot{x}_2(0) + \bar{\Gamma}_{23}(t) \sin (\delta) + \bar{\Gamma}_{24}(t) \cos (\delta),$$

(20)

where the functions $\bar{\Gamma}_i(t)$ are given in Appendix A.
Letting $t = n2\pi/\omega$ in equations (20), using the conditions (15)-(19), and letting $\Gamma_i$ represent $\tilde{\Gamma}_i(n2\pi/\omega)$, one obtains

$$b = \Gamma_1 b + \Gamma_2 x_2(0) + \Gamma_3 x_1(0) + \Gamma_4 x_2(0) + \Gamma_5 \sin(\delta) + \Gamma_6 \cos(\delta),$$

$$x_2(0) = \Gamma_7 b + \Gamma_8 x_2(0) + \Gamma_9 x_1(0) + \Gamma_{10} x_2(0) + \Gamma_{11} \sin(\delta) + \Gamma_{12} \cos(\delta),$$

$$\dot{x}_1(0) = -\alpha(\Gamma_{13} b + \Gamma_{14} x_2(0) + \Gamma_{15} x_1(0) + \Gamma_{16} x_2(0) + \Gamma_{17} \sin(\delta) + \Gamma_{18} \cos(\delta)),$$

$$\dot{x}_2(0) = \Gamma_{19} b + \Gamma_{20} x_2(0) + \Gamma_{21} \dot{x}_1(0) + \Gamma_{22} \dot{x}_2(0) + \Gamma_{23} \sin(\delta) + \Gamma_{24} \cos(\delta).$$

Solving for $\dot{x}_1(0)$ in equations (21) and using equation (17), one obtains the following expression for the impact velocity,

$$\dot{x}_1(n2\pi/\omega) = -\frac{(A_1 A_2 + A_4 A_3)b}{(A_1^2 + A_2^2)} \pm \sqrt{\left(\frac{(A_1 A_2 + A_4 A_3)b}{(A_1^2 + A_2^2)}\right)^2 - \frac{(A_2^2 + A_3^2)b - A_3^2}{(A_1^2 + A_2^2)}},$$

where the quantities $A_i$ are given in Appendix A. The phase corresponding to this impact velocity is given by

$$\sin(\delta) = \frac{A_1 \dot{x}_1(n2\pi/\omega) + A_2 b}{A_3} \quad \text{and} \quad \cos(\delta) = \frac{A_4 \dot{x}_1(n2\pi/\omega) + A_5 b}{A_3}.$$

The position and velocity of the mass $M_2$ at the instant of impact are given by

$$x_2(0) = \left\{ b[\Gamma_7 (1 - \Gamma_{22}) + \Gamma_{10} \Gamma_{19}] - \alpha \dot{x}_1(n2\pi/\omega)[\Gamma_9 (1 - \Gamma_{22}) + \Gamma_{10} \Gamma_{21}] + [\Gamma_{11} (1 - \Gamma_{22}) + \Gamma_{10} \Gamma_{23}] \sin(\delta) + [\Gamma_{12} (1 - \Gamma_{22}) + \Gamma_{10} \Gamma_{24}] \cos(\delta)\right\} /[1 - (1 - \Gamma_{22}) (1 - \Gamma_{21}) - \Gamma_{10} \Gamma_{20}],$$

$$\dot{x}_2(0) = \left\{ b[\Gamma_{19} (1 - \Gamma_{22}) - \Gamma_{17} \Gamma_{20}] - \alpha \dot{x}_1(n2\pi/\omega)[\Gamma_{21} (1 - \Gamma_{22}) + \Gamma_{20} \Gamma_{21}] + [\Gamma_{23} (1 - \Gamma_{22}) + \Gamma_{11} \Gamma_{20}] \sin(\delta) + [\Gamma_{24} (1 - \Gamma_{22}) + \Gamma_{12} \Gamma_{20}] \cos(\delta)\right\} /[1 - (1 - \Gamma_{22}) (1 - \Gamma_{21}) - \Gamma_{10} \Gamma_{20}],$$

respectively.

These results generate periodic solutions corresponding to one impact during $n$ cycles of the forcing for integer values of $n$. The existence of periodic impacts requires that the impact velocity given by equation (22) be real. It should be noted, however, that even some real impact velocities may be physically inadmissible, as they correspond to solutions where the impacting mass penetrates into the wall ($x_i(t) > b$ for $0 < t < n2\pi/\omega$). Also, some of the solutions may be unstable. These aspects are studied further in the next section.

4. STABILITY ANALYSIS

The stability of periodic impacts can be analyzed by examining Figure 2, where a possible periodic motion of the impacting mass is sketched. The solid line indicates the unperturbed periodic motion under consideration. This motion is perturbed at point $s$ and develops according to the dashed path. The perturbed solutions in successive intervals are termed $\tilde{x}_1^{(i)}, \tilde{x}_2^{(i)}, \ldots, \tilde{x}_1^{(p)}$, and the corresponding phase shifts $\Delta \delta^{(1)}, \Delta \delta^{(2)}, \ldots, \Delta \delta^{(p)}$. For each of these solutions the origin of time is chosen at the impact. If the perturbed solution $\tilde{x}_1^{(p)}(t)$ approaches the unperturbed solution $x_1(t)$ as $p$ increases the solution is stable; otherwise, it is unstable.
The perturbed displacements and velocities in the $p$th interval after the disturbance are

$$
\ddot{x}_i^{(p)}(t) = x_i(t) + \Delta \ddot{x}_i^{(p)}(t), \quad \dddot{x}_i^{(p)}(t) = \dot{x}_i(t) + \Delta \dddot{x}_i^{(p)}(t), \quad \text{for } i = 1, 2.
$$

(26, 27)

where the deviations $\Delta x_i^{(p)}(t)$ and $\Delta \dot{x}_i^{(p)}(t)$ can be obtained by differentiating the expressions (4) and (5) for displacements and the corresponding expressions for velocities with respect to the initial conditions:

$$
\Delta x_i^{(p)}(t) = \frac{\partial x_i(t)}{\partial x_1(0)} \Delta x_1^{(p)}(0) + \frac{\partial x_i(t)}{\partial x_2(0)} \Delta x_2^{(p)}(0) + \frac{\partial x_i(t)}{\partial \dot{x}_1(0)} \Delta \dot{x}_1^{(p)}(0)
$$

$$
+ \frac{\partial x_i(t)}{\partial \dot{x}_2(0)} \Delta \dot{x}_2^{(p)}(0) + \frac{\partial x_i(t)}{\partial \ddot{x}_1(0)} \Delta \ddot{x}_1^{(p)}(0) + \frac{\partial x_i(t)}{\partial \dddot{x}_1(0)} \Delta \dddot{x}_1^{(p)}(0)
$$

$$
= \Gamma_1(t) \Delta x_1^{(p)}(0) + \Gamma_2(t) \Delta x_2^{(p)}(0) + \Gamma_3(t) \Delta \dot{x}_1^{(p)}(0) + \Gamma_4(t) \Delta \dot{x}_2^{(p)}(0)
$$

$$
+ (\Gamma_5(t) \cos(\delta) - \Gamma_6(t) \sin(\delta)) \Delta \ddot{x}_1^{(p)}(0),
$$

(28)

$$
\Delta \dot{x}_i^{(p)}(t) = \frac{\partial \dot{x}_i(t)}{\partial x_1(0)} \Delta x_1^{(p)}(0) + \frac{\partial \dot{x}_i(t)}{\partial x_2(0)} \Delta x_2^{(p)}(0) + \frac{\partial \dot{x}_i(t)}{\partial \dot{x}_1(0)} \Delta \dot{x}_1^{(p)}(0)
$$

$$
+ \frac{\partial \dot{x}_i(t)}{\partial \dot{x}_2(0)} \Delta \dot{x}_2^{(p)}(0) + \frac{\partial \dot{x}_i(t)}{\partial \ddot{x}_1(0)} \Delta \ddot{x}_1^{(p)}(0) + \frac{\partial \dot{x}_i(t)}{\partial \dddot{x}_1(0)} \Delta \dddot{x}_1^{(p)}(0)
$$

$$
= \Gamma_7(t) \Delta x_1^{(p)}(0) + \Gamma_8(t) \Delta \dot{x}_1^{(p)}(0) + \Gamma_9(t) \Delta \ddot{x}_1^{(p)}(0) + \Gamma_{10}(t) \Delta \dddot{x}_1^{(p)}(0)
$$

$$
+ (\Gamma_{11}(t) \cos(\delta) - \Gamma_{12}(t) \sin(\delta)) \Delta \ddot{x}_2^{(p)}(0),
$$

(29)

$$
\Delta \ddot{x}_i^{(p)}(t) = \frac{\partial \ddot{x}_i(t)}{\partial x_1(0)} \Delta x_1^{(p)}(0) + \frac{\partial \ddot{x}_i(t)}{\partial x_2(0)} \Delta x_2^{(p)}(0) + \frac{\partial \ddot{x}_i(t)}{\partial \dot{x}_1(0)} \Delta \dot{x}_1^{(p)}(0)
$$

$$
+ \frac{\partial \ddot{x}_i(t)}{\partial \dot{x}_2(0)} \Delta \dot{x}_2^{(p)}(0) + \frac{\partial \ddot{x}_i(t)}{\partial \ddot{x}_1(0)} \Delta \ddot{x}_1^{(p)}(0) + \frac{\partial \ddot{x}_i(t)}{\partial \dddot{x}_1(0)} \Delta \dddot{x}_1^{(p)}(0)
$$

$$
= \Gamma_{13}(t) \Delta x_1^{(p)}(0) + \Gamma_{14}(t) \Delta \dot{x}_1^{(p)}(0) + \Gamma_{15}(t) \Delta \ddot{x}_1^{(p)}(0) + \Gamma_{16}(t) \Delta \dddot{x}_1^{(p)}(0)
$$

$$
+ (\Gamma_{17}(t) \cos(\delta) - \Gamma_{18}(t) \sin(\delta)) \Delta \ddot{x}_2^{(p)}(0),
$$

(30)

$$
\Delta \dddot{x}_i^{(p)}(t) = \frac{\partial \dddot{x}_i(t)}{\partial x_1(0)} \Delta x_1^{(p)}(0) + \frac{\partial \dddot{x}_i(t)}{\partial x_2(0)} \Delta x_2^{(p)}(0) + \frac{\partial \dddot{x}_i(t)}{\partial \dot{x}_1(0)} \Delta \dot{x}_1^{(p)}(0)
$$

$$
+ \frac{\partial \dddot{x}_i(t)}{\partial \dot{x}_2(0)} \Delta \dot{x}_2^{(p)}(0) + \frac{\partial \dddot{x}_i(t)}{\partial \ddot{x}_1(0)} \Delta \ddot{x}_1^{(p)}(0) + \frac{\partial \dddot{x}_i(t)}{\partial \dddot{x}_1(0)} \Delta \dddot{x}_1^{(p)}(0)
$$

$$
= \Gamma_{19}(t) \Delta x_1^{(p)}(0) + \Gamma_{20}(t) \Delta \dot{x}_1^{(p)}(0) + \Gamma_{21}(t) \Delta \ddot{x}_1^{(p)}(0) + \Gamma_{22}(t) \Delta \dddot{x}_1^{(p)}(0)
$$

$$
+ (\Gamma_{23}(t) \cos(\delta) - \Gamma_{24}(t) \sin(\delta)) \Delta \ddot{x}_2^{(p)}(0),
$$

(31)
Applying the boundary and continuity conditions (15)–(19) to the perturbed solution of Figure 2

\[ \dot{x}^{(p)}(0) = b, \quad \ddot{x}^{(p)}[(1/\omega)(n2\pi + \Delta \delta^{(p+1)} - \Delta \delta^{(p)})] = b, \quad (32, 33) \]

\[ \ddot{x}^{(p+1)}(0) = -\alpha \ddot{x}^{(p)}[(1/\omega)(n2\pi + \Delta \delta^{(p+1)} - \Delta \delta^{(p)})], \quad (34) \]

\[ \dot{x}^{(p+1)}(0) = \dot{x}^{(p)}[(1/\omega)(n2\pi + \Delta \delta^{(p+1)} - \Delta \delta^{(p)})], \quad (35) \]

\[ \ddot{x}^{(p+1)}(0) = \ddot{x}^{(p)}[(1/\omega)N2\pi + \Delta \delta^{(p+1)} - \Delta \delta^{(p)})]. \quad (36) \]

Expanding the expressions (33)–(36) about the argument \( n2\pi \), retaining terms up to the first order, and using equations (15)–(19) and (28)–(32), one obtains equations (B2)–(B5) of Appendix B. Furthermore, upon following Kobrinskii [17] and assuming growth according to

\[ \Delta \dot{x}^{(p)}(0) = \bar{x}_1 \lambda^p, \quad \Delta \dot{x}^{(p)}(0) = \bar{x}_2 \lambda^p, \quad \Delta \dot{x}^{(p)}(0) = \bar{x}_2 \lambda^p, \quad \Delta \delta^{(p)} = \bar{\delta} \lambda^p, \quad (37-40) \]

a linear system of equations (B6)–(B9) arises. The eigenvalues of this system are determined by requiring the determinant to be zero:

\[
\begin{vmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} \\
  a_{21} & a_{22} & a_{23} & a_{24} \\
  a_{31} & a_{32} & a_{33} & a_{34} \\
  a_{41} & a_{42} & a_{43} & a_{44}
\end{vmatrix} = 0. \quad (41)
\]

The elements of this determinant are given in equations (B10)–(B13) of Appendix B. If all eigenvalues of this system are inside the unit circle, then the periodic motion is stable; otherwise, it is unstable. In Appendix B, Schur–Cohn’s inequalities [18–20] are used to test these stability conditions.

5. RESULTS AND DISCUSSION

In this section some results are presented in the form of bifurcation diagrams (Figure 3), time histories (Figure 4), phase plane diagrams (Figure 5), Poincaré sections (Figure 6) and stability diagrams (Figures 7, 8 and 11). In these figures a given system with \( m_2 = 0.5 \), \( k_2 = 0.5 \), \( f_2 = 0 \), \( \zeta = 0.10 \) and \( \alpha = 0.80 \) has been chosen to be analyzed. A bifurcation diagram for the system is shown in Figure 3. This diagram is obtained by using equations (4)–(14) together with the impact law. The system has been analyzed for 300 impacts corresponding to each forcing frequency. The first 200 impacts are ignored, as they are considered to be the transient part of the solution. The impact velocities corresponding to the next 100 impacts are plotted for each frequency. In all the bifurcation diagrams the initial conditions are chosen to be \( (x_1(0) = x_2(0) = -1, \dot{x}_1(0) = \dot{x}_2(0) = 1) \) at the lowest forcing frequency. As the frequency is increased (in steps), the initial conditions used at the current frequency are the same as the final state of the system corresponding to the previous frequency. One observes periodic windows with one impact velocity, separated by regions of other periodic or chaotic motions. This behaviour is similar to that observed in the corresponding one-degree-of-freedom systems [6, 7]. The largest impact velocities in the periodic windows are quite close to an integral multiple of \( 2\omega_1 \).

Some time histories are shown in Figure 4. Figures 4(a)–(c) correspond to motions in the first three periodic windows (counted from the left in Figure 3) and Figure 4(d) corresponds to a long-periodic or chaotic motion. From these figures one can observe that
the period of the motion in the periodic window number $j$ is $j$ times the period of the forcing. This feature has been consistently observed when periodic windows are found around integer multiples of $2\omega_1$. The same motions are plotted in the phase-plane in Figure 5.

Some Poincaré sections for the system are shown in Figure 6. The Poincaré section is taken at a constant value of the forcing phase $(\omega t + d) \mod (2\pi)$. Since the system is five-dimensional in the state space, $(x_1, x_1, x_2, \dot{x}_2, (\omega t + d) \mod (2\pi))$, the Poincaré section will be four-dimensional, $\{(x_1, \dot{x}_1, x_2, \dot{x}_2, (\omega t + d) \mod (2\pi)) | (\omega t + d) \mod (2\pi) = 3 \}$. This section is then projected to the $(x_1, \dot{x}_1)$ plane and is hereinafter called the projected Poincaré section. In Figure 6(a) the point marked by a star shows that the frequency of the
motion is equal to the forcing frequency. In Figures 6(b)-(d) are shown long-periodic or chaotic motions in different regions.

Stable and physically admissible impact velocities corresponding to some periodic motions are shown in Figure 7. This figure is obtained by using the results from the stability analysis in section 4. In Figure 8 four stability diagrams for different periodic motions are plotted in the $b$-$\omega$ plane. This figure shows that stable periodic motions corresponding to higher values of $n$ appear at higher forcing frequencies, and near $b$-values close to zero. It may be observed that there are overlapping regions in $n=1$ and 2. In Figure 8(e) the first five stable periodic motions are plotted. The dashed line in this figure
Figure 7. Stable and physically admissible periodic solutions for \( n = 1-4 \) for the system \( f_2 = 0, m_2 = 0.5, k_2 = 0.5, \zeta = 0.1, b = 0, \alpha = 0.8 \).

Figure 8. Stability diagram for different periods \( n \) in the \( b - \omega \) plane for the system \( f_2 = 0, m_2 = 0.5, k_2 = 0.5, \zeta = 0.1, \alpha = 0.8 \). (a) \( n = 1 \); (b) \( n = 2 \); (c) \( n = 3 \); (d) \( n = 4 \); (e) \( n = 1-5 \).
is the boundary curve for steady state response with no impacts and the dotted line shows the gap \( b \) required for continuous contact. It is mainly between these curves that more complex behaviour can be expected. However, periodic solutions can exist even outside these boundary curves.

In Figure 9 is shown a three-dimensional stability diagram, for \( n=1 \), obtained by plotting the impact velocity on the third axis. This diagram is useful when the distribution of the impact velocity is of interest over the whole \( b-\omega \) plane.

![Stability diagram for period \( n=1 \) in the \( b-\omega \) plane with impact velocity on the third axis for the system \( f_2=0, m_2=0.5, k_2=0.5, \zeta=0.1, a=0.8 \).](image)

Figure 9. Stability diagram for period \( n=1 \) in the \( b-\omega \) plane with impact velocity on the third axis for the system \( f_2=0, m_2=0.5, k_2=0.5, \zeta=0.1, a=0.8 \).

Studying the influence of the clearance, one finds that, for large values of \( b \), periodic impacts can be found for \( n=1 \) only around the first natural frequency. At other forcing frequencies the system vibrates without impacts. When \( b \) decreases, the periodic windows move towards higher frequencies and new \( n \)-periodic motions appear. At \( b \) near zero (a commonly studied case) long-periodic or chaotic motions can be observed between the \( n \)-periodic windows. For negative values of \( b \) periodic motions can be observed only at high forcing frequencies as the impacting mass tends to stay in contact with the wall, at low frequencies.

The two-degree-of-freedom model studied here involves seven system parameters: \( \omega, m_2, k_2, f_2, \zeta, a \) and \( b \). Due to this relatively large number of parameters the detailed influence of each parameter on the system dynamics is not presented here. However, it is of special interest to acquire an overall picture of the system dynamics for some extreme values of parameters, especially those which relate to the degenerated case of a single-degree-of-freedom system. Thus, in Figures 10(a)–(h) some bifurcations diagrams corresponding to some extreme parameter values are presented. In most cases one observes the typical behaviour with periodic windows, with one impact velocity, separated by other periodic or chaotic regions, as also observed in the corresponding one-degree-of-freedom systems. However, there are some important differences. From Figure 10(a) it can be observed that shifting the excitation to the mass \( M_2 \), i.e., choosing \( f_2=1 \), influences the impact velocities significantly. Impact velocities around the peak corresponding to \( n=1 \) are slightly increased, whereas much lower impact velocities are obtained for \( n>1 \). When the coefficient of restitution is low, Figure 10(b), no major differences can be observed. Low damping, Figure 10(c), leads to enlarged areas of chaotic motions. Increased damping
generally results in lower impact velocities and larger regions of periodic motions with one impact in a period of the motion as seen in Figure 10(d). The long-periodic or chaotic regions shrink. Despite the impacts, the antiresonance phenomenon, characteristic of multi-degree-of-freedom systems, is maintained. The region of minimum impact velocity near $\omega = 1.4$ in Figures 7(a), 10(b) and 10(c) is due to such an antiresonance. The antiresonance phenomena become more pronounced when the damping is low. This behaviour, like the influence of the second natural frequency, is of course not present in the corresponding case of a single mass.

For low $m_2$ and large $k_2$ the behaviour is similar to that of a one-degree-of-freedom system, as seen in Figures 10(e) and 10(h) respectively. When $m_2$ becomes large, however, the long-periodic or chaotic motions dominate, and only the first periodic window can be found to survive; see Figure 10(f). When $k_2$ decreases the motion in the periodic windows is changed to two impacts in one period of the motion, especially for low forcing frequencies; see Figure 10(g). For higher forcing frequencies the periodic windows are essentially
unchanged. This represents a system with a very stiff spring between the masses. The periods of the motion in each of the periodic windows, with two impact velocities are, however, equal to those of the periodic windows, with one impact velocity, in the previous cases. It turns out that the time interval between the first and second impacts is close to half of the period corresponding to the natural frequency of the subsystem of two masses and the stiff spring.

The similarities with the one-degree-of-freedom system can especially be found in the two cases of low \( m_2 \) (Figure 10(e)) and high \( k_2 \) (Figure 10(h)). These results are in agreement with the results presented earlier for single-degree-of-freedom systems by, e.g., Isomäki [6] and Thomson [7]. As in reference [7] transition to chaos can be observed by period doubling sequences in some frequency regions. Similarities in the chaotic attractors can be observed by comparing the projected Poincaré sections in Figure 11 in this paper with Figure 4 in reference [8]. These attractors are from the chaotic region between the periodic windows \( n = 4 \) and \( n = 5 \).

6. CONCLUSIONS

In conclusion, we have shown that the method of analyzing the stability of periodic motion with equal impact velocity [17-20] holds even for a damped two-degree-of-freedom system. The rapid increase in the number of equations indicates that the method will probably not be practical for systems with more than two degrees of freedom. The two-degree-of-freedom system shows dynamics similar to that of a corresponding one-degree-of-freedom system: i.e., periodic windows with one impact in the period of the motion separated by regions of other periodic or chaotic motion, for large variations of the parameters. However, there are some significant deviations, and important differences in system behaviour at some extreme values of the parameters. Such cases are high values of \( m_2 \) (Figure 9(f)) or low values of \( k_2 \) (Figure 9(g)). The first case corresponds to high mass concentration on the non-impacting mass, and the second case is high stiffness of the

![Figure 11. Projected Poincaré sections of the impacting mass for different systems between the periodic solutions \( n = 4 \) and \( n = 5 \). For each case the initial conditions are \( x_1(0) = -1, x_2(0) = -1, x_3(0) = 1, x_4(0) = 1 \). (a) \( f_2 = 0, m_2 = 0.1, k_2 = 0.5, \zeta = 0.1, b = 0, \alpha = 0.8, \omega = 6.15 \); (b) \( f_2 = 0, m_2 = 0.3, k_2 = 0.5, \zeta = 0.1, b = 0, \alpha = 0.8, \omega = 5.90 \); (c) \( f_2 = 0, m_2 = 0.5, k_2 = 0.5, \zeta = 0.1, b = 0, \alpha = 0.8, \omega = 5.30 \); (d) \( f_2 = 1, m_2 = 0.5, k_2 = 0.5, \zeta = 0.1, b = 0, \alpha = 0.8, \omega = 5.35 \).]
spring between the masses compared to that of the spring attached to the wall. In the second case periodic windows are at the same positions as in the one-degree-of-freedom system, but the motion has changed into two impacts in one period of the motion. The period of the motion in each window is, however, the same as in the one-degree-of-freedom case. When the forcing frequency becomes high, close to the second natural frequency, the behaviour in the periodic windows is of the same kind as for the one-degree-of-freedom system.

Another observation of interest is the existence of areas in the stability diagrams where periodic solutions corresponding to different values of $n$ coexist. Such areas are mainly concentrated as thin strips around the upper left boundaries of regions corresponding to $n=1$ and $n=2$; see Figures 8(a)–(d). This implies that for a given system stable periodic motions with different kinds of periodicities can be obtained by changing the initial conditions.

In designing impact tools it is of great interest to achieve the desired impact velocities. In order to facilitate such design, diagrams of the type shown in Figure 11 may be useful. Thus, if periodic impacts with large impact velocities are desired, one would choose large $b$ in the region $n=1$. Larger impact velocities can also be obtained by appropriately choosing the location of the forcing; compare Figures 3 and 9(a). Increased damping generally leads to lower impact velocities.

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REFERENCES

PERIODIC AND CHAOTIC BEHAVIOUR


APPENDIX A

Given below are the expressions for $\bar{f}_i(t)$, $i=1-24$ and $\Lambda_j$, $j=1-5$:

\[
\bar{f}_1(t) = \frac{(1 - \omega_1^2)}{(\omega_1^2 - \omega_2^2)} e^{-\xi_1 \omega_1 t} \left[ \frac{\xi_1 \omega_1}{\omega_1} \sin (\omega d_1 t) + \cos (\omega d_1 t) \right] - \frac{(1 - \omega_1^2)}{(\omega_1^2 - \omega_2^2)} e^{\xi_2 \omega_1 t} \left[ \frac{\xi_2 \omega_2}{\omega_2} \sin (\omega d_2 t) + \cos (\omega d_2 t) \right], \quad (A1)
\]

\[
\bar{f}_2(t) = -\frac{e^{-\xi_1 \omega_1 t}}{(\omega_1^2 - \omega_2^2)} \left[ \frac{\xi_1 \omega_1}{\omega_1} \sin (\omega d_1 t) + \cos (\omega d_1 t) \right] + \frac{e^{-\xi_2 \omega_1 t}}{(\omega_1^2 - \omega_2^2)} \left[ \frac{\xi_2 \omega_2}{\omega_2} \sin (\omega d_2 t) + \cos (\omega d_2 t) \right], \quad (A2)
\]

\[
\bar{f}_3(t) = \frac{(1 - \omega_1^2)}{(\omega_1^2 - \omega_2^2)} \left[ e^{-\xi_1 \omega_1 t} \sin (\omega d_1 t) - \frac{e^{-\xi_2 \omega_1 t}}{\omega_2} \sin (\omega d_2 t) \right], \quad (A3)
\]

\[
\bar{f}_4(t) = \frac{1}{(\omega_1^2 - \omega_2^2)} \left[ -\frac{e^{-\xi_1 \omega_1 t}}{\omega_1} \sin (\omega d_1 t) + \frac{e^{-\xi_2 \omega_1 t}}{\omega_2} \sin (\omega d_2 t) \right], \quad (A4)
\]

\[
\bar{f}_5(t) = -\left[ e^{-\xi_1 \omega_1 t} \sin (\omega d_1 t) \frac{\xi_1 \omega_1}{\omega_1} + e^{-\xi_2 \omega_1 t} \cos (\omega d_1 t) - \cos (\omega t) \right] \times \frac{(1 - \omega_1^2 f_2)(1 - m_2)}{1 + m_2 \omega_1^2 (\omega_1^2 - 2)} \left[ 2 \xi_1 \omega_1 + \frac{2 \xi_1 \omega_1}{(\omega_1^2 - \omega_2^2) + (2 \xi_1 \omega_1)^2} \right] - \left[ e^{-\xi_1 \omega_1 t} \sin (\omega d_1 t) \frac{\omega}{\omega_1} - \sin (\omega t) \right]
\]
\[
\begin{align*}
\tilde{F}_6(t) &= \left[ e^{-\xi_1\omega t} \sin (\omega d_1 t) \frac{\xi_1\omega}{\omega_{d_1}} + e^{-\xi_1\omega t} \cos (\omega d_1 t) - \cos (\omega t) \right] \\
&\times \left[ (1 - \omega_1^2 f_2)(1 - m_2) \right] \left[ \frac{\omega}{\omega_{d_1}} - \sin (\omega t) \right] \\
&\times \left[ \frac{(1 - \omega_1^2 f_2)(1 - m_2)}{1 + m_2\omega_1^2(\omega_1^2 - 2)} \right] \left[ \frac{2\zeta_1\omega_1\omega}{(\omega_1^2 - \omega^2)^2 + (2\zeta_1\omega_1\omega)^2} \right] \\
&\times \left[ e^{-\xi_1\omega t} \sin (\omega d_1 t) \frac{\omega}{\omega_{d_1}} - \sin (\omega t) \right] \\
&\times \left[ \frac{(1 - \omega_1^2 f_2)(1 - m_2)}{1 + m_2\omega_1^2(\omega_1^2 - 2)} \right] \left[ \frac{2\zeta_1\omega_1\omega}{(\omega_1^2 - \omega^2)^2 + (2\zeta_1\omega_1\omega)^2} \right] \\
&\times \left[ e^{-\xi_1\omega t} \sin (\omega d_2 t) \frac{\omega}{\omega_{d_2}} - \sin (\omega t) \right] \\
&\times \left[ \frac{(1 - \omega_1^2 f_2)(1 - m_2)}{1 + m_2\omega_1^2(\omega_1^2 - 2)} \right] \left[ \frac{2\zeta_1\omega_1\omega}{(\omega_1^2 - \omega^2)^2 + (2\zeta_1\omega_1\omega)^2} \right] \\
&\times \left[ e^{-\xi_1\omega t} \sin (\omega d_1 t) \frac{\omega}{\omega_{d_1}} - \sin (\omega t) \right] \\
&\times \left[ \frac{(1 - \omega_1^2 f_2)(1 - m_2)}{1 + m_2\omega_1^2(\omega_1^2 - 2)} \right] \left[ \frac{2\zeta_1\omega_1\omega}{(\omega_1^2 - \omega^2)^2 + (2\zeta_1\omega_1\omega)^2} \right] \\
&\times \left[ e^{-\xi_1\omega t} \sin (\omega d_1 t) \frac{\omega}{\omega_{d_1}} - \sin (\omega t) \right] \\
&\times \left[ \frac{(1 - \omega_1^2 f_2)(1 - m_2)}{1 + m_2\omega_1^2(\omega_1^2 - 2)} \right] \left[ \frac{2\zeta_1\omega_1\omega}{(\omega_1^2 - \omega^2)^2 + (2\zeta_1\omega_1\omega)^2} \right] \\
\end{align*}
\]
\( \bar{F}_9(t) = \frac{(1 - \omega_1^2)(1 - \omega_2^2)}{(\omega_1^2 - \omega_2^2)} \left[ e^{-\zeta_1 \omega_1 t} \sin(\omega_1 t) - e^{-\zeta_2 \omega_2 t} \sin(\omega_2 t) \right], \)  
(A9)

\( \bar{F}_{10}(t) = \frac{(1 - \omega_1^2)}{(\omega_1^2 - \omega_2^2)} \left[ -e^{-\zeta_1 \omega_1 t} \sin(\omega_1 t) + e^{-\zeta_2 \omega_2 t} \sin(\omega_2 t) \right], \)  
(A10)

\( \bar{F}_{11}(t) = -\left[ e^{-\zeta_1 \omega_1 t} \sin(\omega_1 t) \frac{\zeta_1 \omega_1}{\omega_1} + e^{-\zeta_1 \omega_1 t} \cos(\omega_1 t) - \cos(\omega t) \right] \times (1 - \omega_1^2) \left[ \frac{(1 - \omega_1^2 f_2)(1 - m_2)}{1 + m_2 \omega_1^2(\omega_1^2 - 2)} \right] \left[ \frac{2\zeta_1 \omega_1 \omega}{(\omega_1^2 - \omega^2)^2 + (2\zeta_1 \omega_1 \omega)^2} \right] \)

\( \times (1 - \omega_1^2) \left[ \frac{(1 - \omega_1^2 f_2)(1 - m_2)}{1 + m_2 \omega_1^2(\omega_1^2 - 2)} \right] \left[ \frac{-(\omega_1^2 - \omega^2)}{(\omega_1^2 - \omega^2)^2 + (2\zeta_1 \omega_1 \omega)^2} \right] \)

\( \times (1 - \omega_1^2) \left[ \frac{(1 - \omega_1^2 f_2)(1 - m_2)}{1 + m_2 \omega_2^2(\omega_2^2 - 2)} \right] \left[ \frac{-\omega_2^2 + \omega^2}{(\omega_2^2 - \omega^2)^2 + (2\zeta_2 \omega_2 \omega)^2} \right] \)

\( \times (1 - \omega_2^2) \left[ \frac{(1 - \omega_2^2 f_2)(1 - m_2)}{1 + m_2 \omega_2^2(\omega_2^2 - 2)} \right] \left[ \frac{2\zeta_2 \omega_2 \omega}{(\omega_2^2 - \omega^2)^2 + (2\zeta_2 \omega_2 \omega)^2} \right], \)  
(A11)

\( \bar{F}_{12}(t) = -\left[ e^{-\zeta_1 \omega_1 t} \sin(\omega_1 t) \frac{\zeta_1 \omega_1}{\omega_1} + e^{-\zeta_1 \omega_1 t} \cos(\omega_1 t) - \cos(\omega t) \right] \times (1 - \omega_1^2) \left[ \frac{(1 - \omega_1^2 f_2)(1 - m_2)}{1 + m_2 \omega_1^2(\omega_1^2 - 2)} \right] \left[ \frac{(\omega_1^2 - \omega^2)}{(\omega_1^2 - \omega^2)^2 + (2\zeta_1 \omega_1 \omega)^2} \right] \)

\( \times (1 - \omega_1^2) \left[ \frac{(1 - \omega_1^2 f_2)(1 - m_2)}{1 + m_2 \omega_1^2(\omega_1^2 - 2)} \right] \left[ \frac{2\zeta_1 \omega_1 \omega}{(\omega_1^2 - \omega^2)^2 + (2\zeta_1 \omega_1 \omega)^2} \right] \)

\( \times (1 - \omega_1^2) \left[ \frac{(1 - \omega_1^2 f_2)(1 - m_2)}{1 + m_2 \omega_1^2(\omega_1^2 - 2)} \right] \left[ \frac{-\omega_2^2 + \omega^2}{(\omega_2^2 - \omega^2)^2 + (2\zeta_2 \omega_2 \omega)^2} \right] \)

\( \times (1 - \omega_2^2) \left[ \frac{(1 - \omega_2^2 f_2)(1 - m_2)}{1 + m_2 \omega_2^2(\omega_2^2 - 2)} \right] \left[ \frac{2\zeta_2 \omega_2 \omega}{(\omega_2^2 - \omega^2)^2 + (2\zeta_2 \omega_2 \omega)^2} \right] \)
\[
\begin{align*}
\tilde{F}_{13}(t) &= -\frac{(1 - \omega_1^2)}{(\omega_1^2 - \omega_2^2)} \left[ \frac{\omega_1^2 \zeta_2^2 + \omega_d \omega_1}{\omega_d} \right] e^{-\zeta_1 \omega_1 t} \sin (\omega_d t) \\
&\quad + \frac{(1 - \omega_1^2)}{(\omega_1^2 - \omega_2^2)} \left[ \frac{\omega_2^2 \zeta_2^2}{\omega_d} + \omega_d \right] e^{-\zeta_2 \omega_2 t} \sin (\omega_d t), \\
\tilde{F}_{14}(t) &= \frac{1}{(\omega_1^2 - \omega_2^2)} \left[ \frac{\omega_1^2 \zeta_1^2 + \omega_d \omega_1}{\omega_d} \right] e^{-\zeta_1 \omega_1 t} \sin (\omega_d t) \\
&\quad - \frac{1}{(\omega_1^2 - \omega_2^2)} \left[ \frac{\omega_1^2 \zeta_1^2}{\omega_d} + \omega_d \right] e^{-\zeta_1 \omega_1 t} \sin (\omega_d t), \\
\tilde{F}_{15}(t) &= \left\{ \right. \\
&\quad -\left. \frac{\omega_1^2 \zeta_1^2}{\omega_d} + \omega_d \right \} e^{-\zeta_1 \omega_1 t} \sin (\omega_d t) + \omega \sin (\omega t) \\
&\quad \times \left[ \frac{(1 - \omega_1^2 f_2)(1 - m_2)}{1 + m_2 \omega_1^2 (\omega_1^2 - 2)} \right] \left[ \frac{2 \zeta_1 \omega_1 \omega_2}{(\omega_1^2 - \omega_2^2)^2 + (2 \zeta_1 \omega_1 \omega_2)^2} \right] \\
&\quad - \left\{ \right. \\
&\quad -\left. \frac{\omega_1^2 \zeta_1^2}{\omega_d} + \omega_d \right \} e^{-\zeta_1 \omega_1 t} \sin (\omega_d t) \right\} \cos (\omega t) \\
&\quad \times \omega \left[ \frac{(1 - \omega_1^2 f_2)(1 - m_2)}{1 + m_2 \omega_1^2 (\omega_1^2 - 2)} \right] \left[ \frac{-(\omega_1^2 - \omega_2^2)}{(\omega_1^2 - \omega_2^2)^2 + (2 \zeta_1 \omega_1 \omega_2)^2} \right] \\
&\quad - \left\{ \right. \\
&\quad -\left. \frac{\omega_1^2 \zeta_1^2}{\omega_d} + \omega_d \right \} e^{-\zeta_1 \omega_1 t} \sin (\omega_d t) \right\} \cos (\omega t) \\
&\quad \times \omega \left[ \frac{(1 - \omega_1^2 f_2)(1 - m_2)}{1 + m_2 \omega_1^2 (\omega_1^2 - 2)} \right] \left[ \frac{-(\omega_1^2 - \omega_2^2)}{(\omega_1^2 - \omega_2^2)^2 + (2 \zeta_1 \omega_1 \omega_2)^2} \right]
\end{align*}
\]
\[
\tilde{F}_{18}(t) = - \left[ \frac{\omega_2^2 \gamma_2^2}{\omega_{d2}} + \omega_{d2} \right] e^{-\zeta_1 \omega_2 t} \sin(\omega_{d2} t) + \omega \sin(\omega t)
\]
\[
\times \left[ \frac{(1 - \omega_1^2 f_2)(1 - m_2)}{1 + m_2 \omega_2^2 (\omega_2^2 - 2)} \right] \left[ \frac{2\zeta_2 \omega_2 \omega}{(\omega_2^2 - \omega^2)^2 + (2\zeta_2 \omega_2 \omega)^2} \right],
\]
(A17)

\[
\tilde{F}_{19}(t) = - \left[ \frac{\omega_2^2 \gamma_2^2}{\omega_{d1}} + \omega_{d1} \right] e^{-\zeta_1 \omega_2 t} \sin(\omega_{d1} t) + \omega \sin(\omega t)
\]
\[
\times \left[ \frac{(1 - \omega_1^2 f_2)(1 - m_2)}{1 + m_2 \omega_1^2 (\omega_1^2 - 2)} \right] \left[ \frac{2\zeta_1 \omega_1 \omega}{(\omega_1^2 - \omega^2)^2 + (2\zeta_1 \omega_1 \omega)^2} \right] - \cos(\omega t)
\]
\[
\times \omega \left[ \frac{(1 - \omega_1^2 f_2)(1 - m_2)}{1 + m_2 \omega_1^2 (\omega_1^2 - 2)} \right] \left[ \frac{2\zeta_1 \omega_1 \omega}{(\omega_1^2 - \omega^2)^2 + (2\zeta_1 \omega_1 \omega)^2} \right] - \cos(\omega t)
\]
\[
\times \left[ \frac{\omega_2^2 \gamma_2^2}{\omega_{d2}} + \omega_{d2} \right] e^{-\zeta_2 \omega_2 t} \sin(\omega_{d2} t) + \omega \sin(\omega t)
\]
\[
\times \left[ \frac{(1 - \omega_2^2 f_2)(1 - m_2)}{1 + m_2 \omega_2^2 (\omega_2^2 - 2)} \right] \left[ \frac{2\zeta_2 \omega_2 \omega}{(\omega_2^2 - \omega^2)^2 + (2\zeta_2 \omega_2 \omega)^2} \right],
\]
(A18)

\[
\tilde{F}_{20}(t) = \frac{(1 - \omega_1^2)(1 - \omega_2^2)}{(\omega_1^2 - \omega_2^2)} \left[ \frac{\omega_1^2 \gamma_1^2}{\omega_{d1}} + \omega_{d1} \right] e^{-\zeta_1 \omega_1 t} \sin(\omega_{d1} t)
\]
\[
+ \frac{(1 - \omega_1^2)(1 - \omega_2^2)}{(\omega_1^2 - \omega_2^2)} \left[ \frac{\omega_2^2 \gamma_2^2}{\omega_{d2}} + \omega_{d2} \right] e^{-\zeta_2 \omega_2 t} \sin(\omega_{d2} t),
\]
(A19)

\[
\tilde{F}_{21}(t) = \left[ e^{-\zeta_1 \omega_1 t} \cos(\omega_{d1} t) - \frac{\omega_1^2 \gamma_1^2}{\omega_{d1}} e^{-\zeta_1 \omega_1 t} \sin(\omega_{d1} t) \right] \frac{(1 - \omega_1^2)(1 - \omega_2^2)}{(\omega_1^2 - \omega_2^2)}
\]
\[
- \left[ e^{-\zeta_2 \omega_2 t} \cos(\omega_{d2} t) - \frac{\omega_2^2 \gamma_2^2}{\omega_{d2}} e^{-\zeta_2 \omega_2 t} \sin(\omega_{d2} t) \right] \frac{(1 - \omega_1^2)(1 - \omega_2^2)}{(\omega_1^2 - \omega_2^2)},
\]
(A21)
\[
\tilde{F}_{22}(t) = -\left[ e^{-\zeta_1 \omega t} \cos (\omega_d t) - \frac{\omega_1 \zeta_1}{\omega_d} e^{-\zeta_1 \omega t} \sin (\omega_d t) \right] \frac{(1 - \omega_1^2)}{(\omega_1^2 - \omega_d^2)} \\
+ \left[ e^{-\zeta_2 \omega t} \cos (\omega_d t) - \frac{\omega_2 \zeta_2}{\omega_d} e^{-\zeta_2 \omega t} \sin (\omega_d t) \right] \frac{(1 - \omega_2^2)}{(\omega_2^2 - \omega_d^2)}, \tag{A22}
\]

\[
\tilde{F}_{23}(t) = -\left\{ \left[ \frac{\omega_1^2 \zeta_1^2}{\omega_d} + \omega_d \right] e^{-\zeta_1 \omega t} \sin (\omega_d t) + \omega \sin (\omega t) \right\} \\
\times (1 - \omega_1^2) \left[ \frac{(1 - \omega_1^2 f_2)(1 - m_2)}{1 + m_2 \omega_1^2 (\omega_1^2 - 2)} \right] \frac{2 \zeta_1 \omega_1 \omega}{(\omega_1^2 - \omega^2)^2 + (2 \zeta_1 \omega_1 \omega)^2}, \\
- \left\{ e^{-\zeta_1 \omega t} \cos (\omega_d t) - \frac{\omega_1 \zeta_1}{\omega_d} e^{-\zeta_1 \omega t} \sin (\omega_d t) \right\} - \cos (\omega t) \\
\times \omega (1 - \omega_1^2) \left[ \frac{(1 - \omega_1^2 f_2)(1 - m_2)}{1 + m_2 \omega_1^2 (\omega_1^2 - 2)} \right] \frac{-(\omega_1^2 - \omega^2)}{(\omega_1^2 - \omega^2)^2 + (2 \zeta_1 \omega_1 \omega)^2}, \\
- \left\{ e^{-\zeta_2 \omega t} \cos (\omega_d t) - \frac{\omega_2 \zeta_2}{\omega_d} e^{-\zeta_2 \omega t} \sin (\omega_d t) \right\} - \cos (\omega t) \\
\times \omega (1 - \omega_2^2) \left[ \frac{(1 - \omega_2^2 f_2)(1 - m_2)}{1 + m_2 \omega_2^2 (\omega_2^2 - 2)} \right] \frac{-(\omega_2^2 - \omega^2)}{(\omega_2^2 - \omega^2)^2 + (2 \zeta_2 \omega_2 \omega)^2}, \\
- \left\{ \frac{\omega_2^2 \zeta_2^2}{\omega_d} + \omega_d \right\} e^{-\zeta_2 \omega t} \sin (\omega_d t) + \omega \sin (\omega t) \right\} \\
\times (1 - \omega_2^2) \left[ \frac{(1 - \omega_2^2 f_2)(1 - m_2)}{1 + m_2 \omega_2^2 (\omega_2^2 - 2)} \right] \frac{2 \zeta_2 \omega_2 \omega}{(\omega_2^2 - \omega^2)^2 + (2 \zeta_2 \omega_2 \omega)^2}, \tag{A23}
\]

\[
\tilde{F}_{24}(t) = -\left\{ \left[ \frac{\omega_1^2 \zeta_1^2}{\omega_d} + \omega_d \right] e^{-\zeta_1 \omega t} \sin (\omega_d t) + \omega \sin (\omega t) \right\} \\
\times (1 - \omega_1^2) \left[ \frac{(1 - \omega_1^2 f_2)(1 - m_2)}{1 + m_2 \omega_1^2 (\omega_1^2 - 2)} \right] \frac{2 \zeta_1 \omega_1 \omega}{(\omega_1^2 - \omega^2)^2 + (2 \zeta_1 \omega_1 \omega)^2}, \\
- \left\{ e^{-\zeta_1 \omega t} \cos (\omega_d t) - \frac{\omega_1 \zeta_1}{\omega_d} e^{-\zeta_1 \omega t} \sin (\omega_d t) \right\} - \cos (\omega t) \\
\times \omega (1 - \omega_1^2) \left[ \frac{(1 - \omega_1^2 f_2)(1 - m_2)}{1 + m_2 \omega_1^2 (\omega_1^2 - 2)} \right] \frac{2 \zeta_1 \omega_1 \omega}{(\omega_1^2 - \omega^2)^2 + (2 \zeta_1 \omega_1 \omega)^2}, \\
- \left\{ e^{-\zeta_2 \omega t} \cos (\omega_d t) - \frac{\omega_2 \zeta_2}{\omega_d} e^{-\zeta_2 \omega t} \sin (\omega_d t) \right\} - \cos (\omega t) \\
\times \omega (1 - \omega_2^2) \left[ \frac{(1 - \omega_2^2 f_2)(1 - m_2)}{1 + m_2 \omega_2^2 (\omega_2^2 - 2)} \right] \frac{2 \zeta_2 \omega_2 \omega}{(\omega_2^2 - \omega^2)^2 + (2 \zeta_2 \omega_2 \omega)^2}.
\]
\[
\Lambda_1 = \sum_{j=1}^{2} \frac{\mu_j^2}{\omega_2} + \omega_{d_2} \right] e^{-c_2 \omega_{d_2} t} \sin(\omega_{d_2} t) + \omega \sin(\omega t) \right] \times (1 - \omega^2) \left( \frac{(1 - \omega_2^2 f_0)(1 - m_2)}{1 + m_2 \omega_1^2 (\omega_2^2 - 2)} \right) \left( \frac{\omega_2^2 - \omega^2}{(\omega_2^2 - \omega^2)^2 + (2c_2 \omega_2 \omega_2)^2} \right).
\]

\[
\Lambda_2 = \sum_{j=1}^{2} \frac{\mu_j^2}{\omega_2} + \omega_{d_2} \right] e^{-c_2 \omega_{d_2} t} \sin(\omega_{d_2} t) + \omega \sin(\omega t) \right] \times (1 - \omega^2) \left( \frac{(1 - \omega_2^2 f_0)(1 - m_2)}{1 + m_2 \omega_1^2 (\omega_2^2 - 2)} \right) \left( \frac{\omega_2^2 - \omega^2}{(\omega_2^2 - \omega^2)^2 + (2c_2 \omega_2 \omega_2)^2} \right).
\]

\[
\Lambda_3 = \sum_{j=1}^{2} \frac{\mu_j^2}{\omega_2} + \omega_{d_2} \right] e^{-c_2 \omega_{d_2} t} \sin(\omega_{d_2} t) + \omega \sin(\omega t) \right] \times (1 - \omega^2) \left( \frac{(1 - \omega_2^2 f_0)(1 - m_2)}{1 + m_2 \omega_1^2 (\omega_2^2 - 2)} \right) \left( \frac{\omega_2^2 - \omega^2}{(\omega_2^2 - \omega^2)^2 + (2c_2 \omega_2 \omega_2)^2} \right).
\]

\[
\Lambda_4 = \sum_{j=1}^{2} \frac{\mu_j^2}{\omega_2} + \omega_{d_2} \right] e^{-c_2 \omega_{d_2} t} \sin(\omega_{d_2} t) + \omega \sin(\omega t) \right] \times (1 - \omega^2) \left( \frac{(1 - \omega_2^2 f_0)(1 - m_2)}{1 + m_2 \omega_1^2 (\omega_2^2 - 2)} \right) \left( \frac{\omega_2^2 - \omega^2}{(\omega_2^2 - \omega^2)^2 + (2c_2 \omega_2 \omega_2)^2} \right).
\]
\[- \Gamma_2 \Gamma_{11} \Gamma_{16} \Gamma_{21} + \Gamma_4 \Gamma_{11} \Gamma_{14} \Gamma_{21} + \Gamma_3 \Gamma_{10} \Gamma_{14} \Gamma_{23} - \Gamma_3 \Gamma_{11} \Gamma_{14} \Gamma_{22} - \Gamma_3 \Gamma_{17} \Gamma_{22} + \Gamma_3 \Gamma_{11} \Gamma_{16} \Gamma_{20} + \Gamma_3 \Gamma_{16} \Gamma_{23} + \Gamma_2 \Gamma_9 \Gamma_{16} \Gamma_{23} - \Gamma_4 \Gamma_3 \Gamma_{14} \Gamma_{23} + \Gamma_4 \Gamma_8 \Gamma_{15} \Gamma_{23} - \Gamma_5 \Gamma_8 \Gamma_{16} \Gamma_{21} + \Gamma_3 \Gamma_9 \Gamma_{14} \Gamma_{22} - \Gamma_5 \Gamma_8 \Gamma_{16} \Gamma_{20} - \Gamma_5 \Gamma_{10} \Gamma_{14} \Gamma_{21} + \Gamma_5 \Gamma_{10} \Gamma_{15} \Gamma_{20} + \Gamma_3 \Gamma_{11} \Gamma_{14} - \Gamma_3 \Gamma_8 \Gamma_{16} \Gamma_{23} + \Gamma_3 \Gamma_8 \Gamma_{17} \Gamma_{22}, \]

(A28)

\[ A_5 = \Gamma_4 \Gamma_{13} \Gamma_{23} + \Gamma_{17} + \Gamma_8 \Gamma_{17} \Gamma_{22} - \Gamma_5 \Gamma_{13} \Gamma_{22} + \Gamma_5 \Gamma_7 \Gamma_{14} - \Gamma_7 \Gamma_{22} + \Gamma_5 \Gamma_{13} - \Gamma_5 \Gamma_{17} \Gamma_{23} - \Gamma_1 \Gamma_{17} + \Gamma_1 \Gamma_{17} \Gamma_{22} + \Gamma_2 \Gamma_{11} \Gamma_{13} - \Gamma_1 \Gamma_{17} \Gamma_{20}\]

(A29)

**APPENDIX B**

This appendix contains some intermediate steps pertaining to the section on stability analysis.

Equations (15), (26), (28) and (32) give

\[ x_1(0) + \Delta x_1^{(p)}(0) = b, \quad \Delta x_1^{(p)}(0) = 0. \]

(B1)

Taylor expansions about the argument \(n2\pi\) are carried out in equations (33)–(36) and terms up to the first order are retained. Then equations (16), (26), (28) and (33) give

\[ \dot{x}_1^{(p)}(n2\pi/\omega) + \ddot{x}_1^{(p)}(n2\pi/\omega)[(\Delta \delta^{(p+1)} - \Delta \delta^{(p)})] = b, \]

\[ \Delta x_1^{(p)}(n2\pi/\omega) + \ddot{x}_1^{(p)}(n2\pi/\omega)[(\Delta \delta^{(p+1)} - \Delta \delta^{(p)})] = 0, \]

(B2)

equations (17), (27), (29) and (34) give

\[ \dot{x}_1(0) + \Delta \dot{x}_1^{(p+1)}(0) = -\alpha[x_1^{(p)}(n2\pi/\omega) - \ddot{x}_1^{(p)}(n2\pi/\omega)](\Delta \delta^{(p+1)} - \Delta \delta^{(p)}), \]

\[ \Delta \dot{x}_1^{(p+1)}(0) = -\alpha[x_1(n2\pi/\omega) + \Delta \dot{x}_1^{(p)}(n2\pi/\omega) + \ddot{x}_1(n2\pi/\omega)](\Delta \delta^{(p+1)} - \Delta \delta^{(p)}), \]

\[ \Delta \dot{x}_1^{(p+1)}(0) + \alpha[\Delta \dot{x}_1^{(p)}(n2\pi/\omega) + \ddot{x}_1(n2\pi/\omega)](\Delta \delta^{(p+1)} - \Delta \delta^{(p)}) = 0, \]

(B3)

equations (18), (26), (30) and (35) give

\[ \dot{x}_2^{(p+1)}(0) = \ddot{x}_2^{(p)}(n2\pi/\omega) + \dot{x}_2^{(p)}(n2\pi/\omega)(\Delta \delta^{(p+1)} - \Delta \delta^{(p)}), \]

\[ x_2(0) + \Delta x_2^{(p+1)}(0) = x_2(n2\pi/\omega) + \Delta x_2^{(p)}(n2\pi/\omega) + \dot{x}_2(n2\pi/\omega)(\Delta \delta^{(p+1)} - \Delta \delta^{(p)}), \]

\[ \Delta x_2^{(p+1)}(0) - \Delta x_2^{(p)}(n2\pi/\omega) - \dot{x}_2(n2\pi/\omega)(\Delta \delta^{(p+1)} - \delta^{(p)}) = 0, \]

(B4)
and equations (19), (27), (31) and (36) give
$$\dot{x}_2(0) + \Delta \dot{x}_2^p(0) = \ddot{x}_2(n2\pi/\omega) + \dddot{x}_2(n2\pi/\omega)(\Delta \delta^{(p+1)} - \Delta \delta^{(p)}),$$
$$\dot{x}_2(0) + \Delta \dot{x}_2^p(0) = \ddot{x}_2(n2\pi/\omega) + \dddot{x}_2(n2\pi/\omega)(\Delta \delta^{(p+1)} - \Delta \delta^{(p)}),$$
$$\Delta \dot{x}_2^p(0) - \Delta \dot{x}_2^p(n2\pi/\omega) - \dot{x}_2(n2\pi/\omega)(\Delta \delta^{(p+1)} - \Delta \delta^{(p)}) = 0. \quad (B5)$$

By assuming growth according to equations (37)-(40), equations (B2)-(B5) give
$$r \dot{x}_2(0) + r \ddot{x}_2(0) + \cdots = r \dot{x}_2(n2\pi/\omega)(\lambda - 1)) \Delta \delta^{(p)} = 0, \quad (B6)$$

$$((\lambda \ddot{\bar{x}}_8(0) - \bar{G}_8) \Delta \dot{x}_2^p(0) + (\lambda \ddot{\bar{x}}_9(0) - \bar{G}_9) \Delta \dot{x}_2^p(0) + (\lambda \ddot{\bar{x}}_{10}(0) - \bar{G}_{10}) \Delta \dot{x}_2^p(0)$$
$$+ [(\lambda \ddot{\bar{x}}_{11}(0) - \bar{G}_{11}) \cos(\delta) - (\lambda \ddot{\bar{x}}_{12}(0) - \bar{G}_{12}) \sin(\delta) - \dot{x}_2(n2\pi/\omega)(\lambda - 1)) \Delta \delta^{(p)} = 0, \quad (B7)$$

$$((\lambda \ddot{\bar{x}}_{20}(0) - \bar{G}_{20}) \Delta \dot{x}_2^p(0) + (\lambda \ddot{\bar{x}}_{21}(0) - \bar{G}_{21}) \Delta \dot{x}_2^p(0) + (\lambda \ddot{\bar{x}}_{22}(0) - \bar{G}_{22}) \Delta \dot{x}_2^p(0)$$
$$+ [(\lambda \ddot{\bar{x}}_{23}(0) - \bar{G}_{23}) \cos(\delta) - (\lambda \ddot{\bar{x}}_{24}(0) - \bar{G}_{24}) \sin(\delta) - \dot{x}_2(n2\pi/\omega)(\lambda - 1)) \Delta \delta^{(p)} = 0. \quad (B8)$$

The elements of the fourth order determinant in equation (41) are
$$a_{11} = \bar{G}_2, \quad a_{12} = \bar{G}_3, \quad a_{13} = \bar{G}_4,$$
$$a_{14} = (\bar{G}_5 \cos(\delta) - \bar{G}_6 \sin(\delta) + \dot{x}_2(n2\pi/\omega)(\lambda - 1)), \quad (B10)$$
$$a_{21} = (\bar{G}_{14}(0) + a \bar{G}_{14}), \quad a_{22} = (\bar{G}_{15}(0) + a \bar{G}_{15}), \quad a_{23} = (\bar{G}_{16}(0) + a \bar{G}_{16}),$$
$$a_{24} = [(\bar{G}_{17}(0) + a \bar{G}_{17}) \cos(\delta) - (\bar{G}_{18}(0) + a \bar{G}_{18}) \sin(\delta) + \dot{x}_2(n2\pi/\omega)(\lambda - 1)), \quad (B11)$$
$$a_{31} = (\lambda \ddot{\bar{x}}_{8}(0) - \bar{G}_8), \quad a_{32} = (\lambda \ddot{\bar{x}}_{9}(0) - \bar{G}_9), \quad a_{33} = (\lambda \ddot{\bar{x}}_{10}(0) - \bar{G}_{10}),$$
$$a_{34} = [(\lambda \ddot{\bar{x}}_{11}(0) - \bar{G}_{11}) \cos(\delta) - (\lambda \ddot{\bar{x}}_{12}(0) - \bar{G}_{12}) \sin(\delta) - \dot{x}_2(n2\pi/\omega)(\lambda - 1)), \quad (B12)$$
and
$$a_{41} = (\lambda \ddot{\bar{x}}_{20}(0) - \bar{G}_{20}), \quad a_{42} = (\lambda \ddot{\bar{x}}_{21}(0) - \bar{G}_{21}), \quad a_{43} = (\lambda \ddot{\bar{x}}_{22}(0) - \bar{G}_{22}),$$
$$a_{44} = [(\lambda \ddot{\bar{x}}_{23}(0) - \bar{G}_{23}) \cos(\delta) - (\lambda \ddot{\bar{x}}_{24}(0) - \bar{G}_{24}) \sin(\delta) - \dot{x}_2(n2\pi/\omega)(\lambda - 1)). \quad (B13)$$

The characteristic equation of the determinant is of the form
$$\lambda^4 + a_3 \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0 = 0. \quad (B14)$$

To determine if the roots of the characteristic equation are inside the unit circle, Schur-Cohns inequalities will be used. Accordingly, the solution is stable if
$$|a_0| < 1, \quad |b_1| < 1, \quad \left| \frac{b_2 - b_1 b_3}{1 - b_1^2} \right| < 1, \quad \left| \frac{b_3 - b_1 b_2}{1 + b_2 - b_1 (b_1 + b_3)} \right| < 1, \quad (B15-B18)$$
where the constants $b_1$, $b_2$ and $b_3$ are given by
$$b_1 = \frac{a_1 - a_2 a_0}{1 - a_0^2}, \quad b_2 = \frac{a_2}{1 + a_0}, \quad b_3 = \frac{a_3 - a_1 a_0}{1 - a_0^2}. \quad (B19-B21)$$
A one-dimensional model for the transition from periodic to chaotic motions in granular shear flows

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A simple one-dimensional mechanical model to simulate some aspects of the dynamics of granular flow is suggested. The model consists of a visco-elastic packet bouncing between two oscillating walls. The motion of the walls is prescribed to be harmonic. The amplitude and frequency of the wall motion are related to the concentration and shear rate, respectively. The dynamics of the system is studied for various parameter combinations and it is shown that periodic as well as chaotic motions are possible for different parameters and initial conditions. Results are presented in the form of time histories, power spectral densities, phase diagrams, Poincaré plots and bifurcation diagrams. This simple one-dimensional model presents many features that are analogous to those observed in the two-dimensional simple shear flow of disks.

1. Introduction

In a recent study (Babić et al., 1990), a computer simulated dense simple shear flow of uniform disks has been investigated. The stress tensor was obtained as a function of the shear rate. The stress is generated by visco-elastic/frictional contacts between neighboring disks. These contacts are modeled by linear springs and dashpots which activate when distance between neighboring disk centers becomes less than a diameter. Parameters included in the simulation are the concentration of disks, \( c \), the mass per unit area \( \rho_s \) and diameter \( D \) of a typical disk, the spring and dashpot coefficients, \( K \) and \( C \), respectively, the coefficient of friction, \( \mu \), and the shear rate, \( \dot{\gamma} \).

After non-dimensionalization, the above parameters reduce to the concentration \( c \), the dimensionless shear rate, \( B = \dot{\gamma} / \sqrt{K/m} \), the coefficient of friction, \( \mu \), and the dimensionless dashpot coefficient, \( \zeta = C / 2\sqrt{mK} \). The mass and diameter of the disk are used to construct the dimensionless length, \( x/D \), and stress, \( \tau_{ij}^* = \tau_{ij} / \rho_s D^2 \dot{\gamma}^2 \). Among these parameters, the effect of \( c \) and \( B \) on \( \tau_{ij}^* \) has been studied in detail.

It was found that at very high \( c \), the disk assembly moves in blocks of hexagonal packing, joined by a ‘failure layer’, where a row of disks would roll from hexagonal to cubic and back to hexagonal packing between the top and bottom layers (Figs. 1a–1d, reproduced from Figs. 21a–21d in Babić et al. (1990)).

Through this rolling, the prescribed shear rate is achieved. The stress tensor developed in this assembly thus undergoes a periodic state, with the period specified by the given shear rate. The highest stress state corresponds to the highest compression when the rolling layer is in cubic packing. The lowest stress state corresponds to the lowest compression when the rolling layer is in the hexagonal packing.

When \( c \) decreases, this simple periodic behavior becomes less pronounced, and eventually dis-
appears totally. In the low $c$ extreme, the time series of the stress looks completely random. Transition from the periodic to random behavior appears to be quite sudden when $B$ has low values. At high $B$, (the highest one given in Babic et al. (1990) was 0.0707), this transition no longer exists.

Similar phenomena have been proposed to study the sand dune motions (Bagnold, 1966). A recent computer simulation of constant normal stress instead of constant shear area as the present case also shows abrupt transition between random motion and periodic 'stick–slip' motions (Thompson and Grest, 1991).

It is important to understand the above phenomenon in order to study the flows of a granular material. In the past, constitutive relations for granular materials have been studied in two extremes: the grain inertia regime and the quasi-static regime (Savage, 1982). The first of these two corresponds to a granular flow where binary collisions are responsible for the stress generation. The second corresponds to flows where prolonged contact forces are responsible for the internal stress. The resulting stresses either depend on the square of the shear rate, as in the first case, or are independent of the shear rate, as in the second case. In Babic et al. (1990), empirical equations that unify the two extremes have been established for one set of material properties. However, the underlying physics that governs the transition region is not understood and therefore cannot be theoretically quantified for arbitrary parameters. The purpose of this study is to explore the possibility of understanding the multi-dimensional system of disks by a simpler dynamic system, such that the essence of the simple shear flow of disks can be captured in the simplified system. Since the dynamics of the simpler system is amenable to mathematical analysis, the results will then help to understand the simple shear flow of disks.

As described above, the simulated assembly of disks is a complicated spring–dashpot–mass system. There have been a great deal of studies in the past couple of decades on this subject.

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Fig. 1. A time sequence of a dense simple shear flow of disks.
Most of these studies are for single or double mass systems. These systems may be simple-looking, however, but it has been repeatedly demonstrated that for most non-linear systems (including piecewise linear systems), there are regions in the parameter space where stable periodic motion exists. Outside these regions, however, systems may develop into bifurcated states leading to chaotic motion. Based on this observation, it is tempting to simplify the complicated disk assembly studied in Babić et al. (1990) in order to analyze it with standard methods for non-linear dynamic systems. It is expected that through this simplified analysis, the transition behavior mentioned above can be better understood.

2. A simplified model

In Figs. 1a–1d, a sequence of four pictures of the disk assembly under simple shear motion is shown. The concentration is 0.9. The highest possible concentration without compression is $c_0 = 0.907$, corresponding to the close hexagonal packing. In Figs. 1a–1d, it can be seen that a rolling layer is bounded on the top and bottom by two hexagonal zones. These two zones are moving relative to the rolling layer with equal speed in the opposite directions. The velocity is given by the prescribed shear rate. As the concentration decreases, the hexagonal zone loses its structure, the rolling layer also loses its periodic motion. The whole disk assembly becomes chaotic. These two phenomena, structured hexagonal zone and periodic rolling layer, are interrelated.

We will study the following simplified system. This system is intended to characterize the transitional behavior of the rolling layer from periodic to chaotic motions.

Consider a mass $m$ as shown in Fig. 2. This mass, which has no physical dimension, is initially located at the origin of a coordinate system. There are a pair of identical spring/dashpot in parallel attached to the top and bottom of this mass. Attached to the end of each pair of the spring/dashpot, there is a flat board with no mass and no physical dimension. The springs are non-linear in the sense that they exert no force when in tension and are linear when in compression. The natural length of each spring is $D/2$, where $D$ is the disk diameter in Fig. 1. The spring constant is $k$. The dashpots are also non-linear in the sense that they are active only when the corresponding spring is in compression. The dashpot coefficient is $d$. Such a spring–dashpot–mass system is placed in between two flat oscillating walls and is constrained to move in the vertical direction. The vertical motion of the walls are prescribed.

The parameters in this system include the following: the original locations of the walls and the subsequent motion; the initial velocity of the mass; the mass; the natural length of the spring; its stiffness; and the dashpot coefficient. Because only vertical motion is considered here, the coefficient of friction in the system given in Fig. 1 is removed from this simplified system.

The analogy between this simplified system in Fig. 2 and the system in Fig. 1 is quite obvious. The mass $m$ in Fig. 2 represents a typical disk in the rolling layer. The walls represent the adjacent hexagonal zones. Their oscillatory motion is an immediate result of the forced relative translation and the finite size of the disks in the hexagonal zones. Such motion causes the rolling layer to be compressed periodically from minimum (Fig. 1a) to maximum (Fig. 1c). The frequency of the wall oscillation in Fig. 2 is directly related to the shear rate in Fig. 1. The mean position and the amplitude of the oscillation are functions of the con-
centration in Fig. 1. The initial conditions on the mass correspond to the initial position and vertical component of the velocity of a disk in the rolling layer. The initial condition of the walls represents the initial configuration of the hexagonal zones.

We will formulate the equation of motion for the system given in Fig. 2 below.

Let \( x \) be the position of mass \( m \), \( x_1 \) and \( x_2 \) be the positions of the top and bottom walls, respectively. Denote the force from the top wall by \( F_1 \) and that from the bottom wall by \( F_2 \). Without losing generality, the origin of the coordinate system is set at the initial position of the mass, and positive means upward in all cases. We adopt the dimensionless variables \( \bar{x}' = x/D \), \( t' = t\sqrt{k/m} \) and dimensionless parameters \( \omega' = \omega\sqrt{k/m} \), \( \xi = d/2\sqrt{mk} \), \( \bar{x}_0 = \bar{x}_0/D\sqrt{k/m} \), \( \ell' = \ell/D \), \( a' = a/D \), where \( \omega \) is the wall frequency, \( d \) is the damping coefficient, \( \bar{x}_0 \) is the initial velocity, \( \ell \) is the mean position of the wall and \( a \) is the amplitude of the wall oscillation. In terms of the dimensionless parameters, the equation of motion for the mass is

\[ \ddot{x}' = F_1' + F_2' , \]  

where

\[ F_1' = -\frac{1}{2} + (x_1' - x') + 2\xi(x_1' - \dot{x}') , \]
\[ x_1' - x' < \frac{1}{2} , \]
\[ = 0 , \text{ otherwise} , \]
\[ F_2' = \frac{1}{2} - (x_2' - x_2) - 2\xi(x' - \dot{x}_2) , \]
\[ x' - x_2' < \frac{1}{2} , \]
\[ = 0 , \text{ otherwise} . \]

The wall motions are specified as

\[ x_1'(t) = \ell' + a' \cos \omega' t' , \]
\[ x_2'(t) = -\ell' - a' \cos \omega' t' . \]  

In the above, the mean position of the walls as defined by \( \mp \ell' \) and the amplitude of the wall oscillation \( a' \) are derived from the disk concentration \( c \) in Fig. 1. In Fig. 3, disk O represents a typical disk in the rolling layer. Disks A and B represent the layer above. The maximum and minimum displacements of O are \( d_1 - \frac{1}{2}D \) and \( d_2 - \frac{1}{2}D \), respectively. From simple geometric relations, we have \( h_1/D = \sqrt{\frac{3}{2}}c_0/c , \ h_2/D = \sqrt{\frac{1}{2}}c_0/c \). (We used the relation \( D + s)/D = (c_0/c \) in deriving \( h_1 \) and \( h_2 \).) Therefore, \( d_1/D = (h_1/D) - (h_2/D) + \frac{1}{2} , \ d_2/D = h_1/D - \frac{1}{2} . \)

Hence, the mean position of the wall is \( \ell' = (d_1 + d_2)/2D = \sqrt{\frac{3}{2}}c_0/c - \frac{1}{2}\sqrt{1 - \frac{1}{2}c_0/c} \) and the amplitude of oscillation is \( a' = \frac{1}{2}(1 - \sqrt{1 - \frac{1}{2}c_0/c}) . \) The frequency of the wall oscillation is determined in such way that the period of a rolling cycle in Fig. 1 equals to the period of the wall oscillation in Fig. 2. Hence, \( (D + s)\sqrt{3}/(D + s) = \omega/2\pi , \) or \( \omega' = \pi\sqrt{3}B \).

Equation (1) is solved semi-analytically. Trajectories of the mass packet are obtained analytically for cases when contact is with one wall, both walls and the free flight case. The full dynamic behavior of the mass packet is then followed by a computer code which checks for the occurrence of each case and its duration.

3. Results

Figure 4 shows typical time histories, power spectral densities, phase diagrams and Poincaré plots for two values of the dimensionless forcing frequency, \( \omega' = 0.015 \) and \( \omega' = 0.1 \). The system in Fig. 2 has three inherent frequencies, \( 1/2\pi , \sqrt{2}/2\pi , \) and \( \omega'/2\pi . \) The first one corresponds to the natural frequency of the mass packet when it is in contact with only one wall. The second one corresponds to contact with both walls. The third one is the wall frequency.

Case (a) corresponds to a periodic motion. The large spikes in the time history indicate the free
flight phases of the mass-packet without internal vibrations. The high frequency motion between the spikes indicates damped harmonic oscillations of the mass, while the mass-packet is compressed between the two walls. The power spectral density shows that the second natural frequency and the forcing frequency are the dominating frequencies in the motion, as expected. The first natural frequency corresponds to impacts with only one of the walls. At high concentrations this frequency is seldom excited. The spirals in the central part of the phase diagram also indicate

![Graphs showing periodic and chaotic motions](image)

Fig. 4. Periodic and chaotic motions ($c = 0.8$; $\xi' = 0.01$; $x'(0) = 0$; $\dot{x}''(0) = 1$): (a) $\omega' = 0.015$; and (b) $\omega' = 0.1$. 
damped oscillations when the packet is compressed between the walls. The loop on the right-hand side indicates free flights (with constant velocity) followed by impacts and rebounds with the upper wall. Similarly, the loop on the left side implies corresponding motion towards the lower wall. The Poincaré plot is obtained by strobing the position and velocity at a constant

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**Fig. 5.** Bifurcation and period count diagrams ($x'(0) = 0; \dot{x}'(0) = 1$): (a) $c = 0.8, \zeta = 0.01$; (b) $c = 0.8, \zeta = 0.07$; (c) $c = 0.9, \zeta = 0.01$; (d) $c = 0.9, \zeta = 0.07$. 
forcing phase angle \( \omega' \mod 2\pi = 6.0 \). It simply contains two isolated points, indicating periodic motion. The period of this motion is twice the period of the wall motion.

Case (b) indicates chaotic behavior. This is suggested by the time history as well as by the richness of the frequency content in the power spectral density. There is however a dominating frequency, which is the wall frequency. The black area in the phase diagram and the fractal structure in the Poincaré plot again suggest the chaotic nature of the motion.

In order to explore the whole range of wall frequencies we present some bifurcation diagrams in Fig. 5. The upper diagrams are obtained by recording the velocity of the mass when the phase angle \( \omega' \mod 2\pi \) for the wall motion is 6.0. The transient motion is removed by ignoring values up to \( t' = 2000 \). The following one hundred values are plotted for each frequency in the range \( 0 < \omega' < 1 \). In order to clearly isolate the areas of periodic/chaotic motion in the bifurcation diagrams, the number of distinct states (velocity and displacement) corresponding to each frequency are counted. This count is plotted on the lower part of Figs. 5a–5d.

Figure 5 shows that periodic motions with different types of periodicities and chaotic motions are possible in different frequency ranges. In general, lower velocities are expected in the low frequency range. A comparison of cases (a) and (c) shows that as the concentration \( c \) increases, the range of possible chaos decreases. This tendency is even stronger for the high damping case, as seen by comparing (b) and (d). Large damping in general leads to increase areas of periodic motion as seen by comparing (a) and (b) or (c) and (d).

Finally, Fig. 6 shows the effect of different combinations of the damping and concentration for two different wall frequencies. The lower frequency case provides a larger region for which low period motion is possible.

4. Discussion and conclusions

Due to its simplicity, this one-dimensional (1D) system allows for a more complete understanding of its development from periodic to chaotic motion. Different roles played by the controlling parameters \( c, \zeta \) and \( \omega' \) can be analyzed quantitatively. One no doubt wonders how closely the dynamics of this simple 1D system is related to that of the two-dimensional (2D) case which we intended to understand.

An important phenomenon is present in both 1 and 2D cases. That is, a decrease in \( c \) or \( \zeta \) causes periodic motion to become chaotic. This observation suggests the close relation between the two systems. However, it is evident that the 1D case is much harder to become chaotic than the 2D one. This is seen from the fact that in case (c) of Fig. 4, the 1D motion is periodic but for a
much greater dissipation in the corresponding 2D system chaotic motion has been observed.

Two-dimensional systems are easier to become chaotic for the following reasons. For the same stiffness and damping coefficient, the 1D case is stiffer and more dissipative, since it only acts in the vertical direction. The layers adjacent to the rolling layer in a 2D shear flow have freedom to move in the vertical direction. This freedom is absent in our present way of modeling the wall motion. Allowing such freedom of the wall motion should increase the tendency towards chaotic behavior.

In the 2D system, horizontal motion is present. This motion, depending on its direction, may advance or delay the impact with the adjacent layers. In the 1D case, such an effect is analogous to an increased randomness of impacts with the walls, which reduces the stability of any possible periodic motions. Moreover, disks in the 2D case are not arranged with fixed spacing. No matter how dense the packing, some randomness is always present. This randomness corresponds to fluctuations of the spacing between the walls and their oscillation amplitude and frequency. Thus, to model the 2D disk flow more closely, randomness in the wall spacing, oscillation amplitude and frequency should be included.

The present model also does not describe the wall motion closely in the chaotic zones. The chaotic motion of the 1D mass corresponds to the chaotic rolling layer and the disintegration of the hexagonal zones in 2D. Disintegration of the hexagonal zones implies that the wall motion in the 1D model becomes random. The magnitude of the randomness is greater than what would be present merely due to fluctuations in disk spacing. Analysis including feedback from the rolling layer to the hexagonal zones remains as future work. The present result is thus capable to determine the onset of chaos. The structure of the details in the chaotic zones must include the above-mentioned modifications.

It is also important to point out that the energy in 1D system is a good indicator of how the system dynamics agrees with what has been observed in granular shear flows. In Fig. 5, comparing the bifurcation diagrams in cases (a) and (c) for low wall frequencies (which corresponds to the physical case of granular shear flows), the kinetic energy is much smaller in the high concentration case. This suggests that the system is dominated by contact forces, and the average force in the system should be rate-independent. In case (a), the spread of the velocity grows as the wall frequency increases, which indicates rate-dependent average force.

In conclusion, the simplified mechanical model according to Fig. 2 is clearly capable of showing periodic as well as chaotic behavior. Blow-ups of the bifurcation diagrams in Fig. 5 show sequences of both period doubling and halving before the system finally moves into chaotic zones. As compared to some other impact oscillators, (e.g., Thomson and Ghaffari, 1982), this system shows a richer variety of possible periodic motions. The system has a tendency to switch between these motions for slight variations in system parameters or initial conditions.

Studying the details of the dynamic evolution of this simple system gives insight into the transition behavior of 2D granular shear flows. Some of the characteristics in granular flows are already present in the simple model. It is promising that future work in this direction will enhance our understanding of the constitutive behavior of granular flows.

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Stability and bifurcations of a stationary state for an impact oscillator

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The motion of a vibroimpacting one-degree-of-freedom model is analyzed. This model is motivated by the behavior of a shearing granular material, in which a transitional phenomenon is observed as the concentration of the grains decreases. This transition changes the motion of a granular assembly from an orderly shearing between two blocks sandwiching a single layer of grains to a chaotic shear flow of the whole granular mass. The model consists of a mass-spring-dashpot assembly that bounces between two rigid walls. The walls are prescribed to move harmonically in opposite phases. For low wall frequencies or small amplitudes, the motion of the mass is damped out, and it approaches a stationary state with zero velocity and displacement. In this paper, the stability of such a state and the transition into chaos are analyzed. It is shown that the state is always changed into a saddle point after a bifurcation. For some parameter combinations, horseshoe-like structures can be observed in the Poincaré sections. Analyzing the stable and unstable manifolds of the saddle point, transversal homoclinic points are found to exist for some of these parameter combinations. © 1994 American Institute of Physics.

I. INTRODUCTION

Vibrating systems that undergo repeated impacts between parts or surroundings are of considerable interest in engineering applications. The impacts may be intended, as in applications like percussive drilling, or they may be undesirable as in gear transmissions and linkages. Vibroimpacting systems are inherently nonlinear due to the occurrence of impacts. The dynamics of a number of systems of this kind has been analyzed recently by using the theory of dynamical systems. Commonly, single-degree-of-freedom linear oscillators with one amplitude constraint are studied. See, e.g., analytical studies by Shaw and Holmes,\textsuperscript{1} numerical work by Thompson,\textsuperscript{2} and experimental investigations by Moon.\textsuperscript{3} Examples of systems with more than one amplitude constraint can be found in Refs. 4–8.

In this paper we deal with the dynamical analysis of a one-degree-of-freedom vibroimpacting model with two amplitude constraints. This model is motivated by a different kind of application than the ones mentioned above. In a recent computer simulation of shearing granular materials, it was found that two distinctly different regimes of shearing exist.\textsuperscript{9} One regime can be described as an orderly shear motion between two rigid blocks of granular material, separated by a single layer of grains. The actual shearing motion is localized at this single layer of grains. As the concentration of the grains reduces, the orderly motion evolves into a state where all grains move randomly—much the same as a sheared air mass, in which all molecules undergo their thermal motion, while on average there is a mean shear. This phenomenon is also commonly observed in shearing sand, where homogeneous shearing occurs at low shear rates and a narrow band of localized shearing occurs at high shear rates. Various models have been proposed to study the motion of a granular material. See, e.g., Ref. 10. Commonly, these models have utilized a spring and dashpot pair to simulate the dissipative contact force between impacting grains. The contact forces are determined by the spring and dashpot coefficients when the neighboring grains center of mass distance is less than the grain diameter. Otherwise, the contact force is modeled as zero.

In this paper, a one-degree-of-freedom vibroimpacting system will be investigated to study the shearing motion of a granular system. This model was first introduced in Ref. 11. In that study, it was shown that the dynamics of the proposed model could lead to periodic as well as chaotic motions of the mass. This mass represented a typical grain in the shear layer and the adjacent layers were represented by two walls oscillating in opposite phases. It was also shown that for low wall frequencies and high concentrations the motion of the mass is damped out, and it approaches a stationary state at zero displacement and velocity. Similar models of an impacting ball between two oscillating walls have also been used to study the problem of noise in gears and linkages.\textsuperscript{12–15} In all these studies, however, the walls were moving in phase and therefore the stationary state studied here does not exist.

In this paper, the stability of the stationary state is studied analytically by making some simplifying assumptions about motions close to such a state. Closed form expressions for the eigenvalues of the Poincaré map are derived. The results also provide a detailed understanding of the stability regions in the parameter space. Such regions are represented by stability diagrams of various parameter combinations. Regions of possible chaos are studied, with the help of Poincaré sections. The existence of a Smale horseshoe kind of maps\textsuperscript{16,17} is shown for some wall frequencies by detecting transversal homoclinic points.
II. THE MODEL

In Fig. 1 the one-degree-of-freedom model is shown. The model is an assembly consisting of linear springs and dashpots attached to a mass \( m \). The stiffness of the springs is \( k \) and the damping constant of the dashpots is \( d \). The natural length of this assembly is \( D \) when no forces are acting on it, and the position of the mass is given by \( x \). This assembly is moving (in the vertical direction \( x \)) between two rigid walls, at the positions \( x_1 \) and \( x_2 \), oscillating harmonically with the frequency \( \omega_0 \), in opposite phases. The mean positions of these walls are at plus-minus \( l \), respectively, and the amplitude of their oscillations is \( a \). The values of \( l \) and \( a \) are related to a "concentration" parameter \( c \), as defined in Ref. 11, in the following way:

\[
\begin{align*}
\frac{l}{D} &= \frac{3c_0^2}{4c} \frac{1}{2} \sqrt{1 - \frac{c_0}{4c}} \\
n &= \frac{D}{2} \left( 1 + \frac{1}{2} \sqrt{1 - \frac{c_0}{4c}} \right),
\end{align*}
\]

where \( c_0 = 0.9069 \) is the maximum possible value of \( c \).

When the distance between either of the walls and the mass is less than \( D/2 \), the corresponding spring and the dashpot become active. The assembly will move in free flight if both springs are inactive.

The equation of motion for the mass is

\[
m \ddot{x} = F_1 + F_2,
\]

where

\[
F_1 = \begin{cases} 
-d(x-x_1) - k \left( \frac{D}{2} - x_1 \right); & x_1 - x < \frac{D}{2}, \\
0; & x_1 - x \geq \frac{D}{2},
\end{cases}
\]

is the force from the top wall, and

\[
F_2 = \begin{cases} 
-d(x-x_2) - k \left( \frac{D}{2} - x_2 \right); & x_2 - x < \frac{D}{2}, \\
0; & x_2 - x \geq \frac{D}{2},
\end{cases}
\]

is the force from the bottom wall. An overdot represents differentiation with respect to time \( t \). The positions of the oscillating upper wall \( x_1 \) and the lower wall \( x_2 \) are given by

\[
\begin{align*}
x_1 &= l + a \cos(\omega t), \\
x_2 &= -l - a \cos(\omega t).
\end{align*}
\]

According to Eqs. (4) and (5), the forces \( F_1 \) and \( F_2 \) are active only in compression. Hence, Eq. (3) results in four linear equations: (i) free flight, (ii) contact with the upper wall, (iii) contact with the lower wall, and (iv) contact with both walls. The distances between the mass and the walls determine which of these equations is valid at a given instant. It should be noted that Eqs. (4) and (5) imply kinematic contact conditions, and hence traction forces at contact exit. The choice of kinematic contact conditions (rather than kinetic ones) is motivated by their simplicity, as they enable a much deeper closed form stability analysis. Also, the difference between kinematic and kinetic conditions is very small, especially for systems with low damping, studied here. The difference vanishes in the undamped case.\(^{18}\)

Introducing the nondimensional quantities,

\[
\begin{align*}
x' &= \frac{x}{D}, & x_1' &= \frac{x_1}{D}, & x_2' &= \frac{x_2}{D}, & t' &= t \sqrt{\frac{k}{m}}, \\
\omega' &= \frac{\omega}{\sqrt{km}}, & \xi &= \frac{d}{2 \sqrt{km}}.
\end{align*}
\]

into Eqs. (3)–(6), results in the following nondimensional equation of motion,

\[
\ddot{x}' = F_1' + F_2',
\]

with the nondimensional wall forces,

\[
\begin{align*}
F_1' &= \begin{cases} 
-2\xi(x' - x_1') - \left( x' + \frac{1}{2} - x_1' \right); & x_1' - x' < \frac{1}{2}, \\
0; & x_1' - x' \geq \frac{1}{2},
\end{cases} \\
F_2' &= \begin{cases} 
-2\xi(x' - x_2') - \left( x' - \frac{1}{2} - x_2' \right); & x_2' - x' < \frac{1}{2}, \\
0; & x_2' - x' \geq \frac{1}{2},
\end{cases}
\]

and the nondimensional positions of the walls,

\[
\begin{align*}
x_1' &= l' + a' \cos(\omega' t'), \\
x_2' &= -l' - a' \cos(\omega' t').
\end{align*}
\]

In Eqs. (8)–(10), an overdot above a symbol stands for differentiation with respect to nondimensional time \( t' \).
The system is one dimensional, but has two natural frequencies, \( \omega_1 = 1 \) and \( \omega_2 = \sqrt{2} \), due to contact with one of the walls and contact with both walls, respectively. There are three dimensions in the state space: displacement \( x' \), velocity \( x'' \), and the phase of the oscillating walls \( \phi = \omega t' \). Since the motion of the walls is periodic of period \( \tau = 2\pi/\omega' \), the phase can be restricted to the interval \([0, 2\pi]\), and is described by the circle \( S^1 \) with a period \( \tau \). Thus, points in the three-dimensional state space are given by \((x', x'', \phi) \in R^2 \times S^1 \). The Poincaré section \( \Sigma \) is chosen as the cross section of this state space for a constant value of the phase \( \phi_p \),

\[
\Sigma = \{x', x'', \phi | \phi = \phi_p \}.
\]

Points in the Poincaré section are the intersections of a trajectory with the plane \( \Sigma \) positioned at a constant phase \( \phi_p \). The Poincaré map is then the map for an intersection of a trajectory with \( \Sigma \) to the next intersection with \( \Sigma \).

### III. EXPECTED LOW-FREQUENCY MOTIONS

Some of the results in here are presented as bifurcation diagrams, where the wall frequency is varied. For obtaining these diagrams the system is started with given initial conditions at the lowest frequency. The transient part is neglected and the following 100 velocities, \( x', x'' \), in the Poincaré section are plotted at the current frequency \( \omega' \). By numerical experiments, the transient part is fixed at 80 000 time units in \( t' \). The frequency is then increased one step, and the state of the system from the previous frequency is taken as initial conditions for the new frequency. The same procedure is then followed for the new frequency. These bifurcation diagrams are of interest, since they provide information about the kind of bifurcations, and also give an overview of the kinds of expected motions; i.e., stationary, periodic, or chaotic. This procedure follows only one stable branch of steady-state motions. Other stable solutions can be found by using other initial conditions. For each frequency in this diagram, a limited number of points indicates that the motion is periodic and that the period of the motion is equal to the number of points times the period of the forcing \( \tau \). It is important to notice that the state variable, \( x' \), is not represented in these bifurcation diagrams. Therefore closer investigations must be made if the exact period is of interest.

In Fig. 2 one such bifurcation diagram is shown for \( c = 0.8 \) and \( \zeta = 0.01 \). At the lowest frequency, \( \omega' = 0.005 \), the system is initially given a zero velocity \((x' = 0)\) and a displacement close to the stationary state \((x' = 1 \times 10^{-6})\). One can see that for the lowest frequencies the velocities in the Poincaré sections are close to zero, which indicates that the motion is damped out to a stationary state. The most interesting feature in this figure is that a sudden change in behavior can be observed for frequencies above \( \omega' = 0.01 \). The stationary state is no longer stable for all frequencies and the mass starts to oscillate. Regions of points close to the stationary state can, however, be observed after the first bifurcation. The absence of a fine structure indicates that smaller intervals of the frequency are necessary in order to find the structure of the bifurcations.

In Fig. 3 a blowup for the interval \( \omega' = [0.0135, 0.0145] \) is shown. In this blowup regions of long-periodic or chaotic motion are visible. Again, one can see that stable stationary states exist between windows (in frequency) of periodic and chaotic motions. Another interesting observation is the sudden jumps in the bifurcation diagrams, see, e.g., \( \omega' = 0.0147 \). These indicate that the step in frequency has been chosen too large, so that the disturbance causes a jump to another attractor. The symmetry of the motion of the walls, \( x'_1 = -x'_2 \) and \( x''_1 = -x''_2 \), implies that corresponding to every attractor, there exists another one, where the sign of position and velocity is reversed. However, for the case of symmetric attractors both of these attractors will coincide to form a single attractor.

In the bifurcation diagrams of Figs. 2 and 3 one can observe the transition from periodic to chaotic motions. It can be observed that the loss of stability occurs alternatively by pitchfork bifurcations and by period doubling bifurcations. This is explained by the oscillating nature of the eigenvalues, as seen in Fig. 6. The eigenvalues undergo a sign change at each consecutive bifurcation, which results in a

![FIG. 2. Bifurcation diagram for the system: \( c=0.8, \zeta=0.01 \) in the frequency interval \( \omega'=[0.005,0.015] \).](image)

![FIG. 3. Bifurcation diagram for the system: \( c=0.8, \zeta=0.01 \) in the frequency interval \( \omega'=[0.0135,0.0145] \).](image)
change in the nature of bifurcation. The process of period doublings is also shown in Figs. 4(a) and 4(b). The route to chaos by sequences of period doublings has repeatedly been observed for different kinds of vibroimpacting systems; see, e.g., Ref. 2.

In the low-frequency region, the main change of the motion is determined by the stability of the stationary state. In the following section the stability aspects of the stationary state are analyzed.

IV. STABILITY ANALYSIS
A. Eigenvalues and eigenvectors
For cases with high concentrations and low wall frequencies, the motion of the mass will be damped toward a stationary state at \( x' = x'' = 0 \), as shown in Fig. 2. When the frequency is increased, this stationary state will bifurcate into periodic and chaotic motions. These bifurcations are of interest in order to understand the transitional behavior from periodic to chaotic motions. The first step is to determine the conditions for loss of stability and the nature of bifurcations.

In this section, the stability of this stationary state is analyzed using the Poincaré map. Such an analysis is complicated by the fact that between two consecutive Poincaré sections the system can go through four different phases, namely (i) free flight, (ii) contact with the upper wall, (iii) contact with the lower wall, and (iv) a phase of contact with both walls. The sequence of these phases and the duration in each phase must be found. In here, some assumptions on motions close to the stationary state are made to simplify the stability analysis. The assumptions are based on observations of the motion in low-frequency regions.

In Fig. 5(a) a time history is shown, for the system \( c = 0.8, \; \zeta = 0.01 \), at a frequency \( (\omega' = 0.015) \) for which the stationary state is unstable. The unstable frequency is chosen to show a trajectory repelling from the stationary state. By choosing initial conditions close to the stationary state \( (x' = 1 \times 10^{-6}, \; x'' = 0) \) the behavior of a trajectory close to the stationary state can be studied. The dotted lines indicate instants at which the mass assembly makes and loses contact with both the walls when the mass is positioned at \( x' = x'' = 0 \).

In Figs. 5(b) and 5(c) two blowups are shown for two such times. One can observe that the point of losing and making contact are well defined. These figures indicate that it should be a good approximation to assume that the stability depends mainly on the two modes: free flight and contact with both walls, for trajectories close to the stationary state. Moreover, the duration for each mode can be calculated from the points where the system establishes contact and loses contact with both walls. In the low-frequency region the sequence of modes is as follows. The system goes from contact with both walls into free flight followed by a short duration of contact with one of the walls and then directly into contact with both walls. During each period of the walls there are two main modes of interest: (i) the free flight mode, which causes expansion in the system; and (iv) the mode of contact with both walls, which causes contraction toward the stationary state. The assumption here is that it is the balance between these two modes that determine if the system is stable or unstable. The narrow region of contact with one of the walls is neglected. With this assumption, the stability of the stationary state can be determined analytically. The duration of these modes can be calculated geometrically by analyzing the motions close to the stationary state. Knowing that the displacement is zero for the stationary solution, the nondimensional diameter of the assembly is unity and that the motion of the walls are prescribed by \( x_1' \), \( x_2' \), the times of making and losing contact with both walls can be calculated.

Thus, the duration of free flight, mode (i), is given by

\[
t'_i = \frac{1}{\omega'} \arccos \left( \frac{0.5 - t''}{a_i} \right),
\]

and the duration of contact with both walls, mode (iv), by

\[
t'_w = \frac{2\pi}{\omega'} - t'_i.
\]

Choosing the phase of the Poincaré section to be the point where contacts with both walls are lost, \( \phi_p = (t'_i + t'_w) \omega' \), the complete map can be constructed by

![FIG. 4. Bifurcation diagram for \( c = 0.8, \; \zeta = 0.01 \). (a) \( \omega' = [0.014 \; 0.014 \; 0.320] \); (b) blowup of the lower right corner in (a), \( \omega' = [0.014 \; 0.320 \; 0.14323] \).](image)

![FIG. 5. Motion for the system: \( c = 0.8, \; \zeta = 0.01 \) with the initial conditions \( x' = 1 \times 10^{-6}, \; x'' = 0 \) for a frequency \( \omega' = 0.015 \) when the stationary state is unstable. In (a) the time history is shown for \( t' = [0, \; 2000] \), and in (b) and (c) blowups are shown for a point of making and losing contact with both walls, respectively.](image)
combining two maps: free flight and contact with both walls. In matrix form, the map of free flight, \( J_t \), can be written as
\[
\begin{bmatrix}
\dot{x}' \\
\dot{y}'
\end{bmatrix}_{\phi_0} = [J_t] \begin{bmatrix}
x' \\
y'
\end{bmatrix}_{\phi_0 + t' \omega'}.
\] 
(15)

To find the map of contact with both walls, \( J_{iv} \), the equation of motion must first be solved. The equation of motion for the phase of contact with both walls is found from Eqs. (8)–(11), which yield an equation for damped oscillations,
\[
x'' + 4 \zeta x' + 2 x' = 0.
\] 
(16)

Solving (16), we obtain the map for the region of contact with both walls, \( J_{iv} \). It is given by
\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix}_{\phi} = J_{iv} \begin{bmatrix}
x' \\
y'
\end{bmatrix}_{\phi_0} = e^{-2 \xi t_{iv}} \begin{bmatrix}
2 \xi \sin(\omega_d t_{iv}) + \cos(\omega_d t_{iv}) \\
\sin(\omega_d t_{iv}) - \frac{2 \xi \sin(\omega_d t_{iv})}{ \omega_d}
\end{bmatrix} \begin{bmatrix}
x' \\
y'
\end{bmatrix}_{\phi_0} = e^{-2 \xi t_{iv}} \begin{bmatrix}
j_{11} \\
j_{12}
\end{bmatrix} \begin{bmatrix}
x' \\
y'
\end{bmatrix}_{\phi_0}.
\] 
(17)

where the frequency of damped oscillations,
\[
\omega_d = \sqrt{2} \sqrt{1 - 2 \zeta^2},
\] 
(18)

has been used. Notice that \( t' \omega' \) is a constant phase. The Poincaré map \( J \) is found by combining the map of free flight, \( J_t \), with the map of contact with both walls, \( J_{iv} \). Thus
\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix}_{t_{iv} + n \tau} = [J_{iv}] \begin{bmatrix}
x' \\
y'
\end{bmatrix}_{t_{iv} + (n-1) \tau} = e^{-2 \xi t_{iv}} \begin{bmatrix}
j_{11} \\
j_{12}
\end{bmatrix} \begin{bmatrix}
x' \\
y'
\end{bmatrix}_{t_{iv} + (n-1) \tau},
\] 
(19)

where
\[
\begin{align*}
j_{11} &= \frac{2 \xi}{ \omega_d} \sin(\omega_d t_{iv}) + \cos(\omega_d t_{iv}), \\
j_{12} &= \left( \frac{2 \xi}{ \omega_d} \sin(\omega_d t_{iv}) + \cos(\omega_d t_{iv}) \right) t' + \frac{\sin(\omega_d t_{iv})}{ \omega_d}, \\
j_{21} &= -\left( \frac{4 \xi^2}{ \omega_d} + \omega_d^2 \right) \sin(\omega_d t_{iv}), \\
j_{22} &= -\left( \frac{4 \xi^2}{ \omega_d} + \omega_d^2 \right) \sin(\omega_d t_{iv}) t' + \cos(\omega_d t_{iv}).
\end{align*}
\] 
(20)

Since the fixed point under consideration is stationary at \( x' = \dot{x}' = 0 \), the map (19) also describes the map for the perturbed solution,
\[
\begin{bmatrix}
\delta x' \\
\delta x'
\end{bmatrix}_{t_{iv} + n \tau} = [J] \begin{bmatrix}
\delta x' \\
\delta x'
\end{bmatrix}_{t_{iv} + (n-1) \tau},
\] 
(21)

Assuming that the eigenvalues of \( J \) are distinct, the orbit is described by
\[
\begin{bmatrix}
\delta x' \\
\delta x'
\end{bmatrix}_{t_{iv} + n \tau} = \lambda_n^n \begin{bmatrix}
e_1 \\
e_2
\end{bmatrix},
\] 
(22)

where \( \lambda_n = \cos(\omega_d t_{iv}) - \frac{4 \xi^2}{ \omega_d} + \omega_d^2 \sin(\omega_d t_{iv}) t'_i \),
\[
\Lambda = \cos(\omega_d t_{iv}) - \frac{4 \xi^2}{ \omega_d} + \omega_d^2 \sin(\omega_d t_{iv}) t'_i \]
\] 
(24)

The corresponding eigenvectors are given by
\[
\begin{bmatrix}
e_1 \\
e_2
\end{bmatrix}_{n \tau} = \begin{bmatrix}
1 \\
\lambda_n - j_{11} e^{-2 \xi t_{iv}}
\end{bmatrix}, \quad n = 1, 2.
\] 
(25)

In Fig. 6, the two eigenvalues (solid and dashed line) are shown for the case with \( c = 0.8, \xi = 0.01 \) in the interval \( \omega' = [0, 0.015] \). The dotted lines mark the boundary of stability characterized by eigenvalues with absolute values equal to unity. One can observe that one of the eigenvalues is oscillating with an amplitude that increases with the frequency, while the other eigenvalue is close to zero. Due to this oscillation, regions of stationary state can be found between the unstable regions. This feature was observed in the bifurcation diagram of Fig. 3, where stationary states existed between windows of periodic and chaotic motion. Comparison of Fig. 6 with the bifurcation diagram of Fig. 3 shows that not all of the stable stationary states can be seen in the bifurcation diagram. The reason is that numerical simulations can only follow a single branch, where multiple solutions
may exist. The missing stationary states can be reached by redoing the calculations, using other initial conditions.

In Figs. 7(a) and 7(b) points of stable stationary solutions are shown as black dots in the $\zeta$, $\epsilon$ plane for two wall frequencies: $\omega' = 0.05$ in (a) and $\omega' = 0.10$ in (b). These figures show that the region of stability decreases with increased frequency and that stable solutions are concentrated at high damping and high concentration. It is interesting to observe the fine structure with stripes of solutions that are stable, even for cases of low damping and low concentration.

In Fig. 8 stability diagrams are shown in the $\xi$, $\omega'$ plane. The regions of isolated points in Fig. 8(a) are due to a finer structure than what could be observed at this resolution. This finer structure is shown in the blowup in Fig. 8(b). A similar diagram in $c$, $\omega'$ plane is shown in Figs. 9(a) and 9(b). In order to understand the structures in the stability diagrams, a more detailed analysis of the bifurcations of the stationary state is needed.

### B. Undamped case

For the undamped case, $\zeta = 0$. Equation (23) is simplified to

$$\lambda_1 = (\Lambda + \sqrt{\Lambda^2 - 1}),$$

$$\lambda_2 = \lambda_1 - \sqrt{\Lambda^2 - 1},$$

$$\Lambda = \cos(2\omega') - \sqrt{2} \sin(2\omega') \omega'. \quad (26)$$

From the above equation, one can observe that complex eigenvalues will be obtained when $-1 < \Lambda < 1$. Therefore the analysis is divided into three regions: $\Lambda < -1$, $-1 < \Lambda < 1$, and $\Lambda > 1$.

1. **Case of $\Lambda < -1$**

When $\Lambda < -1$ the second eigenvalue, $\lambda_2$, will always be unstable. The first eigenvalue, $\lambda_1$, will always be less than zero, since $|\Lambda| > \sqrt{\Lambda^2 - 1}$. The remaining task is to determine if $\exists \Lambda$, such that $\lambda_1 < -1$. Assume $\exists \Lambda$, such that $\lambda_1 < -1$, and find these values of $\Lambda$:

$$-1 > \Lambda + \sqrt{\Lambda^2 - 1} \Rightarrow \Lambda > -1,$$

which leads to a contradiction. This implies that $|\lambda_1| < 1$ and $|\lambda_2| > 1$ if $\Lambda < -1$.

2. **Case of $-1 < \Lambda < 1$**

For the case of $\Lambda = -1$, it is clear that both eigenvalues will be equal to $-1$, and for $\Lambda = 1$ both eigenvalues will be equal to $1$. But for all $-1 < \Lambda < 1$ the eigenvalues will be complex. The stability is then determined by analyzing if the absolute value of the eigenvalues are inside the unit circle. The absolute value of the eigenvalues is found as

$$|\lambda_1| = |\lambda_2| = \sqrt{1 - \Lambda^2} = 1.$$

This implies that $|\lambda_1| = |\lambda_2| = 1$ if $-1 < \Lambda < 1$, and hence all points in the interval will be on the boundary of stability.

3. **Case of $\Lambda > 1$**

When $\Lambda > 1$, the first eigenvalue, $\lambda_1$, will always be unstable. The second eigenvalue, $\lambda_2$, will always be greater
than zero since \( |\Lambda| > \sqrt{\Lambda^2 - 1} \). The remaining task is to determine if \( \Lambda > A \), such that \( \lambda_2 > 1 \). Assume \( A \), such that \( \lambda_2 > 1 \), and find these values of \( A \):
\[
1 < A - \sqrt{\Lambda^2 - 1} = \Lambda < 1,
\]
which leads to a contradiction. This implies that \( |\lambda_1| > 1 \) and \( |\lambda_2| < 1 \) unless \( \Lambda > 1 \).

C. Damped case

For the damped case, only small corrections are needed for the analysis of the undamped case. In Eq. (23), there is an additional factor \( e^{-2\omega t} \). This factor will always reduce the eigenvalues, since
\[
\zeta > 0 \quad \text{and} \quad t' = 0 \Rightarrow e^{-2\omega t} < 1.
\]
Therefore the boundary of stability will be crossed for \( |\Lambda| > 1 \). For the damped case of Sec. IV B 1, the boundary of stability will be crossed when
\[
1 = e^{-2\omega t} (A - \sqrt{\Lambda^2 - 1}),
\]
which gives
\[
\Lambda = \left(1 - e^{-2\omega t} + \frac{1}{2}ight).
\]
The damped case of Sec. IV B 3 gives the same result, with opposite sign. Therefore, the damped system bifurcates when
\[
|\Lambda| = \left(1 - e^{-2\omega t} + \frac{1}{2}ight),
\]
and has unstable stationary state for
\[
|\Lambda| > \left(1 - e^{-2\omega t} + \frac{1}{2}ight).
\]
Otherwise the stationary state will be stable.

D. The parameter \( \Lambda \)

For the case of low damping, assuming \( \omega d' = \sqrt{2} \) and \( \xi^2 \approx 0 \), Eq. (24) is simplified to
\[
\Lambda = \cos(\sqrt{2} t') - \frac{t'}{\sqrt{2}} \sin(\sqrt{2} t').
\]
The time \( t' \) is large in the region of interest (low-frequency range) when compared with the cosine term in Eq. (35). Therefore, due to the oscillation of the sine and cosine, \( \Lambda \) will periodically satisfy Eq. (34) and cause instability. When damping is included, all \( |\Lambda| \leq 1 \) will be stable. Equation (35) shows that there will always exist stable regions for
\[
\sqrt{2} t' = n \pi, \quad n = 0, 1, 2, \ldots
\]
The time \( t' \) is a function of the frequency and concentration according to Eqs. (13) and (14). If only one of them is varied in the stability diagram, then lines of stable region will occur when condition (36) is satisfied. This explains the thin lines in the stability diagrams (Figs. 7 and 8), where one of the parameters was varied, and also the thin curves (Fig. 9) when both parameters were changed.

V. CHAOS

In this section the attractors are studied in the frequency regions of long-periodic or chaotic motions. The aim is to explore the development of the attractor at the lowest frequencies and to search for transversal homoclinic connections between the stable and unstable manifold. Figure 10 presents the attractors of the first five long-periodic or chaotic regions given in Fig. 3. The last figure, Fig. 10(f), shows the attractor at a slightly higher frequency. For the system, \( c = 0.8, \xi = 0.01 \), these Poincaré sections are obtained for the phase, where contacts with both walls are lost, \( \phi_p = (t'/2 + t') \omega' = 4.72785 \), and for the forcing frequencies
\[
(a) \omega' = 0.01378, \quad (b) \omega' = 0.0139, \quad (c) \omega' = 0.014055, \quad (d) \omega' = 0.014195, \quad (e) \omega' = 0.014338, \quad \text{and} \quad (f) \omega' = 0.015300.
\]
The figures show how the attractor grows with increasing frequency to form a horseshoe-like structure. Every second region forms a double attractor mirrored through the stationary state. In Fig. 10(e), a horseshoe-like attractor can be observed.

To analyze if this attractor is of the Smale horseshoe type, the Poincaré map from one Poincaré section, for the phase \( \phi_p = \phi_p \), to the next is studied numerically. At this Poincaré section, a set of equidistant points are chosen to cover the attractor of Fig. 10(e). See Fig. 11(a). In Figs. 11(a)–11(d) three points are marked, so that they can be

![FIG. 10. Poincaré sections at the phase \( \phi_p = (t'/2 + t') \omega' \) for \( c = 0.8, \xi = 0.01 \) at different frequencies.](image)
followed in the Poincaré sections. The points are the lower left corner marked “X,” one point in the center marked “O,” and the point in the upper right corner marked “*.” Recall that the phase $\phi = \phi_p$ is at the point where the stationary state loses contact with both walls. After this phase, the motion will be free flight so the set of points will be stretched to larger displacements into a long thin strip, as shown in Fig. 11(b) for the phase $\phi = 1.5$. At a short instant later, the points of highest velocity will get into contact with the upper wall. In this phase, the displacement will decrease and the velocity will change sign. In Fig. 11(c) the phase where the stationary state gets into contact with both walls is chosen to show how the impact influences the set. At this phase all points are impacting with the upper wall, and points with the lowest velocity are the ones that have been in contact for the longest time. The absolute value of the velocity, of the points that have been in contact with the upper wall for the longest time, is increased due to the additional momentum received from the wall. In this phase, the thin strip of points becomes folded, due to the impact with the upper wall. This phase is followed by a phase of contact with both walls. The set of points rotates around and converges toward the stationary state, in damped oscillations, until the next Poincaré sections at phase $\phi = \phi_p + 2\pi$ in Fig. 11(d). Analyzing the map for each point from Figs. 11(a)–11(d), one can observe that it is only the velocity distribution in (a) that governs where each point will be mapped in (d). Points with equal velocity in (a) will be mapped close to each other in (d), while points with different velocities will be stretched apart. The conclusion is that a horseshoe-like structure can be formed in the Poincaré sections when the motion is of a long period or chaotic nature. The development is similar to the known horseshoe described by Smale.16-17 For the attractors at low frequencies, Figs. 10(a)–10(f), the rotation in the phase of contact with both walls accounts for the existence of a single horseshoe for some frequencies and a double horseshoe for some others. If the rotation ends in the same quadrant as the initial set a single horseshoe will be observed. Otherwise, the initial set in quadrant 1 will be mapped to quadrant 3 by the Poincaré map and back to quadrant 1 in the next Poincaré map. For higher frequencies, Fig. 10(f), the stationary state will be included in the attractor, and hence both parts of the attractor will be connected.

For chaotic systems, the necessary condition of stretching and folding is often proven by confirming the existence of transversal homoclinic points. These transversal homoclinic points imply that there exists an embedded horseshoe-like map in the map $\Sigma$. In this case, these homoclinic points are searched by numerical simulation of the stable $W^s$ and unstable $W^u$ manifold of the saddle point at the stationary state. In the direction of each eigenvector, a set of points is put close to the stationary state. By forward and backward iteration the manifolds are detected, with the method described in Ref. 19. In Figs. 12(a) and 12(b) two figures are shown of the manifolds at two different frequencies, $\omega^s = 0.014338$ and $\omega^u = 0.015300$, respectively. In (a), the manifolds are almost touching each other, and in (b) the
manifolds intersect transversally. In the figures the stable manifold $W^s$ is the horizontal lines and the unstable manifold $W^u$ is the $s$ curve. Starting with a set of points close to the stationary state, the unstable manifold will be stretched in the direction of the attractor, and after a few iterations the following orbits will be on the attractor. Parts of the stable manifold is found by running the time backward for a few iterations. Starting with a set of points close to the stationary state the stable manifold expands horizontally. In case (a) no intersections occur between the stable and unstable manifolds, whereas in case (b) the two intersect transversally. The numbers (with sign) near the end of the lines mark the direction for continuation of the stable manifold, as it oscillates back and forth over the figure. An interesting observation is that no transversal homoclinic points are found in Fig. 12(a), which corresponds to long periodic or chaotic motions, as observed confirmed in Figs. 10 and 11. However, transversal homoclinic points are observed in Fig. 12(b), where the frequency is just slightly higher. There is a strong attraction toward the stationary point during the phase of contact with both walls. Consequently, a strong expansion is expected when the system is simulated backward in time. The stable manifold can therefore be determined with good accuracy for only a limited time. The long-time behavior of the stable manifolds has not been established. Hence, one cannot be sure that the manifolds of Fig. 12(a) will not intersect if it is possible to simulate the system for a still longer duration.

VII. CONCLUSIONS

This simple one-dimensional model shows a rich variety of motions, including periodic and chaotic motions, even for frequencies as low as about 1% of the lowest natural frequency of the system. (See the bifurcation diagram in Fig. 3.) From this diagram, it can be observed that for the lowest frequencies the motion of the mass will be damped toward a stationary state at $x = x' = 0$. The reason for this stationary state is that, for high concentrations and low wall frequencies, the time of contact with both walls will be long compared with the natural period. The dashes will then damp out the oscillations during the phase of simultaneous contact with both walls. When the frequency is increased, this stationary state will bifurcate into periodic and chaotic motions.

By observations of time histories of the mass assembly, at these low wall frequencies, reasonable assumptions about the motion can be made to facilitate the stability analysis. The observation is that the mass will follow a sequence of free flight—contact with one of the walls—contact with both walls. The duration of contact with one wall is short (compared with the time in the other modes) and directly followed by the mode of contact with both walls. The assumption is that the stability depends only on the balance between an expansion from the stationary state (free flight) and a contraction toward the stationary state (contact with both walls). The duration in each mode can then be determined by assuming that the displacement is close to zero during the whole forcing cycle. The stability analysis is compared with numerical simulations to confirm the validity of the assumptions. From the results of this stability analysis, one can conclude that stable regions will mainly be found at high concentration and high damping. Furthermore, when the frequency is increased, the region of stability shrinks, as expected. The oscillating eigenvalues cause thin regions of stability separated by periodic and chaotic windows. These thin regions can be observed in all the stability diagrams. Analysis of Eq. (36) shows that such regions occur at specific parameter combinations of concentration and wall frequency.

In the analysis of the bifurcations, it is found that the stationary state will always bifurcate into a saddle point. These saddle points are of interest, since they form a necessary condition for the Smale horseshoe map. For the undamped case, no stable solutions exist, but for some parameter combinations the eigenvalues are on the boundary of stability. For the damped case there will always exist parameter combinations, such that stable stationary states are generated. Observation of Poincaré sections in the long-periodic or chaotic windows shows that with increasing frequency the attractor grows to a horseshoe-like shape (see Fig. 10). Numerical simulation of the Poincaré map, from the point of losing contact with both walls, shows that a rectangular region of points will (i) be stretched into a long thin strip, due to the constant velocity in the region of free flight; (ii) be folded in the region of impact with one of the walls to the shape of a horseshoe; and (iii) rotate and converge toward the stationary state in the region of contact with both walls, so that the set of points is mapped into the original rectangle.

By studying the stable and unstable manifolds of the stationary state, transversal homoclinic points can be found for some parameter ranges. This shows that for some parameters the map of long-periodic or chaotic regions is similar to the Smale horseshoe map in the low-frequency region.

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Experimental and numerical studies of shear layers in a granular shear cell

By Jan-Olov Aidanpää¹, Hayley H. Shen² and Ram B. Gupta³

ABSTRACT: The stability of a shear layer inside a granular material in a gravity field is studied experimentally and numerically. A shear cell is built of transparent acrylic to visualize the motion of the granular material. This shear cell consists of two concentric cylinders containing layers of uniform spheres in the annular space between the cylinders. The shearing motion of the spheres is produced by rotating the bottom boundary of the cell. Friction of the cylinder walls resists the shear motion, thus creates a single shear layer adjacent to the bottom boundary, while the rest of the layers above move with constant speed as a solid body. As the rotation speed of the bottom boundary increases, two layers adjacent to the bottom boundary begin to shear. This shearing zone quickly thickens and dilates as the rotational speed increases. The transition of this shear motion from a single layer to many layers shearing is studied by video recording. The initiation of this transition is observed to depend on the material properties and the number of layers overlain the shear layer. A one-dimensional numerical model is constructed to bring insight into this transitional phenomenon.

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INTRODUCTION

Granular materials display both solid-like and fluid-like behaviors. For a given granular material, the constitutive relation depends on the state of motion. At very low deformation rates, stresses are rate-independent. The granular material is solid-like. When the deformation rate increases, the stresses become rate-dependent. The granular material becomes fluid-like. In one extreme, when a granular material is under slow shear, Mohr-Coulumb law has successfully described stresses generated in this state. This approach has been widely applied to describe many hopper flows. In the other extreme, when a granular material is in rapid flows, recent research effort has resulted in a significant understanding of its constitutive relation. For a broad overview of granular flows see Campbell (1990), Savage (1984, 1989) and Jenkins (1987).

Strict solid or fluid phases are not the only ways that granular materials can behave. In many situations both fluid-like and solid-like behavior may occur simultaneously, such as a funnel flow in hoppers. When transition from fluid-like to solid-like behavior occurs, undesired phenomena such as clogging are not unusual. One way to predict granular flow behavior is by molecular dynamics simulations (Campbell, 1989, 1990; Hopkins et al., 1991; Savage, 1992; Walton et al., 1986, 1991). With this technique the motion of all particles and their interaction can be followed in detail. Experimental devices (annular shear cells) for testing dry granular materials have been developed by Novosad (1964), Bridgewater (1972) and Savage (1978). All experiments had a rotating bottom while the top was fixed. Shear stress was calculated by measuring the torque at the top. In the experiments by Novosad (1964) and Bridgewater (1972) the normal stress was fixed by loading at the top and the granular material could expand vertically. These experiments showed small variation of the shear stress with the shear rate. Savage (1978) performed experiments for granular materials under a fixed concentration. Savage and Sayed (1980, 1984) and Sayed (1981) continued these experiments with more extensive studies. Their results showed that normal and shear stresses were dependent on the square of the shear rate.

Experimental studies of velocity profiles in vertical parallel sided channels of rectangular cross section have been performed by Toyama (1970/1971), Takahashi and Yanai (1973), Savage (1979) and Nedderman and Laohakul (1980). All of these experiments showed a velocity profile with a rigid plug flow core and shear zones at the boundaries. Many of these studies also found that the shear zone to be about 4 to 8 particle diameters thick for a wide range of materials. Savage (1979) showed that for narrow channels the shear zone spanned over the whole width but for wider channels the plug flow was commonly observed. Bridgewater (1980) suggested an analytical explanation of why the shear zone is about 10 particle diameters thick. Other cases of this solid- and fluid-like flow are found in snow avalanches where measurements from the field (Dent and Lang, 1982) have shown that the major deformation takes place at
the base of the avalanche, resulting in large velocity gradients at the base, while the top part moves as a solid body.

Because of the co-existence of the fluid-like and the solid-like behavior, knowing how to describe the two extreme cases of granular flows is not sufficient. For the purpose of prediction and control of many industrial and geological processes, we need to know the criterion that determines when a granular material will switch from the slow state to the rapid state.

Bagnold (1954) was the first to formulate a theory for granular materials that can switch from one constitutive law to the other. His study was for a granular mixture consisting of a viscous fluid and neutrally buoyant solids. The corresponding flows are either in the "macro-viscous" regime, where the stress depends on the first power of the strain-rate, or the rapid flow regime, where a square dependence is observed. The criterion for the transition from one to the other is defined by the so-called Bagnold number, which, like Reynold's number, represents a ratio of inertial to viscous effects. As this number increases, the granular material switches from the macro-viscous regime to the rapid flow regime. For dry granular flows, the fluid viscosity is much less dominant than the interparticle forces. Yet rate-independent constitutive law exists in slow shear. Therefore for dry granular materials, there must exist other criterion for the transition from one regime to the other. Furthermore, in the dry case, the range of power law dependence between the stress and the strain-rate is much greater than that for a liquid/solid mixture.

Evidence of the transition from one regime to other in a dry granular flow and the co-existence of both regimes in the same granular flow has been found in a number of recent studies. Zhang and Campbell (1992) investigated a shear flow of uniform disks in a gravity field by a computer simulation. They found that the flowing material consisted of a rapid shear zone at the top, adjacent to the moving wall, and a solid non-moving zone at the bottom, adjacent to the fixed wall. In a laboratory experiment of a shear cell, Craig (1994) observed a sudden expansion of the granular material as the shear rate increases. This expansion is accompanied by a sudden change of the stress level. Although the shear cell is opaque thus prevents direct observation of the internal motion, this expansion strongly suggests a change of the flow characteristics. Babic et al. (1990) studied a simple shear motion of uniform disks without gravity in a computer simulation and found that concentration plays a role in the flow regime. For fixed shear-rate and material properties, at a very high solid concentration, the shearing motion only occurred in one shear layer, sandwiched between two blocks formed by adjacent materials. In this state, the stresses were rate-independent. By decreasing the concentration, random motion and collisions among grains resulted in uniform shearing of the whole granular mass. In this state, the stresses where rate-dependent.
The fact that granular materials can have various constitutive laws makes it important to determine which parameters control the transition, and how these parameters are inter-related in controlling the transition. In this study, a shear flow created by a transparent shear cell is investigated. The cell consists of two concentric acrylic (Plexiglas) cylinders with an annular space slightly larger than the diameter of the spheres that form the granular material. These spheres pile up inside the shear cell vertically. The top of the material is a free surface. The shear motion is created by rotating the bottom boundary of this cell. The layers of uniform spheres above this boundary are resisted by the wall friction. In this way, only the materials next to the bottom boundary shear at first. As the rotational velocity increase, the stability of this shear layer is destroyed and a chaotic motion involving many layers of spheres begins. The stability of the shear layer is studied for different materials and under different amount of mass overlain the shearing bottom. Video recording of the detailed flow kinematics is used to determine the transitional behavior as the rotational velocity increases. The main interest is to observe the onset of the "rapid flow" condition associated with the disintegration of the orderly single layer shear, and the evolution into a chaotic, collisional state.

A one-dimensional mathematical model is proposed at the end to yield analytic insight into the transitional behavior. This mathematical model is a candidate for further investigation, aimed at obtaining a quantitative criterion for the transitional behavior in a granular flow, where gravity force is significant.

The study reported here addresses one question. That is, in shearing granular materials under gravity force, such as at the bottom of an avalanche, what types of flow do we have? A frictional slide with very narrow shear zone of a few layers of grains, or a chaotic shear zone with many layers of grains? This study investigates the possibility of both regimes, and the dependence of the regime on the material properties and the total mass overlain the sheared bottom.

**EXPERIMENT**

Fig. 1 shows the experimental setup with a blow up of the lower rotating boundary. The granular material, consisting of identical spheres, is located between two fixed, concentric cylinders, which were made of a transparent acrylic material (Plexiglas). The mean diameter between the cylinders is 0.081 m and the gap between the cylinders is 0.0045 m, which is slightly larger than one sphere diameter (0.0040 m). Uniform spheres of various materials are packed at the highest concentration inside the gap between the cylinders. The number of layers of spheres varies from 10 to 30 in steps of 5. The cylinders are placed vertically in the gravity field and the shearing motion is performed with the help of a rotating bottom which is driven by a variable speed motor.
The rotating bottom is roughened by gluing spheres identical to those in the cell. The speed of rotation is measured with a tachometer. The experiments are recorded with a video camera. In each experiment the granular material is initially packed densely without any voids. The speed is slowly increased from 100 rpm in steps of 50 rpm. After each step the motion of the granular material is recorded for 3 minutes. In order to ensure repeatability, the whole setup is disconnected and then put together again, after each experiment.

Fig. 1. Experimental Setup of the Annular Shear Cell
A typical scenario of one such experiment can be described as follows. At low rpm, a single layer of the spheres just above the bottom moves at half of the speed of the bottom boundary. All mass above this single shear layer remains structured and nearly stationary. A snapshot of this stage is shown in Fig. 2(a). Visible vibration of small amplitude is present in all layers. Increasing the rpm, vertical vibration in all layers increases. Most noticeable is the topmost layer. Spheres in this top layer bounce up and down at random. Further increase of the rpm, leads to a stick-slip motion in the second layer just above the boundary. From there, a small increase of the rpm causes a chaotic state, which involves many layers adjacent to the rotating boundary. A snapshot of this situation is shown in Fig. 2(b). In this chaotic state, the layering structure next to the rotating boundary disappears. The evolution from a single layer shearing to a chaotic, multiple layer shearing occurs in a narrow range of rpm values. Occurrence of such a qualitative change is here in after termed as a transition. An important parameter influencing the transitional behavior is the number of layers overlain the bottom boundary. In the case of 10 layers, all of the spheres are mixing in an irregular fashion. When the number of layers is increased to 20 and above, only a few layers of spheres adjacent to the rotating boundary participate in the mixing. All other spheres on the top remain layered. As the number of layers increases, the rotational velocity for transition to initiate also increases. Furthermore, the chaotic mixing region becomes smaller. When the number of layers goes beyond 40, the rotational velocity required to cause the transition suddenly jumps to a much higher value. In some cases no transition could be observed in the test range.

Fig. 2. Snapshot from the Video Recording, (a) Single Shear Layer and (b) Multiple Layer Shearing
The main results from this experiment will be shown in terms of the velocity distribution and the transition frequency for different materials and number of layers in the shear cell. In order to follow the motion of the spheres, a vertical line was drawn on the outer cylinder. The horizontal velocity and vertical position of spheres passing this line were measured every third second. Ten measurements, on the bottom boundary and ten layers above it, were made for each test. Fig. 3 shows velocity profiles for ten layers of cellulose acetate at four different rotational velocities: (a) 100 rpm (b) 200 rpm (c) 300 rpm and (d) 400 rpm. In (a) and (b) the shearing layer moves with a velocity which is nearly half of the bottom boundary's velocity and the layers above the shearing layer are almost stationary. In Fig. 3(c) the stick-slip motion of the layer just above the shear layer is apparent. The vertical vibration of all layers except the first two layers next to the bottom can also be seen. On increasing the rotational velocity, the layered structure is broken down and the particles start to mix between the layers, as seen in (c). At higher velocities shown in (d), the spheres are still mixing and large spread in the horizontal velocities can be observed for all layers.

![Fig. 3. Experimental Velocity Profile for 10 Layers of Cellulose Acetate at Rotational Velocities: (a) 100 rpm, (b) 200 rpm, (c) 300 rpm and (d) 400 rpm](image)

In Fig. 4 the number of layers is increased to 20, and the rotational velocities are:
(a) 200 rpm (b) 400 rpm (c) 600 rpm and (d) 800 rpm. Again, in (a) and (b) the shearing layer is moving with a velocity which is about half of the wall velocity and the layers above the shearing layer have approximately zero displacement and velocity. In (c) the layered structure is broken for the first 3 or 4 layers above the rotating boundary. All other layers remain structured. In (d) the same layers are still mixing but with a more random like velocity distribution. Similar results were observed for other materials. An error estimation of the measurements in Figs 3 and 4 gives an error in position < 0.0005 m and an error in velocities < 0.1 m/s.

![Graphs showing velocity profile](image)

**Fig. 4.** Experimental Velocity Profile for 20 Layers of Cellulose Acetate at Rotational Velocities: (a) 200 rpm, (b) 400 rpm, (c) 600 rpm and (d) 800 rpm

A series of experiments were performed to determine the critical rotational velocities, i.e., the velocities required to initiate the transition. In each case the wall velocity at which the layer above the shearing layer starts to slip is noted. Two types of boundaries were studied. In one case, the bottom boundary was prepared by gluing the spheres as densely as possible. It required 63 spheres. In the second case, 58 spheres were glued randomly to the boundary. Fig. 5 shows the dependence of critical rotational velocity on the number of layers. Both boundary types are studied for four different materials, (a) polystyrene, (b) polypropylene, (c) acrylic and (d) cellulose acetate. The
dotted line represents the average for the case of 58 spheres and the mark "x" shows individual measurement. The solid line represents the average for the case of 63 spheres and the mark "o" shows individual measurement. In all cases the critical rotational velocity increases as the number of layers is increased. When the number of layers is about 40, the increase in the critical velocity becomes more pronounced. For the dense bottom case with 40 layers no data points are shown. The critical velocity in this case lies outside the studied range for the majority of the studied materials.

![Graphs showing critical velocities as a function of the number of layers for different materials.](image)

Fig. 5. Critical Velocities as Function of the Number of Layers for the Studied Materials

Separate experiments were performed to determine the material parameters, namely the mass, stiffness, damping constant and the coefficient of restitution for the four different materials. These experiments with estimated errors are described in Appendices I and II and the values are listed in Table 1 below. Here \( m \) denotes the mass of one sphere, \( \alpha \) is the coefficient of restitution, \( \zeta \) is the damping ratio and \( k \) is stiffness. The two parameters \( \zeta \) and \( k \) are used to model the impact process between two spheres, as will be discussed in the next section. In the model a linear spring with constant stiffness is used. The stiffness of a sphere, however, depends on the load. Assuming the load to be the weight of the layers above the shearing layer, \( k_{10} \) and \( k_{30} \) represent the stiffness for the cases of 10 layers and 30 layers respectively. The names of the materials
are shortened to PS for polystyrene, PP for polypropylene, AC for acrylic and CA for cellulose acetate.

<table>
<thead>
<tr>
<th>material</th>
<th>m [kg]</th>
<th>α</th>
<th>ζ</th>
<th>$k_{10}$ [N/m]</th>
<th>$k_{30}$ [N/m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>PS</td>
<td>$3.67 \times 10^{-5}$</td>
<td>0.92</td>
<td>0.019</td>
<td>$3.92 \times 10^{4}$</td>
<td>$5.67 \times 10^{4}$</td>
</tr>
<tr>
<td>PP</td>
<td>$2.98 \times 10^{-5}$</td>
<td>0.79</td>
<td>0.052</td>
<td>$1.61 \times 10^{4}$</td>
<td>$2.32 \times 10^{4}$</td>
</tr>
<tr>
<td>AC</td>
<td>$4.11 \times 10^{-5}$</td>
<td>0.90</td>
<td>0.024</td>
<td>$3.72 \times 10^{4}$</td>
<td>$5.36 \times 10^{4}$</td>
</tr>
<tr>
<td>CA</td>
<td>$4.11 \times 10^{-5}$</td>
<td>0.84</td>
<td>0.039</td>
<td>$2.50 \times 10^{4}$</td>
<td>$3.61 \times 10^{4}$</td>
</tr>
</tbody>
</table>

THE MODEL

The purpose of this section is to provide a theoretical framework for the transitional behavior observed in the above experiment. The motion of the shear layer is studied numerically by using a simple mechanical model to simulate the most important features of the shear cell.

As observed in the experiments the shear layer occurs adjacent to the bottom boundary, the wall. Its motion is influenced by the wall velocity, the number of layers, and the material properties of the spheres. Below the transitional velocity, the shear cell may be viewed as consisting of three parts: the moving wall, the shear layer and the rest of the layers above the shear layer. The wall is prescribed to move horizontally with a given constant velocity. The shear layer is moving with half of this velocity and the top mass is nearly stationary. The wall is in direct contact with the shear layer. Due to the bumpiness created by the glued spheres, this wall periodically pushes the shear layer over it. The motion of the shear layer is also influenced by the layers above it. Due to the relative motion between the shear layer and the top mass, the shear layer is also pushed downward by the top mass.

It is convenient to choose a moving reference frame which follows the motion of the shear layer in the horizontal direction. The origin of this frame in the vertical direction is fixed at the undisturbed center of the shear layer, or one diameter above the center of the spheres glued on the bottom wall. With respect to this reference frame, the shear layer is stationary in the horizontal direction and it vibrates in the vertical direction.
direction due to the excitations from the wall and the layers above it. The wall moves at a constant speed, and the loose granular top mass moves in the opposite direction at the same speed. The chaotic motion after the onset of the transition involves vigorous vertical motion. To determine the onset of transition, it is thus reasonable to study the vertical component of the motion first. Ignoring the horizontal component of the motion of a typical sphere in each layer, the vibrations in the vertical direction are studied by the model described below.

\[ x_i' = \frac{x_i}{D}, \quad t' = t \sqrt{\frac{k}{m}}, \quad \omega' = \omega \sqrt{\frac{m}{k}}, \quad \zeta = \frac{d}{2 \sqrt{km}}, \quad g' = \frac{mg}{kD}, \quad F\mu' = \frac{F\mu}{kD} \]  

Fig. 6. The N-degree of Freedom Model

Fig. 6 shows an N-degree of freedom model to simulate the mechanics of the shear cell. The reference frame described above is used and one sphere from each layer is modelled. Adopting the non-dimensional quantities,
for \( i=1,2, \ldots N \) and \( j=1 \) and 2, the Eq. of motion for the \( i \)th mass is

\[
\ddot{x}_i' = F_i' - F_{i+1}' - g' - F_{\mu_i}
\]

\( i=1,2, \ldots N \) and \( j=1 \) and 2,

where

\[
F_i' = \begin{cases} 
(y_1' + 1 - x_1') + 2\zeta (y_1' - \dot{x}_1') & ; x_1' - y_1' < 1 \\
0 & ; x_1' - y_1' \geq 1 
\end{cases}
\]

and

\[
F_j' = \begin{cases} 
(x_j' + 1 - y_j') + 2\zeta (x_j' - \dot{y}_j') & ; y_j' - x_j' < 1 \\
0 & ; y_j' - x_j' \geq 1 
\end{cases}
\]

and for \( j=3,4, \ldots, N \)

\[
F_j' = \begin{cases} 
(x_{j-1}' + 1 - x_j') + 2\zeta (x_{j-1}' - \dot{x}_j') & ; x_j' - x_{j-1}' < 1 \\
0 & ; x_j' - x_{j-1}' \geq 1 
\end{cases}
\]

and \( F_{N+1}' = 0 \). The friction force on each particle is

\[
F_{\mu_i}' = \begin{cases} 
\mu g' (N+1-i) & ; \dot{x}_i' > 0 \\
-\mu g' (N+1-i) & ; \dot{x}_i' \leq 0 
\end{cases}
\]

These equations imply that the springs and dampers generate force only when the sphere is in compression and the friction force \( F_{\mu_i}' \) is acting in the opposite direction of the velocity of mass \( i \). The friction force on sphere \( i \) is assumed to be linearly increasing with the number of layers above layer \( i \) and \( \mu \) is a constant. The friction forces were found to be the most difficult to define in the model and also to determine in
experiments. In the experiment, some spheres are sliding while others are rolling against each other and against the inner and the outer cylinder. Therefore the frictional forces are not simply proportional to the load. Another complication is that friction coefficients are difficult to measure, especially between two spheres. Therefore, a simple linear assumption was made to study the influence of friction.

The shear motion between the shear layer and the wall and the mass above is modeled by the vibrating wall \( y'_1 \) and a vibrating contact point \( y'_2 \). Since the mass above the shear layer is moving in the horizontal direction the shear layer will experience the mass above as a bumpy profile but the position of the contact point will depend on the position of the layer \( i=2 \). An approximation of the bumpy boundary profile is provided by the absolute value of a cosine function. Therefore the positions \( y'_1 \) and \( y'_2 \) are given by

\[
\begin{align*}
y'_1 &= -1 - 2a' \left( 1 - |\cos(0.5\omega' t')| \right) \\
y'_2 &= x'_2 + 2a' \left( 1 - |\cos(0.5\omega' t')| \right)
\end{align*}
\]

Recall that the choice of reference frame results in equal speed but opposite sign for the bottom wall and the layers above the shearing layer. Hence, the frequency \( \omega' \) is equal for \( y'_1 \) and \( y'_2 \) but in opposite phase. The amplitude \( 2a' \) is the distance from the peak to the bottom position of a sphere next to the bottom boundary made with densely glued spheres. Thus

\[
a' = \frac{1}{2} \left( 1 - \sqrt{\frac{3}{4}} \right)
\]

The frequency of the oscillation is related to the boundary rotational velocity \( V \) and to the frequency of the boundary rotation \( \omega_b \) in the following way:

\[
\frac{\omega' D}{\pi} \frac{k}{m} = V = \frac{\omega_b 0.081}{2}
\]

The motion of the system can now be simulated by solving these equations numerically. The stiffness \( k \) is estimated in Appendix I by using the weight of the upper layers (Nmg), as the force acting on the shear layer. This model is compared with the

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experiment by making a simulation that follows the experimental procedure. These "transitional diagrams" are similar to the numerical bifurcation diagrams described in e.g. Parker et al. (1989). To obtain these diagrams a set of initial conditions are chosen and the Eqs. of motion are solved, using the lowest value for the wall frequency. The transient part of the motion is neglected and the following 50 displacements at a constant phase (Poincaré section at the phase $2\pi$) of the bottom oscillation are plotted at the current frequency. By numerical experiments, the transient part was fixed at 80000 non-dimensional time units $t'$. The frequency is then increased by one step and the final state of the system from the previous frequency is taken as initial conditions for the new frequency. The same procedure is repeated for the next frequency. These transitional diagrams are of interest as they provide information about the kind of expected motions; i.e., stationary, periodic or chaotic. For each frequency in these diagrams, if the number of points is finite then it indicates that the motion is periodic under that corresponding frequency.

In order to make comparison with the experimental results, some physical criterion for the transitional behavior of the shear layer are need. It is reasonable to assume that the layered structure will disintegrate if the amplitude of vibrations of the shear layer becomes large. As the lower limit of this transitional behavior, the first visible place, in the transitional diagram, where the contact is lost between the shear layer and the wall is chosen. This point is when the displacement becomes larger than zero. As the upper limit of the transitional behavior, the frequency when the maximum vibration amplitude exceeds 10% of the sphere diameter $D$ is chosen. The 10% criterion is selected based on two facts. One, spheres must have vertical freedom exceeding at least 10% of the diameter to allow free horizontal passage between adjacent layers. This free passage between adjacent layers is a reasonable equivalence with mixing between adjacent layers - a phenomenon that occurs past the transition. Two, in the video recordings, only small vibrations, less than 10% of $D$, were observed before the transition. Beyond transition, greater than 10% vibrations begin to appear.

Using the material properties in Table 1, the transitional diagram can be generated for each material. Figs. 7 and 8 show the transitional diagrams for 10 and 30 layers respectively. Only the shear layer is plotted. In both figures, $\mu=0.1$ and (a) shows polystyrene, (b) polypropylene, (c) acrylic and (d) cellulose acetate. In these diagrams one can observe a periodic point at zero displacement for low frequencies (which corresponds to low shearing velocities). At these frequencies the shear layer will always be in contact with the wall at the studied phase. As the velocity was increased this periodic point loses its stability and starts to vibrate with displacements larger than zero. Further increase of the velocity results in long period or chaotic motion. It is expected that the experimentally observed transition will occur in the interval between this first break down frequency and the frequency at which the maximum displacement is more than 10% of $D$. 14
Fig. 7. Transitional Diagram for 10 Layers and $\zeta u=0.1$: (a) Polystyrene, (b) Polypropylene, (c) Cellulose Acetate and (d) Acrylic

Fig. 8. Transitional Diagram for 30 Layers and $\zeta u=0.1$: (a) Polystyrene, (b) Polypropylene, (c) Cellulose Acetate and (d) Acrylic
Table 2. Experimental and numerical transitional frequencies

<table>
<thead>
<tr>
<th>material</th>
<th>N</th>
<th>$c_\mu$</th>
<th>transitional frequency, $\omega'$</th>
<th>corresponding rpm’s</th>
<th>experimental rpm’s</th>
</tr>
</thead>
<tbody>
<tr>
<td>PS</td>
<td>10</td>
<td>0.01</td>
<td>0.0025-0.0045</td>
<td>25-44</td>
<td>340</td>
</tr>
<tr>
<td>PS</td>
<td>10</td>
<td>0.1</td>
<td>0.0045-0.0070</td>
<td>44-69</td>
<td></td>
</tr>
<tr>
<td>PS</td>
<td>30</td>
<td>0.01</td>
<td>0.00300-0.0125</td>
<td>35-148</td>
<td></td>
</tr>
<tr>
<td>PS</td>
<td>30</td>
<td>0.1</td>
<td>0.0175-0.0300</td>
<td>207-354</td>
<td>1074</td>
</tr>
<tr>
<td>PP</td>
<td>10</td>
<td>0.01</td>
<td>0.0045-0.0070</td>
<td>31-49</td>
<td></td>
</tr>
<tr>
<td>PP</td>
<td>10</td>
<td>0.1</td>
<td>0.0065-0.0140</td>
<td>45-98</td>
<td>400</td>
</tr>
<tr>
<td>PP</td>
<td>30</td>
<td>0.01</td>
<td>0.0045-0.0075</td>
<td>38-63</td>
<td></td>
</tr>
<tr>
<td>PP</td>
<td>30</td>
<td>0.1</td>
<td>0.0675</td>
<td>565</td>
<td>492</td>
</tr>
<tr>
<td>AC</td>
<td>10</td>
<td>0.01</td>
<td>0.0030-0.0055</td>
<td>27-50</td>
<td></td>
</tr>
<tr>
<td>AC</td>
<td>10</td>
<td>0.1</td>
<td>0.0050-0.0085</td>
<td>45-77</td>
<td>304</td>
</tr>
<tr>
<td>AC</td>
<td>30</td>
<td>0.01</td>
<td>0.0030-0.0055</td>
<td>33-60</td>
<td></td>
</tr>
<tr>
<td>AC</td>
<td>30</td>
<td>0.1</td>
<td>0.0225-0.0925</td>
<td>244-1003</td>
<td>642</td>
</tr>
<tr>
<td>CA</td>
<td>10</td>
<td>0.01</td>
<td>0.0040-0.0055</td>
<td>30-41</td>
<td></td>
</tr>
<tr>
<td>CA</td>
<td>10</td>
<td>0.1</td>
<td>0.0070-0.0160</td>
<td>52-118</td>
<td>348</td>
</tr>
<tr>
<td>CA</td>
<td>30</td>
<td>0.01</td>
<td>0.0040-0.0065</td>
<td>36-58</td>
<td></td>
</tr>
<tr>
<td>CA</td>
<td>30</td>
<td>0.1</td>
<td>0.0600-&gt;0.1</td>
<td>534-&gt;890</td>
<td>494</td>
</tr>
</tbody>
</table>

The intervals of the transition frequencies for all materials are listed in Table 2 below. Two cases of layer thickness, 10 and 30, and two wall friction, $c_\mu=0.01$ and 0.1, are presented. In this table the rpm values are calculated from the non-dimensional transitional frequencies according to Eq. (10). The last column to the right, show the averaged value of the experimentally found transitional frequencies in rpm. The table shows that for high friction $c_\mu=0.1$ the transitional frequency is strongly dependent of the number of layers. But for the case of low friction only PS shows a clear N-dependence. Similar simulations for the case of negligible friction gives small
dependence on the number of layers. Thus wall friction seems to be the main reason for which an increase in transition frequency would result from increased number of layers. This may be contrary to intuition, namely the weight on top of the shear layer is the main reason for stable and unstable shear.

Although the quantitative comparison between the experiment and the numerical model is weak, the model does seem to capture the qualitative behavior of the shear stability. The qualitative agreement is studied further for PS in Fig. 9. In Fig. 9 (a) the case cμ=0.01 is compared with the experimentally found results from Fig. 5. The experimentally found results are marked "+' for 58 spheres glued to the wall and "o" for 63 spheres glued to the wall. The numerical results are marked "x" and the marks are connected with solid lines for both the lower frequency limit and the upper frequency limit. In Fig. 9 (b) the same Fig. is shown for cμ=0.1. In this Fig. the upper limit of the transitional point for 40 layers was found to be outside the studied interval. One can observe that wall friction plays a major role in the numerical result.

![Fig. 9. Qualitative Agreement for PS Between Experimental and Numerical Results for the Upper and Lower Limit: (a) cμ=0.01 and (b) cμ=0.1](image)

In Figs. 10 (a) to (d) the time histories are shown at four different frequencies for PS, 10 layers and cμ=0.1. The frequencies are: ω = (a) 0.0015 (b) 0.0045 (c) 0.0070 and (d) 0.0100. Recall that earlier results showed that the first transition occur at ω= 0.0045 and the 10% vibration amplitude occur at ω= 0.0070. For low frequencies the motion is periodic and all particles go up and down in phase with the bottom boundary, such as in Fig. 10(a). In Fig. 10(b) the frequency has increased to the frequency of the first visible transition, ω= 0.0045. Now the motion is more random like as one moves upward with the number of layers.
DISCUSSIONS

The experimental results show that transition of a shear layer for a granular flow in a gravity field depends on several things. The properties of the granular material, the velocity of the shear layer, the friction with the side walls, and the number of layers overlain the shear layer. These results may be used to investigate the regime of shear under an avalanche or a landslide. In which, the normal component of the weight in the
sliding material is equivalent with the overlain material in this study. The velocity of the slide is equivalent with the bottom boundary’s velocity. This study shows that both the frictional property and the amount of the overlain material in an avalanche affect the type of flow at the shearing bottom.

A discrete one-dimensional N-degrees-of-freedom model consisting of one sphere from each layer was made. Studying the motion of the sphere next to the bottom boundary, a periodic point at zero displacement was found for low boundary velocities. At these frequencies the shear layer will always be in contact with the wall at the studied phase. As the velocity was increased this periodic point lost its stability and further increase of the velocity resulted in long period or chaotic motion. It is expected that the experimentally observed transition will occur after this periodic point lost its stability. As an upper limit of the transitional frequency the frequency when the amplitude exceeds 10% of the sphere diameter is chosen. For the case no friction, the transition frequency was independent of the number of layers. Introducing friction to the models, qualitative agreement of the transition frequency (as function of the number of layers) could be achieved. One case (PS with $\mu=0.1$) was studied in more detail. For the case of PS and more than 10 layers good agreement can be found by choosing $\mu=0.1$.

The experimentally observed randomly bouncing of the topmost particles was also captured in the numerical model. It was found that frequencies close to the first visible transition initiates the randomly bouncing of spheres on top. These results also suggest that the first visible transition is not what causes the experimentally observed transition. The upper limit of the transitional frequency seems to be a more plausible reason for the experimentally observed transition. Namely, it is more likely that the transition is caused by high vibration amplitudes in the shear layer. In this study maximum displacement amplitude >10% D has been chosen as the upper limit for transitional frequency. A more rigorous argument maybe necessary to improve this criterion.

Uncertainties in this model will be discussed as follows. First, the choice of the friction force and its distribution over the layers. This friction force could not be measure in the apparatus used in this paper. Therefore, the influence of friction can only be studied here qualitatively. Second, the stiffness can cause large errors. The stiffness of a sphere is strongly nonlinear, but in the model a constant value was assumed. This constant value was taken from the nonlinear expression of the stiffness at the load $Nmg$. This value is extrapolated (with use of the equations in Appendix I) far beyond the measured data points of the stiffness. This low value of the loading will cause an area of contact with the radius of $\frac{1}{100}$ of the sphere radius. The surface of the spheres was not of the precision that they could be assumed perfectly round at this scale. Third, in the one-dimensional model all particles were assumed to move in phase. Due to this symmetry, every particle will affect only one particle in each layer. In the experiment, a small amount of random will destroy this symmetry and result in horizontal impacts.
between particles. For the creation of a mixing chaotic region it is essential that more space is generated in the bottom layers. In the experimental apparatus the radius was small (63 spheres densely glued to the wall). This gives the system small possibilities to make space in the horizontal direction, so in order to create a transition particles need to be pushed through the layers above. If the radius was larger it would be possible to create space in the horizontal direction. Transition could then possibly occur at lower frequencies or vibration amplitudes. Still one would expect that the numerically found first transition would be a lower bound for possible transition.

All the above discussed uncertainties imply that errors in the numerical solution were expected. Qualitatively the N-degree of freedom model with friction agrees with experimental observations.

CONCLUSIONS

The motion of a granular material in a shear cell has been studied experimentally and numerically. A transparent shear cell was built in which the granular material was contained between two concentric cylinders. The shearing motion was provided by rotating the bottom boundary with a constant velocity. The experiments show that for low velocities a shear layer is generated in the first layer above the bottom boundary. This shear layer moves with a velocity which is half of the bottom velocity while the layers above it have negligible horizontal motion. Particles in all layers show small vertical vibrations. An exception to this is the topmost layer which bounces randomly with much higher amplitudes. Increasing the rotational velocity, a critical value is reached at which the single shearing layer motion transforms to a chaotic state, where mixing between layers starts to occur. This may involve several layers adjacent to the moving wall. The thickness of the mixing chaotic region and the critical velocity were found to depend on the number of layers in the shear cell. Two different bottom wall profiles were studied. One with maximum packing of spheres (63) and the second with 58 spheres randomly distributed. This change in the boundary type had only a minor influence on the critical velocity. From the numerical model, the critical velocity of the boundary shear is found to depend most sensitively on the material properties of the shearing spheres and the friction of the side walls.

Because of many uncertainties in the measured parameters, a quantitative comparison between the experiment and the numerical model cannot be made. Nevertheless, the numerical model proposed in this study was able to reproduce qualitatively the experimental observations. It is believed that this model can be used as a versatile tool, to help determine the flow regimes for shearing granular materials in a gravity field.
ACKNOWLEDGEMENTS

This work has been carried out at the division of Solid Mechanics at Luleå University of Technology and at Clarkson University, Potsdam, New York. The work at Clarkson University was supported by Sweden-America foundation, with funds made available by Thanks to Scandinavia Inc, Monte M. and Susan Hurowitz Scholarship, The Swedish Institute and by The Royal Swedish Academy of Science’s, Claes Adelskölld's fund.
APPENDIX I. SPHERE STIFFNESS

Stiffness of the spheres is an important parameter in the model. As the force displacement relationship for a sphere is not linear, the stiffness depends on the force level. According to Hertz theory of elastic contact, for the case of an elastic sphere of diameter $D$ in contact with a rigid surface, the relationship between the displacement $x$ and the applied force $F$ is given by

$$x = D - \sqrt{\frac{D^2 - 4(c_1F)}{c_1^2}}$$  \hspace{1cm} (11)

where

$$c_1 = \frac{3D(1-v^2)}{8E}$$  \hspace{1cm} (12)

In the above, $E$ is the Young’s modulus and $v$ is the Poisson’s ratio. Differentiating (11) with respect to $F$, one obtains the stiffness $K$

$$K = \left(\frac{dx}{dF}\right)^{-1} = \frac{3}{4} \sqrt{\frac{D^2 - 4(c_1F)}{c_1^2}} \left(\frac{F}{c_1^2}\right)^{\frac{1}{3}}$$  \hspace{1cm} (13)

Thus the stiffness can be determined by Eq. (12) and Eq. (13) if the material parameters $E$ and $v$ are known. Due to large uncertainties in the values of $E$ available in the literature, a device was built for measuring the stiffness directly. This device is shown in Fig. 11, where the sphere (a) is compressed by the bar (b) due to a force applied at (c) and the displacement is measured by the gauge (d). The bar is supported at its center of mass by ball bearings (e). The device is made of steel, so it may be considered as rigid in comparison with the plastic spheres.

Two sets of loads, in the range of (1 to 4) N and (2 to 8) N, were applied. Three sets of displacements were measured corresponding to each load set and for each of the four materials. The results are shown in Fig. 12 (a) to (d), where each set of displacements is marked with a dotted line through the measured values. These Figs. show a non zero force at zero displacement. This is due to the fact that the gauge was reset to zero after application of the first load. Thus the first data point was neglected and the stiffness $K$ was calculated for each load increment. For each measured stiffness the corresponding value of the constant $c_1$ was calculated by using (12) in the form

22
\[ c_1 = \left( \frac{3F^{\frac{1}{3}}}{8K^2} \left( \sqrt{9F^2 + 4K^2D^2} - 3F \right) \right)^{\frac{3}{2}} \]  \hspace{1cm} (14)

Fig. 11. Device for Measuring Ball Stiffness.

Fig. 12. Displacement-Force Diagrams for (a) PS, (b) PP, (c) AC and (d) CA
By averaging these $c_1$ values, the Young's modulus $E$ can be calculated by (12). Using $D = 0.004$ m and $v = 0.35$, we get

- polystyrene $E = 1.18 \pm 0.42$ Gpa
- polypropylene $E = 0.34 \pm 0.08$ Gpa
- acrylic $E = 1.03 \pm 0.34$ Gpa
- cellulose acetate $E = 0.57 \pm 0.19$ Gpa

Fig. 13. Stiffness-Force Diagrams for (a) PS, (b) PP, (c) AC and (d) CA

Fig. 13 shows the experimentally determined values of stiffness $K$, marked "x", as well as the stiffness curves obtained by using (13) in combination with the measured average $c_1$. The values of $E$ mentioned above together with Eq. (12) and Eq. (13) where used to estimate the stiffness for all force levels.
APPENDIX II. SPHERE DAMPING

In this section the coefficient of restitution and the damping coefficient are studied for the plastic spheres. The coefficient of restitution is determined by dropping the spheres against a smooth marble surface. The marble surface is considered as stiff compared with the plastic spheres. By measuring the amplitudes of the bouncing spheres, the coefficient of restitution for different impact velocities is calculated. In the experiment, some of the impacts were not perfectly vertical and hence lower values of the coefficient of restitution were measured. To erase some of these errors 10 percent of the data points containing the lowest values of coefficient of restitution are ignored.

When only the sphere is moving the coefficient of restitution $\alpha$ is defined as

$$V_b = \alpha V_a$$  \hspace{1cm} (16)

with $V_b$ as velocity of rebound and $V_a$ as velocity of approach. The coefficient $\alpha$ can also be expressed in terms of amplitudes with the drop amplitude $h_d$ and the rebound amplitude $h_r$,

$$\alpha = \sqrt{\frac{2gh_r}{2gh_d}} = \sqrt{\frac{h_r}{h_d}}$$  \hspace{1cm} (17)

where $g$ is the acceleration due to gravity. In Fig. 14 the coefficient of restitution is plotted against the impact velocity. Fig. 14 (a) is polystyrene, (b) is polypropylene, (c) is acrylic and (d) is cellulose acetate. The averaged values of these coefficient of restitution are

- polystyrene $\alpha = 0.921$
- polypropylene $\alpha = 0.793$
- acrylic $\alpha = 0.897$
- cellulose acetate $\alpha = 0.842$  \hspace{1cm} (18)

These values are chosen as the coefficient of restitution for the materials for the case of impacts with a stiff surface. From the figures one can observe that the velocity dependence can be neglected and therefore the coefficient of restitution can be considered as a constant.

In these experiments the spheres are impacting against a stiff surface. A model of the impact process is made by a mass $m$ connected with a spring $k$ and a damper $c$ in parallel on the bottom. Neglecting gravity the equation of motion during the impact
Fig. 14. Coefficient of Restitution at Different Impact Velocities: (a) PS, (b) PP, (c) AC and (d) CA

is

$$ m\ddot{x} + c\dot{x} + kx = 0 $$  \hspace{1cm} (19)

Rewriting Eq. (19) with the parameters natural frequency $\omega_n$ and damping ratio $\zeta$ gives

$$ \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 $$  \hspace{1cm} (20)

The coefficient of restitution in Eq. (16) is modelled with the damper. With $V_a$ as the initial velocity and equilibrium position as the initial position the velocity $V_b$ is calculated after a half period of the oscillation

$$ V_b = -e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} V_a $$  \hspace{1cm} (21)

The above is rewritten to
\[ \zeta_1 = \frac{-\ln(\alpha)}{\sqrt{\ln(\alpha)^2 + \pi^2}} \]  \hspace{1cm} (22)

For the case of two impacting spheres there will be two sets of a parallel spring and damper in serial between the mass centers. The total damping and stiffness will then be decreased to \( c/2 \) and \( k/2 \) respectively. The total damping ratio \( \zeta \) becomes

\[ \zeta = \frac{\zeta_1}{\sqrt{2}} = \frac{-\ln(\alpha)}{\sqrt{2(\ln(\alpha)^2 + \pi^2)}} \]  \hspace{1cm} (23)

The results in Eq. (18) together with Eq. (23) give the damping ratios for the case of impact between two spheres

<table>
<thead>
<tr>
<th>Material</th>
<th>( \zeta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>polystyrene</td>
<td>0.019 ± 0.004</td>
</tr>
<tr>
<td>polypropylene</td>
<td>0.052 ± 0.006</td>
</tr>
<tr>
<td>acrylic</td>
<td>0.024 ± 0.005</td>
</tr>
</tbody>
</table>
| cellulose acetate| 0.039 ± 0.004     |  \hspace{1cm} (24)

These values with the estimated errors given in Eq. (24) are valid for impact velocities in the studied velocity range 1-3 m/s.
APPENDIX III. REFERENCES


APPENDIX IV . NOTATION

Primed variables and constants defines non-dimensional quantities and dotted variables are derivatives with respect to time.

\( \alpha = \) amplitude of the bumpy boundaries defined in Eq. (9)
\( \beta = \) constant used in Appendix I
\( \gamma = \) frictional constant defined in Eq. (6)
\( \delta = \) damping coefficient
\( D = \) sphere diameter
\( E = \) Young's modulus used in Appendix I
\( F = \) force used in Appendix I
\( F_i = \) contact forces acting on the spheres, \( i = 1, 2, \ldots, N \)
\( F_{\mu_i} = \) friction force on sphere \( i, i = 1, 2, \ldots, N \)
\( g = \) acceleration due to gravity
\( h_d = \) drop amplitude used in Appendix II
\( h_r = \) rebound amplitude used in Appendix II
\( k = \) stiffness of a sphere
\( m = \) mass of a sphere
\( N = \) number of sphere layers
\( t = \) time
\( V = \) velocity of the rotating boundary in Fig. 6
\( V_a = \) velocity of approach used in Appendix II
\( V_r = \) velocity of rebound used in Appendix II
\( x_i = \) position of sphere \( i, i = 1, 2, \ldots, N \)
\( y_j = \) position of bumpy boundary \( j, j = 1 \) and 2
\( \dot{\alpha} = \) coefficient of restitution
\( \zeta = \) damping ratio
\( \zeta_1 = \) damping ratio used in Appendix II
\( \nu = \) Poisson's ratio used in Appendix I
\( \omega = \) frequency
\( \omega_b = \) boundary rotational frequency
\( \omega_n = \) natural frequency used in Appendix II
STABILITY OF A VIBRO-IMPACTING MODEL OF A BRAKE PAD: A POSSIBLE EXPLANATION OF SQUEAL IN DISC BRAKES

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ABSTRACT

A vibro-impacting model is suggested to study some aspects of the dynamics of a brake pad in a disk brake. The disk is assumed to be rigid and the brake pad is modelled as a rectangular mass performing vibrations in a plane which is orthogonal to the plane of contact between the pad and the disk. Thus the model has three degrees of freedom; two components of translation and a rotation around the mass center. The brake pad is supported by springs and dampers at variable positions and it is pressed against the disk by a brake force. Springs with distributed stiffness are used to model the interface between the pad and the disk. Friction is also allowed at this interface. The equilibrium position of the pad is first determined by using a static analysis. The assumption here is that the stability of this position is a crucial factor in explaining the phenomenon of squeal. So a detailed stability analyses is performed. For the case of equal linearized natural frequencies, a large region of the parameter space is found to be unstable. The shape and size of this region are strongly influenced by the location of the pad supports, damping and the coefficient of friction but not so much by the brake force. Unstable regions exist even when the natural frequencies are far apart. One finds a rich variety of motions from periodic to seemingly chaotic, including stationary behavior in slip, vibrations with full contact with the disk with possibilities of impact. Stick-slip behavior is also possible at low disk velocities.
1. INTRODUCTION

It is well known that simple impacting systems [1] or systems with stick slip kind of motion can lead to rich bifurcational and chaotic behavior [2]. It is also known that the frictional characteristics [3, 4] or a follower force [5] can cause instabilities in the dynamic behavior of simple mechanical systems. In disc-brake systems all of these non-linearities may be present simultaneously. It is therefore no surprise that chatter and squeal have been and still are rather intricate problems in the design of brakes in cars and other vehicles. A major difficulty is that sometimes brakes which did not produce squeal during development on laboratory rigs, did so on a vehicle, even when the laboratory and road brake usage conditions were apparently identical. The vibrations are self excited and fugitive in the sense that it is extremely difficult to determine any repeatable test conditions under which a brake can reliably be expected to squeal. In addition it is extremely difficult to experimentally trace the source of the vibrations as all vibration modes are coupled and most of the braking system components in the vicinity of the disk and caliper vibrate.

These undesired vibrations have been a problem since the early days of train brakes [6, 7]. In cars the problem of squeal has been found in both drums and disk brakes. One of the first studies on the subject of friction induced vibrations in vehicle brakes were initiated by the Institution of Automobile Engineers [3, 4, 8]. Experiments were performed on a cantilever beam with a lining in contact with a brake drum. One of the main results was that materials having a coefficient of friction that decreases with increasing velocity were more prone to squeal. After the war the work on brake squeal was carried on by the Motor Industry Research Association. See [9 - 11] for drum brakes and [12, 13] for disk brakes.

Spurr [14], however, showed that even a constant coefficient of frictional could cause stick-slip vibrations. He suggested that squeal might depend upon the actual value of the frictional coefficient rather than the velocity slope. The idea was carried forward by Jarvis and Mills [15] and followed by Earles et al. [16 - 18], Aranov et al. [19], D'Souza et al. [20, 21], Oden et al. [22, 23] and by Tworzydlo et al. [24]. These studies lead to a dynamic coupling theory which proposed that brake squeal is a property of the whole brake system and not only a frictional phenomenon. The general nature was described by Tworzydlo et al. [24] as "The self excited oscillation occur when the natural frequencies of normal and rotational vibrations of the slider are relatively close to one other. Then the existence of friction causes coupling between rotational and normal modes and, as a consequence, frequency coalescence and propagation of self-excited oscillations."
The possibility of both theories was also confirmed by Murakami et al. [25] who found that both the friction velocity characteristic and a geometrically induced instability contributes to an increase in squeal. A third possible explanation for squeal was given by Mottershead et al. [5] who showed that instability could occur for a brake disk even if the friction force was introduced as a simple follower force.

A good description of the disk squeal phenomena was already given in one of the first reports on disk brakes by Fosberry et al. [12, 13]. In these experimental studies it was found that during squeal the brake pad excites the disk at natural frequencies with radial nodes i.e. principal vibrations are at right angle to the plane of the disk. The pad follows the motion of the disk and the position of the pad was found to lie between two consecutive nodes of the disk. The squeal was also found to be largely confined to the disk and caliper assembly and not to the mounting of the brake on the suspension or to the hydraulic stiffness. Changing the disk design generally alters only the frequency of vibrations rather than the occurrence. Exceptions are designs which increase damping in the disk. Fosberry et al. indicated that the flexibility of the mechanism probably could be one explanation of the phenomena. The experiments did not show any close relation between the friction characteristics and squeal.

Squeal was generally found to be unaffected by pad size and brake line pressure, except that the increased clamping of the disk caused a slight increase in the squeal frequency. The occurrence of squeal was also effected by the design of the caliper. Increase of temperature resulted in a decrease in squeal frequency mainly due to the decrease in natural frequency of the disk. Changes in the pad thickness showed that thicker pad (lower stiffness) causes lower squeal frequencies. Decrease in pad width caused a slight decrease in squeal frequency. The effect of pad support was studied both in axial and tangential direction. It was found that the tendency of squeal was strongly influenced by offsetting the pad from piston center or by changes in the side supports. It was also found that damping strongly effects the tendency to squeal. Increasing the rotational flexibility of the brake-cylinder mounting on the caliper tends to promote squeal. Factors which did not appear to affect squeal were the weight of the caliper, hydraulic stiffness, and the over all, torsional stiffness of the brake assembly about the wheel axis.

In this paper the dynamics of a vibro-impacting model of a brake pad on a rigid disk is studied. The work is partly influenced by the observations [12, 13] that the brake pad support is important for the occurrence of squeal. It is assumed that instability of the brake pad can be an explanation to the occurrence of squeal. If the brake pad becomes unstable, vibrations of the disk become a natural consequence. The brake pad is modelled as a three degree of freedom system with adjustable support locations for the brake. The interface between the pad and the disc is modelled as elastic with distributed springs. Friction between the pad and the disk as well as possibility of
impacts between the pad and its surrounding are taken into account. The ideas of dynamical coupling of brake pad modes are studied and regions of instability in the parameter space are determined. Unstable regions are found to exist even when the three linearized natural frequencies are quite different. The results show that the locations of the supports of the brake pad are important and that the possibility of impacts may be an additional factor relevant to the problem of squeal. This gives just another dimension to the complexity of brake squeal. One can be sure that no single factor can explain brake squeal completely. In reality it is likely that squeal is generated by several of these factors but for different parameter combinations. It is also possible that they can occur simultaneously in various combinations.

2. THE MODEL

A model of the brake pad is shown in Figures 1 (a) and 1 (b). The pad is modelled as a rectangular mass, having two translational degrees of freedom, X and Y and a rotational degree of freedom \( \theta \). The rotation \( \theta \) is chosen to be zero when the brake pad is parallel with the brake disk. The amplitude in the Y-direction is limited by a stop. The origin of X, Y is chosen so that the pad is just in contact with the stop as well as with the brake disk. Figure 1 (a) shows the reference configuration and Figure 1 (b) shows a displaced configuration.

![Figure 1](image.png)

Figure 1. The brake pad model with the reference configuration in (a) and in (b) the displaced configuration when a brake force P is applied.

The brake pad is considered as rigid with the mass \( M \) and a moment of inertia \( J \). The brake pad stiffness \( K \) is described by using springs with a uniform stiffness per unit length distributed over the width 2B. These springs are assumed to be in contact with
the brake disk which is moving with a constant velocity $V$. Frictional forces with a coefficient of friction $\mu$ are assumed to exist at the area of contact between the springs and the disk. The unloaded distance between the disk and the mass center of the brake pad is $H_1$ and the height of the pad is $H_1 + H_2$. The motion of the pad in the $X$-direction is restrained by a spring of stiffness $K_1$ and a damper with damping constant $C_1$. This assembly acts at a $Y$-distance $H_4$ from the mass center. Similarly, the rotational motion is constrained by a spring with rotational stiffness $K_2$ and a damper with corresponding constant $C_2$. When a brake force $P$ is applied, the brake pad will move in the $Y$-direction until it hits the stop $S$. After that the motion will be restrained by a spring $K_3$ and a damper $C_3$. This assembly acts at a $X$-distance $H_3$ from the mass center. In the contact area between the brake pad and the disk there arises a normal force $F_n$ in the direction of $X$, a friction force $F_\mu$ in the direction of $Y$ and a torque $M_v$. The normal force $F_n$ is given by

$$F_n = -2KB\left(X - H_1(1 - \cos \theta)\right)$$  \hspace{1cm} (1)

and assuming a stick-slip kind of behavior, the frictional force $F_\mu$ is given by

$$F_\mu = \mu F_n \begin{cases} \text{sign}(V - \dot{Y}) & ; \quad V - \dot{Y} \neq 0 \\ \frac{\mu}{|F_n|} & ; \quad V - \dot{Y} = 0 \end{cases}$$  \hspace{1cm} (2)

where the first equation gives the friction force at slip and the second equation gives a requirement on the friction force at stick. The torque $M_v$ is given by

$$M_v = -\frac{2KB^3\tan \theta}{3} + F_\mu (H_1 - X)$$  \hspace{1cm} (3)

The springs $K$ are active in compression only, which implies that the expressions for the contact forces given above are valid only when the brake pad is compressed over the entire width $2B$. The stop implies a similar condition for spring $K_3$ and the damper $C_3$ which are active in compression only, i. e.,

$$K_3 = C_3 = 0 ; \quad \text{if} \quad Y - B(1 - \cos \theta) - H_3 \sin \theta < 0$$  \hspace{1cm} (4)

The stick-slip condition results in two sets of equations of motion; one for the stick mode and one for the slip mode. The equations of motion for the slip mode are
considered first. For this mode the equation of motion in X-direction is

\[ M \ddot{X} = K_1 \left( \Delta L - X - H_2(1 - \cos \theta) + H_4 \sin \theta \right) - C_1 \left( \dot{X} + \dot{\theta}(H_2 \sin \theta - H_4 \cos \theta) \right) + F_n, \tag{5} \]

where \( \Delta L \) is an initial compression which facilitates the treatment of different brake forces \( P \). A sufficiently large brake force also assures compression in the contact area between the pad and the disk. The equation of motion in Y-direction is

\[ M \ddot{Y} = -K_3 \left( Y - B(1 - \cos \theta) - H_3 \sin \theta \right) - C_3 \left( \dot{Y} - \dot{\theta}(B \sin \theta + H_3 \cos \theta) \right) + F_n \tag{6} \]

and in \( \theta \)-direction we have

\[ J \ddot{\theta} = -K_2 \theta - C_2 \dot{\theta} + M_\theta - \left[ K_1 \left( \Delta L - X - H_2(1 - \cos \theta) + H_4 \sin \theta \right) \right. \]
\[ \left. - C_1 \left( \dot{X} + \dot{\theta}(H_2 \sin \theta - H_4 \cos \theta) \right) \right] (H_4 \cos \theta - H_2 \sin \theta) \]
\[ + \left[ K_3 \left( Y - B(1 - \cos \theta) - H_3 \sin \theta \right) + C_3 \left( \dot{Y} - \dot{\theta}(B \sin \theta + H_3 \cos \theta) \right) \right] \]
\[ \times (B \sin \theta + H_3 \cos \theta) \tag{7} \]

The slip mode will be active as long as \( V - \dot{Y} \neq 0 \).

In the stick mode the velocity \( \dot{Y} \) is equal to \( V \) and hence according to equation (6) the friction force becomes

\[ F_\mu = K_3 \left( Y - B(1 - \cos \theta) - H_3 \sin \theta \right) + C_3 \left( V - \theta(B \sin \theta + H_3 \cos \theta) \right) \; \; \text{if} \; \; |F_\mu| < \mu |F_n| \tag{8} \]

The form of the equations of motion (5) and (7) remains unchanged except that \( \dot{Y} \) is replaced by \( V \) and \( Y \) is replaced by \( Y(t_0) + V(t-t_0) \), where \( t_0 \) is the time at which the stick mode is initiated.

From the numerical solution of the equations of motion one finds that for most of the parameter combinations the vibrations of the brake pad will converge towards a stationary position with slip mode. This stationary position can be determined analytically by using the slip conditions. The equilibrium state \( X_e, Y_e \) and \( \theta_e \) corresponding to the brake force \( P \) can be determined by setting the time derivatives in equations (5-7) to be zero which together with (1-4) gives
\[
K_1(\Delta L - X_e - H_2(1 - \cos \theta_e) + H_4 \sin \theta_e) = P
\]
\[
2KB(X_e - H_1(1 - \cos \theta_e)) = P
\]
\[
K_3(Y_e - B(1 - \cos \theta_e) - H_3 \sin \theta_e) = 2\mu KB(X_e - H_1(1 - \cos \theta_e))
\]

(9)

Assuming small angles \(\theta_e\), an approximate solution to equations (9) is given by

\[
X_e = \frac{P}{2KB}, \quad Y_e = \frac{\mu P}{K_3} + H_3 \theta_e
\]

(10)

\[
\theta_e = \frac{\mu P(H_1 - X_e + H_3) - PH_4}{K_2 + \frac{2KB^3}{3} - \mu PB - PH_2}, \quad \Delta L = \frac{P}{K_1} - H_4 \theta_e + X_e
\]

In a neighborhood of this stationary state the natural frequencies may be defined by studying each coordinate separately and treating the others as constants. Thus, the undamped natural frequencies for motion in different direction are

\[
\Omega_x = \sqrt{\frac{2KB + K_1}{M}}, \quad \Omega_y = \sqrt{\frac{K_3}{M}} \quad \text{and} \quad \Omega_\theta = \sqrt{\frac{K_2 + \frac{2KB^3}{3} + K_4 H_4^2 + K_3 H_3^2}{J}}
\]

(11)

These natural frequencies are of interest as earlier studies [22, 23] have shown that instabilities occur when these natural frequencies are close to each other. The stability of this stationary state is studied in the next section.

Normalizing all the lengths with respect to the brake width \(B\) and introducing the
following non-dimensional quantities

\[ t = T \sqrt{\frac{K_1}{M}} \]

\[ \zeta_1 = \frac{C_1}{2\sqrt{K_1M}}, \quad \zeta_2 = \frac{C_2}{2\sqrt{K_2J}}, \quad \zeta_3 = \frac{C_3}{2\sqrt{K_3M}} \]

\[ \alpha = \sqrt{\frac{K_2M}{K_1J}} \quad \beta = \sqrt{\frac{K_3}{K_1}} \quad \gamma = \frac{2KB}{K_1} \quad \kappa = \frac{MB^2}{I} \quad \rho = \frac{P}{2KB^2} \quad \nu = \frac{V}{B} \sqrt{\frac{M}{K_1}} \]

the non-dimensional equations of motion for the slip mode become

\[ \dot{x} = (A_l - x - h_2(1 - \cos \theta) + h_4 \sin \theta) - 2\zeta_1 \left( x + \theta (h_2 \sin \theta - h_4 \cos \theta) \right) + f_n \]

\[ \dot{y} = -\beta^2 \left( y - (1 - \cos \theta) - h_3 \sin \theta \right) - 2\zeta_3 \beta \left( y - \theta (\sin \theta + h_3 \cos \theta) \right) + f_\mu \]

and

\[ \theta = -\alpha^2 \theta - 2\zeta_2 \alpha \theta + m_v \]

\[ \kappa \left[ (A_l - x - h_2(1 - \cos \theta) + h_4 \sin \theta) - 2\zeta_2 \left( x + \theta (h_2 \sin \theta - h_4 \cos \theta) \right) \right] \left[ h_4 \cos \theta - h_2 \sin \theta \right] \]

\[ + \kappa \left[ \beta^2 \left( y - (1 - \cos \theta) - h_3 \sin \theta \right) + 2\zeta_2 \beta \left( y - \theta (\sin \theta + h_3 \cos \theta) \right) \right] (\sin \theta + h_3 \cos \theta), \]

where the forces and the torque are

\[ f_n = -\gamma \left( x - h_1 (1 - \cos \theta) \right) \]

\[ f_\mu = \mu \left| f_n \right| \text{sign} (v - \dot{y}) \quad \text{if} \quad v - \dot{y} \neq 0 \]

and

\[ m_v = \kappa \left( -\tan \theta \frac{\gamma}{3} - f_\mu (h_1 - x) \right) \]

Equation (4) takes the form

\[ \zeta_2 = \beta = 0 \quad \text{if} \quad y - (1 - \cos \theta) - h_3 \sin \theta < 0. \]
The criteria for stick is
\[ \dot{y} = v \] (20)
and the friction force becomes
\[
f_f = \beta^2 \left( y - (1 - \cos \theta) - h_3 \sin \theta \right) + 2 \zeta_3 \beta \left( v - \dot{\theta} (\sin \theta + h_3 \cos \theta) \right); \quad \text{if } |f_f| \leq \mu |f_n| .
\] (21)
Compression between the pad and the disk over the full width \( 2b \) implies
\[ x - h_1 (1 - \cos \theta) \pm \delta \sin \theta \geq 0 \] (22)
and the approximate solution of the equilibrium position at slip becomes
\[
x_e = p , \quad y_e = \frac{\mu p y}{\beta^2} , \quad h_3 \dot{\theta}_e ,
\]
\[
\theta_e = p \frac{\mu (h_1 - x_e + h_3) - h_4}{\alpha^2 + \frac{1}{3} \mu p - p h_2} , \quad \Delta l = p y - h_4 \dot{\theta}_e - x_e .
\] (23)
The undamped natural frequencies as defined in equation (11) become
\[ \omega_x = \sqrt{1 + \gamma} , \quad \omega_y = \beta \quad \text{and} \quad \omega_\theta = \sqrt{\frac{\alpha^2 + \frac{\kappa \gamma}{3} + \kappa h_4^2 + \beta^2 \kappa h_3^2}{\beta^2}} . \] (24)

In order to keep the number of parameters to a minimum, a typical brake pad was chosen to be analyzed. The following data was gathered by measurements on such a pad.

\[
B = 0.050 \quad [m] \quad H_1 = 0.010 \quad [m] \quad \Rightarrow h_1 = 0.20
\]
\[ H_2 = 0.006 \quad [m] \quad \Rightarrow h_2 = 0.12 \]
\[ M = 0.41 \quad [kg] \quad J = 3.50 \times 10^{-4} \quad [kg \ m^2] \quad \Rightarrow \kappa = 2.929
\]
\[ K = 5.0 \times 10^9 \quad [N/m] \] (25)

These values were kept constant while the other parameters are varied over a wide range. In addition to the above the coefficient of friction, the brake force and the damping ratios were also kept constant in most of the simulations, with
Here in after the parameters given in equations (25) and (26) have been used if nothing else is stated. The main task is to find out how the natural frequencies influence the dynamics of the brake pad.

3. STABILITY ANALYSIS

Numerical simulations of the equations of motion show that for most parameter combinations the solution is attracted towards the stationary point in slip mode, given by equation (23). If the stationary state becomes unstable, self excited motions may be generated, possibly causing the brake disk to squeal. Therefore the stability of the stationary state is crucial for the function of the brake. We start by picking a known critical combination of parameters, i.e., when all natural frequencies are equal, as it certainly leads to unstable behavior. Then some other interesting points in the parameter space are identified and investigated further.

The equations of motion (13-15) can be written as first order differential equations

\[
\begin{align*}
\dot{x} &= v_x \\
\dot{y} &= v_y \\
\dot{\theta} &= \omega \\
\dot{v}_x &= f_x(x, v_x, y, v_y, \theta, \omega) \\
\dot{v}_y &= f_y(x, v_x, y, v_y, \theta, \omega) \\
\dot{\omega} &= f_\theta(x, v_x, y, v_y, \theta, \omega)
\end{align*}
\]  

By linearizing the vector field at the stationary position, the time evolution in a neighborhood of the stationary point is given by

\[
\begin{bmatrix}
\delta x \\
\delta y \\
\delta \theta \\
\delta \omega
\end{bmatrix} = \left[ Df(x, y, \theta) \right] \begin{bmatrix}
\delta x \\
\delta y \\
\delta \theta \\
\delta \omega
\end{bmatrix}
\]

\[
\begin{bmatrix}
\delta x \\
\delta y \\
\delta \theta \\
\delta \omega
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\delta x \\
\delta y \\
\delta \theta \\
\delta \omega
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
\delta x \\
\delta y \\
\delta \theta \\
\delta \omega
\end{bmatrix}
\]
where
\[
D_{21} = -(1 + \gamma) \quad D_{22} = -2\zeta_1 \quad D_{25} = h_4 \cos \theta_e - h_2 \sin \theta_e + \gamma h_1 \sin \theta_e \\
D_{26} = -2\zeta_1 (h_3 \sin \theta_e - h_4 \cos \theta_e) \quad D_{41} = \mu \gamma \quad D_{43} = -\beta^2 \quad D_{44} = -2\zeta_3 \beta \\
D_{45} = \beta^2 (\sin \theta_e + h_3 \cos \theta_e) - \mu \gamma h_1 \sin \theta_e \quad D_{46} = 2\zeta_3 \beta (\sin \theta_e + h_3 \cos \theta_e) \\
D_{61} = \kappa (h_4 \cos \theta_e - h_2 \sin \theta_e) - \mu \kappa \gamma (-2x_2 + h_1 (2 - \cos \theta_e)) \\
D_{62} = 2\kappa \zeta_1 (h_4 \cos \theta_e - h_2 \sin \theta_e) \quad D_{63} = \kappa \beta^2 (h_3 \cos \theta_e + \sin \theta_e) \\
D_{64} = 2\zeta_3 \kappa \beta (h_3 \cos \theta_e + \sin \theta_e) \\
D_{65} = -\alpha^2 - \kappa (h_2 \sin \theta_e - h_4 \cos \theta_e)^2 \\
\quad + \kappa \frac{\gamma}{3 \cos^2 \theta_e} - \mu \kappa \gamma h_1 \sin \theta_e (h_1 - x_2) - \kappa \beta^2 (\sin \theta_e + h_3 \cos \theta_e)^2 \\
\quad + \kappa \beta^2 (y_2 - (1 - \cos \theta_e) - h_3 \sin \theta_e) (\cos \theta_e - h_3 \sin \theta_e) \\
D_{66} = -2\zeta_2 \alpha - 2\kappa \zeta_1 (h_2 \sin \theta_e - h_4 \cos \theta_e)^2 - 2\kappa \beta \zeta_3 (\sin \theta_e + h_3 \cos \theta_e)^2
\]

The stability can be determined from the eigenvalues \( \lambda \) of \( Df \). If \( \text{Re}[\lambda] < 0 \) for all of the eigenvalues the stationary point will be asymptotically stable.

The stability is first investigated for the case of equal linearized natural frequencies. According to equation (24) this condition is satisfied for \( \gamma = 1, \beta = \sqrt{2} \) and

\[
\alpha = \sqrt{\beta^2 - \kappa (\frac{\gamma}{3} + h_4^2 + \beta^2 h_5^2)}
\]

In Figure 2 (a) a stability diagram in the \( h_3 - h_4 \) plane is shown. One can observe that there exists a large area of unstable points in this plane when all natural frequencies are equal. But, even for this case there do exist some stable regions. Especially the region close to \( h_3 = h_4 = 0 \) is of special interest as it can be a possible design point. Changing the value of the force \( p \) has no effect on the stability for these parameter combinations. Increasing the damping ratios \( \zeta \) or decreasing the coefficient of friction \( \mu \), however, results in larger areas of stability. See Figure 2 (b). Even if the dependence on the coefficient of friction is strong there do exist unstable regions even for \( \mu = 0.1 \) but increasing the damping ratio one finds that above \( \zeta = 0.05 \) the whole region is stable. From Figure 2 (a) four unstable points were chosen for deeper investigation. These points were \( (h_3, h_4) = (-0.05, \)
In Figures 2 (c) to 2 (f) these points are shown in α-β plane.

Figure 2. Stability diagrams for the systems: (a) $\alpha = \sqrt{\beta^2 - \kappa} (\gamma/3 + h_3^2 + \beta^2 h_4^2)$, $\beta = \sqrt{2}$; (b) $\mu = 0.2$, $\alpha = \sqrt{\beta^2 - \kappa} (\gamma/3 + h_3^2 + \beta^2 h_4^2)$, $\beta = \sqrt{2}$; (c) $h_3 = -0.05$, $h_4 = 0.05$; (d) $h_3 = 0.05$, $h_4 = 0.05$; (e) $h_3 = 0.05$, $h_4 = -0.05$; (f) $h_3 = 0.05$, $h_4 = -0.05$. 

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In all of these diagrams the point $a=1.0015$, $\beta=1.4142$ corresponds to the reference points chosen from Figure 2 (a). Around the points corresponding to equal natural frequencies one finds a large area of instability, as expected. But, there are other parameter combinations which give instability even when the natural frequencies are far apart. In Figures 2 (c) and (f) the unstable region for $\beta=1.4142$ exists up to $a=2.15$. For $\zeta=0.001$ the same region is unstable up to $a=19$. This is a point where the natural frequencies are $[\omega_x, \omega_y, \omega_\theta]=[1.4, 1.4, 19.5]$. In Figures 2 (d) and (e) the unstable region for $\beta=1.4142$ exists down to $a=0$. The natural frequencies corresponding to this point are $[\omega_x, \omega_y, \omega_\theta]=[1.4, 1.4, 1.0]$. In Figures 2 (e) and (f) the unstable region for $a=1$ exists down to $\beta=0$. The natural frequencies at this point are $[\omega_x, \omega_y, \omega_\theta]=[1.4, 0.0, 1.4]$. The unstable region for high values of $\beta$ and $a=1$ in Figure 2 (f) is remarkable. This region was found to be unstable even for $\beta>10000$. For $\beta=10000$ the natural frequencies are $[\omega_x, \omega_y, \omega_\theta]=[1.4, 10000, 855]$, even for $\zeta=0.01$. These results show that avoiding equal natural frequencies is no guarantee for avoiding squeal.

4. RESULTS AND CONCLUSIONS

The equations of motion (13- 21) were integrated numerically. The system response for the first 10000 time units was neglected as it was considered to be the transient part of the solution. Response for the next 2000 time units is studied with regard to periodicity etc. Simulations show that for low disk velocities the pad exhibits a stick-slip kind of motion, whereas for higher disk velocities the pad is always in the slip mode. For some parameter combinations impacts between the pad and its surroundings can occur both in $x$ and in $y$ directions. Motions corresponding to four special points in the $h_3-h_4$ plane were analyzed in detail. These points are given by $(h_3, h_4)= (\pm0.05, \pm0.05)$ and for each of these points the other parameters are chosen in such a way that the three linearized natural frequencies are equal. Two other points were chosen for closer investigations. These are $(h_3, h_4, a, \beta)=(0.05, -0.05, 1, 10000)$ and $(h_3, h_4, a, \beta)=(0.05, -0.05, 19, 1.414)$ corresponding to both of which the linearized natural frequencies of the system are widely apart. The results are shown in Figures 3(a) to (f). Each of these figures shows seven different configurations of the pad. The dotted configuration represents the reference position corresponding to $x= y= \theta= 0$, and the other six configurations indicate the extreme positions corresponding to the largest and the smallest value of each of the three coordinates $x$, $y$ and $\theta$. Some of these configurations almost overlap despite a strong magnification. The vibrations of a real disk are likely to be influenced by the pad motions, even though the disk is treated as being rigid in our model. So the assumption here is that the pad drives the disk. Cases in Figure 3(a) to (c) indicate the possibility of impacts between the pad and the disk. For parameters used in Figure (a) the motion of the pad is mainly rotational which indicates that a laterally vibrating disk could possibly form a node at this location. Figure (c) shows pad motions in mainly $x$ direction which indicates the possibility of maximum
Figure 3. Position of the brake pad at maximum and minimum displacement in each coordinate. The displacements are magnified 50 000 times in figures (a) to (d), 50 times in (e) and 150 000 times in figure (f). Initial conditions: \( x(0) = y(0) = 0(0) = 0 \). (a) \( \alpha = 1.001, \beta = 2, \xi = -0.05, \eta = 0.05 \); (b) \( \alpha = 1.001, \beta = 2, \xi = 0.05, \eta = 0.05 \); (c) \( \alpha = 1.001, \beta = 2, \xi = 0.05, \eta = 0.05 \); (d) \( \alpha = 1.001, \beta = 2, \xi = 0.05, \eta = -0.05 \); (e) \( \alpha = 1.9, \beta = 1.414, \xi = -0.05, \eta = 0.05, \zeta = 0.05 \); (f) \( \alpha = 1, \beta = 10 000, \xi = 0.05, \eta = -0.05 \).
Figure 4. Time histories and power spectral densities for system: \( h_3 = -0.05, h_4 = 0.05, \alpha = 19, \beta = 1.414, \zeta_1 = \zeta_2 = \zeta_3 = 0.001 \). Initial conditions: \( x(0) = y(0) = \theta(0) = 0 \). (a) time history in x-direction; (b) time history in y-direction; (c) time history in \( \theta \)-direction; (d) time history of the velocity in y-direction; (e) power spectral density of the time history in x-direction; (f) power spectral density of the time history in y-direction; (g) power spectral density of the time history in \( \theta \)-direction.
Figure 5. Time histories and power spectral densities for system: \( h_3 = 0.05, h_4 = -0.05, \alpha = 1, \beta = 10000 \). Initial conditions: \( x(0) = y(0) = \theta(0) = 0 \). (a) time history in \( x \)-direction; (b) time history in \( y \)-direction; (c) time history in \( \theta \)-direction; (d) time history of the velocity in \( y \)-direction; (e) power spectral density of the time history in \( x \)-direction; (f) power spectral density of the time history in \( y \)-direction; (g) power spectral density of the time history in \( \theta \)-direction.
amplitude for the disk vibrations at this location. In case (e) the pad impacts with the stop resulting in large vibration amplitudes. In case (f) the system vibrates without any loss of contact. Figures 4 and 5 show the time histories and power spectral densities for the same parameter combinations as used in cases of Figures 3(e) and (f). Figure 4 shows periodic motions with full contact and Figure 5 shows a stick-slip kind of motion with impacts in the y-direction. It is clear that the dynamics of the system is quite rich including motions of different kinds, as e.g., periodic, chaotic, stick-slip and impacts in both x- and y-direction. For high disk velocities, generally an increase in the brake force results in an increased amplitude of vibrations, the two being almost directly proportional. For low disk velocities, however, if the brake force is increased, the system passes through a transition to stick-slip kind of behavior, beyond which the proportionality between the force and vibration amplitude is no longer valid.

As mentioned earlier the phenomena of squeal may be caused by a number of different factors including geometric non-linearities, possibilities of impacts, velocity dependent frictional forces etc. It is likely that more than one of these factors come into play simultaneously for different parameter combinations. The model studied here confirms that having equal natural frequencies for different modes of vibration is an important reason for squeal [20-23]. But it is not the only explanation as regions of instability are also found even when the different modes of vibrations have quite different frequencies. The model also confirms that the support of the pad, amount of damping, and coefficient of friction are important for the occurrence of squeal [12-14]. The difference in vibration modes of Figure 3 can be an explanation as to why the pad is found to vibrate sometimes at a node [25] and at other times between two consecutive nodes [12, 13].

5. REFERENCES


P. V. Lamarque and C. G. Williams 1938, The Institution of Automobile Engineers, Report No. 8500 B. Brake squeak: the experiences of manufacturers and operators and some preliminary experiments.


