

SEQUENCES OF INDEPENDENT WALSH FUNCTIONS IN BMO

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Abstract: Under examination are the sequences of independent Walsh functions in the space of functions of bounded mean oscillation. We study geometric properties of the subspaces spanned by the sequences; in particular, some necessary and sufficient conditions are found for such a subspace to be complemented.

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1. Introduction and Statement of the Main Results

The space $\text{BMO} = \text{BMO}[0, 1]$ consists of the functions $f \in L_1[0, 1]$ of bounded mean oscillation, that is, satisfying the condition

$$\|f\|_{\text{BMO}} := \sup_I \frac{1}{|I|} \int_I |f(u) - f_I| du < \infty,$$

where the supremum is taken over all intervals $I \subset [0, 1]$ and $f_I := \frac{1}{|I|} \int_I f(u) du$. If we use only the dyadic intervals $I_m^i = ((i-1)2^{-m}, i2^{-m}]$ ($m = 0, 1, 2, \dots$, $i = 1, \dots, 2^m$) in the definition then we obtain the dyadic space BMO_d . Clearly, $\text{BMO} \subset \text{BMO}_d$ and $\|f\|_d := \|f\|_{\text{BMO}_d} \leq \|f\|_{\text{BMO}}$ for all $f \in \text{BMO}$. Moreover, $L_\infty[0, 1] \subset \text{BMO}$ and $\|f\|_{\text{BMO}} \leq \|f\|_{L_\infty}$ for all $f \in L_\infty = L_\infty[0, 1]$. At the same time, $\text{BMO} \neq L_\infty$ and $\text{BMO}_d \neq \text{BMO}$. For example, $\log|s - 1/2|\chi_{[0,1]}(s) \in \text{BMO} \setminus L_\infty$ and $\log|s - 1/2|\chi_{[1/2,1]}(s) \in \text{BMO}_d \setminus \text{BMO}$.

Given an increasing sequence of positive integers $1 \leq p_1 < p_2 < \dots$, consider the sequence of Walsh functions

$$f_k = \prod_{i \in A_k} r_i, \quad A_k \subset \{p_k + 1, p_k + 2, \dots, p_{k+1}\}, \quad (1)$$

where $r_i(t)$ are the Rademacher functions on $[0, 1]$; i.e., $r_i(t) = \text{sgn}[\sin(2^i \pi t)]$ ($i \in \mathbb{N}$).

In view of [1, Theorem 1], the system $\{f_k\}_{k=1}^\infty$ defined by (1) is equivalent in BMO_d to the canonical basis of l_2 ; more exactly, for every finite sequence $a = (a_k)_{k=1}^\infty$, we have

$$\frac{1}{\sqrt{2}} \|a\|_2 \leq \left\| \sum_{k=1}^\infty a_k f_k \right\|_d \leq \sqrt{2} \|a\|_2, \quad (2)$$

where $\|a\|_2 := (\sum_{k=1}^\infty a_k^2)^{1/2}$. In this article it is proven that the subspace $[f_k]$ generated by a sequence $\{f_k\}_{k=1}^\infty$ (i.e. the closed linear span of $\{f_k\}$) is complemented in BMO_d . Recall that the closed subspace Y of a Banach space X is called *complemented* in X if there exists a bounded linear projection $P : X \rightarrow X$ whose range coincides with Y .

The main aim of the present article is the study of a system $\{f_k\}$ defined in (1) in the “usual” BMO space. It is important to note that the last space is invariant under translations in contrast to its

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dyadic version BMO_d and so it plays a more important role in analysis. We demonstrate that (2) may fail in BMO , while geometric properties of the subspace $[f_k]$ (in particular, its complementability) are essentially determined by evenness of the number of elements in the sets A_k . The present results are a natural continuation of those of the article [2] in which similar questions are studied for the Rademacher system. In particular, Theorems 1(a) and 3 can be considered as an extension of Theorems 2 and 4 in [2].

We present the main results of the article, where an expression of the form $f \asymp g$ means that $cf \leq g \leq Cf$ for some constants $c > 0$ and $C > 0$ independent of all or some part of arguments of the norms (seminorms) f and g . Moreover, we assume in what follows that A_k and f_k are defined by (1).

Theorem 1. (a) *If every set A_k contains an odd number of elements then, for some universal constants and every sequence $a = (a_k)_{k=1}^\infty \in l_2$, we have*

$$\left\| \sum_{k=1}^{\infty} a_k f_k \right\|_{\text{BMO}} \asymp \|a\|_2 + \sup_{s \geq 1} \left| \sum_{k=1}^s a_k \right|. \quad (3)$$

(b) *If every set A_k contains an even number of elements then, for some universal constants and every sequence $a = (a_k)_{k=1}^\infty \in l_2$, we have*

$$\left\| \sum_{k=1}^{\infty} a_k f_k \right\|_{\text{BMO}} \asymp \|a\|_2. \quad (4)$$

Theorem 2. *If $\{k_i\}$ ($k_1 < k_2 < \dots$) is the set of all $k \in \mathbb{N}$ for which the number of elements of all A_k is odd then, for some universal constants and every sequence $a = (a_k)_{k=1}^\infty \in l_2$, we have*

$$\left\| \sum_{k=1}^{\infty} a_k f_k \right\|_{\text{BMO}} \asymp \|a\|_2 + \sup_{s \geq 1} \left| \sum_{i=1}^s a_{k_i} \right|. \quad (5)$$

Recall that a sequence $\{x_n\}_{n=1}^\infty$ of elements in a Banach space X is called a *basic sequence* if it is a basis of the closed linear span $[x_n]$. Moreover, a basic sequence $\{x_n\}_{n=1}^\infty$ is referred to as *unconditional* whenever, for every permutation π of positive integers, the sequence $\{x_{\pi(k)}\}_{k=1}^\infty$ is a basic sequence in X as well. As is known (see, for instance, [3, Theorem 1.1]), a basic sequence $\{x_n\}_{n=1}^\infty$ is unconditional in X if and only if the convergence of a series $\sum_{n=1}^\infty a_n x_n$ ($a_n \in \mathbb{R}$) ensures the convergence of a series $\sum_{n=1}^\infty \theta_n a_n x_n$, where $\{\theta_n\}_{n=1}^\infty$ is an arbitrary sequence of signs, i.e., $\theta_n = \pm 1$. Thereby, Theorem 2 yields

Corollary 1. *The following are equivalent:*

- (1) $\{f_k\}_{k=1}^\infty$ is an unconditional basic sequence in BMO ;
- (2) $\{f_k\}_{k=1}^\infty$ is equivalent in BMO to the canonical basis of l_2 ;
- (3) all sets A_k , except possibly finitely many, contain an even number of elements.

The last result gives a necessary and sufficient condition for the subspace $[f_k]$ to be complemented in BMO .

Theorem 3. *The following are equivalent:*

- (1) the subspace $[f_k]$ generated by $\{f_k\}_{k=1}^\infty$ is complemented in BMO ;
- (2) all sets A_k , except possibly finitely many, contain an even number of elements.

2. Proofs

To prove Theorem 1, we need two auxiliary propositions; the former of them is the contents of Problem 12(b) in [4, p. 266] (see its proof with the constants below in [2, Proposition 1]).

Define the functional $A(f) = \sup_{I_1, I_2} |f_{I_1} - f_{I_2}|$, where I_1 and I_2 are adjacent dyadic intervals of the same length.

Proposition 1. For all $f \in L_1[0, 1]$, we have

$$\frac{1}{3}(\|f\|_d + A(f)) \leq \|f\|_{\text{BMO}} \leq 32(\|f\|_d + A(f)). \quad (6)$$

Proposition 2. Let functions f_k be defined in (1) and let a sequence of reals $a = (a_k)$ be finite. Put $f := \sum_{k \geq 1} a_k f_k$.

(i) If every set A_k contains an odd number of elements, then

$$\frac{2}{3} \max_{s \geq 1} \left| \sum_{k=1}^s a_k \right| \leq A(f) \leq 8 \max_{s \geq 1} \left| \sum_{k=1}^s a_k \right|. \quad (7)$$

(ii) If every set A_k contains an even number of elements, then

$$A(f) = 2 \max_{s \geq 1} |a_s|. \quad (8)$$

PROOF. Let I be an arbitrary dyadic interval of length 2^{-r} , i.e., $I = I_r^i = ((i-1)2^{-r}, i2^{-r}]$. Put $m_k := \min A_k$, $M_k := \max A_k$, and $s = s(r) = \max\{k : M_k \leq r\}$. If $M_k > r$ for all k then we set $s = 0$. In this case

$$(f_k)_I = \begin{cases} \operatorname{sgn}(f_k|_I), & k \leq s, \\ 0, & k > s. \end{cases}$$

Indeed, if $k \leq s$ then f_k is constant on I . In the case of $k > s$, we have $f_k = \prod_{i \in B_k \cup C_k} r_i$, and r_i are constant on I for all $i \in B_k$, while $\int_I \prod_{i \in C_k} r_i = 0$. Hence, since r_i , $i \in C_k$, are independent with respect to I , we infer

$$(f_k)_I = \frac{1}{|I|} \int_I f_k(t) dt = \frac{c}{|I|} \int_I \prod_{i \in C_k} r_i(t) dt = \frac{c}{|I|} \prod_{i \in C_k} \int_I r_i(t) dt = 0,$$

where c is the value of $\prod_{i \in B_k} r_i$ on I .

Therefore, if I_1 and I_2 are adjacent dyadic intervals of length 2^{-r} then

$$f_{I_1} - f_{I_2} = \sum_{k=1}^s a_k \operatorname{sgn}(f_k|_{I_1}) - \sum_{k=1}^s a_k \operatorname{sgn}(f_k|_{I_2}) = \sum_{k=1}^s a_k [\operatorname{sgn}(f_k|_{I_1}) - \operatorname{sgn}(f_k|_{I_2})]. \quad (9)$$

Let I be the least dyadic interval containing the union of I_1 and I_2 . Assume that its length is 2^{-j} . The definition of I implies that the union $I_1 \cup I_2$ lies in the middle of I (let I_1 lie on the left). As is easily seen, the index j runs over all integers from 0 to $r-1$ in dependence on I_1 and I_2 .

Below, we need the following notation: If $a, b \in \mathbb{N}$ and $a \leq b$ then $[a, b] := \{i \in \mathbb{N} : a \leq i \leq b\}$. In view of the definition of s , we have $\bigcup_{k=1}^s A_k \subset [1, r]$. Put

$$B := \bigcup_{l=1}^s [m_l, M_l] \quad \text{and} \quad B' := [1, r] \setminus B = \bigcup_{l=1}^{s+1} [M_{l-1} + 1, m_l - 1],$$

where $M_0 = 0$ and $m_{s+1} = r + 1$. Moreover, in what follows, $|A|$ stands for the number of elements in a set $A \subset \mathbb{N}$.

Let us compare the values of f_k , $1 \leq k \leq s$, on the intervals I_1 and I_2 . If $m_k > j+1$, then all r_i with indices from A_k change their signs under the passage from I_1 to I_2 ; in this case $r_i|_{I_1} = -1$ and $r_i|_{I_2} = 1$. Hence, for these k , we have $f_k|_{I_1} = (-1)^{|A_k|}$ and $f_k|_{I_2} = 1$. If $M_k \leq j$ then all r_i with indices from A_k are constant on I and so $f_k|_{I_1} = f_k|_{I_2}$. In dependence on which set B or B' contains the number $j+1$, we examine two cases.

(a) $j + 1 \in B'$. In other words, $M_{l-1} + 1 \leq j + 1 \leq m_l - 1$ for some $l = 1, 2, \dots, s+1$. If $l = s+1$ then $j \geq M_s$ and so from (9) and the above remarks it follows that $f_{I_1} - f_{I_2} = 0$. In the case when the previous inequality is fulfilled for $l = 1, 2, \dots, s$, we see that $M_{l-1} \leq j < m_l - 2$ and, by the same reasons,

$$f_{I_1} - f_{I_2} = \sum_{k=l}^s a_k (f_k|_{I_1} - f_k|_{I_2}) = \sum_{k=l}^s a_k ((-1)^{|A_k|} - 1). \quad (10)$$

Since j takes all values from 0 to $r-1$, equation (10) is true for $l \in \mathcal{F}_1$, where the set \mathcal{F}_1 consists of those numbers k for which the segment $[M_{k-1} + 1, m_k - 1]$ is nonempty (if $2 \leq k \leq s$, then the latter is equivalent to the fact that there is a “gap” between the sets A_{k-1} and A_k ; $1 \in \mathcal{F}_1$ if $1 \notin A_1$).

(b) $j + 1 \in B$. In this case $m_l \leq j + 1 \leq M_l$ for some $l = 1, 2, \dots, s$ and the functions r_i change their signs under the passage from I_1 to I_2 if $i \in A_l^{(j)} := \{p \in A_l : p > j\}$ and they are constant on I if $i \in A_l \setminus A_l^{(j)}$. Hence, $f_l|_{I_2} = (-1)^{|A_l^{(j)}|} f_l|_{I_1}$. Since $j + 1 < m_{l+1}$ and $j \geq M_{l-1}$, taking into account the above remarks, by (9), we establish

$$f_{I_1} - f_{I_2} = a_l f_l|_{I_1} (1 - (-1)^{|A_l^{(j)}|}) + \sum_{k=l+1}^s a_k ((-1)^{|A_k|} - 1). \quad (11)$$

Assume that all numbers $|A_k|$ are odd. First of all, (10) implies that

$$|f_{I_1} - f_{I_2}| = 2 \left| \sum_{k=l}^s a_k \right|, \quad l \in \mathcal{F}_1. \quad (12)$$

If there exists j such that $m_l \leq j + 1 \leq M_l$ and either $f_l|_{I_1} = -1$ or $|A_l^{(j)}|$ is even, then (11) ensures (12) (in the second case l should be replaced with $l+1$). Denote by \mathcal{F} the set of l such that (12) holds. Clearly, $\mathcal{F} \supset \mathcal{F}_1$.

It is easy to see that, for $l = 1, 2, \dots, s$, we have one more possibility. If $j = m_l - 1$ then $A_l^{(j)} = A_l$. Since $r_{m_l}|_{I_1} = 1$ and $r_i|_{I_1} = -1$, if $i \in A_l$, $i \neq m_l$ then $f_l|_{I_1} = 1$. Hence, (11) yields

$$|f_{I_1} - f_{I_2}| = 2 \left| \sum_{k=l+1}^s a_k - a_l \right|, \quad l = 1, 2, \dots, s. \quad (13)$$

For a given pair of adjacent dyadic intervals I_1 and I_2 , the quantity $|f_{I_1} - f_{I_2}|$ is determined by equalities (12) or (13). Therefore, according to the definition of the functional $A(f)$ we obtain

$$A(f) = 2 \max \left(\max_{1 \leq l \leq s < \infty} \left| \sum_{k=l+1}^s a_k - a_l \right|, \max_{\substack{1 \leq l \leq s < \infty \\ l \in \mathcal{F}}} \left| \sum_{k=l}^s a_k \right| \right).$$

Moreover, every finite sequence (a_k) satisfies the inequality

$$\frac{1}{3} \max_{1 \leq l \leq s < \infty} \left| \sum_{k=l}^s a_k \right| \leq \max_{1 \leq l \leq s < \infty} \left| \sum_{k=l+1}^s a_k - a_l \right| \leq 2 \max_{1 \leq l \leq s < \infty} \left| \sum_{k=l}^s a_k \right|. \quad (14)$$

Indeed, on the one hand, for arbitrary $1 \leq l \leq s$, we have

$$\left| \sum_{k=l}^s a_k \right| \leq \left| \sum_{k=l+1}^s a_k - a_l \right| + 2|a_l| \leq 3 \max_{1 \leq l \leq s < \infty} \left| \sum_{k=l+1}^s a_k - a_l \right|$$

(letting in the case $s = l$ the sum $\sum_{k=l+1}^s a_k$ to be equal to zero). Conversely, for $1 \leq l \leq s$, we infer

$$\left| \sum_{k=l+1}^s a_k - a_l \right| \leq \left| \sum_{k=l+1}^s a_k \right| + |a_l| \leq 2 \max_{1 \leq l \leq s < \infty} \left| \sum_{k=l}^s a_k \right|.$$

Moreover,

$$\max_{s \geq 1} \left| \sum_{k=1}^s a_k \right| \leq \max_{1 \leq l \leq s < \infty} \left| \sum_{k=l}^s a_k \right| \leq 2 \max_{s \geq 1} \left| \sum_{k=1}^s a_k \right|. \quad (15)$$

The left inequality in (15) is obvious and the right one follows from the estimate

$$\left| \sum_{k=l}^s a_k \right| \leq \left| \sum_{k=1}^s a_k - \sum_{k=1}^{l-1} a_k \right| \leq \left| \sum_{k=1}^s a_k \right| + \left| \sum_{k=1}^{l-1} a_k \right| \leq 2 \max_{s \geq 1} \left| \sum_{k=1}^s a_k \right|$$

which is valid for arbitrary $1 \leq l \leq s$. As a result, the left inequality in (7) follows from the above inequality for $A(f)$ and the left inequality in (14). To obtain the right inequality in (7), it suffices to use the right-hand sides in (14) and (15).

The case when every set A_k contains an even number of elements is much simpler. In fact, equalities (10) and (11) ensure that either $f_{I_1} - f_{I_2} = 0$ or $|f_{I_1} - f_{I_2}| = 2|a_l|$, $l = 1, 2, \dots, s$. Thereby, (8) is proved.

PROOF OF THEOREM 1. It suffices to apply Propositions 1 and 2 together with (2).

PROOF OF THEOREM 2. Let $f := \sum_{k=1}^{\infty} a_k f_k$. Relations (10) and (11) and the same arguments as in the proof of Proposition 2 imply that

$$A(f) \asymp \max \left(\sup_{k=1,2,\dots} |a_k|, \sup_{s \geq 1} \left| \sum_{i=1}^s a_k \right| \right),$$

where $\{k_i\}$ ($k_1 < k_2 < \dots$) is the set of all $k \in \mathbb{N}$ such that $|A_k|$ is odd. Thereby, Proposition 1 and (2) justify the claim.

To prove Theorem 3, we need one more auxiliary statement about the block bases of a sequence $\{f_k\}_{k=1}^{\infty}$ in BMO.

Denote by $U = \{u_n\}_{n=1}^{\infty}$ an arbitrary block basis of the sequence $\{f_k\}_{k=1}^{\infty}$, i.e.,

$$u_n = \sum_{k=s_n+1}^{s_{n+1}} a_k f_k, \quad n = 1, 2, \dots,$$

where $1 \leq s_1 < s_2 < \dots$ and $a_k \in \mathbb{R}$. Moreover, let

$$\gamma_n(U) = \sum_{k=s_n+1}^{s_{n+1}} a_k, \quad n = 1, 2, \dots$$

Proposition 3. *If a sequence $\{f_k\}_{k=1}^{\infty}$ contains infinitely many elements with an odd $|A_k|$, then the closed linear span $[f_k] \subset \text{BMO}$ contains a subspace E isomorphic to c_0 and complemented in $[f_k]$.*

PROOF. Let \mathcal{E} be the set of all $k \in \mathbb{N}$ for which $|A_k|$ is odd. By condition, $|\mathcal{E}| = \infty$. Propositions 1 and 2 together with (2) ensure the existence of a block basis $U = \{u_n\}_{n=1}^{\infty}$ satisfying the conditions

- (a) $\|u_n\|_{\text{BMO}} = 1$, $n = 1, 2, \dots$;
- (b) $\|u_n\|_d \leq \sqrt{2} \left(\sum_{k=s_n+1}^{s_{n+1}} a_k^2 \right)^{1/2} \leq 2^{-n-5}$, $n = 1, 2, \dots$;
- (c) $\gamma_n(U) = 0$, $n = 1, 2, \dots$;
- (d) $\{k : a_k \neq 0\} \subset \mathcal{E}$.

Indeed, assume that we already constructed u_1, \dots, u_{n-1} . Given $n \in \mathbb{N}$, there exist a sufficiently large $m_n \in \mathbb{N}$ and numbers $b_k \geq 0$, $k = 1, 2, \dots, m_n$ such that

$$\left(\sum_{k=1}^{m_n} b_k^2 \right)^{1/2} \leq \frac{2}{9} \cdot 2^{-n-7} \quad \text{and} \quad \sum_{k=1}^{m_n} b_k = 1. \quad (16)$$

Put $a'_k = b_k$ if $k = 1, 2, \dots, m_n$ and $a'_k = -b_{k-m_n}$, if $k = m_n + 1, \dots, 2m_n$. Choose a sufficiently large number $s_{n+1} > s_n$ such that the segment $\{s_n + 1, \dots, s_{n+1}\}$ of positive integers contains at least $2m_n$ elements of \mathcal{E} , and introduce new indices for a'_k from the intersection $\{s_n + 1, \dots, s_{n+1}\} \cap \mathcal{E}$ preserving the order. For the remaining numbers k from the set $\{s_n + 1, \dots, s_{n+1}\}$, we put $a'_k = 0$. If $u'_n := \sum_{k=s_n+1}^{s_{n+1}} a'_k f_k$, then, in view of (2) and the first relation in (16), we find that

$$\|u'_n\|_d \leq \sqrt{2} \cdot 2 \left(\sum_{k=1}^{m_n} b_k^2 \right)^{1/2} \leq \frac{2}{9} \cdot 2^{-n-5}.$$

Moreover, since the definition of a'_k and the second equality in (16) imply that

$$\max_{s_n+1 \leq k \leq s_{n+1}} \left| \sum_{k=s_n+1}^s a'_k \right| = \sum_{k=1}^{m_n} b_k = 1,$$

Propositions 1 and 2 yield

$$\frac{2}{3} \leq A(u'_n) \leq 8 \quad \text{and} \quad \frac{2}{9} \leq \|u'_n\|_{\text{BMO}} \leq 288.$$

Thereby, it is easy to see that the function $u_n := \frac{u'_n}{\|u'_n\|_{\text{BMO}}}$ meets (a), (b), and (d). Condition (c) holds since $\sum_{k=s_n+1}^{s_{n+1}} a'_k = 0$ by construction.

Let us show that the subspace $E := [u_n]$, spanned by the functions of this block basis in BMO, is isomorphic to c_0 .

Take $f \in [u_n]$. If $f = \sum_{n=1}^{\infty} \beta_n u_n$ ($\beta_n \in \mathbb{R}$) then

$$f = \sum_{n=1}^{\infty} \left(\sum_{k=s_n+1}^{s_{n+1}} \beta_n a_k f_k \right) = \sum_{k=1}^{\infty} \gamma_k f_k,$$

where $\gamma_k = \beta_n a_k$ for $k = s_n + 1, \dots, s_{n+1}$. Assuming that $p, q \in \mathbb{N}$ satisfy the inequalities $s_{n-1} \leq p < s_n < s_{n+l} < q \leq s_{n+l+1}$ with some positive integers n and l , we estimate the sum $\sum_{k=p}^q \gamma_k$. In view of (d), (c), and (a), together with Propositions 1 and 2(i), we infer

$$\begin{aligned} \left| \sum_{k=p}^q \gamma_k \right| &= \left| \sum_{k=p}^{s_n} \gamma_k + \sum_{k=s_n+1}^{s_{n+l}} \gamma_k + \sum_{k=s_{n+l}+1}^q \gamma_k \right| \\ &= \left| \sum_{k=p}^{s_n} \beta_{n-1} a_k + \sum_{i=n}^{n+l-1} \sum_{k=s_i+1}^{s_{i+1}} \beta_i a_k + \sum_{k=s_{n+l}+1}^q \beta_{n+l} a_k \right| \\ &\leq |\beta_{n-1}| \left| \sum_{k=p}^{s_n} a_k \right| + \sum_{i=n}^{n+l-1} |\beta_i| \left| \sum_{k=s_i+1}^{s_{i+1}} a_k \right| + |\beta_{n+l}| \left| \sum_{k=s_{n+l}+1}^q a_k \right| \\ &\leq \sup_n |\beta_n| \left(\left| \sum_{k=p}^{s_n} a_k \right| + \left| \sum_{k=s_{n+l}+1}^q a_k \right| \right) \leq \sup_n |\beta_n| \left(\frac{3}{2} A(u_{n-1}) + \frac{3}{2} A(u_{n+l}) \right) \\ &\leq \frac{9}{2} (\|u_{n-1}\|_{\text{BMO}} + \|u_{n+l}\|_{\text{BMO}}) \|\{\beta_n\}\|_{c_0} = 9 \|\{\beta_n\}\|_{c_0}. \end{aligned}$$

Applying Theorem 1(a), inequality (2), and properties (a) and (b), we find that

$$\begin{aligned} \|f\|_{\text{BMO}} &\leq C \left(\|f\|_d + \sup_{q \geq 1} \left| \sum_{k=1}^q \gamma_k \right| \right) \leq C \left(\left\| \sum_{n=1}^{\infty} \beta_n u_n \right\|_d + 9 \|\{\beta_n\}\|_{c_0} \right) \\ &\leq C \left(\sum_{n=1}^{\infty} 2^{-n} + 9 \right) \|\{\beta_n\}\|_{c_0} = 10C \|\{\beta_n\}\|_{c_0}. \end{aligned}$$

On the other hand, Proposition 1, (a), and (b) imply that

$$A(u_n) \geq \frac{1}{32} \|u_n\|_{\text{BMO}} - \|u_n\|_d \geq \frac{1}{32} - 2^{-n-5} \geq \frac{1}{64}$$

for all $n \in \mathbb{N}$. Hence, applying Propositions 1 and 2 once again (also see inequalities (7) and (15)), we arrive at the inequality

$$\|f\|_{\text{BMO}} \geq \frac{1}{3} A(f) \geq \frac{1}{9} \sup_{n \in \mathbb{N}} \sup_{s_n+1 \leq p < q \leq s_{n+1}} \left| \sum_{k=p}^q \beta_n a_k \right| \geq \frac{1}{72} A(u_n) \|\{\beta_n\}\|_{c_0} \geq \frac{1}{72 \cdot 64} \|\{\beta_n\}\|_{c_0}.$$

Thus, the system $\{u_n\}$ is equivalent in BMO to the canonical basis of c_0 , and so E is isomorphic to c_0 . Since $[f_k]$ is separable, by the Sobczyk theorem (see, for instance, [5, Corollary 2.5.9]), E is complemented in $[f_k]_{k=1}^\infty$ and the proposition is proved.

PROOF OF THEOREM 3. First, assume that all sets A_k , except possibly finitely many, contain an even number of elements. By Corollary 1, $\{f_k\}_{k=1}^\infty$ is equivalent in BMO to the canonical basis of l_2 . On the other hand, one of the consequences of the classical John–Nirenberg inequality (see, for instance, [6, Remark 3.19]) is the embedding $\text{BMO} \subset L_p[0, 1]$ for $1 \leq p < \infty$. Since $\{f_k\}_{k=1}^\infty$ is an orthonormal system in $L_2[0, 1]$, the corresponding orthogonal projection P is bounded in $L_2[0, 1]$, its range is the subspace $[f_k]$, and the sequence $\{f_k\}_{k=1}^\infty$ in $L_2[0, 1]$ is also equivalent to the canonical basis of l_2 . Hence, for every function $f \in \text{BMO}$, we have

$$\|Pf\|_{\text{BMO}} \asymp \|Pf\|_{L_2} \leq \|P\| \|f\|_{L_2} \leq C \|P\| \|f\|_{\text{BMO}},$$

i.e., P is bounded in BMO and its range coincides with the closed linear span $[f_k]$ in this space. Thereby, the subspace $[f_k]$ is complemented in BMO.

We prove the converse. Assume that $[f_k]$ is complemented in BMO and the set \mathcal{E} of the numbers $k \in \mathbb{N}$ with an odd $|A_k|$ is infinite. Let $P_1 : \text{BMO} \rightarrow [f_k]$ be a bounded linear projection whose range coincides with $[f_k]$. By Proposition 3, there exists a subspace E complemented in $[f_k]$ and isomorphic to c_0 . Let $P_2 : [f_k] \rightarrow E$ be a bounded linear projection whose range is E . In this case $P := P_2 \circ P_1$ is a linear projection bounded in BMO with the range E . Hence, BMO contains a complemented subspace isomorphic to c_0 . Since BMO is a dual space (more exactly, $\text{BMO} = (\text{Re } H_1)^*$; see, for instance, [3, Theorem 5.5]), it contradicts to the known Bessaga–Pełczyński result that a dual space does not contain a complemented subspace isomorphic to c_0 (see [7, Corollary 4]). Thus, if the set \mathcal{E} of numbers $k \in \mathbb{N}$ with an odd $|A_k|$ is infinite, the subspace $[f_k]$ is not complemented in BMO, and the theorem is proved. \square

References

1. Müller P. F. and Schechtman G., “On complemented subspaces of H^1 and VMO,” Lecture Notes Math., **1376**, 113–125 (1989).
2. Astashkin S. V., Leibov M., and Maligranda L., “Rademacher functions in BMO,” Studia Math., **205**, No. 1, 83–100 (2011).
3. Kashin B. S. and Saakyan A. A., Orthogonal Series, Amer. Math. Soc., Providence (1989).
4. Garnett J. B., Bounded Analytic Functions, Springer-Verlag, New York (2007).
5. Albiac F. and Kalton N. J., Topics in Banach Space Theory (Graduated Texts in Mathematics; V. 233), Springer-Verlag, New York (2006)
6. Korenovskii A., Mean Oscillations and Equimeasurable Rearrangements of Functions, Springer-Verlag, Berlin and Heidelberg (2007).
7. Bessaga C. and Pełczyński A., “Some remarks on conjugate spaces containing subspaces isomorphic to the space c_0 ,” Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., **6**, 249–250 (1958).

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