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INTERPOLATION BETWEEN L_1 AND L_p, $1 < p < \infty$

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Abstract. We show that if X is a rearrangement invariant space on $[0,1]$ that is an interpolation space between L_1 and L_∞ and for which we have only a one-sided estimate of the Boyd index $\alpha(X) > 1/p, 1 < p < \infty$, then X is an interpolation space between L_1 and L_p. This gives a positive answer for a question posed by Semenov. We also present the one-sided interpolation theorem about operators of strong type $(1,1)$ and weak type $(p,p), 1 < p < \infty$.

1. Introduction

Let X be a rearrangement invariant (r. i.) function space on $I = [0,1]$. If any linear operator T is bounded in the spaces L_p and L_q $(1 \leq p < q \leq \infty)$, and the space X is such that the Boyd indices satisfy the estimates

$$\frac{1}{q} < \alpha(X) \leq \beta(X) < \frac{1}{p},$$

then the operator T is also bounded in X. We can say, for short, that X is an interpolation space between L_p and L_q. This theorem was proved by Boyd [B67] in 1967 under the assumption that the space X has the Fatou property (cf. also Boyd [B69], Bennett-Sharpley [BS], Th. 5.16, and Lindenstrauss-Tzafriri [LT], Th. 2.b.11) when the space X has either the Fatou property or is separable (cf. also [JMST], p. 215). For arbitrary r. i. X this can be proved by using Calderón's estimate, as in the Boyd proof, and then Semenov's Lemma 4.7 from [KPS]. Implicitly this result also follows from Th. 6.12 in [KPS].

In the case when $q = \infty$, i.e., one space is the extreme space L_∞, and we have a *one-sided* estimate for an r. i. space X,

$$\beta(X) < \frac{1}{p},\ 1 \leq p < \infty,$$

we obtain that X is an interpolation space between L_p and L_∞ (see [M81], Th. 4.6, which is proved even for Lipschitz operators).

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E. M. Semenov posed the following question about the extreme space L_1: *Let X be an r. i. space on $[0,1]$ with either the Fatou property or absolutely continuous norm. Is the one-sided estimate on the Boyd index $\alpha(X) > \frac{1}{p}$ enough for the interpolation property between L_1 and $L_p, 1 < p < \infty$?*

We first show, by using duality arguments, that the answer is positive.

After this was answered a more general question appeared. Namely, is it true that if X is an interpolation space between L_1 and L_∞ and we have a one-sided estimate $\alpha(X) > \frac{1}{p}$, then X is an interpolation space between L_1 and L_p?

The answer to this question is also positive, and the proof is suprisingly not very complicated.

Finally, we were able to also get the one-sided interpolation theorem for operators of strong type $(1,1)$ and weak type $(p,p), 1 < p < \infty$.

The paper is organized as follows. In Section 2 we collect some necessary definitions and notation.

Section 3 contains two proofs of the answer to the Semenov question. The first proof is obtained via associated duality arguments, and the second one by using estimates typical in the real interpolation theory.

The main result of the paper is Theorem 2, which shows that a one-sided estimate $\alpha(X) > \frac{1}{p}$ on an r. i. space and an interpolation property of X between L_1 and L_∞ are enough for the interpolation property of X between L_1 and L_p. The second assumption that X is an interpolation space between L_1 and L_∞ is necessary (cf. Example 1).

Section 4 deals with operators of strong type $(1,1)$ and weak type $(p,p), 1 < p < \infty$. We are proving a one-sided interpolation theorem for such operators.

2. Definitions and notation

We first recall some basic definitions. If a Banach space $X = (X, \|\cdot\|)$ of all (classes of) measurable functions $x(t)$ on $I = [0,1]$ is such that there exists $u \in X$ with $u > 0$ a.e. on I and $\|x\| \le \|y\|$ whenever $|x| \le |y|$, we say that X is a *Banach function space* (on $I = [0,1]$). A Banach function space X on $I = [0,1]$ is said to be a *rearrangement invariant* (r. i.) space provided $x^*(t) \le y^*(t)$ for every $t \in [0,1]$ and $y \in X$ imply $x \in X$ and $\|x\|_X \le \|y\|_X$, where x^* denotes the decreasing rearrangement of $|x|$. We always have the imbeddings $L_\infty[0,1] \subset X \subset L_1[0,1]$. By X^0 we will denote the closure of $L_\infty[0,1]$ in X.

If χ_A denotes the characteristic function of a measurable set A in I, then clearly $\|\chi_A\|_X$ depends only on $m(A)$. The function $\varphi_X(t) = \|\chi_A\|_X$, where $m(A) = t, t \in I$, is called the *fundamental function* of X.

Given $s > 0$, the dilation operator σ_s given by $\sigma_s x(t) = x(t/s)\chi_I(t/s), t \in I$ is well defined in every r. i. space X and $\|\sigma_s\|_{X\to X} \le \max(1,s)$. The *Boyd indices of* X are defined by (cf. [KPS], [LT], [BS])

$$\alpha(X) = \lim_{s\to 0} \frac{\ln \|\sigma_s\|_{X\to X}}{\ln s},\ \beta(X) = \lim_{s\to\infty} \frac{\ln \|\sigma_s\|_{X\to X}}{\ln s}.$$

In general, $0 \le \alpha(X) \le \beta(X) \le 1$. It is easy to see that $\bar{\varphi}_X(t) \le \|\sigma_t\|_{X\to X}$ for any $t > 0$, where $\bar{\varphi}_X(t) = \sup_{0<s<1, 0<st<1} \frac{\varphi_X(st)}{\varphi_X(s)}$. A Banach function space X with a norm $\|\cdot\|_X$ has

(a) the *Fatou property* if for any increasing positive sequence $0 \le x_n \nearrow x, x_n \in X$ with $\sup_n \|x_n\|_X < \infty$ we have that $x \in X$ and $\|x_n\|_X \nearrow \|x\|_X$;

(b) *absolutely continuous* norm if for any $x \in X$ and every sequence x_n of measurable functions on $I = [0, 1]$ satisfying $|x| \geq x_n \downarrow 0$ we have $\|x_n\|_X \to 0$.

Note that X is separable if and only if the norm of X is an absolutely continuous norm. From the Calderón-Mitjagin theorem it follows that the r. i. space X with either the Fatou property or the separable property is an interpolation space with respect to L_1 and L_∞, i.e., if a linear operator T is bounded in L_1 and L_∞, then T is bounded in X and $\|T\|_{X \to X} \leq C \max(\|T\|_{L_1 \to L_1}, \|T\|_{L_\infty \to L_\infty})$ for some $C \geq 1$ ([KPS], [BS]).

The associated space X' to a Banach function space X is the space of all (classes of) measurable functions $x(t)$ on $I = [0, 1]$ such that $\int_0^1 |x(t)y(t)|dt < \infty$ for every $y \in X$ endowed with the norm

$$\|x\|_{X'} = \sup\{\int_0^1 |x(t)y(t)|dt : \|y\|_X \leq 1\}.$$

X' is a Banach function space. We have the embedding $X \subset X''$ with $\|x\|_{X''} \leq \|x\|_X$ for every $x \in X$. Moreover, $X = X''$ with equality of the norms if and only if X has the Fatou property (cf. [KPS], [LT]). If a Banach function space X is separable, then the embedding $X \subset X''$ is isometric and $X' = X^*$. If X is an r. i. space, then the associated space X' is also a r. i. space.

Among classical r. i. spaces with the Fatou property we mention Lorentz spaces $L_{p,q}$, Lorentz spaces Λ_φ and $\Lambda_{p,\varphi}$, Marcinkiewicz spaces M_φ and Orlicz spaces L_Φ. Typical separable spaces are M_φ^0 and E_Φ, the closures of L_∞ in Marcinkiewicz space M_φ and Orlicz space L_Φ, respectively.

For other general properties of lattices of measurable functions and r. i. spaces, we refer to the books [LT], [KPS], [BS].

3. Strong interpolation of L_1 and L_p, $1 < p < \infty$

Our first proof will use the notion of the associated operator. This notion was considered in Banach function spaces with the Fatou property, for example, by Gribanov [G].

Lemma 1. *If X is a separable Banach function space on $I = [0, 1]$ and $T : X \to X$ is a bounded linear operator, then there exists an associated operator $T' : X' \to X'$, which is linear and bounded, given by*

$$\int_0^1 x(s)T'y(s)ds = \int_0^1 Tx(s)y(s)ds \tag{1}$$

for all $x \in X$ and $y \in X'$.

The proof is clear since the dual space $X^\star$ coincides isometrically with the associated space X'. Note that T' is unique and $\|T'\|_{X' \to X'} = \|T\|_{X \to X}$.

Theorem 1. *Let $1 < p < \infty$. If an r. i. space X has either the Fatou property or is separable and $\alpha(X) > \frac{1}{p}$, then X is an interpolation space between L_1 and L_p.*

Proof. We will first show that if $T : L_1 \to L_1$ is a bounded linear operator such that $T = T_{|L_p} : L_p \to L_p$ is bounded, then

$$T = T_{|X^0} : X^0 \to X'' \tag{2}$$

is also bounded, where X^0 means the closure of L_∞ in X.

Let T_1' be the associated operator to a linear bounded operator $T : L_1 \to L_1$, and let T_2' be the associated operator to $T_{|L_p} : L_p \to L_p$. Then, for all $x \in L_p$ and $y \in L_\infty$, we have

$$\int_0^1 Tx(s)y(s)ds = \int_0^1 x(s)T_1'y(s)ds = \int_0^1 x(s)T_2'y(s)ds.$$

Hence, $T'_{2|L_\infty} = T_1'$ and we can consider $T' = T_1' = T_2'$. Then

(3) $\quad T' : L_{p'} \to L_{p'}$ and $T' : L_\infty \to L_\infty$ is bounded, where $1/p' + 1/p = 1$.

Since X is isometrically embedded into X'', it follows that $\beta(X') = 1 - \alpha(X) < 1 - \frac{1}{p}$ (cf. [KPS], Th. 4.11), and by the Boyd interpolation theorem we have that X' is an interpolation space between $L_{p'}$ and L_∞. Therefore,

$$T' : X' \to X' \text{ is bounded.} \tag{4}$$

In view of (3) and Lemma 1 there exists the second associated operator $T'' : L_p \to L_p$, which is bounded.

We can extend T'' to the whole space L_1. In fact, if $x \in L_\infty$ and $y \in L_p$, then by (3),

$$\begin{aligned} \|T''y\|_1 &= \sup_{\|x\|_\infty \le 1} \int_0^1 x(s)T''y(s)ds \\ &= \sup_{\|x\|_\infty \le 1} \int_0^1 T'x(s)y(s)ds \le \|T'\|_{L_\infty \to L_\infty} \|y\|_1. \end{aligned}$$

Since L_p is dense in L_1, it follows that $T'' : L_1 \to L_1$ is bounded. The uniqueness shows that $T'' = T$. On the other hand, $X' \subset L_{p'}$ and for all $y \in X'$ and $x \in L_p$ we have by (4) that

$$\begin{aligned} \|Tx\|_{X''} &= \sup_{\|y\|_{X'} \le 1} \int_0^1 Tx(s)y(s)ds \\ &= \sup_{\|y\|_{X'} \le 1} \int_0^1 x(s)T'y(s)ds \le \|T'\|_{X' \to X'} \|x\|_X. \end{aligned}$$

Thus,

$$\|Tx\|_{X''} \le \|T'\|_{X' \to X'} \|x\|_X$$

for all $x \in L_p \subset X$, which gives (2).

Now, let X be a separable r. i. space. For every $x \in X$ we can find a sequence $\{x_n\} \subset L_p$ such that $x_n \to x$ in X. From (2) we obtain $Tx_n \to Tx$ in X''. Moreover, $\{Tx_n\} \subset L_p \subset X$. Since $\{Tx_n\}$ is a Cauchy sequence in X'' and X is isometrically embedded in X'', it follows that $\{Tx_n\}$ is a Cauchy sequence in X and so $Tx \in X$.

Suppose that the r. i. space X has the Fatou property or, equivalently, $X = X''$. Let $y = Tx$, where $x \in X$ and $T : L_1 \to L_1$ is a bounded linear operator such that $T = T_{|L_p} : L_p \to L_p$ is bounded. Then

$$K(t, y; L_1, L_p) \le CK(t, x; L_1, L_p) \;\forall\; t \in (0, 1]. \tag{5}$$

There is a sequence of step-functions $\{y_m\}$ such that $y_m \uparrow |y|$ a.e. on $[0, 1]$. We can take, also, the truncations

$$x_n(s) = \min\{|x(s)|, n\}, \;\; n = 1, 2, \ldots.$$

Since $x_n \uparrow |x|$ a.e. on $[0,1]$, it follows that $x_n^* \uparrow x^*$ a.e. on $[0,1]$ (cf. [KPS], p. 67). By using (5) and the Holmstedt formula (cf. [H], Th. 4.1),

$$K(t,x;L_1,L_p) \approx \int_0^{t^{p'}} x^*(s)ds + t\left(\int_{t^{p'}}^1 x^*(s)^p ds\right)^{1/p}$$

with $x \in L_1$ and $0 < t \le 1$, we can show that for every $m = 1, 2, \ldots$ there exists $n_m \in \mathbf{N}$ such that

$$K(t, y_m; L_1, L_p) \le C_1 K(t, x_{n_m}; L_1, L_p) \ \forall\ t \in (0,1]$$

with a constant $C_1 > 0$ independent of m. Since $\alpha(X^0) = \alpha(X) > \frac{1}{p}$ (cf. [KPS], p. 143), then, by above, the separable space X^0 is an interpolation space for the couple (L_1, L_p). But (L_1, L_p) is a K-monotone couple (see [C], Th. 4). Therefore, from the last inequality we have

$$\|y_m\|_X = \|y_m\|_{X^0} \le C_2\|x_{n_m}\|_{X^0} \le C_2\|x\|_X$$

with some constant $C_2 > 0$. Since $X = X''$, it follows that

$$\|y\|_X \le C_2\|x\|_X,$$

and this completes the proof. $\square$

Remark 1. Using results from [Ms], the proof of Theorem 1 can be a little shorter. Once we have proved that the associated operator T' is bounded in the spaces L_∞ and $L_{p'}$, and the second associated operator T'' exists, then from Theorem 3.5(b) in [Ms] it follows that $T'' = T$ is bounded in X''. In this paper we can also find some sufficient conditions under which from the interpolation property of the space X between spaces X_0, X_1 follows the interpolation property of its associated space X' between the associated spaces X_0', X_1'.

Theorem 2. *Let $1 < p < \infty$. If X is an interpolation r. i. space between L_1 and L_∞, and $\alpha(X) > \frac{1}{p}$, then X is an interpolation space between L_1 and L_p.*

Proof. If $T : L_1 \to L_1$ is a bounded linear operator such that $T = T_{|L_p} : L_p \to L_p$ is bounded, then

$$K(t^{1-1/p}, Tx; L_1, L_p) \le C_3 K(t^{1-1/p}, x; L_1, L_p) \le C_3 K(t^{1-1/p}, x; L_1, L_{p,1})$$

for any $x \in L_1$ and all $0 < t \le 1$, where $L_{p,1}$ is the Lorentz space generated by the norm

$$\|x\|_{p,1} = \int_0^1 t^{1/p-1}x^*(t)dt.$$

Using Holmstedt's formulas (cf. [H], Th. 4.1 and Th. 4.2) we obtain an estimate

$$\int_0^t (Tx)^*(s)ds \le C_4\left(\int_0^t x^*(s)ds + t^{1-1/p}\int_t^1 s^{1/p-1}x^*(s)ds\right) \tag{6}$$

for any $x \in L_1$ and all $0 < t \le 1$. For the linear operator

$$Mx(t) = t^{-1/p}\int_t^1 s^{1/p-1}x(s)ds, \ 0 < t \le 1,$$

we obtain, by using the Fubini theorem,

$$\begin{aligned}\int_0^t (Mx^*)^*(s)ds &= \int_0^t s^{-1/p}\left(\int_s^1 u^{1/p-1}x^*(u)du\right)ds \\ &= \int_0^t \left(\int_0^u s^{-1/p}ds\right)u^{1/p-1}x^*(u)du \\ &\qquad + \int_t^1 \left(\int_0^t s^{-1/p}ds\right)u^{1/p-1}x^*(u)du \\ &= p'\left(\int_0^t x^*(u)du + t^{1-1/p}\int_t^1 u^{1/p-1}x^*(u)du\right) \\ &\approx K(t^{1-1/p}, x; L_1, L_{p,1}).\end{aligned}$$

Therefore, in view of (6),

$$\int_0^t (Tx)^*(s)ds \le \frac{C_4}{p'}\int_0^t (Mx^*)^*(s)ds \tag{7}$$

for all $0 < t \le 1$. We show that in any r. i. space X with $\alpha(X) > 1/p$ we have the estimate

$$\|Mx^*\| \le C_5\|x\| \quad \text{for } x \in X. \tag{8}$$

Note that Boyd proved (see [B68], Th. 1, [B69], Lemma 2, and [BS], Th. 5.15) the boundedness of the operator M in r. i. spaces X with the Fatou property. He showed that the operator M is bounded in X if and only if the lower Boyd index $\alpha(X) > 1/p$. In particular, this result gives the estimate (8) but only for r. i. spaces with the Fatou property.

To show (8) in general we first note that the assumption $\alpha(X) > 1/p$ is equivalent to the property that there exist $\varepsilon > 0$ and $A > 0$ such that

$$\|\sigma_s\|_{X\to X} \le As^{1/p+\varepsilon} \quad \text{for all } 0 < s \le 1. \tag{9}$$

For $t \in (0,1)$ we have

$$\begin{aligned}Mx^*(t) &= t^{-1/p}\int_t^1 s^{1/p-1}x^*(s)ds = \int_1^{1/t} s^{1/p-1}x^*(st)ds \\ &= \int_1^\infty s^{1/p-1}x^*(st)\chi_{(1,1/t)}(s)ds \\ &\le \sum_{n=1}^\infty \int_{2^{n-1}}^{2^n} s^{1/p-1}x^*(2^{n-1}t)\chi_{(1,1/t)}(s)ds \\ &\le \sum_{n=1}^\infty 2^{(n-1)(1/p-1)}x^*(2^{n-1}t)\int_{2^{n-1}}^{2^n}\chi_{(1,1/t)}(s)ds \\ &\le \sum_{n=1}^\infty 2^{(n-1)(1/p-1)}x^*(2^{n-1}t)2^{n-1}\chi_{(0,1)}(2^{n-1}t) \\ &= \sum_{n=1}^\infty 2^{(n-1)/p}x^*(2^{n-1}t)\chi_{(0,1)}(2^{n-1}t).\end{aligned}$$

The space X is complete. So we can apply the triangle inequality to infinite sums in X and using (9) we obtain

$$\begin{aligned}
\|Mx^*\| &\le \|\sum_{n=1}^{\infty} 2^{(n-1)/p} x^*(2^{n-1}t)\chi_{(0,1)}(2^{n-1}t)\| \\
&\le \sum_{n=1}^{\infty} 2^{(n-1)/p} \|x^*(2^{n-1}t)\chi_{(0,1)}(2^{n-1}t)\| \\
&\le \sum_{n=1}^{\infty} 2^{(n-1)/p} \|\sigma_{2^{-n+1}}\|_{X\to X}\|x\| \\
&\le A\sum_{n=1}^{\infty} 2^{(n-1)/p} 2^{(-n+1)(1/p+\varepsilon)}\|x\| \\
&= A\sum_{n=1}^{\infty} 2^{-n\varepsilon+\varepsilon}\|x\| = A\frac{2^\varepsilon}{2^\varepsilon - 1}\|x\|,
\end{aligned}$$

and the proof of the estimate (8) is complete. □

Recall now the Calderón-Mitjagin theorem (see [KPS], Th. 4.3) which says that an r. i. space X is an interpolation space between L_1 and L_∞ if and only if the following condition is satisfied: there exists a constant $B > 0$ such that if $y \in X, x \in L_1$ and

$$\int_0^t x^*(s)ds \le \int_0^t y^*(s)ds \quad \text{for all } 0 < t \le 1, \tag{10}$$

then $x \in X$ and $\|x\| \le B\|y\|$.

Putting together the estimates (7), (8) and the Calderón-Mitjagin theorem we obtain that if $x \in X$, then $Tx \in X$ and $\|Tx\| \le C_6\|x\|$. Therefore, X is an interpolation space between L_1 and L_p, and the proof of Theorem 2 is complete.

We give a counterexample showing that in Theorem 2 we cannot omit the assumption that the r. i. X is an interpolation space between L_1 and L_∞.

Example 1. This is the Russu example of the space with a slight modification of function ψ (see [R], Th. 1, or [KPS], Lemma 5.5). Let ψ be an increasing concave function on $(0, 1]$ with $\psi(0^+) = 0, \lim_{t\to 0^+} \frac{\psi(2t)}{\psi(t)} = 1$ and its upper dilation exponent $\delta_\psi = \lim_{t\to\infty} \frac{\ln \bar\psi(t)}{\ln t} = 0$. As a concrete ψ we can take, for example, $\psi(t) = \ln^{-1} \frac{e^2}{t}$.

In the Marcinkiewicz space M_ψ endowed with the norm

$$\|x\|_{M_\psi} = \sup_{0<t\le 1} \frac{1}{\psi(t)} \int_0^t x^*(s)ds$$

we consider a linear space

$$\widetilde{G} = \{x \in L_1 : \sup_{0<t\le 1} \frac{x^*(t)}{\psi'(t)} < \infty\}$$

and as the required space G we take the closure of $\widetilde{G}$ in M_ψ. G is an r. i. space. Since $\|\sigma_t\|_{G\to G} = \|\sigma_t\|_{M_\psi\to M_\psi} = t\bar\psi(1/t)$, it follows that $\alpha(G) = \alpha(M_\psi) = 1-\delta_\psi = 1 > 1/p$, for any $1 < p < \infty$. The space G is not an interpolation space between L_1 and L_∞ (see [R], Th. 2, or [KPS], Lemma 5.5); moreover, G is not an interpolation space between L_1 and L_p.

Remark 2. We can similarly, as in the proof of Theorem 2, show a more general result: *Let $1 \leq r < p < \infty$. If X is an interpolation r. i. space between L_r and L_∞, and $\alpha(X) > \frac{1}{p}$, then X is an interpolation space between L_r and L_p.* In the proof, it is enough to see that there exists a constant $B > 0$ such that

$$\int_0^t (Tx)^*(s)^r ds \leq B \left[\int_0^t x^*(s)^r ds + tMx^*(t)^r\right] \leq B \int_0^t \left[x^*(s) + Mx^*(s)\right]^r ds$$

for any $x \in L_r$ and for all $0 < t \leq 1$ and to use the fact that the couple (L_r, L_∞) is K-monotone (see [LS], Th. 2).

4. Strong type $(1,1)$ and weak type (p,p) interpolation

We observed, after the proof of Theorem 2 was completed, that an even more general result is possible to prove. A linear operator T is said to be of *weak type* $(p,p), 1 \leq p < \infty$ if T is bounded from $L_{p,1}$ into $L_{p,\infty}$, where the spaces $L_{p,1}$ and $L_{p,\infty}$ on $I = [0,1]$ are generated by the functionals

$$\|x\|_{p,\infty} = \sup_{t \in I} t^{1/p} x^*(t), \; \|x\|_{p,1} = \int_I t^{1/p-1} x^*(t) dt.$$

A bounded linear operator $T : L_p \to L_p$ is said to be of *strong type* (p,p) and, of course, every operator of strong type (p,p) is also of weak type (p,p) but not vice versa since $L_{p,1} \subset L_p \subset L_{p,\infty}$.

Boyd [B69] proved in 1969 that any linear operator T that is of weak types (p,p) and $(q,q), 1 \leq p < q < \infty$, is bounded in an r. i. space X if and only if $\frac{1}{q} < \alpha(X) \leq \beta(X) < \frac{1}{p}$. His result is proved for r. i. spaces X with the Fatou property (see also Bennett-Sharpley [BS], Th. 5.16 and Lindenstrauss-Tzafriri [LT], Theorems 2.b.11 and 2.b13). This result however is true for arbitrary r. i. spaces X (cf. our discussion in the Introduction or Th. 6.12 in [KPS]). In the case when $q = \infty$, i.e., for any operator T of weak type (p,p) and strong type $(\infty,\infty), 1 \leq p < \infty$, the one-sided estimate $\beta(X) < \frac{1}{p}$ characterizes the boundedness of T in X (see [M81], Th. 4.6; cf. also [M85], Remark 5.9(a)).

In the case of another extremal space L_1, the one-sided interpolation theorem about operators in L_1 and of weak type $(p,p), 1 < p < \infty$ needs some extra assumptions.

Theorem 3. *Let $1 < p < \infty$. Any linear operator T that is of strong type $(1,1)$ and weak type (p,p) is bounded in an r. i. space X if and only if $\alpha(X) > \frac{1}{p}$ and X is an interpolation space between L_1 and L_∞.*

Proof. The sufficiency follows immediately from the proof of Theorem 2 since the estimate (6) is still true. We must only show that if $\alpha(X) > \frac{1}{p}$, then $L_{p,\infty} \subset X$. By Theorem 5.5 in [KPS], it is enough to prove that

$$\int_0^1 x^*(s) \varphi_X(s) s^{-1} ds \leq C_7 \|x\|_{p,\infty}$$

for all $x \in L_{p,\infty}$. The assumption $\alpha(X) > \frac{1}{p}$ gives (9) and since the estimate $\|\sigma_s\| \geq \varphi_X(s)/\varphi_X(1)$ is clear, we obtain

$$\begin{aligned}\int_0^1 x^*(s)\varphi_X(s)s^{-1}ds &\leq A\varphi_X(1)\int_0^1 x^*(s)s^{1/p+\varepsilon-1}ds \\ &\leq A\varphi_X(1)\|x\|_{p,\infty}\int_0^1 s^{\varepsilon-1}ds = \frac{A\varphi_X(1)}{\varepsilon}\|x\|_{p,\infty}.\end{aligned}$$

Necessity. The operator M is of strong type $(1,1)$ and of weak type (p,p). Moreover, $\|Mx\|_{p,\infty} \leq \|x\|_{p,1}$ for every $x \in L_{p,1}$ and $\|Mx\|_1 \leq p'\|x\|_1$ for every $x \in L_1$. Then M is bounded in X by assumption. If $0 < \varepsilon < 1/\|M\|_{X\to X}$, then the operator $(I_X - \varepsilon M)^{-1}$ exists and is bounded in X. Moreover,

$$(I_X - \varepsilon M)^{-1}x(t) = \sum_{n=0}^{\infty} \varepsilon^n M^n x(t),$$

where the series converges in the operator norm and M^n denotes the nth iteration of M. We have

$$M^n x(t) = t^{-1/p}\int_t^1 s^{1/p-1}\frac{\ln^{n-1}\frac{s}{t}}{(n-1)!}x(s)ds,$$

and the operator $\widetilde{M}$ given by

$$\begin{aligned}\widetilde{M}x(t) &= M(I_X - \varepsilon M)^{-1}x(t) = \sum_{n=0}^{\infty}\varepsilon^n M^{n+1}x(t) \\ &= t^{-1/p}\int_t^1 s^{1/p-1}\left(\sum_{n=0}^{\infty}\frac{\varepsilon^n \ln^n \frac{s}{t}}{n!}\right)x(s)ds \\ &= t^{-1/p}\int_t^1 s^{1/p-1}(\frac{s}{t})^{\varepsilon}x(s)ds\end{aligned}$$

is bounded in X. Since, for $0 < \tau < 1$,

$$\begin{aligned}\widetilde{M}x^*(t) &\geq t^{-1/p}\int_t^{t/\tau} s^{1/p-1}(\frac{s}{t})^{\varepsilon}x^*(s)ds\ \chi_{(0,1)}(t/\tau) \\ &\geq x^*(t/\tau)\chi_{(0,1)}(t/\tau)t^{-1/p}\int_t^{t/\tau} s^{1/p-1}(\frac{s}{t})^{\varepsilon}ds \\ &= \sigma_\tau x^*(t)\frac{p}{1+\varepsilon p}\tau^{-1/p-\varepsilon}(1-\tau^{1/p+\varepsilon}),\end{aligned}$$

it follows that

$$\begin{aligned}\|\sigma_\tau x\| &= \|\sigma_\tau x^*\| \leq (\frac{1}{p}+\varepsilon)\frac{\tau^{1/p+\varepsilon}}{1-\tau^{1/p+\varepsilon}}\|\widetilde{M}x^*\| \\ &\leq (\frac{1}{p}+\varepsilon)\frac{\tau^{1/p+\varepsilon}}{1-\tau^{1/p+\varepsilon}}\|\widetilde{M}\|_{X\to X}\|x\|,\end{aligned}$$

and so

$$\alpha(X) \geq 1/p + \varepsilon > 1/p.$$

We should also show that the r. i. space X must be an interpolation space between L_1 and L_∞. If $T : L_1 \to L_1$ is bounded and $T = T_{|L_\infty} : L_\infty \to L_\infty$ is also bounded, then by the Riesz-Thorin interpolation theorem T is of strong type (p,p), which implies that T is of weak type (p,p). The assumption on X gives that T is also bounded in X, and the proof is complete. $\square$

Remark 3. Theorem 3 can be generalized (cf. Remark 2): *Let* $1 \leq r < p < \infty$. *Any linear operator* T *that is of strong type* (r,r) *and weak type* (p,p) *is bounded in an r. i. space* X *if and only if* $\alpha(X) > \frac{1}{p}$ *and* X *is an interpolation space between* L_r *and* L_∞.

Remark 4. Theorems 2 and 3 can also be proved when the r. i. spaces are on the interval $(0,\infty)$. We then only need to control that $\alpha(X) > \frac{1}{p}$ implies $L_1 \cap L_{p,\infty} \subset X \subset L_1 + L_{p,1}$, which is in fact true. Note also that Theorems 2 and 3 are valid for quasilinear operators.

References

[BS] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, Boston, 1988. MR 89e:46001

[B67] D. W. Boyd, *Spaces between a pair of reflexive Lebesgue spaces*, Proc. Amer. Math. Soc. **18** (1967), 215-219. MR 35:3427

[B68] D. W. Boyd, *The spectral radius of averaging operators*, Pacific J. Math. **24** (1968), 19-28. MR 36:4360

[B69] D. W. Boyd, *Indices of function spaces and their relationship to interpolation*, Canad. J. Math. **21** (1969), 1245-1254. MR 54:909

[C] M. Cwikel, *Monotonicity properties of interpolation spaces*, Ark. Mat. **14** (1976), 213-236. MR 56:1095

[G] Ju. I. Gribanov, *Banach function spaces and integral operators. II.*, Izv. Vysš. Učebn. Zaved. Matematika 1966, no. 6 (55), 54-63 (Russian). MR 35:3489

[H] T. Holmstedt, *Interpolation of quasi-normed spaces*, Math. Scand. **26** (1970), 177-199. MR 54:3440

[JMST] W. B. Johnson, B. Maurey, G. Schechtman, and L. Tzafriri, *Symmetric Structures in Banach Spaces*, Memoirs Amer. Math. Soc. **19**, 1979. MR 82j:46025

[KPS] S. G. Krein, Yu. I. Petunin, and E. M. Semenov, *Interpolation of Linear Operators*, Nauka, Moscow, 1978 (Russian); English transl. in Amer. Math. Soc., Providence, RI, 1982. MR 84j:46103

[LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces, II. Function Spaces*, Springer-Verlag, Berlin-New York, 1979. MR 81c:46001

[LS] G. G. Lorentz and T. Shimogaki, *Interpolation theorems for the pairs of spaces* (L^p, L^∞) *and* (L^1, L^q), Trans. Amer. Math. Soc. **159** (1971), 207-221. MR 52:1347

[M81] L. Maligranda, *A generalization of the Shimogaki theorem*, Studia Math. **71** (1981), 69-83. MR 83d:46034

[M85] L. Maligranda, *Indices and interpolation*, Dissertationes Math. **234** (1985), 1-54. MR 87k:46059

[Ms] M. Mastyło, *Interpolation of linear operators in the Köthe dual spaces*, Ann. Mat. Pura Appl. **154** (1989), 231-242. MR 91f:46105

[R] G. I. Russu, *Symmetric spaces of functions that do not have the majorization property*, Mat. Issled. **4** (1969), 82-93 (Russian). MR 43:2490

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