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MATHEMATICS

Representability of Cones in Weighted Lebesgue Spaces and Extrapolation Operators on Cones

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The role played by exact estimates of classical operators in harmonic analysis and adjacent areas is well known. In recent years, because of new problems of analysis, estimates of operators on some cones in spaces rather than on the entire spaces have become very popular (see, e.g., [1–4]). On the other hand, in the theory of integral operators with positive kernels, the extrapolation theorem of Schur (see, e.g., [5]) is well known; it says that an integral operator $Kx(t) = \int k(t, s)x(s)ds$ with $k(t, s) \geq 0$ is bounded in L^p if and only if there exists a positive function $u(t)$ such that it is finite almost everywhere and the operator is bounded in the pairs K: $L_u^\infty \to L_u^\infty$ and K: $L_v^1 \to L_v^1$, where $v = u^{1/p-1}$. In relation to various problems of analysis, the interest in extrapolation theorems has increased [6–8]. For this reason, it is natural to pass from the Lebesgue space L^p to cones of Lebesgue spaces in these theorems.

In this paper, we suggest a reduction of estimating operators on cones to estimating them on new spaces, which are constructed from the cones and the initial spaces, for the most important cones in the Lebesgue spaces. Such a reduction makes it possible to apply the whole apparatus developed for obtaining exact estimates on weight Lebesgue spaces to obtain exact estimates of operators on cones. Using the reduction, we prove a new extrapolation theorem for a certain class of operators defined on cones in Lebesgue spaces.

Let $S(\mu) = S(R_+, \Sigma, \mu)$ [where $R_+ = (0, +\infty)$] be the space of measurable functions x: $R_+ \to R$. Recall that a Banach space $X = (X, \|\cdot|X\|)$ consisting of measurable functions is said to be ideal [9] if, for any $y \in X$ and any measurable x such that $|x(t)| \leq |y(t)|$ almost everywhere on R_+, $x \in X$ and $\|x|X\| \leq \|y|X\|$. As usual, the symbol L^p (where $1 \leq p \leq \infty$) denotes the classical Lebesgue space.

Let w: $R_+ \to R_+$ be a positive function (weight). For an ideal space X, we use X_w to denote the new ideal space with norm $\|x|X_w\| = \|wx|X\|$.

Definition 1. Let X be an ideal space in $S(\mu)$, and let K be a cone in $S(\mu)$. As usual, $K \cap X$ denotes the intersection of the cone K with the cone X_+.

Let $K(\downarrow)$ denote the cone in $S(\mu)$ consisting of the functions x: $R_+ \to R_+$ that do not increase, i.e., satisfy the condition $x(t+h) \leq x(t)$ for $h \geq 0$, and let $K(\uparrow)$ be the cone of nondecreasing functions in $S(\mu)$; by $K(\downarrow, \uparrow)$ we denote the cone in $S(\mu)$ consisting of the concave functions x: $R_+ \to R_+$ satisfying the additional conditions $\lim\limits_{t \to 0} x(t) = 0$ and $\lim\limits_{t \to 0} t^{-1} x(t) = 0$.

Theorem 1. *Suppose that* $p \in (1, \infty)$ *and* w *is a weight function such that*

$$\int_1^\infty w^p(s)ds = \infty \tag{1}$$

and

$$\int_0^t w^p(s)ds < \infty \quad \text{for any} \quad t > 0 \tag{2}$$

Let Q *be the operator defined by*

$$Qx(t) = \int_t^\infty x(\tau)d\tau.$$

Finally, let v *be a new function for which*

$$\left\|\kappa(0, t)w|L^p\right\| \cdot \left\|\kappa(t, \infty)\frac{1}{v}\middle|L^{p'}\right\| \equiv 1 \tag{3}$$

[$\kappa(D)$ *denotes the characteristic function of the set* D].

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Then, the following assertions are valid: the operator Q acts and is bounded in the pair

$$Q\colon (L_v^p)_+ \to K(\downarrow) \cap L_w^p; \tag{4}$$

there exists a constant $c > 0$ such that, for any $y \in K(\downarrow) \cap L_w^p$ with $\|y|L_p^w\| = 1$, there exists a function $x \in L_v^p$ with $\|x|L_v^p\| = 1$ such that

$$(Qx)(t) \ge cy(t) \tag{5}$$

for any $t \in (0, \infty)$.

Note that condition (4) holds by virtue of the classical estimates of the operator Q in the spaces L_u^p [the weight v in (3) was chosen so as to ensure this]; see, e.g., [4, 10]. For $y \in K(\downarrow) \cap L_w^p$, the function x in (5) is defined constructively.

Remark 1. Theorem 1 has a complete analogue for the cones $K(\uparrow)$, $K(\varphi, \downarrow) = \{x\colon R_+ \to R_+\colon \varphi(t)x(t)\downarrow\}$, and $K(\varphi, \uparrow) = \{x\colon R_+ \to R_+\colon \varphi(t)x(t)\uparrow\}$ with the only difference that, instead of Q, the operators

$$Px(t) = \int_0^t x(s)ds, \quad Q_\varphi x(t) = \frac{1}{\varphi(t)}\int_t^\infty x(s)ds,$$

$$P_\varphi x(t) = \frac{1}{\varphi(t)}\int_0^t x(s)ds$$

should be considered.

Let us exemplify the applications of Theorem 1.

Theorem 2. *Suppose that $1 \le p_0 < p_1 < \infty$ and u and w are weight functions satisfying conditions* (1) *and* (2).

Then,

$$K(\downarrow) \cap L_u^{p_0} \ne K(\downarrow) \cap L_u^{p_1};$$

i.e., these cones do not coincide for any weight functions satisfying the assumptions of the theorem.

We say that an operator $T\colon S(\mu) \to S(\mu)$ is sublinear if $|T(x + y)(t)| \le T|x|(t) + T|y|(t)$ and $|T(\lambda x)(t)| \le \lambda|Tx(t)|$ for $\lambda \ge 0$.

Theorem 1 immediately implies the following assertion.

Theorem 3. *Suppose that $p \in (1, \infty)$ and w is a weight function satisfying conditions* (1) *and* (2). *Let Y be an ideal Banach space in $S(\mu)$.*

A sublinear operator T acts and is bounded as an operator from $K(\downarrow) \cap L_w^p$ to Y if and only if the superposition operator TQ acts and is bounded as an operator from L_v^p to Y.

Using the technique for estimating operators $L\colon L_w^p \to Y$ (see, e.g., [2, 4, 11, 12]), we can obtain various estimates for operators on the cone of monotone functions in Lebesgue spaces by applying Theorem 3.

Consider the cone $K(\downarrow, \uparrow)$. For nonnegative functions on R_+, we define the operators

$$Q_1x(t) = t\int_t^\infty x(s)ds \quad \text{and} \quad P_1x(t) = \int_0^t sx(s)ds.$$

It is easy to show by simple integration by parts that, if a function $x \in K(\downarrow, \uparrow)$ has absolutely continuous first derivative, then

$$x(t) = \int_0^t\left(\int_s^\infty z(\tau)d\tau\right)ds$$

$$= t\int_t^\infty z(s)ds + \int_0^t sz(s)ds = Q_1z(t) + P_1z(t),$$

where $z(s)$ is a nonnegative function. We can set $z(s) \equiv -x''(s)$.

Theorem 4. *Suppose that $p \in (1, \infty)$ and w is a weight function such that, for any $t \in R_+$,*

$$\int_0^\infty\left(\min\left\{1, \frac{s}{t}\right\}\right)^p w^p(s)ds < \infty. \tag{6}$$

Consider the cone $K(\downarrow, \uparrow) \cap L_w^p$.

Let w_0 and w_1 be the new weight functions defined by the equalities

$$\left\|\kappa(t, \infty)\frac{1}{w_0(s)}\middle|L^{p'}\right\| \cdot \|\kappa(0, t)sw(s)|L^p\| \equiv 1, \tag{7}$$

and

$$\left\|\kappa(0, t)\frac{s}{w_1(s)}\middle|L^{p'}\right\| \cdot \|\kappa(t, \infty)w(s)|L^p\| \equiv 1 \tag{8}$$

for all $t > 0$, and let

$$v(t) = \max\{w_0(t), w_1(t)\}.$$

Then, the following assertions are valid: the sum of operators $Q_1 + P_1$ acts and is bounded in the pair

$$(Q_1 + P_1)\colon (L_v^p)_+ \to K(\downarrow, \uparrow) \cap L_w^p; \tag{9}$$

for any $y \in K(\downarrow, \uparrow) \cap L_w^p$ with $\|y|L_w^p\| = 1$, there exists a function $x \in L_v^p$ with $\|x|L_v^p\| = 1$ such that

$$((Q_1 + P_1)x)(t) \ge cy(t) \tag{10}$$

for all $t \in (0, \infty)$, *where* $c > 0$ *is a constant, if and only if*

$$\inf_t \left\{ \left(\left\| \kappa(0,t) \frac{s}{v(s)} \middle| L^{p'} \right\| + t \left\| \kappa(t,\infty) \frac{s}{v(s)} \middle| L^{p'} \right\| \right) \times \left(\left\| \kappa(0,t) \frac{s}{t} w(s) \middle| L^{p} \right\| + \left\| \kappa(t,\infty) w(s) | L^{p} \right\| \right) \right\} < \infty. \quad (11)$$

Condition (6) ensures that the extreme functions $\min\left\{1, \frac{s}{t}\right\}$ belong to the cone $K(\downarrow, \uparrow)$ of the space L_w^p. Condition (7) is necessary and sufficient for the boundedness of Q_1 as an operator from $L_{w_0}^p$ in L_w^p, and condition (8) is necessary and sufficient for the boundedness of P_1 as an operator from $L_{w_1}^p$ to L_w^p. Therefore, if v is chosen as in the statement of the theorem, then condition (9) does hold.

Condition (11) in Theorem 4 simply ensures the possibility that condition (10) holds for the family of extreme functions $\min\left\{1, \frac{s}{t}\right\}$ of the cone $K(\downarrow, \uparrow)$.

For a function $y \in K(\downarrow, \uparrow) \cap L_w^p$, the function x in (10) is defined constructively.

Theorem 4 has an analogue for the cones $K(\varphi, \psi) = \{x: R_+ \to R_+: x(t) \cdot \varphi$ is nondecreasing and $\psi(t) \cdot x(t)$ is nonincreasing$\}$.

Condition (11) does not always hold. There are various sufficient conditions under which (11) holds. In particular, on the power scale, i.e., for $w(t) = t^\alpha$, conditions (11) are satisfied for $\alpha \in \left(-\frac{1}{p} - 1, -\frac{1}{p}\right)$.

The following theorem exemplifies the applications of Theorem 4.

Theorem 5. *Suppose that* $p \in (1, \infty)$ *and* w *is a weight function satisfying* (6). *Let* w_0, w_1, *and* v *be functions such that condition* (11) *holds for this* w, *and let* Y *be an ideal Banach space in* $S(\mu)$.

A sublinear operator T *acts and is bounded as an operator from* $K(\downarrow, \uparrow) \cap L_w^p$ *to* Y *if and only if the superposition operator* $T(Q_1 + P_1)$ *acts and is bounded as an operator from* L_v^p *to* Y.

Now, we proceed to extrapolation theorems for operators on cones. We need some additional constructions.

Let X_0 and X_1 be two ideal spaces in $S(\mu)$.

Take $0 < \theta < 1$. Consider the new ideal space $X_0^\theta X_1^{1-\theta}$ (the Calderon–Lozanovskii construction) consisting of those $x \in S(\mu)$ for which the norm

$$\|x\|_{X_0^\theta X_1^{1-\theta}} = \inf\{\lambda > 0: |x(t)| \le \lambda \cdot |x_0(t)|^\theta |x_1(t)|^{1-\theta} \text{ for any } t \in \Omega; \quad \|x_0\|_{X_0} \le 1, \quad \|x_1\|_{X_1} \le 1\} \quad (12)$$

is finite. The space $X_0^\theta X_1^{1-\theta}$ was introduced by Calderon [13] in studying the complex interpolation method.

If K is a cone in $S(\mu)$, then we can define a new cone $(K \cap X_0)^\theta (K \cap X_1)^{1-\theta}$, by analogy with $X_0^\theta X_1^{1-\theta}$, namely, by taking only decompositions into elements of the cone in (12). The following theorem is an interpolation result; it is well known for the cone of nonnegative functions (see, e.g., [14, 15]).

Theorem 6. *Suppose that* T *is a positive operator and* K_0 *and* K_1 *are two cones in* $S(\mu)_+$. *Let* X_0, X_1, Y_0, *and* Y_1 *be ideal Banach spaces in* $S(\mu)$. *Suppose that an operator* T *acts and is bounded as an operator* $T: X_i \cap K_0 \to Y_i \cap K_1$ *for* $i = 0, 1$. *Let* $\theta \in (0, 1)$.

Then, the operator T *acts and is bounded as an operator* $T: (K_0 \cap X_0)^\theta (K_0 \cap X_1)^{1-\theta} \to (K_1 \cap Y_0)^\theta (K_1 \cap Y_1)^{1-\theta}$.

Remark 2. As is usual in interpolation theory, for an arbitrary cone K, the equality $(K \cap L_{v_0}^1)^\theta (K \cap L_{v_1}^\infty)^{1-\theta} = K \cap ((L_{v_0}^1)^\theta (L_{v_1}^\infty)^{1-\theta})$ does not always hold, even for the cone $K(\downarrow)$.

Theorem 7. *Suppose that* $p \in (1, \infty)$, w *is a weight function satisfying conditions* (1) *and* (2), *and* v *is a function constructed for* w *according to* (3). *Let* $\theta = \frac{1}{p}$. *Suppose that* T *is a linear positive operator* T *acting and bounded in the pair*

$$T: K(\downarrow) \cap L_w^p \to L_w^p.$$

Then, there exist functions v_0, v_1, u_0, *and* u_1 *such that*

$$v_0^\theta(t) \cdot v_1^{1-\theta}(t) \equiv v(t), \quad u_0^\theta(t) \cdot u_1^{1-\theta}(t) \equiv u(t); \quad (13)$$

the operator TQ *acts and is bounded in the pairs*

$$TQ: L_{v_0}^1 \to L_{u_0}^1, \quad TQ: L_{v_1}^\infty \to L_{u_1}^\infty.$$

Combining Theorems 6 and 7, we obtain the following extrapolation theorem for operators on the cone $K(\downarrow)$.

Theorem 8. *Suppose that* $p \in (1, \infty)$; w *is a weight function satisfying condition* (6); w_0, w_1, *and* v *are functions constructed for* w; *and condition* (11) *holds. Let* $\theta = \frac{1}{p}$. *Suppose that* T *is a linear positive operator acting and bounded in the pair*

$$T: K(\downarrow, \uparrow) \cap L_w^p \to L_u^p.$$

Then, there exist functions v_0, v_1, u_0, *and* u_1 *such that relations* (13) *hold and the operator* $T(Q_1 + P_1)$ *acts and is bounded in the pairs*

$$T(Q_1 + P_1)\colon L^1_{v_0} \to L^1_{u_0}, \quad T(Q_1 + P_1)\colon L^\infty_{v_1} \to L^v_{u_1}.$$

Combining Theorems 6 and 7, we obtain an extrapolation theorem for operators on the cone $K(\downarrow, \uparrow)$.

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