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Distribution and rearrangement estimates of the maximal function and interpolation

by

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Abstract. There are given necessary and sufficient conditions on a measure $d\mu(x) = w(x)\,dx$ under which the key estimates for the distribution and rearrangement of the maximal function due to Riesz, Wiener, Herz and Stein are valid. As a consequence, we obtain the equivalence of the Riesz and Wiener inequalities which seems to be new even for the Lebesgue measure. Our main tools are estimates of the distribution of the averaging function f^{**} and a modified version of the Calderón–Zygmund decomposition. Analogous methods allow us to obtain K-functional formulas in terms of the maximal function for couples of weighted L_p-spaces.

0. Introduction. The Hardy–Littlewood maximal function

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\,dy$$

plays a very important role in the study of differentiation, singular integrals and almost everywhere convergence [1].

A great interest in estimates of the rearrangement and distribution of the maximal function started after the classical paper of Hardy–Littlewood (1930). They defined the maximal operator (in the one-dimensional case) and proved boundedness of the maximal operator in $L_p(\mathbb{R}^1)$ for $p > 1$.

The first important step in this direction was done by F. Riesz (1932). He proved first the nice geometrical "sunrise" lemma and used it to show that

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([1]) For notations used in this introduction (and the rest of the paper) we refer to the "conventions" at the end of this section.

in the one-dimensional case (more precisely he proved it for f defined on $[0,1]\subset\mathbb{R}^1$ and for the one-sided maximal function) we have the inequality

$$(0.1)\qquad (Mf)^*(t)\le Af^{**}(t),$$

with a constant $A>0$ independent of f and $t>0$, from which the weak type estimate follows immediately (cf. also [3]). In the present paper the inequality (0.1), even in the n-dimensional case, will be referred to as the *Riesz inequality*.

It was Wiener (1939) who, using the arguments of the Vitali covering lemma, proved in $\mathbb{R}^n$ the key property of M, namely that it is of weak type $(1,1)$, i.e. that

$$|\{x\in\mathbb{R}^n:Mf(x)>\lambda\}|\le\frac{B}{\lambda}\int_{\mathbb{R}^n}|f(x)|\,dx\quad\forall\lambda>0,$$

with a constant $B>0$ independent of f and λ. He also proved the stronger inequality

$$(0.2)\quad |\{x\in\mathbb{R}^n:Mf(x)>\lambda\}|\le\frac{2B}{\lambda}\int_{\{x\in\mathbb{R}^n:|f(x)|>\lambda/2\}}|f(x)|\,dx\quad\forall\lambda>0,$$

which we will call the *Wiener inequality*.

Let us note here that the Wiener inequality is equivalent to the n-dimensional Riesz inequality. This unexpected equivalence, which we could not find in the literature, follows from our Theorem 1.

On the other hand, the "reverse" inequalities to (0.1) and (0.2) were found to be true much later. Namely, in 1969 E. Stein, in connection with the study of integrability of the maximal function Mf (by using the Calderón–Zygmund decomposition lemma) proved that the reverse inequality to (0.2) is valid:

$$(0.3)\quad \frac{1}{\lambda}\int_{\{x\in\mathbb{R}^n:|f(x)|>\lambda\}}|f(x)|\,dx\le C|\{x\in\mathbb{R}^n:Mf(x)>\lambda\}|\quad\forall\lambda>0,$$

with a constant $C>0$ independent of f and λ.

In 1968 C. Herz, under the influence of the forthcoming paper of Stein, proved that

$$(0.4)\qquad f^{**}(t)\le D(Mf)^*(t)\quad\forall t>0,$$

with a constant $D>0$ independent of f and t (for another proof see [2] and [3], Th. 3.8). We will call the inequalities (0.3) and (0.4) the *Stein* and *Herz inequalities*, respectively.

The main purpose of this paper is to find necessary and sufficient conditions on a measure w in $\mathbb{R}^n$ such that the inequalities (0.1)–(0.4) are valid for the weighted maximal operator M_wf. We consider the case when the positive measure w on $\mathbb{R}^n$ is absolutely continuous with respect to the Lebesgue measure, i.e. $w(A)=\int_A w(x)\,dx$ with $w\in L_1^{\rm loc}(\mathbb{R}^n)$; then the maximal function is defined by

$$M_wf(x)=\sup_{Q\ni x}\frac{1}{w(Q)}\int_Q|f(y)|w(y)\,dy,$$

where the supremum is taken over all cubes $Q\subset\mathbb{R}^n$ which contain x with sides parallel to the coordinate axes. The above described locally integrable positive function w will be called a *weight*. In particular, the results proved in this paper show that for such a general measure the following holds:

THEOREM A. *The Riesz inequality is equivalent to the Wiener inequality and they are true if and only if the maximal operator M_w is of weak type $(1,1)$.*

THEOREM B. *The Stein inequality and the Herz inequality are valid without any restriction on the measure w.*

We note here that the above Theorem B cannot be obtained from the proofs of Stein, Herz and Bennett–Sharpley since they are using the Calderón–Zygmund decomposition lemma or the covering lemma with dyadic cubes. Both of them require the "doubling" property $w(2Q)\le dw(Q)$ of the measure w. In the proof given in this paper we use the Besicovitch covering lemma and a modified Calderón–Zygmund decomposition lemma.

Theorems A and B show that the equivalence

$$(0.5)\qquad f_w^{**}(t)\approx(M_wf)_w^*(t)$$

holds if and only if the operator M_w is of weak type $(1,1)$. However, for quite many measures w in $\mathbb{R}^n$, $n\ge2$, and even for very simple measures like $w(x,y)=e^{-(x^2+y^2)/2}$ (cf. [11]) or $w(x,y)=e^{x+y}$ in $\mathbb{R}^2$ (see Example 3 in Section 2), the maximal operators M_w corresponding to these measures are NOT of weak type $(1,1)$.

In this connection, there appeared the question to make an "improvement" of the operator $(M_wf)_w^*$ such that the equivalence (0.5) will still be true.

The maximal function M_wf is the pointwise supremum of the family of linear averaging operators S_π taken over all packings $\pi=\{Q_i\}_{i=1}^{|\pi|}$, i.e.

$$M_wf(x)=\sup_\pi S_\pi(|f|)(x),$$

where

$$S_\pi(f)(x)=\sum_{i=1}^{|\pi|}\Big[\frac{1}{w(Q_i)}\int_{Q_i}f(y)w(y)\,dy\Big]\chi_{Q_i}(x).$$

In the equivalence (0.5), on the right-hand side we have $(M_w f)^*_w(t) = (\sup_\pi S_\pi(|f|))^*_w(t)$, i.e. first we take the supremum over all π and then we make the rearrangement. A new maximal function can be defined by taking first the rearrangement of $S_\pi(|f|)$ and then the supremum over all π. The importance of this function can be seen in the following statement which follows from our results:

THEOREM C. *For every measure w on $\mathbb{R}^n$ we have the equivalence*

$$f^{**}_w(t) \approx \sup_\pi[(S_\pi(|f|))^*_w(t)] \quad \forall 0 < t < w(\mathbb{R}^n).$$

Since the K-functional for the couple $(L_1(w), L_\infty)$ is equal to $K(t, f; L_1(w), L_\infty) = tf^{**}_w(t)$ it is possible, by using the Holmstedt formula (cf. [4]), to obtain a description of the K-functional for the couple $(L_{p_0}(w_0), L_{p_1}(w_1))$, $p_0 \neq p_1$, in terms of the new maximal function defined above.

Conventions. Throughout this paper we use the following notions and notations: All *cubes* are cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes. A *packing* $\pi = \{Q_i\}_{i=1}^{|\pi|}$ means a finite collection of non-overlapping cubes in $\mathbb{R}^n$.

For a fixed weight function w and any measurable function f on $\mathbb{R}^n$ we define the *distribution function*, the *rearrangement function*, and the *average of the rearrangement function*, respectively, as

$$d^w_f(\lambda) = w(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}), \quad \lambda > 0,$$
$$f^*_w(t) = \inf\{\lambda > 0 : d^w_f(\lambda) \le t\}, \quad t > 0,$$

and

$$f^{**}_w(t) = \frac{1}{t}\int_0^t f^*_w(s)\,ds, \quad t > 0.$$

If $w = 1$, i.e. we have the usual Lebesgue measure, we write simply $d_f(\lambda)$, $f^*(t)$ and $f^{**}(t)$, respectively.

An equivalence $f(t) \approx g(t)$ means that there are constants $C, D > 0$ such that $Cf(t) \le g(t) \le Df(t)$ for all $t > 0$; an equivalence $f(t) \approx g(t)$ $\forall 0 < t < a$ means that there are constants $C, D > 0$ such that $Cf(t) \le g(t) \le Df(t)$ for all $t \in (0, a)$.

A function on $(0, \infty)$ is said to be *positive* or *decreasing* if it is non-negative or non-increasing, respectively.

1. Distribution and rearrangement function inequalities. The concept of rearrangement function was introduced and used by Hardy and Littlewood [7]. On the other hand, for a long time most authors, such as

Wiener, Stein, Burkholder and others, have preferred to work with distribution functions in the theory of maximal functions. Indeed, covering lemmas lead immediately to an estimate for the distribution function of the maximal function. We will show that there was no reason for this preference. The distribution inequalities are equivalent to rearrangement inequalities. The main tool in our proof is the following lemma on the precise estimates concerning the distribution of values of an integral and the average of a decreasing function.

LEMMA 1. *Let f be a positive decreasing function on $(0, \infty)$ and let $0 < \alpha < 1$. Then*

$$(1.1)\quad \frac{1}{\lambda}\int_{\{t:f(t)>\lambda\}} f(t)\,dt \le \left|\left\{t > 0 : \frac{1}{t}\int_0^t f(s)\,ds > \lambda\right\}\right| \le \frac{1}{1-\alpha}\cdot\frac{1}{\lambda}\int_{\{t:f(t)>\alpha\lambda\}} f(t)\,dt$$

for all $\lambda > 0$. The constants 1 and $1/(1-\alpha)$ are the best possible.

Proof. We prove the first inequality of (1.1). Note that $\int_0^t f(s)\,ds$ is an increasing concave function of $t > 0$. Define

$$t_* = t_*(\lambda) = \sup\left\{t > 0 : \int_0^t f(s)\,ds > \lambda t\right\}, \quad \sup\emptyset = 0,$$

and

$$t(\lambda) = |\{t > 0 : f(t) > \lambda\}| = \inf\{t > 0 : f(t) \le \lambda\}, \quad \inf\emptyset = \infty.$$

For fixed $\lambda > 0$ there are three possibilities for $t_* = t_*(\lambda)$:

(i) $t_* = \infty$. Then $\int_0^t f(s)\,ds > \lambda t$ for all $t > 0$, and $|\{t > 0 : (1/t)\int_0^t f(s)\,ds > \lambda\}| = \infty$. Thus we have nothing to prove.

(ii) $0 < t_* < \infty$. Since f is decreasing it follows that $f(t_*)t_* \le \int_0^{t_*} f(s)\,ds = \lambda t_*$, which gives $f(t_*) \le \lambda$. Thus $t(\lambda) \le t_*$ and

$$\frac{1}{\lambda}\int_{\{t:f(t)>\lambda\}} f(t)\,dt = \frac{1}{\lambda}\int_0^{t(\lambda)} f(t)\,dt \le \frac{1}{\lambda}\int_0^{t_*} f(t)\,dt = t_* = \left|\left\{t > 0 : \frac{1}{t}\int_0^t f(s)\,ds > \lambda\right\}\right|.$$

(iii) $t_* = 0$. In this case, $\int_0^t f(s)\,ds \le \lambda t$ for all $t > 0$. Since f is decreasing we obtain $tf(t) \le \int_0^t f(s)\,ds \le \lambda t$, or $f(t) \le \lambda$ for all $\lambda > 0$. This means that

$$\frac{1}{\lambda} \int\limits_{\{t: f(t) > \lambda\}} f(t)\,dt = 0 = \left|\left\{t > 0 : \frac{1}{t}\int\limits_0^t f(s)\,ds > \lambda\right\}\right|,$$

and the first inequality is proved.

We prove the second inequality of (1.1). For fixed $\lambda > 0$ and $0 < \alpha < 1$ we have four cases for $t_* = t_*(\lambda)$:

(i) $t_* = \infty$. Then $(1/t)\int_0^t f(s)\,ds > \lambda$ for every $t > 0$. If $f(t) > \alpha\lambda$ for all $t > 0$, then $\int_{\{t: f(t) > \alpha\lambda\}} f(t)\,dt = \infty$, and we have nothing to prove. Assume that there is $t_0 > 0$ such that $f(t_0) \le \alpha\lambda$. Then, for $t > t_0$,

$$\begin{aligned}\lambda &< \frac{1}{t}\int\limits_0^t f(s)\,ds = \frac{1}{t}\int\limits_0^{t_0} f(s)\,ds + \frac{1}{t}\int\limits_{t_0}^t f(s)\,ds \\ &\le \frac{1}{t}\int\limits_0^{t_0} f(s)\,ds + \frac{1}{t}\int\limits_{t_0}^t (\alpha\lambda)\,ds = \frac{1}{t}\int\limits_0^{t_0} f(s)\,ds + \frac{t - t_0}{t}\alpha\lambda.\end{aligned}$$

Now let $t \to \infty$ to obtain $\lambda \le \alpha\lambda$, which contradicts the assumption $0 < \alpha < 1$.

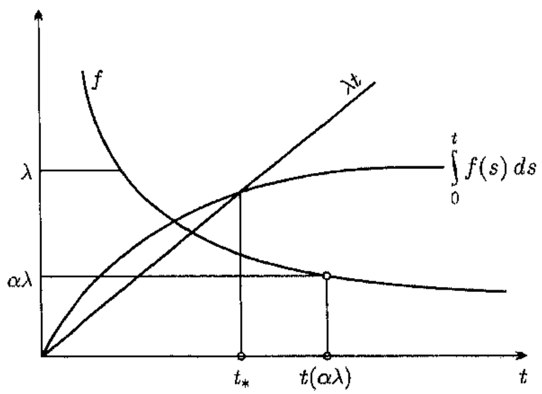

Fig. 1

(ii) $0 < t_* \le t(\alpha\lambda)$ (Fig. 1). Then

$$\frac{1}{\lambda}\int\limits_0^{t_*} f(t)\,dt \le \frac{1}{\lambda}\int\limits_0^{t(\alpha\lambda)} f(t)\,dt$$

and, thus

$$\begin{aligned}\left|\left\{t > 0 : \frac{1}{t}\int\limits_0^t f(s)\,ds > \lambda\right\}\right| = t_* = \frac{1}{\lambda}\int\limits_0^{t_*} f(t)\,dt &\le \frac{1}{\lambda}\int\limits_0^{t(\alpha\lambda)} f(t)\,dt \\ &= \frac{1}{\lambda}\int\limits_{\{t: f(t) > \alpha\lambda\}} f(t)\,dt.\end{aligned}$$

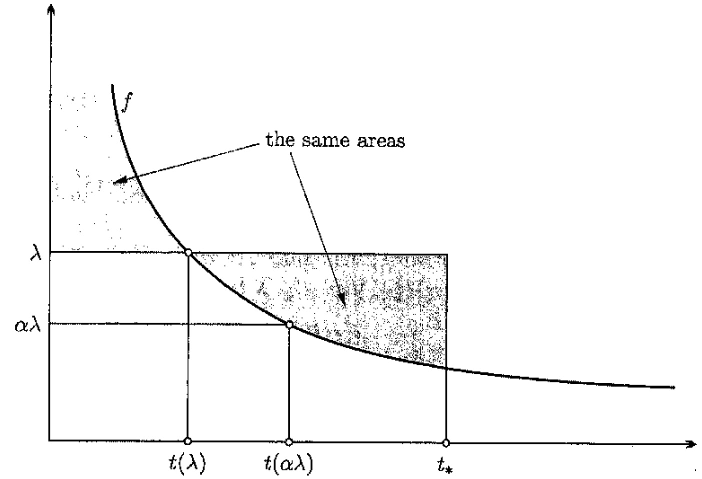

Fig. 2

(iii) $t(\alpha\lambda) < t_* < \infty$ (Fig. 2). Then $\int_0^{t_*} f(s)\,ds = \lambda t_*$ and so

$$\begin{aligned}\left|\left\{t > 0 : \frac{1}{t}\int\limits_0^t f(s)\,ds > \lambda\right\}\right| = t_* &= \frac{1}{\lambda}\int\limits_0^{t_*} f(s)\,ds \\ &= \frac{1}{\lambda}\int\limits_0^{t(\alpha\lambda)} f(s)\,ds + \frac{1}{\lambda}\int\limits_{t(\alpha\lambda)}^{t_*} f(s)\,ds.\end{aligned}$$

Since $f(s) \le \alpha\lambda$ for $s \in [t(\alpha\lambda), t_*]$ it follows that

$$\begin{aligned}\int\limits_{t(\alpha\lambda)}^{t_*} f(s)\,ds \le \alpha\lambda[t_* - t(\alpha\lambda)] &= \frac{\alpha}{1-\alpha}(1-\alpha)\lambda[t_* - t(\alpha\lambda)] \\ &= \frac{\alpha}{1-\alpha} \times \text{area of the rectangle } [t(\alpha\lambda), t_*] \times [\alpha\lambda, \lambda] \\ &\le \frac{\alpha}{1-\alpha} \times \text{area of one of the shaded regions [see Figure 2]}\end{aligned}$$

$$= \frac{\alpha}{1-\alpha} \int_0^{t(\lambda)} [f(s)-\lambda]\,ds \le \frac{\alpha}{1-\alpha} \int_0^{t(\lambda)} f(s)\,ds$$
$$\le \frac{\alpha}{1-\alpha} \int_0^{t(\alpha\lambda)} f(s)\,ds.$$

Therefore

$$t_* \le \frac{1}{\lambda} \int_0^{t(\alpha\lambda)} f(s)\,ds + \frac{\alpha}{1-\alpha}\cdot\frac{1}{\lambda} \int_0^{t(\alpha\lambda)} f(s)\,ds$$
$$= \frac{1}{1-\alpha}\cdot\frac{1}{\lambda} \int_0^{t(\alpha\lambda)} f(s)\,ds = \frac{1}{1-\alpha}\cdot\frac{1}{\lambda} \int_{\{t:f(t)>\alpha\lambda\}} f(t)\,dt.$$

(iv) $t_* = 0$. Then, as we have seen before, $|\{t>0:(1/t)\int_0^t f(s)\,ds>\lambda\}| = 0$ and the inequality (1.1) is proved.

It remains to point out optimal functions showing that the constants 1 and $1/(1-\alpha)$ are the best possible. For $c > \lambda$ let

$$f(t) = \begin{cases} c & \text{for } 0<t\le 1,\\ 0 & \text{for } t>1.\end{cases}$$

Then

$$\left|\left\{t>0:\frac{1}{t}\int_0^t f(s)\,ds>\lambda\right\}\right| = \left|\left\{t>0: c\chi_{(0,1]}(t)+\frac{c}{t}\chi_{(1,\infty)}(t)>\lambda\right\}\right|$$
$$= 1+\left(\frac{c}{\lambda}-1\right) = \frac{c}{\lambda} = \frac{1}{\lambda}\int_{\{t:f(t)>\lambda\}} f(t)\,dt,$$

which gives equality in the first inequality of (1.1).

On the other hand, if we take $c>\lambda$ and

$$f(t) = \begin{cases} c & \text{for } 0<t\le 1,\\ \alpha\lambda & \text{for } t>1,\end{cases}$$

then

$$\left|\left\{t>0:\frac{1}{t}\int_0^t f(s)\,ds>\lambda\right\}\right| \Big/ \left[\frac{1}{\lambda}\int_{\{t:f(t)>\alpha\lambda\}} f(t)\,dt\right]$$
$$= t_* \Big/ \left[\frac{1}{\lambda}\int_0^1 c\,dt\right] = \frac{c-\alpha\lambda}{(1-\alpha)\lambda}\cdot\frac{\lambda}{c} = \frac{c-\alpha\lambda}{(1-\alpha)c} \to \frac{1}{1-\alpha}$$

as $c\to\infty$. Thus, also the second inequality is sharp and the proof is complete.

We recall the following properties of the rearrangement function which will be necessary in our proofs later on. The functions f and f_w^* are equimeasurable, i.e.,

$$(1.2)\qquad w(\{x\in\mathbb{R}^n:|f(x)|>\lambda\}) = |\{t>0:f_w^*(t)>\lambda\}| \quad \forall\lambda>0,$$

and

$$(1.3)\qquad \int_{\{x\in\mathbb{R}^n:|f(x)|>\lambda\}} |f(x)|w(x)\,dx = \int_{\{t>0:f_w^*(t)>\lambda\}} f_w^*(t)\,dt \quad \forall\lambda>0.$$

Immediately from (1.3) and Lemma 1 applied to the decreasing function $f_w^*(t)$ we obtain the following sharp estimates:

Corollary 1. *Let f be a Lebesgue measurable function on $\mathbb{R}^n$ and let $0<\alpha<1$. Then*

$$\frac{1}{\lambda}\int_{\{x\in\mathbb{R}^n:|f(x)|>\lambda\}} |f(x)|w(x)\,dx \le |\{t>0:f_w^{**}(t)>\lambda\}|$$
$$\le \frac{1}{(1-\alpha)\lambda}\int_{\{x\in\mathbb{R}^n:|f(x)|>\alpha\lambda\}} |f(x)|w(x)\,dx$$

for all $\lambda>0$. Both estimates are sharp.

Now we are ready to present our main result in this section.

Theorem 1. *Let g be a positive function on $(0,\infty)$ such that $g(t)=g^*(t)$.*

(a) *The inequality*

$$(1.4)\qquad g(t)\le Cf_w^{**}(t)$$

holds, for a certain $C>0$ and all $t>0$, if and only if there are constants $C_1, C_2>0$ such that

$$(1.5)\qquad |\{t>0:g(t)>\lambda\}| \le \frac{C_1}{\lambda}\int_{\{x\in\mathbb{R}^n:|f(x)|>\lambda/C_2\}} |f(x)|w(x)\,dx$$

for all $\lambda>0$.

(b) *The inequality*

$$(1.6)\qquad f_w^{**}(t)\le Cg(t)$$

holds, for a certain $C>0$ and all $t>0$, if and only if there are constants $C_1, C_2>0$ such that

$$(1.7)\qquad \frac{1}{\lambda}\int_{\{x\in\mathbb{R}^n:|f(x)|>\lambda\}} |f(x)|w(x)\,dx \le C_1|\{t>0:g(t)>\lambda/C_2\}|$$

for all $\lambda>0$.

Proof. (1.4)⇒(1.5). By using the assumption (1.4), the second inequality from Lemma 1 for f_w^* and $\alpha = 1/2$, and the equality (1.3) we obtain

$$\begin{aligned}|\{t>0: g(t)>\lambda\}| &\le |\{t>0: f_w^{**}(t)>\lambda/C\}|\\ &\le \frac{2C}{\lambda}\int\limits_{\{t>0: f_w^*(t)>\lambda/(2C)\}} f_w^*(t)\,dt\\ &= \frac{2C}{\lambda}\int\limits_{\{x\in\mathbb{R}^n: |f(x)|>\lambda/(2C)\}} |f(x)|w(x)\,dx\end{aligned}$$

and (1.5) is proved with $C_1 = C_2 = 2C$.

(1.5)⇒(1.4). Using the assumption (1.5), the equality (1.3) and the first inequality from Lemma 1 for f_w^* we find

$$\begin{aligned}|\{t>0: g(t)>\lambda\}| &\le \frac{C_1}{\lambda}\int\limits_{\{x\in\mathbb{R}^n: |f(x)|>\lambda/C_2\}} |f(x)|w(x)\,dx\\ &= \frac{C_1}{\lambda}\int\limits_{\{t>0: f_w^*(t)>\lambda/C_2\}} f_w^*(t)\,dt\\ &\le \frac{C_1}{C_2}|\{t>0: f_w^{**}(t)>\lambda/C_2\}|\end{aligned}$$

and so

$$\begin{aligned}g(t) = g^*(t) &\le C_2 f_w^{**}(\min(1, C_2/C_1)t)\\ &\le \max(C_1, C_2) f_w^{**}(t).\end{aligned}$$

(1.6)⇒(1.7). Similarly, by using the equality (1.3), the first inequality from Lemma 1 for f_w^* and the assumption (1.6) we get

$$\begin{aligned}\frac{1}{\lambda}\int\limits_{\{x\in\mathbb{R}^n: |f(x)|>\lambda\}} |f(x)|w(x)\,dx &= \frac{1}{\lambda}\int\limits_{\{t>0: f_w^*(t)>\lambda\}} f_w^*(t)\,dt\\ &\le |\{t>0: f_w^{**}(t)>\lambda\}|\\ &\le |\{t>0: g(t)>\lambda/C\}|\end{aligned}$$

and (1.7) holds with $C_1 = 1$, $C_2 = C$.

(1.7)⇒(1.6). Taking $\alpha = 1/(C_1+1) < 1$ we have $\alpha C_1/(1-\alpha) = 1$ and, thus, by using the second inequality from Lemma 1 for f_w^* and $\alpha = 1/(C_1+1)$, the equality (1.3) and the assumption (1.7) we obtain

$$\begin{aligned}|\{t>0: f_w^{**}(t)>\lambda\}| &\le \frac{1}{(1-\alpha)\lambda}\int\limits_{\{t>0: f_w^*(t)>\alpha\lambda\}} f_w^*(t)\,dt\\ &= \frac{1}{(1-\alpha)\lambda}\int\limits_{\{x\in\mathbb{R}^n: |f(x)|>\alpha\lambda\}} |f(x)|w(x)\,dx\\ &\le C_1\frac{\alpha}{1-\alpha}|\{t>0: g(t)>\alpha\lambda/C_2\}|\\ &= |\{t>0: g(t)>\alpha\lambda/C_2\}|.\end{aligned}$$

This gives

$$f_w^{**}(t) = (f_w^{**})^*(t) \le \frac{C_2}{\alpha}g^*(t) = \frac{C_2}{\alpha}g(t) = C_2(C_1+1)g(t),$$

and the proof is complete.

COROLLARY 2. *Let w_1, w_2 be two weights on $\mathbb{R}^n$. The equivalence $f_{w_2}^{**}(t) \approx g_{w_1}^*(t)$ holds if and only if there are constants $C_1, C_2, C_3, C_4 > 0$ such that*

$$\begin{aligned}\frac{C_1}{\lambda}\int\limits_{\{x\in\mathbb{R}^n: |f(x)|>\lambda/C_2\}} |f(x)|w_2(x)\,dx &\le w_1(\{x\in\mathbb{R}^n: |g(x)|>\lambda\})\\ &\le \frac{C_3}{\lambda}\int\limits_{\{x\in\mathbb{R}^n: |f(x)|>\lambda/C_4\}} |f(x)|w_2(x)\,dx\end{aligned}$$

for all $\lambda > 0$.

Proof. Apply Theorem 1 with g equal to $g_{w_1}^*$ and $w = w_2$.

COROLLARY 3. *Let φ be a strictly increasing continuous function on $[0,\infty)$ with an inverse satisfying $\varphi^{-1}(2t) \le A\varphi^{-1}(t)$ for all $t > 0$. Assume that f and g are positive decreasing functions on $(0,\infty)$. Then*

$$\frac{1}{t}\int_0^t \varphi(f(s))\,ds \approx \varphi(g(t))$$

if and only if there are constants $C_1, C_2, C_3, C_4 > 0$ such that

$$\begin{aligned}C_1\int\limits_{\{t: f(t)>\lambda/C_2\}} \varphi(f(t))\,dt &\le \varphi(\lambda)|\{t>0: g(t)>\lambda\}|\\ &\le C_3\int\limits_{\{t: f(t)>\lambda/C_4\}} \varphi(f(t))\,dt\end{aligned}$$

for all $\lambda > 0$.

Proof. Let $s = \varphi(\lambda)$. Then, according to Corollary 2 applied to $f_{w_2}^* =$

$\varphi(f)$ and $g^*_{w_1} = \varphi(g)$, we obtain

$$\begin{aligned}|\{t>0: g(t)>\lambda\}| &= |\{t>0:\varphi(g(t))>s\}|\\ &\approx \frac{1}{s}\int\limits_{\{t>0:\varphi(f(t))>s/D\}}\varphi(f(t))\,dt\\ &= \frac{1}{\varphi(\lambda)}\int\limits_{\{t>0:f(t)>\varphi^{-1}(\varphi(\lambda)/D)\}}\varphi(f(t))\,dt\\ &\approx \frac{1}{\varphi(\lambda)}\int\limits_{\{t:f(t)>\lambda/C\}}\varphi(f(t))\,dt.\end{aligned}$$

COROLLARY 4. *Let w_1, w_2 be two weights on $\mathbb{R}^n$. If $p>1$ and, for some positive constants C_1 and C_2,*

$$w_1(\{x\in\mathbb{R}^n : |g(x)|>\lambda\}) \le \frac{C_1}{\lambda}\int\limits_{\{x\in\mathbb{R}^n:|f(x)|>\lambda/C_2\}}|f(x)|w_2(x)\,dx \quad \forall\lambda>0,$$

then

$$\int\limits_{\mathbb{R}^n}|g(x)|^p w_1(x)\,dx \le C_1C_2^{p-1}\frac{p}{p-1}\int\limits_{\mathbb{R}^n}|f(x)|^p w_2(x)\,dx.$$

Proof. We have, according to Corollary 2,

$$\begin{aligned}\int\limits_{\mathbb{R}^n}|g(x)|^p w_1(x)\,dx &= p\int\limits_0^\infty \lambda^{p-1}w_1(\{x\in\mathbb{R}^n:|g(x)|>\lambda\})\,d\lambda\\ &\le C_1p\int\limits_0^\infty\lambda^{p-2}\Big[\int\limits_{\{x\in\mathbb{R}^n:|f(x)|>\lambda/C_2\}}|f(x)|w_2(x)\,dx\Big]d\lambda\\ &= C_1p\int\limits_{\mathbb{R}^n}\Big(\int\limits_0^{C_2|f(x)|}\lambda^{p-2}\,d\lambda\Big)|f(x)|w_2(x)\,dx\\ &= C_1C_2^{p-1}\frac{p}{p-1}\int\limits_{\mathbb{R}^n}|f(x)|^p w_2(x)\,dx.\end{aligned}$$

2. On the Riesz–Wiener inequality for the maximal function. In this section we will generalize the Riesz–Wiener inequalities to more general measures. For $f\in L_1^{\mathrm{loc}}(\mathbb{R}^n, w\,dx)$ and $x\in\mathbb{R}^n$, define

$$M_wf(x) = \sup_{Q\ni x}\frac{1}{w(Q)}\int\limits_Q|f(y)|w(y)\,dy,$$

where the supremum is taken over all cubes $Q\subset\mathbb{R}^n$ containing x such that $w(Q)>0$.

THEOREM 2. *The following statements are equivalent:*

(i) *M_w is of weak type $(1,1)$, i.e.*

$$w(\{x\in\mathbb{R}^n : M_wg(x)>\lambda\}) \le \frac{C}{\lambda}\int\limits_{\mathbb{R}^n}|g(x)|w(x)\,dx \quad \forall g\in L_1(w)\ \forall\lambda>0,$$

(ii) $w(\{x\in\mathbb{R}^n : M_wf(x)>\lambda\})$

$$\le \frac{C}{(1-\alpha)\lambda}\int\limits_{\{x\in\mathbb{R}^n:|f(x)|>\alpha\lambda\}}|f(x)|w(x)\,dx$$

$\forall f\in L_1(w)+L_\infty$ $\forall\lambda>0$, *and all* $0<\alpha<1$,

(iii) $$(M_wf)^*_w(t)\le Df^{**}_w(t) \quad \forall f\in L_1(w)+L_\infty\ \forall t>0.$$

Proof. (i)⇒(ii). Put

$$f = f\chi_{\{|f|>\alpha\lambda\}} + f\chi_{\{|f|\le\alpha\lambda\}} = f_0+f_1.$$

Then $M_wf(x)\le M_wf_0(x)+M_wf_1(x)$ and so

$$\begin{aligned}w(\{x\in\mathbb{R}^n : M_wf(x)>\lambda\}) &\le w(\{x\in\mathbb{R}^n : M_wf_0(x)>(1-\alpha)\lambda\})\\ &\quad + w(\{x\in\mathbb{R}^n : M_wf_1(x)>\alpha\lambda\}).\end{aligned}$$

Since $M_wf_1(x)\le\|f_1\|_{L_\infty}\le\alpha\lambda$ a.e. it follows that the measure of the second set is zero and we obtain

$$w(\{x\in\mathbb{R}^n : M_wf(x)>\lambda\}) \le w(\{x\in\mathbb{R}^n : M_wf_0(x)>(1-\alpha)\lambda\}),$$

which by the assumption that M_w is of weak type $(1,1)$ can be estimated by

$$\frac{C}{(1-\alpha)\lambda}\int\limits_{\mathbb{R}^n}|f_0(x)|w(x)\,dx = \frac{C}{(1-\alpha)\lambda}\int\limits_{\{x\in\mathbb{R}^n:|f(x)|>\alpha\lambda\}}|f(x)|w(x)\,dx.$$

(ii)⇒(iii). Applying Theorem 1(a) with $g(t) = (M_wf)^*_w(t)$ to the assumption (ii) we obtain

$$(M_wf)^*_w(t) \le \max(C/(1-\alpha), 1/a)f^{**}_w(t).$$

Taking the infimum over all $0<\alpha<1$ we get $(M_wf)^*_w(t)\le(C+1)f^{**}_w(t)$.

(iii)⇒(i). From the well-known fact

$$\sup_{\lambda>0}\lambda w(\{x\in\mathbb{R}^n : h(x)>\lambda\}) = \sup_{t>0}th^*_w(t)$$

and the assumption (iii) it follows that, for all $\lambda>0$,

$$\begin{aligned}\lambda w(\{x\in\mathbb{R}^n : M_wf(x)>\lambda\}) &\le D\sup_{t>0}\int\limits_0^t f^*_w(s)\,ds = D\int\limits_0^\infty f^*_w(s)\,ds\\ &= D\int\limits_{\mathbb{R}^n}|f(x)|w(x)\,dx,\end{aligned}$$

and the proof is finished.

In connection with Theorem 2 we will now discuss the following important problem:

PROBLEM 1. For which w is the maximal operator M_w of weak type $(1,1)$?

We say that the measure $w(A)=\int_A w(x)\,dx$ with $w\in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ satisfies the *doubling condition* and we write $w\in D$ if $w(2Q)\le dw(Q)$ for every cube Q, with a certain constant $d>0$ independent of Q.

EXAMPLE 1 (cf. [6, pp. 142–144] or [11]). If either $w\in D$ or $n=1$ (the one-dimensional case), then M_w is of weak type $(1,1)$.

EXAMPLE 2 (Sjögren [11]). The maximal operator M_w generated by the Gaussian measure $w(x,y)=e^{-(x^2+y^2)/2}$ in $\mathbb{R}^2$ is not of weak type $(1,1)$. Note that $w(\mathbb{R}^2)<\infty$.

EXAMPLE 3. The maximal operator M_w generated by the measure $w(x,y)=e^{x+y}$ in $\mathbb{R}^2$ is not of weak type $(1,1)$. Note that $w(\mathbb{R}^2)=\infty$.

Proof. It is enough to prove that

$$\sup\{w(\{(x,y)\in\mathbb{R}^2 : M_wf(x,y)>1\}) : f\in L_1(w) \text{ and } \|f\|_{L_1(w)}=1\}=\infty. \tag{2.1}$$

In order to prove this we first observe that if for $a\in\mathbb{R}$ we define $S_a=\{(x,y)\in\mathbb{R}^2 : x\le a,\ y\le -a\}$ and $H=\{(x,y)\in\mathbb{R}^2 : x+y\le 0\}$, then

$$w(S_a)=\iint_{S_a} e^{x+y}\,dx\,dy=\int_{-\infty}^{a} e^x\,dx\int_{-\infty}^{-a} e^y\,dy=1$$

and

$$w(H)=\iint_H e^{x+y}\,dx\,dy=\int_{-\infty}^{\infty} e^x\Big(\int_{-\infty}^{-x} e^y\,dy\Big)\,dx=\infty.$$

Let (x_0,y_0) be an arbitrary point in $\mathbb{R}^2$ such that $x_0+y_0<0$ and $f(x,y)=e^{-x_0-y_0}\delta_{(x_0,y_0)}(x,y)$, where $\delta_{(x_0,y_0)}$ is the δ-function at this point, i.e.

$$\iint_{\mathbb{R}^2} f(x,y)e^{x+y}\,dx\,dy=1 \quad\text{and}\quad \operatorname{supp} f=(x_0,y_0).$$

Then, since any cube Q such that $(x_0,y_0)\in Q\subset H$ is contained in some S_a, we have

$$\frac{1}{w(Q)}\iint_Q f(x,y)e^{x+y}\,dx\,dy>1.$$

This means that $M_wf>1$ on the union of all the above cubes Q containing (x_0,y_0). The measure of this union tends to the measure of H (which is equal to ∞) as $x_0+y_0\to-\infty$. Thus (2.1) holds and the proof is complete.

EXAMPLE 4 (Vargas [14]). There is a non-doubling measure on $\mathbb{R}^n$, $n>1$, such that the maximal operator M_w generated by this measure is of weak type $(1,1)$. Take, for example, the measure $w(x)=(1+|x|^\alpha)^{-1}$ in $\mathbb{R}^n$ with $\alpha\ge n$.

3. On the Stein–Herz inequality for the maximal function. In order to be able to extend the Stein inequality to the weighted case we need to modify the Calderón–Zygmund decomposition lemma using the *centered maximal function* defined by

$$M_w^{\mathrm{c}}f(x)=\sup_{r>0}\frac{1}{w(Q(x,r))}\int_{Q(x,r)}|f(y)|w(y)\,dy,$$

where $Q(x,r)$ denotes the cube with center at x and side-length $2r$.

LEMMA 2 (Modified Calderón–Zygmund lemma). *Let $f\in L_1(w)+L_\infty$ and*

$$\lambda>\lim_{t\to w(\mathbb{R}^n)} f_w^{**}(t). \tag{3.1}$$

Put $\Omega=\{x\in\mathbb{R}^n : M_w^{\mathrm{c}}f(x)>\lambda\}$. Then

(i) $\|f\chi_{\mathbb{R}^n\setminus\Omega}\|_{L_\infty}\le\lambda$.

(ii) *For every $x\in\Omega$ there exists a cube Q_x with center at x such that*

$$\lambda<\frac{1}{w(Q_x)}\int_{Q_x}|f(y)|w(y)\,dy\le 2\lambda.$$

Proof. (i) By using the Lebesgue theorem we obtain

$$\begin{aligned}\lim_{r\to0+}\frac{1}{w(Q(x,r))}\int_{Q(x,r)}|f(y)|w(y)\,dy &=\lim_{r\to0+}\frac{1}{|Q(x,r)|}\int_{Q(x,r)}|f(y)|w(y)\,dy\Big/\Big[\frac{1}{|Q(x,r)|}\int_{Q(x,r)}w(y)\,dy\Big]\\ &=|f(x)|w(x)/w(x)=|f(x)|\quad\text{a.e.}\end{aligned}$$

and so

$$\|f\chi_{\mathbb{R}^n\setminus\Omega}\|_{L_\infty}\le\|(M_w^{\mathrm{c}}f)\chi_{\mathbb{R}^n\setminus\Omega}\|_{L_\infty}\le\lambda.$$

(ii) The assumption (3.1) gives that for some $0<t_\lambda<w(\mathbb{R}^n)$ we have

$$\lambda>\frac{1}{t_\lambda}\int_0^{t_\lambda} f_w^*(s)\,ds.$$

Now, if $w(Q(x,r)) \ge t_\lambda$, then

$$\frac{1}{w(Q(x,r))}\int\limits_{Q(x,r)}|f(y)|w(y)\,dy \le \frac{1}{w(Q(x,r))}\int\limits_0^{w(Q(x,r))} f_w^*(s)\,ds \le \frac{1}{t_\lambda}\int\limits_0^{t_\lambda} f_w^*(s)\,ds < \lambda.$$

We note that the function

$$\varphi_x(r) = \frac{1}{w(Q(x,r))}\int\limits_{Q(x,r)}|f(y)|w(y)\,dy$$

as a function of $r>0$ has the following properties:

(a) $\varphi_x(r) < \lambda$ when $r \to \infty$,
(b) $\varphi_x(r)$ is continuous and
(c) $\sup_{r>0}\varphi_x(r) > \lambda$ for each $x \in \Omega$.

These properties of φ_x give (ii) and the proof is complete.

THEOREM 3. *If* $w(\mathbb{R}^n) = \infty$, *then the inequalities*

(i) $$\frac{C'}{\lambda}\int\limits_{\{x\in\mathbb{R}^n:|f(x)|>\lambda\}}|f(x)|w(x)\,dx \le w(\{x\in\mathbb{R}^n : M_w f(x) > \lambda\}) \quad \forall\lambda>0,$$

and

(ii) $$f_w^{**}(t) \le D'(M_w f)_w^*(t) \quad \forall t>0,$$

are valid. The constants C' and D' are only dependent on the dimension n.

Proof. First, note that the assumption $w(\mathbb{R}^n)=\infty$ implies that if $\lambda < \lim_{t\to w(\mathbb{R}^n)} f_w^{**}(t) = \lim_{t\to\infty} f_w^{**}(t)$, then both sides of (i) are infinite.

In fact, if t is sufficiently large and $0<t_0\le t<\infty$, then

$$\lambda < \frac{1}{t}\int\limits_0^{t_0} f_w^*(s)\,ds + \frac{1}{t}\int\limits_{t_0}^{t} f_w^*(s)\,ds \le \frac{1}{t}\int\limits_0^{t_0} f_w^*(s)\,ds + \frac{t-t_0}{t}f_w^*(t_0)$$

and letting $t\to\infty$ we get $f_w^*(t_0) > \lambda$. Therefore, $|f(x)|>\lambda$ on a set of infinite measure and both expressions in (i) are equal to ∞.

Thus it is enough to consider the case when $\lambda > \lim_{t\to\infty} f_w^{**}(t)$ since the case when $\lambda = \lim_{t\to\infty} f_w^{**}(t)$ can be obtained by taking limits. Using the

Lebesgue differentiation theorem we get the following estimate:

$$\frac{1}{\lambda}\int\limits_{\{x\in\mathbb{R}^n:|f(x)|>\lambda\}}|f(x)|w(x)\,dx \le \frac{1}{\lambda}\int\limits_{\{x\in\mathbb{R}^n:M_w^c f(x)>\lambda\}}|f(x)|w(x)\,dx = \frac{1}{\lambda}\int\limits_{\Omega}|f(x)|w(x)\,dx,$$

where $\Omega = \{x\in\mathbb{R}^n : M_w^c f(x) > \lambda\}$.

Below $\{Q_x\}_{x\in\Omega}$ is a family of cubes from Lemma 2(ii).

Let Q be an arbitrary cube in $\mathbb{R}^n$. Then

$$\frac{1}{\lambda}\int\limits_{\Omega\cap Q}|f(x)|w(x)\,dx$$

can be estimated by the Besicovitch covering theorem applied to the family of cubes $\{Q_x\}_{x\in\Omega\cap Q}$. Therefore, there exists a finite number (depending only on the dimension n) of packings $\pi_1,\dots,\pi_N$ of cubes $\pi_k = \{Q_{x_i,k}\}$ containing only cubes from the family $\{Q_x\}_{x\in\Omega\cap Q}$ and such that

$$\Omega\cap Q \subset \bigcup_{i,k} Q_{x_i,k}.$$

By using Lemma 2 (modified Calderón–Zygmund decomposition) we obtain

$$\frac{1}{\lambda}\int\limits_{\Omega\cap Q}|f(x)|w(x)\,dx \le 2\sum_{k=1}^{N}\Big(\sum_{Q_{x_i,k}\in\pi_k} w(Q_{x_i,k})\Big) \le 2N\max_{1\le k\le N}\sum_i w(Q_{x_i,k}).$$

For $z\in\bigcup_{x\in\Omega}Q_x$ we have

$$M_w f(z) \ge \frac{1}{w(Q_x)}\int\limits_{Q_x}|f(y)|w(y)\,dy > \lambda,$$

which gives

$$\sum_i w(Q_{x_i,k}) \le w(\{z\in Q : M_w f(z) > \lambda\})$$

for $1\le k\le N$ and, thus,

$$\frac{1}{\lambda}\int\limits_{\Omega\cap Q}|f(x)|w(x)\,dx \le 2Nw(\{z\in Q : M_w f(z) > \lambda\}).$$

Since the cube Q was arbitrary we obtain (i).

Now, according to Theorem 1(b) with $g(t) = (M_w f)_w^*(t)$, (i) implies (ii) and we are done.

Finally, we note that if the inequality (ii) holds, then Theorem 1(b) with $g(t) = (M_w f)^*_w(t)$ gives the following inequality:

$$\frac{1}{\lambda} \int_{\{x\in\mathbb{R}^n : |f(x)|>\lambda\}} |f(x)|w(x)\,dx \le w(\{x \in \mathbb{R}^n : M_w f(x) > \lambda/D'\}) \quad \forall\lambda > 0.$$

Therefore (i) and (ii) are "almost" equivalent.

Remark 1. If $w(\mathbb{R}^n) < \infty$ and $\lambda < \lim_{t\to w(\mathbb{R}^n)} f^{**}_w(t)$, then the inequality (i) in Theorem 3 is in general not true. For example, if we take $f(x) = c$, then for $\lambda < c$ the inequality (i) has the form

$$\frac{C'c}{\lambda} w(\mathbb{R}^n) \le w(\mathbb{R}^n)$$

and this is not true for small values of $\lambda > 0$.

Remark 2. If $w(\mathbb{R}^n) < \infty$ and if for $t \ge w(\mathbb{R}^n)$ we define $(M_w f)^*_w(t) = \lim_{s\to w(\mathbb{R}^n)} (M_w f)^*_w(s)$, then we have the following inequality corresponding to (i) of Theorem 3:

$$(3.2)\qquad \frac{C'}{\lambda} \int_{\{x\in\mathbb{R}^n : |f(x)|>\lambda\}} |f(x)|w(x)\,dx \le |\{t > 0 : (M_w f)^*_w(t) > \lambda\}| \quad \forall\lambda > 0.$$

Indeed, if $\lambda \ge f^{**}_w(w(\mathbb{R}^n))$, then the proof is the same as that of Theorem 3. For $\lambda < f^{**}_w(w(\mathbb{R}^n))$ we have

$$\lim_{r\to\infty} \frac{1}{w(Q(x,r))} \int_{Q(x,r)} |f(y)|w(y)\,dy = f^{**}_w(w(\mathbb{R}^n)),$$

which gives $M_w f(x) > \lambda$ for all $x \in \mathbb{R}^n$ and from equimeasurability we obtain $(M_w f)^*_w(t) > \lambda$ for all $t \ge w(\mathbb{R}^n)$. Thus $|\{t > 0 : (M_w f)^*_w(t) > \lambda\}| = \infty$.

Next we point out the following consequences of our Theorems 2 and 3 and Example 1:

COROLLARY 5. *Let $f \in L_1(w) + L_\infty$. The equivalence*

$$(M_w f)^*_w(t) \approx f^{**}_w(t) \quad \forall 0 < t < w(\mathbb{R}^n)$$

holds if and only if the maximal operator M_w is of weak type $(1,1)$.

COROLLARY 6 (The Riesz–Herz equivalence). *Let $f \in L_1(w) + L_\infty$. If $w \in D$, then*

$$(M_w f)^*_w(t) \approx f^{**}_w(t).$$

Note here that $w \in D$ implies that $w(\mathbb{R}^n) = \infty$.

4. The K-functional for the couple $(L_1(w), L_\infty)$. First of all, let us note that calculations of the K-functional for the couple $(L_1(w), L_\infty)$ are only necessary when $0 < t < w(\mathbb{R}^n) = \int_{\mathbb{R}^n} w(x)\,dx$. Indeed, by the well-known Peetre formula (cf. [4])

$$(4.1)\qquad K(t, f; L_1(w), L_\infty) = \int_0^t f^*_w(s)\,ds = t f^{**}_w(t), \quad 0 < t < \infty,$$

and since $f^*_w(s) = 0$ for $s \ge w(\mathbb{R}^n)$ it follows that for $t \ge w(\mathbb{R}^n)$ we have

$$(4.2)\qquad K(t, f; L_1(w), L_\infty) = \|f\|_{L_1(w)} = \lim_{s\to w(\mathbb{R}^n)} K(s, f; L_1(w), L_\infty).$$

From the equality (4.1) and Corollary 6 we see that the equivalence

$$(4.3)\qquad K(t, f; L_1(w), L_\infty) \approx t(M_w f)^*_w(t) \quad \forall 0 < t < w(\mathbb{R}^n)$$

is valid if and only if the maximal operator M_w is of weak type $(1,1)$.

Moreover, we have seen that for quite a few measures w in $\mathbb{R}^n$ $(n \ge 2)$ the maximal operator M_w is *not* of weak type $(1,1)$. Here we will make an "improvement" of the maximal operator M_w such that we can have an equivalence of the type (4.3) also in cases when M_w is not of weak type $(1,1)$.

For the formulation of our main result in this section we need some notions. Let $\pi = \{Q_i\}_{i=1}^{|\pi|}$ be a packing, i.e., a finite collection of non-overlapping cubes in $\mathbb{R}^n$. Consider the linear averaging operator S_π transforming every function $f \in L_1(w) + L_\infty$ into a step function, defined by

$$S_\pi(f)(x) = \sum_{i=1}^{|\pi|} \left[\frac{1}{w(Q_i)} \int_{Q_i} f(y)w(y)\,dy\right] \chi_{Q_i}(x).$$

The maximal function $M_w f$ can be obtained as the pointwise supremum of the family of the linear averaging operators S_π, $M_w f(x) = \sup_\pi S_\pi(|f|)(x)$ and so

$$(4.4)\qquad (M_w f)^*_w(t) = (\sup_\pi S_\pi(|f|))^*_w(t).$$

Now we introduce a modified "maximal function" F_f defined by

$$(4.5)\qquad (F_f)^*_w(t) = \sup_\pi [(S_\pi(|f|))^*_w(t)],$$

which is different from (4.4) in that the order of taking supremum and rearrangement is interchanged.

The importance of this definition can be seen in the following result:

THEOREM 4. *If $f \in L_1(w) + L_\infty$, then*

$$(4.6)\qquad K(t, f; L_1(w), L_\infty) \approx t(F_f)^*_w(t) \quad \forall 0 < t < w(\mathbb{R}^n).$$

Proof. Since, for every packing $\pi = \{Q_i\}_{i=1}^{|\pi|}$, the operator $S_\pi(|f|)$ is sublinear and bounded (with norm 1) in the couple $(L_1(w), L_\infty)$ it follows that

$$K(t, S_\pi(|f|); L_1(w), L_\infty) \le K(t, f; L_1(w), L_\infty).$$

Therefore, by using the equality (4.1), we obtain

$$K(t, f; L_1(w), L_\infty) \ge K(t, S_\pi(|f|); L_1(w), L_\infty) = \int_0^t (S_\pi(|f|))_w^*(s)\,ds$$
$$\ge t(S_\pi(|f|))_w^*(t)$$

for every packing $\pi = \{Q_i\}_{i=1}^{|\pi|}$. Thus

$$K(t, f; L_1(w), L_\infty) \ge t(F_f)_w^*(t)$$

and we have proved the inequality in one direction.

In order to prove the reverse inequality $K(t, f; L_1(w), L_\infty) \le Ct(F_f)_w^*(t)$ we decompose $\mathbb{R}^n$ into two subsets

$$\Omega_0 = \left\{x \in \mathbb{R}^n : \sup_{r>0} \frac{1}{w(Q(x,r))} \int_{Q(x,r)} |f(y)|w(y)\,dy > (F_f)_w^*(t)\right\}$$

and $\Omega_1 = \mathbb{R}^n \setminus \Omega_0$, and consider the decomposition $f = f\chi_{\Omega_0} + f\chi_{\Omega_1}$. By using the Lebesgue theorem we find that

$$\lim_{r\to 0^+} \frac{1}{w(Q(x,r))} \int_{Q(x,r)} |f(y)|w(y)\,dy$$
$$= \lim_{r\to 0^+} \frac{1}{|Q(x,r)|} \int_{Q(x,r)} |f(y)|w(y)\,dy \Big/ \left[\frac{1}{|Q(x,r)|} \int_{Q(x,r)} w(y)\,dy\right]$$
$$= |f(x)|w(x)/w(x) = |f(x)| \quad \text{a.e.}$$

and, thus,

$$\|f\chi_{\Omega_1}\|_{L_\infty} \le (F_f)_w^*(t). \tag{4.7}$$

It remains to show that

$$\|f\chi_{\Omega_0}\|_{L_1(w)} \le Ct(F_f)_w^*(t). \tag{4.8}$$

To prove (4.8) we shall construct below, for every $x \in \Omega_0$, a cube Q_x with center at x such that

$$(F_f)_w^*(t) < \frac{1}{w(Q_x)} \int_{Q_x} |f(y)|w(y)\,dy \le 2(F_f)_w^*(t). \tag{4.9}$$

If such a family of cubes is constructed the proof of (4.8) is the following. This family of cubes $\{Q_x\}_{x\in\Omega_0}$ will have the following property:

(*) If $\pi = \{Q_{x_i}\}$ is an arbitrary packing from the family $\{Q_x\}_{x\in\Omega_0}$, then

$$\sum_{Q_i\in\pi} w(Q_{x_i}) \le t.$$

Indeed, if $\sum_{Q_i\in\pi} w(Q_{x_i}) > t$, then, by using (4.9), we obtain

$$S_\pi(|f|)(x) = \sum_{Q_i\in\pi} \left[\frac{1}{w(Q_{x_i})} \int_{Q_{x_i}} |f(y)|w(y)\,dy\right] \chi_{Q_{x_i}}(x)$$
$$> (F_f)_w^*(t) \Big(\sum_{Q_i\in\pi} \chi_{Q_{x_i}}(x)\Big).$$

Thus, for $\lambda \le (F_f)_w^*(t)$,

$$w(\{x \in \mathbb{R}^n : S_\pi(|f|)(x) > \lambda\}) \ge \sum_{Q_i\in\pi} w(Q_{x_i}) > t,$$

which gives $(S_\pi(|f|))_w^*(t) > \lambda$ and so $(S_\pi(|f|))_w^*(t) > (F_f)_w^*(t)$, but this contradicts the definition of $(F_f)_w^*(t)$.

Let now Q be an arbitrary cube in $\mathbb{R}^n$. Then the set $Q \cap \Omega_0$ is bounded and we can apply the Besicovitch covering theorem to the family of cubes $\{Q_x\}_{x\in Q\cap\Omega_0}$. Therefore, there exist a finite number of packings $\pi_1, \ldots, \pi_N$, depending only on the dimension n, containing only cubes from the family $\{Q_x\}_{x\in Q\cap\Omega_0}$ and such that

$$Q \cap \Omega_0 \subset \bigcup_{k=1}^N \bigcup_{Q_x\in\pi_k} Q_x.$$

Thus, by using (4.9) and property (*) just proved, we obtain

$$\|f\chi_{Q\cap\Omega_0}\|_{L_1(w)} \le \sum_{k=1}^N \Big(\sum_{Q_{x_i}\in\pi_k} \|f\chi_{Q_{x_i}}\|_{L_1(w)}\Big)$$
$$\le 2(F_f)_w^*(t) \sum_{k=1}^N \Big(\sum_{Q_{x_i}\in\pi_k} w(Q_{x_i})\Big)$$
$$\le 2(F_f)_w^*(t) \sum_{k=1}^N t = 2N(F_f)_w^*(t)t.$$

Since the cube Q was arbitrary we obtain

$$\|f\chi_{\Omega_0}\|_{L_1(w)} \le 2Nt(F_f)_w^*(t).$$

This gives the required estimate (4.8).

Now we must only construct a family of cubes $\{Q_x\}_{x\in\Omega_0}$ with centers at the points x of Ω_0 such that the inequalities (4.9) hold. First, we observe that if $w(Q) > t$ (such cubes exist because $t < w(\mathbb{R}^n)$), then for a packing π containing only one cube Q we have

$$(S_Q(|f|))^*_w(t) = \frac{1}{w(Q)} \int_Q |f(y)|w(y)\,dy \le (F_f)^*_w(t).$$

Therefore, the function

$$\varphi_x(r) = \frac{1}{w(Q(x,r))} \int_{Q(x,r)} |f(y)|w(y)\,dy$$

of r is not greater than $(F_f)^*_w(t)$ for sufficiently large r. By using the continuity of $\varphi_x(r)$ in r and the fact that

$$\sup_{r>0} \varphi_x(r) > (F_f)^*_w(t) \quad \text{for } x \in \Omega_0$$

we conclude that for any $\varepsilon > 0$ and $x \in \Omega_0$ there exists $r = r_\varepsilon(x)$ such that

$$\varphi_x(r_\varepsilon(x)) \subset ((F_f)^*_w(t), (1+\varepsilon)(F_f)^*_w(t)),$$

which implies that it is possible to construct cubes satisfying the inequalities (4.9).

Remark 3. Since, on the right-hand side of (4.9), instead of the constant 2 we can take any number $q > 1$ it follows that

$$t(F_f)^*_w(t) \le K(t, f; L_1(w), L_\infty) \le (N+1)t(F_f)^*_w(t)$$

where the constant N is the constant from the Besicovitch covering theorem.

We also point out the following consequence of the equality (4.3) and Theorem 4:

COROLLARY 7. *If $f \in L_1(w) + L_\infty$ and $w \in D$, then*

$$(M_w f)^*_w(t) \approx (F_f)^*_w(t).$$

Using the above Theorem 4 we can also write a formula for the K-functional of the couple $(L_{p_0}(w_0), L_{p_1}(w_1))$, $0 < p_0 < p_1 \le \infty$. We need the following definitions: for $0 < p < \infty$ and a weight function w on $\mathbb{R}^n$ the weighted space $L_p(w)$ is the space generated by the quasi-norm

$$\|f\|_{L_p(w)} = \Big(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx \Big)^{1/p},$$

and $(F^p_f)^*_w(t) = [(F_{|f|^p})^*_w(t)]^{1/p}$.

THEOREM 5. (a) *Let $0 < p < \infty$. If $f \in L_p(w) + L_\infty$, then*

$$K(t^{1/p}, f; L_p(w), L_\infty) \approx t^{1/p}(F^p_f)^*_w(t) \quad \forall 0 < t < w(\mathbb{R}^n). \tag{4.10}$$

(b) *For $0 < p_0 < p_1 < \infty$ and two weight functions w_0, w_1 on $\mathbb{R}^n$ we put*

$$w_2 = (w_1/w_0)^{1/(p_1-p_0)} \quad \text{and} \quad w = (w_0^{p_1} w_1^{-p_0})^{1/(p_1-p_0)}.$$

If $f \in L_{p_0}(w_0) + L_{p_1}(w_1)$, then

$$\begin{aligned} K(t^{1/p_0-1/p_1}, f; L_{p_0}(w_0), L_{p_1}(w_1)) &\approx t^{1/p_0-1/p_1}\Big(\int_t^{w(\mathbb{R}^n)} (F^{p_0}_{fw_2})^*_w(s)^{p_1}\,ds\Big)^{1/p_1} \\ &\quad + t^{1/p_0-1/p} w(\mathbb{R}^n)^{1/p_1-1/p_0}\|f\|_{L_{p_0}(w_0)}. \end{aligned} \tag{4.11}$$

Proof. (a) We have the equivalence

$$K(t^{1/p}, f; L_p(w), L_\infty) \approx (K(t, |f|^p; L_1(w), L_\infty))^{1/p},$$

which was proved, even for more general spaces, in [9] for $p \ge 1$ but the same proof gives the result for every $p > 0$. Moreover, by using our Theorem 4, we obtain

$$(K(t, |f|^p; L_1(w), L_\infty))^{1/p} \approx [t(F_{|f|^p})^*_w(t)]^{1/p} = t^{1/p}(F^p_f)^*_w(t),$$

and the assertion follows.

(b) First, note that

$$K(t, f; L_{p_0}(w_0), L_{p_1}(w_1)) = K(t, fw_2; L_{p_0}(w), L_{p_1}(w)).$$

Then, since (cf. [4], Th. 5.2.1)

$$(L_{p_0}(w), L_\infty)_{\theta_1,p_1} = L_{p_1}(w), \quad \theta_1 = 1 - p_0/p_1,$$

it follows from the Holmstedt reiteration formula (cf. [4], Corollary 3.6.2) that

$$\begin{aligned} K(u, g; L_{p_0}(w), L_{p_1}(w)) &= K(u, g; L_{p_0}(w), (L_{p_0}(w), L_\infty)_{\theta_1,p_1}) \\ &\approx u\Big(\int_{u^{1/\theta_1}}^\infty (s^{-\theta_1}K(s, g; L_{p_0}(w), L_\infty))^{p_1}\frac{ds}{s}\Big)^{1/p_1}. \end{aligned}$$

Putting together the formulas above we obtain

$$\begin{aligned} K(t^{1/p_0-1/p_1}, f; L_{p_0}(w_0), L_{p_1}(w_1)) &= K(t^{1/p_0-1/p_1}, fw_2; L_{p_0}(w), L_{p_1}(w)) \\ &\approx t^{1/p_0-1/p_1}\Big(\int_{t^{1/p_0}}^\infty (s^{-\theta_1}K(s, fw_2; L_{p_0}(w), L_\infty))^{p_1}\frac{ds}{s}\Big)^{1/p_1}. \end{aligned}$$

Now, there are three possibilities for $w(\mathbb{R}^n)$ and t:

(i) $w(\mathbb{R}^n) = \infty$ and $0 < t < w(\mathbb{R}^n) = \infty$. Then, using (a), we obtain

$$t^{1/p_0-1/p_1}\Big(\int_{t^{1/p_0}}^{\infty}(s^{-\theta_1}K(s,fw_2;L_{p_0}(w),L_\infty))^{p_1}\frac{ds}{s}\Big)^{1/p_1}$$
$$\approx t^{1/p_0-1/p_1}\Big[\int_{t^{1/p_0}}^{\infty}(s^{1-\theta_1}(F_{fw_2}^{p_0})_w^*(s^{p_0})^{p_1}\frac{ds}{s}\Big]^{1/p_1}$$
$$\approx t^{1/p_0-1/p_1}\Big[\int_{t}^{\infty}(F_{fw_2}^{p_0})_w^*(s)^{p_1}\,ds\Big]^{1/p_1}.$$

(ii) $w(\mathbb{R}^n) < \infty$ and $t \ge w(\mathbb{R}^n)$. By the Hölder inequality we have

$$\|f\|_{L_{p_0}(w_0)} \le w(\mathbb{R}^n)^{1/p_0-1/p_1}\|f\|_{L_{p_1}(w_1)} \quad \forall f \in L_{p_1}(w_1),$$

i.e., $L_{p_0}(w_0) + L_{p_1}(w_1) = L_{p_0}(w_0)$ and then, for $f \in L_{p_0}(w_0)$,

$$K(t^{1/p_0-1/p_1}, f; L_{p_0}(w_0), L_{p_1}(w_1)) \approx \|f\|_{L_{p_0}(w_0)} \quad \text{for } t \ge w(\mathbb{R}^n).$$

On the other hand,

$$\|fw_2\|_{L_{p_0}(w)} \le w(\mathbb{R}^n)^{1/p_0}\|fw_2\|_{L_\infty} \quad \forall fw_2 \in L_\infty$$

gives $L_{p_0}(w) + L_\infty = L_{p_0}(w)$ and so

$$K(s, fw_2; L_{p_0}(w), L_\infty) \approx \|fw_2\|_{L_{p_0}(w)} = \|f\|_{L_{p_0}(w_0)} \quad \text{for } s > w(\mathbb{R}^n)^{1/p_0}.$$

Thus

$$t^{1/p_0-1/p_1}\Big(\int_{t^{1/p_0}}^{\infty}(s^{-\theta_1}K(s,fw_2;L_{p_0}(w),L_\infty))^{p_1}\frac{ds}{s}\Big)^{1/p_1}$$
$$\approx t^{1/p_0-1/p_1}\Big(\int_{t^{1/p_0}}^{\infty}(s^{-\theta_1}\|f\|_{L_{p_0}(w_0)})^{p_1}\frac{ds}{s}\Big)^{1/p_1}$$
$$= t^{1/p_0-1/p_1}t^{-\theta_1/p_0}\|f\|_{L_{p_0}(w_0)}/(\theta_1 p_1)$$
$$= \|f\|_{L_{p_0}(w_0)}/(\theta_1 p_1).$$

The above calculations show that formula (4.11) is true in this case.

(iii) $w(\mathbb{R}^n) < \infty$ and $0 < t < w(\mathbb{R}^n)$. Then the result from (a),

$$K(s, fw_2; L_{p_0}(w), L_\infty) \approx s(F_{fw_2}^{p_0})_w^*(s^{p_0}) \quad \forall 0 < s^{p_0} \le w(\mathbb{R}^n),$$

together with the known property of the K-functional

$$K(s, fw_2; L_{p_0}(w), L_\infty) \approx \|fw_2\|_{L_{p_0}(w)} = \|f\|_{L_{p_0}(w_0)} \quad \text{for } s^{p_0} > w(\mathbb{R}^n),$$

gives

$$K(t^{1/p_0-1/p_1}, f; L_{p_0}(w_0), L_{p_1}(w_1))$$
$$\approx t^{1/p_0-1/p_1}\Big(\int_{t^{1/p_0}}^{\infty}(s^{-\theta_1}K(s,fw_2;L_{p_0}(w),L_\infty))^{p_1}\frac{ds}{s}\Big)^{1/p_1}$$
$$\approx t^{1/p_0-1/p_1}\Big[\int_{t^{1/p_0}}^{w(\mathbb{R}^n)^{1/p_0}}(s^{1-\theta_1}(F_{fw_2}^{p_0})_w^*(s^{p_0}))^{p_1}\frac{ds}{s} + \int_{w(\mathbb{R}^n)^{1/p_0}}^{\infty}(s^{1-\theta_1}\|f\|_{L_{p_0}(w_0)})^{p_1}\frac{ds}{s}\Big]^{1/p_1}$$
$$= t^{1/p_0-1/p_1}\Big[\int_{t}^{w(\mathbb{R}^n)}(F_{fw_2}^{p_0})_w^*(s)^{p_1}\,ds + \frac{w(\mathbb{R}^n)^{-\theta_1 p_1/p_0}}{\theta_1 p_1}\|f\|_{L_{p_0}(w_0)}^{p_1}\Big]^{1/p_1}$$
$$\approx t^{1/p_0-1/p_1}\Big[\Big(\int_{t}^{w(\mathbb{R}^n)}(F_{fw_2}^{p_0})_w^*(s)^{p_1}\,ds\Big)^{1/p_1} + w(\mathbb{R}^n)^{1/p_1-1/p_0}\|f\|_{L_{p_0}(w_0)}\Big].$$

The proof is complete.

Remark 4. The formula (4.11), in the case when $w(\mathbb{R}^n) = \infty$, has surprisingly only one term in contrast to the usual Holmstedt two-term formula (cf. [4], Th. 3.6.1):

$$K(t^{1/p_0-1/p_1}, f; L_{p_0}(w_0), L_{p_1}(w_1)) \approx t^{1/p_0-1/p_1}\Big(\int_{t}^{\infty}(F_{fw_2}^{p_0})_w^*(s)^{p_1}\,ds\Big)^{1/p_1}.$$

Remark 5. Using more general formulas of Holmstedt type (see [1]) it is possible to obtain formulas for the K-functional in terms of the maximal function for many other spaces, for example for weighted Lorentz spaces.

Addendum. During the refereeing process of this paper we were kindly informed on November 11, 1995, by Professor María J. Carro, Universitat de Barcelona, Spain, that she and Professor Javier Soria in the paper *The Hardy–Littlewood maximal function and weighted Lorentz spaces*, J. London Math. Soc. (to appear) have obtained a result similar to our Lemma 1. More exactly, their Theorem 2.1 is almost the same as our Lemma 1 except that they have the constant 1/2 instead of our (sharp) constant 1 in the first inequality of (1.1).

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Convergence of conditional expectations for unbounded closed convex random sets

by

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Abstract. We discuss here several types of convergence of conditional expectations for unbounded closed convex random sets of the form $E^{\mathcal{B}_n}X_n$ where $(\mathcal{B}_n)$ is a decreasing sequence of sub-σ-algebras and (X_n) is a sequence of closed convex random sets in a separable Banach space.

1. Introduction. The Mosco convergence of sequences of sets or functions is known to be a useful tool in the approximation of optimization problems and variational inequalities (see e.g. [A, Mo, We]). Often these problems are considered in the presence of a parameter ω whose value depends on the outcome of a random experiment.

The present paper precisely concerns Mosco convergence in such a stochastic context. Indeed, our main contribution consists in the study of almost sure Mosco convergence for sequences of random sets of the form $E^{\mathcal{B}_n}X_n$, where $(\mathcal{B}_n)_{n\geq 1}$ is a decreasing sequence of sub-σ-algebras and (X_n) a sequence of Banach-valued closed convex random sets (recall that a *random set* is a random variable whose values are subsets of some given space).

It is worthwhile to observe that, even for real-valued random variables, results of such kind are not completely standard. That is why we provide a short and self-contained treatment of the problem in this special case (in Section 4.A). This is done in the same spirit as in the papers by Szynal and Zięba [SZ] and by Zięba [Zi].

On the other hand, we stress the fact that the values of the random sets we deal with are not assumed to be bounded. So, specific results borrowed from [He1, 2] are needed; they are recalled in Section 3 for convenience. Our main results and their proofs are presented in Sections 4.B and 4.C.

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