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CESÀRO FUNCTION SPACES FAIL THE FIXED POINT PROPERTY

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ABSTRACT. The Cesàro sequence spaces $ces_p, 1 < p < \infty$, are reflexive but they have the fixed point property. In this paper we prove that in contrast to these sequence spaces the corresponding Cesàro function spaces Ces_p on both $[0, 1]$ and $[0, \infty)$ for $1 < p < \infty$ are not reflexive and they fail to have the fixed point property.

1. INTRODUCTION

Let $1 \leq p \leq \infty$. The *Cesàro sequence space* ces_p is defined as the set of all real sequences $x = \{x_k\}$ such that

$$\|x\|_{c(p)} = \left(\sum_{n=1}^{\infty} (\frac{1}{n} \sum_{k=1}^{n} |x_k|)^p \right)^{1/p} < \infty \text{ when } 1 \leq p < \infty,$$

and

$$\|x\|_{c(\infty)} = \sup_{n \in \mathbf{N}} \frac{1}{n} \sum_{k=1}^{n} |x_k| < \infty \text{ when } p = \infty.$$

The *Cesàro function spaces* $Ces_p = Ces_p(I)$ are the classes of Lebesgue measurable real functions f on $I = [0, 1]$ or $I = [0, \infty)$ such that the corresponding norms are finite, where

$$\|f\|_{C(p)} = \left(\int_I (\frac{1}{x} \int_0^x |f(t)| \, dt)^p \, dx \right)^{1/p} \text{ for } 1 \leq p < \infty,$$

and

$$\|f\|_{C(\infty)} = \sup_{x \in I, \ x > 0} \frac{1}{x} \int_0^x |f(t)| \, dt < \infty \text{ for } p = \infty.$$

The Cesàro sequence spaces ces_p were investigated in the seventies by Shiue, Leibowitz and Jagers. In particular, they proved that $ces_1 = \{0\}$, ces_p are separable reflexive Banach spaces for $1 < p < \infty$ and the l^p spaces are continuously embedded

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into ces_p for $1 < p \leq \infty$ with strict embeddings. Also if $1 < p < q \leq \infty$, then $ces_p \subset ces_q$ with continuous strict embeddings. Bennett [3] proved that ces_p for $1 < p < \infty$ are not isomorphic to any l^q space with $1 \leq q \leq \infty$ (see also [15] for another proof). Moreover, Maligranda-Petrot-Suantai [15] proved recently that Cesàro sequence spaces ces_p for $1 < p < \infty$ are not uniformly nonsquare; that is, there are sequences $\{x_n\}$ and $\{y_n\}$ on the unit sphere such that $\lim_{n\to\infty} \min(\|x_n + y_n\|_{c(p)}, \|x_n - y_n\|_{c(p)}) = 2$. They even proved that these spaces are not B-convex. We refer here to [3], [15] and the references given there.

Several geometric properties of the Cesàro sequence spaces ces_p were studied in recent years by many mathematicians, and in 1999-2000 it was also proved by Cui-Hudzik [5], Cui-Hudzik-Li [6] and Cui-Meng-Płuciennik [7] that Cesàro sequence spaces ces_p for $1 < p < \infty$ have the fixed point property (cf. also [4, Part 9]).

Cesàro function spaces $Ces_p[0,\infty)$ for $1 \leq p \leq \infty$ were considered by Shiue [16], Hassard-Hussein [12] and Sy-Zhang-Lee [17]. They proved that $Ces_1[0,\infty) = \{0\}$ and $Ces_p[0,\infty)$ for $1 < p < \infty$ are separable Banach spaces and that $Ces_\infty[0,\infty)$ is a nonseparable Banach space. The space $Ces_\infty[0,1]$ is known as the Korenblyum-Krein-Levin space already introduced in 1948.

By the Hardy inequality the $L^p(I)$ spaces are continuously embedded into $Ces_p(I)$ for $1 < p \leq \infty$ with strict embedding, where $I = [0,1]$ or $I = [0,\infty)$ (cf. [11, Theorem 327] and [13, Theorem 2]). Also if $1 < p < q \leq \infty$, then $Ces_q[0,1] \subset Ces_p[0,1]$ with continuous strict embedding. Moreover, $Ces_1[0,1]$ is a weighted $L^1_w[0,1]$ space with the weight $w(t) = \ln\frac{1}{t}$ for $0 < t \leq 1$. In fact,

$$\int_0^1 \left(\frac{1}{x}\int_0^x |f(t)|\,dt\right)dx = \int_0^1 \left(\int_t^1 \frac{1}{x}\,dx\right)|f(t)|\,dt = \int_0^1 |f(t)| \ln\frac{1}{t}\,dt. \tag{1}$$

We will show that, in contrast to Cesàro sequence spaces, the Cesàro function spaces $Ces_p(I)$ on both $I = [0,1]$ and $I = [0,\infty)$ for $1 < p < \infty$ are not reflexive and that they do not have the fixed point property.

A Banach space X has the *fixed point property* (FPP) [resp. *weak fixed point property* (WFPP)] if every nonexpansive mapping of every closed bounded convex [resp. nonempty weakly compact convex] subset K of X into itself has a fixed point. Recall that T is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$.

The spaces c_0 and l^1 both fail to have the FPP with their classical norms, but they have the WFPP. The space $L^1[0,1]$ does not have the WFPP, as was proved by Alspach [1].

Our proof that the Cesàro function spaces $Ces_p(I)$ on $I = [0,1]$ with $1 \leq p \leq \infty$ and on $I = [0,\infty)$ with $1 < p \leq \infty$ fail to have the fixed point property will be carried out by showing that these spaces contain an asymptotically isometric copy of l^1.

A Banach space X *contains an asymptotically isometric copy of* l^1 if there exists a null sequence (ε_n) in $(0,1)$ and a sequence (x_n) in X such that

$$\sum_{n=1}^{\infty}(1-\varepsilon_n)|\alpha_n| \leq \|\sum_{n=1}^{\infty}\alpha_n x_n\|_X \leq \sum_{n=1}^{\infty}|\alpha_n|$$

for all $(\alpha_n) \in l^1$ of scalars. This notion was introduced by Dowling and Lennard in [9], where they proved that such spaces fail to have the FPP.

2. Main results

Cesàro sequence spaces $ces_p, 1 < p < \infty$, are reflexive but not B-convex and they have the fixed point property. In contrast to these sequence spaces the corresponding Cesàro function spaces $Ces_p(I)$ on both $I = [0,1]$ and $I = [0,\infty)$ for $1 < p < \infty$ are not reflexive and they do not have the fixed point property. Our main result reads:

Theorem 1. *Let $1 \leq p \leq \infty$. The Cesàro function space $Ces_p[0,1]$ contains an asymptotically isometric copy of l^1; that is, there exist a sequence $\{\varepsilon_n\} \subset (0,1), \varepsilon_n \to 0$ as $n \to \infty$ and a sequence of functions $\{f_n\} \subset Ces_p[0,1]$ such that for arbitrary $\{\alpha_n\} \in l^1$ we have*

$$\sum_{n=1}^{\infty}(1-\varepsilon_n)|\alpha_n| \leq \|\sum_{n=1}^{\infty}\alpha_n f_n\|_{C(p)} \leq \sum_{n=1}^{\infty}|\alpha_n|. \tag{2}$$

Before the proof of this theorem we prove the following auxiliary result.

Lemma 1. *Let $0 < a < b < 1, f \in Ces_p[0,1]$ and $\operatorname{supp} f := \{t \in [0,1] : f(t) \neq 0\} \subset [a,b]$. Then*

$$\left(b^{1-p}-1\right)^{1/p}\|f\|_1 \leq (p-1)^{1/p}\|f\|_{C(p)} \leq \left(a^{1-p}-1\right)^{1/p}\|f\|_1, \tag{3}$$

for $1 < p < \infty$ and

$$\ln\frac{1}{b}\|f\|_1 \leq \|f\|_{C(1)} \leq \ln\frac{1}{a}\|f\|_1,\ \ \frac{1}{b}\|f\|_1 \leq \|f\|_{C(\infty)} \leq \frac{1}{a}\|f\|_1, \tag{4}$$

where $\|f\|_1 = \int_0^1 |f(t)|\,dt$.

Proof. It is obvious that for any $0 < x \leq 1$ we have

$$\frac{1}{x}\|f\|_1\chi_{[b,1]}(x) \leq F_f(x) := \frac{1}{x}\int_0^x |f(t)|\,dt \leq \frac{1}{x}\|f\|_1\chi_{[a,1]}(x).$$

Since, for every $c \in (0,1), \int_c^1 t^{-p}\,dt = \frac{c^{1-p}-1}{p-1}$ and $\int_c^1 t^{-1}\,dt = \ln\frac{1}{c}$ we obtain (3) and (4) for $p = 1$. In the case of $p = \infty$ we see that

$$\frac{1}{b}\|f\|_1 \leq \|F_f\|_{L^\infty[0,1]} \leq \frac{1}{a}\|f\|_1,$$

and the lemma is proved. □

Proof of Theorem 1. For $1 < p < \infty$ we set

$$g_n = \chi_{[a_n,a_{n+1})},\ \text{with}\ a_n = 2^{1/(1-p)}\left(1-\frac{1}{2^n}\right),\ n = 1,2,\ldots.$$

Since $\|g_n\|_1 = a_{n+1} - a_n = 2^{1/(1-p)}\cdot 2^{-n-1}$ and $a_n^{1-p} - 1 = \frac{2}{(1-2^{-n})^{p-1}} - 1$, then, by Lemma 1 (see the second estimate in (3)), this yields that

$$2^{-1/(1-p)}2^{n+1}(p-1)^{1/p}\|g_n\|_{C(p)} \leq \left(\frac{2}{(1-2^{-n})^{p-1}}-1\right)^{1/p}.$$

Let $f_n = g_n/\|g_n\|_{C(p)}$ and $\alpha_n \in \mathbb{R}$ for $n = 1,2,\ldots$. Since $\operatorname{supp} g_n \subset [2^{1(1-p)-1}, 2^{1/(1-p)})$ for every $n \in \mathbb{N}$ it follows from Lemma 1 (see the first estimate in (3)) that

$$\begin{aligned}\|\sum_{n=1}^{\infty}\alpha_n f_n\|_{C(p)} &\geq \frac{\|\sum_{n=1}^{\infty}\alpha_n f_n\|_1}{(p-1)^{1/p}}\\ &= \sum_{n=1}^{\infty}\frac{|\alpha_n|2^{1/(1-p)}}{2^{n+1}(p-1)^{1/p}\|g_n\|_{C(p)}}\\ &\geq \sum_{n=1}^{\infty}\left(\frac{2}{(1-2^{-n})^{p-1}}-1\right)^{-1/p}|\alpha_n|.\end{aligned}$$

Denote

$$\varepsilon_n = 1-\left(\frac{2}{(1-2^{-n})^{p-1}}-1\right)^{-1/p}.$$

Then $\{\varepsilon_n\}\subset(0,1)$ and $\varepsilon_n\to 0$ as $n\to\infty$. This means that the left-hand side of (2) is proved. The right-hand side of (2) is obvious since $\|f_n\|_{C(p)}=1$.

In the case $p=\infty$ we take $f_n = g_n/\|g_n\|_{C(\infty)}$, where $g_n=\chi_{[a_n,a_{n+1})}$ with $a_n = 1-2^{-n}, n=1,2,\ldots$. Then $\|g_n\|_1 = 2^{-n-1}$ and, by Lemma 1 (see the second estimate in (4)) $\|g_n\|_{C(\infty)}\leq\frac{1}{1-2^{-n}}2^{-n-1}$ or $2^{n+1}\|g_n\|_{C(\infty)}\leq\frac{1}{1-2^{-n}}$. Since $\operatorname{supp} g_n\subset[1/2,1)$, for every $n\in\mathbb{N}$, it follows from Lemma 1 (see the first estimate in (4)) that:

$$\begin{aligned}\|\sum_{n=1}^{\infty}\alpha_n f_n\|_{C(\infty)} &\geq \|\sum_{n=1}^{\infty}\alpha_n f_n\|_1\\ &= \sum_{n=1}^{\infty}\frac{|\alpha_n|}{2^{n+1}\|g_n\|_{C(\infty)}}\geq\sum_{n=1}^{\infty}(1-2^{-n})|\alpha_n|,\end{aligned}$$

and $\varepsilon_n = 2^{-n}$ is a required sequence.

In the case $p=1$ we take $g_n=\chi_{[a_n,a_{n+1})}$, where $a_n=\frac{1}{e}(1-2^{-n}), n=1,2,\ldots$ and argue in a similar way. The proof is complete. □

The analogous result holds for Cesàro function spaces on $[0,\infty)$.

Theorem 2. *Let $1<p\leq\infty$. The Cesàro function space $Ces_p[0,\infty)$ contains an asymptotically isometric copy of l^1.*

Proof. We consider only the case $1<p<\infty$ (the case $p=\infty$ can be proved similarly as in Theorem 1). We take $g_n=\chi_{[a_n,a_{n+1})}$ with $a_n=1-2^{-n},\ n=1,2,\ldots$ and continue the proof as in Theorem 1, observing that the estimate corresponding to (3) for $Ces_p[0,\infty)$ will be (for $0<a<b<\infty$)

$$b^{1/p-1}\|f\|_1\leq(p-1)^{1/p}\|f\|_{C(p)}\leq a^{1/p-1}\|f\|_1,\ \text{with } \|f\|_1=\int_0^{\infty}|f(t)|\,dt,$$

and then for $f_n = g_n/\|g_n\|_{C(p)}$ we have that

$$\begin{aligned}\|\sum_{n=1}^{\infty}\alpha_n f_n\|_{C(p)} &\geq \frac{\|\sum_{n=1}^{\infty}\alpha_n f_n\|_1}{(p-1)^{1/p}} \\ &= \sum_{n=1}^{\infty}\frac{|\alpha_n|}{2^{n+1}(p-1)^{1/p}\|g_n\|_{C(p)}} \geq \sum_{n=1}^{\infty} a_n^{1-1/p}|\alpha_n| \\ &= \sum_{n=1}^{\infty}(1-2^{-n})^{1-1/p}|\alpha_n| = \sum_{n=1}^{\infty}(1-\varepsilon_n)|\alpha_n|,\end{aligned}$$

and $\varepsilon_n = 1-(1-2^{-n})^{1-1/p}$ is a required sequence. The proof is complete. □

Remark 1. It is obvious from Theorem 1 and Theorem 2 that the Cesàro function spaces $Ces_p(I)$ for $1 < p \leq \infty$ are not reflexive.

Dowling-Lennard [9] proved that if a Banach space X contains an asymptotically isometric copy of l^1, then there exists a nonexpansive mapping defined on a closed bounded convex subset of X without a fixed point, i.e., that X fails to have the fixed point property (see also [10, Theorem 2.3 and Corollary 2.11]). By the Dilworth-Girardi-Hagler result [8, Theorem 2] the dual space X^* does not have the fixed point property since they proved there that X contains an asymptotically isometric copy of l^1 if and only if the dual space X^* contains an isometric copy of $L^1[0,1]$. Combining these results with our Theorem 1 and Theorem 2 we obtain immediately our main result on the fixed point property of Cesàro function spaces and their dual spaces.

Theorem 3. *If either $1 \leq p \leq \infty$ and $I = [0,1]$ or $1 < p \leq \infty$ and $I = [0,\infty)$, then the Cesàro function spaces $Ces_p(I)$ and their dual spaces $Ces_p(I)^*$ fail to have the fixed point property.*

Theorem 3 gives information about the fixed point property, and therefore it is natural to ask what one can say about the weak fixed point property.

Note that the space $Ces_1[0,1]$ is isometric to $L^1[0,1]$ by the equality (1), and by the Alspach result [1] $L^1[0,1]$ fails to have WFPP; therefore $Ces_1[0,1]$ also fails to have WFPP.

By combining Theorem 1, Theorem 2 and the Dilworth-Girardi-Hagler result [8, Corollary 13] we have the following result:

Proposition 1. *The dual spaces to the Cesàro function spaces $Ces_p(I)^*$ do not have the weak fixed point property.*

Proposition 2. *The Cesàro function spaces $Ces_p(I)$ for $1 < p < \infty$ are not isomorphic to any $L^q(I)$ space for $1 \leq q \leq \infty$. In particular, they are not isomorphic to $L^1(I)$.*

Of course, $Ces_p(I)$ for $1 < p < \infty$, as nonreflexive and separable spaces, cannot be isomorphic to any $L^q(I)$ with $1 < q < \infty$ or $q = \infty$. The statement for $q = 1$ will be proved in the forthcoming paper [2].

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