ON THE WEIGHTED HARDY TYPE INEQUALITY IN A FIXED DOMAIN
FOR FUNCTIONS VANISHING ON THE PART OF THE BOUNDARY

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Abstract. We derive and discuss a new two-dimensional weighted Hardy-type inequality in a rectangle for the class of functions from the Sobolev space $H^1$ vanishing on small alternating pieces of the boundary.

1. Introduction

Inequalities of Hardy-type are very important for many applications. These inequalities are important tools e.g. for deriving some estimates for operator norms, for proving some embedding theorems, for estimating eigenvalues, etc.

The following basic one-dimensional Hardy-type inequality is well known:

$$\int_0^a \left( \frac{u'}{t} \right)^p dt \leq \left( \frac{p}{p-1} \right)^p \int_0^a (u')^p dt,$$

where $u \in L^p(0, a)$, $u' \in L^p(0, a)$, $p > 1$, $u(0) = 0$.

This inequality could be generalized to the multidimensional weighted form:

$$\left( \int_{\mathbb{R}^n} V(x) |u(x)|^q dx \right)^{\frac{1}{q}} \leq C \left( \int_{\mathbb{R}^n} W(x) |\nabla u(x)|^p dx \right)^{\frac{1}{p}},$$

where $\n \in \mathbb{Z}_+$, $u(x) \in C_0^\infty(\mathbb{R}^n)$, $V(x) \geq 0, W(x) \geq 0, p, q \geq 1$, and the constant $C$ depends only on $V(x)$ and $W(x)$. There are several results concerning weighted Hardy-type inequality (see e.g. the books [10], [11] and [14] and the references given there).

For the case $n = 1, 1 < p \leq q < \infty$ we have the following necessary and sufficient condition for the validity of (1.2):

$$A_M := \sup_{x > 0} \left( \int_x^{\infty} V(t) \left( \int_0^t W^{1-p^+}(t) dt \right)^{\frac{1}{p^+}} dt \right)^{\frac{1}{q}} < \infty,$$

where $p^+ = \frac{p}{p-1}$.

Let us mention also the following result of V. Maz’ya (see [12, Corollary of Theorem 1.4.1.2, Theorem 1.4.2.2 ]):


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Theorem 1.1. Let $1 < p < q < \infty$, $p < n$ or $1 = p \leq q < \infty$. Then the Hardy inequality (1.2) with $W(x) \equiv 1$ holds for every $u \in C^\infty_0(\mathbb{R}^n)$ with a finite constant $C > 0$ if and only if

$$B := \sup_{x \in \mathbb{R}^n} \sup_{R > 0} \left( \frac{1}{R} \int_{B_R(x)} V(y) \, dy \right) < \infty,$$

where $B_R(x)$ is a ball of the radius $R$ centered at the point $x$.

The weighted Hardy-type inequality can be generalized to domains in $\mathbb{R}^n$. It was first done by J. Nečas in [13]. He proved that if $\Omega$ is a bounded domain with Lipschitz boundary, $1 < p < \infty$, $\alpha < p - 1$, then for $u \in C^\infty_0(\Omega)$ the inequality

$$\int_{\Omega} |u(x)|^p \rho^{-p+\alpha}(x) \, dx \leq C \int_{\Omega} |\nabla u(x)|^p \rho^\alpha(x) \, dx \quad (1.4)$$

holds, where $\rho(x) = \text{dist}(x, \partial \Omega)$. After that this inequality was generalized by A. Kufner in [9] to domains with the Hölder boundary and later by A. Wannebo (see [16]) to domains with the generalized Hölder condition. All results related to (1.4) in the case $\alpha = 0$ was described in [7].

The aim of this paper is to prove a Hardy-type inequality (1.4) with $p = 2$ for functions from $H^1$, vanishing on small alternating pieces of the boundary of the domain. It is assumed for the simplicity that $\Omega$ is a rectangle in $\mathbb{R}^2$.

Such a result is completely new in the theory of Hardy-type inequalities and it gives us possibility to apply the tools of homogenization theory to obtain the asymptotics of the best constant in the Hardy-type inequality.

An analogous result was obtained earlier in [1] and in [2] for Friedrichs inequality which can be regarded as a special case of weighted Hardy-type inequalities when we assume that the weight functions equals to 1.

The paper is organized as follows: in Section 2 we give some necessary definitions and formulate auxiliary lemmas. The main results are presented and proved in Section 3 and Section 4 is reserved for some concluding remarks.

2. Preliminaries

Let $\Omega \subset \mathbb{R}^2$ be the rectangle $[0, a] \times [0, b]$:

$$\Omega = \{(x_1, x_2) : 0 \leq x_1 \leq a, 0 \leq x_2 \leq b\}.$$ 

Assume that $\varepsilon > 0$ is a small positive parameter, $\varepsilon = \frac{b}{N}$, $N \gg 1$, $N \in \mathbb{N}$, and $\delta = \text{const}, 0 < \delta < 1$. Moreover, let

$$\Gamma := \{(x_1, x_2) \in \partial \Omega : x_1 = 0\}.$$ 

We suppose that $\Gamma$ is represented in the form:

$$\Gamma = \Gamma_\varepsilon \cup \gamma_\varepsilon, \quad \Gamma_\varepsilon = \bigcup_{i} (\Gamma_i^\varepsilon), \quad \gamma_\varepsilon = \bigcup_{i} (\gamma_i^\varepsilon), \quad \Gamma_i^\varepsilon \cap \gamma_i^\varepsilon = \emptyset,$$
\[ \text{mes } \Gamma_i^e = \epsilon \delta, \quad \text{mes } (\Gamma_i^e \cup \gamma_i^e) = \epsilon, \]

where \( \Gamma_i^e \) and \( \gamma_i^e \) are alternating (see Figure 2).

Figure 1: The domain \( \Omega \).

Denote by
\[
\Pi_1^i := \{(x_1, x_2) \in \Omega : 0 < x_1 < a, x_2 \in \Gamma_i^e\}, \\

\Pi_2^i := \{(x_1, x_2) \in \Omega : 0 < x_1 < a, x_2 \in \gamma_i^e\}, \\

\Pi_1 = \bigcup_i \Pi_1^i, \quad \Pi_2 = \bigcup_i \Pi_2^i.
\]

Fix a parameter \( \theta > 0 \). Define the set \( \Omega^\theta := \{x = (x_1, x_2) \in \Omega : x_1 > \theta\} \). The sets \( \Pi_1^1, \Pi_2^1, \Pi_1^0 \) and \( \Pi_2^0 \) are defined analogously. Moreover, we use the notation

\[ B(x, r) := \{(y_1, y_2) \in \mathbb{R}^2 : (y_1 - x_1)^2 + (y_2 - x_2)^2 \leq r^2\}, \]

and the average value of the function \( u \) over \( B(\cdot, r) \in \mathbb{R}^2 \) is defined as

\[ u_B := \frac{1}{\pi r^2} \int_B u(x) \, dx. \]

Let \( u \) be a locally integrable function on \( \mathbb{R}^2 \). The maximal functions \( M(u) \) and \( M_R(u) \) of \( u \) are defined by

\[ M(u)(x) := \sup_{r > 0} u_B, \quad M_R(u)(x) := \sup_{0 < r \leq R} u_B. \]

Let us define the Sobolev space

\[ H^1(\Omega, \Gamma_e) = \{u_e \in H^1(\Omega) : u_e|_{\Gamma_e} = 0\}. \]

Analogously,

\[ C^\infty(\Omega, \Gamma_e) = \{u_e \in C^\infty(\Omega) : u_e = 0 \text{ in a neighborhood of } \Gamma_e = 0\}. \]
Let \( x \in \Omega \), \( \rho(x) = \text{dist}(x, \Gamma) \). Define the following functions:

\[
\begin{align*}
    r_1(x) &= \rho(x), \\
r_2(x) &= \text{dist}(x, \Gamma_\varepsilon) := \inf_{y \in \Gamma_\varepsilon} \text{dist}(x, y).
\end{align*}
\]

According to the geometrical construction of the domain, \( r_1(x) < r_2(x) < \rho(x) + (1 - \delta) \varepsilon \) (see Figure 2).

![Figure 2: Periodical structure of the domain.](image)

We need to derive the following auxiliary Lemma of independent interest:

**Lemma 2.1.** Let \( u_\varepsilon \in H^1(\Omega, \Gamma_\varepsilon) \). Then the Friedrichs type inequality

\[
\int_{\Pi_2} u_\varepsilon^2 \, dx \leq K(a, \varepsilon, \delta) \int_{\Omega} |\nabla u_\varepsilon|^2 \, dx,
\]

holds with \( K(a, \varepsilon, \delta) = 2 \left( a^2 \frac{1-\delta}{\delta} + \varepsilon^2 (1-\delta)^2 \right) \).

**Proof.** Fix the point \((x_1, x_2)\) in \(\Pi_1\). By using the Newton-Leibnitz formula, we have

\[
u_{\varepsilon}(x_1, x_2) = u_{\varepsilon}(x_1, x_2) - u(0, x_2) = \int_0^{x_1} \frac{\partial u_{\varepsilon}}{\partial x_1} \, dx_1 \leq \int_0^a \frac{\partial u_{\varepsilon}}{\partial x_1} \, dx_1.
\]

Hence,

\[
u_{\varepsilon}^2(x_1, x_2) \leq \left( \int_0^a \frac{\partial u_{\varepsilon}}{\partial x_1} \, dx_1 \right)^2 \leq a^2 \int_0^a |\nabla u_{\varepsilon}|^2 \, dx_1.
\]

Then, by integrating the last inequality with respect to \(x_2\) and after that with respect to \(x_1\) over \(\Pi_1\), we obtain that

\[
\int_{\Pi_1} u_{\varepsilon}^2 \, dx \leq a^2 \int_{\Pi_1} |\nabla u_{\varepsilon}|^2 \, dx.
\]

(2.2)
Now choose the point \((x_1, \tilde{x}_2) \in \Pi_2^i\) such that \(\tilde{x}_2 = \frac{1-\delta}{\delta} x_2 + \epsilon \delta\). This means that \((x_1, \tilde{x}_2) \in \Pi_2^i\) if and only if \((x_1, x_2) \in \Pi_1^i\). By again using the Newton-Leibnitz formula, we find that
\[
 u_\epsilon(x_1, \tilde{x}_2) = u(x_1, x_2) + \int_{x_2}^{\tilde{x}_2} \frac{\partial u_\epsilon}{\partial x_2} d\tilde{x}_2.
\]
Consequently,
\[
 u_\epsilon^2(x_1, \tilde{x}_2) \leq 2u^2(x_1, x_2) + 2(\tilde{x}_2 - x_2) \int_{x_2}^{\tilde{x}_2} |\nabla u_\epsilon|^2 d\tilde{x}_2.
\]
Integrating the last inequality over \(\Pi_2^i\) and substituting the integral on the right-hand side by the greater integral, we get that
\[
 \int_{\Pi_2^i} u_\epsilon^2 dx \leq 2 \left( a^2 \frac{1-\delta}{\delta} + \epsilon^2 (1-\delta)^2 \right) \int_{\Pi_1^i \cup \Pi_2^i} |\nabla u_\epsilon|^2 dx.
\]
Finally, by applying the estimate (2.2) to the first integral on the right-hand side and substituting both integrals by the greater integral, we obtain that
\[
 \int_{\Pi_2^i} u_\epsilon^2 dx \leq 2 \left( a^2 \frac{1-\delta}{\delta} + \epsilon^2 (1-\delta)^2 \right) \int_{\Omega} |\nabla u_\epsilon|^2 dx. \tag{2.3}
\]
By summarizing up the inequalities (2.2) and (2.3) with respect to \(i\), we obtain the desired estimate:
\[
 \int_{\Pi_2^i} u_\epsilon^2 dx = \int_{\bigcup \Pi_2^i} u_\epsilon^2 dx \leq 2 \left( a^2 \frac{1-\delta}{\delta} + \epsilon^2 (1-\delta)^2 \right) \int_{\bigcup \Pi_1^i \cup \Pi_2^i} |\nabla u_\epsilon|^2 dx
\]
\[
 = 2 \left( a^2 \frac{1-\delta}{\delta} + \epsilon^2 (1-\delta)^2 \right) \int_{\Omega} |\nabla u_\epsilon|^2 dx. \quad \Box
\]

We also need the following well-known Lemmas:

**Lemma 2.2.** Let \(u \in W_1^1(B)\). Then
\[
 |u(x) - u_B| \leq 2 \int_B \frac{|\nabla u(y)|}{|x-y|} dy.
\]
For the proof see in [3, Lemma 7.16].

The following important inequality was derived in [4]:
Lemma 2.3. Let $B = B(x, r) \subset \Omega$, $u \in C^\infty(\Omega)$, $\Gamma \subset \partial \Omega$. Then
\[
\inf_{y \in \Gamma \cap B} \int_B \frac{|\nabla u(z)|}{|y - z|} \, dz \leq \int_B \frac{|\nabla u(z)|}{|x - z|} \, dz.
\]

Lemma 2.4. If $0 < \alpha < 2$ and $r > 0$, then there exists a constant $c_2$, $c_2 \leq \left( \frac{1}{2} \right)^{\alpha - 2}$, such that for each $x \in \mathbb{R}^2$,
\[
\int_{B(x,r)} \frac{|u(y)|}{|x - y|^{2-\alpha}} \, dy \leq c_2 r^\alpha M(u)(x).
\]

For the proof we refer to [15, Lemma 2.8.3]. We will use this result for the case $\alpha = 1$. Here, as usual, $M(u)$ stands for the Hardy-Littlewood maximal operator. Moreover, the following important theorem (the Hardy-Littlewood theorem on Maximal Operator) will be used:

Theorem 2.1. If $u \in L_2(\mathbb{R}^2)$, then there exist a constant $c_3 > 0$ such that
\[
\|M(u)\|_2 \leq c_3 \|u\|_2.
\]

For the proof see e.g. in [15, Theorem 2.8.2].

3. The main results

Consider the function
\[
\rho_{\varepsilon}(x) = \begin{cases} 
\rho(x), & \text{if } x \in \Pi_1, \\
\rho(x) + (1 - \delta) \frac{\varepsilon}{2}, & \text{if } x \in \Pi_2.
\end{cases}
\]  
(3.1)

Our first main result is the following pointwise inequality:

Theorem 3.1. Let $u_\varepsilon \in C^\infty(\Omega, \Gamma_\varepsilon)$. Then there exist a constant $C$, $C \leq 4$, such that the pointwise inequality
\[
|u_\varepsilon(x)| \leq C \rho_\varepsilon(x) M_{\rho_\varepsilon(x)} |\nabla u_\varepsilon| \chi_{B(x, \rho_\varepsilon(x))}(x)
\]  
(3.2)

holds, where $\overline{\mathbf{x}} \in \Gamma$ is satisfying that $|x - \overline{\mathbf{x}}| = \rho(x)$.

Proof. Choose the point $x \in \Omega$ and denote by $B := B(\overline{\mathbf{x}}, \rho_\varepsilon(x))$, where $\rho_\varepsilon(x)$ is defined in (3.1).

Then $B \cap \Gamma_\varepsilon \neq \emptyset$ for each $x \in \Omega$. Extend the function $u_\varepsilon$ in $\mathbb{R}^2 \setminus \overline{\Omega}$ by reflecting it across the boundary. By applying Lemma 2.2 to the extended $u_\varepsilon$, we have for any $y \in B \cap \Gamma_\varepsilon$:
\[
|u_\varepsilon(x)| = |u_\varepsilon(x) - u_\varepsilon(y)| \leq |u_\varepsilon(x) - u_\varepsilon B| + |u_\varepsilon(y) - u_\varepsilon B|
\]  
\[
\leq 2 \left( \int_B |\nabla u_\varepsilon(z)| \, dz + \int_B \frac{|\nabla u_\varepsilon(z)|}{|x - z|} \, dz \right).
\]  
(3.3)
Hence, by applying Lemma 2.3 to (3.3), we obtain that
\[ |u(x)| \leq 2 \int_B \frac{|\nabla u(z)|}{|x-z|} \, dz. \] (3.4)

Finally, taking into account Lemma 2.4 and (3.1), we have that
\[ |u(x)| \leq 2c_2 \rho(x) M_{\rho(x)} (|\nabla u|)(x) \]
\[ = \begin{cases} 4 \rho(x) M_{\rho(x)} (|\nabla u|), & x \in \Pi_1 \cap B(x, \rho(x)), \\ 4 \left( \rho(x) + \frac{(1-\delta)}{2} \right) M_{\rho(x)} + \frac{(1-\delta)}{2} \rho(x) (|\nabla u|), & x \in \Pi_2 \cap B(x, \rho(x) + \frac{(1-\delta)}{2}). \end{cases} \] (3.5)

The proof is complete. \( \square \)

The next two theorems generalize some well-known classical Hardy-type for \( p = 2 \) inequalities to a much more wide class of functions.

**Theorem 3.2.** Let \( \rho(x) \) be the function defined in (3.1) and \( 0 \leq \alpha < \alpha_0 \). Then the estimate
\[ \int_{\Omega} \rho^{-2+\alpha}(x) u^2 \, dx \leq C_1 \int_{\Omega} \rho^{\alpha}(x) |\nabla u| \, dx \] (3.6)
holds for each fixed \( \varepsilon \) for all functions \( u \in H^1(\Omega, \Gamma) \), where the constant \( C_1 \) does not depend on \( u \) and on \( \varepsilon \).

**Proof.** Fix \( u \in C^\infty(\Omega, \Gamma) \) extended in \( \mathbb{R}^n \setminus \Omega \). According to (3.2) the inequality
\[ \frac{|u(x)|}{\rho(x)} \leq 4M_{\rho(x)} (|\nabla u|)(x) \]
holds for all \( x \in \Omega \). Then we have that
\[ \int_{\Omega} \frac{|u(x)|^2}{\rho^2(x)} \, dx \leq 16 \int_{\Omega} M^2_{\rho(x)} (|\nabla u|)(x) \, dx. \]
The statement in Theorem 3.1 implies that
\[ \int_{\Omega} M^2_{\rho(x)} (|\nabla u|)(x) \, dx \leq c_3 \int_{\Omega} |\nabla u(x)|^2 \, dx. \]
Thus, it yields that
\[ \int_{\Omega} \left( \frac{|u(x)|}{\rho(x)} \right)^2 \, dx \leq C_1 \int_{\Omega} |\nabla u(x)|^2 \, dx, \]
where \( C_1 = 16c_3 \).
Hence, the inequality (3.6) holds with $\alpha = 0$. The next step is to prove (3.6) for $\alpha > 0$. Choose $\sigma > 0$ and put $\nu_\epsilon = |u_\epsilon|\rho_\epsilon^\sigma$. It is not difficult to derive that

$$|\nabla u_\epsilon|^2 = \left( \frac{\partial u_\epsilon}{\partial x_1} \rho_\epsilon^\sigma + \sigma \rho_\epsilon^{\sigma-1} \frac{\partial \rho_\epsilon}{\partial x_1} u_\epsilon \right)^2 + \rho_\epsilon^{2\sigma} \left( \frac{\partial u_\epsilon}{\partial x_2} \right)^2 \leq 2\rho_\epsilon^{2\sigma} |\nabla u_\epsilon|^2 + 2\sigma^2 \rho_\epsilon^{2\sigma-2} u_\epsilon^2.$$  \hfill (3.7)

By now applying (3.6) with $\alpha = 0$ to $v_\epsilon$, we obtain that

$$\int_\Omega \rho_\epsilon^{-2+2\sigma} u_\epsilon^2 dx \leq C_1 \left( \int_\Omega \rho_\epsilon^{2\sigma} |\nabla u_\epsilon|^2 dx + \sigma^2 \int_\Omega \rho_\epsilon^{2(\sigma-1)} |u_\epsilon|^2 dx \right).$$

If $1 - C_1 \sigma^2 > 0$, then we have that

$$\int_\Omega \rho_\epsilon^{-2+2\sigma} u_\epsilon^2 dx \leq \frac{2C_1}{1-2C_1 \sigma^2} \int_\Omega \rho_\epsilon^{2\sigma} |\nabla u_\epsilon|^2 dx.$$

Substituting $\sigma$ by $\frac{A}{2}$ and denoting $C_1$ by $\frac{2C_1}{1-2C_1 \sigma^2}$, we prove inequality (3.6). Finally, by approximating the functions from $H^1(\Omega, \Gamma_\epsilon)$ by smooth functions belonging to $C^\infty(\Omega, \Gamma_\epsilon)$, we can complete the proof. \hfill $\Box$

Our final main result reads:

**THEOREM 3.3.** Let $\rho(x) = \text{dist}(x, \Gamma)$,

$$K := K(a, \epsilon, \delta, \theta) = 4 + \frac{2}{\theta^2} \left( a^2 \frac{1-\delta}{\delta} + \epsilon^2 (1-\delta)^2 \right)$$

and $0 \leq \alpha < \alpha_0 := \sqrt{\frac{2}{K}}$. Then the estimate

$$\int_{\Omega^\theta} \rho^{-2+\alpha}(x) u_\epsilon^2(x) dx \leq C(a, \epsilon, \delta, \theta, \alpha) \int_\Omega \rho^\alpha(x) |\nabla u_\epsilon(x)|^2 dx \hfill (3.8)$$

holds for each fixed $\theta > 0$ for all functions $u_\epsilon \in H^1(\Omega, \Gamma_\epsilon)$, where the constant

$$C(a, \epsilon, \delta, \theta, \alpha) = \frac{4K}{2 - K\alpha^2}.$$

**Proof.** First we note that

$$\int_{\Pi_1} \left( \frac{u_\epsilon}{\rho} \right)^2 dx \leq 4 \int_{\Pi_1} |\nabla u_\epsilon|^2 dx \leq 4 \int_\Omega |\nabla u_\epsilon|^2 dx. \hfill (3.9)$$

Indeed, if $u_\epsilon(0) = 0$, then, according to the classical one-dimensional inequality (1.1), we have that

$$\int_0^a u_\epsilon^2 dx_1 \geq \frac{1}{4} \int_0^a \left( \frac{u_\epsilon}{x_1} \right)^2 dx_1.$$
Using this fact, we obtain that
\[
\int_{\Pi_1} |\nabla u_\varepsilon|^2 \, dx \geq \int_{\Gamma_\varepsilon} \int_0^a \left( \frac{\partial u_\varepsilon}{\partial x_1} \right)^2 \, dx \geq \frac{1}{4} \int_{\Gamma_\varepsilon} \int_0^a \left( \frac{u_\varepsilon}{x_1} \right)^2 \, dx
\]
\[= \frac{1}{4} \int_{\Pi_1} \left( \frac{u_\varepsilon}{\rho} \right)^2 \, dx.
\]
In particular, the estimate (3.9) holds in \( \Pi_1^\theta \). The next step is to prove that
\[
\int_{\Pi_2^\theta} \left( \frac{u_\varepsilon}{\rho} \right)^2 \, dx \leq \frac{K(a, \varepsilon, \delta)}{\theta^2} \int_\Omega |\nabla u_\varepsilon|^2 \, dx.
\] (3.10)

Using the Friedrichs inequality (2.1) and taking into account the fact that \( \rho(x) > \theta \) when \( x \in \Pi_2^\theta \), we obtain the following estimate:
\[
\int_{\Pi_2^\theta} \left( \frac{u_\varepsilon}{\rho} \right)^2 \, dx \leq \frac{1}{\theta^2} \int_{\Pi_2} u_\varepsilon^2 \, dx \leq \frac{K(a, \varepsilon, \delta)}{\theta^2} \int_\Omega |\nabla u_\varepsilon|^2 \, dx.
\]

Now, summarizing the inequalities (3.9) and (3.10), we derive the desired estimate:
\[
\int_{\Omega^\theta} \left( \frac{u_\varepsilon}{\rho} \right)^2 \, dx = \int_{\Pi_1^\theta} \left( \frac{u_\varepsilon}{\rho} \right)^2 \, dx + \int_{\Pi_2^\theta} \left( \frac{u_\varepsilon}{\rho} \right)^2 \, dx
\]
\[\leq \left( 4 + \frac{K(a, \varepsilon, \delta)}{\theta^2} \right) \int_\Omega |\nabla u_\varepsilon|^2 \, dx.
\]

We have derived the inequality (3.8) for the case \( \alpha = 0 \). The proof of (3.8) for the case \( \alpha > 0 \) is identically to the proof of the second part of Theorem 3.2, so we omit the details. \( \square \)

4. Concluding remarks

**Remark 4.1.** The condition \( \delta = 1 \) in the definition (3.1) corresponds to the case \( \Gamma_\varepsilon = \Gamma \).

**Remark 4.2.** If \( \alpha = 0 \), then (3.6) takes the form
\[
\int_\Omega \left( \frac{u_\varepsilon(x)}{\rho_\varepsilon(x)} \right)^2 \, dx \leq C_1 \int_\Omega |\nabla u_\varepsilon(x)|^2 \, dx,
\] (4.1)
while (3.8) becomes
\[
\int_{\Omega^\theta} \left( \frac{u_\varepsilon(x)}{\rho(x)} \right)^2 \, dx \leq C(a, \varepsilon, \delta, \theta, \alpha) \int_\Omega |\nabla u_\varepsilon(x)|^2 \, dx.
\]
We conjecture that these Hardy-type inequalities hold also when $p = 2$ is replaced by any $p > 1$ but then another type of proof must be found.

**Remark 4.3.** In this paper we have succeeded to prove a weighted Hardy-type inequality in a fixed domain for functions vanishing on a part of the boundary. We can see several open questions equipped with this result. For instance, one interesting problem is to try to find a weighted Hardy-type inequality for perforated domains in the case when the size of perforation depends on the small parameter.

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**References**


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