TORSION-BENDING-SHEAR INTERACTION FOR CONCRETE BEAMS

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INTRODUCTION

During the last decade much research has been carried out regarding various aspects of the influence of torsion on concrete beams (1,5,24,25). Many approaches have been proposed but so far no rational, simple method has been presented for the analysis of the total interaction between torsion, bending, and shear. The object and scope of this paper is to examine such a method (4).

The pioneering work on the interaction of torsion with bending and shear was carried out in Moscow by Lessig, Chinenkov, Lyalin, and Gvozdev. They observed failure mechanisms which were characterized by skew bending, and studied equilibrium conditions and expressions for external and internal energy for the observed mechanisms. Their first results were published in 1958 (17). The method was reviewed in the United States in 1960 (2) and later was expanded and modified by its originators (9), as well as by Collins, et al. (3), McMullen and Warvaruk (19), and Goode and Helmy (8).

Another approach for the study of the interaction of torsion with bending and shear is based on the truss analogy. This method was for pure torsion presented by Rausch in 1929 (22). In 1969, the approach was expanded to combined torsion and bending by Lampert, Lüchinger, and Thürlimann (13,15,16,18), and in 1971 to combined torsion, bending, and shear by Elfgren (4). The truss method is also feasible for a study of the stiffness properties in the cracked stage and has been used in this context by Karlsson (10,11,12) and by Lampert (14). Interaction formulas based on experimental work for torsion-bending and for torsion-bending-shear have been published, e.g., by Rajagopalan, Behera, and Ferguson (21).

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As an introduction and in order to demonstrate some fundamental properties of the interaction between torsion and bending, a simplified, basic failure mechanism will be presented first. The studied mechanism will then be refined and expanded to include the influence of vertical shear. This leads to a rational, straightforward torsion-bending-shear interaction in closed form. The proposed method will be examined and compared to some test results.

**Basic Simplified Failure Mechanism**

Consider a rectangular beam cross section as given in Fig. 1. The height and the width of the beam are $h$ and $b$, respectively. The cross-sectional area

![Rectangular Cross Section to Be Studied](image)

**FIG. 1.—Rectangular Cross Section to Be Studied**

of each of the two **longitudinal top** reinforcement bars is $A_t$, and the cross-sectional area of each of the two **longitudinal bottom** reinforcement bars is $A_b$. The top longitudinal reinforcement bars are weaker than, or of the same strength as, the bottom longitudinal bars. The transverse reinforcement consists of closed web bars (stirrups) with the cross-sectional area, $A_w$, and the spacing, $s$. The height and the width of the stirrups are $h'$ and $b'$, respectively.

Let the beam be loaded with a torsional moment, $T$, and a bending moment,
FIG. 3.—Basic, Simplified Failure Mechanism for Beam Loaded in Combined Torsion and Bending; at Failure Beam Rotates Around Hinge Along Compression Zone in Top of Beam

FIG. 4.—Torsion-Bending Interaction According to Basic, Simplified Failure Mechanism
M. When the loads are increased inclined cracks will appear as in Fig. 2. At failure the reinforcement is assumed to be yielding in three sides, whereas a compression zone is formed in the fourth side. A skew bending failure surface can then be studied. The surface is on three sides defined by an inclined spiraling crack. On the fourth side, the top of the beam, the ends of the crack are joined by a compression zone as shown in Fig. 2. A detailed picture of the assumed failure surface is given in Fig. 3.

To facilitate the deductions, some simplifying assumptions will now be made. The inclination of the cracks for real beams varies around 45°. However, in this simplified basic mechanism the inclination is assumed to have exactly the value 45°. The height of the concrete compression zone is usually small and the center of the zone can for that reason be approximately located to the level of the stirrups in the top of the beam. It is further assumed that the height, \( h'' \), which is the distance from the stirrups in the bottom of the beam to the neutral axis, can be approximated to the height, \( h' \), which is the vertical distance between the stirrup legs.

**Deduction of Interaction Equations.**—The ultimate load can now be deduced from two moment equations around axes running through the center of the compression zone:

1. The first moment axis is parallel to the beam axis and the corresponding equation expresses the equivalence of the moment of the internal forces and the applied torsional moment, \( T \). The first equation gives (see Fig. 3)

\[
T = A_w \sigma_w' \left( \frac{b'}{s} h' + \frac{2h''}{s} \frac{b'}{2} \right) \tag{1}
\]

in which \( A_w \sigma_w' \) = the force in one web bar (stirrup leg); \( A_w \) = the cross-sectional area of the bar; and \( \sigma_w' \) = the yield stress. In the parenthesis, the first term is the number of the horizontal stirrup legs multiplied with their lever arm, \( h' \). The second term in the parenthesis is the number of stirrup legs in the two vertical sides multiplied with their lever arm, \( b'/2 \).

2. The second moment axis is horizontal and perpendicular to the beam axis and the corresponding equation expresses the equivalence of the moment of the internal forces and the applied bending moment, \( M \). The second equation gives (see Fig. 3)

\[
M = 2A_{ib} \sigma_{ib}' h' - A_w \sigma_w' \frac{2h''}{s} \left( \frac{b'}{2} + \frac{h'}{2} \right) \tag{2}
\]

The first term is the contribution to the moment-carrying capacity from the two bottom longitudinal bars with the area, \( A_{ib} \), and the yield stress, \( \sigma_{ib}' \). The second term is the negative contribution from the vertical stirrup legs. Their number is \( 2h''/s \) and their moment arm is \( (b' + h')/2 \).

For \( h'' = h' \), Eqs. 1 and 2 are simplified to

\[
T = 2b' h' \frac{A_w \sigma_w'}{s} \tag{3}
\]
\[ M = 2A_{lb} \sigma_{lb}^r h' - A_w \sigma_w^r \frac{h'}{s} (b' + h') \] 

(4)

Eq. 3 can be inserted into Eq. 4 to obtain a relationship between the torsional moment, \( T \), and the bending moment, \( M \), failure

\[ M = 2A_{lb} \sigma_{lb}^r h' - T \frac{h' + b'}{2b'} \] 

(5)

Eq. 5 governs the interaction between torsion and bending at failure. The
equation can be shown in an interaction diagram as in Fig. 4.

For negative or small positive bending moments another equation can be
governing. To deduce this equation, a new failure mechanism has to be studied.

For a negative bending moment, \(-M\), the concrete compression zone will
form in the bottom of the beam (see Fig. 5). With the same assumptions as
before, Eq. 3 will still be valid. However, Eq. 4 will change to

\[
-M = 2A_h \sigma_h h' - A_w \sigma_w \frac{h'}{s} (b' + h')
\]  
\[\text{(6)}\]

Insertion of Eq. 3 leads to a new relationship between the torsional moment,
\(T\), and the bending moment, \(M\)

\[
M = -2A_h \sigma_h h' + T \frac{h' + b'}{2b'}
\]  
\[\text{(7)}\]

Eq. 7 is also shown in the interaction diagram in Fig. 4. For negative and
small positive bending moments \(M\), the compression zone will form in the
bottom of the beam and Eq. 7 is governing. For high positive bending moments
\(M\), the compression zone will form in the top of the beam and Eq. 5 will
be governing.

For the assumed failure surfaces, the maximum torsional moment, \(T\), is also
governed directly by the yielding capacity of the stirrups. This restriction is
expressed by Eq. 3 and a line representing Eq. 3 has been drawn in Fig. 4.
The studied beam cross section can carry load combinations falling inside the
lines defined by Eqs. 3, 5, and 7.

Two interesting facts may be noticed in Fig. 4. The bending moment capacity
is reduced when a torsional moment, \(T\), is present. On the other hand, a small
positive bending moment, \(M\), may increase the torsional capacity.

The interaction described by the basic failure mechanism will now be studied
for a refined and expanded failure mechanism, where a shear force also can
be taken into consideration.

**Refined Failure Mechanism**

For a beam loaded with a pure torsional moment, \(T\), diagonal tensile stresses
will spiral around the beam faces as shown in Fig. 6(a). For a beam loaded
with a vertical shear force, \(V\), diagonal tensile stresses will also appear but
they will be parallel on the two vertical faces of the beam according to Fig.
6(b). Consequently, for a beam loaded in combined torsion and shear the diagonal
tensile stresses will be additive on one of the vertical faces and subtractive
on the other [see Fig. 6(c)]. On the horizontal faces the diagonal tensile stresses
will be due to torsion only.

A beam loaded with a torsional moment, \(T\), a bending moment, \(M\), and
a vertical shear force, \(V\), will now be studied. The beam is reinforced in the
same way as the beam in Fig. 1 with four longitudinal bars and closed stirrups.
The four sides of the beam are denoted \(r\) (right-hand side), \(b\) (bottom), \(l\) (left-hand
side), and \(t\) (top).

At failure an inclined failure surface, as shown in Fig. 7, may develop. Due
to the different shear stresses and the different diagonal tensile stresses in
FIG. 7.—Failure Mechanism for Combined Torsion, Bending, and Shear According to Mode \( t \) (Compression Zone in Top of Beam)

FIG. 8.—Forces Along Horizontal Cut in Beam Side
the different faces of the beam, the inclination of the cracks and of the concrete compressive struts between the cracks will vary from face to face. The inclination will also change as the loading is increased. Sometimes the inclination of the cracks will not change as much as the inclination of the concrete compressive struts. This is due to difficulties for new cracks to develop when old cracks already have formed. In the deductions, the inclination of the compressive struts will be studied, and not the inclination of the cracks. The inclination of the compressive struts to the horizontal beam axis is denoted \( \alpha_{a}, \alpha_{b}, \) and \( \alpha_{t} \) for the right-hand face, bottom face, and left-hand face, respectively.

Assumptions.—To simplify the deductions, the following assumptions are made:

1. The amount of longitudinal reinforcement bars and of stirrups is such that both reinforcement categories reach their yield stresses at failure. (The beams can then be said to be underreinforced for the loading case considered.)
2. No secondary failures will appear. This means, e.g., that the reinforcement is properly anchored and can support the concrete compressive struts so that no corner in the cross section is caused to spall.
3. The tensile strength of the concrete is neglected.
4. Shear forces carried by dowel action of the reinforcement are neglected as well as shear forces carried by the concrete compression zone.
5. The height of the concrete compression zone is usually small and the center of the zone can be located approximately to the level of the stirrups in the top of the beam. It is further assumed that the height, \( h'' \), which is the distance from the stirrups in the bottom of the beam to the neutral axis can be approximated with the height, \( h' \), which is the vertical distance between the stirrup legs.

Deduction of Equilibrium Equations.—The ultimate load-carrying capacity can now be deduced from equilibrium equations. As in the study of the basic failure mechanism, different positions of the concrete compression zone is possible. First a mechanism with the concrete compression zone in the top of the beam as in Fig. 7 will be studied. This failure mode is called mode 1 as the concrete compression zone is formed in the top of the beam. Equilibrium equations for moments around two axes running through the center, \( C \), of the compression zone will be studied as well as a vertical projection equation (see Fig. 7):

1. The first moment axis is parallel to the beam axis and the corresponding equation expresses the equivalence of the moments of the internal forces and the applied torsional moment, \( T \). The first equation gives (see Fig. 7 and compare with Eq. 1):

\[
T = A_w \sigma_y \left[ \frac{b' \cot \alpha_{b}}{s} h' + \frac{h''}{s} (\cot \alpha_{t} + \cot \alpha_{e}) \frac{b'}{2} \right] \quad \ldots \ldots \ldots \ldots (8)
\]

In the parenthesis the first term is the number of the horizontal stirrup legs multiplied with their lever arm, \( h' \). The second term is the number of stirrup legs in the two vertical sides multiplied with their lever arm, \( b'/2 \).

2. The second moment axis is horizontal and perpendicular to the beam axis and the corresponding equation expresses the equivalence of the moment of
the internal forces and the applied bending moment at point C. As a vertical shear force, \( V \), also is present the bending moment is varying. It has the value, \( M \), at the reference point, R, where the longitudinal bottom reinforcement bars are intersecting the inclined curved surface. The distance in the longitudinal direction of the beam between the points, C and R, is \( a_1 \) and consequently the applied bending moment at point C will be \( M + V a_1 \). The second equation gives (see Fig. 7 and compare with Eq. 2):

\[
M + V a_1 = 2A_{1b} \sigma_{1b} \frac{h'' \cot \alpha_i}{s} \left( \frac{b'}{2} \cot \alpha_b + \frac{h''}{2} \cot \alpha_r \right) - A_w \sigma_w \frac{h'' \cot \alpha_r}{s} \left( \frac{b'}{2} \cot \alpha_b + \frac{h''}{2} \cot \alpha_i \right) 
\]

The second and the third terms after the equal sign are the negative contributions from the vertical stirrup legs on the left and right-hand side of the beam, respectively.

3. The vertical projection equation expresses the equivalence of the internal forces and the applied vertical shear force, \( V \). It is assumed that the concrete compression zone does not carry any part of the vertical shear force. Usually the concrete compression zone and aggregate interlock are supposed to carry a certain part of the vertical shear force. For simplicity their contributions are neglected here. The third equation gives (see Fig. 7)

\[
V = A_w \sigma_w \frac{h''}{s} (\cot \alpha_i - \cot \alpha_r) 
\]

Inclination of Concrete Compressive Struts.—The value of the inclination, \( \alpha \), of the concrete compression struts will now be examined. To do this, the conditions for equilibrium are studied along a horizontal cut in one side of the beam (see Fig. 8). The conditions are studied along a distance, \( s \), equal to the stirrup spacing. Let \( F_c \) = the compressive force in the concrete strut over the length, \( s \); let \( F_w \) = the force in one stirrup; and let \( \tau \) = the shear stress along the horizontal cut. After cracking, at the stages close to failure, the torsional moment and the vertical shear force are mostly carried by the outer portions of the beam cross section. Accordingly the inner portions can be neglected and the study can be restricted to the outer portion of the beam, e.g., a wall with the thickness, \( t \).

One horizontal and one vertical equilibrium equation can now be written for the studied horizontal cut: \( \tau ts = F_c \cos \alpha \) and \( F_w = F_c \sin \alpha \). The two equations give

\[
\cot \alpha = \frac{\tau ts}{F_w} 
\]

At yielding of the reinforcement, the force, \( F_w \), will be the same in all beam sides. The wall thickness, \( t \), and the stirrup spacing, \( s \), will also be constants. Consequently, \( \cot \alpha \) according to Eq. 11 will be directly proportional to the shear stresses, \( \tau \). The shear stresses are in their turn proportional to the torsional moment, \( T \), and to the vertical shear force, \( V \). Now let \( \alpha_T = \) the angle of inclination for a beam loaded with a pure torsional moment, \( T \), and let \( \alpha_V \)
= the angle of inclination, for a beam loaded with a vertical shear force, \( V \) (and no torsion). For the case with combined loading with torsion and shear, three equations can now be formulated for the inclinations on the different sides of the beam (see Figs. 6 and 7).

In side \( l \) the torsional moment, \( T \), and the vertical shear force, \( V \), act in the same direction. In side \( b \) only the torsional moment, \( T \), gives rise to shear stresses and in side \( r \), finally, the torsional moment, \( T \), and the vertical shear force, \( V \), act in opposite directions. This gives

\[
\cot \alpha_l = \cot \alpha_T + \cot \alpha_V \\
\cot \alpha_b = \cot \alpha_T \\
\cot \alpha_r = \cot \alpha_T + \cot \alpha_V
\]

The horizontal distance, \( a_1 \), along the beam axis between the midpoints, \( C \) and \( R \), on the top and bottom sides of the inclined curved surface (see Fig. 7) can now be expressed in a simple way

\[
a_1 = \frac{1}{2} (h'' \cot \alpha_r + b' \cot \alpha_b + h'' \cot \alpha_l) - \left( h'' \cot \alpha_r + \frac{b'}{2} \cot \alpha_b \right) = h'' \cot \alpha_V
\]

**Interaction Equation.**—For \( h'' = h' \), the equilibrium equations, Eqs. 8, 9, and 10, can be rewritten using the information of Eqs. 12 and 13

\[
T = 2b'h' \frac{A_w \sigma_w^y}{s} \cot \alpha_T 
\]

\[
M = 2A_{lb} \sigma_{lb}^y h' - \frac{A_w \sigma_w^y}{2s} \left[ h' (\cot \alpha_T + \cot \alpha_V)(b' \cot \alpha_T + h' \cot \alpha_T - h' \cot \alpha_V) + h' \cot \alpha_T \right] - Vh' \cot \alpha_V
\]

\[
V = 2h' \frac{A_w \sigma_w^y}{s} \cot \alpha_V
\]

Eq. 15 can further be rewritten as

\[
2A_{lb} \sigma_{lb}^y h' = M + A_w \sigma_w^y \frac{h'}{s} \left[ (b' + h') \cot^2 \alpha_T - h' \cot^2 \alpha_V \right] + Vh' \cot \alpha_V
\]

or, with Eq. 16, as

\[
1 = \frac{M}{2A_{lb} \sigma_{lb}^y h'} + \frac{A_w \sigma_w^y}{2A_{lb} \sigma_{lb}^y s} \left[ (b' + h') \cot^2 \alpha_T + h' \cot^2 \alpha_V \right]
\]

Finally, the Eqs. 14 and 16 can be inserted to give a relationship between
the torsional moment, $T$, the bending moment, $M$, and the vertical shear force, $V$, at failure

$$1 = \frac{M}{2A_{lb} \sigma_{lb} ^ y h'} + \frac{T^2}{(2b' + h') \frac{b'}{2A_{ib} \sigma_{ib} ^ y A_w \sigma_{w} ^ y}} + \frac{V^2}{(2h')^2 \frac{h'}{2A_{lb} \sigma_{lb} ^ y A_w \sigma_{w} ^ y}}$$  \tag{18}

From Eq. 18 the load-carrying capacity in mode $t$ can be evaluated for pure bending, $M'_o$, for pure torsion $T'_o$ and for pure shear $V'_o$:

$$M'_o = 2A_{lb} \sigma_{lb} ^ y h'$$  \tag{19a}

$$T'_o = 2b' \frac{h'}{A_w \sigma_{w} ^ y} \sqrt{\frac{2A_{lb} \sigma_{lb} ^ y}{b' + h'}} \frac{s}{A_w \sigma_{w} ^ y}$$  \tag{19b}

$$V'_o = 2h' \frac{A_w \sigma_{w} ^ y}{s} \sqrt{\frac{2A_{lb} \sigma_{lb} ^ y}{h'}} \frac{s}{A_w \sigma_{w} ^ y}$$  \tag{19c}

**Comparison with Basic Simplified Failure Mechanism.**—The expression for $M'_o$ in Eq. 19a can be obtained from the simplified basic mechanism earlier studied (compare with Eq. 5). The expression for $T'_o$ in Eq. 19b differs from Eq. 3 only by the factor under the square root sign. This factor corrects the ultimate torsional capacity for different ratios of longitudinal to transverse reinforcement. From a comparison with Eqs. 3 and 14 it can be seen that the correction factor is an expression for the cot $\alpha_T$: $\cot \alpha_T = \sqrt{2A_{lb} \sigma_{lb} ^ y / (b' + h')} [s / (A_w \sigma_{w} ^ y)]$.

The expression for $V'_o$ is basically the well-known equation from the original truss analogy according to Ritter (23) and Mörsch (20)

$$V'_o = 2h' \frac{A_w \sigma_{w} ^ y}{s}$$  \tag{20}

The equation is only supplemented by an expression that corrects for variations in the inclination, $\alpha_v$, of the concrete compressive struts. From a comparison with Eq. 16 it can be seen that $\cot \alpha_v = \sqrt{(2A_{lb} \sigma_{lb} ^ y / h')} [s / (A_w \sigma_{w} ^ y)]$.

The expression given in Eq. 20 is correct for a failure mechanism with a crack inclination of $\alpha_v = 45^\circ$. The expression can be obtained directly from Eq. 10 for $\alpha = -\alpha_r = 45^\circ$.

The expression for $V'_o$ in Eq. 20 is analogous to the expression for $T$ in Eq. 3 in the basic simplified failure mechanism. In fact, the simplified failure mechanism can also form the basis for a shear-bending interaction equation. For $\alpha_v = 45^\circ$ and $\alpha_T = 0^\circ$, Eqs. 17a and 16 give

$$M = 2A_{lb} \sigma_{lb} ^ y h' - \frac{V h'}{2}$$  \tag{21}

Eq. 21 is analogous to Eq. 5. Both equations are valid when the compression zone forms in the top of the beam. For a negative bending moment, $-M$, the compression zone will form in the bottom of the beam. The interaction equation for bending and shear will then take the following form (compare with Eq. 7):
\[ M = -2A_h \sigma_h^y h' + \frac{V h'}{2} \quad \ldots \ldots \ldots \ldots \quad (22) \]

Eqs. 20, 21, and 22 will form a simplified bending-shear interaction diagram in the same way as Eqs. 3, 5, and 7 form a simplified bending-torsion interaction diagram (see Fig. 4).

**Graphic Representation of Interaction Equations.**—Using Eq. 19, Eq. 18 can be rewritten in the following simple dimensionless form

\[ \frac{M}{M_o^t} + \left( \frac{T}{T_o^t} \right)^2 + \left( \frac{V}{V_o^t} \right)^2 = 1 \quad \ldots \ldots \ldots \ldots \quad (23) \]

An example of the torsion-bending interaction according to Eqs. 18 and 23 is shown in Fig. 9(a). The interaction curve is a second-degree parabola. For comparison, the interaction Eqs. 3 and 5 for the earlier treated basic failure mechanism are also shown. It can be seen that they give a lower load-carrying capacity. This is due to the fact that the inclination, \( \alpha \), was given the value 45\(^\circ\) in the basic mechanism. In the expanded mechanism, on the other hand, the angle is free to adjust to the value which is best suited to carry the applied load.

In Fig. 9(b) the full interaction surface for mode \( t \) according to Eqs. 18 and 23 is shown. In order to get the same dimensions on all axes, the vertical shear force, \( V \), has been multiplied with the height, \( h' \), of the beam cross section. The interaction between torsion \( T \) and vertical shear \( V \) is governed by an ellipse and the interaction between shear \( V \) and bending \( M \) is governed by a second-degree parabola.

**Other Modes of Failure.**—As in the basic failure mechanism, modes of failure other than mode \( t \) also have to be considered. For negative or small positive bending moments the compression zone will form in the bottom of the beam. Accordingly, this mode is called mode \( b \). The following interaction equation can be derived with the same method as for mode \( t \)

\[ \frac{M}{M_o^b} + \left( \frac{T}{T_o^b} \right)^2 + \left( \frac{V}{V_o^b} \right)^2 = 1 \quad \ldots \ldots \ldots \ldots \quad (24a) \]

in which

\[ M_o^b = -2A_h \sigma_h^y h' \]

\[ T_o^b = 2b'h' \frac{A_w \sigma_w^y}{s} \sqrt{\frac{2A_h \sigma_h^y}{b' + h'} \frac{s}{A_w \sigma_w^y}} \]

\[ V_o^b = 2h' \frac{A_w \sigma_w^y}{s} \sqrt{\frac{2A_h \sigma_h^y}{h'} \frac{s}{A_w \sigma_w^y}} \quad \ldots \ldots \ldots \ldots \quad (24b) \]

An example of the interaction surface for mode \( b \) is shown in Fig. 10. In the same way as for mode \( t \), the interaction between torsion and bending and between shear and bending are governed by second-degree parabolas. However, the parabolas have their vertices on the negative bending axis instead of on the positive bending axis as in mode \( t \). The interaction between torsion and vertical shear is governed by an ellipse just as the case was for mode \( t \).

For some loading cases the compression zone can also form in one of the
FIG. 9.—Interaction Surface for Mode $t$: (a) Torsion-Bending; (b) Torsion-Bending-Shear

FIG. 10.—Interaction Surface for Mode $b$ and Mode $t$: (a) Torsion-Bending; (b) Torsion-Bending-Shear

FIG. 11.—Failure Mechanism for Combined Torsion, Bending, and Shear According to Mode $s$
vertical sides of the beam. Accordingly, this failure mode is called mode $s$. It can be decisive for beams subjected to high shear stresses from both torsion and vertical shear. On one vertical side these shear stresses will be additive and yielding will start here, while on the other vertical side the shear stresses will be subtractive and here a compression zone will form. The corresponding

![Diagram showing interaction surfaces for modes $s$, $b$, and $t$.](image)

**FIG. 12.** Interaction Surface for Mode $s$, Mode $b$, and Mode $t$

![Diagram showing interaction diagrams for a beam with shear span $M/V = a$: (a) Three-Dimensional Diagram; (b) Influence of Shear in Torsion-Bending Diagram](image)

**FIG. 13.** Interaction Diagrams for Beam with Shear Span $M/V = a$: (a) Three-Dimensional Diagram; (b) Influence of Shear in Torsion-Bending Diagram

failure surface is shown in Fig. 11. The following interaction equation can be deduced (4):

\[
\left( \frac{T}{T_o^s} \right)^2 + \left( \frac{V}{V_o^s} \right)^2 + \frac{TV}{T_o^s V_o^s} \frac{2h'}{\sqrt{h'(h' + b')}} = 1 \quad \ldots \quad (25a)
\]

in which

\[
T_o^s = 2b'h' \frac{A_w \sigma^y_w}{s} \sqrt{\frac{A_{lt} \sigma^y_{lt} + A_{lb} \sigma^y_{lb}}{b' + h'} \frac{s}{A_w \sigma^y_w}} \quad \ldots \quad (25b)
\]

\[
V_o^s = 2h' \frac{A_w \sigma^y_w}{s} \sqrt{\frac{A_{lt} \sigma^y_{lt} + A_{lb} \sigma^y_{lb}}{b' + h'} \frac{s}{A_w \sigma^y_w}} \quad \ldots \quad (25b)
\]
An example of an interaction surface that corresponds to Eq. 25a is shown in Fig. 12. The interaction surface for mode \( s \) is a cylinder with an elliptic base in the torsion-shear plane. The surface is not influenced by the value of the bending moment, \( M \). In Fig. 12 it can be seen how the elliptic cylinder of mode \( s \) cuts a slice from the interaction surface defined by mode \( t \) and mode \( b \).

![Diagram showing interaction surface](image)

**FIG. 14.—Nondimensional Interaction Diagrams for Test Series 2, 4, and 5 (Ref. 5)**

For many beams loaded in bending and shear, there is a constant ratio between the bending moment, \( M \), and the vertical shear force, \( V \). If the length of the shear span is denoted \( a \), this relationship can be written

\[
M = Va
\]

which corresponds to a vertical plane in the torsion-bending-shear diagram. An example is shown in Fig. 13(a). It can be seen that the plane intersects with the general interaction surface so that a special interaction curve is formed for the studied load case. In Fig. 13(b) this interaction curve is compared to the interaction curve for the loading case torsion and bending only. The differences between the two curves are most pronounced in the top and in the right end of the diagrams. The top is cut off by mode \( s \) and in the right end the bending capacity is reduced by the influence of the shear force.

**Concentrated Loads.**—In the vicinity of concentrated loads the interaction equations of the preceding sections are not always valid. This is due to the
fact that the vertical shear force in these regions does not influence the stress in the longitudinal reinforcement as has been presumed in the deductions. Every one of the interaction equations, Eq. 17 and Eqs. 23-25, can be interpreted as expressions for the manner in which the forces, \( A, \sigma_i \), in the two most strained longitudinal reinforcement bars are composed of contributions from the bending moment, \( M \), the torsional moment, \( T \), and the vertical shear force, \( V \). However, the vertical shear force does not give such contributions close to concentrated loads. Instead, the influence of vertical shear forces are reduced in such areas and the interaction curve will run somewhere between the curves for torsion-bending-shear and the curve for torsion-bending.

**Analysis**

The main feature of the present method is that the influence of the vertical shear is incorporated in the analysis in a simple rational way. This has been done by a study of equilibrium equations for assumed failure surfaces. For the special case of loading in torsion and bending only, the results are in agreement with what has been developed previously with the skew bending method and with the truss method.

Several simplifications have been made in order to facilitate the derivations. One of them is that the lengths of the moment lever arms have been approximated to the distances, \( b' \) and \( h' \), between the vertical and the horizontal legs of the stirrups. It is obvious that the lengths of the lever arms are functions of the amount of reinforcement and of the concrete compressive strength, as the case is in beams loaded in pure bending. In a further refinement of the theory this variation should be taken into consideration (4).

Another approximation is that only the contribution from the stirrups have been considered in the calculations of the vertical shear-carrying capacity. This is a rather crude assumption and it is clear that the theory with such an assumption cannot predict accurately the ultimate load-carrying capacity in bending and shear (with no torsion). On the other hand, the assumption gives conservative estimations of the load-carrying capacity in shear and as the different contributions making up the capacity in bending and shear are still not fully evaluated it may be justified with some rough conservative assumptions in order to get an easy and comprehensible model for the torsion-bending-shear interaction in cases when torsion is predominant.

It should be pointed out again that the theory is only valid for underreinforced beams, i.e., beams where both the longitudinal reinforcement bars and the stirrups yield before failure. This, of course, sets forth restrictions on the total amount of reinforcement as well as on the ratio between the amount of longitudinal bars and the amount of stirrups. The maximum amount of reinforcement depends on the concrete compression strength and can be determined from a study of the possibility of a primary concrete crushing failure (7). The ratio between the amount of longitudinal reinforcement bars and the amount of stirrups ideally should have such a value that \( \cot \alpha_T = 1 \). However, if \( \cot \alpha_T \) has a value within the range \( 0.5 < \cot \alpha_T < 2 \), yielding usually is obtained in both categories of reinforcement.

The accuracy of the presented method has been checked against results from test series carried out at Chalmers University of Technology (4,5) and at the
University of Alberta (19). Some results are shown in Fig. 14. There are some differences between the theoretical curve and the test results. These are mainly due to the simplifying approximations made in the deduction of the theoretical curves. In general, the available test results support the present method. However, more tests are needed to better establish the interaction between torsion and shear for different bending moments.

CONCLUSIONS

Based on the results of this investigation the following conclusions seem to be justified for rectangular, underreinforced beams with closed stirrups:

1. The ultimate strength in combined torsion-bending-shear can, after some simplifying assumptions, be evaluated in a rational way from a study of the equilibrium of external and internal forces for inclined failure surfaces.

2. The concrete compression zone can form in the top, in the bottom, or in one of the vertical sides of the beam. This leads to three different modes of failure.

3. For the cases in which the compression zone forms in the top (mode \( t \)) or bottom (mode \( b \)) side of the beam, the torsion-bending and the shear-bending interaction are governed by second-degree parabolas, whereas the torsion-shear interaction is governed by an ellipse.

4. For the case when the compression zone forms in one of the vertical sides of the beam (mode \( s \)) the interaction surface is a cylinder with an elliptic base in the torsion-shear plane. The interaction is not influenced by the value of the bending moment, \( M \).

5. The interaction surfaces for the three modes together form an interaction surface which governs the load-carrying capacity for a beam.

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APPENDIX I.—REFERENCES

4. Elfgren, L., “Reinforced Concrete Beams Loaded in Combined Torsion, Bending
and Shear. A Study of the Ultimate Load-Carrying Capacity,” thesis presented to Chalmers University of Technology, at Göteborg, Sweden, in 1972, in partial fulfillment of the requirements for the degree of Doctor of Engineering (Division of Concrete Structures Publication 71:3).


APPENDIX II.—NOTATION

The following symbols are used in this paper:

\[ A = \text{cross-sectional area of reinforcement bar;} \]
\[ a = \text{shear span;} \]
\[ a_1 = \text{distance;} \]
\[ b = \text{total width of rectangular section;} \]
\[ b' = \text{horizontal distance between vertical stirrup legs;} \]
\[ F = \text{force;} \]
\[ h = \text{total depth of rectangular section;} \]
\[ h' = \text{vertical distance between horizontal stirrup legs;} \]
\[ h'' = \text{vertical distance from stirrups in bottom (top) of beam to neutral axis in top (bottom) of beam;} \]
\[ M = \text{bending moment;} \]
\[ s = \text{spacing between stirrups;} \]
\[ T = \text{torsional moment;} \]
\[ t = \text{wall thickness;} \]
\[ V = \text{vertical shear force;} \]
\[ \alpha = \text{angle of inclination for concrete compressive struts;} \]
\[ \sigma = \text{normal stress.} \]

Subscripts

\[ b = \text{bottom;} \]
\[ c = \text{concrete;} \]
\[ l = \text{left, longitudinal;} \]
\[ lb = \text{longitudinal bottom reinforcement;} \]
\[ lt = \text{longitudinal top reinforcement;} \]
\[ o = \text{ultimate value of moment or force under loading;} \]
\[ r = \text{right;} \]
\[ T = \text{torsion;} \]
\[ t = \text{top;} \]
\[ V = \text{vertical shear;} \]
\[ w = \text{stirrup reinforcement (web).} \]
Superscripts

\[ b = \text{mode } b \text{ (compression zone in bottom of beam)}; \]
\[ s = \text{mode } s \text{ (compression zone in one vertical side of beam)}; \]
\[ t = \text{mode } t \text{ (compression zone in top of beam)}; \text{ and } \]
\[ y = \text{yield}. \]
ABSTRACT: A rational, simple method is presented for analysis of the torsion-bending-shear interaction at failure for reinforced concrete beams. Equilibrium equations are studied for observed failure mechanisms. Three mechanisms can occur depending on where the concrete compression zone is formed: in the top (mode t), in the bottom (mode b), or in one vertical side (mode s). In modes t and b, the torsion-bending and the shear-bending interaction are governed by second-degree parabolas, whereas the torsion-shear interaction is governed by an ellipse. In mode s, the interaction surface is a cylinder with an elliptic base in the torsion-shear plane. The interaction is in this mode not influenced by the value of the bending moment, M. Approximations regarding the lengths of the moment lever arms and regarding the shear-carrying capacity are examined. Some test results are presented to demonstrate the accuracy of the method. The main feature of the method is that the vertical shear forces are considered in a simple but rational way.