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Orlicz spaces which are AM-spaces

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Abstract. The Orlicz function and sequence spaces which are AM-spaces are characterized for both the Luxemburg-Nakano and the Amemiya (Orlicz) norm.

1. Preliminaries. Let (Ω, Σ, μ) be a positive complete σ-finite measure space and $L^0 = L^0(\mu)$ the space of all (equivalence classes of) Σ-measurable real functions on Ω.

Consider an *Orlicz function* $\varphi : [0, \infty) \to [0, \infty]$, i.e., a *convex nondecreasing function vanishing at zero* (not identically 0 or ∞ on $(0, \infty)$) and define the functional $I_\varphi : L^0(\mu) \to [0, \infty]$ by the formula

$$I_\varphi(x) = \int_\Omega \varphi(|x(t)|) d\mu.$$

The *Orlicz space* $L_\varphi(\mu)$ is defined by $L_\varphi(\mu) = \{x \in L^0(\mu) : I_\varphi(x/\lambda) < \infty \text{ for some } \lambda > 0\}$. This space is a Banach space with the following three norms: the *Luxemburg-Nakano norm*

$$\|x\|_\varphi = \inf\{\lambda > 0 : I_\varphi(x/\lambda) \leqq 1\},$$

the *Amemiya norm*

$$\|x\|_\varphi^A = \inf_{k>0} \frac{1}{k}(1 + I_\varphi(kx))$$

and the *Orlicz norm*

$$\|x\|_\varphi^0 = \sup\{|\int_\Omega x(t)y(t)d\mu| : y \in L_{\varphi^*}, I_{\varphi^*}(y) \leqq 1\},$$

where the function $\varphi^* : [0, \infty) \to [0, \infty]$ is defined by the formula

$$\varphi^*(u) = \sup\{uv - \varphi(v) : v \geqq 0\}$$

and called *complementary* to φ in the sense of Young (see [5], [8], [9], [10]). For the counting measure μ on N we obtain the *Orlicz sequence space* $l_\varphi = \{x = (x_n) : I_\varphi(x/\lambda) = \sum_{n=1}^{\infty} \varphi(|x_n|/\lambda) < \infty \text{ for some } \lambda > 0\}$. It is well known that $\|x\|_\varphi \leqq \|x\|_\varphi^0 \leqq 2\|x\|_\varphi$

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and $\|x\|_\varphi \leqq 1$ if and only if $I_\varphi(x) \leqq 1$ (cf. [5], [8], [9] and [10]). It is also known that $\|x\|_\varphi^0 = \|x\|_\varphi^A$ for any $x \in L_\varphi$ (cf. [4]).

Baron and Hudzik [1] have proved that the only Orlicz spaces L_φ that are abstract L_p spaces $(1 < p < \infty)$, i.e. $\|x+y\|_\varphi^p = \|x\|_\varphi^p + \|y\|_\varphi^p$ for $x, y \in L_\varphi, x \perp y$, are the Lebesgue spaces L_p.

In this paper we will consider the limit case, namely we will solve the problem which Orlicz function spaces L_φ or Orlicz sequence spaces l_φ are AM-spaces. We will solve this problem for both the Luxemburg-Nakano and the Amemiya (Orlicz) norm. As we will see, Orlicz spaces L_φ and l_φ can be AM-spaces iff they are isometric to L_∞ or l_∞ under the isometry $\lambda\,\mathrm{Id}$ for some $\lambda > 0$.

Recall that a subspace $(X, \|\cdot\|)$ of $L^0(\mu)$ is said to be a *Banach function space* if it is a Banach space satisfying the following condition: if $x \in L^0, y \in X$ and $|x(t)| \leqq |y(t)|$ μ-a.e., then $x \in X$ and $\|x\| \leqq \|y\|$.

A Banach function space $X = (X, \|\cdot\|)$ is an *AM-space* if

$$\|\max(x, y)\| = \max(\|x\|, \|y\|) \text{ for all } 0 \leqq x, y \in X. \tag{1}$$

An equivalent and very useful condition on AM-space says that a Banach function space is an AM-space if and only if

$$\|x + y\| = \max(\|x\|, \|y\|) \text{ for all } x, y \in X \text{ with } x \perp y, \tag{2}$$

where $x \perp y$ means that $\mu(\operatorname{supp} x \cap \operatorname{supp} y) = 0$ and the support $\operatorname{supp} x$ of a function $x \in X$ is defined (up to a set of measure zero) by the formula $\operatorname{supp} x = \{t \in \Omega : x(t) \neq 0\}$.

A Banach function space X has the *Fatou property* if $0 \leqq x_n \uparrow x$ with $x_n \in X, x \in L^0$ and $\sup_n \|x_n\| < \infty$ imply $x \in X$ and $\|x\| = \lim_{n\to\infty} \|x_n\|$ (see [7] and [8]).

The Orlicz space $L_\varphi(\mu)$ with both the Luxemburg-Nakano and the Amemiya norm is a Banach function space with the Fatou property.

2. Orlicz spaces over a nonatomic measure space which are AM-spaces. First of all we will prove that if an Orlicz function is finite-valued then the Orlicz function space cannot be an AM-space. We need to define for an Orlicz function φ the following two parameters:

$$u_0(\varphi) = \sup\{u \geqq 0 : \varphi(u) = 0\} \text{ and } u_\infty(\varphi) = \sup\{u > 0 : \varphi(u) < \infty\}. \tag{3}$$

From the definition of Orlicz function we have $u_0(\varphi) \leqq u_\infty(\varphi), u_0(\varphi) < \infty$ and $u_\infty(\varphi) > 0$.

Theorem 1. (i) *If the Orlicz space $L_\varphi(\mu)$ with either the Luxemburg-Nakano or the Amemiya norm on a nonatomic measure space (Ω, Σ, μ) is an AM-space, then $u_\infty(\varphi) < \infty$.*

(ii) *If $u_\infty(\varphi) < \infty$, then $L_\varphi(\mu) \subset L_\infty(\mu)$ and $\|x\|_\infty \leqq u_\infty(\varphi)\|x\|_\varphi$.*

Proof. Let $u_\infty(\varphi) = \infty$, i.e., let φ be a finite-valued function. Take disjoint $A, B \in \Sigma$ and a number $c > u_0(\varphi)$ such that $\varphi(c)\mu(A) = 1$ and $\varphi(c)\mu(B) = 1$. Define

$$x = c\chi_A, \quad y = c\chi_B.$$

Then $I_\varphi(x) = \int_A \varphi(c)d\mu = \varphi(c)\mu(A) = 1$ and so $\|x\|_\varphi = 1$. Similarly, $\|y\|_\varphi = 1$ but

$$I_\varphi(x+y) = \int_A \varphi(c)d\mu + \int_B \varphi(c)d\mu = \varphi(c)\mu(A) + \varphi(c)\mu(B) = 2$$

gives that $\|x+y\|_{\varphi} > 1$, and the Orlicz space $L_{\varphi}(\mu)$ with the Luxemburg-Nakano norm $\|\cdot\|_{\varphi}$ does not satisfy (2) which means that it is not an AM-space.

Consider now the Orlicz space with the Amemiya norm and assume that $u_{\infty}(\varphi) = \infty$. For any $c > u_0(\varphi)$ there exists $\varepsilon > 0$ such that $(1+\varepsilon)(u_0(\varphi)+\varepsilon) < c$. Choose $A \in \Sigma$ such that $0 < \mu(A) < \infty$ and $I_{\varphi}(c\chi_A) = \varphi(c)\mu(A) \leqq \varepsilon$. This is possible since our measure is nonatomic. Then

$$\begin{aligned}\|\chi_A\|_{\varphi}^{A} &= \inf_{k>0} \frac{1}{k}(1 + I_{\varphi}(k\chi_A)) \leqq \frac{1}{c}(1 + I_{\varphi}(c\chi_A)) \\ &= \frac{1}{c}(1 + \varphi(c)\mu(A)) \leqq (1+\varepsilon)/c < 1/(u_0(\varphi)+\varepsilon).\end{aligned}$$

Consider now two cases.

I. There is $k_0 > 0$ such that $\|\chi_A\|_{\varphi}^{A} = \dfrac{1}{k_0}(1 + I_{\varphi}(k_0\chi_A))$.

We will show that $I\varphi(k_0\chi_A) > 0$. If $u_0(\varphi) = 0$, then this is obviously true. Let $u_0(\varphi) > 0$ and assume for the contrary that $I_{\varphi}(k_0\chi_A) = 0$. Then it must be $k_0 \leqq u_0(\varphi)$, and so $\|\chi_A\|_{\varphi}^{A} = 1/k_0 \geqq 1/u_0(\varphi)$, a contradiction. Therefore we have indeed $I_{\varphi}(k_0\chi_A) > 0$. This implies that $0 < \varphi(k_0) < \infty$. Let $B \subset A, B \in \Sigma$, be such that $\mu(B) = \mu(A \backslash B) = \mu(A)/2$. Then

$$\begin{aligned}\|\chi_A\|_{\varphi}^{A} &= [1 + I_{\varphi}(k_0\chi_A)]/k_0 > [1 + I_{\varphi}(k_0\chi_B)]/k_0 \geqq \\ &\geqq \inf_{k>0} \frac{1}{k}[1 + I_{\varphi}(k\chi_B)] = \|\chi_B\|_{\varphi}^{A}.\end{aligned}$$

We can prove in the same way that

$$\|\chi_A\|_{\varphi}^{A} > \|\chi_{A\backslash B}\|_{\varphi}^{A},$$

and consequently that

$$\|\chi_B + \chi_{A\backslash B}\|_{\varphi}^{A} = \|\chi_A\|_{\varphi}^{A} > \max\{\|\chi_B\|_{\varphi}^{A}, \|\chi_{A\backslash B}\|_{\varphi}^{A}\}.$$

This yields that equality (2) does not hold, which gives that $(L_{\varphi}(\mu), \|\cdot\|_{\varphi}^{A})$ is not an AM-space.

II. Assume that $\|\chi_A\|_{\varphi}^{A} < [1 + I_{\varphi}(k\chi_A)]/k$ for any $k > 0$. Then

$$\|\chi_A\|_{\varphi}^{A} = \lim_{k\to\infty} \frac{1}{k} I_{\varphi}(k\chi_A) = \mu(A) \lim_{k\to\infty} (\varphi(k)/k).$$

Of course case II is possible only when $\lim\limits_{u\to\infty} \varphi(u)/u < \infty$. Then for $B \subset A, B \in \Sigma$ with $\mu(B) = \mu(A\backslash B) = \mu(A)/2$, we have

$$\begin{aligned}\|\chi_A\|_{\varphi}^{A} &= \mu(A) \lim_{k\to\infty} (\varphi(k)/k) = 2\mu(B) \lim_{k\to\infty} (\varphi(k)/k) \\ &\geqq 2 \inf_{k>0} \frac{1}{k}[1 + I_{\varphi}(k\chi_B)] > \inf_{k>0} \frac{1}{k}[1 + I_{\varphi}(k\chi_B)] = \|\chi_B\|_{\varphi}^{A}.\end{aligned}$$

In the analogous way we can prove that $\|\chi_A\|_{\varphi}^{A} > \|\chi_{A\backslash B}\|_{\varphi}^{A}$. Therefore

$$\|\chi_B + \chi_{A\backslash B}\|_{\varphi}^{A} = \|\chi_A\|_{\varphi}^{A} > \max\{\|\chi_B\|_{\varphi}^{A}, \|\chi_{A\backslash B}\|_{\varphi}^{A}\}.$$

This means that $(L_{\varphi}(\mu), \|\cdot\|_{\varphi}^{A})$ is not an AM-space, and the proof is complete.

(ii) (cf. [2]). For $0 \neq x \in L_\varphi(\mu)$ let $A = \{t \in \Omega : |x(t)| > u_\infty(\varphi)\|x\|_\varphi\}$. Since $\varphi(x\chi_A/\|x\|_\varphi) = \infty$ it follows that $\infty \cdot \mu(A) = I_\varphi(x\chi_A/\|x\|_\varphi) \leqq I_\varphi(x/\|x\|_\varphi) \leqq 1$. This gives that $\mu(A) = 0$, i.e.,

$$|x(t)| \leqq u_\infty(\varphi)\|x\|_\varphi \quad \mu\text{-a.e. on } \Omega,$$

and (ii) follows.

Theorem 2. *Let (Ω, Σ, μ) be a nonatomic measure space. The following assertions are equivalent:*

(i) *An Orlicz space $L_\varphi(\mu))$ with the Luxemburg-Nakano norm is an AM-space.*
(ii) *$u_\infty(\varphi) < \infty$ and $\varphi(u_\infty(\varphi)) = 0$ if $\mu(\Omega) = \infty$ or $\varphi(u_\infty(\varphi))\mu(\Omega) \leqq 1$ if $\mu(\Omega) < \infty$.*
(iii) *$L_\varphi(\mu) = L_\infty(\mu)$ and there is a constant $k > 0$ such that $\|x\|_\varphi = k\||x\|_\infty$ for every $x \in L_\varphi(\mu)$.*

Proof. (i) $\Rightarrow$ (ii). The fact that $u_\infty(\varphi) < \infty$ follows from Theorem 1. Assume then that $\varphi(u_\infty(\varphi))\mu(\Omega) > 1$, where $0 \cdot \infty = 0$ by definition. Then $\varphi(u_\infty(\varphi)) > 0$ must hold. Take $A, B \subset \Omega$, $A, B \in \Sigma$ such that $A \cap B = \emptyset, \varphi(u_\infty(\varphi))\mu(A) = 1$ and $0 < \varphi(u_\infty(\varphi))\mu(B) \leqq 1$.

Define

$$x = u_\infty(\varphi)\chi_A, \quad y = u_\infty(\varphi)\chi_B.$$

Since $I_\varphi(x) = 1$, we get directly $\|x\|_\varphi = 1$. The conditions $I_\varphi(y) \leqq 1$ and $I_\varphi(y/\lambda) = \infty$ for any $\lambda \in (0,1)$ imply that $\|y\|_\varphi = 1$. However, $I_\varphi(x+y) > 1$, whence it follows that $\|x+y\|_\varphi > 1$, i.e.

$$\|x+y\|_\varphi > \max\{\|x\|_\varphi, \|y\|_\varphi\}$$

and (i) does not hold. This finishes the proof of the implication.

(ii) $\Rightarrow$ (iii). We will prove first that (ii) implies that $L_\varphi(\mu) = L_\infty(\mu)$. Assume that $x \in L_\infty(\mu)$. Then, by (ii),

$$I_\varphi(u_\infty(\varphi)x/\|x\|_\infty) \leqq 1, \quad \text{i.e. } x \in L_\varphi(\mu)$$

and

$$\|x\|_\varphi \leqq u_\infty(\varphi)^{-1}\|x\|_\infty.$$

Assume now that $x \in L_\varphi(\mu)$, i.e., there is $\lambda > 0$ such that $d = I_\varphi(\lambda x) < \infty$. Then by the convexity of I_φ, we obtain

$$I_\varphi(\lambda x/\max\{1,d\}) \leqq I_\varphi(\lambda x)/\max\{1,d\} \leqq 1.$$

This means that

$$\lambda|x(t)|/\max\{1,d\} \leqq u_\infty(\varphi) \quad \mu\text{-a.e. in } \Omega,$$

i.e. $x \in L_\infty(\mu)$. Since

$$I_\varphi(u_\infty(\varphi)x/(\lambda\|x\|_\infty)) = \infty \quad \forall \lambda \in (0,1),$$

we get

$$\|x\|_\varphi \geqq u_\infty(\varphi)^{-1}\|x\|_\infty.$$

Since the opposite inequality was also proved we obtain the equality

$$\|x\|_\varphi = u_\infty(\varphi)^{-1}\|x\|_\infty \quad \forall x \in L_\varphi(\mu).$$

The implication (iii) $\Rightarrow$ (i) follows immediately from the fact that $L_\infty(\mu)$ is an AM-space. This finishes the proof of the theorem.

The next result concerns Orlicz spaces with the Amemiya norm.

Theorem 3. *Let (Ω, Σ, μ) be a nonatomic measure space. Then an Orlicz space $L_\varphi(\mu)$ with the Amemiya norm is an AM-space if and only if*

$$u_0(\varphi) > 0, u_\infty(\varphi) < \infty \quad \textit{and} \quad u_0(\varphi) = u_\infty(\varphi). \tag{4}$$

Proof. Sufficiency. Assume that φ satisfies condition (4), i.e.,

$$\varphi(u) = \begin{cases} 0 & \text{for } 0 \leqq u \leqq u_0, \\ \infty & \text{for } u > u_0, \end{cases}$$

for some $u_0 > 0$. Then $L_\varphi(\mu) = L_\infty(\mu), \|x\|_\varphi = u_0^{-1}\|x\|_\infty$ for every $x \in L_\varphi(\mu)$ and

$$\begin{aligned}\|x\|_\varphi^A &= \inf_{k>0, I_\varphi(kx)<\infty} \frac{1}{k}(1 + I_\varphi(kx)) = \inf\{1/k : k > 0 \quad \text{and} \quad I_\varphi(kx) < \infty\} \\ &= \inf\{1/k : k > 0 \text{ and } k|x(t)| \leqq u_0\,\mu\text{-a.e. in } \Omega\} = u_0^{-1}\|x\|_\infty.\end{aligned}$$

It is obvious that the last equality implies that $(L_\varphi(\mu), \|\cdot\|_\varphi^A)$ is an AM-space.

Necessity. From Theorem 1 we have that $u_\infty(\varphi) < \infty$. If $u_0(\varphi) = 0$, then the Amemiya norm $\|\cdot\|_\varphi^A$ is strictly monotone, i.e., $0 \leqq x \leqq y, x \neq y$ μ-a.e. imply $\|x\|_\varphi^A < \|y\|_\varphi^A$ (see [3]), so $\|\cdot\|_\varphi^A$ does not satisfy condition (2). For the sake of completeness, we will repeat here the proof of strict monotonicity of $\|\cdot\|_\varphi^A$ from [3]. The assumption $\lim_{u\to\infty} \varphi(u)/u = \infty$, which follows by $u_\infty(\varphi) < \infty$, gives that $\|y\|_\varphi^A = [1 + I_\varphi(k_0 y)]/k_0$ for some positive k_0 (cf. [10]). Then, since the convex function φ is superadditive (cf. [5], 1.19),

$$I_\varphi(k_0 y) = I_\varphi(k_0(y - x) + k_0 x) \geqq I_\varphi(k_0(y - x)) + I_\varphi(k_0 x)$$

and so

$$\begin{aligned}\|y\|_\varphi^A &= [1 + I_\varphi(k_0 y)]/k_0 \geqq [1 + I_\varphi(k_0(y - x)) + I_\varphi(k_0 x)]/k_0 \\ &= [1 + I_\varphi(k_0 x)]/k_0 + I_\varphi(k_0(y - x))/k_0 \\ &\geqq \|x\|_\varphi^A + I_\varphi(k_0(y - x))/k_0 > \|x\|_\varphi^A.\end{aligned}$$

The last strict inequality follows from the facts that $u_0(\varphi) = 0$ (or equivalently $\varphi(u) > 0$ for $u > 0$) and $x \neq y$.

Assume now that $u_0(\varphi) > 0$ and $u_0(\varphi) < u_\infty(\varphi)$. Take $\varepsilon > 0$ such that $(1 + \varepsilon)u_0(\varphi) < u_\infty(\varphi) - \varepsilon$ and choose $A \in \Sigma$ with $0 < \mu(A) < \infty$ and such that

$$I_\varphi((u_\infty(\varphi) - \varepsilon)\chi_A) = \varphi(u_\infty(\Phi) - \varepsilon)\mu(A) \leqq \varepsilon.$$

Then

$$\begin{aligned}\|\chi_A\|_\varphi^A &= \inf_{k>0} \frac{1}{k}(1 + I_\varphi(k\chi_A)) \\ &\leqq [1 + I_\varphi((u_\infty(\varphi) - \varepsilon)\chi_A)]/(u_\infty(\varphi) - \varepsilon) \\ &= [1 + \varphi(u_\infty(\varphi) - \varepsilon)\mu(A)]/(u_\infty(\varphi) - \varepsilon) \leqq (1 + \varepsilon)/(u_\infty(\varphi) - \varepsilon) \\ &< 1/u_0(\varphi),\end{aligned}$$

and, similarly as in the proof of Theorem 1,

$$\|\chi_B + \chi_{A\backslash B}\|_\varphi^A = \|\chi_A\|_\varphi^A > \max\{\|\chi_B\|_\varphi^A, \|\chi_{A\backslash B}\|_\varphi^A\},$$

where $B \subset A, B \in \Sigma$ is such that $\mu(B) = \mu(B\backslash A) = \mu(A)/2$.

This yields that equality (2) does not hold, and the proof is finished.

Remark 1. If $\mu(\Omega) = \infty$, then conditions $u_\infty(\varphi) < \infty$ and $\varphi(u_\infty(\varphi)) = 0$ from Theorem 2 and $u_0(\varphi) > 0, u_\infty(\varphi) < \infty$ and $u_0(\varphi) = u_\infty(\varphi)$ from Theorem 3 are equivalent. This means that in the case of nonatomic infinite measure space the Orlicz space with the Luxemburg-Nakano norm is an AM-space if and only if the Orlicz space with the Amemiya norm is also an AM-space, and this is equivalent to the fact that $\varphi(u) = 0$ for $0 \leqq u \leqq u_0$ and $\varphi(u) = \infty$ for $u > u_0$ for some $u_0 > 0$. The difference can appear only in the case when $\mu(\Omega) < \infty$.

Example 1. For a fixed $c > 0$ and $p \geqq 1$ let

$$\varphi(u) = \begin{cases} u^p & \text{for} \quad 0 \leqq u \leqq c, \\ \infty & \text{for} \quad u > c. \end{cases}$$

Then $u_0(\varphi) = 0, u_\infty(\varphi) = c$ and $L_\varphi([a,b]) = L_p([a,b]) \cap L_\infty([a,b]) = L_\infty([a,b])$ with

$$\|x\|_\varphi = \max\{\|x\|_p, c^{-1}\|x\|_\infty\} \quad \text{and} \quad \|x\|_\varphi^A = \|x\|_p + c^{-1}\|x\|_\infty.$$

Note that if $(b-a)c^p \leqq 1$, then $\|x\|_\varphi = c^{-1}\|x\|_\infty$.

3. Orlicz sequence spaces which are AM-spaces. In Orlicz sequence spaces the case of Luxemburg-Nakano norm is easy again.

Theorem 4. *An Orlicz sequence space l_φ with the Luxemburg-Nakano norm is an AM-space if and only if*

(4) $$u_0(\varphi) > 0, u_\infty(\varphi) < \infty \quad \textit{and} \quad u_0(\varphi) = u_\infty(\varphi).$$

Proof. Assume that (4) does not hold, i.e., either $u_0(\varphi) = 0$ or $u_0(\varphi) \neq u_\infty(\varphi)$. If $u_0(\varphi) = 0$, then l_φ is strictly monotone (see [6]), so it cannot be an AM-space. If $u_0(\varphi) \neq u_\infty(\varphi)$, then there exists $u > 0$ such that $0 < \varphi(u) < \infty$. Then we can find $v \in [0,u]$ and a natural number n such that $n\varphi(v) = 1$. Define

$$x = (\overbrace{v, \ldots, v}^{n\text{-times}}, 0, 0, \ldots), y = (\overbrace{0, \ldots, 0}^{n\text{-times}}, \overbrace{v, \ldots, v}^{n\text{-times}}, 0, 0, \ldots).$$

We have $I_\varphi(x) = I_\varphi(y) = n\varphi(v) = 1$ and so $\|x\|_\varphi = \|y\|_\varphi = 1$. Moreover, $I_\varphi(x+y) = 2n\varphi(v) = 2$. Thus $\|x+y\|_\varphi > 1 = \max(\|x\|_\varphi, \|y\|_\varphi)$, which means that l_φ with the Luxemburg-Nakano norm is not AM-space.

The case of Orlicz sequence space with the Orlicz norm contains more possibilities. Denote by φ'_+ the right derivative of φ.

Theorem 5. *The following are equivalent:*

(i) *An Orlicz sequence space l_φ with the Orlicz norm is an AM-space.*
(ii) *$l_\varphi = l_\infty$ and there is a constant $c > 0$ such that $\|x\|_\varphi^0 = c\|x\|_\infty$ for any $x \in l_\varphi$.*

(iii) $u_0(\varphi)\varphi'_+(u_0(\varphi)) \geqq 1$.
(iv) φ^* *is linear on the interval* $[0, u_1]$, *where* $\varphi^*(u_1) = 1$.
(v) $l_{\varphi^*} = l_1$ *and there is a constant* $k > 0$ *such that* $\|x\|_{\varphi^*} = k\|x\|_1$ *for any* $x \in l_{\varphi^*}$.

Proof. (i) $\Rightarrow$ (ii). Note that (i) implies that $u_0(\varphi) > 0$, because conversely l_φ is strictly monotone (see [3]), so it cannot be an AM-space. This also follows by the fact that if l_φ is an AM-space, then by virtue of the Fatou property of l_φ, we have $\chi_N \in l_\varphi$, i.e., $l_\infty \subset l_\varphi$ but this yields that $u_0(\varphi) > 0$. Indeed, if $(l_\varphi, \|\cdot\|_\varphi^0)$ is an AM-space, then for any $k, n \in N, n > k$, we have

$$\left\|\sum_{i=k}^{n} e_i\right\|_\varphi^0 = \|\max(e_k, e_{k+1}, \ldots, e_n)\|_\varphi^0$$
$$= \max(\|e_k\|_\varphi^0, \|e_{k+1}\|_\varphi^0, \ldots, \|e_n\|_\varphi^0) = c.$$

Therefore by the Fatou property of $\|\cdot\|_\varphi^0$, we get that $\sum\limits_{i=k}^{\infty} e_i \in l_\varphi$ and $\left\|\sum\limits_{i=k}^{\infty} e_i\right\|_\varphi^0 = c$ for any $k \in N$. Hence we can easily get that

$$\|\chi_A\|_\varphi^0 = \left\|\sum_{i \in A} e_i\right\|_\varphi^0 = c \text{ for any } A \subset N, A \neq \emptyset.$$

Now, we will show that $l_\varphi \subset l_\infty$. Let $x \in l_\varphi$. If $x \notin l_\infty$, then for any $k \in N$ there exist $n_k \in N$ such that $|x_{n_k}| > k$. Therefore, for each $k \in N$,

$$\|x\|_\varphi^0 \geqq \||x_{n_k}|e_{n_k}\|_\varphi^0 > k\|e_{n_k}\|_\varphi^0 = kc.$$

By the arbitrariness of $k \in N$ we get $\|x\|_\varphi^0 = \infty$, a contradiction. Thus $l_\varphi \subset l_\infty$. We even will show that $l_\varphi = l_\infty$ and $\|x\|_\varphi^0 = c\|x\|_\infty$. For any $x \in l_\varphi, x \neq 0$, we have $\|x/\|x\|_\infty\|_\varphi^0 \leqq \|\chi_{\mathrm{supp}\,x}\|_\varphi^0 = c$, i.e. $\|x\|_\varphi^0 \leqq c\|x\|_\infty$.

On the other hand, take any $\lambda \in (0,1)$ and any $x \in l_\varphi, x \neq 0$. There exists $n \in N$ such that $|x_n| > \lambda\|x\|_\infty$, whence

$$\|x/\|x\|_\infty\|_\varphi^0 \geqq \|\lambda e_n\|_\varphi^0 = \lambda\|e_n\|_\varphi^0 = \lambda c,$$

and by arbitrariness of $\lambda \in (0,1)$, $\|x\|_\varphi^0 \geqq c\|x\|_\infty$. Thus $\|x\|_\varphi^0 = c\|x\|_\infty$.

(ii) $\Rightarrow$ (i). This implication is obvious.

(ii) $\Leftrightarrow$ (v). Since l_∞, l_1 and $(l_\varphi, \|\cdot\|_\varphi^0), (l_{\varphi^*}, \|\cdot\|_{\varphi^*})$ are two couples of mutually dual spaces in the sense of Köthe (for the Köthe duality see e.g. [7]), we deduce that (ii) is equivalent to (v).

(iii) $\Rightarrow$ (iv). Let q denote the generalized inverse function of φ'_+, i.e.,

$$q(t) = \sup\{s > 0 : \varphi'_+(s) < t\} \text{ with } \operatorname{supp}\emptyset = 0.$$

Then we have in our case $q(t) = u_0(\varphi)$ for $t \in [0, \varphi'_+(u_0(\varphi))]$. Therefore $\varphi^*(u) = \int\limits_0^u q(t)dt$ is linear on the interval $[0, \varphi'_+, (u_0(\varphi))]$ and

$$\varphi^*(\varphi'_+(u_0(\varphi))) = u_0(\varphi)\varphi'_+(u_0(\varphi)) \geqq 1.$$

Thus (iv) holds with $u_1 \leqq \varphi'_+(u_0(\varphi))$.

The implication (iv) $\Rightarrow$ (iii) can be proved analogously.

(iv) $\Rightarrow$ (v). Assumption (iv) gives that $\varphi^*(u) = u/u_1$ for $u \in [0, u_1]$. We will show that if $x \in l_{\varphi^*}, x \neq 0$, then $\|x\|_{\varphi^*} = u_1^{-1}\|x\|_1$.

We have $I_{\varphi^*}(x/\|x\|_{\varphi^*}) \leqq 1$. This implies $\varphi^*(|x_n|/\|x\|_{\varphi^*}) \leqq 1$ for all $n \in N$, and so $|x_n|/\|x\|_{\varphi^*} \leqq u_1$, which gives $\varphi^*(|x_n|/\|x\|_{\varphi^*}) = |x_n|/(\|x\|_{\varphi^*}u_1)$. By summation we obtain

$$\begin{aligned}1 \geqq I_{\varphi^*}(x/\|x\|_{\varphi^*}) &= \sum_{n=1}^{\infty} \varphi^*(|x_n|/\|x\|_{\varphi^*}) \\ &= \sum_{n=1}^{\infty} |x_n|/(\|x\|_{\varphi^*}u_1) = \|x\|_1/(\|x\|_{\varphi^*}u_1),\end{aligned}$$

i.e. $\|x\|_{\varphi^*} \geqq \|x\|_1/u_1$.

On the other hand,

$$I_{\varphi^*}(xu_1/\|x\|_1) = \sum_{n=1}^{\infty} \varphi^*(|x_n|u_1/\|x\|_1) = \sum_{n=1}^{\infty} |x_n|/\|x\|_1 = 1,$$

and so $\|xu_1/\|x\|_1\|_{\varphi^*} \leqq 1$, which gives $\|x\|_{\varphi^*} \leqq \|x\|_1/u_1$. Therefore,

$$\|x\|_{\varphi^*} = \|x\|_1/u_1.$$

(v) $\Rightarrow$ (iv). Note first that condition (v) implies that there is $u_1 > 0$ such that $\varphi^*(u_1) = 1$. Denote $\Psi = \varphi^*$ and assume for the contrary that $\Psi(u_\infty(\Psi)) < 1$.

Defining $x = (u_\infty(\Psi), 0, 0, \ldots)$, we get $I_\Psi(x) = \Psi(u_\infty(\Psi)) < 1$ and for any $\lambda \in (0,1)$, we have $I_\Psi(x/\lambda) = \Psi(u_\infty(\Psi)/\lambda) = \infty$, whence $\|x\|_\Psi = 1$. Let $b > 0$ be such that $\Psi(u_\infty(\Psi)) + \Psi(b) \leqq 1$ and define $y = (u_\infty(\Psi)), b, 0, 0, \ldots)$. Then $\|y\|_\Psi = 1$ and $\|x\|_1 = u_\infty(\Psi), \|y\|_1 = u_\infty(\Psi) + b > u_\infty(\Psi)$, and so l_Ψ and l_1 cannot be isometric under the isometry $\lambda\,\mathrm{Id}$ for some $\lambda > 0$. So, we have proved that condition (v) implies that $\Psi(u_\infty(\Psi)) \geqq 1$. Assume without loss of generality that $\Psi(1) = 1$ (since we can take a new function $\phi(u) = \Psi(uu_1)$ for which $\phi(1) = 1$ and $\|\cdot\|_\phi = u_1\|\cdot\|_\Psi$). Then we need to prove that Ψ is linear on the interval $[0,1]$. Assume for the contrary that Ψ is not linear on the interval $[0,1]$. Then $\Psi(1/2) < \Psi(1)/2 = 1/2$. Therefore, defining $x = (1/2, 1/2, 0, 0, \ldots)$, we get $\|x\|_1 = 1$ but $I_\Psi(x) = 2\Psi(1/2) < 1$, whence it follows that $\|x\|_\Psi < 1$. This shows that l_Ψ is not then isometric to l_1 under the identity mapping. It is obvious that if $\Psi(1) = 1$ and Ψ is linear on $[0,1]$, then $\|x\|_\Psi = \|x\|_1$ for any $x \in l_\Psi$. We can prove in the same way that $\|x\|_\Psi = k\|x\|_1$ for any $x \in l_\Psi$ if and only if $\Psi(1/k) = 1$ and Ψ is linear on the interval $[0, 1/k]$.

Example 2. For a fixed $c > 1$ let $\varphi(u) = 0$ for $0 \leqq u \leqq 1/c, \varphi(u) = cu - 1$ for $1/c \leqq u \leqq 1$ and $\varphi(u) = \infty$ for $u > 1$. Then $l_\varphi = l_\infty$ with $\|x\|_\varphi^A = c\|x\|_\infty$. On the other hand, for any nonempty finite subset A of N we have $\|\chi_A\|_\varphi = \max\{1, c|A|/(1 + |A|)\}$, which shows that l_φ with the Luxemburg-Nakano norm is not an AM-space.

Remark 2. Let us define for any Orlicz function φ, the subspace E_φ of L_φ as the closure of the set of simple functions in the space L_φ. In the sequence case let us define h_φ to be the closure in l_φ of the space of all sequences with finite number of coordinates different from zero. Consider the spaces E_φ and h_φ with the Luxemburg-Nakano and the Amemiya norm induced from L_φ (resp. l_φ). These norms are order continuous in E_φ and h_φ but they do not have the Fatou property. Sine l_∞ is not order continuous the equalities $E_\varphi = L_\infty$ and

$l_\varphi = l_\infty$ are impossible. Note that $l_\infty = l_\varphi$ and $c_0 = h_\varphi$ isometrically when $\varphi(u) = 0$ for $0 \leqq u \leqq 1$ and $\varphi(u) = \infty$ for $u \leqq 1$. It is obvious that both l_∞ and c_0 are AM-spaces. So, it is natural to ask when E_φ and h_φ are AM-spaces. Note that if we replace equalities $E_\varphi = L_\infty$ and $l_\varphi = l_\infty$ by the inclusions $E_\varphi \subset L_\infty$ and $l_\varphi \subset l_\infty$, respectively, then all the theorems remain valid for E_φ and h_φ in place of L_φ and l_φ, respectively. The sufficiency is obvious and in the necessity part we always constructed simple functions or sequences with finite number of coordinates different from zero, which were in fact in E_φ or h_φ, respectively.

References

[1] K. BARON and H. HUDZIK, Orlicz spaces which are L^p-spaces. Aequationes Math. **48,** 254–261 (1994).

[2] H. HUDZIK, Orlicz spaces of essentially bounded functions and Banach-Orlicz algebras. Arch. Math. **44,** 535–538 (1985).

[3] H. HUDZIK and W. KURC, Monotonicity properties of Musielak-Orlicz spaces and best approximation in Banach lattices. J. Approx. Theory, to appear.

[4] H. HUDZIK and L. MALIGRANDA, Amemiya norm and Orlicz norm are equal, to appear.

[5] M. A. KRASNOSEL'SKII and J. B. RUTICKII, Convex Functions and Orlicz Spaces. Groningen 1961.

[6] W. KURC, Strictly and uniformly monotone Musielak-Orlicz spaces and applications to best approximation. J. Approx. Theory **69,** 173–187 (1992).

[7] J. LINDENSTRAUSS and L. TZAFRIRI, Classical Banach Spaces II. Function Spaces. Berlin-Heidelberg-New York 1979.

[8] L. MALIGRANDA, Orlicz Spaces and Interpolation. Seminars in Math. **5,** Campinas 1989.

[9] J. MUSIELAK, Orlicz Spaces and Modular Spaces. LNM 1034. Berlin-Heidelberg-New York 1983.

[10] M. M. RAO and Z. D. REN, Theory of Orlicz Spaces. Pure Appl. Math. **146,** New York 1991.

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