ON \( n \)-TH JAMES AND KHINTCHINE CONSTANTS OF BANACH SPACES

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Abstract. For any Banach space \( X \) the \( n \)-th James constants \( J_n(X) \) and the \( n \)-th Khintchine constants \( K_{p,q}^n(X) \) are investigated and discussed. Some new properties of these constants are presented. The main result is an estimate of the \( n \)-th Khintchine constants \( K_{p,q}^n(X) \) by the \( n \)-th James constants \( J_n(X) \). In the case of \( n = 2 \) and \( p = q = 2 \) this estimate is even stronger and improves an earlier estimate proved by Kato-Maligranda-Takahashi [25].

Introduction

Several constants of a Banach space \( X = (X, \| \cdot \|) \) are used in the description of its geometric properties. The James constant \( J(X) \), the Jordan-von Neumann constant \( C_{NJ}(X) \), the \( n \)-th James constants \( J_n(X) \) and the \( n \)-th Khintchine constants \( K_{p,q}^n(X) \) are examples of such constants. We will derive some properties of these constants and also investigate the relations among them.

Our main results are about estimates of the Jordan-von Neumann constant by the James constant and also the \( n \)-th Khintchine constants by the \( n \)-th James constants.

In Section 1 we collect and discuss some properties of the constants \( J(X) \) and \( C_{NJ}(X) \). Moreover, we prove a new estimate which improves the Kato-Maligranda-Takahashi [25] estimate (see Theorem 1). In Section 2 we consider the \( n \)-th James constants \( J_n(X) \) and collect their properties as the measure of \( B \)-convexity of a Banach space \( X \). We calculate them for \( L^p \) spaces by using Clarkson’s inequalities (see Theorem 2). The \( n \)-th strong James constants \( J^s_n(X) \) are also considered and a non-trivial estimate of \( J_n(X) \) by \( J^s_n(X) \) is proved (see Theorem 3). Our conjecture is that \( J^s_n(X) < J_n(X) \) for \( n \geq 3 \).

In Section 3 the \( n \)-th Khintchine constants \( K_{p,q}^n(X) \) are considered with their properties. We also calculate or estimate these constants for some classes of Banach spaces.

In Section 4 the main result on an estimate of the \( n \)-th Khintchine constants \( K_{p,q}^n(X) \) by the \( n \)-th James constants \( J_n(X) \) is presented and proved (see Theorem 4). Finally, several facts concerning the constants \( J_n(X) \) and their close relation to the notion of
type of an infinite dimensional Banach space $X$ are pointed out (see e. g. Proposition 4). In particular, we prove the following formula: $p(X) = \sup \{ \frac{\ln n}{\mu_n(X)}, n \geq 2 \}$, where $p(X) = \sup \{ p : X \text{ is of type } p \}$ (see Theorem 5).

Finally, in the last Section 5 we investigate the relation between the $n$-th James and the $n$-th Khintchine constants for isomorphic Banach spaces (see Theorem 6).

Throughout this paper we assume that $X = (X, \| \cdot \|)$ is a real Banach space with $\dim X \geq 2$. $B_X$ will denote the closed unit ball $\{ x \in X : \| x \| \leq 1 \}$ of $X$ and $S_X = \{ x \in X : \| x \| = 1 \}$ is its unit sphere.

1. An estimate of the Jordan-von Neumann constant by the James constant

The James non-square constant of a Banach space $X$ is the number $J(X)$ defined by

$$J(X) = \sup \{ \min(\| x + y \|, \| x - y \|) : x, y \in B_X \},$$

and the Jordan-von Neumann constant $C_{NJ}(X)$ of $X$ is defined by

$$C_{NJ}(X) = \sup \left\{ \frac{\| x + y \|^2 + \| x - y \|^2}{2(\| x \|^2 + \| y \|^2)} : x, y \in X \text{ not both zero} \right\}.$$

These constants have been studied by several authors (see e.g. Casini [4], Gao-Lau [11], [12], Kato-Maligranda-Takahashi [25] and Kato-Maligranda [24]). Let us collect some properties of these constants:

(i) $J(X) = \sup \{ \min(\| x + y \|, \| x - y \|) : x, y \in S_X \}.$

(ii) $\sqrt{2} \leq J(X) \leq 2$ and $1 \leq C_{NJ}(X) \leq 2$.

(iii) $X$ is Hilbert space $\implies J(X) = \sqrt{2}$ and the converse is not true; $C_{NJ}(X) = 1 \iff X$ is a Hilbert space.

(iv) $J(X) < 2 \iff C_{NJ}(X) < 2 \iff$ the space $X$ is uniformly non-square, i.e., there exists a $\delta \in (0, 1)$ such that for any $x, y \in S_X$ either $\| x + y \|/2 \leq 1 - \delta$ or $\| x - y \|/2 \leq 1 - \delta$.

(v) $J(X^{**}) = J(X)$, $\max \{ \sqrt{2}, 2J(X) - 2 \} \leq J(X^*) \leq J(X)/2 + 1$, and there exists a two-dimensional Banach space $X$ such that $J(X^*) \neq J(X)$, where $X^*$ and $X^{**}$ are dual and the second dual of $X$; $C_{NJ}(X^*) = C_{NJ}(X)$.

(vi) If $1 \leq p < \infty$ and $\dim L^p(\mu) \geq 2$, then $J(L^p(\mu)) = \max \{ 2^{1/p}, 2^{1-1/p} \}$ and $C_{NJ}(L^p(\mu)) = \max \{ 2^{2/p-1}, 2^{1-2/p} \} = 2^{1-2/p}$.

Kato-Maligranda-Takahashi [25] proved that

$$J(X)^2/2 \leq C_{NJ}(X) \leq \frac{J(X)^2}{(J(X) - 1)^2 + 1}. \tag{1}$$

Moreover, if $X$ is not uniformly non-square, then we have equalities in (1) and there exists a two-dimensional Banach space $X$ for which $J(X)^2/2 < C_{NJ}(X)$. 
Thus,

\[ \|x + y\| \leq \frac{x}{\|x\|} + \frac{y}{\|y\|} + \left(1 - \frac{x}{\|x\|}\right)\|y\| \]

\[ \leq \|x\| \left(\frac{x}{\|x\|}\right) + \frac{y}{\|y\|} + \left(1 - \frac{x}{\|x\|}\right)\|y\| \]

\[ = \|x\| \left(\frac{x}{\|x\|}\right) + \frac{y}{\|y\|} + \|y\| - \|x\| \]

\[ = \|y\| + \left(\frac{x}{\|x\|} + \frac{y}{\|y\|} - 1\right)\|x\|, \]

and, similarly, \[ \|x - y\| \leq \|y\| + \left(\frac{x}{\|x\|} - \frac{y}{\|y\|} - 1\right)\|x\|, \]

which gives

\[ \min (\|x + y\|, \|x - y\|) \leq \|y\| + \left[\min \left(\frac{x}{\|x\|} + \frac{y}{\|y\|}, \frac{x}{\|x\|} - \frac{y}{\|y\|}\right) - 1\right]\|x\| \]

\[ \leq \|y\| + \left[\|J(X) - 1\|\|x\|\right]. \]

Our improvement of (1) reads:

**Theorem 1.** For any Banach space \(X\) we have

\[ C_{NJ}(X) \leq \frac{J(X)^2}{4} + 1 + \frac{J(X)}{4} \left(\sqrt{J(X)^2 - 4J(X)} + 8 - 2\right) \]

\[ \leq \frac{J(X)^2}{2} + 2 - J(X) \leq \frac{J(X)^2}{(J(X) - 1)^2 + 1}. \]

**Proof.** If \(J(X) = 2\), then we have equalities in (3). Therefore, we assume that \(J(X) < 2\). To prove the first inequality in (3), let \(x, y \in X\), not both zero and \(\|x\| \leq \|y\|\). Then for \(x' = \frac{x}{\|x\|} + \frac{y}{\|y\|}\), \(y' = \frac{y}{\|x\|} + \frac{y}{\|y\|}\), we have that

\[ A := \frac{\|x + y\|^2 + \|x - y\|^2}{2 (\|x'\|^2 + \|y'\|^2)} = \frac{\|x' + y'\|^2 + \|x' - y'\|^2}{2 (\|x'\|^2 + \|y'\|^2)}. \]

Since \(\|x' \pm y'\| = \frac{\|x \pm y\|}{\|x\| + \|y\|} \leq 1\) it follows that

\[ A \leq \frac{1 + \|x' - y'\|^2}{2 (\|x'\|^2 + \|y'\|^2)} \quad \text{and} \quad A \leq \frac{\|x' + y'\|^2 + 1}{2 (\|x'\|^2 + \|y'\|^2)}. \]

Thus,

\[ A \leq \frac{1 + \min (\|x' + y'\|, \|x' - y'\|)}{2 (\|x'\|^2 + \|y'\|^2)}. \]
and, hence, by Lemma 1,
\[ \mathcal{A} \leq \frac{1 + [ \| \mathbf{y}' \| + (J(X) - 1) \| \mathbf{x}' \| ]^2}{2 (\| \mathbf{x}' \|^2 + \| \mathbf{y}' \|^2)}. \]

Note that \( \| \mathbf{x}' \| + \| \mathbf{y}' \| = 1 \) and \( \| \mathbf{x}' \| = \frac{\| \mathbf{x} \|_{\| \mathbf{x} \| + \| \mathbf{y} \|}}{\| \mathbf{x} \| + \| \mathbf{y} \|} \leq \frac{1}{2} \), which gives that
\[ \mathcal{A} \leq \frac{1 + (1 + (J(X) - 2) \| \mathbf{x}' \|)^2}{2 (\| \mathbf{x}' \|^2 + (1 - \| \mathbf{x}' \|)^2)}. \]

Now, consider the function
\[ f(u) = \frac{1 + [1 + (J - 2)u]^2}{u^2 + (1 - u)^2} \quad \text{for} \quad u \in [0, \frac{1}{2}] \quad \text{with} \quad J < 2. \]

Note that the derivative
\[ f'(u) = \frac{2J [(2 - J)u^2 - (4 - J)u + 1]}{[u^2 + (1 - u)^2]^2} \]

is zero at \( u_1 = \frac{4 - J - \sqrt{J^2 - 4J + 8}}{2(2 - J)} \in (0, \frac{1}{2}) \) and \( u_2 = \frac{4 - J + \sqrt{J^2 - 4J + 8}}{2(2 - J)} > 1 \). Therefore, for \( u \in [0, \frac{1}{2}] \), we have that
\[ f(u) \leq f(u_1) = \frac{J^2}{2} + 2 + \frac{J}{2} \sqrt{J^2 - 4J + 8} - J. \]

The second estimate in (3) follows easily from the following equivalences:
\[ \frac{J^2}{4} + 1 + \frac{J}{4} \left( \sqrt{J^2 - 4J + 8} - 2 \right) \leq \frac{J^2}{2} + 2 - J \]
\[ \iff J \sqrt{J^2 - 4J + 8} \leq J^2 - 2J + 4 \]
\[ \iff J^2 (J^2 - 4J + 8) \leq (J^2 - 4J + 4)^2 \]
\[ \iff 16J \leq 4J^2 + 16 \]
\[ \iff 0 \leq 4(J - 2)^2. \]

To prove the third estimate in (3) consider the function
\[ g(t) = \frac{t^2}{2} + 2 - t - \frac{t^2}{(t - 1)^2 + 1} \quad \text{for} \quad t \in [\sqrt{2}, 2] \]

and observe that the derivative \( g'(t) = t - 1 + 2 \frac{t(t - 1)^2}{(t - 1)^2 + 1} > 0 \) and, thus, \( g \) is increasing with \( g(t) \leq g(2) = 0 \). The proof of (3) is complete.

The estimates (3) were proved independently in 2003 by Maligranda [31, Theorem 1] and Nikolova-Persson-Zachariades [37, p. 8] (see also [38]) as an improvement.
of the upper Kato-Maligranda-Takahashi estimate in (1). Maligranda even formulated
the following conjecture (cf. [31] and [32]):

\[
\text{The estimate } C_{NJ}(X) \leq \frac{J(X)^2}{4} + 1 \text{ holds for any Banach space } X. \quad (4)
\]

Recently Saejung [42] published a paper with a contribution to the proof of this
Maligranda conjecture but his “proof” contains only a proof of the first estimate in
(3) and, thus, the conjecture is not really proved. Also Takahashi [43] announced the
estimate (3). Up to now we were able only to prove estimates (3) and therefore we can
still ask to prove or disprove the Maligranda conjecture.

2. B-convexity and the \( n \)-th James constants

Let us start with the notion of the uniformly non-\( l_1^n \) and B-convexity of a Banach
space \( X \). These notions were introduced by James [15] and Beck [1].

For every natural number \( n \geq 2 \) we say, as in Giesy-James [14], that a Banach space
\( X \) is uniformly non-\( l_1^n \) if there exists a \( \delta \in (0, 1) \) such that for every \( x_1, \ldots, x_n \in BX \)
it holds that \( \| \sum_{k=1}^{n} \theta_k x_k \| \leq n(1 - \delta) \) for some choice of signs \( \theta_1, \theta_2, \ldots, \theta_n \).

A Banach space \( X \) is called B-convex if it is uniformly non-\( l_1^n \) for some \( n \geq 2 \).

In the connection to these two notions we consider the \( n \)-th James constants (or
the measure of uniformly non-\( l_1^n \), or sometimes called the measure of B-convexity). For
given \( n \in \mathbb{N} \) the \( n \)-th James constant \( J_n(X) \) of a Banach space \( X \) is defined by

\[
J_n(X) := \sup \left\{ \min_{\theta_k = \pm 1} \left\| \sum_{k=1}^{n} \theta_k x_k \right\| : x_1, \ldots, x_n \in BX \right\}.
\]

Note that \( J_1(X) = 1 \), \( J_2(X) \) is just the James constant \( J(X) \) discussed in Section
1 and \( J_n(l_1) = J_n(l_1^n) = n \) for \( m \geq n \) (which can be seen by considering unit
vectors). We have also equality \( J_n(X) = \inf \{ C > 0 : \min_{\theta_k = \pm 1} \| \sum_{k=1}^{n} \theta_k x_k \| \leq C \sup_{\| x \| = 1} \| x \| \} \) for all \( x_1, x_2, \ldots, x_n \in X \).

It is clear that \( X \) is uniformly non-\( l_1^n \) if and only if \( J_n(X) < n \) and \( X \) is B-convex
if and only if \( J_n(X) < n \) for some \( n \geq 2 \).

The \( n \)-th James constants were studied by several authors e.g. Giesy [13, p.
117], Pisier [40, pp. 1-3], Wojciechowski [46, pp. 340-343], Kalton [20, p. 248],
Kalton-Peck-Roberts [22, pp. 98-99], Kadets-Kadets [18, pp. 83-84], [19, p. 69] and
Diestel-Jarchow-Tonge [6, pp. 261-266]. It seems that these constants for the first time
explicitly appeared in 1966, in the paper by Giesy [13, p. 117], where he investigated
the numbers \( G_n(X) = \sup \{ \min_{\theta_k = \pm 1} \frac{1}{n} \| \sum_{k=1}^{n} \theta_k x_k \| : x_1, \ldots, x_n \in BX \} \).

Let us collect some properties of \( n \)-th James constants:

\begin{enumerate}
\item[(i)] \( 1 \leq J_n(X) \leq n \); if \( \dim X = \infty \), then \( J_n(X) \geq \sqrt{n} \).
\item[(ii)] \( J_n(X) \) is increasing in \( n \) and subadditive in \( n \), that is, \( J_{m+n}(X) \leq J_m(X) + J_n(X) \) for all \( m, n \in \mathbb{N} \); in particular, \( J_{n+1}(X) \leq J_n(X) + 1 \).
\end{enumerate}
(iii) \( J_n(X) \) is submultiplicative sequence, i.e., \( J_{mn}(X) \leq J_m(X)J_n(X) \) for all \( m, n \in \mathbb{N} \).

(iv) If \( X \) is a Hilbert space and \( \dim X \geq n \), then \( J_n(X) = \sqrt{n} \); the converse is not true in general.

Before we give the proof of these properties let us prove a useful result concerning \( n \)-James constants for finitely representable spaces. The notion of finitely representable spaces was introduced by James in [16].

A Banach space \( X \) is said to be finitely representable in a Banach space \( Y \) if, for every \( \varepsilon > 0 \) and for every finite-dimensional subspace \( X_0 \) of \( X \), there exists a subspace \( Y_0 \) of \( Y \) and an isomorphism \( T \) from \( X_0 \) onto \( Y_0 \) such that

\[
\frac{1}{1 + \varepsilon} \|x\| \leq \|Tx\| \leq (1 + \varepsilon)\|x\| \quad \text{for every } x \in X_0.
\]

It is well-known that an infinite dimensional Banach space \( X \) is \( B \)-convex if and only if \( l^1 \) is finitely representable in \( X \) (see e.g. [6, p. 262] or [18, p. 69]).

**Proposition 1.** If \( X \) is finitely representable in \( Y \), then \( J_n(X) \leq J_n(Y) \) for every \( n \geq 2 \).

**Proof.** Let \( \varepsilon > 0 \). Then there exists \( x_1, \ldots, x_n \in B_X \) such that \( J_n(X) - \varepsilon \leq \|\sum_{k=1}^n \theta_k x_k\| \) for every \( \theta_k = \pm 1 \). Let \( X_0 = [(x_k)]_{k=1}^n \). Since \( X \) is finitely representable in \( Y \), there exists a linear one-to-one operator \( T \) from \( X_0 \) into \( Y \) such that \( \frac{1}{1 + \varepsilon} \|x\| \leq \|Tx\| \leq (1 + \varepsilon)\|x\| \) for every \( x \in X_0 \).

We put \( y_k = T(\frac{1}{1 + \varepsilon} x_k) \) for \( k = 1, \ldots, n \). Then \( y_k \in B_Y \) for every \( k = 1, \ldots, n \) and for each \( \theta_k = \pm 1 \) we have that

\[
\left\| \sum_{k=1}^n \theta_k y_k \right\| \leq \frac{1}{1 + \varepsilon} \left( \sum_{k=1}^n \theta_k x_k \right) \leq \frac{1}{(1 + \varepsilon)^2} (J_n(X) - \varepsilon).
\]

Thus, \( J_n(Y) \geq \frac{1}{(1 + \varepsilon)^2} (J_n(X) - \varepsilon) \) for every \( \varepsilon > 0 \) and, hence, \( J_n(Y) \geq J_n(X) \).

**Proof.** (i) The first part is clear. If \( \dim X = \infty \) then, according to Dvoretzky’s theorem (see e.g. [7], [44]), it yields that \( l^2 \) is finitely representable in \( X \). Hence, from Proposition 1 we obtain that

\[
J_n(X) \geq J_n(l^2) \geq \min_{\theta_k = \pm 1} \left\| \sum_{k=1}^n \theta_k e_k \right\| = \sqrt{n},
\]

where \( e_k \) are unit vectors.

(ii) The first part is easy to prove by just taking zero as the \( (n + 1) \)-element. For the proof of the second part we assume that \( m, n \in \mathbb{N} \) and then for arbitrary \( \varepsilon > 0 \) there exist \( x_0^0, \ldots, x_{m+n}^0 \in B_X \) such that \( \min_{\theta_k = \pm 1} \left\| \sum_{k=1}^{m+n} \theta_k x_k^0 \right\| > J_{m+n}(X) - \varepsilon \), i.e. for any choice of signs \( \theta_k = \pm 1 \) we have

\[
\left\| \sum_{k=1}^m \theta_k x_k^0 \right\| + \left\| \sum_{k=m+1}^{m+n} \theta_k x_k^0 \right\| > J_{m+n}(X) - \varepsilon.
\]
This means that

$$\min_{\theta_k = \pm 1} \left\| \sum_{k=1}^{m} \theta_k x_k^0 \right\| > J_{m+n}(X) - \min_{\theta_k = \pm 1} \left\| \sum_{k=m+1}^{m+n} \theta_k x_k^0 \right\| - \varepsilon,$$

or

$$J_m(X) \geq \min_{\theta_k = \pm 1} \left\| \sum_{k=1}^{m} \theta_k x_k^0 \right\| > J_{m+n}(X) - \min_{\theta_k = \pm 1} \left\| \sum_{k=m+1}^{m+n} \theta_k x_k^0 \right\| - \varepsilon.$$

We have

$$J_m(X) \geq \min_{\theta_k = \pm 1} \left\| \sum_{k=1}^{m+n} \theta_k x_k^0 \right\| \geq J_{m+n}(X) - J_m(X) - \varepsilon$$

and, thus, $$J_{m+n}(X) \leq J_m(X) + J_n(X)$$ since $$\varepsilon > 0$$ was arbitrary.

(iii) For the proof see Pisier [40, pp. 2-3], Kalton [20, p. 248], Woyczyński [46, pp. 340-341], Kalton-Peck-Roberts [22, p. 99] and Diestel-Jarchow-Tonge [6, p. 261].

(iv) By the parallelogram law we have

$$\min_{\theta_k = \pm 1} \left\| \sum_{k=1}^{n} \theta_k x_k \right\|^2 = \sum_{\theta_k = \pm 1} \left\| \sum_{k=1}^{n} \theta_k x_k \right\|^2 = \sum_{\theta_k = \pm 1} \sum_{k=1}^{n} \left\| x_k \right\|^2 \leq n,$$

for $$x_1, \ldots, x_n \in B_X$$, and so $$J_n(X) \leq \sqrt{n}$$. The equality follows from (i) in the infinite-dimensional case. If $$\dim X = n$$, then $$X$$ is isometric to $$l^p_n$$ and $$J_n(X) = J_n(l^p_n) = \sqrt{n}$$, which can be seen by considering the unit vectors. The proof is complete.

Some results concerning the $$n$$-th James constants for $$L^p$$ spaces are presented in the following theorem (cf. also Proposition 4):

**THEOREM 2.** If $$1 \leq p \leq \infty$$, then $$J_n(L^p(\mu)) \leq \max(n^{1/p}, n^{1-1/p})$$. Moreover, if $$1 \leq p \leq 2$$ and $$\dim L^p(\mu) \geq n$$, then $$J_n(L^p(\mu)) = n^{1/p}$$; if $$2 < p \leq \infty$$ and $$\dim L^p(\mu) = \infty$$, then $$\sqrt{n} \leq J_n(L^p(\mu)) \leq n^{1-1/p}$$.

**Proof.** First, let $$1 \leq p \leq 2$$. In this case we have Clarkson’s inequality (cf. [5] or [33])

$$\|x + y\|_p^p + \|x - y\|_p^p \leq 2 \left(\|x\|_p^p + \|y\|_p^p \right),$$

which is valid for all $$x, y \in L^p(\mu)$$. Then, by induction,

$$\sum_{\theta_k = \pm 1} \left\| \sum_{k=1}^{n} \theta_k x_k \right\|_p^p \leq 2^n \sum_{k=1}^{n} \left\| x_k \right\|_p^p \text{ for all } x_1, \ldots, x_n \in X.$$

In fact, by the Clarkson inequality and the induction assumption (for $$n-1$$) we obtain that

$$\sum_{\theta_k = \pm 1} \left\| \sum_{k=1}^{n} \theta_k x_k \right\|_p^p = \sum_{\theta_k = \pm 1} \left\| \sum_{k=1}^{n-1} \theta_k x_k + x_n \right\|_p^p + \sum_{\theta_k = \pm 1} \left\| \sum_{k=1}^{n-1} \theta_k x_k - x_n \right\|_p^p \leq 2 \sum_{\theta_k = \pm 1} \left( \left\| \sum_{k=1}^{n-1} \theta_k x_k \right\|_p^p + \left\| x_n \right\|_p^p \right)$$
This implies that if 
and, hence, 

In fact, by the first and second estimates above and the induction we obtain that 

from which we obtain the estimate 

This implies that if \( x_1, x_2, \ldots, x_n \in B_{L_p} \), then \( \min_{\theta_k \pm 1} \| \sum_{k=1}^{n} \theta_k x_k \|_p \leq n^{1/p} \) and, hence, \( J_n(L^p(\mu)) \leq n^{1/p} \).

Since \( \dim L^p(\mu) \geq n \) we can find at least \( n \) pairwise disjoint subsets \( A_1, \ldots, A_n \) of \( \Omega \) such that \( 0 < \mu(A_k) < \infty \) for \( k = 1, \ldots, n \). Define \( x_k = \frac{x_{A_k}}{\mu(A_k)^{1/p}} \) for \( k = 1, \ldots, n \). Then, for every choice of signs \( \theta_1, \ldots, \theta_n \), we have that \( \| \sum_{k=1}^{n} \theta_k x_k \|_p = n \) and, hence, \( J_n(L^p(\mu)) \geq n^{1/p} \).

For \( 2 \leq p < \infty \) we use another Clarkson’s inequality (cf. [5] or [33])

from which we obtain the estimate

or

These estimates give the required estimate for \( n \)-elements

In fact, by the first and second estimates above and the induction we obtain that

\[
\min_{\theta_k = \pm 1} \left\| \sum_{k=1}^{n} \theta_k x_k \right\|_p = \min_{\theta_k = \pm 1} \left\{ \left\| \sum_{k=1}^{n-1} \theta_k x_k + x_n \right\|_p, \left\| \sum_{k=1}^{n-1} \theta_k x_k - x_n \right\|_p \right\} \\
\leq \min_{\theta_k = \pm 1} \left( \left\| \sum_{k=1}^{n-1} \theta_k x_k \right\|_p + \left\| x_n \right\|_p \right)^{p-1} \\
= \min_{\theta_k = \pm 1} \left[ \min \left( \left\| \sum_{k=1}^{n-2} \theta_k x_k + x_{n-1} \right\|_p + \left\| x_n \right\|_p \right)^{p-1}, \left\| \sum_{k=1}^{n-2} \theta_k x_k - x_{n-1} \right\|_p + \left\| x_n \right\|_p \right)^{p-1} \right] \\
\leq \min_{\theta_k = \pm 1} \left( \left\| \sum_{k=1}^{n-1} \theta_k x_k \right\|_p + \left\| x_n \right\|_p \right)^{p-1} \]
with the norm of the first equality can be proved by considering two extreme cases. The proof is complete.

Hence, if \( x_1, x_2, \ldots, x_n \in B_{L^p} \), then \( \min_{\theta_k = \pm 1} \| \sum_{k=1}^n \theta_k x_k \|_p \leq n^{1-1/p} \) and we conclude that \( J_n(L^p(\mu)) \leq n^{1-1/p} \).

The estimate from below \( J_n(L^p(\mu)) \geq \sqrt{n} \) follows from the property (i). The proof is complete.

**PROBLEM 1.** Find the exact formula for \( J_n(L^p(\mu)) \) when \( p > 2 \) and \( n \geq 3 \).

It is well-known (see [6]) that a Banach space \( X \) is B-convex if and only if its dual space \( X^* \) is B-convex. In this connection and in view of the estimates (v) between the James constants \( J(X) \) and \( J(X^*) \) we can ask the following question:

**PROBLEM 2.** Find some relations between \( J_n(X) \) and \( J_m(X^*) \) for \( m, n \in \mathbb{N} \) and \( n \geq 3 \).

We can also consider the \( n \)-th strong James constants of a Banach space \( X \) defined by

\[
J_n^s(X) := \sup \left\{ \min_{\theta_k = \pm 1} \left\| \sum_{k=1}^n \theta_k x_k \right\| : x_1, \ldots, x_n \in S_X \right\}.
\]

Then, obviously \( J_n^s(X) \leq J_n(X) \) and \( J_3^s(X) < J_3(X) \) for \( X = l_2^\infty \) (that is, when \( X = \mathbb{R}^2 \) with the norm of \( x = (x_1, x_2) \) equal to \( \|x\| = \max\{|x_1|, |x_2|\} \)). In fact,

\[
J_3^s(l_2^\infty) = 1 \quad \text{and} \quad J_3(l_2^\infty) \geq 2.
\]

The first equality can be proved by considering two extreme cases \( x_1 = (1, a), x_2 = (1, b), x_3 = (1, c) \) and \( x_1 = (1, a), x_2 = (1, b), x_3 = (c, 1) \) and the second estimate we obtain by taking \( x_1 = (1, 1), x_2 = (-1, 1), x_3 = (0, \varepsilon) \) with \( 0 < \varepsilon < 1 \) since then

\[
J_3(l_2^\infty) \geq \min\{\|x_1 + x_2 + x_3\|, \|x_1 + x_2 - x_3\|, \|x_1 - x_2 + x_3\|, \|x_1 - x_2 - x_3\|\}
\]

\[
= \min\{2 + \varepsilon, 2 - \varepsilon, 2\} = 2 - \varepsilon,
\]

and our claim follows by letting \( \varepsilon \to 0^+ \).

We don’t know any example of a Banach space \( X \) such that \( \dim X \geq 3 \) and \( J_3^s(X) < J_3(X) \) but we guess that only \( J_2^s(X) = J_2(X) \) for \( \dim X \geq 2 \). We easily see that \( J_n^s(l_1^m) = J_n^s(l_1^1) = n \) for \( m \geq n \) and in the paper [34] it was proved that for the Cesàro sequence spaces \( ces_p, 1 < p \leq \infty \) we have the equalities \( J_n^s(ces_p) = n \) for all natural \( n \geq 2 \), which means that they are not B-convex.

Our main result in this Section is to estimate \( J_n(X) \) by \( J_n^s(X) \) constants.
THEOREM 3. For any Banach space $X$ and $n \geq 2$ we have the estimate
\[ J_n(X) \leq \frac{1}{2} \left[ J_n^*(X) - 1 + \sqrt{(2n - J_n^*(X) - 1)^2 + 4n} \right]. \tag{7} \]

Proof. Denote for simplicity $J_n^*(X) = a$. Let $c = c(a)$ be a number from $[0, 1]$ which we will determine later on in a suitable way. For fixed $x_1, \ldots, x_n \in B_X$ consider two cases:

I. $\min_{k=1,2,\ldots,n} \|x_k\| \leq c$.

Then, for every choice of signs $\theta_k$, we have that $\| \sum_{k=1}^n \theta_k x_k \| \leq c + n - 1$.

II. $\min_{k=1,2,\ldots,n} \|x_k\| > c$.

There is a choice of signs $\theta_k$ such that $\left\| \sum_{k=1}^n \theta_k \frac{x_k}{\|x_k\|} \right\| \leq J_n^*(X) = a$. Since
\[ \left\| \sum_{k=1}^n \theta_k \left( \frac{x_k}{\|x_k\|} - x_k \right) \right\| \leq \sum_{k=1}^n \|x_k\| \left( \frac{1}{\|x_k\|} - 1 \right) \leq \sum_{k=1}^n \left( \frac{1}{c} - 1 \right) = n \left( \frac{1}{c} - 1 \right) \]

it follows that for this choice of signs
\[ \left\| \sum_{k=1}^n \theta_k x_k \right\| \leq \left\| \sum_{k=1}^n \theta_k \frac{x_k}{\|x_k\|} \right\| + \left\| \sum_{k=1}^n \theta_k \left( \frac{x_k}{\|x_k\|} - x_k \right) \right\| \leq \left\| \sum_{k=1}^n \theta_k \frac{x_k}{\|x_k\|} \right\| + n \left( \frac{1}{c} - 1 \right) \leq a + n \left( \frac{1}{c} - 1 \right). \]

Putting these two cases together we obtain that
\[ \left\| \sum_{k=1}^n \theta_k x_k \right\| \leq \max \left\{ c + n - 1, a + n \left( \frac{1}{c} - 1 \right) \right\} \]

and taking the supremum over all $x_1, \ldots, x_n \in B_X$ we get that
\[ J_n(X) \leq \max \left\{ c + n - 1, a + n \left( \frac{1}{c} - 1 \right) \right\}. \]

Denote the right hand side by $f(c)$ and let us look for the minimum of this function on $[0, 1]$. The minimum is attained at $c = c_0$ when $c + n - 1 = a + \frac{n}{c} - n$. Considering the function
\[ g(c) = c^2 - (a + 1 - 2n)c - n, \ c \in [0, 1] \]
we have that $g(c_0) = 0, g(0) = -n < 0$ and $g(1) = n - a - 1 < 0$. Thus $c_0 \in (0, 1]$; more precisely, if $a < n$, then $0 < c_0 < 1$ and if $a = n$, then $c_0 = 1$. Also $c_0 = \left[ a + 1 - 2n + \sqrt{(2n - a - 1)^2 + 4n} \right] / 2$ and, hence,
\[ J_n(X) \leq c_0 + n - 1 = \left( a - 1 + \sqrt{(2n - a - 1)^2 + 4n} \right) / 2, \]
and the proof is complete.

Immediately from Theorem 3 (since \( a < n \) implies \( c_0 < 1 \)) we see that in the
definition of uniformly non-\( l^n \) space we can have elements from the unit sphere or from
the unit ball (see also Kamińska-Turett [23, Lemma 2]).

**COROLLARY 1.** For any Banach space \( X \) and \( n \geq 2 \) it yields that \( J_n(X) < n \) if
and only if \( J^n_n(X) < n \).

We pose the following conjecture:

**CONJECTURE.** If \( J_n(X) < n \), then \( J^n_n(X) < J_n(X) \) for \( n \geq 3 \) and \( \dim X \geq 3 \).

### 3. Type and the \( n \)-th Khintchine constants of Banach spaces

For given \( n \in \mathbb{N} \), \( 0 < p, q \leq \infty \) and a Banach space \( X \), we define the \( n \)-th
Khintchine constants \( K^n_{p,q}(X) \) to be the smallest of all numbers \( C \geq 1 \) such that

\[
\left( \int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\|^q dt \right)^{\frac{1}{q}} \leq C \left( \sum_{k=1}^n \|x_k\|^p \right)^{\frac{1}{p}}
\]

for every choice \( x_1, \ldots, x_n \in X \), where \( \{r_k\}_{k=1}^n \) are the Rademacher functions. If
\( X = \mathbb{R} \) with the absolute value as the norm, then we will write \( K^n_{p,q}(\mathbb{R}) \). Moreover, for
\( p = q \) we denote these constants by \( t_{p,n}(X) \) as the numbers connected with the type
\( p \) of the space \( X \) and if \( p = q = 2 \) we denote them shortly by \( t_n(X) \) as the numbers
connected with the type 2 of the space \( X \). Note that \( t_2(X) = \sqrt{C_{NJ}(X)} \).

**REMARK 1.** For \( 0 < q < \infty \) we have equality

\[
\int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\|^q dt = \frac{1}{2^n} \sum_{\theta_k = \pm 1} \left\| \sum_{k=1}^n \theta_k x_k \right\|^q.
\]

A Banach space \( X \) is of type \( p \), \( 1 \leq p \leq 2 \), if \( T_{p,q}(X) := \sup_{n \in \mathbb{N}} K^n_{p,q}(X) < \infty \).

Pisier [39] proved in 1973 that a Banach space \( X \) has non-trivial type, i.e., is of type
\( p > 1 \) if and only if it is B-convex.

The Khintchine constants \( K^n_{p,q}(X) \) for some choices of \( p, q \) were studied by
several authors, e.g. Pisier [39], [40] in 1973, Enflo-Lindenstrauss-Pisier [8] in 1975
and Figiel-Lindenstrauss-Milman [10] in 1977 considered \( t_n(X) \), Maurey-Pisier [33]
\( t_{p,n}(X) \) for \( 1 \leq p \leq 2 \). Let us collect some properties of these constants in the next
proposition:

**PROPOSITION 2.** The following properties for the \( n \)-th Khintchine constants hold

(i) \( 1 \leq K^n_{p,q}(\mathbb{R}) \leq K^n_{p,q}(X) \leq n^{(1-1/p)_+} \).

(ii) \( K^n_{p,q}(X) \) are increasing in \( n, p, q, K^n_{1,1}(X) = 1 \) and \( K^n_{p,q}(X) \leq n^{\frac{2}{p} - \frac{1}{q} K^n_{r,q}(X)} \)
for all \( 0 < q \leq \infty, 0 < r \leq p \leq \infty \).
(iii) If $1 \leq q < \infty$, then $K_{p,q}^n(X)$ are subadditive in $n$, that is, $K_{p,q}^{m+n}(X) \leq K_{p,q}^m(X) + K_{p,q}^n(X)$ for all $m, n \in \mathbb{N}$.

(iv) If $0 < p < q < \infty$, then $K_{p,q}^n(X)$ are submultiplicative in $n$, that is, $K_{p,q}^m(X) \leq K_{p,q}^m(X) K_{p,q}^n(X)$ for all $m, n \in \mathbb{N}$.

(v) If $X$ is a Hilbert space, then $K_{p,q}^n(X) = 1$ for $0 < p, q \leq 2$; $K_{2,2}^n(X) = 1$ for $n \geq 2$ if and only if $X$ is a Hilbert space.

(vi) If $1 \leq r < \infty$ and $0 < p \leq r \leq q < \infty$, then $K_{p,q}^n(L^r(\mu)) = K_{p,q}^n(\mathbb{R})$. In particular, if $1 \leq r \leq \infty$ and $0 < p \leq r \leq q \leq 2$, then $K_{p,q}^n(L^r(\mu)) = 1$.

Proof. (i) There exists $x \in X$ with $\|x\| = 1$. Take $x_k = a_k x$, $k = 1, \ldots, n$, with arbitrary $a_k \in \mathbb{R}$. Then $K_{p,q}^n(X) \geq K_{p,q}^n(\mathbb{R})$. Moreover, for every $x_1, x_2, \ldots, x_n \in X$,

$$
\left( \int_0^1 \left\| \sum_{k=1}^n r_k(t)x_k \right\| q \, dt \right)^{1/q} \leq \sum_{k=1}^n \|x_k\| \leq n^{(1-1/p)_+} \left( \sum_{k=1}^n \|x_k\|^p \right)^{1/p}.
$$

(ii) This follows from the properties of the $\ell_p^n$ and $L^p[0,1]$ spaces.

(iii) This statement for $t_n(X)$ can be found in Pisier [40, p. 7] and for $t_{p,n}(X)$ in Woyczyński [46, p. 344]. We can easily see that the subadditivity of $K_{p,q}^n(X)$ holds if $q \geq 1$. This follows directly from the following two estimates

$$
\left( \int_0^1 \left\| \sum_{k=1}^{m+n} r_k(t)x_k \right\| q \, dt \right)^{1/q} \leq \left( \int_0^1 \left\| \sum_{k=1}^m r_k(t)x_k \right\| q \, dt \right)^{1/q} + \left( \int_0^1 \left\| \sum_{k=m+1}^{m+n} r_k(t)x_k \right\| q \, dt \right)^{1/q}
$$

and

$$
\left( \int_0^1 \left\| \sum_{k=m+1}^{m+n} r_k(t)x_k \right\| q \, dt \right)^{1/q} \leq K_{p,q}^n(X) \left( \sum_{k=m+1}^{m+n} \|x_k\|^p \right)^{1/p}.
$$

(iv) This statement for $t_n(X)$ was proved by Pisier [39, pp. 991-992], [40, pp. 7-8], Enflo-Lindenstrauss-Pisier [8, p. 200] and Figiel-Lindenstrauss-Milman [10, p. 82] (see also Beauzamy [2, p. 313], Diestel-Jarchow-Tonge [6, p. 265], Wojtaszczyk [45, p. 142]). For $t_{p,n}(X)$ it was proved by Maurey-Pisier [33, p. 71], Woyczyński [46, p. 345] and Milman-Schechtman [36, p. 86] (see also Benyamini-Lindenstrauss [3, p. 443]).

We modify the proof in [36] and [3], where the statement (iv) was proved for $p = q$, and prove it for $p \leq q$. Let $m, n \in \mathbb{N}$ and $x_1, \ldots, x_{mn} \in X$. For each $k = 1, \ldots, n$ and $t \in [0,1]$ define

$$
y_k(t) := \sum_{j=(k-1)m+1}^{km} r_j(t)x_j.
$$

Then

$$
\int_0^1 \|y_k(t)\|^q \, dt \leq K_{p,q}^m(X)^q \left( \sum_{j=(k-1)m+1}^{km} \|x_j\|^p \right)^{q/p}.
$$
The products \( \{r_k(s)r_j(t)\} \) have the same joint distribution as \( \{r_j(t)\} \). Hence, by the Minkowski inequality (since \( q/p \geq 1 \)),

\[
\int_0^1 \left\| \sum_{j=1}^{mn} r_j(t)x_j \right\|^q dt = \int_0^1 \int_0^1 \left\| \sum_{k=1}^n r_k(s)y_k(t) \right\|^q ds dt \\
\leq K^n_{p,q}(X)^q \int_0^1 \left( \sum_{k=1}^n \left\| y_k(t) \right\|^p \right)^{q/p} dt \\
\leq K^n_{p,q}(X)^q \left[ \sum_{k=1}^n \left( \int_0^1 \left\| y_k(t) \right\|^q dt \right)^{p/q} \right]^{q/p} \\
\leq K^n_{p,q}(X)^q \left\{ \sum_{k=1}^n \left[ K^m_{p,q}(X)^q \left( \sum_{j=(k-1)m+1}^{km} \left\| x_j \right\|^p \right)^{q/p} \right] \right\}^{q/p} \\
= K^n_{p,q}(X)^q K^m_{p,q}(X)^q \left( \sum_{j=1}^{mn} \left\| x_j \right\|^p \right)^{q/p},
\]

i.e., \( K^n_{p,q}(X) \leq K^n_{p,q}(X) K^n_{p,q}(X) \).

(v) If \( X \) is a Hilbert space, then \( K^n_{2,2}(X) = 1 \) and the rest of the proof follows from the properties (i) and (ii).

(vi) In fact, by using the Minkowski inequality twice we obtain that

\[
\int_0^1 \left\| \sum_{k=1}^n r_k(t)x_k \right\|^q dt = \int_0^1 \left( \int_{\Omega} \left\| \sum_{k=1}^n r_k(t)x_k(s) \right\|^r d\mu(s) \right)^{q/r} dt \\
\leq \left[ \int_{\Omega} \left( \int_0^1 \left\| \sum_{k=1}^n r_k(t)x_k(s) \right\|^q dt \right)^{r/q} d\mu(s) \right]^{q/r} \\
\leq K^n_{p,q} \left[ \int_{\Omega} \left( \sum_{k=1}^n \left| x_k(t) \right|^p \right)^{r/p} d\mu(s) \right]^{q/r} \\
\leq K^n_{p,q} \left[ \sum_{k=1}^n \left( \int_{\Omega} \left| x_k(t) \right|^r d\mu(s) \right)^{p/r} \right]^{q/p} \\
= K^n_{p,q} \left( \sum_{k=1}^n \left\| x_k \right\|_r^p \right)^{q/p}.
\]

Hence \( K^n_{p,q}(L^r(\mu)) \leq K^n_{p,q} \) and the reversed inequality follows from (i). The proof is complete.
COROLLARY 2. For any Banach space \( X \), \( 0 < p \leq q < \infty \) and \( n \geq 2 \) we have
\[
K_{p,q}^{n+1}(X) \leq K_{p,q}^{2n}(X) K_{p,q}^n(X) \quad \text{and} \quad K_{p,q}^n(X) \leq [K_{p,q}^2(X)]^{n-1}. \quad (9)
\]

In fact, using properties (ii) and (iv) in Proposition 2 we obtain that
\[
K_{p,q}^{n+1}(X) \leq K_{p,q}^{2n}(X) \leq K_{p,q}^2(X) K_{p,q}^n(X)
\]
and then
\[
K_{p,q}^n(X) \leq K_{p,q}^2(X) K_{p,q}^{n-1}(X) \leq [K_{p,q}^2(X)]^{2} K_{p,q}^{n-2}(X) \leq \ldots \leq [K_{p,q}^2(X)]^{n-1}.
\]

PROBLEM 3. Is \( K_{p,q}^n(X) \) a submultiplicative function of \( n \) for \( p > q > 0 \)?

Let us note that Pisier [41] proved the equality \( K_{2,2}^{n}(p_{m}^0) = [\min(m,n)]^{1/p-1/2} \) for \( 1 \leq p \leq 2 \) and by using this equality he proved that for any \( n \geq 2 \) and any \( C \in [1, \sqrt{n}] \) there exists a Banach space \( X \) such that \( K_{2,2}^{n}(X) = C \).

4. An estimate of the \( n \)-th Khintchine constants by the \( n \)-th James constants

Our main result is an estimate of the \( n \)-th Khintchine constants by the \( n \)-th James constants. For the proof we need the following crucial lemma (corresponding to Lemma 1), which has been proved in Kutzarova-Nikolova-Zachariades [29, Lemma 6].

LEMMA 2. Let \( X \) be a normed space and \( n \geq 2 \). Then, for every \( x_1, x_2, \ldots, x_n \in X \), with \( ||x_n|| \leq ||x_k|| \) for \( k = 1, 2, \ldots, n - 1 \), there exist \( \theta_1, \ldots, \theta_n \in \{-1, 1\} \) such that
\[
\left\| \sum_{k=1}^{n} \theta_k x_k \right\| \leq \sum_{k=1}^{n-1} ||x_k|| + [J_{n}^{p}(X) - n + 1] ||x_n||.
\]

Proof (see also [29]). If \( x_n = 0 \) the statement in the lemma is clear. Let \( x_n \neq 0 \). Then there exists a choice of signs \( \theta_1, \ldots, \theta_n \) such that \( \left\| \sum_{k=1}^{n} \theta_k \frac{x_k}{||x_k||} \right\| \leq J_{n}^{p}(X) \) and, hence, we have that
\[
\left\| \sum_{k=1}^{n} \theta_k x_k \right\| = \left\| \sum_{k=1}^{n} \theta_k \left( 1 - \frac{||x_n||}{||x_k||} \right) x_k + \sum_{k=1}^{n} \theta_k \frac{x_n}{||x_k||} x_k \right\|
\]
\[
\leq \sum_{k=1}^{n} \left( 1 - \frac{||x_n||}{||x_k||} \right) ||x_k|| + ||x_n|| \left\| \sum_{k=1}^{n} \theta_k \frac{x_k}{||x_k||} \right\|
\]
\[
\leq \sum_{k=1}^{n-1} ||x_k|| + [J_{n}^{p}(X) - n + 1] ||x_n||.
\]

Our main result reads:
THEOREM 4. Let $X$ be a Banach space $X$, $1 \leq p, q \leq \infty$ and $n \geq 2$. Then

\[
\frac{J_n(X)}{n^\frac{1}{p}} \leq K_{p,q}^n(X) \leq \frac{1}{2}\left[\frac{(n-1)p'}{q'}(n-1) + \frac{q'}{q}c_n^n\right]^{\frac{1}{p'}},
\]

where $c_n = a_n^q + 2^{n-1} - 1$ and $a_n = [J_n(X) - n + 1]_+$. For $p = 1$ the right hand side in (10) if it is as usual interpreted is equal to 1.

Proof. It is clear that

\[
\min_{\theta_k = \pm 1} \left\| \sum_{k=1}^n \theta_k x_k \right\| \leq \left( \frac{1}{2^n} \sum_{\theta_k = \pm 1} \left\| \sum_{k=1}^n \theta_k x_k \right\|^q \right)^{\frac{1}{q}} \leq K_{p,q}^n(X) \left( \sum_{k=1}^n \|x_k\|^p \right)^{\frac{1}{p'}} \leq K_{p,q}^n(X) n^\frac{1}{p'},
\]

which gives the left hand side inequality of (10). It is also clear that $J_n(X) \leq K_{\infty,q}^n(X)$.

To prove the upper estimate in (10) we may suppose, without loss of generality, that $x_1, \ldots, x_n \in X$ are not all zero, $\|x_n\| \leq \|x_k\|$ for every $k = 2, 3, \ldots, n-1$ and $\sum_{k=1}^{n-1} \|x_k\| = 1$.

Let $1 < p < \infty$ and $1 \leq q < \infty$. By using Lemma 2 and Minkowski’s inequality, we obtain that

\[
\frac{1}{2^n} \left[ \sum_{\theta_k = \pm 1} \left\| \sum_{k=1}^n \theta_k x_k \right\|^q \right]^{\frac{1}{q}} \leq \left[ \left( \sum_{k=1}^{n-1} \|x_k\| + a_n \|x_n\| \right)^q + (2^{n-1} - 1) \left( \sum_{k=1}^{n-1} \|x_k\| + \|x_n\| \right)^q \right]^{\frac{1}{q}} \leq 2^{n-1} \left[ \sum_{k=1}^{n-1} \|x_k\|^q \right]^{\frac{1}{q}} + \left[ a_n^q \|x_n\|^q + (2^{n-1} - 1) \|x_n\|^q \right]^{\frac{1}{q}} = 2^{\frac{n-1}{q}} + \frac{c_n^q}{q} \|x_n\|.
\]

Hence

\[
K_{p,q}^n(X) \leq \frac{2^{\frac{n-1}{q}} + \frac{c_n^q}{q} \|x_n\|}{2^{\frac{n-1}{q}} \left( \sum_{k=1}^n \|x_k\|^p \right)^{\frac{1}{p'}}}.
\]

Moreover, according to Hölder’s inequality,

\[
1 = \sum_{k=1}^{n-1} \|x_k\| \leq (n-1)^{\frac{1}{p'}} \left( \sum_{k=1}^{n-1} \|x_k\|^p \right)^{\frac{1}{p'}},
\]
where $\frac{1}{p} + \frac{1}{p'} = 1$, and, thus, $\sum_{k=1}^{n-1} \|x_k\|^{p'} \geq (n-1)^{1-p}$. We conclude that

$$K_{p,q}^n(X) \leq \frac{2^{\frac{n-1}{q}} + c_n^q \|x_n\|}{2^{\frac{n-1}{q}} \left(\|x_n\|^p + (n-1)^{1-p}\right)^\frac{1}{p}}.$$  

We consider the function $f(t) = \frac{2^{\frac{n-1}{q}} + c_n^q t}{2^{\frac{n-1}{q}} \left(t^p + (n-1)^{1-p}\right)^\frac{1}{p}}$ for $t \geq 0$. It is easy to see that $f(t) \leq f(t_0)$, where $t_0 = \frac{\frac{q}{p} c_n^q}{(n-1)^{2-p}}$. Thus, we obtain that

$$K_{p,q}^n(X) \leq \frac{1}{2^{\frac{n-1}{q}}} \left[2^{\frac{n-1+p'}{q}} (n-1) + c_n^q \right]^\frac{1}{p}.$$  

Therefore the right hand side inequality of (10) holds.

If $p = 1$, then the estimate $K_{1,q}^n(X) \leq 1$ is clear and the right hand side is equal to $2^{(1-n)/q} \max\{2^{(n-1)/q}, \ldots, 2^{(n-1)/q}, c_n^1\} = 1$.

If $p = \infty$, then $p' = 1$ and $1 = \sum_{k=1}^{n-1} \|x_k\| \leq (n-1) \max_{k=1,\ldots,n-1} \|x_k\|$ from which it follows that

$$\max_{k=1,\ldots,n} \|x_k\| = \max_{k=1,\ldots,n-1} \|x_k\| \geq \frac{1}{n-1}$$

and, thus,

$$K_{\infty,q}^n(X) \leq \frac{2^{\frac{n-1}{q}} + \frac{1}{q} \|x_n\|}{2^{\frac{n-1}{q}} \max_{k=1,\ldots,n} \|x_k\|} \leq \frac{2^{\frac{n-1}{q}} + \frac{1}{q} \|x_n\|}{2^{\frac{n-1}{q}} \frac{1}{n-1}}.$$  

This gives the required estimate $K_{\infty,q}^n(X) \leq n - 1 + \frac{c_n^q}{2^{\frac{n-1}{q}}}$.

If $q = \infty$, then

$$\max_{\theta_k = \pm 1} \left\|\sum_{k=1}^{n} \theta_k x_k\right\| \leq \sum_{k=1}^{n-1} \|x_k\| + \|x_n\| = 1 + t,$$

and

$$K_{p,\infty}^n(X) \leq \frac{1 + t}{\sum_{k=1}^{n} \|x_k\|^{p-1}/p} \leq \frac{1 + t}{\left[\sum_{k=1}^{n} \|x_k\|^{p-1}/p + (n-1)^{1/p}\right]} := g(t).$$

The function $g$ has maximum at $t_0 = \frac{1}{n-1}$ and, hence, $g(t) \leq g(t_0) = n^{1/p'}$, which is again the right hand side of (10) and the proof is complete.

We will now point out some direct consequences of Theorem 4.

**Corollary 3.** If $n \geq 2$, then

$$\frac{J_n(X)}{\sqrt{n}} \leq t_n(X) = K_{2,2}^n(X) \leq 2^{\frac{1-n}{2}} \left\{2^{n-1} n - 1 + [J_n^2(X) - n + 1] \right\}.$$  

In particular, \( \frac{J(X)^2}{2} \leq C_{NJ}(X) = K_{2,2}^2(X)^2 \leq \frac{J(X)^2}{2} + 2 - J(X) \).

Pisier [39] proved that a Banach space \( X \) is uniformly non-\( l_n^1 \) if and only if \( K_{2,2}^n(X) < \sqrt{n} \). By using the the estimates (10) we can state similar result for the \( n \)-th Khintchine constants \( K_{p,q}^n(X) \).

**Corollary 4.** An infinite dimensional Banach space \( X \) is uniformly non-\( l_n^1 \) if and only if \( K_{p,q}^n(X) < n^{1/p'} \) for \( 1 < p \leq \infty, 1 \leq q < \infty \).

**Corollary 5.** Let \( X \) be a Banach space. For \( n \geq 2 \) and \( 1 < p \leq \infty, 1 \leq q < \infty \) fixed we have that \( J_n(X) = n \) if and only if \( K_{p,q}^n(X) = n^{1/p'} \).

The assertion of the following proposition is known for \( p = 1 \) and means that the Banach space \( X \) is B-convex (cf. [14] and [6]). Note that a Banach space \( X \) is B-convex if and only if \( \lim_{n \to \infty} \frac{J_n(X)}{n^r} = 0 \). We extend this result to \( 1 \leq p < 2 \).

**Proposition 3.** Let \( X \) be an infinite dimensional Banach space and \( 1 \leq p < 2 \). The following conditions are equivalent

(i) \( X \) is of type strictly bigger than \( p \).

(ii) \( \lim_{n \to \infty} \frac{J_n(X)}{n^{1/p}} = 0 \).

(iii) \( \inf_{n \geq 2} \frac{J_n(X)}{n^{1/p}} < 1 \).

(iv) \( p \) is not finitely representable in \( X \).

**Proof.** (i) \( \Rightarrow \) (ii). Let \( X \) be of type \( r \) for some \( p < r \leq 2 \). Then, according to the first estimate in (10), we obtain that \( \sup_{n \geq 2} \frac{J_n(X)}{n^r} < \infty \). Hence, \( \lim_{n \to \infty} \frac{J_n(X)}{n^r} = \lim_{n \to \infty} \frac{J_n(X)}{n^r} \cdot \frac{n^{r - \frac{1}{p}}}{n^{r - \frac{1}{p}}} = 0 \).

(ii) \( \Rightarrow \) (iii). This implication is obvious.

(iii) \( \Rightarrow \) (iv). Since \( J_n(p^r) = n^{1/p} \) we conclude that \( p^r \) is not finitely representable in \( X \). In fact, if it is so, then by Proposition 1 we will have that \( n^{1/p} = J_n(p^r) \leq J_n(X) \) for every \( n \), which contradicts the assumption (iii).

(iv) \( \Rightarrow \) (i). We use the Maurey-Pisier theorem (cf. [33]): If \( X \) is an infinite dimensional Banach space, then \( p^r(X) \) is finitely representable in \( X \) and even more: \( l^r \) is finitely representable in \( X \) for every \( r \in [p(X), 2] \), where

\[ p(X) := \sup\{p \geq 1 : X \text{ is of type } p\}. \]

From this fact we conclude that \( p < p(X) \) and, thus, \( X \) is of type strictly bigger than \( p \). The proof is complete.

**Corollary 6.** If \( X \) be an infinite dimensional Banach space, \( 1 < p \leq 2 \) and \( \sup_{n \geq 2} \frac{J_n(X)}{n^{1/p}} < \infty \), then \( X \) is of type \( r \) for every \( 1 < r < p \).

**Proof.** The result follows from Proposition 3 since for \( 1 < r < p \) we have that \( \lim_{n \to \infty} \frac{J_n(X)}{n^{1/r}} = \lim_{n \to \infty} \frac{J_n(X)}{n^{1/p}} n^{1/p - 1/r} = 0 \).
THEOREM 5. If $X$ is an infinite dimensional Banach space, then

$$
\lim_{n \to \infty} \frac{\ln n}{\ln J_n(X)} = \sup_{n \geq 2} \frac{\ln n}{\ln J_n(X)} = p(X).
$$

Proof. Define $l(X) := \sup_{n \geq 2} \frac{\ln n}{\ln J_n(X)}$ and note also that from the estimates $\sqrt{n} \leq J_n(X) \leq n$ we obtain that $1 \leq \frac{\ln n}{\ln J_n(X)} \leq 2$ and, hence, $1 \leq l(X) \leq 2$.

We suppose that $p(X) < l(X)$. Let $p(X) < r < l(X)$. Then there exists $m \geq 2$ such that $r < \frac{\ln m}{\ln J_m(X)}$. We conclude that $\frac{J_m(X)}{m^r} < 1$ and, thus, according to Proposition 3, $X$ is of type bigger than $r$ which is a contradiction. Hence $l(X) \leq p(X)$. Now we suppose that $\liminf_{n \to \infty} \frac{\ln n}{\ln J_n(X)} < p(X)$. Let $\liminf_{n \to \infty} \frac{\ln n}{\ln J_n(X)} = r$ such that $r < 1$ and, hence, $(\sqrt{n})^{\frac{1}{n}}$ which means that $(\frac{J_n(X)}{n^{\frac{1}{n}}})$ does not converge to 0. Thus, again by using Proposition 3, we find that $X$ is of type bigger than $s$ which is a contradiction. Hence $p(X) \leq \liminf_{n \to \infty} \frac{\ln n}{\ln J_n(X)}$ and so

$$
p(X) \leq \liminf_{n \to \infty} \frac{\ln n}{\ln J_n(X)} \leq \limsup_{n \to \infty} \frac{\ln n}{\ln J_n(X)} \leq \sup_{n \geq 2} \frac{\ln n}{\ln J_n(X)} = l(X) \leq p(X),
$$

which means that

$$
\lim_{n \to \infty} \frac{\ln n}{\ln J_n(X)} = \sup_{n \geq 2} \frac{\ln n}{\ln J_n(X)} = p(X)
$$

and the proof is complete.

We should mention here that the first equality in (11) follows also from the well-known property of submultiplicative sequences. In fact, if $\{a_n\}$ is a submultiplicative sequence, then $\lim_{n \to \infty} \frac{\ln a_n}{\ln n}$ exists and is equal to $\inf_{n \geq 2} \frac{\ln a_n}{\ln n}$.

Note that Woyczyński [46, p. 347] proved the following related result: if $X$ is an infinite dimensional Banach space and $0 < p < \infty$, then $p(X) = \lim_{n \to \infty} \frac{\ln n}{\ln [n^{1/p} K_{p,2}^n(X)]}$, and Milman-Schechtman [36, p. 87] for $p(X) \leq p \leq 2$ that

$$
p(X) = \lim_{n \to \infty} \frac{\ln n}{\ln [n^{1/p} K_{p,2}^n(X)]}
$$

or, equivalently,

$$
\lim_{n \to \infty} \frac{\ln K_{p,2}^n(X)}{\ln n} = \frac{1}{p(X)} - \frac{1}{p}.
$$

PROPOSITION 4. If $p \geq 2$, then

$$
J_n(L^p(\mu)) \leq \min \left[ n^{1-1/p}, \left( \int_0^1 \left( \sum_{k=1}^n r_k(t) \right)^p \, dt \right)^{1/p} \right].
$$
The first estimate was proved already in Theorem 2. The other estimate follows from Theorem 4 and the Figiel-Iwaniec-Pelczyński estimate [9]:

\[ J_n(L_p(\mu)) \leq n^{1/p} K^n_{p,p}(L_p(\mu)) = K^n_{p,p}(\mathbb{R}) \leq \left( \int_0^1 \left| \sum_{k=1}^n r_k(t) \right|^p dt \right)^{1/p}. \]

Note that for \( n = 2 \) and \( n = 3 \) the constant \( n^{1-1/p} \) is smaller than \( (\int_0^1 \left| \sum_{k=1}^n r_k(t) \right|^p dt)^{1/p} \) and for large \( n \) we have reverse inequality. Moreover, \( \lim_{n \to \infty} \frac{J_n(L_p)}{n} = \frac{1}{2} \).

5. Banach-Mazur distance and stability under norm perturbations

For isomorphic Banach spaces \( X \) and \( Y \), the Banach-Mazur distance between \( X \) and \( Y \), denoted by \( d(X, Y) \), is defined to be the infimum of \( \| T \| \| T^{-1} \| \) taken over all bicontinuous linear operators \( T \) from \( X \) onto \( Y \) (cf. [39]).

We follow considerations in the paper by Kato-Maligranda-Takahashi [25], where the results for \( n = 2 \) were proved.

**Theorem 6.** If \( X \) and \( Y \) are isomorphic Banach spaces, then for any \( n \geq 2 \)

\[
\frac{J_n(X)}{d(X, Y)} \leq J_n(Y) \leq J_n(X)d(X, Y) \quad \text{and} \quad \frac{K^n_{p,q}(X)}{d(X, Y)} \leq K^n_{p,q}(Y) \leq K^n_{p,q}(X)d(X, Y).
\]

In particular, if the spaces \( X \) and \( Y \) are isometric, then for any \( n \geq 2 \) \( J_n(X) = J_n(Y) \) and \( K^n_{p,q}(X) = K^n_{p,q}(Y) \).

**Proof.** Let \( x_1, \ldots, x_n \in B_X \) be arbitrary. For each \( \epsilon > 0 \) there exists an isomorphism \( T \) from \( X \) onto \( Y \) such that \( \| T \| \| T^{-1} \| \leq (1 + \epsilon)d(X, Y) \). Put

\[ y_k = \frac{T x_k}{\| T \|}, \quad k = 1, \ldots, n. \]

Then \( y_k \in B_Y, k = 1, \ldots, n \) since \( \| y_k \| = \frac{\| T x_k \|}{\| T \|} \leq \| x_k \| \leq 1 \) and \( x_k = \| T \|^{-1} (y_k) \).

We obtain

\[
\min_{\theta_k = \pm 1} \left\| \sum_{k=1}^n \theta_k x_k \right\|_X = \| T \| \min_{\theta_k = \pm 1} \left\| T^{-1} \left( \sum_{k=1}^n \theta_k y_k \right) \right\|_X
\]

\[
\leq \| T \| \| T^{-1} \| \min_{\theta_k = \pm 1} \left\| \sum_{k=1}^n \theta_k y_k \right\|_Y
\]

\[
\leq \| T \| \| T^{-1} \| J_n(Y) \leq (1 + \epsilon)d(X, Y)J_n(Y),
\]

and, since \( x_1, \ldots, x_n \in B_X \) were arbitrary, \( J_n(X) \leq (1 + \epsilon)d(X, Y)J_n(Y) \), which gives the first estimate. The second estimate follows by just interchanging \( X \) and \( Y \). The other two estimates for \( K^n_{p,q}(\cdot) \) constants can be proved similarly.

**Corollary 7.** If \( X \) and \( Y \) are isomorphic Banach spaces and \( X \) is B-convex, then also \( Y \) is B-convex.
COROLLARY 8. Let $X = (X, \| \cdot \|)$ be a non-trivial Banach space and let $X_1 = (X, \| \cdot \|_1)$, where $\| \cdot \|_1$ is an equivalent norm on $X$ satisfying, for some $a, b > 0$ and all $x \in X, a\|x\| \leq \|x\|_1 \leq b\|x\|$. Then

$$\frac{a}{b} J_n(X) \leq J_n(X_1) \leq \frac{b}{a} J_n(X) \quad \text{and} \quad \frac{a}{b} K^n_{p,q}(X) \leq K^n_{p,q}(X_1) \leq \frac{b}{a} K^n_{p,q}(X).$$

The proof follows immediately from Theorem 6 and the fact that $d(X, X_1) \leq b/a$.

We illustrate the above corollary by the following example:

EXAMPLE 1. For $1 \leq p \leq 2$ and $\lambda \geq 1$ let $X_{\lambda,p}$ be the space $L^p[0,1]$ with the norm $\|x\|_{\lambda,p} = \max\{\|x\|_p, \lambda \|x\|_1\}$. Then

$$J_n(X_{\lambda,p}) = \min\{n, \lambda n^{1/p}\}, \quad \text{and} \quad K^n_{p,p}(X_{\lambda,p}) = \min\{n^{-1/p}, \lambda\}.$$

The inequalities from above follow from the estimates $\|x\|_p \leq \|x\|_{\lambda,p} \leq \lambda \|x\|_p$ for all $x \in L^p$ and Corollary 8. The equalities we are getting by taking functions $x_{k,n} = a\chi_{\left(\frac{k}{n}, \frac{k+1}{n}\right)}$ for $k = 1, 2, \ldots, n, n = 2, 3, \ldots$ with $a = \min(n^{1/p}, n/\lambda)$, since

$$\|x_{k,n}\|_{\lambda,p} = \max\{\frac{a}{n^{1/p}}, \lambda \frac{a}{n}\} = \frac{a}{\min(n^{1/p}, n/\lambda)} = 1$$

and

$$\sum_{\theta_k = \pm 1} \left\| \sum_{k=1}^n \theta_k x_{k,n} \right\|_{\lambda,p} = \max(a, \lambda a) = \lambda a.$$

Thus $J_n(X_{\lambda,p}) \geq \lambda a$ and $K^n_{p,p}(X_{\lambda,p}) \geq \frac{a^2}{n^{1/p}} = \min\{\lambda, n^{-1/p}\}$, which means we have estimates from below, and consequently equalities.

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