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Review

A short proof of some recent results related to Cesàro function spaces

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Abstract

We give a short proof of the recent results that, for every $1 \leq p < \infty$, the Cesàro function space $Ces_p(I)$ is not a dual space, has the weak Banach–Saks property and does not have the Radon–Nikodym property.

Keywords: Cesàro function spaces; Weak Banach–Saks property; Bochner function spaces; Subspaces; Dual Banach space; Radon–Nikodym property

Contents

The main purpose of this paper is to give a short proof of the following recent results related to the Cesàro function spaces: for every $1 \leq p < \infty$ the space $\mathrm{Ces}_p = \mathrm{Ces}_p(I)$ has the weak Banach–Saks property [2, Theorem 8], does not have the Radon–Nikodym property and it is not a dual space [10, Corollaries 5.1 and 5.5].

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The Cesàro function spaces $\mathrm{Ces}_p = \mathrm{Ces}_p(I)$ $(1 \le p < \infty)$, where $I = [0, 1]$ or $I = [0, \infty)$, are the classes of all Lebesgue measurable real functions f on I such that

$$\|f\|_{C(p)} = \left[\int_I \left(\frac{1}{x}\int_0^x |f(t)|\,dt\right)^p dx\right]^{1/p} < \infty.$$

A Banach space X is said to have the *weak Banach–Saks property* if every weakly null sequence in X, say (x_n), contains a subsequence (x_{n_k}) whose first arithmetical means converge strongly to zero, that is, $\lim_{m\to\infty} \frac{1}{m}\left\|\sum_{k=1}^m x_{n_k}\right\|_X = 0$.

It is known that uniformly convex spaces, c_0, l^1 and L^1 have the weak Banach–Saks property. We should mention that the result on L^1 space, proved by Szlenk [17] in 1965, was a very important break-through in studying the weak Banach–Saks property. On the other hand, spaces $C[0, 1]$, l^∞ and $L^\infty[0, 1]$ do not have the weak Banach–Saks property (see Schreier [16] and Diestel [7, p. 85]).

In 1982, Rakov [15, Theorem 1] proved that a Banach space with non-trivial type (or equivalently B-convex) has the weak Banach–Saks property (cf. also Tokarev [18, Theorem 1]). Recently Dodds–Semenov–Sukochev [8] investigated the weak Banach–Saks property of rearrangement invariant spaces and Astashkin–Sukochev [3] have got a complete description of Marcinkiewicz spaces with the latter property.

The spaces $\mathrm{Ces}_p[0, 1]$ for $1 \le p < \infty$ are neither B-convex (they have trivial type) nor rearrangement invariant. Nevertheless, by studying the dual space $\mathrm{Ces}_p[0, 1]^*$, Astashkin–Maligranda [2, Theorem 8] proved that these spaces have the weak Banach–Saks property. Here, we present another simpler proof of this result which does not use any knowledge of the structure of the latter dual space.

Theorem 1. *For every* $1 \le p < \infty$ *the Cesàro function space* $\mathrm{Ces}_p(I)$ *has the weak Banach–Saks property.*

The proof will be based on the following simple observation. Recall that *the space with mixed norm* $L^p(I)[L^1[0, 1]]$ consists of all classes of Lebesgue measurable functions $x(s, t)$ on $I \times [0, 1]$ such that for a.e. $s \in I$ the function $x(s, \cdot) \in L^1[0, 1]$ and the function $\|x(s, \cdot)\|_{L^1[0,1]} \in L^p(I)$ with the norm $\|x\|_{L^p(I)[L^1[0,1]]} = \|\, \|x(s, \cdot)\|_{L^1[0,1]} \|_{L^p(I)}$ (see, for example, [11, Section 11.1, p. 400]).

Lemma 2. *For every* $1 \le p < \infty$ *the space* $\mathrm{Ces}_p(I)$ *is isometric to a closed subspace of the mixed norm space* $L^p(I)[L^1[0, 1]]$.

Proof. In fact, the mapping $f(t) \longmapsto \mathcal{S}f(x, t) = f(xt)$ is such an isometry from $\mathrm{Ces}_p(I)$ into $L^p(I)[L^1[0, 1]]$ since

$$\begin{aligned}\|f\|_{C(p)} &= \left\|\frac{1}{x}\int_0^x |f(t)|\,dt\right\|_{L^p(I)} = \left\|\int_0^1 |f(tx)|\,dt\right\|_{L^p(I)} \\ &= \left\|\, \|\mathcal{S}f(x, \cdot)\|_{L^1[0,1]}\right\|_{L^p(I)}. \quad \square\end{aligned}$$

Proof of Theorem 1. Firstly, we note that the Bochner vector-valued Banach space $L^p(I, L^1[0, 1])$ coincides with the mixed norm space $L^p(I)[L^1[0, 1]]$ (see [9, Theorem 1.1], [5, Theorem 2.2]; cf. also [13, pp. 282–283]). Moreover, by the Szlenk theorem [17], the space $L^1(I)[L^1[0, 1]] = L^1(I \times [0, 1])$ has the weak Banach–Saks property. Therefore, applying the

Cembranos theorem [6, Theorem C] (see also [12, pp. 295–302]), we see that the same is true also for the space $L^p(I)[L^1[0,1]]$. Note also that, by this theorem, the Bochner vector-valued Banach space $L^1(\mu, X)$ (equivalently, $L^p(\mu, X)$, $1 < p < \infty$), where X is a Banach space, has the weak Banach–Saks property if and only if X has the weak Komlós property. The latter means that for any weakly null sequence (f_n) in $L^1(\mu, X)$ there exist a subsequence (f_{n_k}) and $f \in L^1(\mu, X)$ such that for any further subsequence (h_k) of (f_{n_k}) the sequence $\frac{1}{m}\sum_{k=1}^m h_k$ converges to f μ-almost everywhere. Since, due to Lemma 2, the Cesàro function space $Ces_p(I)$ is isometric to a closed subspace of $L^p(I)[L^1[0,1]]$ and any closed subspace inherits the weak Banach–Saks property, then $\mathrm{Ces}_p(I)$ has this property as well. The proof is complete. □

The following results were proved in [10] (see Corollaries 5.1 and 5.5) by using an isometric representation of the dual space of $\mathrm{Ces}_p(I)$, $1 \le p < \infty$. Here, we show that they are rather simple consequences of well-known classical theorems.

Theorem 3. *Let $1 \le p < \infty$. Then*

(a) $\mathrm{Ces}_p(I)$ *is not a dual space;*
(b) $\mathrm{Ces}_p(I)$ *does not have the Radon–Nikodym property.*

Firstly, we prove the following auxiliary statement.

Lemma 4. *For every $1 \le p < \infty$ there is a norm $\|\cdot\|^*_{C(p)}$ equivalent to the usual norm in $\mathrm{Ces}_p(I)$ such that the space $(\mathrm{Ces}_p(I), \|\cdot\|^*_{C(p)})$ contains a closed subspace isometric to the space $L^1[0,1]$.*

Proof. For arbitrary $f \in \mathrm{Ces}_p := \mathrm{Ces}_p[0,1]$ we set

$$\|f\|^*_{C(p)} := \left\|f \cdot \chi_{[0,1/4]\cup[3/4,1]}\right\|_{C(p)} + \|f \cdot \chi_{(1/4,3/4)}\|_{L^1}.$$

Since

$$\begin{aligned}\left\|f \cdot \chi_{(1/4,3/4)}\right\|^p_{C(p)} &= \int_{1/4}^{3/4}\left(\frac{1}{x}\int_{1/4}^{x}|f(s)|\,ds\right)^p dx + \int_{3/4}^{1}\left(\frac{1}{x}\int_{1/4}^{3/4}|f(s)|\,ds\right)^p dx \\ &\le \left(\frac{1}{2}4^p + \frac{1}{4}\left(\frac{4}{3}\right)^p\right)\left(\int_{1/4}^{3/4}|f(s)|\,ds\right)^p \le 4^p\|f \cdot \chi_{(1/4,3/4)}\|^p_{L^1},\end{aligned}$$

we have

$$\|f\|_{C(p)} \le 4\|f\|^*_{C(p)} \quad (f \in \mathrm{Ces}_p).$$

Conversely,

$$\left\|f \cdot \chi_{(1/4,3/4)}\right\|^p_{C(p)} \ge \int_{3/4}^{1}\left(\frac{1}{x}\int_{1/4}^{3/4}|f(s)|\,ds\right)^p dx \ge \frac{1}{4}\|f \cdot \chi_{(1/4,3/4)}\|^p_{L^1},$$

whence for every $f \in \mathrm{Ces}_p$

$$\|f\|^*_{C(p)} \le \|f\|_{C(p)} + \|f \cdot \chi_{(1/4,3/4)}\|_{L^1} \le 5\|f\|_{C(p)}.$$

Therefore, the norms $\|\cdot\|^*_{C(p)}$ and $\|\cdot\|_{C(p)}$ are equivalent on $\mathrm{Ces}_p := \mathrm{Ces}_p[0,1]$. Since the mapping

$$f(t) \longmapsto \mathcal{H}f(t) := \begin{cases} 2f(2t - 1/2) & \text{if } 1/4 < t < 3/4 \\ 0 & \text{if } 0 \le t \le 1/4 \text{ or } 3/4 \le t \le 1, \end{cases}$$

is a linear isometry from $L^1[0,1]$ onto the subspace of $(\mathrm{Ces}_p[0,1], \|\cdot\|^*_{C(p)})$ consisting of all functions with support from the interval $(1/4, 3/4)$, we obtain the result in the case $I = [0,1]$. If $I = [0,\infty)$ the proof follows in the same way. □

Proof of Theorem 3. Assume that (a) is not valid and Ces_p is a dual space. Since this property is invariant with respect to isomorphisms (i.e., one-to-one linear operators T such that both T and its inverse T^{-1} are bounded), we see that $(\mathrm{Ces}_p(I), \|\cdot\|^*_{C(p)})$, where $\|\cdot\|^*_{C(p)}$ is the norm from Lemma 4, is also a dual space. Then, since it is separable, by the classical Bessaga–Pełczyński result [4], the space $(\mathrm{Ces}_p(I), \|\cdot\|^*_{C(p)})$ has the Krein–Milman property (i.e., every closed bounded set in this space is the closed convex hull of its extreme points). Since $(\mathrm{Ces}_p(I), \|\cdot\|^*_{C(p)})$ contains a closed subspace isometric to the space $L^1[0,1]$, the latter contradicts to the fact that the closed unit ball in $L^1[0,1]$ has no extreme points. Therefore, (a) is proved.

It is well-known that every Banach space which has the Radon–Nikodym property possesses also the Krein–Milman property [7, Theorem 6.5.1] (see also [1, p. 118], [14, p. 229] and the references given there). Thus, we obtain (b), and the proof is complete. □

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References

[1] F. Albiac, N.J. Kalton, Topics in Banach Space Theory, in: Graduate Texts in Mathematics, vol. 233, Springer, New York, 2006.
[2] S.V. Astashkin, L. Maligranda, Structure of Cesàro function spaces, Indag. Math. (N.S.) 20 (3) (2009) 329–379.
[3] S.V. Astashkin, F.A. Sukochev, Banach–Saks property in Marcinkiewicz spaces, J. Math. Anal. Appl. 336 (2) (2007) 1231–1258.
[4] C. Bessaga, A. Pełczyński, On extreme points in separable conjugate spaces, Israel J. Math. 4 (1966) 262–264.
[5] A.V. Bukhvalov, Spaces with mixed norm, Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. 19 (1973) 5–12; English transl. Vestn. Leningr. Univ., Math. 6 (1979) 303–311.
[6] P. Cembranos, The weak Banach–Saks property on $L^p(\mu, E)$, Math. Proc. Cambridge Philos. Soc. 115 (2) (1994) 283–290.
[7] J. Diestel, Geometry of Banach Spaces, Springer-Verlag, Berlin, Heidelberg, New York, 1975.
[8] P.G. Dodds, E.M. Semenov, F.A. Sukochev, The Banach–Saks property in rearrangement invariant spaces, Studia Math. 162 (3) (2004) 263–294.
[9] H.W. Ellis, A note on Banach function spaces, Proc. Amer. Math. Soc. 9 (1958) 75–81.
[10] A. Kamińska, D. Kubiak, On the dual of Cesàro function spaces, Nonlinear Anal. 75 (5) (2012) 2760–2773.
[11] L.V. Kantorovich, G.P. Akilov, Functional Analysis, Nauka, Moscow, 1977. English transl. Pergamon Press, Oxford, Elmsford, New York 1982.
[12] P.-K. Lin, Köthe-Bochner Function Spaces, Birkhäuser, Boston, 2004.
[13] L. Maligranda, Calderón-Lozanovskiĭ construction for mixed norm spaces, Acta Math. Hungar. 103 (4) (2004) 279–302.
[14] A. Pietsch, History of Banach Spaces and Linear Operators, Birkhäuser, Boston, 2007.
[15] S.A. Rakov, The Banach–Saks exponent of some Banach spaces of sequences, Mat. Zametki 32 (5) (1982) 613–625; English transl. Math. Notes 32 (5) (1982) 791–797.
[16] J. Schreier, Ein Gegenbeispiel zur Theorie der schwachen Konvergenz, Studia Math. 2 (1930) 58–62.
[17] W. Szlenk, Sur les suites faiblement convergentes dans l'espace L, Studia Math. 25 (1965) 337–341.
[18] E.V. Tokarev, The Banach–Saks property in Banach lattices, Sibirsk. Mat. Zh. 24 (1) (1983) 187–189 (in Russian).