Graphs with Hamiltonian Balls

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Abstract
For a vertex $u$ of a graph $G$ and an integer $r$, the ball of radius $r$ centered at $u$ is the subgraph $G_r(u)$ induced by the set of all vertices of $G$ whose distance from $u$ does not exceed $r$. We investigate the set $\mathcal{H}$ of connected graphs $G$ with at least 3 vertices such that every ball of radius 1 in $G$ has a Hamilton cycle. We prove that every graph $G$ in $\mathcal{H}$ with $n$ vertices has at least $2n - 3$ edges, and every such graph with $2n - 3$ edges is isomorphic to a triangulation of a polygon. We show that some well-known conditions for hamiltonicity of a graph $G$ also guarantee that $G$ has the following property: for each vertex $u$ of $G$ and each integer $r \geq 1$, the ball $G_r(u)$ has a Hamilton cycle.

1. Introduction

Interconnection between local and global properties of mathematical objects has always been a subject of investigations in different areas of mathematics. Usually by local properties of a mathematical object, for example a function, we mean its properties in balls with small radii. If a considered mathematical object is a graph, balls of radius $r$ are defined only for integer $r$. For a vertex $u$ of a graph $G$ the ball of radius $r$ centered at $u$ is a subgraph of $G$ induced by the set $M_r(u)$ of all vertices of $G$ whose distance from $u$ does not exceed $r$. This ball we denote by $G_r(u)$. In fact, for each vertex $u$ of a connected graph $G$ there is an integer $r(u)$ such that $G$ is a ball of radius $r(u)$ centered at $u$. Note that our definition of the ball is different from the usual definition (see, for example, [12]) where the set $M_r(u)$ is considered as the ball of radius $r$ centered at $u$.

The following problem arises naturally.

**Problem 1.1.** If each ball of radius 1 in a graph $G$ enjoys a given property $P$, does $G$ have the same property?

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For different properties the answers are different. For example, it is known [4] that if $G_1(u)$ is $k$-connected for each vertex $u$, then $G$ is also $k$-connected. On the other hand, for each positive integer $m \geq 3$ there exists a graph $G$ such that the chromatic number $\chi(G_1(u))$ is 2 for each vertex $u$ but $\chi(G) = m$ (see [15,24]).

We consider Problem 1.1 in the case when $P$ is the property of a graph to be hamiltonian. (A graph is called hamiltonian if it has a Hamilton cycle, that is, a cycle containing all the vertices of a graph). Let $\mathcal{H}$ denote the set of all connected graphs $G$ with at least 3 vertices such that any ball of radius 1 in $G$ is hamiltonian. The set $\mathcal{H}$ contains, in particular, locally hamiltonian graphs investigated in [7,19,22,23]: a graph $G$ is called locally hamiltonian if, for each vertex $u$ of $G$, the subgraph induced by the set of vertices adjacent to $u$ is hamiltonian.

In Section 3 we give some general properties of graphs in $\mathcal{H}$. We show that every graph $G \in \mathcal{H}$ with $n$ vertices has at least $2n - 3$ edges and every graph $G \in \mathcal{H}$ with $n$ vertices and $2n - 3$ edges is a maximal outerplanar graph, that is, a graph isomorphic to a triangulation of a polygon.

We introduce and investigate a new property of a graph $G$: every ball of any radius in $G$ is hamiltonian. Graphs with this property we call uniformly hamiltonian. Clearly, all uniformly hamiltonian graphs are in $\mathcal{H}$ but not every hamiltonian graph in $\mathcal{H}$ is uniformly hamiltonian. For example, the graph $G$ in Fig.1 is hamiltonian and belongs to $\mathcal{H}$ but the subgraph $G_2(x_0)$ is not hamiltonian.

![Fig.1](image)

In Section 4 we show that some well-known conditions for hamiltonicity of a graph $G$ guarantee also uniform hamiltonicity of $G$.

2. Definitions and auxiliary results

We use [3] for terminology and notation not defined here and consider simple graphs only. Let $V(G)$ and $E(G)$ denote, respectively, the vertex set and edge set of a graph $G$, and let $d(u,v)$ denote the distance between vertices $u$ and $v$ in $G$. For each vertex $u$ of $G$ and a positive integer $r$, we denote by $N_r(u)$ and $M_r(u)$ the sets of all $v \in V(G)$ with $d(u,v) = r$ and $d(u,v) \leq r$, respectively. The set $N_1(u)$ is called the *neighbourhood* of a vertex $u$. The subgraph of $G$ induced by the set $N_1(u)$ is denoted by $\langle N_1(u) \rangle$. A graph $G$ is called *locally $k$-connected* if, for each vertex $u \in V(G)$, the subgraph $\langle N_1(u) \rangle$ is $k$-connected. In other words, $G$ is locally $k$-connected if, for each vertex $u \in V(G)$, the ball $G_1(u)$ is $(k+1)$-connected.
Let $G$ be a connected graph and $v$ be a vertex in a ball $G_r(u), r \geq 1$. We call $v$ an interior vertex of $G_r(u)$ if $G_1(v)$ is a subgraph of $G_r(u)$. Clearly, every vertex in $G_{r-1}(u)$ is interior for $G_r(u)$ and if $G = G_r(u)$ then all vertices in $G$ are interior vertices.

Let $C$ be a cycle of a graph. We denote by $\overrightarrow{C}$ the cycle $C$ with a given orientation, and by $\overleftarrow{C}$ the cycle $C$ with the reverse orientation. If $u, v \in V(C)$ then $u\overrightarrow{C}v$ denote the consecutive vertices of $C$ from $u$ to $v$ in the direction specified by $\overrightarrow{C}$. The same vertices, in reverse order, are given by $v\overleftarrow{C}u$. We use $u^+$ to denote the successor of $u$ on $\overrightarrow{C}$ and $u^-$ to denote its predecessor.

Analogous notation is used with respect to paths instead of cycles.

A planar graph $G$ is called maximal planar if no edges can be added to $G$ without losing planarity. A planar graph is outerplanar if it can be embedded in the plane so that all its vertices lie on the exterior face. A maximal outerplanar graph, or a mop, is an outerplanar graph such that the addition of an edge between any two non-adjacent vertices results in a non-outerplanar graph. In other words, a mop is a graph which is isomorphic to a triangulation of a polygon. Every mop has the unique Hamilton cycle which forms the exterior face. It is also known [9] that a mop with $n$ vertices has $2n - 3$ edges and at least two vertices of degree 2. Hence every mop $G$ with $n \geq 4$ vertices can be obtained from some mop $H$ with $n - 1$ vertices by adding a new vertex adjacent to two consecutive vertices on the Hamilton cycle of $H$. (Then we say that $G$ is obtained from $H$ by an elementary extension). This property implies the following lemma which usually is used as a recursive characterization of a mop [14,20].

**Lemma 2.1.** A graph $G \neq K_3$ is a mop if and only if it can be obtained from the triangle $K_3$ by a sequence of elementary extensions.

### 3. Some properties of graphs in $\mathcal{H}$

**Proposition 3.1.** Every graph $G \in \mathcal{H}$ has at least $2|V(G)| - 3$ edges.

*Proof.* Let $C(v)$ denote a Hamilton cycle of a graph $G_1(v)$ for each $v \in V(G)$. Consider a vertex $x_0$ of $G$. If $G = G_1(x_0)$ then $|E(G)| \geq 2|N_1(x_0)| - 1 = 2|V(G)| - 3$ and the assertion of the theorem is true. Let $G_1(x_0) \neq G$ and $r$ be an integer such that $G = G_r(x_0)$ and $G \neq G_{r-1}(x_0)$. First we will describe an algorithm which for each vertex $v \in V(G)$ constructs an edge set $E(v)$. During this algorithm a vertex is considered to be unscanned or scanned. Initially all vertices of $G$ are unscanned. A vertex $v$ becomes scanned if $E(v)$ is already constructed.

Step 1. Let $C(x_0) = x_0x_1...x_mx_0$. Put $E(x_0) = \emptyset, E(x_m) = \{x_mx_0\}$ and $E(x_i) = \{x_0x_i, x_ix_{i+1}\}$ for $i = 1, ..., m - 1$. The vertices $x_0, x_1, ..., x_m$ are now considered to be scanned.

Step j (j \geq 2). Assume that all vertices in $M_{j-1}(x_0)$ have already been scanned. Consider an unscanned vertex $z \in N_j(x_0)$ and a vertex $y = y(z)$ in $N_{j-1}(x_0)$ which is adjacent to $z$. There is a path $P = P(y, z)$ in the cycle $C(y)$ such that $z$ lies on $P$, all internal vertices of $P$ are in $N_j(x_0)$ and the origin and the
terminus of $P$ are in $N_{j-1}(x_0)$. Without loss of generality we assume that the origin of $P$ is not $y$. Let $\overrightarrow{P}$ denote an oriented path obtained from $P$ by orienting the edges in the direction from the origin to the terminal vertex. For each unscanned vertex $v$ on $\overrightarrow{P}$ we put

$$E(v) = \{uv \in E(G) / u \in N_{j-1}(x_0)\} \cup \{vv^{-1}(\overrightarrow{P})\}$$

where $v^{-1}(\overrightarrow{P})$ is the predecessor of $v$ on $\overrightarrow{P}$. All vertices of $\overrightarrow{P}$ are now scanned.

If there remains any unscanned vertex in $N_j(x_0)$ then we repeat Step j. Otherwise go to Step j+1 if $j < r$ and stop if $j = r$.

Clearly, $|E(v)| \geq 2$ for each $v \in V(G) \setminus \{x_0, x_m\}$ and $E(v_1) \cap E(v_2) = \emptyset$ for each pair of distinct vertices $v_1, v_2 \in V(G)$. Therefore

$$|E(G)| \geq 2|N_1(x_0)| - 1 + 2 \sum_{j=2}^{r} |N_j(x_0)| = 2|V(G)| - 3.$$ 

The proof is complete.

![Diagram](image.png)

**Fig. 2**

It is known [4] that in a connected graph $G$ local $(k - 1)$-connectedness, $k \geq 2$, implies $k$-connectedness of $G$. Now by using the same argument, we will show that indeed in such graphs all balls are $k$-connected. Note that not every ball of a locally $(k - 1)$-connected graph is locally $(k - 1)$-connected. For example, the graph $G$ in Fig. 2 is locally connected but in the ball $G_2(u)$ the neighbourhood of the vertex $v$ induces a disconnected subgraph.

**Proposition 3.2.** If every ball of radius 1 in a connected graph $G$ is $k$-connected, $k \geq 2$, then balls of any radius in $G$ are $k$-connected.

**Proof.** It is clear that $k$-connectedness of $G_1(u)$ implies that $|M_r(u)| \geq k + 1$ for each $u \in V(G)$ and each $r \geq 2$. Suppose that for a vertex $u$ and an integer $r \geq 2$ the ball $G_r(u)$ is not $k$-connected. Then there exists a subset $S \subseteq M_r(u)$ such that $|S| \leq k - 1$ and $G_r(u) - S$ is disconnected. Among all such sets $S$, let $S_0$ be one of minimum cardinality. Clearly, $S_0$ contains an interior vertex $v$ of $G_r(u)$ because otherwise $G_r(u) - S_0$ is connected. The minimality of $S_0$ implies that there are two neighbours $w_1$ and $w_2$ of $v$ such that $w_1$ and $w_2$ belong to different components in $G_r(u) - S$. Then the set $S_0 \cap M_1(v)$ separates $w_1$ and $w_2$ in $G_1(v)$.
and $|S_0 \cap M_1(u)| \leq k - 1$. This contradicts $k$-connectedness of $G_1(v)$. Hence all the balls in $G$ are $k$-connected.

**Corollary 3.3.** Every ball in a graph $G \in \mathcal{H}$ is 2-connected.

![Graph](image)

**Fig. 3.**

Not every graph in $\mathcal{H}$ is hamiltonian. Moreover $\mathcal{H}$ contains non-hamiltonian graphs $G$ such that every ball of $G$, except $G$ itself, is hamiltonian. Two examples of such graphs are given in Fig. 3. Non-hamiltonicity of the graph with nine vertices is evident. Non-hamiltonicity of the second graph follows from the fact that removing the twelve “light vertices” separates it into thirteen components, each containing a single “dark vertex”.

However, all graphs in $\mathcal{H}$ which are minimal concerning the number of vertices are hamiltonian and even uniformly hamiltonian.

**Proposition 3.4.** Every graph $G \in \mathcal{H}$ with $|E(G)| = 2|V(G)| - 3$ is uniformly hamiltonian. Moreover such graphs are mops.

**Proof.** Let $G \in \mathcal{H}$ and $|E(G)| = 2|V(G)| - 3$. Consider a vertex $x_0$ of $G$. If $G = G_1(x_0)$ then the proposition is evident.

Assume that $G \neq G_1(x_0)$ and $r$ be an integer such that $G_r(x_0) = G$ and $G_{r-1}(x_0) \neq G$. Using the same argument as in the proof of Theorem 3.1 we obtain that

$$|E(G)| \geq 2|N_1(x_0)| - 1 + 2 \sum_{j=2}^{r} |N_j(x_0)| = 2|V(G)| - 3$$

On the other hand we have that $|E(G)| = 2|V(G)| - 3$. This implies that

(i) $|E(G)| = 2|N_1(x_0)| - 1 + 2 \sum_{j=2}^{r} |N_j(x_0)|$,

(ii) $|E(v)| = 2$ for each $v \in N_j(x_0)$ and $j \geq 2$,

Now by induction on $j$ we will show that the graph $G_j(x_0)$ is a mop. For $j = 1$ this is evident. Suppose that $G_{j-1}(x_0)$ is a mop for some $j \geq 2$ and $\overline{G}$ is the
Hamilton cycle of $G_{j-1}(x_0)$ with a given orientation. Consider the set $\{\overrightarrow{P_1}, ..., \overrightarrow{P_k}\}$ of all distinct oriented paths which have been constructed in Step j of the algorithm described in the proof of Theorem 3.1. Clearly, $\overrightarrow{P_i} = b_i \overrightarrow{Q_i} a_i$, where $a_i, b_i \in N_{j-1}(x_0)$ and $V(\overrightarrow{Q_i}) \subseteq N_j(x_0)$. Furthermore, (ii) implies that $b_i$ is the predecessor or the successor of $a_i$ on $\overrightarrow{C}$ and $V(\overrightarrow{Q_i}) \subseteq N_j(x_0) \cap N_1(a_i)$.

Using this observation and (i)-(ii) we deduce the following property.

**Property.** The terminal vertex $v$ of $\overrightarrow{Q_i}$ is adjacent in $G_j(x_0)$ only to two vertices: $a_i$ and the predecessor of $v$ on $\overrightarrow{Q_i}$. Each non-terminal vertex $v$ of $\overrightarrow{Q_i}$ is adjacent in $G_j(x_0)$ only to three vertices: $a_i$, the predecessor and the successor of $v$ on $\overrightarrow{Q_i}$.

Using this property and taking into consideration that the neighbourhood of each vertex $v \in N_j(x_0)$ induces a path we obtain that $\{a_i, b_i\} \neq \{a_s, b_s\}$ for $1 \leq i < s \leq k$.

It is easy to see now that $G_j(x_0)$ can be obtained from $G_{j-1}(x_0)$ by a sequence of elementary extensions. Since $G_{j-1}(x_0)$ is a mop, we conclude, by Lemma 2.1, that $G_j(x_0)$ also is a mop.

Thus, every ball of $G$ is a mop, that is, a hamiltonian graph. Therefore $G$ is uniformly hamiltonian.

**Corollary 3.5.** A connected graph $G$ with $n \geq 3$ vertices is a mop if and only if it has $2n - 3$ edges and, for each vertex $u$ of $G$, the neighbourhood $N_1(u)$ induces a path.

**Proof.** If $G$ is a mop then it is connected, $|V(G)| \geq 3$, $|E(G)| = 2|V(G)| - 3$ and it is not difficult to see that for each vertex $x \in V(G)$ the neighbourhood $N_1(x)$ induces a path. Conversely, suppose that $G$ is a connected graph with $|V(G)| \geq 3$ and $|E(G)| = 2|V(G)| - 3$ such that for each vertex $u$ of $G$, the neighbourhood $N_1(u)$ induces a path. Then a subgraph $G_1(u)$ is hamiltonian for each $u \in V(G)$ and $G \in \mathcal{H}$. Therefore, by Proposition 3.1, $G$ is a mop.

For comparison, we note a characterisation of a maximal planar graph given by Skupien [23]: A connected graph $G$ with $n \geq 3$ vertices is a maximal planar graph if and only if $G$ is locally hamiltonian and has $3n - 6$ edges.

### 4. Some classes of uniformly hamiltonian graphs

Here we will consider some classes of hamiltonian graphs in $\mathcal{H}$. We will show that graphs in these classes are also uniformly hamiltonian.

1. **Powers of graphs**

For a connected graph $G$ and an integer $t \geq 2$, $G^t$ is the graph with $V(G^t) = V(G)$ where two vertices $u$ and $v$ are adjacent if and only if the distance between $u$ and $v$ in $G$ does not exceed $t$. The graphs $G^2$ and $G^3$ are called, respectively, the *square* and the *cube* of $G$.

**Proposition 4.1.** For every connected graph $G$ with at least 3 vertices the cube $G^3$ is a uniformly hamiltonian graph.
Proof. By the result of Karagounis [10] and Sekanina [21], the cube of every connected graph with at least 3 vertices is hamiltonian. Hence \((G_{3r}(u))^3\) is hamiltonian for each vertex \(u\) and each \(r \geq 1\). Denote the graph \(G^3\) by \(H\). Clearly, \((G_{3r}(u))^3\) is a spanning subgraph of \(H_r(u)\) because \(V(H_r(u)) = V(G_{3r}(u)) = V((G_{3r})^3)\). Then hamiltonicity of \((G_{3r}(u))^3\) implies hamiltonicity of \(H_r(u)\) for each \(u \in V(G)\) and each \(r \geq 1\). Therefore \(H = G^3\) is uniformly hamiltonian.

It is well-known, due to Fleischner[8], that the square of every 2-connected graph is hamiltonian. But in the general case even hamiltonicity of a graph \(G\) does not guarantee uniform hamiltonicity of \(G^2\). For example, if \(G\) is obtained from a cycle \(x_1x_2...x_{2n}x_1\) by joining the vertices \(x_1\) and \(x_n\), \(n \geq 8\) and \(H = G^2\) then the subgraph \(H_2(x_1)\) is not hamiltonian. We will indicate two cases when the square of a graph is uniformly hamiltonian.

Proposition 4.2. The square of every cycle is uniformly hamiltonian.

Proposition 4.3. If \(G\) is a connected, locally connected graph with \(|V(G)| \geq 3\), then \(G^2\) is uniformly hamiltonian.

Proof. Denote the graph \(G^2\) by \(H\). We have that \(V(H_r(u)) = V(G_{2r}(u)) = V((G_{2r})^2)\) and \((G_{2r}(u))^2\) is a spanning subgraph of \(H_r(u)\) for each vertex \(u\) and each \(r \geq 1\). Since \(G\) is locally connected, \(G_1(u)\) is 2-connected for each vertex \(u \in V(G)\). Then, by Proposition 3.2, the ball \(G_{2r}(u)\) is 2-connected for each \(u \in V(G)\) and each integer \(r \geq 1\). Therefore, by the result of Fleischner [8], \((G_{2r}(u))^2\) is hamiltonian. This implies that the ball \(H_r(u)\) is also hamiltonian for each vertex \(u \in V(G)\) and each \(r \geq 1\). Therefore, \(H = G^2\) is uniformly hamiltonian.

2. Graphs with local Chvatal–Erdös condition

Let \(\alpha(G)\) and \(k(G)\) denote the independence number and connectivity of a graph \(G\), respectively. The following theorem is well-known.

Theorem A (Chvatal and Erdös [5]). A graph \(G\) with at least three vertices is hamiltonian if \(\alpha(G) \leq k(G)\).

A local variation of the result of Chvatal and Erdös was obtained by Khachatrian [11]: A connected graph \(G\) is hamiltonian if \(|V(G)| \geq 3\) and there is a positive integer \(r \geq 1\) such that \(\alpha(G_{r+1}(u)) \leq k(G_r(u))\) for each vertex \(u \in V(G)\). Now we will show that in the case \(r = 1\) this condition implies uniform hamiltonicity of \(G\).

Theorem 4.3. A connected graph \(G\) with \(|V(G)| \geq 3\) is uniformly hamiltonian if \(\alpha(G_2(x)) \leq k(G_1(x))\) for each vertex \(x \in V(G)\).

Proof. Suppose that there is a vertex \(x \in V(G)\) and an integer \(r \geq 1\) such that the ball \(G_r(x)\) is not hamiltonian. The condition \(\alpha(G_2(x)) \leq k(G_1(x))\) implies that \(x\) lies on a triangle. Among all cycles in \(G_r(x)\) which contain \(x\), let \(C\) be one of maximum length. Consider a vertex \(y \in M_r(x) \setminus V(C)\) and a shortest \((x,y)\)-path in \(G_r(x)\). Clearly, there are two adjacent vertices \(v\) and \(u\) on this path such
that $v \notin V(C)$, $v \in V(C)$ and $u$ is an interior vertex of $G_r(x)$. Let $\mathcal{C}$ be the cycle $C$ with a given orientation. We have that $2 \leq \alpha(G_2(u)) \leq k(G_1(u))$ since $vu^+ \notin E(G)$. Then, by Menger's theorem [13], in $G_1(u)$ there are $k$ internally disjoint $(v, u^+)$-paths $Q_1, ..., Q_k$, where $k = k(G_1(u))$. Maximal of $C$ implies that each $Q_i$ has at least one common vertex with $C$. This means that there are paths $P_1, ..., P_k$ having initial vertex $v$ that are pairwise disjoint, apart from $v$, and that share with $C$ only their terminal vertices $v_1, ..., v_k$, respectively. Furthermore, maximality of $C$ implies that $vv_i^+ \notin E(G)$ for each $i = 1, ..., k$. Then there is a pair $i, j$ such that $1 \leq i < j \leq k$ and $v_i^+ v_j^+ \in E(G)$. (Otherwise in $G_2(v)$ there are $k + 1$ mutually non-adjacent vertices $v, v_1, ..., v_k$ which contradicts the condition $\alpha(G_2(v)) \leq k(G_1(v)))$. Since $u$ is an interior vertex of $G_r(x)$, the paths $P_1, ..., P_k$ lie in $G_r(x)$.

Now by deleting the edges $v_i v_i^+$ and $v_j v_j^+$ from $C$ and adding the edge $v^+_i v^+_j$ together with the paths $P_i$ and $P_j$, we obtain in $G_r(v)$ a cycle that is longer than $C$ and contains $x$; a contradiction. Therefore, $C$ is a Hamilton cycle of $G_r(x)$.

3. Claw-free graphs

A graph $G$ is called claw-free if $G$ has no induced subgraph isomorphic to $K_{1,3}$. In terms of balls this means that for each vertex $x \in V(G)$ the ball $G_1(x)$ does not contain three mutually non-adjacent vertices. The following result is well-known.

**Theorem B** (Oberly and Sumner [16]). A connected, locally connected, claw-free graph $G$ with $|V(G)| \geq 3$ is hamiltonian.

Clearly, local connectedness of a claw-free graph $G$ is equivalent to the condition $\alpha(G_1(w)) \leq 2 \leq k(G_1(w))$ for each vertex $w \in V(G)$. Taking Theorem A into consideration we can reformulate Theorem B in the following way: In a connected, claw-free graph $G$ hamiltonicity of balls of radius 1 implies hamiltonicity of $G$. The next result shows that indeed this implies hamiltonicity of all the balls of $G$.

**Theorem 4.4.** A connected, claw-free graph $G$ with $|V(G)| \geq 3$ is locally connected if and only if it is uniformly hamiltonian.

**Proof.** If $G$ is uniformly hamiltonian then, clearly, it is locally connected. Conversely, suppose that $G$ is a connected, locally connected, claw-free graph but some ball $G_r(x)$ is not hamiltonian. Among all cycles in $G_r(x)$ which contain $x$, let $C$ be one of maximum length. Clearly, there are two adjacent vertices $v_1$ and $u$ such that $u \notin V(C)$, $v_1 \in V(C)$ and $v_1$ is an interior vertex of $G_r(x)$. Let $\mathcal{C}$ be the cycle $C$ with a given orientation, and let $v_1, ..., v_n$ be the vertices of $C$ occurring on $\mathcal{C}$ in the order of their indices. Since $G$ is claw-free, $v_j^+ v_j^+ \in E(G)$ for each $v_j \in V(C) \cap N_1(u)$. The subgraph $< N_1(v_1) >$ is connected because $G$ is locally connected. Consider a shortest $(u, v_1^+)$-path $Q$ in $< N_1(v_1) >$. Let $Q = u_1 u_2 ... u_t$ where $u_1 = v_1^+$ and $u_t = u$. Since $Q$ is a shortest path, $u_2 \in V(C)$. Let $u_2 = v_{i_2}$. Since $G$ is claw-free, $t \leq 4$. Moreover, $t = 4$ and $u_3 \in V(C)$. (If $t = 3$ then there is a cycle longer than $C$, which is obtained from $C$ by deleting edges $v_{i_2} v_{i_2}, v_{i_2} v_{i_2}^+, v_{i_1} v_{i_1}^+$ and adding edges $v_1 u, uv_{i_2}$ and $v_{i_2} v_{i_2}^+$; a contradiction. If $t = 4$ and $u_3 \notin V(C)$

192
then we also can extend \( C \) and obtain a contradiction, by taking instead of \( u \) the vertex \( u_3 \) and using the same argument).

Let \( u_3 = u_{i^3} \). Consider a subgraph \( H \) induced by the set \( \{u, v_1, v_{1^3}^-, v_{i^2} \} \). Then \( v_{1^3}^- v_{i^2} \in E(G) \) since \( G \) is claw-free and \( uv_{1^3}^-, uv_{i^2} \notin E(G) \). This implies that without loss of generality we can consider the case \( 1 < i^2 < i^3 \) only.

Clearly, \( v_1 v_{i^2}^- \notin E(G) \) because otherwise the cycle

\[
v_1^{-} v_{1^3}^+ \overline{C} v_{i^2}^- v_1 u v_{i^3} v_{i^2} \overline{C} v_{i^3}^- v_{i^3}^+ \overline{C} v_1^-,
\]

is longer than \( C \) and contains \( x \). Furthermore, \( v_1 v_{i^2}^+ \notin E(G) \) because otherwise the cycle

\[
v_1^- v_{1^3}^+ \overline{C} v_{i^2} v_{i^3} u v_1 v_{i^2}^+ \overline{C} v_{i^3}^- v_{i^3}^+ \overline{C} v_1^-,
\]

is longer than \( C \) and contains \( x \). Then \( v_{i^2}^- v_{i^2}^+ \in E(G) \) because otherwise the set \( \{v_1, v_{i^2}^-, v_{i^2}, v_{i^2}^+\} \) induces a graph \( K_{1,3} \). Now we obtain a cycle

\[
v_1 u v_{i^3} v_{i^2} v_{1^3}^- v_{i^2}^+ \overline{C} v_{i^3}^- v_{i^3}^+ \overline{C} v_1,
\]

which is longer than \( C \) and contains \( x \); a contradiction. Therefore, \( C \) is a Hamilton cycle of \( G_r(x) \).

4. Graphs with local Ore’s condition

A graph \( H \) is said to satisfy Ore’s condition if \( |V(H)| \geq 3 \) and \( d_H(u) + d_H(v) \geq |V(H)| \) for each pair of non-adjacent vertices \( u \) and \( v \) of \( H \). It is well-known that every graph with Ore’s condition is hamiltonian [17]. The following result was obtained in [2]

**Theorem C** (Asratian and Khachatrian [2]). Let \( G \) be a connected graph with at least three vertices where for each vertex \( x \in V(G) \), the ball \( G_1(x) \) satisfies Ore’s condition. Then \( G \) is hamiltonian.

Graphs satisfying the condition of Theorem C are called graphs with local Ore’s condition. Some properties of such graphs were investigated in [1]. Now we will indicate some classes of graphs with local Ore’s condition which are also uniformly hamiltonian.

It is known due to Ore [18] that a graph \( G \) on \( n \geq 3 \) vertices is hamiltonian if it has at least \( \frac{(n-1)(n-2)}{2} + 2 \) edges. The next result shows that indeed such graphs are also uniformly hamiltonian.

**Theorem 4.5.** A graph \( G \) with \( n \geq 3 \) vertices is uniformly hamiltonian if \( |E(G)| \geq \frac{(n-1)(n-2)}{2} + 2 \).

**Proof.** Assume that \( G \neq K_n \). First we will show that \( G_1(x) \) is hamiltonian for each vertex \( x \) of \( G \). Let \( E' \) denote the set of all pairs of non-adjacent vertices of \( G \). Clearly, \( |E'| \leq n - 3 \). If \( G_1(x) \) is not a complete graph consider two non-adjacent vertices \( u \) and \( v \) in \( G_1(x) \). Let

\[
E'(x) = \{uv\} \cup \{xy/y \in V(G) \setminus M_1(x)\}.
\]

193
Then $|E' \setminus E'(x)| \leq d(x) - 3$ since $E'(x) \subseteq E'$, $|E'(x)| = n - d(x)$ and $|E'| \leq n - 3$. Hence,

$$d_{G_1}(x)(u) + d_{G_1}(x)(v) \geq 2(|M_1(x)| - 2) - (d(x) - 3) = |M_1(x)|,$$

which means that $G_1(x)$ satisfies Ore's condition for each $x$. The graph $G$ is hamiltonian, by Ore's result [18] (and also by Theorem C). Furthermore, $G_2(x) = G$ for each $x \in V(G)$. (Otherwise there are two non-adjacent vertices $u$ and $v$ of $G$ with distance $d(u, v) \geq 3$ and then $|E(G)| < \frac{(n-2)(n-3)}{2} + n = \frac{(n-1)(n-2)}{2} + 2$; a contradiction.) Hence, $G$ is a hamiltonian graph where for each vertex $x \in V(G)$, the subgraph $G_1(x)$ is hamiltonian and $G_2(x) = G$. Therefore $G$ is uniformly hamiltonian.

**Theorem 4.6.** A graph $G$ on $n \geq 4$ vertices is uniformly hamiltonian if

$$d(x) + d(y) \geq \frac{3n - 3}{2}$$

for each pair of non-adjacent vertices $x$ and $y$ of $G$.

**Proof.** Assume that $G \neq K_n$. Clearly, the distance between any two non-adjacent vertices in $G$ is 2 and, therefore, $G_2(x) = G$ for each $x \in V(G)$. We will show that $G_1(x)$ satisfies Ore condition for each $x \in V(G)$. Suppose that for a vertex $x$ the subgraph $G_1(x)$ contains two non-adjacent vertices $u$ and $v$ such that $d_{G_1}(x)(u) + d_{G_1}(x)(v) < |M_1(x)|$. Then

$$\frac{3n - 3}{2} \leq d_G(u) + d_G(v) < |M_1(x)| + 2(n - |M_1(x)|)$$

which implies that $d(x) \leq \frac{n}{2}$. Therefore there is a vertex $y$ which is not adjacent to $x$ and $d(y) \leq n - 2$. Thus

$$d_G(x) + d_G(y) \leq (n - 2) + \frac{n}{2} = \frac{3n - 4}{2},$$

a contradiction. Therefore, $G_1(x)$ satisfies Ore's condition. The graph $G$ is hamiltonian by Ore's result [17] (and also by Theorem C). Thus, $G$ is a hamiltonian graph where for each $x \in V(G)$, the subgraph $G_1(x)$ is hamiltonian and $G_2(x) = G$. Therefore $G$ is uniformly hamiltonian.

Now we will show that the bound in Theorem 4.6 is sharp. Let $n \geq 4$ be an even integer, $U = \{u_1, u_2, ..., u_{n-2}\}$ and $V = \{v_1, v_2, ..., v_{n-2}\}$. Consider a graph $G$ with vertex set $U \cup V \cup \{u, v\}$, such that $U \cup \{u\}$ induces a complete subgraph, $V \cup \{v\}$ induces a complete subgraph, $u$ is adjacent to $v$, and each vertex of $U$ is adjacent to each vertex of $V$. Clearly, $d_G(u) = d_G(v) = \frac{n-2}{2} + 1 = \frac{n}{2}$ and $d_G(w) = n - 2$ for each vertex $w \in U \cup V$. Therefore the degree sum of any two
non-adjacent vertices in $G$ is $(n - 2) + \frac{n}{2} = \frac{3n - 4}{2}$. However $G$ is not uniformly hamiltonian since the subgraphs $G_1(u)$ and $G_1(v)$ are not hamiltonian.

A graph $G$ is said to satisfy Dirac’s condition if $|V(G)| \geq 3$ and $d(x) \geq \frac{|V(G)|}{2}$ for each vertex $x \in V(G)$. Graphs with this condition are hamiltonian [6].

**Proposition 4.7.** For every $n \geq 1$ there exists a graph $G$ with the condition $d(x) \geq \frac{1}{2}|V(G)| + n$ for each $x \in V(G)$, which is not uniformly hamiltonian.

**Proof.** Let $F_1, ..., F_{n+2}, H_1, ..., H_{n+2}$ be disjoint complete graphs each on $n + 1$ vertices. Construct a graph $G$ by joining each vertex of $F_i$ with each vertex of $H_j$ for $i, j = 1, ..., n + 2$. Clearly, $G$ is a $k$-regular graph with $|V(G)| = 2(n + 1)(n + 2)$ and $k = (n + 1)(n + 2) + n = \frac{1}{2}|V(G)| + n$. Consider a vertex $x \in V(H_1)$. Then $G_1(x) = H_1 \cup F_1 \cup ... \cup F_{n+2}$. If we delete $n + 1$ vertices of $H_1$ from $G_1(x)$ we obtain $n + 2$ components. Hence $G_1(x)$ is not hamiltonian. Therefore, $G$ is not uniformly hamiltonian. ■

Proposition 4.7 shows that Dirac’s condition is weak for uniform hamiltonicity. The next result gives a Dirac-type condition which guarantees uniform hamiltonicity of a graph.

**Theorem 4.8.** A graph $G$ with $n \geq 3$ vertices is uniformly hamiltonian if $d(x) \geq \frac{2n - 1}{3}$ for each vertex $x$ of $G$.

**Proof.** Assume that $G \neq K_n$. Clearly, $G_2(x) = G$ for each $x \in V(G)$ and $G$ is hamiltonian [6]. Hence it is sufficient to show that $G_1(x)$ satisfies Ore condition for each $x \in V(G)$.

Suppose that for a vertex $x$ the ball $G_1(x)$ contains two non-adjacent vertices $u$ and $v$ such that $d_{G_1(x)}(u) + d_{G_1(x)}(v) < |M_1(x)|$. Then

$$\frac{4n - 2}{3} \leq d_G(u) + d_G(v) < |M_1(x)| + 2(n - |M_1(x)|)$$

which implies that $d(x) < \frac{2n - 1}{3}$; a contradiction. Therefore, $G_1(x)$ satisfies local Ore’s condition and $G$ is uniformly hamiltonian. ■

Now we will consider uniform hamiltonicity of complete $m$-partite graphs with $m \geq 3$.

The complete $m$-partite graph $K_{n_1, ..., n_m}$ where $m \geq 3$ is that graph whose vertex set is partitioned into sets $V_1, ..., V_m$ so that $|V_i| = n_i$ for each $i = 1, ..., m$ and so that $uv$ is an edge of the graph if and only if $u$ and $v$ belong to distinct partite sets $V_i$ and $V_j$.

**Theorem 4.9.** Let $G = K_{n_1, ..., n_m}$ be an $m$-partite complete graph where $m \geq 3$ and $n_1 \leq n_2 \leq ... \leq n_m$. Then $G$ is uniformly hamiltonian if and only if $|V(G)| \geq 2n_m + n_{m-1} - 1$, and this condition is equivalent to local Ore’s condition for $G$.

**Proof.** Let $V_1, V_2, ..., V_m$ denote partite sets of $G$ and $|V_i| = n_i$ for $i = 1, ..., m$. Suppose that $|V(G)| \geq 2n_m + n_{m-1} - 1$. We will show that for each $u \in V(G)$ the ball $G_1(u)$ satisfies Ore’s condition. Consider two non-adjacent vertices $x$ and $y$ in

195
\( G_1(u) \). Then \( x, y \in V_i \) and \( u \in V_j \) for some \( i \neq j \), \( |V(G_1(u))| = |V(G)| - |V_j| + 1 \) and \( |V(G)| - 2|V_i| - |V_j| + 1 \geq |V(G)| - 2n_m - n_{m-1} + 1 \geq 0 \). Therefore \[
 d_{G_1(u)}(x) + d_{G_1(u)}(y) = 2(|V(G)| - |V_i| - |V_j| + 1) \geq |V(G_1(u))|.
\]
Thus, \( G_1(u) \) satisfies Ore's condition and, therefore, is hamiltonian. Then, by Theorem C, \( G \) also is hamiltonian. This implies that \( G \) is uniformly hamiltonian since \( G_2(u) = G \) for each \( u \in V(G) \).

Conversely, suppose \( G \) is uniformly hamiltonian. Consider a vertex \( v \in V_{m-1} \) and a Hamilton cycle \( C \) of \( G_1(v) \). Then \( 1 + n_1 + ... + n_{m-2} \geq n_m \) since no two vertices of \( V_m \) appear consecutively on \( C \). Therefore, \( |V(G)| \geq 2n_m + n_{m-1} - 1 \).

As we have shown above, this implies that \( G_1(u) \) satisfies Ore's condition for each \( u \in V(G) \).

Taking Theorems 4.5, 4.6, 4.8 and 4.9 into consideration we formulate the following conjecture.

**Conjecture.** Every graph with local Ore's condition is uniformly hamiltonian.

Finally we show that graphs with local Ore's condition have a ball property which is close to uniform hamiltonicity.

**Theorem 4.10.** Let \( G \) be a graph with \( |V(G)| \geq 3 \) which satisfies local Ore's condition. Then for each vertex \( x \in V(G) \) and each integer \( r \geq 1 \) the ball \( G_r(x) \) has the following property: every longest cycle in \( G_r(x) \) contains all interior vertices of \( G_r(x) \).

**Proof.** Since \( G \) satisfies local Ore's condition, \( d_{G_1(w)}(u) + d_{G_1(w)}(v) \geq |M_1(w)| \) for each \( w \in V(G) \) and each pair of non-adjacent vertices \( u, v \in N_1(w) \). Clearly,

\[
 d_{G_1(w)}(u) + d_{G_1(w)}(v) = |M_1(w) \cup N_1(u) \cup N_1(v)| + |M_1(w) \cup (N_1(u) \cup N_1(v))|.
\]

Then Ore's condition for the ball \( G_1(w) \) is equivalent to the condition

\[
(1) \quad |M_1(w) \cap N_1(u) \cap N_1(v)| \geq |M_1(w) \setminus (N_1(u) \cup N_1(v))|.
\]

for each pair of non-adjacent vertices \( u, v \in M_1(w) \). Now consider a longest cycle \( C \) in a ball \( G_r(x) \). Suppose that \( C \) does not contain all interior vertices of \( G_r(x) \). Then there is an interior vertex \( v \) of \( G_r(x) \) outside \( C \) with \( N_1(v) \cap V(C) \neq \emptyset \). Let \( w_1, ..., w_k \) be the vertices of \( W = N_1(v) \cap V(C) \) occurring on \( C \) in the order of their indices. Then the set \( W^+ = \{w_1^+, ..., w_k^+\} \) is independent, since any two vertices in \( W^+ \) are non-adjacent. (Otherwise, \( W^+ \) contains two adjacent vertices \( w_i^+ \) and \( w_j^+ \) and then \( G_r(x) \) has a cycle \( w_i^+ w_j \uparrow C w_i^+ w_j^+ C w_i \) which is longer than \( C \); a contradiction.)

Since \( d(v, w_i^+) = 2 \) for each \( i = 1, ..., k \), we obtain from (1) that

\[
(2) \quad \sum_{i=1}^{k} |M_1(w_i) \cap N_1(w_i^+) \cap N_1(v)| \geq \sum_{i=1}^{k} |M_1(w_i) \setminus (N_1(w_i^+) \cup N_1(v))|.
\]

Let \( e(W, W^+) \) denote the number of edges in \( G \) with one end in \( W \) and the other in \( W^+ \). Clearly, \( M_1(w_i) \cap N_1(w_i^+) \cap N_1(v) \subseteq V(C) \) for each \( i = 1, ..., k \) because \( C \) is a longest cycle of \( G_r(x) \) and \( M_1(v) \subseteq M_r(x) \). Then
\[
(3) \sum_{i=1}^{k} |M_1(w_i) \cap N_1(w_i^+) \cap N_1(v)| = e(W, W^+)
\]

and

\[
(4) \sum_{i=1}^{k} |M_1(w_i) \setminus (N_1(w_i^+) \cup N_1(v))| \geq e(W, W^+) + k
\]

because \( v \notin W^+ \) and \( v \in M_1(w_i) \setminus (N_1(w_i^+) \cup N_1(v)) \) for each \( i = 1, \ldots, k \). But (3) and (4) contradict (2). Therefore, \( C \) contains all interior vertices of the ball \( G_r(x) \).

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References


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