# On Complementability of Subspaces Generated by Contractions and Shifts of Functions 

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Suppose that $E$ is an ri. space on $[0,1]$ (the definition is given below), $a \in E, 1 \leq k \leq 2^{n}$, and $n=0,1, \ldots$. We set

$$
a_{n, k}(t)=\left\{\begin{array}{l}
a\left(2^{n} t-k+1\right) \text { if } \frac{k-1}{2^{n}} \leq t \leq \frac{k}{2^{n}} \\
0 \text { for other } t \in[0,1]
\end{array}\right.
$$

and $Q_{n, a}=\operatorname{span}\left\{a_{n, k}, 1 \leq k \leq 2^{n}\right\}$. We denote the set of $a \in E$ such that $Q_{n, a}$ is uniformly complemented in $E$, i.e., there exist projectors $P_{n}$ from $E$ onto $Q_{n, a}$ with $\sup \left\|P_{n}\right\|<\infty$, by $\mathfrak{l}(E)$. This paper studies the set $\mathfrak{M}(E)$ and the class of ri. spaces $E$ coinciding with $\mathfrak{N}(E)$. It turns out that $\mathfrak{N}(E)$ is closely related to the space of tensor multipliers acting in $E$.

Below, we give the necessary definitions.
A Banach space $E$ of measurable functions on $[0,1]$ is called symmetric or rearrangement invariant (ri.) if
(i) $|x(t)| \leq|y(t)|$ and $y \in E$ imply $x \in E$ and $\|x\|_{E} \leq\|y\|_{E} ;$
(ii) the equimeasurability of functions $x$ and $y$ and the inclusion $y \in E$ imply $x \in E$ and $\|x\|_{E}=\|y\|_{E}$.

Following [1], we assume that $E$ is separable or dual to a separable space.

Examples of r.i. spaces are $L_{p}$ with $1 \leq p \leq \infty$ and the Orlicz, Lorentz, and Marcinkiewicz spaces. Let $\Omega$ denote the set of increasing concave functions $\varphi(t)$ on $[0,1]$ such that $\varphi(0)=\varphi(+0)=0$ and $\varphi(1)=1$. Each

[^0]function $\varphi \in \Omega$ generates a Lorentz space $\Lambda(\varphi)$ and a Marcinkiewicz space $M(\varphi)$ with norms
\[

$$
\begin{gathered}
\|x\|_{\Lambda(\varphi)}=\int_{0}^{1} x^{*}(t) d \varphi(t) \\
\|x\|_{M(\varphi)}=\sup _{0<s \leq 1} \frac{1}{\varphi(s)} \int_{0}^{s} x^{*}(t) d t
\end{gathered}
$$
\]

where $x^{*}(t)$ is the permutation of $|x(t)|$ in decreasing order. The space $\Lambda(\varphi)$ is separable, and $(\Lambda(\varphi))^{*}=M(\varphi)$.

If $\Phi(t)$ is a convex increasing function on $[0, \infty)$ and

$$
\lim _{t \rightarrow 0} \frac{\Phi(t)}{t}=\lim _{t \rightarrow \infty} \frac{t}{\Phi(t)}=0
$$

then the Orlitz space $L_{\Phi}$ is, by definition, the set of functions such that $\Phi(\varepsilon x) \in L_{1}$ for some $\varepsilon>0$ with norm

$$
\|x\|_{L_{\Phi}}=\inf \left\{\lambda: \lambda>0, \int_{0}^{1} \Phi\left(\frac{|x(t)|}{\lambda}\right) d t \leq 1\right\}<\infty .
$$

The condition $\Phi \in \Delta_{2}$ means that $\Phi(2 t) \leq C \Phi(t)$ for some constant $C>0$ and all $t>1$.

In any r.i. space $E$, the operator family

$$
\sigma_{\tau} x(t)=\left\{\begin{array}{l}
x(t / \tau), \quad 0 \leq t \leq \min (1, \tau) \\
0, \quad \min (1, \tau)<t \leq 1
\end{array}\right.
$$

acts continuously. The numbers

$$
\alpha_{E}=\lim _{\tau \rightarrow 0} \frac{\ln \left\|\sigma_{\tau}\right\|_{E}}{\ln \tau}, \quad \beta_{E}=\lim _{\tau \rightarrow \infty} \frac{\ln \left\|\sigma_{\tau}\right\|_{E}}{\ln \tau}
$$

are called the Boyd indices of the space $E$. We always have $0 \leq \alpha_{E} \leq \beta_{E} \leq 1$. We use the Calderon-Lozanovskii construction [2]. If $E_{0}$ and $E_{1}$ are r.i. spaces and $0<\theta<1$, then $E_{0}^{1-\theta} E_{1}^{\theta}$ denotes the space of functions with the norm

$$
\|x\|=\inf _{\left\|x_{0}\right\|_{E_{0}}=\left\|x_{1}\right\|_{E_{1}}=1} \sup _{0 \leq t \leq 1} \frac{|x(t)|}{\left|x_{0}(t)\right|^{1-\theta}\left|x_{1}(t)\right|^{\theta}}
$$

For any r.i. space $E$, the continuous embeddings $L_{\infty} \subset E \subset L_{1}$ hold. For an r.i. space $E, E^{0}$ denotes the closure of $L_{\infty}$ in $E$. If $E$ is separable, then $E^{0}=E$; if $E \neq L_{\infty}$, then $E^{0}$ is separable. The equality of two r.i. spaces means that they coincide as sets. By the closed graph theorem, their norms are equivalent. By $E^{\prime}$ we denote the space of functions for which

$$
\|x\|_{E^{\prime}}=\sup _{\|y\|_{E^{\prime}} \leq 1} \int_{0}^{1} x(t) y(t) d t<\infty .
$$

The space $E^{\prime}$ is isometrically embedded in $E^{*}$. If $E$ is separable, then $E^{\prime}=E^{*}$ and the norms are equal.

There exists a measure-preserving one-to-one mapping of $[0,1]$ onto the square $[0,1] \times[0,1]$; therefore, for any ri. space $E$ on $[0,1]$, the measure-preserving mapping generates a space of functions on the square isomeric to this space. We denote this function space on the square by the same symbol $E$. This allows us to consider the tensor multiplier $(x \otimes y)(t, s)=x(t) y(s)$ as a bilinear operator in $L_{p}$, where $1 \leq p \leq \infty$. Every r.i. space $E$ generates a space $M(E)$ of tensor multipliers with the norm

$$
\|x\|_{M_{(E)}}=\sup _{\|y\|_{E} \leq 1}\|x \otimes y\|_{E} .
$$

Obviously, the embeddings $L_{\infty} \subset \mathfrak{M}(E) \subset E$ hold. Tensor products of r.i. spaces are considered in $[3,4]$ and elsewhere, and the space of tensor multipliers are studied in [5]. In more detail, the ri. spaces are studied in [1, 6]. Subspaces generated by shifts of one function from an ri. space on $[0, \infty$ ) are examined in [7].

The central result of this paper is the following theorem.

Theorem 1. Let E be an ri. space. Then,

$$
\mathfrak{M}\left(E^{0}\right) \subset \mathfrak{M}(E) \subset \mathfrak{M}(E)
$$

Theorem 1 solves the problem of finding $\mathfrak{l}(E)$ for separable r.i. spaces. In this case, $\mathfrak{P}(E)$ coincides with $\mathfrak{M}(E)$. Using the results describing $\mathfrak{M}(E)$ [3-5], we obtain the following corollary.

Corollary 1. (i) If $1<p<\infty$ and $1 \leq q<\infty$, then $\mathfrak{P}\left(L_{p, q}\right)=L_{p, \min (p, q)}$ and $\mathfrak{P}\left(L_{p, \infty}^{0}\right)=L_{p}$.
(ii) If $\varphi \in \Omega$ and $\tilde{\varphi}(t)=\sup _{0<s \leq 1} \frac{\varphi(t s)}{\varphi(s)}$, then

$$
\Lambda(\tilde{\varphi}) \subset \mathfrak{N}(\Lambda(\varphi)) \subset \Lambda(\varphi) .
$$

The equality $\mathfrak{N}(\Lambda(\varphi))=\Lambda(\varphi)$ holds if and only if $\varphi(t s) \leq$ $C \varphi(t) \varphi(s)$ for some $C>0$ and all $t, s \in[0,1]$.
(iii) If $\Phi \in \Delta_{2}$, then $\mathfrak{P}\left(L_{\Phi}\right)=L_{\Phi}$ if and only if $\Phi$ is semimultiplicative, i.e., there exists a $C>0$ such that $\Phi(u v) \leq C \Phi(u) \Phi(v)$ for all $u, v \geq 1$.
(iv) If $E$ is a separable ri. space, then $\mathfrak{P}(E)=L_{\infty}$ if and only if $\alpha_{E}=0$.

For nonseparable r.i. spaces $E$, the set $\mathfrak{M}(E)$ may differ from $\mathfrak{M}_{( }(E)$, and the problem of describing $\mathfrak{N}(E)$ becomes more complicated.

Theorem 2. Suppose that $\varphi \in \Omega$ and $\lim _{t \rightarrow 0} \frac{\varphi(2 t)}{\varphi(t)}=2$. Then, $a \in M(\varphi)$ belongs to $\mathfrak{P}(M(\varphi)$ ) if and only if $a \in$ $L_{\infty}$ or $a \notin M^{0}(\varphi)$, i.e.,

$$
\mathfrak{M}(M(\varphi))=L_{\infty} \cup\left(M(\varphi) \backslash M^{0}(\varphi)\right) .
$$

Theorem 2 shows that the set $\mathfrak{l}(E)$ may be nonlinear if $E$ is nonseparable.

Theorem 3. If $\varphi \in \Omega$ and $\sup _{0<t \leq 1} \frac{\varphi(t)}{\varphi\left(t^{2}\right)}<\infty$, then

$$
\mathfrak{l}(M(\varphi))=\mathfrak{M}(M(\varphi))=M(\varphi)
$$

Let $\mathfrak{l}$ denote the set of r.i. spaces $E$ coinciding with $\mathfrak{l}(E)$. Theorem 3 shows that some nonseparable r.i. spaces belong to the class $\mathfrak{N}$. The class $\mathfrak{N}$ is stable with respect to the complex interpolation method.

Theorem 4. Suppose that $E_{0}$ and $E_{1}$ are separable r.i. spaces; $E_{0}, E_{1} \in \mathfrak{N}$; and $0<\theta<1$. Then, $E_{0}^{1-\theta} E_{1}^{\theta} \in \mathfrak{N}$.

Corollary 1(i) shows that the class $\mathfrak{R}$ is not stable with respect to the real interpolation method. Below, we give yet another description of the class $\mathfrak{N}$.

Theorem 5. Let $E$ be an ri. space. The following conditions are equivalent.
(i) $\mathfrak{M}(E)=E$;
(ii) $\mathfrak{N}(E)=E$;
(iii) There exists a constant $C>0$ such that

$$
\left\|\sum_{k=1}^{2^{n}} c_{n, k} a_{n, k}\right\|_{E} \leq C\|a\|_{E}\left\|\sum_{k=1}^{2^{n}} c_{n, k} \chi_{n, k}\right\|_{E}
$$

for all $a \in E$ and all $c_{n, k} \in \mathbb{R}^{1}$, where $k=1,2, \ldots, 2^{n}$ and $n=0,1, \ldots$; here, $\chi_{n, k}$ is the characteristic function of the interval $\left((k-1) \cdot 2^{-n}, k \cdot 2^{-n}\right)$.

The spaces $L_{p}$ have the following characteristic property in terms of the class $\mathfrak{N}$.

Theorem 6. Let E be an ri. space. The following conditions are equivalent.
(i) $\mathfrak{P}(E)=E$ and $\mathfrak{P}\left(E^{\prime}\right)=E^{\prime}$;
(ii) The tensor product operators $\otimes$ from $E \times E$ to $E$ and from $E^{\prime} \times E^{\prime}$ to $E^{\prime}$ are bounded;
(iii) There exists a $p \in[1, \infty]$ such that $E=L_{p}$.

Thus, if an r.i. space $E$ does not coincide with $L_{p}$ for $1 \leq p \leq \infty$, then the class $\mathfrak{N}$ cannot contain more than one of the spaces $E$ and $E^{\prime}$.

The problem of describing $\mathfrak{P}\left(L_{p, \infty}\right)(1<p<\infty)$ remains unsolved. It is clear only that $\mathfrak{M}\left(L_{p, \infty}\right) \cap$ $L_{p, \infty}^{0}=L_{p}$.

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## REFERENCES

1. Lindenstrauss, J. and Tzafriri, L., Classical Banach Spaces. Function Spaces, Berlin: Springer, 1978.
2. Maligranda, L., Orlicz Spaces and Interpolation: Seminars in Math., Campinas: Univ. Campinas, 1989, vol. 5.
3. O’Neil, R., J. Anal. Math., 1968, vol. 21, pp. 1-276.
4. Milman, M., Proc. Am. Math. Soc., 1981, vol. 83, pp. 743-746.
5. Astashkin, S.V., Funkts. Anal. Ego Prilozh., 1996, vol. 30, no. 4, pp. 58-60.
6. Krein, S.G., Petunin, Yu.I., and Semenov, E.M., Interpolyatsiya lineinykh operatorov (Interpolation of Linear Operators), Moscow: Nauka, 1978.
7. Hernandez, F.L. and Semenov, E.M., J. Funct. Anal., 1999, vol. 169, pp. 52-80.

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