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On a decomposition of some functions

Abstract. A representation of submultiplicative and supermultiplicative functions on $(0, 1)$ is given with some applications to Orlicz spaces.

1. Decomposition theorem. Let I be a subset of $\mathbf{R}_+ = [0, \infty)$ such that $xy \in I$ whenever $x, y \in I$, i.e.,

$$(1) \quad I \cdot I \subset I,$$

and let $f: I \rightarrow \mathbf{R}_+$ be a measurable positive function which is zero at zero if $0 \in I$ satisfying the inequality

$$(2) \quad f(xy) \leq f(x)f(y) \quad \text{for all } x, y \in I.$$

Then f will be called *submultiplicative on I* . If the reverse inequality holds, then we say that f is *supermultiplicative on I* .

Examples of submultiplicative functions on $(0, 1)$, $(0, \infty)$ and $[1, \infty)$ may be found in [7], [12] and [6]. They appear in many places and are related to diverse subjects.

THEOREM 1. (a) If f is a submultiplicative function on $I = (0, 1)$, then

$$\alpha = \lim_{x \rightarrow 0^+} \frac{\ln f(x)}{\ln x}$$

exists and

$$(3) \quad f(x) = x^\alpha g(x)$$

with $g(x) \geq 1$ for $x \in I$ and $\lim_{x \rightarrow 0^+} x^\varepsilon g(x) = 0$ for every $\varepsilon > 0$.

(b) If f is a supermultiplicative function on $I = (0, 1)$, then

$$\beta = \lim_{x \rightarrow 0^+} \frac{\ln f(x)}{\ln x}$$

exists and

$$(4) \quad f(x) = x^\beta h(x)$$

with $h(x) \leq 1$ for $x \in I$ and $\lim_{x \rightarrow 0^+} x^{-\varepsilon} h(x) = \infty$ for every $\varepsilon > 0$.

Proof. Note that if f is submultiplicative on I , then $1/f$ is supermultiplicative on I and vice versa. Therefore, it is enough to prove (a). Let $f(xy) \leq f(x)f(y)$ for $x, y \in I$ and let $F(x) = \ln f(e^{-x})$. Then

$$F(x+y) \leq F(x) + F(y) \quad \text{for } x, y \in (0, \infty).$$

Hence F is a measurable subadditive function on $(0, \infty)$. A result from [7], p. 244, asserts that

$$\lim_{x \rightarrow \infty} F(x)/x = \inf_{x > 0} F(x)/x = -\alpha.$$

Replacing $-x$ by $\ln y$ yields

$$\alpha = \lim_{x \rightarrow \infty} \frac{\ln f(e^{-x})}{-x} = \lim_{y \rightarrow 0^+} \frac{\ln f(y)}{\ln y}.$$

If $0 < g(x_0) < 1$ for some $0 < x_0 < 1$, then for any n

$$\ln f(x_0^n)/\ln x_0^n = \alpha + \ln g(x_0^n)/\ln x_0^n \geq \alpha + \ln g(x_0)/\ln x_0 > \alpha,$$

and so

$$\lim_{x \rightarrow 0^+} \ln f(x)/\ln x \geq \alpha + \ln g(x_0)/\ln x_0 > \alpha.$$

This contradiction means that $g(x) \geq 1$ for $x \in I$.

If $\lim_{x \rightarrow 0^+} x^\varepsilon g(x) > 0$ for some $\varepsilon > 0$, then there exist constants $c > 0$, $x_0 > 0$, with $g(x) \geq cx^{-\varepsilon}$ for $0 < x < x_0$, and

$$\ln f(x)/\ln x = \alpha + \ln g(x)/\ln x \leq \alpha + (\ln c - \varepsilon \ln x)/\ln x$$

so that

$$\lim_{x \rightarrow 0^+} \ln f(x)/\ln x \leq \alpha - \varepsilon,$$

a contradiction. This completes the proof.

Note that if $\lim_{x \rightarrow 0^+} f(x) = 0$ then $\alpha > 0$. Indeed, if $0 < f(x_0) < 1$ for some $0 < x_0 < 1$, then

$$\ln f(x_0^n)/\ln x_0^n \geq \ln f(x_0)/\ln x_0 = \ln f(x_0)/\ln x_0$$

and hence

$$\alpha = \lim_{x \rightarrow \infty} \ln f(x_0^n)/\ln x_0^n \geq \ln f(x_0)/\ln x_0 > 0.$$

Remark 1. The above representation for a bounded supermultiplicative function was stated in [3], p. 147, and used to obtain some estimate of the modulus of convexity of Lorentz sequence spaces. In [1] a representation theorem for supermultiplicative functions on $(0, 1)$ was proved with a nondecreasing factor h . But, as we will see in some examples (submultiplicative case), the factor h is not always a monotonic function.

Remark 2. The above theorem is also true for functions on $I = (1, \infty)$, because f is submultiplicative or supermultiplicative on $(1, \infty)$ if and only if $f_*(x) = 1/f(1/x)$ is supermultiplicative or submultiplicative on $(0, 1)$, respectively.

Let us give some examples (always $p \geq 1$).

EXAMPLE 1. Let I be an interval such that (1) holds and let $f(x)$ be x^p for x rational from I and $2x^p$ for x irrational from I . Then f is submultiplicative on I , $\alpha = p$ and $g(x) = x^{-p}f(x)$ is not monotonic on any subinterval of I .

EXAMPLE 2 (see [12], Ex. 5). Let $f(x) = x^p(1 + |\sin \ln x|)$ or $f(x) = x^p e^{|\sin \ln x|}$. Then f is a submultiplicative continuous increasing function on $(0, \infty)$ with $f(1) = 1$ and with $g(x) = x^{-p}f(x)$ not monotonic on any interval $(0, \varepsilon)$, $0 < \varepsilon \leq 1$.

EXAMPLE 3. Define, for $x \geq 0$ and $n = 1, 2, \dots$,

$$u(x) = \begin{cases} x/n & \text{if } (n-1)n^2 \leq x \leq n^2(n+2), \\ n(n+1)(n+2) - x & \text{if } n^2(n+2) \leq x \leq n(n+1)^2. \end{cases}$$

Then $u(x)/x$ is nonincreasing and so u is subadditive on $(0, \infty)$. This example was introduced in [3], p. 141. Let, for $x \in (0, 1)$ and $p \geq 2$,

$$f(x) = x^p g(x) \quad \text{with } g(x) = \exp\left(u\left(\ln \frac{1}{x}\right)\right).$$

Then f is a submultiplicative increasing convex function on $(0, 1)$, $\alpha = p$, $\lim_{x \rightarrow 0+} g(x) = \infty$ and g is not monotonic on any interval $(0, \varepsilon)$, $0 < \varepsilon < 1$.

EXAMPLE 4. Define, for $x \geq 0$ and $n = 2, 3, \dots$,

$$v(x) = \begin{cases} x/2 & \text{if } 0 \leq x \leq 2, \\ x/n! & \text{if } n! \leq x \leq nn!, \\ n^2 - (n-1)x/n! & \text{if } nn! \leq x \leq (n+1)!. \end{cases}$$

Then $v(x)/x$ is nonincreasing and so v is subadditive on $(0, \infty)$. Let, for $x \in (0, 1)$, and $p \geq 2$,

$$f(x) = x^p g(x) \quad \text{with } g(x) = \exp\left(v\left(\ln \frac{1}{x}\right)\right).$$

Then f is a submultiplicative increasing convex function on $(0, 1)$, $\alpha = p$, $\liminf_{x \rightarrow 0+} g(x) = \exp(\lim_{n \rightarrow \infty} v(n!)) = e$, $\limsup_{x \rightarrow 0+} g(x) = \exp(\lim_{n \rightarrow \infty} v(nn!)) = \infty$ and g is not monotonic on any interval $(0, \varepsilon)$, $0 < \varepsilon < 1$.

2. Vector-valued Orlicz spaces and Orlicz spaces on product spaces. Let us recall some notations from the theory of Orlicz spaces. An Orlicz function φ is a continuous convex increasing function on $[0, \infty)$ so that $\varphi(0) = 0$.

The Orlicz space $L_\varphi = L_\varphi(S)$ on a σ -finite measure space (S, Σ, μ) is the Banach space of Σ -measurable functions (with the usual identification) defined by

$$L_\varphi = \{x: S \rightarrow \mathbb{R} \text{ measurable} \mid m_\varphi(rx) = \int_S \varphi(r|x(s)|) d\mu(s) < \infty \text{ for some } r > 0\}$$

with the Luxemburg-Nakano norm

$$\|x\|_\varphi = \inf\{a > 0: m_\varphi(x/a) \leq 1\}.$$

In recent years a number of papers have appeared in which spaces of vector-valued functions are considered. In the proof of theorems about vector-valued L_p spaces it is often used that $L_p(S_1, L_p(S_2)) = L_p(S_2 \times S_1)$, that is, it is possible to consider that space as an L_p space on a product space. An essential limitation to the extension for Orlicz spaces will be the content of the following theorem.

THEOREM 2. Let $I = (0, 1)$ and let $\varphi_1, \varphi_2, \varphi$ be Orlicz functions. Let $L_{\varphi_1}(I, L_{\varphi_2}(I))$ and $L_\varphi(I \times I)$ be the vector-valued Orlicz space on I and the Orlicz space on $I \times I$. Then

$$(5) \quad L_{\varphi_1}(I, L_{\varphi_2}(I)) = L_\varphi(I \times I)$$

if and only if $L_{\varphi_1}(I) = L_{\varphi_2}(I) = L_\varphi(I) = L_p(I)$ for some $p \geq 1$.

Proof. It is sufficient to prove the necessity. Assume that for some positive constants a and b ,

$$a\|x\|_{L_{\varphi_1}(I, L_{\varphi_2}(I))} \leq \|x\|_{L_\varphi(I \times I)} \leq b\|x\|_{L_{\varphi_1}(I, L_{\varphi_2}(I))}.$$

Then, in particular, for any measurable subsets A and B of I ,

$$a\|1_{A \times B}\|_{L_{\varphi_1}(I, L_{\varphi_2}(I))} \leq \|1_{A \times B}\|_{L_\varphi(I \times I)} \leq b\|1_{A \times B}\|_{L_{\varphi_1}(I, L_{\varphi_2}(I))},$$

i.e.,

$$(6) \quad \frac{a}{\varphi_1^{-1}\left(\frac{1}{mA}\right)\varphi_2^{-1}\left(\frac{1}{mB}\right)} \leq \frac{1}{\varphi^{-1}\left(\frac{1}{mA mB}\right)} \leq \frac{b}{\varphi_1^{-1}\left(\frac{1}{mA}\right)\varphi_2^{-1}\left(\frac{1}{mB}\right)}.$$

First, putting $mA = 1/u$ and $mB = 1$ in (6), and then $mA = 1$ and $mB = 1/u$ we get

$$\frac{\varphi_2^{-1}(1)}{b} \varphi_1^{-1}(u) \leq \varphi^{-1}(u) \leq \frac{\varphi_2^{-1}(1)}{a} \varphi_1^{-1}(u) \quad \text{for } u > 1,$$

and

$$\frac{\varphi_1^{-1}(1)}{b} \varphi_2^{-1}(u) \leq \varphi^{-1}(u) \leq \frac{\varphi_1^{-1}(1)}{a} \varphi_2^{-1}(u) \quad \text{for } u > 1,$$

respectively. The above inequalities mean that $L_{\varphi_1}(I) = L_{\varphi}(I) = L_{\varphi_2}(I)$ and

$$\frac{c}{\varphi^{-1}\left(\frac{1}{mA}\right)\varphi^{-1}\left(\frac{1}{mB}\right)} \leq \frac{1}{\varphi^{-1}\left(\frac{1}{mA mB}\right)} \leq \frac{d}{\varphi^{-1}\left(\frac{1}{mA}\right)\varphi^{-1}\left(\frac{1}{mB}\right)}$$

with $c = a\varphi_1^{-1}(1)\varphi_2^{-1}(1)/b^2$, $d = cb^3/a^3$. Therefore, $f(u) = c/\varphi^{-1}(1/u)$ is a supermultiplicative function on $I = (0, 1)$ and $f(u) = d/\varphi^{-1}(1/u)$ is a submultiplicative function on I . From the representation theorem:

$$\frac{c}{\varphi^{-1}(1/u)} = u^{1/p}h(u) \quad \text{with } h(u) \leq 1, \quad \frac{d}{\varphi^{-1}(1/u)} = u^{1/p}g(u) \quad \text{with } g(u) \geq 1,$$

and so

$$\frac{c}{\varphi^{-1}(1/u)} = u^{1/p}h(u) \leq u^{1/p} \leq u^{1/p}g(u) = \frac{d}{\varphi^{-1}(1/u)} \quad \text{for all } u \in I.$$

Hence, $d^{-p}v^p \leq \varphi(v) \leq c^{-p}v^p$ for $v \in [d, \infty)$ and so $L_{\varphi}(I) = L_p(I)$.

Remark 3. If (S_i, Σ_i, μ_i) , $i = 1, 2$, are nonatomic σ -finite measure spaces, then $L_{\varphi_1}(S_1, L_{\varphi_2}(S_2)) = L_{\varphi}(S_2 \times S_1)$ if and only if $L_{\varphi_1}(S_i) = L_{\varphi_2}(S_i) = L_{\varphi}(S_i) = L_p(S_i)$, $i = 1, 2$, for some $p \geq 1$. The proof is the same as that of the above theorem.

Remark 4. The first part of our Theorem 2, i.e., only the equalities $L_{\varphi_1} = L_{\varphi_2} = L_{\varphi}$, was also proved in [2], [5], [9], [13].

3. Strictly singular inclusions between some Orlicz spaces. Now we will consider the case of Orlicz sequence spaces l_{φ} . Kalton [8] proved that if an Orlicz function φ satisfies the Δ_2 -condition at zero (i.e. $\limsup_{u \rightarrow 0^+} \varphi(2u)/\varphi(u) < \infty$) and $\varphi(u) \geq Cu^p$ for some $C > 0$, $1 \leq p < \infty$ and for every $0 \leq u \leq 1$, then the imbedding $i: l_{\varphi} \hookrightarrow l_p$ is a strictly singular operator (i.e. there is no infinite-dimensional subspace E of l_{φ} such that $i|_E$ is an isomorphism) if and only if

$$(7) \quad s(\varphi, p) = \liminf_{\varepsilon \rightarrow 0^+} \inf_{0 < s \leq 1} \frac{1}{\ln(1/\varepsilon)} \int_{\varepsilon}^1 \frac{\varphi(st)}{s^p t^{p+1}} dt = \infty.$$

For example, if $\varphi_{p,q}(u) = u^p(1 + |\ln u|)^q$ with $q \geq 1$ and $p \geq 3q$ then $\varphi_{p,q}$ is an Orlicz function and

$$s(\varphi_{p,q}, p) \geq s(\varphi_{p,1}, p) = \liminf_{\varepsilon \rightarrow 0^+} \inf_{0 < s \leq 1} \left(\ln \frac{e}{s} + \frac{1}{2} \ln \frac{1}{\varepsilon} \right) = \infty,$$

and so the imbedding $l_{\varphi_{p,q}} \hookrightarrow l_p$ is strictly singular.

Note that $\varphi_{p,q}$ is a submultiplicative function on $(0, \infty)$. In general it is not a simple matter to verify when (7) holds, but using the above representation theorem we prove the strict singularity of the imbedding $l_{\varphi} \hookrightarrow l_p$, provided that φ is a submultiplicative or supermultiplicative function on $(0, 1)$.

THEOREM 3. Let φ be an Orlicz function satisfying the Δ_2 -condition at zero and such that $\varphi(u) \geq Cu^p$ for some $C > 0$, $1 < p < \infty$ and every $0 \leq u \leq 1$. Let $\limsup_{u \rightarrow 0^+} \varphi(u)/u^p = \infty$, i.e., φ is not equivalent at zero to u^p . If φ is either a supermultiplicative function on $(0, 1)$ or a submultiplicative function on $(0, 1)$ with $\lim_{u \rightarrow 0^+} \varphi(u)/u^p = \infty$, then the imbedding $i: l_\varphi \hookrightarrow l_p$ is strictly singular.

Proof. First, if $\varphi(u)$ is equivalent at zero to u^q with $1 \leq q < p$ then $i: l_\varphi \hookrightarrow l_p$ is strictly singular.

If φ is a submultiplicative function on $(0, 1)$ not equivalent at zero to u^q for any $q \geq 1$, then by assumption and the representation theorem

$$Cu^p \leq \varphi(u) = u^\beta h(u) \leq u^\beta \quad \text{for } 0 \leq u \leq 1,$$

and so $\beta < p$ (if $\beta = p$, then $\varphi(u)$ is equivalent at zero to u^p). The exponent β is precisely the Matuszewska-Orlicz index β_φ (see [4], [12]). Therefore, being the intervals associated to the functions φ and u^p disjoint, it follows from a result of Lindberg [10] that l_φ and l_p are totally incomparable, i.e., they have no isomorphic infinite-dimensional subspaces and so any bounded linear operator from l_φ into l_p is strictly singular. In particular, the imbedding $l_\varphi \hookrightarrow l_p$ is strictly singular.

If φ is a submultiplicative function on $(0, 1)$ not equivalent at zero to u^q for any $q \geq 1$, then by Theorem 1(a) we have

$$u^\alpha \leq u^\alpha g(u) = \varphi(u) \quad \text{for } 0 \leq u \leq 1,$$

and by the assumption,

$$\alpha = \lim_{u \rightarrow 0^+} \frac{\ln \varphi(u)}{\ln u} \leq \lim_{u \rightarrow 0^+} \frac{\ln(Cu^p)}{\ln u} = p.$$

If $\alpha < p$ then the imbedding $l_\alpha \hookrightarrow l_p$ is strictly singular and since the imbedding $l_\varphi \hookrightarrow l_\alpha$ is continuous, it follows that the imbedding $l_\varphi \hookrightarrow l_p$ is strictly singular. We have used here the well-known fact that the composition of a strictly singular operator with a bounded operator is also strictly singular.

Let $\alpha = p$ and $\lim_{u \rightarrow 0^+} g(u) = \infty$. The function

$$\tilde{g}(u) = \inf_{0 < s \leq 1} g(su) = \inf_{0 < t \leq u} g(t)$$

is nonincreasing, $\tilde{g}(u) \leq g(u)$ and $\lim_{u \rightarrow 0^+} \tilde{g}(u) = \infty$. Therefore,

$$\begin{aligned} s(\varphi, p) &= \liminf_{\varepsilon \rightarrow 0^+} \inf_{0 < s \leq 1} \frac{1}{\ln(1/\varepsilon)} \int_{\varepsilon}^1 (g(st)/t) dt \geq \liminf_{\varepsilon \rightarrow 0^+} \frac{1}{\ln(1/\varepsilon)} \int_{\varepsilon}^1 (\tilde{g}(t)/t) dt \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{\int_{\varepsilon}^1 (\tilde{g}(t)/t) dt}{\frac{1}{\ln(1/\varepsilon)}} = \lim_{\varepsilon \rightarrow 0^+} \frac{\tilde{g}(\varepsilon)/\varepsilon}{1/\varepsilon} = \infty, \end{aligned}$$

Hence (7) holds and the imbedding $l_\varphi \hookrightarrow l_p$ is strictly singular.

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On a decomposition of some functions.

Comment. Math. Prace Mat. 30 (1991), no. 2, 285–291.

A representation of submultiplicative and supermultiplicative functions on $(0, 1)$ is given with some applications to Orlicz spaces. For example. Theorem 3: Let φ be an Orlicz function satisfying the Δ_2 -condition at zero and such that $\varphi(u) \geq C u^p$ for some $C > 0$, $1 < p < \infty$ and every $0 \leq u \leq 1$. If φ is either a supermultiplicative or a submultiplicative function on $(0, 1)$ with $\lim_{u \rightarrow 0} \varphi(u)/u^p = \infty$, then the embedding $i: l_\varphi \hookrightarrow l_p$ is strictly singular (i.e., there is no infinite-dimensional subspace E of l_φ such that $i|_E$ is an isomorphism).
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Zbl. Math. 756

Finol, C. E.; Maligranda, L.

On a decomposition of some functions.

Ann. Soc. Math. Pol., Ser. I, Commentat. Math. 30, No. 2, 285–291 (1991).

The authors give a representation of submultiplicative and supermultiplicative functions on $(0, 1)$ with some applications to Orlicz spaces.
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2 Б663. О разложении некоторых функций. On a decomposition of some functions / Finol C. E., Maligranda L. // *Rocz. PTM. Ser. I.*— 1991.— 30, № 2.— С. 285–291.— Англ.

В теореме 1 устанавливаются некоторые свойства функции $\frac{f(x)}{x^\alpha}$, где $\alpha = \lim_{x \rightarrow +0} \frac{\ln f(x)}{\ln x}$, для субмультипликативных и супермультипликативных на отрезке $I = [0, 1]$ функций. В теореме 2 доказывается, что равенство $L_{\varphi_1}(I, L_{\varphi_2}(I)) = L_{\varphi_0}(I \times I)$ возможно тогда и только тогда, когда $L_{\varphi_1}(I) = L_{\varphi_2}(I) = L_{\varphi_0}(I) = L_p$ ($p \geq 1$). Через L_{φ_i} ($i = 0, 1, 2$) обозначаются пространства Орлича, через $L_{\varphi_1}(I, L_{\varphi_2}(I))$ — пространство Орлича L_{φ_1} вектор-функций со значениями в пространстве Орлича L_{φ_2} . В теореме 3 для координатных пространств Орлича l_φ устанавливаются новые условия строгой сингулярности оператора вложения $l_\varphi \hookrightarrow l_p$ (ср. РЖМат, 1977, 11Б751). Библи. 14.
Я. Рутницкий