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Structure of Cesàro function spaces ☆

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ABSTRACT

The structure of the Cesàro function spaces Ces_p on both $[0,1]$ and $[0,\infty)$ for $1 < p \leqslant \infty$ is investigated. We find their dual spaces, which equivalent norms have different description on $[0,1]$ and $[0,\infty)$. The spaces Ces_p for $1 < p < \infty$ are not reflexive but strictly convex. They are not isomorphic to any L^q space with $1 \leqslant q \leqslant \infty$. They have "near zero" complemented subspaces isomorphic to l^p and "in the middle" contain an asymptotically isometric copy of l^1 and also a copy of $L^1[0,1]$. They do not have Dunford–Pettis property but they do have the weak Banach–Saks property. Cesàro function spaces on $[0,1]$ and $[0,\infty)$ are isomorphic for $1 < p \leqslant \infty$. Moreover, we give characterizations in terms of p and q when $\mathrm{Ces}_p[0,1]$ contains an isomorphic copy of l^q.

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Let $1 \leqslant p \leqslant \infty$. The *Cesàro sequence space* ces_p is defined as the set of all real sequences $x = \{x_k\}$ such that

$$\|x\|_{c(p)} = \left[\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p\right]^{1/p} < \infty \quad \text{when } 1 \leqslant p < \infty$$

and

$$\|x\|_{c(\infty)} = \sup_{n\in\mathbf{N}} \frac{1}{n}\sum_{k=1}^{n}|x_k| < \infty \quad \text{when } p = \infty.$$

The *Cesàro function spaces* $\mathrm{Ces}_p = \mathrm{Ces}_p(I)$ are the classes of Lebesgue measurable real functions f on $I = [0,1]$ or $I = [0,\infty)$ such that

$$\|f\|_{C(p)} = \left[\int_I \left(\frac{1}{x}\int_0^x |f(t)|\,dt\right)^p dx\right]^{1/p} < \infty \quad \text{for } 1 \leqslant p < \infty$$

and

$$\|f\|_{C(\infty)} = \sup_{x\in I,\,x>0} \frac{1}{x}\int_0^x |f(t)|\,dt < \infty \quad \text{for } p = \infty.$$

The Cesàro sequence spaces ces_p and ces_∞ appeared in 1968 in connection with the problem of the Dutch Mathematical Society to find their duals. Some investigations of ces_p were done by Shiue [50] in 1970. Then Leibowitz [36] and Jagers [26] proved that $\mathrm{ces}_1 = \{0\}$, ces_p are separable reflexive Banach spaces for $1 < p < \infty$ and the l^p spaces are continuously and strictly embedded into ces_p for $1 < p \leqslant \infty$. More precisely, $\|x\|_{c(p)} \leqslant p'\|x\|_p$ for all $x \in l^p$ with $p' = \frac{p}{p-1}$ when $1 < p < \infty$ and $p' = 1$ when $p = \infty$. Moreover, if $1 < p < q \leqslant \infty$, then $\mathrm{ces}_p \subset \mathrm{ces}_q$ with continuous strict embedding. Bennett [8] proved that ces_p for $1 < p < \infty$ are not isomorphic to any l^q space with $1 \leqslant q \leqslant \infty$ (see also [45] for another proof).

Several geometric properties of the Cesàro sequence spaces ces_p were studied in the last years by many mathematicians (see e.g. [10–16,34]). Some more results on ces_p can be found in two books [8,39].

In 1999–2000 it was proved by Cui and Hudzik [11], Cui, Hudzik and Li [14] and Cui, Meng and Płuciennik [16] that the Cesàro sequence spaces ces_p for $1 < p < \infty$ have the fixed point property (cf. also [10, Part 9]). Maligranda, Petrot and Suantai [45] proved that the Cesàro sequence spaces ces_p for $1 < p < \infty$ are not uniformly non-square, that is, there are sequences $\{x_n\}$ and $\{y_n\}$ on the unit sphere such that $\lim_{n\to\infty} \min(\|x_n + y_n\|_{c(p)}, \|x_n - y_n\|_{c(p)}) = 2$. They even proved that these spaces are not B-convex.

The Cesàro function spaces $\mathrm{Ces}_p[0,\infty)$ for $1 \leqslant p \leqslant \infty$ were considered by Shiue [51], Hassard and Hussein [25] and Sy, Zhang and Lee [54]. The space $\mathrm{Ces}_\infty[0,1]$

appeared already in 1948 and it is known as the Korenblyum, Krein and Levin space K (see [31] and [59]).

Recently, we proved in the paper [4] that, in contrast to Cesàro sequence spaces, the Cesàro function spaces $\mathrm{Ces}_p(I)$ on both $I=[0,1]$ and $I=[0,\infty)$ for $1<p<\infty$ are not reflexive and they do not have the fixed point property. In other paper [5] we investigated Rademacher sums in $\mathrm{Ces}_p[0,1]$ for $1\leqslant p\leqslant\infty$. The description is different for $1\leqslant p<\infty$ and $p=\infty$.

We recall some notions and definitions which we will need later on. By $L^0=L^0(I)$ we denote the set of all equivalence classes of real-valued Lebesgue measurable functions defined on $I=[0,1]$ or $I=[0,\infty)$. A *normed function lattice* or *normed ideal space* $X=(X,\|\cdot\|)$ (on I) is understood to be a normed space in $L^0(I)$, which satisfies the so-called ideal property: if $|f|\leqslant|g|$ a.e. on I and $g\in X$, then $f\in X$ and $\|f\|\leqslant\|g\|$. If, in addition, X is a complete space, then we say that X is a *Banach function lattice* or a *Banach ideal space* (on I). Sometimes we write $\|\cdot\|_X$ to be sure in which space the norm is taken.

For two normed ideal spaces X and Y on I the symbol $X\hookrightarrow Y$ means that $X\subset Y$ and the imbedding is continuous, and the symbol $X\overset{C}{\hookrightarrow}Y$ means that $X\hookrightarrow Y$ with the inequality $\|x\|_Y\leqslant C\|x\|_X$ for all $x\in X$. Moreover, notation $X\simeq Y$ means that these two spaces are isomorphic.

For a normed ideal space $X=(X,\|\cdot\|)$ on I and $1\leqslant p<\infty$ the *p-convexification* $X^{(p)}$ of X is the space of all $f\in L^0(I)$ such that $|f|^p\in X$ with the norm

$$\|f\|_{X^{(p)}}:=\big\||f|^p\big\|_X^{1/p}.$$

$X^{(p)}$ is also a normed ideal space on I.

For a normed ideal space $X=(X,\|\cdot\|)$ on I the *Köthe dual* (or *associated space*) X' is the space of all $f\in L^0(I)$ such that the *associate norm*

$$\|f\|':=\sup_{g\in X,\|g\|_X\leqslant 1}\int_I|f(x)g(x)|\,dx$$

is finite. The Köthe dual $X'=(X',\|\cdot\|')$ is a Banach ideal space. Moreover, $X\subset X''$ with $\|f\|\leqslant\|f\|''$ for all $f\in X$ and we have equality $X=X''$ with $\|f\|=\|f\|''$ if and only if the norm in X has the *Fatou property*, that is, if $0\leqslant f_n\nearrow f$ a.e. on I and $\sup_{n\in\mathbf{N}}\|f_n\|<\infty$, then $f\in X$ and $\|f_n\|\nearrow\|f\|$.

For a normed ideal space $X=(X,\|\cdot\|)$ on I with the Köthe dual X' we have the following Hölder type inequality: if $f\in X$ and $g\in X'$, then fg is integrable and

$$\int_I|f(x)g(x)|\,dx\leqslant\|f\|_X\|g\|_{X'}.$$

A function f in a normed ideal space X on I is said to have *absolutely continuous norm* in X if, for any decreasing sequence of Lebesgue measurable sets $A_n\subset I$ with empty intersection, we have that $\|f\chi_{A_n}\| \to 0$ as $n \to \infty$. The set of all functions in X with absolutely continuous norm is denoted by X_a. If $X_a = X$, then the space X itself is said to have *absolutely continuous norm*. For a normed ideal space X with absolutely continuous norm, the Köthe dual X' and the dual space X^* coincide. Moreover, a Banach ideal space X is reflexive if and only if both X and its associate space X' have absolutely continuous norms.

For general properties of normed and Banach ideal spaces we refer to the books Krein, Petunin and Semenov [32], Kantorovich and Akilov [28], Bennett and Sharpley [9], Lindenstrauss and Tzafriri [38] and Maligranda [43].

The paper is organized as follows: In Section 1 some necessary definitions and notation are collected. In Section 2 some simple results on Cesàro function spaces are presented. In particular, we can see that the Cesàro function spaces $\mathrm{Ces}_p(I)$ are not reflexive but strictly convex for all $1 < p < \infty$.

Sections 3 and 4 contain results on the dual and Köthe dual of Cesàro function spaces. There is a big difference between the cases on $[0, \infty)$ and on $[0, 1]$, as we can see from Theorems 2 and 3. This was also the reason why we put these investigations into two parts. Important in our investigations were earlier results on the Köthe dual $(\mathrm{ces}_p)'$ and remark on the Köthe dual $(\mathrm{Ces}_p[0, \infty))'$ due to Bennett [8]. This remark was recently proved, even for more general spaces, by Kerman, Milman and Sinnamon [30]. Luxemburg and Zaanen [42] gave a description of the Köthe dual $(\mathrm{Ces}_\infty[0, 1])'$.

Section 5 deals with the p-concavity and cotype of Cesàro sequence spaces ces_p and Cesàro function spaces $\mathrm{Ces}_p(I)$. It is shown, in Theorem 4, that they are p-concave for $1 < p < \infty$ with constant one and, thus, they have cotype $\max(p, 2)$.

In Section 6 it is proved, in Theorem 6, that the Cesàro function spaces $\mathrm{Ces}_p(I)$ contain an order isomorphic and complemented copy of l^p. Therefore, they do not have the Dunford–Pettis property. This result and cotype property imply that $\mathrm{Ces}_p(I)$ are not isomorphic to any $L^q(I)$ space for $1 \leqslant q \leqslant \infty$ (Theorem 7).

The authors proved in [4] that “in the middle” Cesàro function spaces $\mathrm{Ces}_p(I)$ contain an asymptotically isometric copy of l^1 and consequently they are not reflexive and do not have the fixed point property. This is a big difference with Cesàro sequence spaces ces_p, which for $1 < p < \infty$ are reflexive and which have the fixed point property.

Section 7 contains the proof that the Cesàro function spaces $\mathrm{Ces}_p[0, 1]$ for $1 \leqslant p < \infty$ have the weak Banach–Saks property. Important role in the proof will be played by the description of the dual space given in Section 4.

In Section 8 we present a construction showing that the Cesàro function spaces $\mathrm{Ces}_p[0, \infty)$ and $\mathrm{Ces}_p[0, 1]$ for $1 < p \leqslant \infty$ are isomorphic. The isomorpisms are different in the cases $1 < p < \infty$ and $p = \infty$.

In Section 9 it is proved that $\mathrm{Ces}_p[0, 1]$ contains an isomorphic copy of l^q if and only if $q \in [1, 2]$ for the case $1 \leqslant p \leqslant 2$ and in the case when $p > 2$ this can happen when either $q \in [1, 2]$ or $q = p$. This result is, in fact, different from the one for $L^p[0, 1]$ space.

The Cesàro function spaces $\mathrm{Ces}_p[0,\infty)$ for $1 \leqslant p \leqslant \infty$ were considered by Shiue [51], Hassard and Hussein [25] and Sy, Zhang and Lee [54]. The space $\mathrm{Ces}_\infty[0,1]$ appeared in 1948 and it is known as the Korenblyum, Krein and Levin space K (see [31] and [59, p. 26 and 61]).

We collect some known or clear properties of $\mathrm{Ces}_p(I)$ for both $I = [0,1]$ and $I = [0,\infty)$ in one place.

Theorem 1.

(a) *If* $1 < p \leqslant \infty$, *then* $\mathrm{Ces}_p(I)$ *are Banach spaces,* $\mathrm{Ces}_1[0,1] = L^1_w$ *with the weight* $w(t) = \ln \frac{1}{t}$, $t \in (0,1]$ *and* $\mathrm{Ces}_1[0,\infty) = \{0\}$.

(b) *The spaces* $\mathrm{Ces}_p(I)$ *are separable for* $1 < p < \infty$ *and* $\mathrm{Ces}_\infty(I)$ *is non-separable.*

(c) *If* $1 < p \leqslant \infty$, *then* $L^p(I) \overset{p'}{\hookrightarrow} \mathrm{Ces}_p(I)$, *where* $p' = \frac{p}{p-1}$ *and the embedding is strict.*

(d) *If* $1 < p < \infty$, *then* $\mathrm{Ces}_p[0,1]_{|[0,a]} \hookrightarrow L^1[0,a]$ *for any* $a \in (0,1)$ *but not for* $a = 1$ *and* $\mathrm{Ces}_p[0,\infty)_{|[0,a]} \hookrightarrow L^1[0,a]$ *for any* $0 < a < \infty$ *but not for* $a = \infty$, *that is,* $\mathrm{Ces}_p[0,\infty) \not\subset L^1[0,\infty)$. *Moreover,* $\mathrm{Ces}_\infty[0,1] \overset{1}{\hookrightarrow} L^1[0,1]$.

(e) *If* $1 < p < q \leqslant \infty$, *then* $\mathrm{Ces}_q[0,1] \overset{1}{\hookrightarrow} \mathrm{Ces}_p[0,1]$ *and the embedding is strict.*

(f) *The spaces* $\mathrm{Ces}_p(I)$ *are not rearrangement invariant.*

(g) *The spaces* $\mathrm{Ces}_p(I)$ *are not reflexive.*

(h) *The spaces* $\mathrm{Ces}_p(I)$ *for* $1 < p < \infty$ *are strictly convex, that is, if* $\|f\|_{C(p)} = \|g\|_{C(p)} = 1$ *and* $f \neq g$, *then* $\|\frac{f+g}{2}\|_{C(p)} < 1$.

Proof. (a), (b) Shiue [51] and Hassard and Hussein [25] proved that $\mathrm{Ces}_p(I)$ are separable Banach spaces for $1 < p < \infty$ and non-separable ones for $p = \infty$. We only show here that $\mathrm{Ces}_1[0,1]$ is a weighted $L^1_w[0,1]$ space with the weight $w(t) = \ln \frac{1}{t}$ for $0 < t \leqslant 1$ and $\mathrm{Ces}_1[0,\infty) = \{0\}$. In fact,

$$\int_0^1 \left(\frac{1}{x} \int_0^x |f(t)|\, dt \right) dx = \int_0^1 \left(\int_t^1 \frac{1}{x}\, dx \right) |f(t)|\, dt = \int_0^1 |f(t)| \ln \frac{1}{t}\, dt. \tag{1}$$

Moreover, if $f \in L^0[0,\infty)$ and $|f(x)| > 0$ for $x \in A$ with $0 < m(A) < \infty$, then there exists sufficiently large $a > 0$ such that $\delta = \int_0^a |f(t)|\, dt > 0$. Therefore, for $b > a$, it yields that

$$\begin{aligned} \int_0^b \left(\frac{1}{x} \int_0^x |f(t)|\, dt \right) dx &\geqslant \int_a^b \left(\frac{1}{x} \int_0^x |f(t)|\, dt \right) dx \\ &\geqslant \int_a^b \left(\frac{1}{x} \int_0^a |f(t)|\, dt \right) dx \end{aligned}$$

$$= \delta \ln \frac{b}{a} \to \infty \quad \text{as } b \to \infty.$$

Thus $f \notin \mathrm{Ces}_1[0, \infty)$.

(c) Considering the Hardy operator $Hf(x) = \frac{1}{x}\int_0^x f(t)\,dt$ and using the Hardy inequality (cf. [24, Theorem 327] and [33, Theorem 2]) we obtain that

$$\|f\|_{C(p)} = \|H(|f|)\|_p \leqslant p' \|f\|_p$$

for all $f \in L^p(I)$, which means that the $L^p(I) \overset{p'}{\hookrightarrow} \mathrm{Ces}_p(I)$ for $1 < p \leqslant \infty$.

The embeddings are strict. For example, $f = \sum_{n=1}^{\infty} \frac{1}{n^{1/p}} \chi_{[n^2-1, n^2)} \in \mathrm{Ces}_p(I) \setminus L^p(I)$ for $I = [0, \infty)$ and $1 < p < \infty$.

(d) If $0 < a < 1$ and $\operatorname{supp} f \subset [0, a]$, then

$$\|f\|_{C(p)} \geqslant \left(\int_a^1 \left(\frac{1}{x} \int_0^x |f(t)|\,dt \right)^p dx \right)^{1/p}$$

$$\geqslant \left(\int_a^1 \left(\frac{1}{x} \int_0^a |f(t)|\,dt \right)^p dx \right)^{1/p} = \int_0^a |f(t)|\,dt \left(\frac{1 - a^{1-p}}{p - 1} \right)^{1/p}.$$

For $a = 1$ this is not the case. In fact, consider function $f(x) = \frac{1}{1-x}$ for $x \in [0, 1)$. Then $\frac{1}{x}\int_0^x f(t)\,dt = \frac{1}{x} \ln \frac{1}{1-x}$ and

$$\|f\|_{C(p)}^p = \int_0^1 \left(\frac{1}{x} \ln \frac{1}{1-x} \right)^p dx = \int_1^{\infty} \left(\frac{t \ln t}{t-1} \right)^p \frac{dt}{t^2}$$

$$\leqslant c + \int_2^{\infty} \frac{(2 \ln t)^p}{t^2}\,dt < \infty$$

and, hence, $f \in \mathrm{Ces}_p[0, 1]$ for any $1 \leqslant p < \infty$ but clearly, $f \notin L^1[0, 1]$.

In the case of $\mathrm{Ces}_p[0, \infty)$ we will have for $0 < a < \infty$ with $\operatorname{supp} f \subset [0, a]$ and $p \in (1, \infty)$,

$$\|f\|_{C(p)} \geqslant \left(\int_a^{\infty} \left(\frac{1}{x} \int_0^x |f(t)|\,dt \right)^p dx \right)^{1/p}$$

$$\geqslant \left(\int_a^{\infty} \left(\frac{1}{x} \int_0^a |f(t)|\,dt \right)^p dx \right)^{1/p} = \int_0^a |f(t)|\,dt \frac{1}{(p-1)a^{1-1/p}}.$$

For the function $f(x) = \frac{1}{x}\chi_{[1,\infty)}(x)$, $x \in (0,\infty)$ we have $\frac{1}{x}\int_0^x f(t)\,dt = \frac{1}{x}\ln x$ $(x \geqslant 1)$ and

$$\|f\|_{C(p)}^p = \int_1^\infty \left(\frac{\ln x}{x}\right)^p dx < \infty.$$

Thus, $f \in \mathrm{Ces}_p[0,\infty)$ for any $1 < p < \infty$, but clearly $f \notin L^1[0,\infty)$.

(e) If $1 < p < q \leqslant \infty$, then $L^q[0,1] \overset{1}{\hookrightarrow} L^p[0,1]$ and the embedding is strict, and, thus,

$$\|f\|_{C(p)} = \|H(|f|)\|_p \leqslant \|H(|f|)\|_q = \|f\|_{C(q)}$$

for all $f \in \mathrm{Ces}_q[0,1]$, that is, $\mathrm{Ces}_q[0,1] \overset{1}{\hookrightarrow} \mathrm{Ces}_p[0,1]$ and the embedding is strict since for positive decreasing functions the norms of Ces_p and L^p are equivalent. The last statement follows from the fact that for a positive decreasing function f on I we have $f(x) \leqslant \frac{1}{x}\int_0^x f(t)\,dt$ for $x \in I$ and so

$$\|f\|_p \leqslant \|Hf\|_p = \|f\|_{C(p)} \leqslant p'\|f\|_p \quad \text{for any } 0 \leqslant f \in L^p(I).$$

(f) Consider $f(x) = \frac{1}{1-x}$ for $x \in [0,1)$. Then, as it was shown in (d), $f \in \mathrm{Ces}_p[0,1]$ for any $1 \leqslant p < \infty$. However, its non-increasing rearrangement $f^*(t) = t^{-1}$ $(0 < x \leqslant 1)$ does not belong to $\mathrm{Ces}_p[0,1]$ for any $1 \leqslant p \leqslant \infty$ and therefore the space $\mathrm{Ces}_p[0,1]$ is not rearrangement invariant for $1 \leqslant p < \infty$. In the case when $p = \infty$ we can take the function $g(x) = \frac{1}{\sqrt{1-x}}$, $x \in [0,1)$ for which $\frac{1}{x}\int_0^x g(t)\,dt = \frac{2}{x}(1 - \sqrt{1-x}) = \frac{2}{1+\sqrt{1-x}}$ and so $\|g\|_{C(\infty)} = 2$ and for its rearrangement $g^*(t) = t^{-1/2}$, $t \in (0,1)$ we have $\|g^*\|_{C(\infty)} = \sup_{t\in(0,1)} 2t^{-1/2} = \infty$, that is, $g^* \notin \mathrm{Ces}_\infty[0,1]$ and the space $\mathrm{Ces}_\infty[0,1]$ is not rearrangement invariant. Similarly, we can consider the case when $I = [0,\infty)$.

(g) If $1 < p < \infty$, then $\mathrm{Ces}_p(I)$ contains a copy of $L^1(I)$ (cf. [4], Lemma 1 for $I = [0,1]$ and Theorem 2 for $I = [0,\infty)$) and therefore, in particular, these spaces cannot be reflexive. Of course, $\mathrm{Ces}_1[0,1] = L^1(\ln 1/t)$ is not reflexive and the space $\mathrm{Ces}_\infty(I)$ does not have absolutely continuous norm and therefore is also not reflexive.

(h) Assume that $\|f\|_{C(p)} = \|g\|_{C(p)} = 1$ and $\|f+g\|_{C(p)} = 2$; then $\|H(|f|)\|_{L^p} = \|H(|g|)\|_{L^p} = 1$ and

$$\begin{aligned} 2 = \|f+g\|_{C(p)} &= \|H(|f+g|)\|_{L^p} \\ &\leqslant \|H(|f|) + H(|g|)\|_{L^p} \leqslant \|H(|f|)\|_{L^p} + \|H(|g|)\|_{L^p} \\ &= \|f\|_{C(p)} + \|g\|_{C(p)} = 2. \end{aligned}$$

Thus $\|H(|f|) + H(|g|)\|_{L^p} = 2$ and by the strict convexity of $L^p(I)$ for $1 < p < \infty$ and the above estimates we obtain that $H(|f|)(x) = H(|g|)(x)$ for almost all x in I. Therefore, $|f(x)| = |g(x)|$ for almost all $x \in I$. We want to show that this implies that $f(x) = g(x)$ for almost all $x \in I$. Assume on the contrary that $f \neq g$ on I, that is, there exists a set $A \subset I$ of positive measure $m(A) > 0$ such that $f(x) \neq g(x)$ for all $x \in A$. Then $f(x) = -g(x)$ and $|f(x)| > 0$ for $x \in A$. Moreover, if $B = \{x \in I\colon m([0, x] \cap (I \setminus A)) < x\}$, then $m(B) > 0$ and

$$\int_0^x \left| \frac{f(t) + g(t)}{2} \right| dt = \int_{[0,x]\cap(I\setminus A)} |f(t)|\, dt < \int_0^x |f(t)|\, dt$$

for all $x \in B$. Therefore,

$$\begin{aligned} 1 = \left\| \frac{f+g}{2} \right\|_{C(p)}^p &= \int_I \left(\frac{1}{x} \int_0^x \left| \frac{f(t)+g(t)}{2} \right| dt \right)^p dx \\ &< \int_I \left(\frac{1}{x} \int_0^x |f(t)|\, dt \right)^p dx = \|f\|_{C(p)}^p = 1, \end{aligned}$$

which is a contradiction and the proof is complete. $\square$

3. THE DUAL SPACES OF THE CESÀRO FUNCTION SPACES $\mathrm{Ces}_p[0, \infty)$

We describe the dual and Köthe dual spaces of $\mathrm{Ces}_p(I)$ for $1 < p < \infty$ in the case $I = [0, \infty)$. The description appeared as remark in Bennett [8] paper but it was proved recently, even for more general spaces, by Kerman, Milman and Sinnamon [30, Theorem D] and they used in the proof some of Sinnamon results [53, Theorem 2.1] and [52, Proposition 2.1 and Lemma 3.2].

We present here another proof following the Bennett's idea for Cesàro sequence spaces together with factorization theorems which are of independent interest. Since the case $I = [0, 1]$ is essentially different it will be considered in the next section.

Theorem 2. *Let $I = [0, \infty)$. If $1 < p < \infty$, then*

$$(\mathrm{Ces}_p)^* = (\mathrm{Ces}_p)' = D(p'), \quad p' = \frac{p}{p-1}, \tag{2}$$

with $\|f\|_{C(p)'} \leqslant p' \|f\|_{D(p')} \leqslant 8p' \|f\|_{C(p)'}$, where the norm in $D(p')$ is given by formula

$$\|f\|_{D(p')} = \|\tilde{f}\|_{L^{p'}} \quad \text{with } \tilde{f}(x) = \operatorname{ess\,sup}_{t \in [x, \infty)} |f(t)|. \tag{3}$$

We need the definition of the $G(p)$ space for $1 \leqslant p < \infty$, which is the p-convexification of $\mathrm{Ces}_\infty[0, \infty)$, that is, its norm is given by the functional

$$\|f\|_{G(p)} = \big\| |f|^p \big\|_{C(\infty)}^{1/p} = \sup_{x>0} \left(\frac{1}{x} \int_0^x |f(t)|^p\, dt \right)^{1/p}.$$

Proposition 1. *If $1 < p < \infty$, then*

$$\mathrm{Ces}_p = L^p \cdot G\big(p'\big), \tag{4}$$

that is, $f \in \mathrm{Ces}_p$ if and only if $f = gh$ with $g \in L^p, h \in G(p')$ and

$$\|f\|_{C(p)} \approx \inf \|g\|_p \|h\|_{G(p')}, \tag{5}$$

where infimum is taken over all factorizations $f = gh$ with $g \in L^p, h \in G(p')$.

Proof. "*Imbedding* $\hookrightarrow$". For $f \in \mathrm{Ces}_p$, $f \not\equiv 0$ let

$$k(x) = \int_x^\infty u^{-p}\left(\int_0^u |f(t)|\,dt\right)^{p-1} du, \quad x > 0.$$

Then $k(x) > 0, k$ is decreasing and by the Hölder–Rogers inequality

$$\begin{aligned} k(x) &= \int_x^\infty u^{-1}\left(\frac{1}{u}\int_0^u |f(t)|\,dt\right)^{p-1} du \\ &\leqslant \left(\int_x^\infty u^{-p}\,du\right)^{1/p}\left(\int_x^\infty\left(\frac{1}{u}\int_0^u |f(t)|\,dt\right)^p du\right)^{1/p'} \\ &= \frac{1}{(p-1)^{1/p}x^{1-1/p}}\|f\|_{C(p)}^{p-1}. \end{aligned}$$

We consider the factorization $f = g \cdot h$, where

$$g(x) = \big(|f(x)|k(x)\big)^{1/p}\,\mathrm{sgn}\, f(x) \quad \text{and} \quad h(x) = |f(x)|^{1/p'}k(x)^{-1/p}.$$

Then

$$\begin{aligned} \|g\|_p^p &= \int_0^\infty |f(x)| \int_x^\infty u^{-p}\left(\int_0^u |f(t)|\,dt\right)^{p-1} du\,dx \\ &= \int_0^\infty u^{-p}\left(\int_0^u |f(t)|\,dt\right)^{p-1}\int_0^u |f(x)|\,dx\,du = \|f\|_{C(p)}^p \end{aligned}$$

and, by the Hölder–Rogers inequality,

$$\begin{aligned} \left(\int_0^x |h(t)|^{p'}\,dt\right)^p &= \left(\int_0^x |f(t)|^{1/p'}|f(t)|^{1/p}k(t)^{-p'/p}\,dt\right)^p \\ &\leqslant \left(\int_0^x |f(t)|\,dt\right)^{p-1}\left(\int_0^x |f(t)|k(t)^{-p'}\,dt\right). \end{aligned}$$

Hence, by the above and using the fact that k is decreasing, it yields that

$$\int_x^\infty \left(s^{-1}\int_0^x |h(t)|^{p'}\,dt\right)^p ds$$

$$\leqslant \int_x^\infty s^{-p}\left[\left(\int_0^x |f(t)|\,dt\right)^{p-1}\int_0^x |f(t)|k(t)^{-p'}\,dt\right]ds$$

$$= k(x)\int_0^x |f(t)|k(t)^{-p'}\,dt$$

$$\leqslant \int_0^x |f(t)|k(t)^{1-p'}\,dt = \int_0^x |h(t)|^{p'}\,dt$$

or, equivalently,

$$\int_x^\infty s^{-p}\,ds\left(\int_0^x |h(t)|^{p'}\,dt\right)^{p-1} \leqslant 1,$$

which means that

$$\left(\int_0^x |h(t)|^{p'}\,dt\right)^{p-1} \leqslant (p-1)x^{p-1}$$

and, hence,

$$\sup_{x>0}\frac{1}{x}\int_0^x |h(t)|^{p'}\,dt \leqslant (p-1)^{1/(p-1)}$$

or $\|h\|_{G(p')} \leqslant (p-1)^{1/p}$. We have proved that

$$\mathrm{Ces}_p \subset L^p \cdot G(p')$$

and

$$\inf\{\|g\|_{L^p}\|h\|_{G(p')}\colon\ f = g\cdot h\} \leqslant (p-1)^{1/p}\|f\|_{C(p)}.$$

"*Imbedding* $\hookleftarrow$". Let $f = g\cdot h$ with $g \in L^p$ and $h \in G(p')$. Then

$$\int_0^x |h(t)|^{p'}\,dt \leqslant \|h\|_{G(p')}^{p'}\int_0^x dt$$

and then, for any positive decreasing function w on $(0, \infty)$, we have by [32, property 18^0, p. 72] that

$$\int_0^x |h(t)|^{p'} w(t)\,dt \leqslant \|h\|_{G(p')}^{p'} \int_0^x w(t)\,dt.$$

By the Hölder–Rogers inequality we find that

$$\begin{aligned}\left(\int_0^x |f(t)|\,dt\right)^p &= \left(\int_0^x |g(t)|w(t)^{-1/p'}|h(t)|w(t)^{1/p'}\,dt\right)^p \\ &\leqslant \int_0^x |g(t)|^p w(t)^{1-p}\,dt \left(\int_0^x |h(t)|^{p'} w(t)\,dt\right)^{p-1} \\ &\leqslant \int_0^x |g(t)|^p w(t)^{1-p}\,dt \|h\|_{G(p')}^p \left(\int_0^x w(t)\,dt\right)^{p-1}\end{aligned}$$

and, thus,

$$\begin{aligned}&\int_0^\infty \left(\frac{1}{x}\int_0^x |f(t)|\,dt\right)^p dx \\ &\quad \leqslant \int_0^\infty x^{-p}\left(\int_0^x |g(t)|^p w(t)^{1-p}\,dt\right)\left(\int_0^x w(t)\,dt\right)^{p-1} dx \|h\|_{G(p')}^p.\end{aligned}$$

Taking in the last estimate $w(t) = t^{-1/p}$ we obtain that

$$\begin{aligned}\|f\|_{C(p)}^p &\leqslant \int_0^\infty x^{-p}\left(\int_0^x |g(t)|^p t^{1-1/p}\,dt\right)\left(\frac{x^{1-1/p}}{1-1/p}\right)^{p-1} dx \|h\|_{G(p')}^p \\ &= \left(p'\right)^{p-1} \int_0^\infty \left(\int_0^x |g(t)|^p t^{1-1/p}\,dt\right) x^{1/p-2}\,dx \|h\|_{G(p')}^p \\ &= \left(p'\right)^{p-1} \int_0^\infty \left(\int_t^\infty x^{1/p-2}\,dx\right) |g(t)|^p t^{1-1/p}\,dt \|h\|_{G(p')}^p \\ &= \left(p'\right)^p \int_0^\infty |g(t)|^p\,dt \|h\|_{G(p')}^p = \left(p'\right)^p \|g\|_p^p \|h\|_{G(p')}^p\end{aligned}$$

or

$$\|f\|_{C(p)} \leqslant p' \|g\|_p \|h\|_{G(p')},$$

that is, $L^p \cdot G(p') \subset \mathrm{Ces}_p$ and

$$\|f\|_{C(p)} \leqslant p' \inf\{\|g\|_p \|h\|_{G(p')}\colon\ f = gh\}.$$

Putting these facts together we have that $\mathrm{Ces}_p \overset{(p-1)^{1/p}}{\hookrightarrow} L^p \cdot G(p') \overset{p'}{\hookrightarrow} \mathrm{Ces}_p$ and the proof of Proposition 1 is complete. $\square$

Proposition 2. *If $1 \leqslant p < \infty$, then*

$$D(p) \cdot G(p) = L^p$$

and

$$\|f\|_{L^p} = \inf\{\|g\|_{D(p)} \|h\|_{G(p)}\colon\ f = gh,\ g \in D(p),\ h \in G(p)\}.$$

Moreover, $G(1)' = D(1)$ with equality of the norms.

Proof. It suffices to prove the statement for $p = 1$ because the general result follows by p-convexification. Suppose that $f = gh$ with $g \in D(1), h \in G(1)$. Then

$$\|f\|_{L^1} = \int_0^\infty |g(t)h(t)|\,dt \leqslant \int_0^\infty \tilde{g}(t)|h(t)|\,dt.$$

Moreover, from the definition of the norm in $G(1)$ it follows that

$$\int_0^t |h(s)|\,ds \leqslant \|h\|_{G(1)} t = \|h\|_{G(1)} \int_0^t \chi_{[0,\infty)}(s)\,ds, \quad t > 0.$$

Therefore, since $\tilde{g}$ decreases it follows by [32, property 18^0, p. 72], we find that

$$\|f\|_{L^1} \leqslant \|h\|_{G(1)} \int_0^\infty \tilde{g}(t)\,dt = \|h\|_{G(1)} \|g\|_{D(1)}.$$

Hence, $D(1) \cdot G(1) \subset L^1$ and

$$\|f\|_{L^1} \leqslant \inf\{\|g\|_{D(1)} \|h\|_{G(1)}\colon\ f = gh,\ g \in D(1),\ h \in G(1)\}.$$

This also means that $G(1) \subset D(1)'$ and $\|h\|_{D(1)'} \leqslant \|h\|_{G(1)}$. We show that we have in fact even equality. If $f \in D(1)'$, then

$$\begin{aligned} \frac{1}{x} \int_0^x |f(t)|\,dt &= \frac{1}{x} \int_0^1 \chi_{[0,x]}(t)|f(t)|\,dt \\ &\leqslant \frac{1}{x} \|\chi_{[0,x]}\|_{D(1)} \|f\|_{D(1)'} = \|f\|_{D(1)'}, \end{aligned}$$

for all $x > 0$, i.e., $f \in G(1)$ and so $D(1)' \subset G(1)$ with $\|f\|_{G(1)} \leqslant \|f\|_{D(1)'}$. Of course, $G(1)' = D(1)'' = D(1)$ since the norm of $D(1)$ has the Fatou property. Finally, if $f \in L^1$, then, by the Lozanovskiĭ factorization theorem ([40, Theorem 6, p. 429]; cf. also [43, p. 185]), we can find $g \in D(1)$ and $h \in D(1)' = G(1)$ such that $f = g \cdot h$ and

$$\|g\|_{D(1)}\|h\|_{G(1)} = \|f\|_{L^1}.$$

This ends the proof of Proposition 2. □

Remark 1. In particular, Proposition 2 shows that $(\mathrm{Ces}_\infty[0,\infty))' = G(1)' = D(1)$. Thus, for the Cesàro function space on $[0,\infty)$ we get the result analogous to the Luxemburg–Zaanen theorem (cf. [42]): $(\mathrm{Ces}_\infty[0,1])' = \tilde{L}^1[0,1]$, where $\|f\|_{\tilde{L}^1} = \|\tilde{f}\|_{L^1[0,1]}$ with $\tilde{f}(x) = \operatorname{ess\,sup}_{t\in[x,1]} |f(t)|$.

Remark 2. For a positive weight function w and $1 \leqslant p < \infty$ let us define the weighted spaces $D(w,p)$ and $G(w,p)$ by the norms $\|f\|_{D(w,p)} = (\int_0^\infty \tilde{f}(x)^p \times w(x)\,dx)^{1/p}$, where $\tilde{f}(x) = \operatorname{ess\,sup}_{t\in[x,\infty)} |f(t)|$, and $\|f\|_{G(w,p)} = \sup_{x>0}(\frac{1}{W(x)} \times \int_0^x |f(t)|^p\,dt)^{1/p}$, $W(x) = \int_0^x w(t)\,dt$, respectively. Proposition 2 is valid for weighted spaces: *If* $1 \leqslant p < \infty$, *then* $D(w,p)\cdot G(w,p) = L^p$ *and* $\|f\|_{L^p} = \inf\{\|g\|_{D(w,p)} \times \|h\|_{G(w,p)}\colon f = gh, g \in D(w,p), h \in G(w,p)\}$.

Proposition 3. *Let* $1 < p < \infty$. *If* $g \in (\mathrm{Ces}_p)'$, *then* $\tilde{g}(x) = \operatorname{ess\,sup}_{t\in[x,\infty)} |g(t)| \in (\mathrm{Ces}_p)'$ *and*

$$\|\tilde{g}\|_{C(p)'} \leqslant 8\|g\|_{C(p)'}.$$

Proof. Let $f \in \mathrm{Ces}_p$, $f \geqslant 0$. Then $\int_0^x f(t)\,dt \to 0$ if $x \to 0^+$. Consider two cases:

(a) If $\int_0^\infty f(s)\,ds = \infty$, then we select a two-sided sequence $\{a_k\}_{k\in\mathbb{Z}}$ such that $0 \leqslant a_k < a_{k+1}$, $a_k \to \infty$ when $k \to \infty$ and

$$\int_{a_{k-1}}^{a_k} f(s)\,ds = 2^k, \quad k \in \mathbb{Z}. \tag{6}$$

(b) If $A = \int_0^\infty f(s)\,ds < \infty$, we find a one-sided sequence $\{a_k\}_{k\leqslant 0}$ such that $0 \leqslant a_k < a_{k+1}$, $a_0 = \infty$ and

$$\int_{a_{k-1}}^{a_k} f(s)\,ds = 2^{k-1}A, \quad k \leqslant 0. \tag{7}$$

By J let us denote either $\mathbb{Z}$ or $\{k \in \mathbb{Z}\colon k \leqslant 0\}$ depending on which of the cases (a) or (b) we have, and let

$$\mathcal{P} = \left\{ k \in J\colon \text{ there is a set } A_k \subset [a_{k-1}, a_k) \text{ such that } m(A_k) > 0 \right.$$
$$\left. \text{and } |g(s)| \geqslant \frac{1}{2}\tilde{g}(a_{k-1}) \text{ for all } s \in A_k \right\}.$$

Note that $\mathcal{P} \neq \emptyset$. In fact, let $k \in J$ be arbitrary and let i be the first “time” such that $i \geqslant k$ and

$$m\left\{ s \in (a_{i-1}, a_i]\colon |g(s)| \geqslant \frac{1}{2}\tilde{g}(a_{k-1}) \right\} > 0.$$

Since $\tilde{g}(a_{i-1}) = \tilde{g}(a_{k-1})$, then $i \in \mathcal{P}$.

Let $\mathcal{P} = \{k_i\}_{i=l}^{m}$, where $k_i < k_j$ $(i < j)$ and l may be $-\infty$. Moreover, it is easily seen that either $m = \infty$ and $k_i \to \infty$ when $i \to \infty$ (in the case (a)) or $k_m = 0$ and $t_{k_m} = \infty$ (in the case (b)).

Define the function

$$\bar{f}(t) = \sum_{i=l}^{m} \int_{\Delta_i} f(s)\,ds \frac{1}{m(A_{k_i})} \chi_{A_{k_i}}(t),$$

where $\Delta_i = (a_{k_i-1}, a_{k_i}]$, and estimate its norm in Ces_p.

Let $\bar{a} = \lim_{i \to -\infty} a_{k_i}$ if $l = -\infty$ and $\bar{a} = a_{k_l}$ if l is finite. If $\bar{a} > 0$, then $\bar{f}(t) = 0$ for all $t \in [0, \bar{a})$. Therefore

$$\int_0^x \bar{f}(t)\,dt = 0 \quad (0 < x \leqslant \bar{a}). \tag{8}$$

Suppose $x > \bar{a}$. Then either (1^o) $t \in \Delta_i$ for some i or (2^o) there is $i < m$ such that $t \in (a_{k_i}, a_{k_{i+1}-1}]$. In the first case, by (6) or (7) it yields that

$$\int_0^x \bar{f}(t)\,dt = \sum_{j=l}^{i-1} \int_{\Delta_j} f(s)\,ds \frac{1}{m(A_{k_j})} m(A_{k_j}) + \frac{m(A_{k_i} \cap (a_{k_i-1}, t])}{m(A_{k_i})} \int_{\Delta_i} f(s)\,ds$$
$$\leqslant \int_0^{a_{k_i}} f(s)\,ds \leqslant 2 \int_0^x f(s)\,ds.$$

Analogously, in the second case we have that

$$\int_0^x \bar{f}(t)\,dt \leqslant \int_0^{a_{k_i}} f(s)\,ds \leqslant \int_0^x f(s)\,ds.$$

The last inequalities and equality (8) show that

$$\|\bar f\|_{C(p)} \leqslant 2\|f\|_{C(p)}. \tag{9}$$

Moreover, for any i running from l to m we find that

$$\begin{aligned}\int_{\Delta_i} \bar f(t)|g(t)|\,dt &= \int_{A_{k_i}} \bar f(t)|g(t)|\,dt \geqslant \frac{1}{2}\tilde g(a_{k_i-1}) \int_{A_{k_i}} \bar f(t)\,dt \\ &= \frac{1}{2}\tilde g(a_{k_i-1}) \int_{\Delta_i} f(t)\,dt. \end{aligned} \tag{10}$$

Since $\tilde g$ decreases, then (10) implies, in particular, that

$$\int_{\Delta_i} \bar f(t)|g(t)|\,dt \geqslant \frac{1}{2}\int_{\Delta_i} f(t)\tilde g(t)\,dt. \tag{11}$$

Note that, by definition of the set $\mathcal{P}$, it yields that $\tilde g(t) \leqslant \tilde g(a_{k_i-1})$ a.e. on the interval $(a_{k_{i-1}}, a_{k_i-1}]$ if $i > l$ and on the interval $(0, a_{k_l-1}]$ if l is finite. Moreover, taking into account (6) or (7) once again, we have that

$$\int_{a_{k_{i-1}}}^{a_{k_i-1}} f(s)\,ds \leqslant \int_{\Delta_i} f(s)\,ds \quad \text{if } i > l$$

and

$$\int_0^{a_{k_l-1}} f(s)\,ds \leqslant \int_{\Delta_l} f(s)\,ds \quad \text{if } l \text{ is finite.}$$

Therefore, by (10), it follows that

$$\begin{aligned}\int_{\Delta_i} \bar f(t)|g(t)|\,dt &\geqslant \frac{1}{2}\tilde g(a_{k_i-1}) \int_{\Delta_i} f(t)\,dt \geqslant \frac{1}{2}\tilde g(a_{k_i-1}) \int_{a_{k_{i-1}}}^{a_{k_i-1}} f(t)\,dt \\ &\geqslant \frac{1}{2}\int_{a_{k_{i-1}}}^{a_{k_i-1}} \tilde g(t) f(t)\,dt, \end{aligned}$$

where $a_{l-1} = 0$ if l is finite.

Since $f = 0$ a.e. on the interval $(0, \bar a]$, when $l = -\infty$ and $\bar a = \lim_{i\to-\infty} a_{k_i} > 0$, then, by summing the last inequalities and inequality (11) over all i, we get that

$$2\int_0^\infty \bar f(t)|g(t)|\,dt \geqslant 2\sum_{i=l}^{m} \int_{\Delta_i} \bar f(t)|g(t)|\,dt \geqslant \frac{1}{2}\int_0^\infty \tilde g(t) f(t)\,dt,$$

whence,

$$\int_0^\infty \tilde{g}(t) f(t)\,dt \leqslant 4 \int_0^\infty \bar{f}(t)|g(t)|\,dt.$$

Combining the last inequality with (9), we obtain that

$$\begin{aligned}
\|\tilde{g}\|_{C(p)'} &= \sup\left\{\int_0^\infty \tilde{g}(t) f(t)\,dt\colon\ \|f\|_{C(p)} \leqslant 1\right\} \\
&\leqslant 4\sup\left\{\int_0^\infty \bar{f}(t)|g(t)|\,dt\colon\ \|f\|_{C(p)} \leqslant 1\right\} \\
&\leqslant 4\sup\left\{\int_0^\infty \bar{f}(t)|g(t)|\,dt\colon\ \|\bar{f}\|_{C(p)} \leqslant 2\right\} = 8\|g\|_{C(p)'}
\end{aligned}$$

and the proof is complete. $\square$

Proof of Theorem 2. Firstly, we show that $D(p') \overset{1}{\hookrightarrow} (L^p \cdot G(p'))'$. In fact, let $f \in D(p')$ and $g \in L^p \cdot G(p')$, then $g = h \cdot k$ with $h \in L^p$ and $k \in G(p')$. By the Hölder–Rogers inequality and the imbedding $D(p') \cdot G(p') \overset{1}{\hookrightarrow} L^{p'}$ proved in Proposition 2 we obtain that

$$\|fg\|_{L^1} = \|fhk\|_{L^1} \leqslant \|h\|_{L^p} \|fk\|_{L^{p'}} \leqslant \|h\|_{L^p} \|k\|_{G(p')} \|f\|_{D(p')},$$

from which it follows that $D(p') \subset (L^p \cdot G(p'))'$ and $\|f\|_{(L^p \cdot G(p'))'} \leqslant \|f\|_{D(p')}$. Since, by Proposition 1 we have equality $\mathrm{Ces}_p = L^p \cdot G(p')$, it follows that

$$D(p') \overset{p'}{\hookrightarrow} (\mathrm{Ces}_p)'.$$

To prove the converse, take $f \in (\mathrm{Ces}_p)'$. Since $\tilde{f} \geqslant |f|$ and $D(p')$ is a Banach lattice, then by Proposition 3, we may (and will) assume that f is a non-negative decreasing function on $(0, \infty)$, i.e., $f = \tilde{f}$. Then, by the Hardy inequality, we find that

$$\begin{aligned}
\|f\|_{D(p')} = \|f\|_{L^{p'}} &= \sup\left\{\int_0^\infty |f(x)g(x)|\,dx\colon\ \|g\|_{L^p} \leqslant 1\right\} \\
&\leqslant p' \sup\left\{\int_0^\infty |f(x)g(x)|\,dx\colon\ \|g\|_{C(p)} \leqslant 1\right\} = p'\|f\|_{(\mathrm{Ces}_p)'}.
\end{aligned}$$

Therefore, $f \in D(p')$ and $(\mathrm{Ces}_p)' \overset{8p'}{\hookrightarrow} D(p')$. $\square$

We describe the dual and Köthe dual of $\mathrm{Ces}_p(I)$ for $1 < p < \infty$ in the case $I = [0,1]$. Surprisingly this will have a different description than in the case $I = [0,\infty)$. For $p = \infty$ the space $\mathrm{Ces}_\infty[0,1]$ introduced by Korenblyum, Kreĭn and Levin [31] we denote by K and its separable part by K_0.

As we already mentioned the Köthe dual space K' was found by Luxemburg and Zaanen [42]: $K' = \tilde{L}^1$ with equality of norms, where

$$\|f\|_{\tilde{L}^1} = \|\tilde{f}\|_{L^1}, \quad \text{with } \tilde{f}(x) = \operatorname*{ess\,sup}_{t\in[x,1]} |f(t)|.$$

Earlier the dual space of K_0 was found by Tandori [56]: $(K_0)^* = \tilde{L}^1$ with equality of norms.

We will find the Köthe dual space $(\mathrm{Ces}_p[0,1])'$ for $1 < p < \infty$. Consider, for $1 < p < \infty$, a Banach ideal space $U(p)$ on $I = [0,1]$ which norm is given by the formula

$$\|f\|_{U(p)} = \left\| \frac{1}{1 - x^{1/(p-1)}}\tilde{f}(x)\right\|_{L^p} = \left[\int_0^1 \left(\frac{\tilde{f}(x)}{1 - x^{1/(p-1)}}\right)^p dx\right]^{1/p}, \tag{12}$$

where $\tilde{f}(x) = \operatorname{ess\,sup}_{t\in[x,1]} |f(t)|$.

Remark 3. Since $\min(1, p-1) \leqslant \frac{1-x}{1-x^{1/(p-1)}} \leqslant \max(1, p-1)$ for all $x \in (0,1)$, then the norm (12) in $U(p)$ is equivalent to the norm

$$\|f\|^0_{U(p)} = \left[\int_0^1 \left(\frac{\tilde{f}(x)}{1-x}\right)^p dx\right]^{1/p}.$$

Theorem 3. *If* $1 < p < \infty$, *then*

$$(\mathrm{Ces}_p)^* = (\mathrm{Ces}_p)' = U(p'), \quad p' = \frac{p}{p-1}, \tag{13}$$

with equivalent norms.

Before the proof of this theorem we prove some auxiliary results of independent interest. First, for $1 < p < \infty$ we define the Banach ideal space $V(p)$ on $I = [0,1]$ generated by the functional

$$\|f\|_{V(p)} = \sup_{0<x\leqslant 1}\left[\frac{(1 - x^{1/(p-1)})^{p-1}}{x}\int_0^x |f(t)|^p\,dt\right]^{1/p}. \tag{14}$$

Proposition 4. *If* $1 < p < \infty$, *then*

$$\mathrm{Ces}_p \subset L^p \cdot V(p'), \quad p' = \frac{p}{p-1}, \tag{15}$$

that is, if $f \in \mathrm{Ces}_p$, *then* $f = gh$ *with* $g \in L^p, h \in V(p')$ *and*

$$\inf\left\{\|g\|_p\|h\|_{V(p')}\colon\ f = g \cdot h, g \in L^p, h \in V\big(p'\big)\right\} \leqslant (p-1)^{1/p}\|f\|_{C(p)}. \tag{16}$$

Proof. The proof is analogous to the proof of Proposition 1 (for the case $I = [0, \infty)$) but we put details to see how the weight $w(x) = (1 - x^{p-1})^{1/(p-1)}$ appeared in the definition of the space $V(p')$. For $f \in \mathrm{Ces}_p$, $f \neq 0$, define

$$k(x) = \int_x^1 u^{-p}\left(\int_0^u |f(t)|\,dt\right)^{p-1} du, \quad x \in [0, 1].$$

Then $k(x) > 0$, k is decreasing and, by the Hölder–Rogers inequality, we find that

$$\begin{aligned} k(x) &= \int_x^1 u^{-1}\left(\frac{1}{u}\int_0^u |f(t)|\,dt\right)^{p-1} du \\ &\leqslant \left(\int_x^1 u^{-p}\,du\right)^{1/p}\left(\int_x^1\left(\frac{1}{u}\int_0^u |f(t)|\,dt\right)^p du\right)^{1/p'} \\ &\leqslant \frac{1}{(p-1)^{1/p}}\left(\frac{1 - x^{p-1}}{x^{p-1}}\right)^{1/p}\|f\|_{C(p)}^{p-1}. \end{aligned}$$

Let

$$g(x) = \big(|f(x)|k(x)\big)^{1/p}\operatorname{sgn} f(x) \quad \text{and} \quad h(x) = |f(x)|^{1/p'}k(x)^{-1/p}, \quad 0 < x < 1.$$

Then $f = g \cdot h$ and

$$\begin{aligned} \|g\|_p^p &= \int_0^1 |f(x)|\int_x^1 u^{-p}\left(\int_0^u |f(t)|\,dt\right)^{p-1} du\,dx \\ &= \int_0^1 u^{-p}\left(\int_0^u |f(t)|\,dt\right)^{p-1}\int_0^u |f(x)|\,dx\,du = \|f\|_{C(p)}^p, \end{aligned}$$

and, by the Hölder–Rogers inequality,

$$\begin{aligned} \left(\int_0^x |h(t)|^{p'}\,dt\right)^p &= \left(\int_0^x |f(t)|^{1/p'}|f(t)|^{1/p}k(t)^{-p'/p}\,dt\right)^p \\ &\leqslant \left(\int_0^x |f(t)|\,dt\right)^{p-1}\left(\int_0^x |f(t)|k(t)^{-p'}\,dt\right). \end{aligned}$$

Hence, by the above and using the fact that k is decreasing, we obtain that

$$\begin{aligned}
&\int_x^1 \left(s^{-1} \int_0^x |h(t)|^{p'}\,dt \right)^p ds \\
&\quad \leqslant \int_x^1 s^{-p} \left[\left(\int_0^x |f(t)|\,dt \right)^{p-1} \int_0^x |f(t)| k(t)^{-p'}\,dt \right] ds \\
&\quad \leqslant \int_x^1 s^{-p} \left[\left(\int_0^s |f(t)|\,dt \right)^{p-1} \int_0^x |f(t)| k(t)^{-p'}\,dt \right] ds \\
&\quad = k(x) \int_0^x |f(t)| k(t)^{-p'}\,dt \\
&\quad \leqslant \int_0^x |f(t)| k(t)^{1-p'}\,dt = \int_0^x |h(t)|^{p'}\,dt
\end{aligned}$$

or, equivalently,

$$\int_x^1 s^{-p}\,ds \left(\int_0^x |h(t)|^{p'}\,dt \right)^{p-1} \leqslant 1,$$

which means that

$$\left(\int_0^x |h(t)|^{p'}\,dt \right)^{p-1} \leqslant (p-1) \frac{x^{p-1}}{1 - x^{p-1}}$$

and, thus,

$$\sup_{x>0} \frac{(1 - x^{p-1})^{1/(p-1)}}{x} \int_0^x |h(t)|^{p'}\,dt \leqslant (p-1)^{1/(p-1)}$$

or $\|h\|_{V(p')} \leqslant (p-1)^{1/p}$. Summing up we have proved that $\mathrm{Ces}_p \subset L^p \cdot V(p')$ and

$$\inf\{\|g\|_{L^p} \|h\|_{V(p')}\colon\ f = g \cdot h\} \leqslant (p-1)^{1/p} \|f\|_{C(p)}. \qquad \square$$

Remark 4. In the above imbedding we cannot take instead of the space $V(p')$, where the weight $w(x) = (1 - x^{p-1})^{1/(p-1)}$ appeared, the corresponding space without this weight, that is, the p'-convexification $K^{(p')}$ of K. This space is too small since if the imbedding $\mathrm{Ces}_p[0,1] \subset L^p \cdot K^{(p')}$ would be valid, then since $L^p \cdot K^{(p')} \subset L^p \cdot L^{p'} \subset L^1[0,1]$ we will have a contradiction because $\mathrm{Ces}_p[0,1]$ is not embedded into $L^1[0,1]$ (cf. Theorem 1(d)) and the problem is "near 1", therefore this weight w is really needed in the imbedding (15).

Proposition 5. *If* $1 < p < \infty$, *then*

(a) $U(p) \cdot V(p) \subset L^p$ *with*

$$\|f\|_{L^p} \leqslant \inf\{\|g\|_{U(p)}\|h\|_{V(p)}\colon\ f = g \cdot h,\ g \in U(p),\ h \in V(p)\}.$$

(b) $U(p) \subset (V(p) \cdot L^{p'})'$ *and* $\|f\|_{(V(p)\cdot L^{p'})'} \leqslant \|f\|_{U(p)}$ *for all* $f \in U(p)$.

Proof. (a) Let $f = g \cdot h$, $g \in U(p)$, $h \in V(p)$. Since $|g| \leqslant \tilde{g}$ it follows that

$$\|f\|_{L^p}^p \leqslant \int_0^1 \tilde{g}(t)^p |h(t)|^p\, dt. \tag{17}$$

On the other hand, by the definition of the norm in $V(p)$ and using the equality

$$\frac{d}{dx}\left(\frac{x}{(1 - x^{1/(p-1)})^{p-1}}\right) = \frac{1}{(1 - x^{1/(p-1)})^p},$$

we obtain that

$$\int_0^x |h(t)|^p\, dt \leqslant \|h\|_{V(p)}^p \frac{x}{(1 - x^{1/(p-1)})^{p-1}}$$
$$= \|h\|_{V(p)}^p \int_0^x \frac{1}{(1 - t^{1/(p-1)})^p}\, dt$$

for all $x \in (0, 1]$. Since $\tilde{g}^p$ decreases, then, by [32, property 18^0, p. 72], the last inequality implies that

$$\int_0^1 \tilde{g}(t)^p |h(t)|^p\, dt \leqslant \|h\|_{V(p)}^p \int_0^1 \left(\frac{\tilde{g}(t)}{1 - t^{1/(p-1)}}\right)^p dt.$$

Therefore, by (17), $f \in L^p$ and

$$\|f\|_{L^p} \leqslant \|g\|_{U(p)} \cdot \|h\|_{V(p)},$$

and the proof of (a) is complete.

(b) For any $f \in U(p)$ and $g \in V(p) \cdot L^{p'}$ we have $g = h \cdot k$ with $h \in V(p)$, $k \in L^{p'}$ and, by the Hölder–Rogers inequality and Proposition 5(a), we obtain that

$$\int_0^1 |fg|\, dx = \int_0^1 |fhk|\, dx \leqslant \left(\int_0^1 |fh|^p\, dx\right)^{1/p} \left(\int_0^1 |k|^{p'}\, dx\right)^{1/p'}$$
$$\leqslant \|f\|_{U(p)}\|h\|_{V(p)}\|k\|_{L^{p'}} = \|h\|_{V(p)}\|k\|_{L^{p'}}\|f\|_{U(p)}$$

or $f \in (V(p) \cdot L^{p'})'$ and $\|f\|_{(V(p)\cdot L^{p'})'} \leqslant \|f\|_{U(p)}$. The proof of (b) is complete. $\square$

Proposition 6. *Let* $1 \leqslant p < \infty$. *If* $g \in (\mathrm{Ces}_p)'$, *then* $\tilde{g}(x) = \operatorname{ess\,sup}_{t \in [x,1]} |g(t)| \in (\mathrm{Ces}_p)'$ *and*

$$\|\tilde{g}\|_{C(p)'} \leqslant 8\|g\|_{C(p)'}.$$

Proof. When $p = 1$, then the assertion is obvious since $\mathrm{Ces}_1 = L^1(\ln 1/t)$ and $(\mathrm{Ces}_1)' = L^\infty(\ln^{-1} 1/t)$. Let $p > 1$ and $f \in \mathrm{Ces}_p$, $f \geqslant 0$. Consider two cases:
(a) If $\int_0^1 f(s)\,ds = \infty$, then we select a two-sided sequence $\{a_k\}_{k\in\mathbb{Z}}$ such that $0 \leqslant a_k < a_{k+1}$, $a_k \to 1$ when $k \to \infty$ and

$$\int_{a_{k-1}}^{a_k} f(s)\,ds = 2^k, \quad k \in \mathbb{Z}. \tag{18}$$

(b) If $A = \int_0^1 f(s)\,ds < \infty$, then we can find an one-sided sequence $\{a_k\}_{k\leqslant 0}$ such that $0 \leqslant a_k < a_{k+1}$, $a_0 = 1$ and

$$\int_{a_{k-1}}^{a_k} f(s)\,ds = 2^{k-1}A, \quad k \leqslant 0. \tag{19}$$

The remaining part of the proof is completely analogous to the proof of Proposition 3 so we omit the details. □

Proof of Theorem 3. "*Imbedding* $\supset$". If $f \in U(p')$, then, by Proposition 5(b) and Proposition 4, we obtain that

$$U(p') \subset \left(V(p') \cdot L^p\right)' \subset (\mathrm{Ces}_p)' \quad \text{and} \quad \|f\|_{C(p)'} \leqslant (p-1)^{1/p}\|f\|_{U(p')}.$$

"*Imbedding* $\subset$". Let $f \in (\mathrm{Ces}_p)'$. Since $\tilde{f} \geqslant |f|$ and $U(p')$ is a Banach lattice, then by Proposition 6 we may (and we will) assume that f is a non-negative decreasing function on $(0,1]$, i.e., $f = \tilde{f}$. Define the weight

$$w(x) = \chi_{[0,1/2]}(x) + (1-x)\chi_{[1/2,1]}(x), \quad 0 < x \leqslant 1.$$

Since $1 - x \leqslant w(x) \leqslant 2(1-x)$ for $x \in (0,1]$, then according to Remark 3 it is enough to prove that for some constant $A_p > 0$ we have that

$$\left\|\frac{f}{w}\right\|_{L^{p'}} = \left[\int_0^1 \left(\frac{f(x)}{w(x)}\right)^{p'} dx\right]^{1/p} \leqslant A_p\|f\|_{C(p)'} \tag{20}$$

since

$$\begin{aligned}\|f\|_{U(p')} &= \left\|\frac{1}{1-x^{1/(p'-1)}} f(x)\right\|_{L^{p'}} \leqslant \max\left(1, p'-1\right)\left\|\frac{1}{1-x} f(x)\right\|_{L^{p'}} \\ &\leqslant 2\max\left(1, p'-1\right)\|f/w\|_{L^{p'}}.\end{aligned}$$

We now prove that if $h \in L^p$, $h \geqslant 0$, then $h/w \in \mathrm{Ces}_p$ and

(21) $$\|h/w\|_{C(p)} \leqslant \big(p' + 2p\big)\|h\|_{L^p}.$$

To prove this we first show that the operator S_w defined by

$$S_w h(x) = \int_0^x \frac{h(t)}{w(t)}\,dt \quad (0 < x \leqslant 1)$$

is bounded in $L^p[0,1]$ for $1 \leqslant p < \infty$. In fact, for $0 < x \leqslant 1/2$ we have that

$$S_w h(x) = \int_0^x h(t)\,dt = \int_{1-x}^1 h(1-t)\,dt \leqslant \int_{1-x}^1 \frac{h(1-t)}{t}\,dt$$

and for $1/2 \leqslant x \leqslant 1$

$$\begin{aligned} S_w h(x) &= \int_0^{1/2} h(t)\,dt + \int_{1/2}^x \frac{h(t)}{1-t}\,dt \\ &= \int_{1/2}^1 h(1-t)\,dt + \int_{1-x}^{1/2} \frac{h(1-t)}{t}\,dt \leqslant \int_{1-x}^1 \frac{h(1-t)}{t}\,dt. \end{aligned}$$

Thus,

$$S_w h(x) \leqslant H'(\bar{h})(1-x) \quad \text{for } 0 < x < 1,$$

where $\bar{h}(t) = h(1-t)$ and H' is the associated Hardy operator, i.e., $H'h(x) = \int_x^1 \frac{h(t)}{t}\,dt$. It is well known that H' is bounded in $L^p[0,1]$ for $1 \leqslant p < \infty$ (cf. [32], pp. 138–139) and, thus,

$$\|S_w h\|_{L^p} \leqslant \big\|H'(\bar{h})\big\|_{L^p} \leqslant \big\|H'\big\|\,\|\bar{h}\|_{L^p} = \big\|H'\big\|\,\|h\|_{L^p}.$$

Since

$$\frac{1}{x} S_w h(x) = \frac{1}{x}\int_0^x \frac{h(t)}{w(t)}\,dt \leqslant \frac{1}{x}\int_0^x h(t)\,dt\,\chi_{[0,\frac12]}(x) + 2 S_w h(x)\chi_{[\frac12,1]}(x)$$

it follows that

$$\begin{aligned} \|h/w\|_{C(p)} &= \left\|\frac{1}{x} S_w h(x)\right\|_{L^p} \leqslant \|Hh\|_{L^p} + 2\|S_w h\|_{L^p} \\ &\leqslant p'\|h\|_{L^p} + 2p\|h\|_{L^p} = \big(p' + 2p\big)\|h\|_{L^p} \end{aligned}$$

and the estimate (21) is proved. Moreover, by using this fact we obtain that

$$\begin{aligned}\left\|\frac{f}{w}\right\|_{L^{p'}} &= \sup\left\{\int_0^1 \frac{f(t)}{w(t)}h(t)\,dt\colon\ h \geqslant 0,\ \|h\|_{L^p} \leqslant 1\right\}\\ &\leqslant \sup\left\{\int_0^1 \frac{f(t)}{w(t)}h(t)\,dt\colon\ h \geqslant 0,\ \left\|\frac{h}{w}\right\|_{C(p)} \leqslant p' + 2p\right\}\\ &\leqslant \left(p' + 2p\right)\|f\|_{C(p)'}\end{aligned}$$

and also the estimate (20) is proved, which shows that $(\mathrm{Ces}_p)' \subset U(p')$ and for every $f \in (\mathrm{Ces}_p)'$

$$\|f\|_{U(p')} \leqslant 16\max\left(1, p' - 1\right)\left(p' + 2p\right)\|f\|_{C(p)'},$$

and the proof is complete. $\square$

Remark 5. Let $1 < p < \infty$. The L^p spaces have the property that the restriction of $L^p[0,\infty)$ to $[0,1]$ gives the space $L^p[0,1]$. The situation is different for Cesàro function spaces. In fact, if $f \in \mathrm{Ces}_p[0,\infty)$ and $\operatorname{supp} f \subset [0,1]$, then

$$\begin{aligned}\|f\|^p_{\mathrm{Ces}_p[0,\infty)} &= \int_0^\infty \left(\frac{1}{x}\int_0^x |f(t)|\,dt\right)^p dx\\ &= \int_0^1 \left(\frac{1}{x}\int_0^x |f(t)|\,dt\right)^p dx + \int_1^\infty \left(\frac{1}{x}\int_0^1 |f(t)|\,dt\right)^p dx\\ &= \|f\|^p_{\mathrm{Ces}_p[0,1)} + \frac{1}{p-1}\|f\|^p_{L^1[0,1]},\end{aligned}$$

which means that

$$\mathrm{Ces}_p[0,\infty)|_{[0,1]} = \mathrm{Ces}_p[0,1] \cap L^1[0,1].$$

Therefore,

$$(\mathrm{Ces}_p[0,1] \cap L^1[0,1])' = \left(\mathrm{Ces}_p[0,\infty)\right)'|_{[0,1]} = D\left(p'\right)|_{[0,1]}$$

or

$$U\left(p'\right) + L^\infty[0,1] = D\left(p'\right)|_{[0,1]}.$$

The last equality can be easily verified. For example, for $f \in D(p')|_{[0,1]}$ we can take as a decomposition $f = g + h$, $g \in U(p')$, $h \in L^\infty[0,1]$ the functions

$$g(x) = (1-x)f(x) \quad \text{and} \quad h(x) = xf(x), \quad x \in [0,1].$$

Then $f = g + h$ and $\tilde{g}(x) = \operatorname{ess\,sup}_{t\in[x,1]}(1-t)|f(t)| \leqslant (1-x)\tilde{f}(x)$, which shows that $g \in U(p')$ since $f \in D(p')$. Moreover,

$$\|h\|_\infty = \operatorname{ess}\sup_{x\in[0,1]} x|f(x)| \leqslant \operatorname{ess}\sup_{x\in[0,1]} x\tilde{f}(x)$$
$$\leqslant \|\tilde{f}\|_{L^1} \leqslant \|\tilde{f}\|_{L^{p'}} = \|f\|_{D(p')},$$

so that $h \in L^\infty[0,1]$.

5. ON p-CONCAVITY, TYPE AND COTYPE OF CESÀRO SEQUENCE AND FUNCTION SPACES

A Banach lattice X is said to be *p-convex* ($1 \leqslant p < \infty$) with constant $K \geqslant 1$, respectively *q-concave* ($1 \leqslant q < \infty$) with constant $L \geqslant 1$ if

$$\left\| \left(\sum_{k=1}^n |x_k|^p \right)^{1/p} \right\| \leqslant K \left(\sum_{k=1}^n \|x_k\|^p \right)^{1/p},$$

respectively

$$\left(\sum_{k=1}^n \|x_k\|^q \right)^{1/q} \leqslant L \left\| \left(\sum_{k=1}^n |x_k|^q \right)^{1/q} \right\|,$$

for every choice of vectors $x_1, x_2, \ldots, x_n$ in X.

Of course, every Banach lattice is 1-convex with constant 1. In particular, ces_p and $\mathrm{Ces}_p(I)$ are 1-convex with constant 1. The spaces $L^p(I)$ are p-convex and p-concave with constant 1.

If the above estimates hold for pairwise disjoint elements $\{x_k\}_{k=1}^n$ in X, that is,

$$\left\| \sum_{k=1}^n x_k \right\| \leqslant K \left(\sum_{k=1}^n \|x_k\|^p \right)^{1/p},$$

respectively

$$\left(\sum_{k=1}^n \|x_k\|^q \right)^{1/q} \leqslant L \left\| \sum_{k=1}^n x_k \right\|,$$

then we say that X satisfies an *upper p-estimate* with constant K and a *lower q-estimate* with constant L, respectively. It is obvious that a p-convex (q-concave) Banach lattice satisfies upper p-estimate (lower q-estimate).

Let $r_n : [0,1] \to \mathbb{R}, n \in \mathbb{N}$, be the Rademacher functions, that is, $r_n(t) = \operatorname{sign}(\sin 2^n \pi t)$. A Banach space X has *type* $1 \leqslant p \leqslant 2$ if there is a constant $K > 0$ such that, for any choice of finitely many vectors $x_1, \ldots, x_n$ from X,

$$\int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\| dt \leqslant K \left(\sum_{k=1}^n \|x_k\|^p \right)^{1/p}.$$

A Banach space X has *cotype* $q \geqslant 2$ if there is a constant $K > 0$ such that, for any choice of finitely many vectors $x_1, \ldots, x_n$ from X,

$$\left(\sum_{k=1}^{n} \|x_k\|^q\right)^{1/q} \leqslant K \int_0^1 \left\|\sum_{k=1}^{n} r_k(t) x_k\right\| dt.$$

In order to complete this definition for $q = \infty$ the left-hand side should be replaced by $\max_{1 \leqslant k \leqslant n} \|x_k\|$.

We say that the space X has *trivial type* or *trivial cotype*, if it does not have any type bigger than one or any finite cotype, respectively.

More information and connections among the above notions may be found in [17] and [38].

Theorem 4. *If* $1 < p < \infty$*, then* $\mathrm{Ces}_p(I)$ *are* p*-concave with constant* 1*, that is,*

$$\left(\sum_{k=1}^{n} \|f_k\|_{C(p)}^p\right)^{1/p} \leqslant \left\|\left(\sum_{k=1}^{n} |f_k|^p\right)^{1/p}\right\|_{C(p)}, \tag{22}$$

for all $f_1, f_2, \ldots, f_n \in \mathrm{Ces}_p(I)$.

Proof. Inequality (22) taken to the power p means that

$$\sum_{k=1}^{n} \int_I \left(\frac{1}{x} \int_0^x |f_k(t)|\, dt\right)^p dx \leqslant \int_I \left[\frac{1}{x} \int_0^x \left(\sum_{k=1}^{n} |f_k(t)|^p\right)^{1/p} dt\right]^p dx.$$

If we show that

$$\sum_{k=1}^{n} \left(\frac{1}{x} \int_0^x |f_k(t)|\, dt\right)^p \leqslant \left[\frac{1}{x} \int_0^x \left(\sum_{k=1}^{n} |f_k(t)|^p\right)^{1/p} dt\right]^p$$

for every $x \in I$, then we are done. The last estimate can also be written as

$$\left[\sum_{k=1}^{n} \left(\int_0^x |f_k(t)|\, dt\right)^p\right]^{1/p} \leqslant \int_0^x \left(\sum_{k=1}^{n} |f_k(t)|^p\right)^{1/p} dt,$$

which is the p-concavity of $L^1[0, x]$ for every $x \in I$.

It is clear that $L^1(J)$, $J = J_x = [0, x]$ is 1-convex with constant 1 and it is well known that then $L^1(J)$ is p-concave with constant 1 (cf. [38, Proposition 1.d.5] or [44, Theorem 4.3]). We can also prove this fact directly as in [44, Theorem 4.3]: by the Hölder–Rogers inequality for $t \in J$ it yields that

$$\sum_{k=1}^{n} |f_k(t)||a_k| \leqslant \left(\sum_{k=1}^{n} |f_k(t)|^p\right)^{1/p} \|\{a_k\}\|_{p'}$$

and, by integrating over J,

$$\int_J \sum_{k=1}^n |f_k(t)||a_k|\,dt \leqslant \|\{a_k\}\|_{p'} \int_I \left(\sum_{k=1}^n |f_k(t)|^p\right)^{1/p} dt$$
$$= \|\{a_k\}\|_{p'} \left\| \left(\sum_{k=1}^n |f_k|^p\right)^{1/p} \right\|_{L^1(J)}.$$

Taking the supremum over all $\{a_k\}$ such that $\|\{a_k\}\|_{p'} \leqslant 1$ we obtain, by the Landau theorem,

$$\sup\left\{ \int_J \sum_{k=1}^n |f_k(t)||a_k|\,dt\colon\ \|\{a_k\}\|_{p'} \leqslant 1 \right\}$$
$$= \sup\left\{ \sum_{k=1}^n |a_k| \int_J |f_k(t)|\,dt\colon\ \|\{a_k\}\|_{p'} \leqslant 1 \right\}$$
$$= \left\| \left\{ \int_J |f_k(t)|\,dt \right\} \right\|_p = \left[\sum_{k=1}^n \left(\int_J |f_k(t)|\,dt \right)^p \right]^{1/p}$$
$$= \left(\sum_{k=1}^n \|f_k\|_{L^1(J)}^p \right)^{1/p}.$$

Thus,

$$\left(\sum_{k=1}^n \|f_k\|_{L^1(J)}^p \right)^{1/p} \leqslant \left\| \left(\sum_{k=1}^n |f_k|^p \right)^{1/p} \right\|_{L^1(J)},$$

and putting these facts together we obtain the estimate (22). □

Theorem 5. *If* $1 < p < \infty$, *then the space* $\mathrm{Ces}_p(I)$ *has trivial type and cotype* $\max(p, 2)$. *The space* $\mathrm{Ces}_\infty(I)$ *has trivial type and trivial cotype.*

Proof. Let $1 < p < \infty$. The space $\mathrm{Ces}_p(I)$ contains a copy of $L^1(I)$ (cf. [4], Lemma 1 for $I = [0, 1]$ and Theorem 2 for $I = [0, \infty)$) which implies that $\mathrm{Ces}_p(I)$ has trivial type.

On the other hand, since, by Theorem 4 the space $\mathrm{Ces}_p(I)$ is p-concave, then by a well-known theorem (cf. Lindenstrauss and Tzafiri [38, p. 100]) it has cotype $\max(p, 2)$. The fact that this space has no smaller cotype follows, for example, from Theorem 6 showing that $\mathrm{Ces}_p(I)$ contains an isomorphic copy of l^p and the fact that the space l^p has cotype $\max(p, 2)$ and this value is the best possible (cf. [38, p. 73] or [44, pp. 91–94]).

For $p = \infty$ the space $\mathrm{Ces}_\infty(I)$ has no absolutely continuous norm and, by the Lozanovskiĭ theorem (see [40, Theorem 5, p. 65]; cf. also [28, Theorem 4 in X.4] and [59, Theorem 4.1]), it contains an isomorphic copy of l^∞, therefore it has trivial type and trivial cotype. The proof is complete. $\square$

Remark 6. Similarly as in Theorem 4 we can prove that the Cesàro sequence spaces ces_p are p-concave with constant 1 since l^1 is p-concave with constant 1. Moreover, similarly as in Theorem 5 we can obtain that the Cesàro sequence spaces ces_p have trivial type and cotype $\max(p, 2)$ for $1 < p < \infty$. Also ces_∞ has trivial type and trivial cotype.

6. COPIES OF l^p SPACES IN THE CESÀRO FUNCTION SPACES Ces_p

The Cesàro function space $Ces_p(I)$ contains a copy of $L^1(I)$ and as we will see in the next theorem also complemented copies of l^p.

Theorem 6. *If $1 < p < \infty$, then $Ces_p(I)$ contains an order isomorphic and complemented copy of l^p.*

Proof. Let $I = [0, 1]$. We shall construct a sequence $\{f_n\}_{n=1}^\infty \subset Ces_p[0, 1]$ with disjoint supports which spans an isomorphic copy of l^p in $Ces_p[0, 1]$ and the closed linear span $[f_n]_{Ces_p}$ is complemented in $Ces_p[0, 1]$. Let us denote

$$f_n = \chi_{[2^{-n-1}, 2^{-n}]} \quad \text{and} \quad F_n(t) = \frac{1}{t}\int_0^t f_n(s)\,ds, \quad n = 1, 2, \ldots.$$

Since

$$F_n(t) = \begin{cases} 0, & \text{if } 0 < t \leqslant 2^{-n-1}, \\ 1 - \frac{1}{2^{n+1}t}, & \text{if } 2^{-n-1} \leqslant t \leqslant 2^{-n}, \\ \frac{1}{2^{n+1}t}, & \text{if } t \geqslant 2^{-n}, \end{cases}$$

it follows that

$$\|f_n\|_{C(p)}^p = \|F_n\|_{L^p}^p = \int_{2^{-n-1}}^{2^{-n}} \left(1 - \frac{1}{2^{n+1}t}\right)^p dt + 2^{-(n+1)p}\,\frac{2^{n(p-1)} - 1}{p-1}.$$

Note that the first term in the above sum is not bigger than 2^{-p-n-1} and the second one satisfies the inequalities

$$\frac{1 - 2^{-p+1}}{p-1} 2^{-p-n} \leqslant 2^{-(n+1)p}\,\frac{2^{n(p-1)} - 1}{p-1} \leqslant \frac{2^{-p-n}}{p-1}.$$

Therefore,

$$\|f_n\|_{C(p)} \approx \|f_n\|_{L^p} \approx 2^{-n/p} \tag{23}$$

with constants which depend only on p. If

$$\bar{f}_n = \frac{f_n}{\|f_n\|_{C(p)}}, \quad n = 1, 2, \dots,$$

then

$$1 = \|\bar{f}_n\|_{C(p)} \approx \|\bar{f}_n\|_{L^p}, \quad n \in \mathbb{N}.$$

Let us denote

$$x(t) = \sum_{n=1}^{\infty} \alpha_n \bar{f}_n, \quad \alpha_n \in \mathbb{R}.$$

Since $\bar{f}_n$ are disjoint functions we may assume that $\alpha_n \geqslant 0$. By Theorem 1(c) (the Hardy inequality) and the above equivalence

$$\|x\|_{C(p)} \leqslant \frac{p}{p-1}\|x\|_{L^p} = \frac{p}{p-1}\left(\sum_{n=1}^{\infty} \alpha_n^p \|\bar{f}_n\|_{L^p}^p\right)^{1/p} \leqslant C_p\left(\sum_{n=1}^{\infty} \alpha_n^p\right)^{1/p}.$$

On the other hand, by Theorem 4, for any $n \in \mathbb{N}$,

$$\left\|\sum_{k=1}^{n} \alpha_k \bar{f}_k\right\|_{C(p)} \geqslant \left(\sum_{k=1}^{n} \|\alpha_k \bar{f}_k\|_{C(p)}^p\right)^{1/p} = \left(\sum_{k=1}^{n} \alpha_k^p\right)^{1/p}$$

and passing to the limit as $n \to \infty$ we arrive at the inequality

$$\|x\|_{C(p)} \geqslant \left(\sum_{k=1}^{\infty} \alpha_k^p\right)^{1/p} \approx \|x\|_{L^p},$$

which together with estimation from above gives us that

$$[\bar{f}_n]_{\mathrm{Ces}_p} \simeq [\bar{f}_n]_{L^p} \simeq l^p. \tag{24}$$

Next, we prove that $[\bar{f}_n]_{\mathrm{Ces}_p}$ is complemented in Ces_p for $1 < p < \infty$. Let $x \in \mathrm{Ces}_p$, $x \geqslant 0$ and $\operatorname{supp} x \subset [2^{-n-1}, 2^{-n}]$, $n \in \mathbb{N}$. Then

$$\frac{1}{t}\int_0^t x(s)\,ds = \frac{1}{t}\int_{2^{-n-1}}^{t} x(s)\,ds\,\chi_{[2^{-n-1},2^{-n}]}(t) + \frac{1}{t}\|x\|_{L^1}\chi_{[2^{-n},1]}(t)$$

and

$$\|x\|_{C(p)}^p = \int_{2^{-n-1}}^{2^{-n}} \left(\frac{1}{t}\int_{2^{-n-1}}^{t} x(s)\,ds\right)^p dt + \|x\|_{L^1}^p \int_{2^{-n}}^{1} t^{-p}\,dt.$$

The first term in the last sum is not bigger than

$$\|x\|_{L^1}^p \int_{2^{-n-1}}^{2^{-n}} t^{-p}\, dt = \frac{2^{p-1}-1}{p-1} 2^{n(p-1)} \|x\|_{L^1},$$

and the second one is equal to

$$\|x\|_{L^1}^p \frac{2^{n(p-1)}-1}{p-1}.$$

Therefore,

$$\|x\|_{C(p)} \approx \|x\|_{L^1} 2^{n(1-1/p)}, \quad n = 1, 2, \ldots, \tag{25}$$

with constants which depend only on p. We consider the orthogonal projector

$$Tx(t) := \sum_{k=1}^{\infty} 2^{k+1} \int_{2^{-k-1}}^{2^{-k}} x(s)\, ds\, \chi_{[2^{-k-1}, 2^{-k}]}(t) \tag{26}$$

and prove that it is bounded in Ces_p.

For arbitrary $x \in \mathrm{Ces}_p$, $x \geqslant 0$ we set $x_k = x\chi_{[2^{-k-1}, 2^{-k}]}$ $(k = 1, 2, \ldots)$. Since

$$Tx_k = \|x_k\|_{L^1} 2^{k+1} \chi_{[2^{-k-1}, 2^{-k}]},$$

then (23) and (25) imply that

$$\|Tx_k\|_{C(p)} = \|x_k\|_{L^1} 2^{k+1} \|f_k\|_{C(p)} \leqslant B\|x_k\|_{L^1} 2^{k+1} 2^{-k/p} \leqslant C\|x_k\|_{C(p)}.$$

Therefore, by (24) and Theorem 4, we have that

$$\begin{aligned}\|Tx\|_{C(p)} &\leqslant C' \left(\sum_{k=1}^{\infty} \|Tx_k\|_{C(p)}^p \right)^{1/p} \leqslant C'C \left(\sum_{k=1}^{\infty} \|x_k\|_{C(p)}^p \right)^{1/p} \\ &\leqslant C'C \left\| \sum_{k=1}^{\infty} x_k \right\|_{C(p)} = C'C\|x\|_{C(p)},\end{aligned}$$

and the proof of the boundedness of T in Ces_p is complete. Since the image of T coincides with $[x_n]_{\mathrm{Ces}_p}$, then Theorem 6 is proved. □

The above theorem shows that the Cesàro function spaces $\mathrm{Ces}_p[0, 1]$ behave "near zero" similar to the l^p spaces. The authors proved in [4] that "in the middle" Cesàro function spaces $\mathrm{Ces}_p(I)$ contain an asymptotically isometric copy of l^1, that is, there exist a sequence $\{\varepsilon_n\} \subset (0, 1)$, $\varepsilon_n \to 0$ as $n \to \infty$ and a sequence of functions $\{f_n\} \subset \mathrm{Ces}_p[0, 1]$ such that, for arbitrary $\{\alpha_n\} \in l^1$, we have that

$$\sum_{n=1}^{\infty} (1-\varepsilon_n)|\alpha_n| \leqslant \left\| \sum_{n=1}^{\infty} \alpha_n f_n \right\|_{C(p)} \leqslant \sum_{n=1}^{\infty} |\alpha_n|. \tag{27}$$

Consequently, these spaces are not reflexive and do not have the fixed point property. This is a big difference with the Cesàro sequence spaces ces_p, which for $1 < p < \infty$ are reflexive and have the fixed point property.

Let us recall that a Banach space X has the *Dunford–Pettis property* if $x_n \to 0$ weakly in X and $f_n \to 0$ weakly in the dual space X^* imply $f_n(x_n) \to 0$. The classical examples of Banach spaces with the Dunford–Pettis property are the AL-spaces and AM-spaces. It is clear that if X^* has the Dunford–Pettis property, then X has itself this property (cf. [2, pp. 334–336]). Of course, the Cesàro sequence spaces ces_p, $1 < p < \infty$, as reflexive spaces do not have the Dunford–Pettis property.

Corollary 1. *If* $1 < p < \infty$*, then* $\operatorname{Ces}_p(I)$ *do not have the Dunford–Pettis property.*

Proof. By Theorem 6, $\operatorname{Ces}_p(I)$ contains a complemented copy of l^p and l^p do not have the Dunford–Pettis property. On the other hand, it is easy to show that, if a Banach space has the Dunford–Pettis property, then its complemented subspace has also the Dunford–Pettis property (cf. Wnuk [58, Lemma 1(i)] or [23, Proposition 11.37]). Thus, $\operatorname{Ces}_p(I)$ do not have the Dunford–Pettis property. $\square$

As it was mentioned before the Cesàro sequence spaces ces_p are not isomorphic to the l^q space for any $1 \leqslant q \leqslant \infty$. An analogous theorem is true for Cesàro function spaces.

Theorem 7. *If* $1 < p \leqslant \infty$*, then* $\operatorname{Ces}_p(I)$ *are not isomorphic to any* $L^q(I)$ *space for any* $1 \leqslant q \leqslant \infty$.

Proof. If $1 < q < \infty$, then $\operatorname{Ces}_p(I)$ has trivial type but $L^q(I)$ has type $\min(q, 2) > 1$ and therefore they cannot be isomorphic. The space $\operatorname{Ces}_p(I)$ for $1 < p < \infty$ is not isomorphic to $L^1(I)$ since $L^1(I)$ has the Dunford–Pettis property but $\operatorname{Ces}_p(I)$, as we have seen in Corollary 1, do not have the Dunford–Pettis property. Also $\operatorname{Ces}_p(I)$ for $1 < p < \infty$ is not isomorphic to $L^\infty(I)$ since the first space is separable and the second one is non-separable.

It only remains to show that $\operatorname{Ces}_\infty(I)$ is not isomorphic to $L^\infty(I)$. Since, by Pełczyński theorem $L^\infty(I)$ is isomorphic to ℓ^∞ (cf. Albiac and Kalton [1, Theorem 4.3.10]), therefore it is enough to show that $\operatorname{Ces}_\infty(I)$ is not isomorphic to ℓ^∞.

We show this for $K = \operatorname{Ces}_\infty[0, 1]$ since for the case of $\operatorname{Ces}_\infty(0, \infty)$ the proof is similar. For fixed $a \in (0, 1)$ define a continuous projection $P : K \to K$ by $Pf = f\chi_{[a,1]}$. Then

$$\int_a^1 |Pf(t)|\,dt \leqslant \int_0^1 |Pf(t)|\,dt \leqslant \|Pf\|_K = \sup_{0<x\leqslant 1} \frac{1}{x} \int_0^x |f(t)\chi_{[a,1]}(t)|\,dt$$
$$= \sup_{a\leqslant x\leqslant 1} \frac{1}{x} \int_a^x |f(t)\chi_{[a,1]}(t)|\,dt \leqslant \frac{1}{a} \int_a^1 |Pf(t)|\,dt.$$

Hence, $P(K)$ is isomorphic to $L^1[a, 1]$, i.e., K contains a complemented copy of a separable space while no separable subspace of ℓ^∞ is complemented in ℓ^∞ because ℓ^∞ is prime, that is, every infinite dimensional complemented subspace of ℓ^∞ is isomorphic to ℓ^∞ (see Lindenstrauss and Tzafriri [37, Theorem 2.a.7], or Albiac and Kalton [1, Theorem 5.6.5]). Therefore, K and ℓ^∞ are not isomorphic. □

7. ON THE WEAK BANACH–SAKS PROPERTY OF THE CESÀRO FUNCTION SPACES

Let us recall that a Banach space X is said to have the *weak Banach–Saks property* if every weakly null sequence in X, say (x_n), contains a subsequence (x_{n_k}) whose first arithmetical means converge strongly to zero, that is,

$$\lim_{m\to\infty} \frac{1}{m}\left\| \sum_{k=1}^{m} x_{n_k} \right\| = 0.$$

It is known that uniformly convex spaces, c_0, l^1 and L^1 have the weak Banach–Saks property, whereas $C[0,1]$ and l^∞ do not have. We should mention that the result on L^1 space, proved by Szlenk [55] in 1965, was a very important breakthrough in studying of the weak Banach–Saks property.

In 1982, Rakov [48, Theorem 1] proved that a Banach space with non-trivial type (or equivalently B-convex) has the weak Banach–Saks property (cf. also [57, Theorem 1]). Recently Dodds, Semenov and Sukochev [19] investigated the weak Banach–Saks property of rearrangement invariant spaces and Astashkin and Sukochev [6] have got a complete description of Marcinkiewicz spaces with the latter property.

The spaces $\mathrm{Ces}_p[0,1]$ for $1 \leqslant p < \infty$ are neither B-convex (they have trivial type) nor rearrangement invariant. Nevertheless, we will prove that Ces_p for all $1 \leqslant p < \infty$ have the weak Banach–Saks property.

Theorem 8. *If $1 \leqslant p < \infty$, then the Cesàro function space $\mathrm{Ces}_p[0,1]$ has the weak Banach–Saks property.*

We begin with some auxiliary notation and results.

If $I = [a, b]$ and $J = [c, d]$ are two closed intervals, then we write $I < J$ if $b \leqslant c$. Let $\{I_n\}_{n=1}^\infty$ be a sequence of closed intervals $I_n = [a_n, b_n] \subset [0,1]$. Then $I_n \to 0$ means that $I_1 > I_2 > \cdots$ and $b_n \to 0^+$. Analogously, $I_n \to 1$ means that $I_1 < I_2 < \cdots$ and $a_n \to 1^-$. Moreover, in what follows $\operatorname{supp} f = \{t\colon\ f(t) \neq 0\}$.

Lemma 1 (Weakly null sequences in $\mathrm{Ces}_p[0,1], 1 < p < \infty$). *Let $\{x_n\}_{n=1}^\infty \subset \mathrm{Ces}_p$. Then $x_n \xrightarrow{w} 0$ in Ces_p if and only if*

(a) *there exists a constant $M > 0$ such that $\|x_n\|_{C(p)} \leqslant M$ for all $n = 1, 2, \ldots,$*

and

(b) *for every set $A \subset [0,1]$ such that $A \subset [h, 1-h]$ for some $h \in (0, \frac{1}{2})$ we have $\int_A x_n(t)\,dt \to 0$ as $n \to \infty$.*

Proof. It is enough to check that the set of all functions of the form

$$a(t) = \sum_{k=1}^{n} a_k \chi_{A_k}(t), \tag{28}$$

where $n \in \mathbb{N}$, $a_k \in \mathbb{R}$ and $A_k \subset [0,1]$ are pairwise disjoint sets such that $A_k \subset [h, 1-h]$ for some $h \in (0, \frac{1}{2})$, is dense in the space $U(p') = (\mathrm{Ces}_p)^* = (\mathrm{Ces}_p)'$, $p' = \frac{p}{p-1}$, with the norm

$$\|y\|_{U(p')} = \left(\int_0^1 \left(\frac{\tilde{y}(t)}{1-t} \right)^{p'} dt \right)^{1/p'}, \quad \tilde{y}(t) = \operatorname{ess} \sup_{s \in [t,1]} |y(s)|.$$

Let $y \in U(p')$ and $\varepsilon > 0$. Note that $\tilde{y}(1^-) = \lim_{t \to 1^-} \tilde{y}(t) = 0$. In fact, if $\tilde{y}(t) \geqslant c > 0$ $(0 < t < 1)$, then since $p' > 1$ we have that $\|y\|_{U(p')}^{p'} \geqslant c \int_0^1 \frac{1}{(1-t)^{p'}} \, dt = \infty$. Therefore, we may choose $\delta \in (0,1)$ and $h \in (0, \delta)$ so that

$$\max \left(\int_0^{\delta} \left(\frac{\tilde{y}(t)}{1-t} \right)^{p'} dt, \int_{1-\delta}^{1} \left(\frac{\tilde{y}(t)}{1-t} \right)^{p'} dt \right) \leqslant \varepsilon^{p'} \tag{29}$$

and

$$\tilde{y}(1-h) \leqslant \varepsilon \cdot \left(\frac{p'-1}{\delta^{1-p'} - 1} \right)^{1/p'}. \tag{30}$$

Since $y \in U(p')$, then $\tilde{y}(t)$ is finite for every $t \in (0,1)$ which implies that $y(t)$ is a bounded measurable function on the interval $[h, 1-h]$. Therefore, there exists a function $a(t)$ of the form (28) such that $\operatorname{supp} a \subset [h, 1-h]$ and

$$\|(y-a)\chi_{[h,1-h]}\|_{L^\infty} \leqslant \varepsilon \cdot \left(\frac{p'-1}{h^{1-p'} - 1} \right)^{1/p'}. \tag{31}$$

By the triangle inequality we have that

$$\begin{aligned} &\|y - a\|_{U(p')} \\ &\quad \leqslant \|y\chi_{[0,h]}\|_{U(p')} + \|(y-a)\chi_{[h,1-h]}\|_{U(p')} + \|y\chi_{[1-h,1]}\|_{U(p')}, \end{aligned} \tag{32}$$

and let us estimate each of the three terms separately. At first, since $0 < h < \delta$, then, by (29),

$$\|y\chi_{[0,h]}\|_{U(p')}^{p'} \leqslant \int_0^{\delta} \left(\frac{\tilde{y}(t)}{1-t} \right)^{p'} dt \leqslant \varepsilon^{p'}. \tag{33}$$

Next, (31) implies

$$\|(y-a)\chi_{[h,1-h]}\|_{U(p')}^{p'} \leqslant \int_0^{1-h} \frac{dt}{(1-t)^{p'}} \cdot \varepsilon^{p'} \cdot \left(\frac{p'-1}{h^{1-p'} - 1} \right) = \varepsilon^{p'}. \tag{34}$$

Finally, (30) and (29) imply that

$$\|y\chi_{[1-h,1]}\|_{U(p')}^{p'} \leqslant \int\limits_0^{1-\delta} \left(\frac{\tilde{y}(1-h)}{1-t}\right)^{p'} dt + \int\limits_{1-\delta}^{1} \left(\frac{\tilde{y}(t)}{1-t}\right)^{p'} dt \leqslant 2\varepsilon^{p'}. \tag{35}$$

Thus, by (32)–(35), we have that $\|y-a\|_{U(p')} \leqslant 4^{1/p'}\varepsilon$, and the proof is complete. $\square$

Corollary 2. *Let* $\{I_n\}_{n=1}^{\infty}$ *be a sequence of intervals from* $[0,1]$ *such that either* $I_n \to 0$ *or* $I_n \to 1$*. Then, for every* $p \in (1,\infty)$*, we have* $\frac{\chi_{I_n}}{\|\chi_{I_n}\|_{C(p)}} \xrightarrow{w} 0$ *in* $\mathrm{Ces}_p[0,1]$.

Following Kadec and Pełczyński [27] (see also [46] and [47]) we will use the following notation: Let X be a Banach function lattice on $[0,1]$. For every $x \in X$ and $\alpha \in (0,1]$ we set

$$\eta(x,\alpha) = \sup_{A\subset[0,1],m(A)=\alpha} \|x\chi_A\|_X.$$

Moreover, if $K \subset X$, then

$$\eta(K,\alpha) = \sup_{x\in K} \eta(x,\alpha),\ \eta(K,0^+) = \lim_{\alpha\to 0^+} \eta(K,\alpha).$$

Lemma 2. *If a Banach function lattice* X *on* $[0,1]$ *satisfies a lower* p*-estimate (*$1 \leqslant p < \infty$*) with constant one, then for any disjointly supported* $x, y \in X$ *and* $\alpha > 0, \beta > 0$ *we have that*

$$\eta(x+y,\alpha+\beta)^p \geqslant \eta(x,\alpha)^p + \eta(y,\beta)^p.$$

Proof. For any $\varepsilon > 0$ choose the sets A and B from $[0,1]$ such that $A \subset \operatorname{supp} x$, $B \subset \operatorname{supp} y$, $m(A) \leqslant \alpha$, $m(B) \leqslant \beta$, and

$$\|x\chi_A\|_X^p \geqslant \eta(x,\alpha)^p - \varepsilon, \qquad \|y\chi_B\|_X^p \geqslant \eta(y,\beta)^p - \varepsilon.$$

Since $m(A\cup B) \leqslant \alpha+\beta$ and X satisfies a lower p-estimate with constant one it follows that

$$\begin{aligned}\eta(x+y,\alpha+\beta) &\geqslant \|(x+y)\chi_{A\cup B}\|_X = \|x\chi_A + y\chi_B\|_X \\ &\geqslant \left(\|x\chi_A\|_X^p + \|y\chi_B\|_X^p\right)^{1/p} \\ &\geqslant \left(\eta(x,\alpha)^p + \eta(y,\beta)^p - 2\varepsilon\right)^{1/p},\end{aligned}$$

and the proof of the lemma follows by letting $\varepsilon \to 0^+$. $\square$

Let X be a Banach function lattice on $[0,1]$ and a set $K \subset X$. We say that K consists of elements having *equicontinuous norms* in X if

$$\lim_{A\subset[0,1],m(A)\to 0} \sup_{x\in K} \|x\chi_A\|_X = 0.$$

An important tool in the proof of Theorem 8 will be the following assertion:

Proposition 7 (Subsequence splitting property). *Let* $1 < p < \infty, \{x_n\}_{n=1}^{\infty} \subset \mathrm{Ces}_p[0,1]$, $\|x_n\|_{C(p)} = 1$ *and* $x_n \xrightarrow{w} 0$ *in* $\mathrm{Ces}_p[0,1]$. *Then there exists a subsequence* $\{x_n'\} \subset \{x_n\}$ *such that*

$$x_n' = y_n + z_n, \quad n = 1, 2, \ldots,$$

where $\{y_n\}_{n=1}^{\infty}$ *consists of elements having equicontinuous norms in* Ces_p *and* $\operatorname{supp} z_n \subset I_n' \cup I_n''$ *with* $\{I_n', I_n''\}_{n=1}^{\infty}$ *being a sequence of pairwise disjoint intervals from* $[0,1]$ *such that* $I_n' \to 0, I_n'' \to 1$. *Moreover,* $y_n \xrightarrow{w} 0, z_n \xrightarrow{w} 0$ *in* Ces_p.

Proof. We set $\eta_0 = \eta(\{x_n\}, 0^+)$. If $\eta_0 = 0$, then the sequence $\{x_n\}$ consists of elements with equicontinuous norms in Ces_p and we have nothing to prove. Therefore, assume that $\eta_0 > 0$. By the definition of η_0, there exists a sequence of sets $A_n \subset [0,1], m(A_n) = \alpha_n \to 0$ and a subsequence of $\{x_n\}$ (which will be denoted also by $\{x_n\}$) such that for all $n \in \mathbb{N}$

$$\|x_n \chi_{A_n}\|_{C(p)} \geqslant \eta_0 - \frac{1}{n}. \tag{36}$$

Let us denote

$$u_n = x_n \chi_{A_n} \quad \text{and} \quad v_n = x_n - u_n. \tag{37}$$

Since $\mathrm{Ces}_p[0,1]$ is p-concave with constant one, then, by Lemma 2, it yields that

$$\eta(v_n, \alpha)^p \leqslant \eta(x_n, \alpha + \alpha_n)^p - \eta(u_n, \alpha_n)^p \leqslant \eta(x_n, \alpha + \alpha_n)^p - \left(\eta_0 - \frac{1}{n}\right)^p.$$

Hence, for $0 < \alpha \leqslant 1/2$ we have that

$$\limsup_{n\to\infty} \eta(v_n, \alpha)^p \leqslant \eta(\{x_n\}, 2\alpha)^p - \eta_0^p.$$

Since Ces_p is a separable space the last inequality implies that

$$\eta\big(\{v_n\}, 0^+\big) = 0, \tag{38}$$

that is, $\{v_n\}$ consists of elements with equicontinuous norms in Ces_p.

According to Lemma 1, for every $h \in (0, \frac{1}{2})$,

$$x_n \chi_{[h,1-h]} \xrightarrow{w} 0 \quad \text{in } \mathrm{Ces}_p\,. \tag{39}$$

Therefore, since $\mathrm{Ces}_p[0,1]_{|[h,1-h]} = L^1[h, 1-h]$ with equivalent norms (see [4, Lemma 1]) we have that $x_n \chi_{[h,1-h]} \xrightarrow{w} 0$ in L^1. Moreover, since $\eta(\{v_n\}, 0^+) = 0$ it follows that

$$\eta_{L^1}\big(\{v_n \chi_{[h,1-h]}\}, 0^+\big) = 0$$

(where η_{L^1} is calculated in the space L^1) and

$$\|v_n \chi_{[h,1-h]}\|_{L^1} \leqslant C \|v_n \chi_{[h,1-h]}\|_{C(p)} \leqslant C.$$

Thus, by the classical Dunford–Pettis criterion (see, for example, [22, Theorem 4.21.2] or [1, Theorem 5.2.9]), the sequence $\{v_n \chi_{[h,1-h]}\}_{n=1}^{\infty}$ is a relatively weakly compact subset of L^1 and, hence, simultaneously in Ces_p. Therefore, there is a subsequence $\{v_{n_k}\} \subset \{v_n\}$ such that $v_{n_k} \chi_{[h,1-h]} \xrightarrow{w} v$, where $v \in \mathrm{Ces}_p$. By combining the last mentioned facts with (39) and with the equality $x_n = u_n + v_n$, we get that $u_{n_k} \chi_{[h,1-h]} \xrightarrow{w} -v$ in Ces_p, and, hence, in L^1. Taking into account the definition of u_n (see (36) and (37)) and using again the Dunford–Pettis criterion we conclude that for every $h \in (0, 1/2)$ there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ (depending on h) such that

$$\|u_{n_k} \chi_{[h,1-h]}\|_{C(p)} \to 0 \quad \text{as } k \to \infty.$$

Since Ces_p is a separable space, then by a standard procedure, we may choose a subsequence of $\{u_{n_k}\}$ (denote it again by $\{u_{n_k}\}$) and pairwise disjoint intervals $\{I_k', I_k''\}_{k \in \mathbb{N}}$, $I_k' \to 0$, $I_k'' \to 1$ such that

(40) $$\|u_{n_k} \chi_{[0,1] \setminus (I_k' \cup I_k'')}\|_{C(p)} \to 0 \quad \text{as } k \to \infty.$$

Setting $x_k' = y_k + z_k$, with

$$y_k = v_{n_k} + u_{n_k} \chi_{[0,1] \setminus (I_k' \cup I_k'')}, \qquad z_k = u_{n_k} \chi_{I_k' \cup I_k''},$$

we see that, by (38) and (40), this representation satisfies all conditions. In particular, according to Lemma 1, we have that $z_k \xrightarrow{w} 0$ and $y_k \xrightarrow{w} 0$. The proof is complete. □

Now, we may proceed with the proof of Theorem 8.

Proof of Theorem 8. Since $\mathrm{Ces}_1[0,1] = L^1(\ln 1/t)$ (with equality of norms) and $L^1(\ln 1/t)$ is isometric to L^1, then in the case $p = 1$ the result follows from the Szlenk theorem [55]. Therefore, we will consider the case when $1 < p < \infty$. Taking into account Proposition 7 it is enough to prove the following: if $\{x_n\} \subset \mathrm{Ces}_p$, $x_n \xrightarrow{w} 0$ and either

(a) $\{x_n\}$ consists of elements with equicontinuous norms

or

(b) $\operatorname{supp} x_n \subset I_n$, where $I_n \to 1$

or

(c) $\operatorname{supp} x_n \subset I_n$, where $I_n \to 0$,

then there is a subsequence $\{x_n'\} \subset \{x_n\}$ such that

(41) $$\frac{1}{m} \left\| \sum_{k=1}^{m} x_k' \right\|_{C(p)} \to 0 \quad \text{as } m \to \infty.$$

Case (a). We will use the following remark from Szlenk paper [55, Remark 1]: a sequence $\{x_n\} \subset X, x_n \xrightarrow{w} 0$ in X (X is a Banach space) contains a subsequence $\{x_n'\}$ such that $\frac{1}{m}\|\sum_{k=1}^m x_k'\|_X \to 0$ as $m \to \infty$ if and only if it contains a subsequence $\{x_{n_k}\}$ such that

$$\lim_{m\to\infty} \sup_{k_1<\cdots<k_m} \left\| \frac{1}{m} \sum_{i=1}^m x_{n_{k_i}} \right\|_X = 0. \tag{42}$$

Let $\{x_n\} \subset \mathrm{Ces}_p, x_n \xrightarrow{w} 0$ and $\varepsilon > 0$. At first, setting

$$A_{n,m} = \{t \in [0,1]\colon |x_n(t)| \geqslant m\}, \quad m, n = 1, 2, \ldots,$$

we prove that

$$\lim_{m\to\infty} \sup_{n\in\mathbb{N}} m(A_{n,m}) = 0. \tag{43}$$

We may assume that $\|x_n\|_{C(p)} = 1 (n = 1, 2, \ldots)$. Therefore,

$$\begin{aligned} 1 = \|x_n\|_{C(p)} \geqslant \|x_n\|_{C(1)} = \|x_n\|_{L^1(\ln 1/t)} = \int_0^1 |x_n(t)| \ln \frac{1}{t}\, dt \\ \geqslant \int_{A_{n,m}} |x_n(t)| \ln \frac{1}{t}\, dt \geqslant m \int_{A_{n,m}} \ln \frac{1}{t}\, dt, \end{aligned}$$

i.e.,

$$\int_{A_{n,m}} \ln \frac{1}{t}\, dt \leqslant \frac{1}{m} \quad \text{for all } n, m \in \mathbb{N}. \tag{44}$$

Assume that (43) does not hold, that is, there exists a $\delta > 0$ such that for every $m \in \mathbb{N}$ there is $n_m \in \mathbb{N}$ such that $m(A_{n_m,m}) > \delta$. Clearly, we may assume that $n_m \to \infty$ as $m \to \infty$. Since

$$m\left(A_{n_m,m} \cap \left[0, 1 - \frac{\delta}{2}\right]\right) > \frac{\delta}{2},$$

then we have that, for any $m \in \mathbb{N}$,

$$\begin{aligned} \int_{A_{n_m,m}} \ln \frac{1}{t}\, dt \geqslant \int_{A_{n_m,m}\cap[0,1-\frac{\delta}{2}]} \ln \frac{1}{t}\, dt \\ \geqslant \ln \frac{2}{2-\delta} m\left(A_{n_m,m} \cap \left[0, 1 - \frac{\delta}{2}\right]\right) > \frac{\delta}{2} \ln \frac{2}{2-\delta}. \end{aligned}$$

The last inequality contradicts (44) and, therefore, (43) is proved.

We recall that $\{x_n\}$ consists of functions having equicontinuous norms. Hence, by (43), for some m_0 and all $n \in \mathbb{N}$,

$$\|x_n \chi_{A_{n,m_0}}\|_{C(p)} < \frac{\varepsilon}{3}. \tag{45}$$

Denote $y_n = x_n \chi_{A_{n,m_0}}$ $(n = 1, 2, \ldots)$. Then $|x_n(t) - y_n(t)| \leqslant m_0$ for $t \in [0, 1]$, so that, in particular, $x_n - y_n \in L^p$ and $\|x_n - y_n\|_{L^p} \leqslant m_0$. Since L^p is a reflexive space for $1 < p < \infty$ and since L^p has the Banach–Saks property (cf. [7, Chapter XII, Theorem 2]), we may choose an increasing sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$ such that $x_{n_k} - y_{n_k} \xrightarrow{w} v$ in L^p, where $v \in L^p$, and (see (42))

$$\lim_{m\to\infty} \sup_{k_1<\cdots<k_m} \left\| \frac{1}{m} \sum_{i=1}^{m} (x_{n_{k_i}} - y_{n_{k_i}}) - v \right\|_{L^p} = 0. \tag{46}$$

Using the imbedding $L^p \subset \mathrm{Ces}_p$ (see Theorem 1(c)) we obtain that $x_{n_k} - y_{n_k} \xrightarrow{w} v$ in Ces_p. Therefore, since $x_{n_k} \xrightarrow{w} 0$ in Ces_p, we get that $y_{n_k} \xrightarrow{w} -v$ in Ces_p. Moreover, from (45) it follows that $\|y_{n_k}\|_{C(p)} < \frac{\varepsilon}{3}$ so that $\|v\|_{C(p)} \leqslant \frac{\varepsilon}{3}$. At last, by Theorem 1(c) and (46), for large enough $m \in \mathbb{N}$,

$$\begin{aligned} &\sup_{k_1<\cdots<k_m} \left\| \frac{1}{m} \sum_{i=1}^{m} (x_{n_{k_i}} - y_{n_{k_i}}) - v \right\|_{C(p)} \\ &\quad \leqslant p' \sup_{k_1<\cdots<k_m} \left\| \frac{1}{m} \sum_{i=1}^{m} (x_{n_{k_i}} - y_{n_{k_i}}) - v \right\|_{L^p} \leqslant \frac{\varepsilon}{3}. \end{aligned}$$

The last inequalities give us that

$$\begin{aligned} &\sup_{k_1<\cdots<k_m} \left\| \frac{1}{m} \sum_{i=1}^{m} x_{n_{k_i}} \right\|_{C(p)} \\ &\quad \leqslant \sup_{k_1<\cdots<k_m} \left(\left\| \frac{1}{m} \sum_{i=1}^{m} (x_{n_{k_i}} - y_{n_{k_i}}) - v \right\|_{C(p)} \right. \\ &\qquad \left. + \frac{1}{m} \left\| \sum_{i=1}^{m} y_{n_{k_i}} \right\|_{C(p)} + \|v\|_{C(p)} \right) \leqslant \varepsilon, \end{aligned}$$

if m is large enough. Thus, $\{x_{n_k}\}$ satisfies condition (42) and in this case everything is proved.

Case (b). Let $\{x_n\} \subset \mathrm{Ces}_p, x_n \xrightarrow{w} 0,$ $\operatorname{supp} x_n \subset I_n = [a_n, b_n] (n = 1, 2, \ldots),$ $I_1 < I_2 < \cdots$ and $a_n \to 1^-$. We may suppose that $x_n \geqslant 0$, $\|x_n\|_{C(p)} = 1$ and $a_1 \geqslant 1/2$.

Since $p > 1$, it is enough to show that $\{x_n\}$ contains a subsequence (for simplicity, it will be denoted also by $\{x_n\}$) such that

$$\left\| \sum_{k=1}^{n} x_k \right\|_{C(p)} \leqslant C n^{1/p}, \tag{47}$$

where $C > 0$ is independent of $n \in \mathbb{N}$. We will choose x_n inductively. Suppose that $m \geqslant 2$ and $x_1, x_2, \dots, x_{m-1}$ are already chosen. Then $a_1 < b_1 \leqslant \dots \leqslant a_{m-1} < b_{m-1} < 1$ are fixed and, since $a_n \to 1^-$, we may take a_m so that

$$1 - a_m \leqslant (1 - b_{m-1}) \cdot 2^{-p}. \tag{48}$$

Then for x_m we take the function corresponding to the interval $I_m = [a_m, b_m]$ (that is, $\operatorname{supp} x_m \subset I_m$). Let's check that inequality (47) holds. For all $n \in \mathbb{N}$ and $t \in (0, 1]$ we have that

$$\begin{aligned}\frac{1}{t}\int_0^t \left|\sum_{k=1}^n x_k(s)\right| ds &= \frac{1}{t}\sum_{m=1}^n \left(\sum_{i=1}^{m-1}\int_{a_i}^{b_i} x_i(s)\,ds + \int_{a_m}^t x_m(s)\,ds\right)\chi_{[a_m,b_m]}(t) \\ &\quad + \frac{1}{t}\sum_{m=1}^n\sum_{i=1}^m \int_{a_i}^{b_i} x_i(s)\,ds\,\chi_{[b_m,a_{m+1}]}(t) \\ &= S_1(t) + S_2(t) + S_3(t),\end{aligned}$$

where $a_{n+1} = 1$ and

$$S_1(t) = \frac{1}{t}\sum_{m=2}^n\sum_{i=1}^{m-1}\int_{a_i}^{b_i} x_i(s)\,ds\,\chi_{[a_m,b_m]}(t),$$

$$S_2(t) = \frac{1}{t}\sum_{m=1}^n\sum_{i=1}^{m}\int_{a_i}^{b_i} x_i(s)\,ds\,\chi_{[b_m,a_{m+1}]}(t),$$

$$S_3(t) = \frac{1}{t}\sum_{m=1}^n\int_{a_m}^{t} x_m(s)\,ds\,\chi_{[a_m,b_m]}(t).$$

Since, by Theorem 3, $(\mathrm{Ces}_p[0,1])^* = (\mathrm{Ces}_p[0,1])' = U(p')$ it follows that, for all $i \in \mathbb{N}$,

$$\begin{aligned}\int_{a_i}^{b_i} x_i(s)\,ds &\leqslant A\|x_i\|_{C(p)} \cdot \|\chi_{[a_i,b_i]}\|_{U(p')} = A\left(\int_0^{b_i} \frac{dt}{(1-t)^{p'}}\right)^{1/p'} \\ &= A(p-1)^{1/p'}\left[\frac{1}{(1-b_i)^{p'-1}} - 1\right]^{1/p'} \leqslant \frac{B}{(1-b_i)^{1/p}},\end{aligned}$$

where $B > 0$ depends only on p. Moreover, by (48), for every $i = 1, 2, \dots, m-1$,

$$\left(\frac{1-a_m}{1-b_i}\right)^{1/p} \leqslant \prod_{j=i}^{m-1}\left(\frac{1-a_{j+1}}{1-b_j}\right)^{1/p} \leqslant 2^{i-m}.$$

Therefore,

$$\begin{aligned}
\|S_1\|_p^p &= \sum_{m=2}^{n}\left(\sum_{i=1}^{m-1}\int_{a_i}^{b_i} x_i(s)\,ds\right)^p \int_{a_m}^{b_m}\frac{dt}{t^p} \\
&\leqslant B^p\sum_{m=2}^{n}\left(\sum_{i=1}^{m-1}(1-b_i)^{-1/p}\right)^p \frac{b_m^{p-1}-a_m^{p-1}}{(p-1)a_m^{p-1}b_m^{p-1}} \\
&\leqslant C_1^p\sum_{m=2}^{n}\left(\sum_{i=1}^{m-1}(1-b_i)^{-1/p}\right)^p (1-a_m) \\
&\leqslant C_1^p\sum_{m=2}^{n}\left(\sum_{i=1}^{m-1}2^{i-m}\right)^p \leqslant C_1^p n,
\end{aligned}$$

so that $\|S_1\|_p \leqslant C_1 n^{1/p}$, where $C_1 > 0$ depends only on p. Similarly,

$$\begin{aligned}
\|S_2\|_p^p &\leqslant B^p\sum_{m=1}^{n}\left(\sum_{i=1}^{m}(1-b_i)^{-1/p}\right)^p \frac{a_{m+1}^{p-1}-b_m^{p-1}}{(p-1)a_{m+1}^{p-1}b_m^{p-1}} \\
&\leqslant C_2^p\sum_{m=1}^{n}\left(\sum_{i=1}^{m}\left(\frac{1-b_m}{1-b_i}\right)^{1/p}\right)^p \\
&\leqslant C_2^p\sum_{m=1}^{n}\left(1+\sum_{i=1}^{m-1}2^{i-m}\right)^p \leqslant (2C_2)^p n,
\end{aligned}$$

which implies that $\|S_2\|_p \leqslant 2C_2 n^{1/p}$, where $C_2 > 0$ depends only on p. Finally, it is easy to see that

$$\|S_3\|_p \leqslant \left(\sum_{m=1}^{n}\|x_m\|_{C(p)}^p\right)^{1/p} = n^{1/p}.$$

Thus, combining the estimates of S_1, S_2 and S_3 we get

$$\left\|\sum_{k=1}^{n} x_k\right\|_{C(p)} \leqslant \sum_{k=1}^{3}\|S_k\|_p \leqslant (1+C_1+2C_2)\cdot n^{1/p},$$

where $C := 1+C_1+2C_2$ is independent of $n \in \mathbb{N}$, that is, inequality (47) is proved.

Case (c). Let $\{x_n\} \subset \mathrm{Ces}_p$, $x_n \xrightarrow{w} 0$, $\operatorname{supp} x_n \subset I_n = [a_n, b_n]$ $(n = 1, 2, \ldots)$, $I_1 > I_2 > \cdots$ and $b_n \to 0^+$. Again we may assume that $x_n \geqslant 0$, $\|x_n\|_{C(p)} = 1$ and $b_1 \leqslant 1/2$. As in the case (b) it is enough to prove inequality (47) for some subsequence of $\{x_n\}$ (it will be denoted also by $\{x_n\}$), which will be chosen inductively.

Suppose that $m \geqslant 2$ and the functions $x_1, x_2, \ldots, x_{m-1}$ are chosen. Then $b_1 > a_1 \geqslant \cdots \geqslant b_{m-1} > a_{m-1} > 0$ are fixed and, since $b_n \to 0^+$, we may take b_m so that

(49) $$b_m \leqslant 2^{-p'} a_{m-1}.$$

Let us show that the corresponding subsequence $\{x_n\}$ satisfies inequality (47). For any $n \in \mathbb{N}$ and $t \in (0, 1]$ we have that

$$\begin{aligned}
&\frac{1}{t}\int_0^t \left|\sum_{k=1}^n x_k(s)\right| ds \\
&= \frac{1}{t}\sum_{j=1}^n \left(\sum_{i=n-j+2}^n \int_{a_i}^{b_i} x_i(s)\,ds + \int_{a_{n-j+1}}^t x_{n-j+1}(s)\,ds\right)\chi_{[a_{n-j+1},b_{n-j+1}]}(t) \\
&\quad + \frac{1}{t}\sum_{j=1}^n \sum_{i=n-j+1}^n \int_{a_i}^{b_i} x_i(s)\,ds\,\chi_{[b_{n-j+1},a_{n-j}]}(t) \\
&= T_1(t) + T_2(t) + T_3(t),
\end{aligned}$$

where $a_0 = 1$ and

$$T_1(t) = \frac{1}{t}\sum_{j=2}^n \sum_{i=n-j+2}^n \int_{a_i}^{b_i} x_i(s)]\,ds\,\chi_{[a_{n-j+1},b_{n-j+1}]}(t),$$

$$T_2(t) = \frac{1}{t}\sum_{j=1}^n \sum_{i=n-j+1}^n \int_{a_i}^{b_i} x_i(s)\,ds\,\chi_{[b_{n-j+1},a_{n-j}]}(t),$$

$$T_3(t) = \frac{1}{t}\sum_{j=1}^n \int_{a_{n-j+1}}^t x_{n-j+1}(s)\,ds\,\chi_{[a_{n-j+1},b_{n-j+1}]}(t).$$

Using again the duality result, as in the proof of (b), we find that

$$\begin{aligned}
\int_{a_i}^{b_i} x_i(s)\,ds &\leqslant A\|x_i\|_{C(p)} \cdot \|\chi_{[a_i,b_i]}\|_{U(p')} \\
&= A(p-1)^{1/p'}\left[\frac{1-(1-b_i)^{p'-1}}{(1-b_i)^{p'-1}}\right]^{1/p'} \\
&\leqslant B' b_i^{1/p'} \qquad (i = 1, 2, \ldots),
\end{aligned}$$

where $B' > 0$ depends only on p. Since, by (49), for any $k < i$

$$\left(\frac{b_i}{a_k}\right)^{1/p'} \leqslant \prod_{m=k}^i \left(\frac{b_m}{a_{m-1}}\right)^{1/p'} \leqslant 2^{k-i-1},$$

then

$$\|T_1\|_p^p = \sum_{j=2}^{n} \left(\sum_{i=n-j+2}^{n} \int_{a_i}^{b_i} x_i(s)\,ds \right)^p \int_{a_{n-j+1}}^{b_{n-j+1}} \frac{dt}{t^p}$$

$$\leqslant (B')^p \sum_{j=2}^{n} \left(\sum_{i=n-j+2}^{n} b_i^{1/p'} \right)^p \frac{b_{n-j+1}^{p-1} - a_{n-j+1}^{p-1}}{(p-1)a_{n-j+1}^{p-1} b_{n-j+1}^{p-1}}$$

$$\leqslant B_1^p \sum_{j=2}^{n} \left(\sum_{i=n-j+2}^{n} b_i^{1/p'} \right)^p a_{n-j+1}^{1-p}$$

$$\leqslant B_1^p \sum_{j=2}^{n} \left(\sum_{i=n-j+2}^{n} 2^{n-i-j+1} \right)^p \leqslant B_1 n,$$

so that $\|T_1\|_p \leqslant B_1 n^{1/p}$, where $B_1 > 0$ depends only on p. Similarly,

$$\|T_2\|_p^p \leqslant (B')^p \sum_{j=1}^{n} \left(\sum_{i=n-j+1}^{n} b_i^{1/p'} \right)^p \frac{a_{n-j}^{p-1} - b_{n-j+1}^{p-1}}{(p-1)a_{n-j}^{p-1} b_{n-j+1}^{p-1}}$$

$$\leqslant B_2^p \sum_{j=1}^{n} \left(\sum_{i=n-j+1}^{n} \left(\frac{b_i}{b_{n-j+1}} \right)^{1/p'} \right)^p$$

$$\leqslant B_2^p \sum_{j=1}^{n} \left(1 + \sum_{i=n-j+2}^{n} 2^{n-i-j+1} \right)^p \leqslant (2B_2)^p n.$$

Hence, $\|T_2\|_p \leqslant 2B_2 n^{1/p}$, where $B_2 > 0$ depends only on p. Moreover, it is clear that

$$\|T_3\|_p \leqslant \left(\sum_{j=1}^{n} \|x_j\|_{C(p)}^p \right)^{1/p} = n^{1/p}.$$

Thus, combining the estimates of T_1, T_2 and T_3 we get that

$$\left\| \sum_{k=1}^{n} x_k \right\|_{C(p)} \leqslant \sum_{k=1}^{3} \|T_k\|_p \leqslant (1 + B_1 + 2B_2) \cdot n^{1/p},$$

where $B := 1 + B_1 + 2B_2$ is independent of $n \in \mathbb{N}$. Since all cases (a)–(c) are examined, the theorem is proved. □

8. THE CESÀRO FUNCTION SPACES $\mathrm{Ces}_p[0,\infty)$ AND $\mathrm{Ces}_p[0,1]$ ARE ISOMORPHIC FOR $1 < p \leqslant \infty$

The main result in this Section is a construction of an isomorphism between the Cesàro function spaces $\mathrm{Ces}_p[0,\infty)$ and $\mathrm{Ces}_p[0,1]$ for $1 < p \leqslant \infty$.

Theorem 9. *If* $1 < p \leqslant \infty$, *then the Cesàro function spaces* $\mathrm{Ces}_p[0,1]$ *and* $\mathrm{Ces}_p[0,\infty)$ *are isomorphic.*

Proof. The proof will go in two parts. Let $1 < p < \infty$. Sy, Zhang and Lee proved in [54] that the norm in $\mathrm{Ces}_p[0,\infty)$ is equivalent to the functional

$$\|f\|_0 = \left[\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} s_k(f)\right)^p + \sum_{m=1}^{\infty}\left(m\sum_{k=m}^{\infty} t_k(f)\right)^p m^{-2}\right]^{1/p}, \tag{50}$$

where

$$s_k(f) = \int_k^{k+1} |f(s)|\,ds \quad \text{and} \quad t_k(f) = \int_{\frac{1}{k+1}}^{\frac{1}{k}} |f(s)|\,ds, \quad k = 1, 2, \ldots.$$

Let's prove the analogous assertion for the space $\mathrm{Ces}_p[0,1]$. At first, if $\frac{1}{m+1} \leqslant x \leqslant \frac{1}{m}$, $m = 1, 2, \ldots$, then

$$\frac{m+1}{2}\int_0^{1/(m+1)} |f| \leqslant \frac{1}{x}\int_0^x |f| \leqslant (m+1)\int_0^x |f| \leqslant 2m\int_0^{1/m} |f|.$$

Therefore,

$$\begin{aligned} 2^{-p}\sum_{m=1}^{\infty}&\left((m+1)\int_0^{1/(m+1)} |f|\right)^p\left(\frac{1}{m} - \frac{1}{m+1}\right) \\ &\leqslant \sum_{m=1}^{\infty}\int_{1/(m+1)}^{1/m}\left(\frac{1}{x}\int_0^x |f|\right)^p dx \\ &= \int_0^1\left(\frac{1}{x}\int_0^x |f|\right)^p dx \end{aligned}$$

and

$$\begin{aligned} \int_0^{1/2}\left(\frac{1}{x}\int_0^x |f|\right)^p dx &= \sum_{m=2}^{\infty}\int_{1/(m+1)}^{1/m}\left(\frac{1}{x}\int_0^x |f|\right)^p dx \\ &\leqslant 2^p\sum_{m=2}^{\infty}\left(m\int_0^{1/m} |f|\right)^p\left(\frac{1}{m} - \frac{1}{m+1}\right). \end{aligned}$$

The first of these inequalities implies that

$$2^{-p}\sum_{m=2}^{\infty}\left(m\int_0^{1/m}|f|\right)^p\left(\frac{1}{m}-\frac{1}{m+1}\right)$$

$$=2^{-p}\sum_{m=1}^{\infty}\left((m+1)\int_0^{1/(m+1)}|f|\right)^p\left(\frac{1}{m+1}-\frac{1}{m+2}\right)$$

$$\leqslant\int_0^1\left(\frac{1}{x}\int_0^x|f|\right)^p dx=\|f\|_{C(p)}^p,$$

and the second one yields that

$$2^{-p}\int_0^{1/2}\left(\frac{1}{x}\int_0^x|f|\right)^p dx\leqslant\sum_{m=2}^{\infty}\left(m\int_0^{1/m}|f|\right)^p\left(\frac{1}{m}-\frac{1}{m+1}\right)\leqslant 2^p\|f\|_{C(p)}^p. \tag{51}$$

Denote

$$\alpha_n=\frac{1}{2}\big(2-n^{1-p}\big),\quad n=1,2,\ldots.$$

It is easy to check that $\frac{1}{2}=\alpha_1\leqslant\alpha_n<\alpha_{n+1}\leqslant 2\alpha_n, n=1,2,\ldots,$ and $\alpha_n\to 1$ as $n\to\infty$. Thus, if $\alpha_n\leqslant x\leqslant\alpha_{n+1}$, then

$$\frac{1}{2\alpha_n}\int_0^{\alpha_n}|f|\leqslant\frac{1}{\alpha_{n+1}}\int_0^{\alpha_n}|f|\leqslant\frac{1}{x}\int_0^x|f|$$

$$\leqslant\frac{1}{\alpha_n}\int_0^{\alpha_{n+1}}|f|\leqslant\frac{2}{\alpha_{n+1}}\int_0^{\alpha_{n+1}}|f|,$$

which implies that

$$2^{-p}\sum_{n=1}^{\infty}\left(\frac{1}{\alpha_n}\int_0^{\alpha_n}|f|\right)^p(\alpha_{n+1}-\alpha_n) \tag{52}$$

$$\leqslant\sum_{n=1}^{\infty}\int_{\alpha_n}^{\alpha_{n+1}}\left(\frac{1}{x}\int_0^x|f|\right)^p dx$$

$$=\int_{1/2}^1\left(\frac{1}{x}\int_0^x|f|\right)^p dx$$

and

$$(53)\qquad \int_{1/2}^{1}\left(\frac{1}{x}\int_0^x |f|\right)^p dx = \sum_{n=1}^{\infty}\int_{\alpha_n}^{\alpha_{n+1}}\left(\frac{1}{x}\int_0^x |f|\right)^p dx$$
$$\leqslant 2^p\sum_{n=1}^{\infty}\left(\frac{1}{\alpha_{n+1}}\int_0^{\alpha_{n+1}} |f|\right)^p(\alpha_{n+1}-\alpha_n).$$

Moreover, since

$$\alpha_{n+1}-\alpha_n = \frac{1}{2}\left(\frac{1}{n^{p-1}}-\frac{1}{(n+1)^{p-1}}\right) = \frac{p-1}{2}\int_n^{n+1}\frac{1}{t^p}\,dt$$

and $\frac{1}{n^p}\geqslant\int_n^{n+1}\frac{1}{t^p}\,dt\geqslant\frac{1}{(n+1)^p}\geqslant\frac{1}{(2n)^p}=2^{-p}\frac{1}{n^p}$, it follows that

$$(54)\qquad \frac{p-1}{2^{p+1}n^p}\leqslant\alpha_{n+1}-\alpha_n\leqslant\frac{p-1}{2n^p},\quad n=1,2,\ldots,$$

and we conclude that

$$\alpha_{n+1}-\alpha_n\leqslant\frac{p-1}{2n^p}\leqslant 4^p\frac{p-1}{2^{p+1}(n+1)^p}$$
$$\leqslant 4^p(\alpha_{n+2}-\alpha_{n+1}),\quad n=1,2,\ldots.$$

Therefore,

$$\sum_{n=1}^{\infty}\left(\frac{1}{\alpha_n}\int_0^{\alpha_n}|f|\right)^p(\alpha_{n+1}-\alpha_n)$$
$$=\left(\frac{1}{\alpha_1}\int_0^{\alpha_1}|f|\right)^p(\alpha_2-\alpha_1)+\sum_{n=1}^{\infty}\left(\frac{1}{\alpha_{n+1}}\int_0^{\alpha_{n+1}}|f|\right)^p(\alpha_{n+2}-\alpha_{n+1})$$
$$\geqslant 4^{-p}\sum_{n=1}^{\infty}\left(\frac{1}{\alpha_{n+1}}\int_0^{\alpha_{n+1}}|f|\right)^p(\alpha_{n+1}-\alpha_n).$$

By combining the last inequality with (52) we obtain that

$$(55)\qquad \sum_{n=1}^{\infty}\left(\frac{1}{\alpha_{n+1}}\int_0^{\alpha_{n+1}}|f|\right)^p(\alpha_{n+1}-\alpha_n)\leqslant 8^p\int_{1/2}^{1}\left(\frac{1}{x}\int_0^x|f|\right)^p dx.$$

From (51), (53) and (55) it follows that

$$\|f\|_{C(p)}^p \leqslant 2^p \sum_{m=2}^{\infty}\left(m \int_0^{1/m} |f|\right)^p \left(\frac{1}{m} - \frac{1}{m+1}\right) + 2^p \sum_{n=1}^{\infty}\left(\frac{1}{\alpha_{n+1}} \int_0^{\alpha_{n+1}} |f|\right)^p (\alpha_{n+1} - \alpha_n)$$

and

$$\begin{aligned}\sum_{m=2}^{\infty}\left(m \int_0^{1/m} |f|\right)^p \left(\frac{1}{m} - \frac{1}{m+1}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{\alpha_{n+1}} \int_0^{\alpha_{n+1}} |f|\right)^p (\alpha_{n+1} - \alpha_n) \\ \leqslant 2^p \int_0^1 \left(\frac{1}{x}\int_0^x |f|\right)^p dx + 8^p \int_{1/2}^1 \left(\frac{1}{x}\int_0^x |f|\right)^p dx \\ \leqslant 2^p\left(4^p+1\right)\|f\|_{C(p)}^p.\end{aligned}$$

Thus, taking into account (54), we obtain that

$$\|f\|_{C(p)} \approx \left[\sum_{n=1}^{\infty}\left(\frac{1}{n\alpha_{n+1}} \int_0^{\alpha_{n+1}} |f(t)|\,dt\right)^p + \sum_{m=2}^{\infty}\left(m \int_0^{1/m} |f(t)|\,dt\right)^p m^{-2}\right]^{1/p}. \tag{56}$$

Note that

$$\int_0^{1/m} |f(t)|\,dt = \sum_{k=m}^{\infty} t_k(f), \quad \text{where } t_k(f) = \int_{1/(k+1)}^{1/k} |f(t)|\,dt,$$

and

$$\int_0^{\alpha_{n+1}} |f(t)|\,dt = \int_0^{1/2} |f(t)|\,dt + \sum_{k=1}^{n} b_k(f), \quad \text{where } b_k(f) = \int_{\alpha_k}^{\alpha_{k+1}} |f(t)|\,dt.$$

Since $\alpha_n \geqslant \frac{1}{2}$ $(n = 1, 2, \ldots)$ it follows that the first sum in (56) does not exceed

$$2^p \sum_{n=1}^{\infty} \left(\int_0^{1/2} |f(t)|\, dt + \sum_{k=1}^{n} b_k(f) \right)^p n^{-p}$$
$$\leqslant 2^{2p} \left[\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} b_k(f) \right)^p + \left(\int_0^{1/2} |f(t)|\, dt \right)^p \sum_{n=1}^{\infty} n^{-p} \right].$$

Because $p > 1$ and the second sum on the right-hand side of (56) contains $(\int_0^{1/2} |f(t)|\, dt)^p$, then

$$\|f\|_{C(p)} \approx \left[\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} b_k(f) \right)^p + \sum_{m=2}^{\infty} \left(m \sum_{k=m}^{\infty} t_k(f) \right)^p m^{-2} \right]^{1/p}. \tag{57}$$

Denote by k_n and l_m one-to-one affine mappings such that

$$\begin{aligned} &k_n : [n, n+1] \to [\alpha_n, \alpha_{n+1}], \\ &l_m : \left[\frac{1}{m+1}, \frac{1}{m} \right] \to \left[\frac{1}{m+2}, \frac{1}{m+1} \right] \end{aligned} \qquad (n, m = 1, 2, \ldots)$$

and define the linear operator T for $f \in \mathrm{Ces}_p[0, 1]$ by

$$Tf(x) = \sum_{n=1}^{\infty} (\alpha_{n+1} - \alpha_n) f\big(k_n(x)\big) \chi_{[n,n+1]}(x)$$
$$+ \sum_{m=1}^{\infty} f\big(l_m(x)\big) \chi_{[\frac{1}{m+1}, \frac{1}{m}]}(x).$$

Since

$$\int_n^{n+1} \big|f\big(k_n(x)\big)\big|\, dx = \frac{1}{\alpha_{n+1} - \alpha_n} \int_{\alpha_n}^{\alpha_{n+1}} |f(t)|\, dt$$

and

$$\int_{1/(m+1)}^{1/m} \big|f\big(l_m(x)\big)\big|\, dx = \frac{m+2}{m} \int_{1/(m+2)}^{1/(m+1)} |f(t)|\, dt$$

for $n, m = 1, 2, \ldots,$ then the equivalences (50) and (57) show that

$$\|Tf\|_{C_p[0,\infty)} \approx \|Tf\|_0$$
$$= \left[\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} s_k(Tf) \right)^p + \sum_{m=1}^{\infty} \left(m \sum_{k=m}^{\infty} t_k(Tf) \right)^p m^{-2} \right]^{1/p}$$

$$\approx \left[\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} b_k(f) \right)^p + \sum_{m=2}^{\infty} \left(m \sum_{k=m}^{\infty} t_k(f) \right)^p m^{-2} \right]^{1/p}$$
$$\approx \|f\|_{C_p[0,1]}.$$

Therefore, $T : \mathrm{Ces}_p[0,1] \to \mathrm{Ces}_p[0,\infty)$ is an isomorphism and the proof for $1 < p < \infty$ is complete.

If $p = \infty$ the construction of isomorphism will be different and the proof is even working for the p-convexifications, that is, if $1 \leqslant p < \infty$, then the spaces $\mathrm{Ces}_\infty^{(p)}[0,1]$ and $\mathrm{Ces}_\infty^{(p)}[0,\infty)$ are isomorphic. In particular, $\mathrm{Ces}_\infty[0,1]$ and $\mathrm{Ces}_\infty[0,\infty)$ are isomorphic.

It is easy to check that

$$\|f\|_{C(\infty)^{(p)}[0,\infty)} \approx \sup_{k\in\mathbb{Z}} \left(2^{-k+1} \int_{\{2^{k-1} < t \leqslant 2^k\}} |f(t)|^p \, dt \right)^{1/p} \tag{58}$$

and

$$\|f\|_{C(\infty)^{(p)}[0,1]} \approx \sup_{k=0,-1,-2,\ldots} \left(2^{-k+1} \int_{\{2^{k-1} < t \leqslant 2^k\}} |f(t)|^p \, dt \right)^{1/p}. \tag{59}$$

Moreover, for every $k \in \mathbb{Z}$,

$$2^{-k+1} \int_{2^{k-1}}^{2^k} |f(t)|^p \, dt = \int_0^1 \left| f\left(2^{k-1}(t+1)\right)\right|^p dt. \tag{60}$$

Define the linear transforms

$$T_1 : \mathrm{Ces}_\infty[0,\infty) \to l^\infty\left(\sum_{k=-\infty}^{\infty} \oplus L^p[0,1] \right), \quad T_1 f = \left(f\left(2^{k-1}(t+1)\right)_{k\in\mathbb{Z}}\right)$$

and

$$T_2 : \mathrm{Ces}_\infty[0,1] \to l^\infty\left(\sum_{k=0}^{-\infty} \oplus L^p[0,1] \right), \quad T_2 f = \left(f\left(2^{k-1}(t+1)\right)_{k=0}^{-\infty}\right).$$

Formulas (58)–(60) show that T_1 and T_2 are isomorphisms. It is obvious that the spaces $l^\infty(\sum_{k=-\infty}^{\infty} \oplus L^p[0,1])$ and $l^\infty(\sum_{k=0}^{-\infty} \oplus L^p[0,1])$ are isomorphic. Therefore, the spaces $\mathrm{Ces}_\infty^{(p)}[0,\infty)$ and $\mathrm{Ces}_\infty^{(p)}[0,1]$ are isomorphic. □

Problem 1. Is the Cesàro function space $\mathrm{Ces}_\infty(I)$ isomorphic with the Cesàro sequence space ces_∞?

In Theorem 6 we proved that $\mathrm{Ces}_p[0,1]$ contains an isomorphic copy of l^p. Now we try to investigate when this is true for the spaces l^q.

Theorem 10.

(a) *If* $1 \leqslant p \leqslant 2$*, then the space* l^q *is embedded isomorphically into* $\mathrm{Ces}_p[0,1]$ *if and only if* $q \in [1,2]$.

(b) *If* $2 < p < \infty$*, then the space* l^q *is embedded isomorphically into* $\mathrm{Ces}_p[0,1]$ *if and only if either* $q \in [1,2]$ *or* $q = p$.

Proof. Firstly, $\mathrm{Ces}_p[0,1]$ contains a copy of $L^1[0,1]$ (cf. [4, Lemma 1]) and in turn l^q is embedded into $L^1[0,1]$ if $1 \leqslant q \leqslant 2$ (cf. [1, Theorem 6.4.18]). Moreover, by Theorem 6, l^p is embedded into $\mathrm{Ces}_p[0,1]$ for every $p \in [1,\infty)$ so we have to prove only the necessity.

In the case when $1 \leqslant p \leqslant 2$ necessity is obvious as a consequence of the fact that $\mathrm{Ces}_p[0,1]$ has cotype 2.

If $p > 2$ noting that $\mathrm{Ces}_p[0,1] \subset \mathrm{Ces}_1[0,1] = L^1(\ln 1/t)$ we consider two cases:

(a) Assume that the norms of the spaces $\mathrm{Ces}_p[0,1]$ and $L^1(\ln 1/t)$ are equivalent on a subspace $X \subset \mathrm{Ces}_p[0,1]$ which is isomorphic to l^q. In other words, X is a subspace of $L^1(\ln 1/t)$. Since the last space has cotype 2, then $q \leqslant 2$.

(b) The norms of the spaces $\mathrm{Ces}_p[0,1]$ and $L^1(\ln 1/t)$ are not equivalent on $X \approx l^q$. Then there is a sequence $\{x_n\} \subset X$ such that $\|x_n\|_{C(p)} = 1$ and $\|x_n\|_{L^1(\ln 1/t)} \to 0$. In particular, $x_n \to 0$ weakly in $L^1(\ln 1/t)$, i.e.,

$$\int_0^1 x_n(t)y(t)\,dt \to 0 \quad \text{for every } y \in L^\infty\big(\ln^{-1} 1/t\big).$$

Denote $\mathcal{F} := \bigcup_{0<\delta<1} L^\infty[0,\delta]$. Obviously, it yields that $\mathcal{F} \subset L^\infty(\ln^{-1} 1/t)$ and $\mathcal{F}$ is dense in $(\mathrm{Ces}_p[0,1])' = U(p')$ (see Theorem 3). Therefore, $\|x_n\|_{C(p)} = 1$ and $x_n \to 0$ weakly in $\mathrm{Ces}_p[0,1]$. By a known result (cf. [37, Proposition 1.a.12]) there exists a subsequence $\{x_n'\} \subset \{x_n\}$ which is equivalent to a seminormalized block basis of the canonical basis of l^q and, consequently, is equivalent to the canonical basis of l^q itself (see [1, Lemma 2.1.1 and Remark 2.1.2]). Moreover, $\|x_n'\|_{C(p)} = 1$ and $x_n' \to 0$ in the Lebesgue measure m. Next, since $\mathrm{Ces}_p[0,1]$ is separable for $1 \leqslant p < \infty$, then applying the Kadec–Pełczyński procedure we may find a subsequence $\{x_n''\} \subset \{x_n'\}$ and a sequence of disjoint sets $A_n \subset [0,1]$ such that $\|x_n'' - x_n''\chi_{A_n}\|_{C(p)} \to 0$. Using a standard argument we can select a subsequence $\{x_{n_k}''\} \subset \{x_n''\}$, which is equivalent to the sequence of disjoint functions $z_k := x_{n_k}''\chi_{A_{n_k}}$. Note that $\{x_{n_k}''\}$ and $\{z_k\}$ as well are equivalent to the canonical basis of l^q. To show that either $q \in [1,2]$ or $q = p$ we consider separately two cases:

(1) firstly, assume that there is $h \in (0, \frac{1}{2})$ such that $\operatorname{supp} z_k \subset [h, 1-h]$ for all $k = 1, 2, \ldots$. Since $\mathrm{Ces}_p[h, 1-h] \simeq L^1[h, 1-h]$ (cf. [4, Lemma 1]), then l^q will be embedded into $L^1[h, 1-h] \simeq L^1[0,1]$, so that $q \in [1,2]$.

(2) otherwise, there is a subsequence $\{z_k'\} \subset \{z_k\}$ such that $\operatorname{supp} z_k \subset I_k$ for some intervals I_k satisfying either $I_k \to 0$ or $I_k \to 1$. Then, using the same arguments as in the proof of Theorem 8, we may select a subsequence $\{z_k''\} \subset \{z_k'\}$ such that

$$\left\| \sum_{k=1}^{m} z_k'' \right\|_{C(p)} \leqslant C m^{1/p},$$

where the constant $C > 0$ does not depend on $m = 1, 2, \ldots$. Since $[z_k''] \simeq l^q$, then we have $q \geqslant p$. On the other hand, $q \leqslant p$ because $\mathrm{Ces}_p[0, 1]$ has cotype p, thus $q = p$ and the proof is complete. $\square$

Let us remind that $L^p[0, 1]$ contains an isomorphic copy of l^q if and only if $q \in [p, 2]$ for the case $1 \leqslant p \leqslant 2$ and in the case when $p > 2$ this can be when either $q = p$ or $q = 2$. We can see then the difference between $L^p[0, 1]$ and $\mathrm{Ces}_p[0, 1]$ spaces. In particular, if $1 < p < \infty$, then $\mathrm{Ces}_p[0, 1]$ contains an isomorphic copy of l^1 but not $L^p[0, 1]$.

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