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A property of interpolation spaces

By

LECH MALIGRANDA

Let A_0, A_1 and A_2 be Banach spaces continuously imbedded in some Hausdorff topological vector space, and let F be an interpolation functor. We consider the question: when is it true that

(1) $$F(\{A_0, A_1 \cap A_2\}) = F(\{A_0, A_1\}) \cap F(\{A_0, A_2\}).$$

Peetre [4] proved that if $\{A_0, A_1\}$ is quasi-linearizable pair, i.e., there exist linear operators $V_0(t)$, $V_1(t)$ (depending on $t > 0$) such that: $V_i(t)\colon A_0 + A_1 \to A_i$, $i = 0, 1$, $V_0(t)a + V_1(t)a = a$ and $\| V_0(t)a \|_{A_0} + t \| V_1(t)a \|_{A_1} \leqq c K(t, a\colon A_0, A_1)$ for $a \in A_0 + A_1$, and if moreover

$$\| V_1(t)a \|_{A_2} \leqq c_2 \| a \|_{A_2} \quad \text{for} \quad a \in A_2,$$

then for $a \in (A_0 + A_1) \cap (A_0 + A_2)$, we have

(2) $$K(t, a\colon A_0, A_1 \cap A_2) \leqq c_3 (K(t, a; A_0, A_1) + K(t, a; A_0, A_2)).$$

From (2) it follows that (1) is true for any K-interpolation functor $F = K_\Phi$ (in particular, for $F = K_{\theta q}$). The couples $\{C, C^1\}$, $\{L_p, W_p^k\}$ are quasi-linearizable and the couple $\{L_{p_0}, L_{p_1}\}$, $p_0 \neq p_1$ is not quasi-linearizable (cf. [3]). Triebel [5] has given an example of Banach spaces for which equality in (1) does not hold when $F = K_{\theta q}$ or $F = C_{[\theta]}$.

In this note we prove affirmative results in the case of Banach lattices on (Ω, μ).

Let $L^0(\Omega, \mu)$ be the topological vector space of all measurable functions on a measure space Ω with a σ-finite measure μ (the topology is the one which induces convergence in measure).

A Banach space $X \subset L^0(\Omega, \mu)$ is called a *Banach lattice on* (Ω, μ) if $|x| \leqq |y|$ μ – a.e. on Ω, $x \in L^0(\Omega, \mu)$ and $y \in X$ imply that $x \in X$ and $\| x \|_X \leqq \| y \|_X$.

Theorem 1. *Let* X_0, X_1 *and* X_2 *be Banach lattices on* (Ω, μ). *Then, for* $x \in (X_0 + X_1) \cap (X_0 + X_2)$, *we have*

(3) $$K(t, x; X_0, X_1 \cap X_2) \leqq 2(K(t, x; X_0, X_1) + K(t, x; X_0, X_2)).$$

Proof. Let $x \in (X_0 + X_1) \cap (X_0 + X_2)$. For each $\varepsilon > 0$ there exist decompositions $x = x_{0t} + x_{1t} = x'_{0t} + x_{2t}$ with $x_{0t}, x'_{0t} \in X_0$ and $x_{1t} \in X_1$, $x_{2t} \in X_2$ such that

$$\|x_{0t}\|_{X_0} + t\,\|x_{1t}\|_{X_1} \leqq (1+\varepsilon)\,K(t, x; X_0, X_1)$$

$$\|x'_{0t}\|_{X_0} + t\,\|x_{2t}\|_{X_2} \leqq (1+\varepsilon)\,K(t, x; X_0, X_2).$$

Put $U_t = \{s \in \Omega\colon |x_{1t}(s)| \leqq |x_{2t}(s)|\ \mu - \text{a.e.}\}$ and define y_{it} by

$$y_{0t}(s) = \begin{cases} x_{0t}(s), & s \in U_t \\ x'_{0t}(s), & s \in \Omega \setminus U_t \end{cases}, \qquad y_{1t}(s) = \begin{cases} x_{1t}(s), & s \in U_t \\ x_{2t}(s), & s \in \Omega \setminus U_t \end{cases}.$$

Then $y_{0t} + y_{1t} = x$ and $|y_{0t}| \leqq |x_{0t}| + |x'_{0t}|$, $|y_{1t}| \leqq \min(|x_{1t}|, |x_{2t}|)$. Furthermore,

$$\|y_{0t}\|_{X_0} \leqq \|x_{0t}\|_{X_0} + \|x'_{0t}\|_{X_0} \leqq (1+\varepsilon)\,K(t, x; X_0, X_1) + (1+\varepsilon)\,K(t, x; X_0, X_2)$$

and

$$t\,\|y_{1t}\|_{X_1 \cap X_2} \leqq t \max(\|x_{1t}\|_{X_1}, \|x_{2t}\|_{X_2}) \leqq (1+\varepsilon)\,K(t, x; X_0, X_1) + (1+\varepsilon)\,K(t, x; X_0, X_2).$$

Consequently, we obtain

$$K(t, x; X_0, X_1 \cap X_2) \leqq \|y_{0t}\|_{X_0} + t\,\|y_{1t}\|_{X_1 \cap X_2} \leqq 2(1+\varepsilon)\,K(t, x; X_0, X_1) + 2(1+\varepsilon)\,K(t, x; X_0, X_2)$$

and the proof is finished.

Now we will investigate equality (1) for the complex interpolation functors and Banach lattices on (Ω, μ).

Let X_0 and X_1 be two Banach lattices on (Ω, μ) and let $0 \leqq \theta \leqq 1$. We denote by $X_0^{1-\theta} X_1^{\theta}$ the Calderón space of all $x \in L^0(\Omega, \mu)$ such that $|x| \leqq \lambda\,|x_0|^{1-\theta}\,|x_1|^{\theta}\ \mu - \text{a.e.}$ on Ω for some constant $\lambda > 0$ and some $x_i \in X_i$ with $\|x_i\|_{X_i} \leqq 1$, $i = 0, 1$. We put $\|x\|_{X_0^{1-\theta} X_1^{\theta}} = \inf \lambda$.

We note that $X_0^{1-\theta} X_1^{\theta}$ is a Banach lattice on (Ω, μ).

Theorem 2. *Let X_0, X_1 and X_2 be Banach lattices on (Ω, μ). Then*

$$X_0^{1-\theta}(X_1 \cap X_2)^{\theta} = X_0^{1-\theta} X_1^{\theta} \cap X_0^{1-\theta} X_2^{\theta}. \tag{4}$$

Proof. It is sufficient to prove that

$$X_0^{1-\theta} X_1^{\theta} \cap X_0^{1-\theta} X_2^{\theta} \subset X_0^{1-\theta}(X_1 \cap X_2)^{\theta}.$$

First we note that if $x_i \in X_i$, $i = 1, 2$, then $\min(|x_1|, |x_2|) \in X_1 \cap X_2$ and

$$\|\min(|x_1|, |x_2|)\|_{X_1 \cap X_2} \leqq \max(\|x_1\|_{X_1}, \|x_2\|_{X_2}).$$

Let $x \in X_0^{1-\theta} X_1^{\theta} \cap X_0^{1-\theta} X_2^{\theta}$ with the norm $\leqq 1$ and let $\varepsilon > 0$. Then there exist $x_0, x'_0 \in X_0$, $x_i \in X_i$, $i = 1, 2$ with norms $\leqq 1$ and

$$|x| \leqq (1+\varepsilon)\,|x_0|^{1-\theta}\,|x_1|^{\theta}, \quad |x| \leqq (1+\varepsilon)\,|x'_0|^{1-\theta}\,|x_2|^{\theta} \quad \mu - \text{a.e.}$$

We have

$$\begin{aligned} |x| &\leqq (1+\varepsilon)\min(|x_0|^{1-\theta}|x_1|^{\theta}, |x_0'|^{1-\theta}|x_2|^{\theta}) \\ &\leqq (1+\varepsilon)\min[\max(|x_0|, |x_0'|)^{1-\theta}|x_1|^{\theta},\ \max(|x_0|, |x_0'|)^{1-\theta}|x_2|^{\theta}] \\ &= (1+\varepsilon)\max(|x_0|, |x_0'|)^{1-\theta}\min(|x_1|, |x_2|)^{\theta}. \end{aligned}$$

Hence $x \in X_0^{1-\theta}(X_1 \cap X_2)^{\theta}$ and the norm is $\leqq 2(1+\varepsilon)$.

From Theorem 2 and Calderón's results (see [2], p. 125) it follows that if Banach lattices X_0, X_1 and X_2 on (Ω, μ) have absolutely continuous [Fatou] norms then (1) is true for the lower [upper] complex interpolation functor $F = C_{[\theta]}\,[= C^{[\theta]}]$.

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Anschrift des Autors:

Lech Maligranda
Institute of Mathematics
Polish Academy of Sciences
Mielżyńskiego 27/29
61-725 Poznań, Poland