Multiplicator space and complemented subspaces of rearrangement invariant space [☆]

S.V. Astashkin,[a] L. Maligranda,[b,*] and E.M. Semenov[c]

[a] *Department of Mathematics, Samara State University, Akad. Pavlova 1, 443011 Samara, Russia*
[b] *Department of Mathematics, Luleå University of Technology, 971 87 Luleå, Sweden*
[c] *Department of Mathematics, Voronezh State University, Universitetskaya pl.1, 394693 Voronezh, Russia*

Abstract

We show that the multiplicator space $\mathscr{M}(X)$ of an rearrangement invariant (r.i.) space X on $[0, 1]$ and the nice part $N_0(X)$ of X, that is, the set of all $a \in X$ for which the subspaces generated by sequences of dilations and translations of a are uniformly complemented, coincide when the space X is separable. In the general case, the nice part is larger than the multiplicator space. Several examples of descriptions of $\mathscr{M}(X)$ and $N_0(X)$ for concrete X are presented.
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0. Introduction

For rearrangement invariant (r.i.) function space X on $I = [0, 1]$, we will consider the multiplicator space $\mathscr{M}(X)$ and the nice part $N_0(X)$ of the space X. The space

*Corresponding author.
E-mail addresses: astashkn@ssu.samara.ru (S.V. Astashkin), lech@sm.luth.se (L. Maligranda), root@func.vsu.ru (E.M. Semenov).
URL: http://www.sm.luth.se/~lech/.

$\mathcal{M}(X)$ is connected with the tensor product of two functions $x(s)y(t), s, t \in [0, 1]$, and $N_0(X)$ is the space given by uniformly bounded sequence in X of projections into $Q_{a,n}$ generated by the dilations and translations of the non-zero, decreasing function $a \in X$ on dyadic intervals $[\frac{k-1}{2^n}, \frac{k}{2^n})$ in $I, k = 1, 2, \ldots, 2^n, \ n = 0, 1, 2, \ldots$. These functions are given by

$$a_{n,k}(t) = \begin{cases} a(2^n t - k + 1) & \text{if } t \in [\frac{k-1}{2^n}, \frac{k}{2^n}), \\ 0 & \text{elsewhere.} \end{cases}$$

The spaces $\mathcal{M}(X)$ and $N_0(X)$ coincide when X is a separable space but in the non-separable case the nice part can be larger than the multiplicator space. Such a description is helpful in the proofs of properties of $N_0(X)$ and it motivates us to investigate more the multiplicator space $\mathcal{M}(X)$. We will describe $\mathcal{M}(X)$ for concrete r.i. spaces X as Lorentz, Orlicz and Marcinkiewicz spaces. Suitable results on $N_0(X)$, especially when X is a Marcinkiewicz space M_φ, are given.

The paper is organized as follows. In Section 1 we collect some necessary definitions and notations.

Section 2 contains results on the multiplicator space $\mathcal{M}(X)$ of a r.i. space X on $[0, 1]$. At first we collect its properties. After that the multiplicator space $\mathcal{M}(X)$ is described for concrete spaces like Lorentz $\Lambda_{p,\varphi}$ spaces, Orlicz L_Φ spaces and Marcinkiewicz M_φ spaces. The main result here is Theorem 1 which gives necessary and sufficient condition for the tensor product operator to be bounded between Marcinkiewicz spaces M_φ.

In Section 3, we consider a subspace $N_0(X)$ of X generated by dilations and translations in r.i. space on $[0, 1]$ of a decreasing function from X. The main result of the paper is Theorem 2 showing that the multiplicator space $\mathcal{M}(X)$ is a subset of the nice part $N_0(X)$ of X and that they are equal when a space X is separable. In the general case, the nice part is larger than the multiplicator space (cf. Example 2). Here we apply results on multiplicators from Section 2 to the description of $N_0(X)$. Special attention is taken about $N_0(X)$ when X is a Marcinkiewicz space M_φ (see Corollary 5 and Theorem 3). Stability properties of the class $\mathcal{N}_0$ with respect to the complex and real interpolation methods are presented. There is also given, in Theorem 7, a characterization of L_p-spaces among the r.i. spaces on $[0, 1]$, which is saying that r.i. space X on $[0, 1]$ coincides with $L_p[0, 1]$ for some $1 \leqslant p \leqslant \infty$ if and only if X and its associated space X' belong to the class $\mathcal{N}_0$.

Finally, in Section 4 we show that, in general, you cannot compare the results on the interval $[0, 1]$ with the results on $[0, \infty)$ and vice versa.

1. Definitions and notations

We first recall some basic definitions. A Banach function space X on $I = [0, 1]$ is said to be a *rearrangement invariant* (r.i.) space provided $x^*(t) \leqslant y^*(t)$ for every $t \in [0, 1]$ and $y \in X$ imply $x \in X$ and $\|x\|_X \leqslant \|y\|_X$, where x^* denotes the decreasing

rearrangement of $|x|$. Always we have imbeddings $L_\infty[0,1] \subset X \subset L_1[0,1]$. By X^0 we will denote the closure of $L_\infty[0,1]$ in X.

An r.i. space X with a norm $\|\cdot\|_X$ has the *Fatou property* if for any increasing positive sequence (x_n) in X with $\sup_n \|x_n\|_X < \infty$ we have that $\sup_n x_n \in X$ and $\|\sup_n x_n\|_X = \sup_n \|x_n\|_X$.

We will assume that the r.i. space X is either separable or it has the Fatou property. Then, as follows from the Calderón–Mityagin theorem [BS,KPS], the space X is an interpolation space with respect to L_1 and L_∞, i.e., if a linear operator T is bounded in L_1 and L_∞, then T is bounded in X and $\|T\|_{X \to X} \leqslant C \max(\|T\|_{L_1 \to L_1}, \|T\|_{L_\infty \to L_\infty})$ for some $C \geqslant 1$.

If χ_A denotes the characteristic function of a measurable set A in I, then clearly $\|\chi_A\|_X$ depends only on $m(A)$. The function $\varphi_X(t) = \|\chi_A\|_X$, where $m(A) = t$, $t \in I$, is called the *fundamental function* of X.

For $s > 0$, the dilation operator σ_s given by $\sigma_s x(t) = x(t/s)\chi_I(t/s), t \in I$ is well defined in every r.i. space X and $\|\sigma_s\|_{X \to X} \leqslant \max(1, s)$. The classical *Boyd indices* of X are defined by (cf. [BS,KPS,LT])

$$\alpha_X = \lim_{s \to 0} \frac{\ln \|\sigma_s\|_{X \to X}}{\ln s}, \quad \beta_X = \lim_{s \to \infty} \frac{\ln \|\sigma_s\|_{X \to X}}{\ln s}.$$

In general, $0 \leqslant \alpha_X \leqslant \beta_X \leqslant 1$. It is easy to see that $\bar\varphi_X(t) \leqslant \|\sigma_t\|_{X \to X}$ for any $t > 0$, where $\bar\varphi_X(t) = \sup_{0 < s < 1, 0 < st < 1} \frac{\varphi_X(st)}{\varphi_X(s)}$.

The associated space X' to X is the space of all (classes of) measurable functions $x(t)$ such that $\int_0^1 |x(t)y(t)| dt < \infty$ for every $y \in X$ endowed with the norm

$$\|x\|_{X'} = \sup\left\{ \int_0^1 |x(t)y(t)|\, dt : \|y\|_X \leqslant 1 \right\}.$$

For every r.i. space X the embedding $X \subset X''$ is isometric. If an r.i. space X is separable, then $X' = X^*$.

Let us recall some classical examples of r.i. spaces. Denote by $\mathscr{C}$ the set of increasing concave functions $\varphi(t)$ on $[0,1]$ with $\varphi(0^+) = \varphi(0) = 0$. Each function $\varphi \in \mathscr{C}$ generates the *Lorentz space* Λ_φ endowed with the norm

$$\|x\|_{\Lambda_\varphi} = \int_0^1 x^*(t)\, d\varphi(t)$$

and the *Marcinkiewicz space* M_φ endowed with the norm

$$\|x\|_{M_\varphi} = \sup_{0 < t \leqslant 1} \frac{1}{\varphi(t)} \int_0^t x^*(s)\, ds.$$

If Φ is a positive convex function on $[0, \infty)$ with $\Phi(0) = 0$, then the *Orlicz space* $L_\Phi = L_\Phi[0,1]$ (cf. [KR,M89]) consists of all measurable functions $x(t)$ on $[0,1]$ for

which the functional $||x||_{L_\Phi}$ is finite, where

$$||x||_{L_\Phi} = \inf\left\{\lambda > 0 : I_\Phi\left(\frac{x}{\lambda}\right) \leqslant 1\right\} \quad \text{with } I_\Phi(x) := \int_0^1 \Phi(|x(t)|)\,dt.$$

An Orlicz space L_Φ is separable if and only if the function Φ satisfies the Δ_2-*condition* (i.e. $\Phi(2u) \leqslant C\Phi(u)$ for every $u \geqslant u_0$ and some constants $u_0 > 0$ and $C > 0$).

The *Lorentz space* $L_{p,q}$, $1 < p < \infty, 1 \leqslant q \leqslant \infty$, is the space generated by the functionals (quasi-norms)

$$||x||_{p,q} = \left(\int_0^1 [t^{1/p} x^*(t)]^q \frac{dt}{t}\right)^{1/q} \quad \text{if } q < \infty$$

and

$$||x||_{p,\infty} = \sup_{0 < t < 1} t^{1/p} x^*(t).$$

For $1 \leqslant p < \infty$ and $\varphi \in \mathscr{C}$ the *Lorentz space* $\Lambda_{p,\varphi}$ is the space generated by the norm

$$||x||_{p,\varphi} = \left(\int_0^1 [x^*(t)]^p\,d\varphi(t)\right)^{1/p}.$$

We will use the *Calderón–Lozanovskiĭ construction* (see [C,M89]). Let (X_0, X_1) be a pair of r.i. spaces on $[0,1]$ and $\rho \in \mathscr{U}$ ($\rho \in \mathscr{U}$ means that $\rho(s,t) = s\rho(t/s)$ for $s > 0$ with an increasing, concave function ρ on $[0, \infty)$ such that $\rho(0) = 0$ and $\rho(0,t) = 0$). By $\rho(X_0, X_1)$ we mean the space of all measurable functions $x(t)$ on $[0,1]$ for which

$$|x(t)| \leqslant \lambda\rho(|x_0(t)|, |x_1(t)|) \quad \text{a.e. on } [0,1]$$

for some $x_i \in X_i$ with $||x_i||_{X_i} \leqslant 1$, $i = 0, 1$, and with the infimum of these λ as the norm $||x||_\rho$. In the case of the power function $\rho_\theta(s,t) = s^{1-\theta}t^\theta$ with $0 \leqslant \theta \leqslant 1$, $\rho_\theta(X_0, X_1)$ is the Calderón construction $X_0^{1-\theta}X_1^\theta$ (see [C,LT,M89]). The particular case $X^{1/p}(L_\infty)^{1-1/p} = X^{(p)}$ for $1 < p < \infty$ is known as the *p-convexification* of X defined as $X^{(p)} = \{x \text{ is measurable on } I : |x|^p \in X\}$ with the norm $||x||_{X^{(p)}} = |||x|^p||_X^{1/p}$ (see [LT,M89]).

For other general properties of lattices of measurable functions and r.i. spaces we refer to books [BS,KPS,LT].

2. Multiplicator space of an r.i. space

Let $X = X(I)$ be an r.i. space on $I = [0,1]$. Then the corresponding r.i. space $X(I \times I)$ on $I \times I$ is the space of measurable functions $x(s,t)$ on $I \times I$ such that $x^\circledast(t) \in X(I)$ with the norm $||x||_{X(I \times I)} = ||x^\circledast||_{X(I)}$, where $x^\circledast$ denotes the decreasing

rearrangement of $|x|$ with respect to the Lebesgue measure m_2 on $I \times I$. For two measurable functions $x = x(s), y = y(t)$ on $I = [0, 1]$ we define the bilinear operator of the tensor product $\otimes$ by

$$(x \otimes y)(s, t) = x(s)y(t), \quad s, t \in I.$$

Definition 1. The *multiplicator space* $\mathscr{M}(X)$ of an r.i. space X on $I = [0, 1]$ is the set of all measurable functions $x = x(s)$ such that $x \otimes y \in X(I \times I)$ for arbitrary $y \in X$ with the norm

$$\|x\|_{\mathscr{M}(X)} = \sup\{\|x \otimes y\|_{X(I \times I)} : \|y\|_X \leqslant 1\}. \tag{1}$$

The multiplicator space $\mathscr{M}(X)$ is an r.i. space on $[0, 1]$ because for the product measure we have

$$m_2(\{(s, t) \in I \times I : |x(s)y(t)| > \lambda\}) = \int_0^1 m(\{s \in I : |x(s)y(t)| > \lambda\}) \, dt.$$

Let us collect some properties of $\mathscr{M}(X)$. First note that for any measurable set A in I the functions $\chi_A \otimes x$ and $\sigma_{m(A)}x$ are equimeasurable, i.e., their distributions are equal

$$d_{\chi_A \otimes x}(\lambda) = m_2(\{(s, t) \in I \times I : \chi_A(s)|x(t)| > \lambda\})$$

$$= m(\{t \in I : |\sigma_{m(A)}x(t)| > \lambda\}) = d_{\sigma_{m(A)}x}(\lambda)$$

for all $\lambda > 0$. In particular, $\|x\|_{\mathscr{M}(X)} \geqslant \|x \otimes \chi_{[0,1]}/\varphi_X(1)\|_X = \|x\|_X/\varphi_X(1)$ gives the imbedding

$$\mathscr{M}(X) \subset X \quad \text{and} \quad \|x\|_X \leqslant \varphi_X(1)\|x\|_{\mathscr{M}(X)} \quad \text{for } x \in \mathscr{M}(X). \tag{2}$$

Moreover, $\mathscr{M}(X) = X$ if and only if the operator $\otimes : X \times X \to X(I \times I)$ is bounded. In particular, $\mathscr{M}(L_{p,q}) = L_{p,q}$ for $1 < p < \infty$ and $1 \leqslant q \leqslant p$ since from the O'Neil theorem (see [O, Theorem 7.4]) the tensor product $\otimes : L_{p,q} \times L_{p,q} \to L_{p,q}(I \times I)$ is bounded.

From the equality $(\chi_{[0,u]} \otimes \chi_{[0,v]})^{\circledast}(t) = \chi_{[0,uv]}(t)$ we obtain that if $X \subset \mathscr{M}(X)$, then fundamental function φ_X is submultiplicative on $[0, 1]$, i.e., there exists a constant $c > 0$ such that $\varphi_X(st) \leqslant c\varphi_X(s)\varphi_X(t)$ for all $s, t \in [0, 1]$.

Some other properties of the multiplicator space $\mathscr{M}(X)$ (cf. [A97] for the proofs):

(a) $\varphi_{\mathscr{M}(X)}(t) = \|\sigma_t\|_{X \to X}, \quad \|\sigma_t\|_{\mathscr{M}(X) \to \mathscr{M}(X)} = \|\sigma_t\|_{X \to X}$ for $0 < t \leqslant 1$ and $\|\sigma_{1/t}\|_{X \to X}^{-1} \leqslant \|\sigma_t\|_{\mathscr{M}(X) \to \mathscr{M}(X)} \leqslant \|\sigma_t\|_{X \to X}$ for $t > 1$. In particular, $\alpha_{\mathscr{M}(X)} = \alpha_X$ and $\beta_{\mathscr{M}(X)} \leqslant \beta_X$.

(b) We have imbeddings $\Lambda_\psi \subset \mathcal{M}(X) \subset L_p$, where $\psi(t) = ||\sigma_t||_{X \to X}$, $0 < t \leqslant 1$, $p = 1/\alpha_X$ and the constants of imbeddings are independent of X. In particular,
  (b_1) $\mathcal{M}(X) = L_\infty$ if and only if $\alpha_X = 0$.
  (b_2) If the operator $\otimes : X \times X \to X(I \times I)$ is bounded, then $X \subset L_{1/\alpha_X}$.
(c) If X is an interpolation space between L_p and $L_{p,\infty}$ for some $1 < p < \infty$, then $\mathcal{M}(X) = L_p$. In particular, $\mathcal{M}(L_{p,q}) = L_p$ for $1 < p < \infty$ and $p \leqslant q \leqslant \infty$.

Note that the operation $\mathcal{M}(X)$ is not monotone, i.e., if X, Y are r.i. spaces on $[0,1]$ such that $X \subset Y$ then, in general, it is not true that $\mathcal{M}(X) \subset \mathcal{M}(Y)$. Namely, consider the r.i. space X on $[0,1]$ constructed by Shimogaki [S]. This space has Boyd lower index $\alpha_X = 0$ with $\varphi_X(t) = t^{1/2}$ and $L_2 \subset X$. On the other hand, $\mathcal{M}(L_2) = L_2$ but $\mathcal{M}(X) = L_\infty$ by (b_1).

Proposition 1. *We have* $\mathcal{M}(\mathcal{M}(X)) = \mathcal{M}(X)$ *with equal norms.*

Proof. It is enough to show the imbedding $\mathcal{M}(X) \subset \mathcal{M}(\mathcal{M}(X))$. Let $x \in \mathcal{M}(X)$ with the norm $||x||_{\mathcal{M}(X)} \leqslant C$. Then

$$||x \otimes u||_{X(I \times I)} \leqslant C||u||_{X(I)} \quad \forall u \in X.$$

In particular, for $u = (y \otimes z)^{\circledast}$ with fixed $y \in \mathcal{M}(X)$ and any $z \in X$ with $||z||_{X(I)} \leqslant 1$ we have

$$||x \otimes (y \otimes z)^{\circledast}||_{X(I \times I)} \leqslant C||(y \otimes z)^{\circledast}||_{X(I)} = C||y \otimes z||_{X(I \times I)}.$$

Since

$$m_2(\{(s,t) \in I \times I : |x(s)(y \otimes z)^{\circledast}(t)| > \lambda\})$$

$$= \int_0^1 m(\{t \in I : |x(s)(y \otimes z)^{\circledast}(t)| > \lambda\}) \, ds$$

$$= \int_0^1 m_2(\{(t,\alpha) \in I \times I : |x(s)y(t)z(\alpha)| > \lambda\}) \, ds$$

$$= m_3(\{(s,t,\alpha) \in I \times I \times I : |x(s)y(t)z(\alpha)| > \lambda\})$$

$$= \int_0^1 m_2(\{(s,t) \in I \times I : |x(s)y(t)z(\alpha)| > \lambda\}) \, d\alpha$$

$$= \int_0^1 m(\{t \in I : |(x \otimes y)^{\circledast}(t)z(\alpha)| > \lambda\}) \, d\alpha$$

$$= m_2(\{(t,\alpha) \in I \times I : (x \otimes y)^{\circledast}(t)z(\alpha)| > \lambda\})$$

for any $\lambda > 0$ it follows that

$$||x \otimes (y \otimes z)^{\circledast}||_{X(I \times I)} = ||(x \otimes y)^{\circledast} \otimes z||_{X(I \times I)}.$$

Taking the supremum over all $z \in X$ with $||z||_{X(I)} \leqslant 1$ we obtain

$$||(x \otimes y)^{\circledast}||_{\mathscr{M}(X)} = \sup\{||(x \otimes y)^{\circledast} \otimes z||_{X(I \times I)} : ||z||_{X(I)} \leqslant 1\}$$

$$= \sup\{||x \otimes (y \otimes z)^{\circledast}||_{X(I \times I)} : ||z||_{X(I)} \leqslant 1\}$$

$$\leqslant C \sup\{||y \otimes z||_{X(I \times I)} : ||z||_{X(I)} \leqslant 1\} = C||y||_{\mathscr{M}(X)}.$$

This means that $x \in \mathscr{M}(\mathscr{M}(X))$ and its norm is $\leqslant C$.   □

Note that if $X = \mathscr{M}(Y)$ for some r.i. space Y, then $X = \mathscr{M}(X)$. Indeed, $\mathscr{M}(X) = \mathscr{M}(\mathscr{M}(Y)) = \mathscr{M}(Y) = X$.

For concrete r.i. spaces, like Lorentz, Orlicz and Marcinkiewicz, we have the following results about multiplicator space. From the above discussion we have that if $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$, then

$$\mathscr{M}(L_{p,q}) = L_{p,\min(p,q)}. \tag{3}$$

Proposition 2 (cf. Astashkin [A97] for $p = 1$). *Let $\varphi \in \mathscr{C}$ and $1 \leqslant p < \infty$. Then*

(i) $\Lambda_{p,\bar{\varphi}} \subset \mathscr{M}(\Lambda_{p,\varphi}) \subset \Lambda_{p,\varphi}$.
(ii) $\mathscr{M}(\Lambda_{p,\varphi}) = \Lambda_{p,\varphi}$ *if and only if φ is submultiplicative on* $[0,1]$.
(iii) *If $\bar{\varphi}(t) = \lim_{s \to 0^+} \frac{\varphi(st)}{\varphi(s)}$, then $\mathscr{M}(\Lambda_{p,\varphi}) = \Lambda_{p,\bar{\varphi}}$.*

The proof follows from [A97] (cf. also [Mi76,Mi78]), property (b) and the fact that $\mathscr{M}(X)^{(p)} = \mathscr{M}(X^{(p)})$, where $X^{(p)}$ is the p-convexification of X.

Proposition 3. *For the Orlicz space $L_\Phi = L_\Phi[0,1]$ we have the following*:

(i) *If $\Phi \notin \Delta_2$, then $\mathscr{M}(L_\Phi) = L_\infty$.*
(ii) *If $\Phi \in \Delta_2$, then $L_{\bar{\Phi}} \subset \mathscr{M}(L_\Phi) \subset L_\Phi$, where $\bar{\Phi}(u) = \sup_{v \geqslant 1} \frac{\Phi(uv)}{\Phi(v)}$, $u \geqslant 1$.*
(iii) *If $\Phi \in \Delta_2$, then $\mathscr{M}(L_\Phi) = L_\Phi$ if and only if Φ is a submultiplicative function for large u, i.e., $\Phi(uv) \leqslant C\Phi(u)\Phi(v)$ for some positive C, u_0 and all $u, v \geqslant u_0$.*

Proof. (i) It follows from property (b_1) and the fact that Boyd index $\alpha_{L_\Phi} = 0$.
(ii) The imbedding $L_{\bar{\Phi}} \subset \mathscr{M}(L_\Phi)$ follows from Ando theorem [A, Theorem 6] on boundedness of tensor product between Orlicz spaces. In fact, if $x(s) \in L_{\bar{\Phi}}$ and

$y(t) \in L_\Phi$, then $I_{\bar{\Phi}}(x/\lambda) + I_\Phi(y/\lambda) < \infty$ for some $\lambda \geqslant 1$, and so

$$\Phi(\lambda^{-2}|x(s)||y(t)|) \leqslant [1 + \bar{\Phi}(|x(s)|/\lambda)][\Phi(1) + \Phi(|y(t)|/\lambda)],$$

from which

$$I_\Phi(\lambda^{-2}x \otimes y) \leqslant [1 + I_{\bar{\Phi}}(x/\lambda)][\Phi(1) + I_\Phi(y/\lambda)] < \infty,$$

that is, $x \otimes y \in L_\Phi(I \times I)$. Therefore, $L_{\bar{\Phi}} \subset \mathcal{M}(L_\Phi)$. The second imbedding follows from (2).

   (iii) It follows directly from (ii) and it was also proved in [A,A82,Mi81,O].   □

The situation is different in the case of Marcinkiewicz spaces.

Theorem 1. *Let $\varphi \in \mathscr{C}$. The following statements are equivalent*:

  (i) $\mathcal{M}(M_\varphi) = M_\varphi$.
 (ii) *The tensor product $\otimes : M_\varphi \times M_\varphi \to M_\varphi(I \times I)$ is bounded.*
(iii) $\varphi' \otimes \varphi' \in M_\varphi$.
 (iv) *There exists a constant $C > 0$ such that the inequality*

$$\sum_{i=1}^{n} \varphi(u_i)\left(\varphi\left(\frac{i}{n}\right) - \varphi\left(\frac{i-1}{n}\right)\right) \leqslant C\varphi\left(\frac{1}{n}\sum_{i=1}^{n} u_i\right) \tag{4}$$

   is valid for any $u_i \in [0, 1]$, $i = 1, 2, \ldots, n$ and every $n \in \mathbf{N}$.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is true for any r.i. space, in particular also for the Marcinkiewicz space M_φ.

   Implication (ii) $\Rightarrow$ (iii) follows from the fact that $\varphi' \in M_\varphi$.

   (iii) $\Rightarrow$ (iv): Given an integer n and a sequence $u_1, u_2, \ldots, u_n \in [0, 1]$, consider the set

$$A = \bigcup_{i=1}^{n}\left(\frac{i-1}{n}, \frac{i}{n}\right) \times (0, u_i) \subset [0, 1] \times [0, 1].$$

Then

$$\int_A (\varphi' \otimes \varphi')\, dm_2 \leqslant C\varphi(m_2(A)),$$

where m_2 is the Lebesque measure on $[0, 1] \times [0, 1]$. Since

$$\int_A \varphi'(t)\varphi'(s)\, dt\, ds = \sum_{i=1}^{n} \varphi(u_i)\left(\varphi\left(\frac{i}{n}\right) - \varphi\left(\frac{i-1}{n}\right)\right)$$

and

$$m_2(A) = \sum_{i=1}^{n} \frac{1}{n} u_i$$

it follows that estimate (4) holds.

(iv) $\Rightarrow$ (ii): Assume that (4) is valid. It is sufficient to prove that the inequality

$$||x \otimes y||_{M_\varphi} \leqslant C$$

holds for $x, y \in M_\varphi$, $||x||_{M_\varphi} = ||y||_{M_\varphi} = 1$ and $x = x^*$, $y = y^*$. Given $x = x^* \in M_\varphi$ with $||x||_{M_\varphi} = 1$ and $\varepsilon > 0$ there exists a strictly decreasing function $z = z^* \in M_\varphi$ such that $||z||_{M_\varphi} \leqslant 1 + \varepsilon$ and $z \geqslant x$. Therefore, we can assume in addition that x and y are strictly decreasing and continuous on $(0, 1]$. We must prove the inequality

$$\int_{A_\tau} x(t) y(s) \, dt \, ds \leqslant C m_2(A_\tau)$$

for any set

$$A_\tau = \{(t, s) \in [0, 1] \times [0, 1] : x(t) y(s) \geqslant \tau\}, \quad \tau > 0.$$

Given $\tau > 0$, there exists a continuous decreasing function $g(t) = g_\tau(t)$ such that $A_\tau = \{(t, s) : g(s) \geqslant t\}$.

Put

$$P_n = \bigcup_{i=1}^{n} \left[0, g\left(\frac{i}{n}\right)\right] \times \left(\frac{i-1}{n}, \frac{i}{n}\right]$$

and

$$Q_n = \bigcup_{i=1}^{n} \left[0, g\left(\frac{i-1}{n}\right)\right] \times \left(\frac{i-1}{n}, \frac{i}{n}\right].$$

Then

$$P_n \subset A_\tau \subset Q_n.$$

The continuity of the function g implies that

$$\lim_{n \to \infty} \mu(Q_n \backslash P_n) = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n} \left(g\left(\frac{i-1}{n}\right) - g\left(\frac{i}{n}\right) \right)$$

$$\leqslant \lim_{n \to \infty} \max_{1 \leqslant i \leqslant n} \left(g\left(\frac{i-1}{n}\right) - g\left(\frac{i}{n}\right) \right) = 0.$$

The function $x(t)y(s)$ belongs to $L_1(m_2)$. Hence

$$\lim_{n \to \infty} \int_{Q_n \backslash P_n} x(t)y(s) \, dt \, ds = 0$$

and

$$\int_{A_\tau} x(t)y(s) \, dt \, ds = \lim_{n \to \infty} \int_{P_n} x(t)y(s) \, dt \, ds$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \int_0^{g(\frac{i}{n})} x(t) \, dt \int_{\frac{i-1}{n}}^{\frac{i}{n}} y(s) \, ds$$

$$\leqslant \lim_{n \to \infty} \sum_{i=1}^{n} \varphi\left(g\left(\frac{i}{n}\right)\right) \left(\int_0^{\frac{i}{n}} y(s) \, ds - \int_0^{\frac{i-1}{n}} y(s) \, ds \right)$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \left(\varphi\left(g\left(\frac{i}{n}\right)\right) - \varphi\left(g\left(\frac{i+1}{n}\right)\right) \right) \int_0^{\frac{i}{n}} y(s) \, ds.$$

Since

$$\varphi\left(g\left(\frac{i}{n}\right)\right) - \varphi\left(g\left(\frac{i+1}{n}\right)\right) > 0$$

and

$$\int_0^{\frac{i}{n}} y(s) \, ds \leqslant \varphi\left(\frac{i}{n}\right)$$

it follows that

$$\int_{A_\tau} x(t)y(s) \, dt \, ds \leqslant \lim_{n \to \infty} \sum_{k=1}^{n} \left(\varphi\left(g\left(\frac{i}{n}\right)\right) - \varphi\left(g\left(\frac{i+1}{n}\right)\right) \right) \varphi\left(\frac{i}{n}\right)$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \varphi\left(g\left(\frac{i}{n}\right)\right) \left(\varphi\left(\frac{i}{n}\right) - \varphi\left(\frac{i-1}{n}\right) \right).$$

Denoting $g\left(\frac{i}{n}\right) = u_i$ and applying (4) we get

$$\int_{A_\tau} x(t)y(s) \, dt \, ds \leqslant \lim_{n \to \infty} \sum_{i=1}^{n} \varphi(u_i) \left(\varphi\left(\frac{i}{n}\right) - \varphi\left(\frac{i-1}{n}\right) \right)$$

$$\leqslant C \lim_{n \to \infty} \varphi\left(\frac{1}{n} \sum_{i=1}^{n} u_i \right) = C \lim_{n \to \infty} \varphi(m_2(P_n)) = C m_2(A_\tau),$$

and the proof is complete. □

Observe that we have even proved that the tensor multiplicator norm in the space M_φ is equal

$$\sup_{0 < u_i \leqslant 1, i=1,2,\ldots,n, n \in \mathbf{N}} \frac{\sum_{i=1}^n \varphi(u_i)(\varphi(\frac{i}{n}) - \varphi(\frac{i-1}{n}))}{\varphi(\frac{1}{n} \sum_{i=1}^n u_i)}.$$

The concavity of φ implies that the supremum is attained on the set of decreasing sequences $1 \geqslant u_1 \geqslant u_2 \geqslant \cdots \geqslant u_n > 0$.

Remark 1. Theorem 1 can be formulated in a more general form. Let $\varphi_1, \varphi_2, \varphi_3 \in \mathscr{C}$. Then the tensor product $\otimes : M_{\varphi_1} \times M_{\varphi_2} \to M_{\varphi_3}(I \times I)$ is bounded if and only if there exists a constant $C > 0$ such that the inequality

$$\sum_{i=1}^n \varphi_1(u_i) \left(\varphi_2 \left(\frac{i}{n} \right) - \varphi_2 \left(\frac{i-1}{n} \right) \right) \leqslant C \varphi_3 \left(\frac{1}{n} \sum_{i=1}^n u_i \right) \tag{5}$$

is true for every integer n and every $u_i \in [0, 1]$, $i = 1, 2, \ldots, n$.

Condition (5) can be also written in the integral form

$$\int_0^1 \varphi_1(u(t)) \varphi_2'(t) \, dt \leqslant C \varphi_3 \left(\int_0^1 u(t) \, dt \right)$$

for all functions $u(t)$ on $[0, 1]$ such that $0 \leqslant u(t) \leqslant 1$. The last integral condition is satisfied when for example

$$\int_0^1 \varphi_1 \left(\frac{s}{t} \right) \varphi_2'(t) \, dt \leqslant C \varphi_3(s)$$

for all s in $[0, 1]$. A similar assumption appeared in papers [Mi76,Mi78].

We will find a condition on $\varphi \in \mathscr{C}$ under which estimate (4) will be true.

Lemma 1. *Let $\varphi \in \mathscr{C}$ and $\varphi(t) \leqslant K \varphi(t^2)$ for some positive number K and for every $t \in [0, 1]$. Then*

$$\sum_{i=1}^n \varphi(u_i) \left(\varphi \left(\frac{i}{n} \right) - \varphi \left(\frac{i-1}{n} \right) \right) \leqslant (K+1) \varphi(1) \varphi \left(\frac{1}{n} \sum_{i=1}^n u_i \right)$$

for every integer n and every $u_i \in [0, 1]$, $i = 1, 2, \ldots, n$.

Proof. The concavity of φ implies that we can suppose the monotonicity $1 \geqslant u_1 \geqslant u_2 \geqslant \cdots \geqslant u_n \geqslant 0$. Denote

$$s = \frac{1}{n} \sum_{i=1}^n u_i.$$

There exists a natural number m, $1 \leqslant m \leqslant n$, such that $u_m \leqslant \sqrt{s}$ and $u_i > \sqrt{s}$ for $i \leqslant m$. Since

$$ns = \sum_{i=1}^{n} u_i \geqslant \sum_{i=1}^{m} u_i \geqslant m\sqrt{s}$$

it yields that $m \leqslant n\sqrt{s}$ and

$$\sum_{i=1}^{m} \varphi(u_i)\left(\varphi\left(\frac{i}{n}\right) - \varphi\left(\frac{i-1}{n}\right)\right) \leqslant \varphi(1)\sum_{i=1}^{m}\left(\varphi\left(\frac{i}{n}\right) - \varphi\left(\frac{i-1}{n}\right)\right) = \varphi(1)\varphi\left(\frac{m}{n}\right)$$

$$\leqslant \varphi(1)\varphi\left(\frac{n\sqrt{s}}{n}\right) = \varphi(1)\varphi(\sqrt{s}) \leqslant K\varphi(1)\varphi(s).$$

If $i > m$, then $\varphi(u_i) \leqslant \varphi(s)$ and so

$$\sum_{i=m+1}^{n} \varphi(u_i)\left(\varphi\left(\frac{i}{n}\right) - \varphi\left(\frac{i-1}{n}\right)\right) \leqslant \varphi(s)\sum_{i=m+1}^{n}\left(\varphi\left(\frac{i}{n}\right) - \varphi\left(\frac{i-1}{n}\right)\right) \leqslant \varphi(1)\varphi(s).$$

Hence

$$\sum_{i=1}^{n} \varphi(u_i)\left(\varphi\left(\frac{i}{n}\right) - \varphi\left(\frac{i-1}{n}\right)\right) \leqslant (K+1)\varphi(1)\varphi(s) = (K+1)\varphi(1)\varphi\left(\frac{1}{n}\sum_{i=1}^{n} u_i\right). \quad \square$$

Immediately from Theorem 1 and Lemma 1 we have the following assertion.

Corollary 1. *Let $\varphi \in \mathscr{C}$ and $\varphi(t) \leqslant K\varphi(t^2)$ for some positive number K and for every $t \in [0,1]$. Then*

$$\mathscr{M}(M_\varphi) = M_\varphi.$$

Let us note that the power function $\varphi(t) = t^\alpha$ with $0 < \alpha < 1$ does not satisfy inequality (4) but there are functions $\varphi \in \mathscr{C}$ which satisfy the estimate $\varphi(t) \leqslant K\varphi(t^2)$ for some positive number K and for every $t \in [0,1]$. This estimate gives, of course, the supermultiplicativity of φ on $[0,1]$.

Example 1. For each $\lambda > 0$ there exists $a = a(\lambda) \in (0,1)$ such that the function

$$\varphi_\lambda(t) = \begin{cases} 0 & \text{if } t = 0, \\ \ln^{-\lambda}\frac{1}{t} & \text{if } 0 < t \leqslant a(\lambda), \\ \text{linear} & \text{if } t \in [a(\lambda), 1], \end{cases}$$

belongs to $\mathscr{C}$. Clearly, $\varphi_\lambda(t) \leqslant 2^\lambda \varphi_\lambda(t^2)$ for every $t \in [0, a(\lambda)]$. Consequently, φ_λ satisfies the conditions of Lemma 1.

Remark 2. There exists a function $\varphi \in \mathscr{C}$ such that the tensor product acts from $M_\varphi \times M_\varphi$ into M_φ and φ does not satisfy the condition $\varphi(t) \leqslant K\varphi(t^2)$ for some positive number K and for every $t \in [0, 1]$. It is enough to take $\varphi(t) = t^\alpha \ln^{-\beta} \frac{a}{t}$ for $0 < \alpha < 1$, $\beta > 1$ and $a > e^{2\beta/(1-\alpha)}$.

We finish this part with the imbeddings of Calderón–Lozanovskiĭ construction on multiplicator spaces.

Proposition 4. *Let X_0, X_1 be r.i. spaces on $[0, 1]$. Then*

(i) $\mathscr{M}(X_0)^{1-\theta} \mathscr{M}(X_1)^\theta \subset \mathscr{M}(X_0^{1-\theta} X_1^\theta)$.

(ii) *If $\rho \in \mathscr{U}$ is a supermultiplicative function on $[0, \infty)$, i.e., there exists a constant $c > 0$ such that $\rho(st) \geqslant c\rho(s)\rho(t)$ for all $s, t \in [0, \infty)$, then*

$$\rho(\mathscr{M}(X_0), \mathscr{M}(X_1)) \subset \mathscr{M}(\rho(X_0, X_1)).$$

Proof. (i) Observe first that $Y \subset \mathscr{M}(X)$ if and only if the operator $\otimes : Y \times X \to X(I \times I)$ is bounded.

Since $\otimes : \mathscr{M}(X_i) \times X_i \to X_i(I \times I)$, $i = 0, 1$, is bounded with the norm $\leqslant 1$ and the Calderón construction is an interpolation method for positive bilinear operators (cf. [C]) it follows that

$$\otimes : \mathscr{M}(X_0)^{1-\theta} \mathscr{M}(X_1)^\theta \times X_0^{1-\theta} X_1^\theta \to X_0(I \times I)^{1-\theta} X_1(I \times I)^\theta = X_0^{1-\theta} X_1^\theta(I \times I)$$

is bounded with the norm $\leqslant 1$. Therefore, $\mathscr{M}(X_0)^{1-\theta} \mathscr{M}(X_1)^\theta \subset \mathscr{M}(X_0^{1-\theta} X_1^\theta)$.

(ii) When ρ is a supermultiplicative function the Calderón–Lozanovskiĭ construction is an interpolation method for positive bilinear operators (see [As97,M] Theorem 2]) and the proof of the imbedding is similar as in (i). □

Note that the inclusions in Proposition 4 can be strict. For the spaces $X_0 = L_{p,q}$, $X_1 = L_{p,\infty}$ with $1 \leqslant q < p < \infty$ we have

$$\mathscr{M}(X_0)^{1-\theta} \mathscr{M}(X_1)^\theta = \mathscr{M}(L_{p,q})^{1-\theta} \mathscr{M}(L_{p,\infty})^\theta = L_{p,q}^{1-\theta} L_p^\theta = L_{p,r},$$

where $1/r = (1-\theta)/q + \theta/p$ and

$$\mathscr{M}(X_0^{1-\theta} X_1^\theta) = \mathscr{M}(L_{p,q}^{1-\theta} L_{p,\infty}^\theta) = \mathscr{M}(L_{p,s}) = L_{p,\min(p,s)},$$

where $1/s = (1-\theta)/q$. The strict imbedding $L_{p,r} \subsetneq L_{p,\min(p,s)}$ gives then the corresponding example.

3. Subspaces generated by dilations and translations in r.i. spaces

Given an r.i. space X on $I = [0, 1]$ let us denote by

$$V_0(X) = \{a \in X : a \neq 0, \ a = a^*\}.$$

For a fixed function $a \in V_0(X)$ and dyadic intervals $\Delta_{n,k} = [\frac{k-1}{2^n}, \frac{k}{2^n})$, $k = 1, 2, \ldots, 2^n$, $n = 0, 1, 2, \ldots$, let us consider the dilations and translations of a function a

$$a_{n,k}(t) = \begin{cases} a(2^n t - k + 1) & \text{if } t \in \Delta_{n,k}, \\ 0 & \text{elsewhere.} \end{cases}$$

Then $\operatorname{supp} a_{n,k} \subset \Delta_{n,k}$ and

$$m(\{t \in \Delta_{n,k} : |a_{n,k}(t)| > \lambda\}) = 2^{-n} m(\{t \in I : |a(t)| > \lambda\}) \quad \text{for all } \lambda > 0.$$

For $a \in V_0(X)$ and $n = 0, 1, 2, \ldots$ we denote by $Q_{a,n}$ the linear span $[\{a_{n,k}\}_{k=1}^{2^n}]$ generated by functions $a_{n,k}$ in X.

Definition 2. For an r.i. function space X on $[0, 1]$ the *nice part* $N_0(X)$ of X is defined by

$$N_0(X) = \{a \in V_0(X) : \text{ there exists a sequence of projections } \{P_n\}_{n=0}^{\infty} \text{ on } X \text{ such that}$$

$\operatorname{Im} P_n = Q_{a,n}$ and $\sup_{n=0,1,\ldots} \|P_n\|_{X \to X} < \infty \}$.

We say that X is a *nice space* (or shortly $X \in \mathcal{N}_0$) if a^* belongs to $N_0(X)$ for every a from X.

We are using here similar notions as in the paper [HS99]. They were considering r.i. space $X = X[0, \infty)$ on $[0, \infty)$, the corresponding set $V(X) = \{a \in X : a \neq 0, \operatorname{supp} a \subset [0, 1), a = a^*\}$ and the set $N(X)$ of all $a \in V(X)$ such that Q_a is a complemented subspace of $X = X[0, \infty)$, where Q_a is the linear closed span generated by the sequence $(a_k)_{k=1}^{\infty}$ with

$$a_k(t) = a(t - k + 1) \quad \text{for } t \in [k - 1, k) \text{ and } a_k(t) = 0 \text{ elsewhere.}$$

If $N(X) = X$, then they write that $X \in \mathcal{N}$ (or say that X is a nice space).

We are putting "sub-zero" notions, that is, $V_0(X)$ and $N_0(X)$, so that we have difference between of our case of r.i. spaces on $[0, 1]$ and their case $[0, \infty)$.

Theorem 2. *Let X be an r.i. space on $[0, 1]$ and let X^0 denote the closure of $L_\infty[0, 1]$ in X. Then we have embeddings*

(i) $\mathcal{M}(X) \subset N_0(X)$,
(ii) $N_0(X^0) \subset \mathcal{M}(X)$.

Before the proof of this theorem we will need some auxiliary results.
Let $a \in V_0(X)$ and $f \in V_0(X')$ be such that

$$\int_0^1 f(t) a(t) \, dt = 1. \tag{6}$$

Define the sequence of natural projections (averaging operators)

$$P_n x(t) = P_{n,a,f} x(t) = \sum_{k=1}^{2^n} \left(2^n \int_{\Delta_{n,k}} f_{n,k}(s)x(s)ds\right) a_{n,k}(t), \quad n = 0, 1, 2, \ldots . \quad (7)$$

Lemma 2. *The sequence of norms* $\{||P_{n,a,f}||_{X \to X}\}_{n=0}^{\infty}$ *is a non-decreasing sequence.*

Proof. For $x = x(t)$ with supp $x \subset \Delta_{n,k}$ we define

$$R_{n,k}x(t) = x\left(2t - \frac{k-1}{2^n}\right), \quad S_{n,k}x(t) = x\left(2t - \frac{k}{2^n}\right).$$

Then

$$\text{supp } R_{n,k}x \subset \Delta_{n+1,2k-1}, \quad \text{supp } S_{n,k}x \subset \Delta_{n+1,2k}$$

and

$$m(\{t \in \Delta_{n+1,2k-1} : |R_{n,k}x(t)| > \lambda\}) = m(\{t \in \Delta_{n+1,2k} : |S_{n,k}x(t)| > \lambda\})$$

$$= \tfrac{1}{2} m(\{t \in I : |x(t)| > \lambda\})$$

for all $\lambda > 0$. Therefore,

$$\int_{\Delta_{n+1,2k-1}} R_{n,k}x(t)\, dt = \int_{\Delta_{n+1,2k}} S_{n,k}x(t)\, dt = \tfrac{1}{2} \int_{\Delta_{n,k}} x(t)\, dt.$$

Moreover,

$$R_{n,k}(f_{n,k}x\chi_{\Delta_{n,k}})(t) = f_{n+1,2k-1}R_{n,k}(x\chi_{\Delta_{n,k}})(t),$$

$$S_{n,k}(f_{n,k}x\chi_{\Delta_{n,k}})(t) = f_{n+1,2k}S_{n,k}(x\chi_{\Delta_{n,k}})(t)$$

and

$$m(\{t \in \Delta_{n+1,j} : a_{n+1,j}(t) > \lambda\}) = \tfrac{1}{2} m(\{t \in \Delta_{n,i} : a_{n,i} > \lambda\})$$

for all $\lambda > 0$ and any $i = 1, 2, \ldots, 2^n$, $j = 1, 2, \ldots, 2^{n+1}$.

Denote $P_n = P_{n,a,f}$. The last equality and the equality of integrals give that the function $P_n x(t)$ is equimeasurable with the function

$$P_{n+1}y(t) = \sum_{k=1}^{2^n} \left(2^{n+1} \int_{\Delta_{n+1,2k-1}} f_{n+1,2k-1}(s)R_{n,k}(x\chi_{\Delta_{n,k}})(s)\, ds\right) a_{n+1,2k-1}(t)$$

$$+ \sum_{k=1}^{2^n} \left(2^{n+1} \int_{\Delta_{n+1,2k}} f_{n+1,2k}(s)S_{n,k}(x\chi_{\Delta_{n,k}})(s)\, ds\right) a_{n+1,2k}(t),$$

where

$$y(t) = \sum_{k=1}^{2^n} [R_{n,k}(x\chi_{\Delta_{n,k}})(t) + S_{n,k}(x\chi_{\Delta_{n,k}})(t)].$$

From the above we can see that y is equimeasurable with x and so

$$||P_n x||_X = ||P_{n+1} y||_X \leqslant ||P_{n+1}|| \ ||y||_X = ||P_{n+1}|| \ ||x||_X,$$

that is, $||P_n|| \leqslant ||P_{n+1}||$.   □

Lemma 3. *Let X be a separable r.i. space. If $a \in N_0(X)$, then there exists a function $f \in N_0(X')$ such that (6) is fulfilled and for the sequence of projections $\{P_{n,a,f}\}$ defined by (7) we have*

$$\sup_{n=0,1,2,...} ||P_{n,a,f}||_{X \to X} < \infty.$$

Proof. Since X is a separable space and $a \in N_0(X)$ it follows that there are functions $g_{n,k} \in X'(k = 1, 2, ..., 2^n, n = 0, 1, 2, ...)$ such that

$$\int_0^1 g_{n,k}(s) a_{n,k}(s)\, ds = 1 \quad \text{and} \quad \int_0^1 g_{n,k}(s) a_{n,j}(s)\, ds = 0, \ j \neq k,$$

and for the projections

$$T_n x(t) = \sum_{k=1}^{2^n} \left(\int_0^1 g_{n,k}(s) x(s)\, ds \right) a_{n,k}(t)$$

we have $\sup_{n=0,1,...} ||T_n||_{X \to X} < \infty$. Let $\{r_i\}_{i=1}^n$ be the first n Rademacher functions on the segment $[0, 1]$. Since X is an r.i. space it follows that for every $u \in [0, 1]$ the norms of the operators

$$T_{n,u} x(t) = \sum_{k=1}^{2^n} r_k(u) \sum_{i=1}^{2^n} r_i(u) \left(\int_{\Delta_{n,i}} g_{n,k}(s) x(s)\, ds \right) a_{n,k}(t)$$

are the same as the norms of T_n. Let us consider the operators

$$S_n x(t) = \int_0^1 T_{n,u} x(t)\, du = \sum_{k=1}^{2^n} \left(\int_{\Delta_{n,k}} g_{n,k}(s) x(s)\, ds \right) a_{n,k}(t).$$

Then

$$||S_n|| \leqslant \sup_{u \in [0,1]} ||T_{n,u}|| = ||T_n|| \leqslant C.$$

Therefore, we can assume that

$$\text{supp } g_{n,k} \subset \Delta_{n,k} \text{ and } g_{n,k} \text{ is decreasing on } \Delta_{n,k}.$$

Moreover, we shift supports of the functions $g_{n,k}(k = 1, 2, \dots, 2^n)$ to the segment $[0, 2^{-n}]$ and consider the averages

$$G_n(t) = 2^{-n} \sum_{k=1}^{2^n} \tau_{-(k-1)2^{-n}} g_{n,k}(t),$$

where $\tau_s g(t) = g(t - s)$.

Then the shifts $h_{n,k}(t) = 2^{-n} \tau_{(k-1)2^{-n}} G_n(t)$ generate operators

$$U_n x(t) = \sum_{k=1}^{2^n} \left(2^n \int_{\Delta_{n,k}} h_{n,k}(s) x(s) \, ds \right) a_{n,k}(t)$$

and we can show that $\|U_n\|_{X \to X} \leqslant C$.

Since $h_{n,k}(t) = (F_n)_{n,k}(t)$, where $F_n(t) = 2^{-n} G_n(2^{-n} t)$ for $0 \leqslant t \leqslant 1$, it follows that

$$\int_0^1 F_n(t) a(t) \, dt = \int_0^{2^{-n}} G_n(t) a_{n,1}(t) \, dt$$

$$= 2^{-n} \sum_{k=1}^{2^n} \int_0^{2^{-n}} \tau_{-(k-1)2^{-n}} g_{n,k}(t) a_{n,1}(t) \, dt$$

$$= 2^{-n} \sum_{k=1}^{2^n} \int_{\Delta_{n,k}} g_{n,k}(t) a_{n,k}(t) \, dt = 1.$$

Let us show that there exists a subsequence $\{F_{n_k}(t)\}$ of $F_n(t)$ which converges at every $t \in (0, 1]$.

Lemma 2 gives that the norm of the one-dimensional operator

$$L_n x(t) = \left(\int_0^1 F_n(s) x(s) \, ds \right) a(t)$$

does not exceed $\|U_n\|_{X \to X}$, and consequently also not C. Therefore,

$$\|F_n\|_{X'} \leqslant \frac{C}{\|a\|_X}. \tag{8}$$

By the definition of F_n we have $F_n^*(t) = F_n(t)$ and

$$1 = \int_0^1 F_n(s) a(s) \, ds \geqslant F_n(t) \int_0^t a(s) \, ds$$

or

$$F_n(t) \leqslant \left(\int_0^t a(s)\, ds \right)^{-1} \quad \text{for all } t \in (0,1].$$

Applying Helly selection theorem (see [N]) we can choose subsequences

$$\{F_n\} \supset \{F_{n,1}\} \supset \{F_{n,2}\} \supset \dots \supset \{F_{n,m}\} \supset \dots$$

that converge on the intervals

$$\left[\frac{1}{2}, 1\right], \left[\frac{1}{3}, 1\right], \dots, \left[\frac{1}{m+1}, 1\right], \dots,$$

respectively. Then the diagonal sequence $f_n(t) = F_{n,n}(t)$ converges at every $t \in (0,1]$ to a function $f(t) = f^*(t)$. Using estimate (8) we obtain

$$\int_0^s f_n(t) a(t)\, dt \leqslant \|f_n\|_{X'} \|a\chi_{(0,s)}\|_X \leqslant \frac{C}{\|a\|_X} \|a\chi_{(0,s)}\|_X.$$

Since X is a separable r.i. space it follows that $\|a\chi_{(0,s)}\|_X \to 0$ as $s \to 0^+$. Therefore, the equalities $f_n^* = f_n$ and $a^* = a$ imply that $\{f_n a\}$ is an equi-integrable sequence of functions on $[0,1]$. Hence (see [E, Theorem 1.21], or [HM, Theorem 6, Chapter V])

$$\int_0^1 f(t) a(t)\, dt = \lim_{n \to \infty} \int_0^1 f_n(t) a(t)\, dt = 1.$$

Let $m \in \mathbf{N}$ be fixed. By the estimate $\|U_n\|_{X \to X} \leqslant C$, the definition of f_n, and Lemma 2 we have

$$\left\| \sum_{k=1}^{2^m} \left(2^m \int_{\Delta_{m,k}} (f_n)_{m,k}(t) x(t)\, dt \right) a_{m,k} \right\|_X \leqslant C \|x\|_X \tag{9}$$

for all $n \geqslant m$ and all $x \in X$.

Suppose that $x(t)$ is a non-negative and non-increasing function on every interval $\Delta_{m,k}, k = 1, 2, \dots, 2^m$. As above, from (8) it follows that $\{(f_n)_{m,k} x\chi_{\Delta_{m,k}}\}_{m=1}^{\infty}$ is an equi-integrable sequence on $\Delta_{m,k}$. Hence

$$\lim_{n \to \infty} \int_{\Delta_{m,k}} (f_n)_{m,k}(t) x(t)\, dt = \int_{\Delta_{m,k}} (f)_{m,k}(t) x(t)\, dt$$

and for all such functions $x(t)$ estimate (9) implies

$$\left\| \sum_{k=1}^{2^m} \left(2^m \int_{\Delta_{m,k}} f_{m,k}(t) x(t)\, dt \right) a_{m,k} \right\|_X \leqslant C \|x\|_X.$$

Since X is an r.i. space we can prove that the above estimate holds for all $x \in X$. The proof is complete. $\square$

Proof of Theorem 2. (i) At first by the result in [A97, Theorem 1.14], we have that $a \in \mathcal{M}(X)$ if and only if there exists a constant $C > 0$ such that

$$\left\| \sum_{k=1}^{2^n} c_{n,k} a_{n,k} \right\|_X \leqslant C \left\| \sum_{k=1}^{2^n} c_{n,k} \chi_{\Delta_{n,k}} \right\|_X \tag{10}$$

for all $c_{n,k} \in \mathbf{R}$ and $k = 1, 2, \ldots, 2^n$, $n = 0, 1, 2, \ldots$.

Suppose that $a \in V_0(X) \cap \mathcal{M}(X)$, that is, estimate (10) holds. If $e(t) \equiv 1$, then the operators

$$P_{n,e} x(t) = \sum_{k=1}^{2^n} \left(2^n \int_{\Delta_{n,k}} x(s) \, ds \right) \chi_{\Delta_{n,k}}(t) \quad (n = 1, 2, \ldots) \tag{11}$$

are bounded projections in every r.i. space X and $\|P_{n,e}\|_{X \to X} \leqslant 1$ (see [KPS, Theorem 4.3]).

Define operators $R_{n,a} : Q_{e,n} = \operatorname{Im} P_{n,e} \to Q_{a,n}$ as follows:

$$R_{n,a} \left(\sum_{k=1}^{2^n} c_{n,k} \chi_{\Delta_{n,k}} \right) = \sum_{k=1}^{2^n} c_{n,k} a_{n,k}.$$

By the assumption $a \in \mathcal{M}(X)$ or equivalently by estimate (10) we have $\|R_{n,a}\|_{Q_{e,n} \to X} \leqslant C$. Therefore, for the operators

$$P_{n,a} = \frac{1}{\|a\|_{L_1}} R_{n,a} P_{n,e}.$$

We have

$$\|P_{n,a}\|_{X \to X} \leqslant C \|a\|_{L_1}^{-1}, \quad n = 1, 2, \ldots .$$

It is easy to check that $P_{n,a}$ are projections and $\operatorname{Im} P_{n,a} = Q_{a,n}$. Therefore, $a \in N_0(X)$.

(ii) If $X = L_\infty$, then $\mathcal{M}(X) = N_0(X^0) = L_\infty$.

If $X \neq L_\infty$, then X^0 is a separable r.i. space. In this case, by Lemma 3, for any $a \in N_0(X^0)$ there exists a function $f \in V_0((X^0)')$ such that (6) is fulfilled and for the projections $P_{n,a,f}$ defined as in (7) we have

$$C = \sup_{n=0,1,2,\ldots} \|P_{n,a,f}\|_{X^0 \to X^0} < \infty. \quad \square$$

If x is a function of the form $x(s) = \sum_{k=1}^{2^n} c_{n,k}\chi_{\Delta_{n,k}}(s)$, then

$$P_{n,a,f}x = \|f\|_{L_1} \sum_{k=1}^{2^n} c_{n,k}a_{n,k}.$$

Therefore,

$$\left\| \sum_{k=1}^{2^n} c_{n,k}a_{n,k} \right\|_{X^0} \leqslant C\|f\|_{L_1}^{-1} \left\| \sum_{k=1}^{2^n} c_{n,k}\chi_{\Delta_{n,k}} \right\|_{X^0},$$

and we obtain (10) for X^0.

If $X^0 = X$, that is, X is a separable r.i. space, then the theorem is proved. If X is a non-separable r.i. space, then X^0 is an isometric subspace of $X = X''$. By using the Fatou property, we can extend the above inequality to the whole space X and obtain (10), which gives that $a \in \mathscr{M}(X)$. The proof of Theorem 2 is complete. □

Immediately from Theorem 2 and the properties of the multiplicator space we obtain the following corollaries.

Corollary 2. *If X is a separable r.i. space, then $\mathscr{M}(X) = N_0(X)$.*

Corollary 3. *If $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, then $N_0(L_{p,q}) = L_{p,q}$ for $1 \leqslant q \leqslant p$ and $N_0(L_{p,q}) = N_0(L_{p,\infty}^0) = L_p$ for $p < q < \infty$.*

Corollary 4. *Let X_0 and X_1 be separable r.i. spaces. If $X_0, X_1 \in \mathscr{N}_0$, then $X_0^{1-\theta}X_1^\theta \in \mathscr{N}_0$.*

Corollaries 3 and 4 show that the class of nice spaces $\mathscr{N}_0$ is stable with respect to the complex method of interpolation but it is not stable with respect to the real interpolation method.

Corollary 5. *If $\varphi \in \mathscr{C}$ and $\varphi(t) \leqslant K\varphi(t^2)$ for some positive number K and for every $t \in [0,1]$, then $N_0(M_\varphi) = M_\varphi$.*

By $\varphi \in \mathscr{C}_0$ we mean $\varphi \in \mathscr{C}$ such that $\lim_{t \to 0^+} \frac{t}{\varphi(t)} = 0$.

Theorem 3. *Let $\varphi \in \mathscr{C}_0$.*

(i) *If*

$$\limsup_{t \to 0^+} \frac{\varphi(2t)}{\varphi(t)} = 2 \tag{12}$$

then

$$L_\infty \subset N_0(M_\varphi) \subset L_\infty \cup (M_\varphi \backslash M_\varphi^0). \tag{13}$$

(ii) *If*

$$\lim_{t \to 0^+} \frac{\varphi(2t)}{\varphi(t)} = 2$$

then

$$N_0(M_\varphi) = L_\infty \cup (M_\varphi \backslash M_\varphi^0).$$

Proof. (i) By Theorem 2 the left part of (13) is valid for any r.i. space. Assumption (12) implies

$$\limsup_{t \to 0^+} \frac{\varphi(t)}{\varphi(ts)} = \frac{1}{s}, \quad 0 < s < 1$$

and so

$$||\sigma_s||_{M_\varphi \to M_\varphi} \geqslant s \limsup_{t \to 0^+} \frac{\varphi(t)}{\varphi(st)} = 1$$

for every $0 < s \leqslant 1$. This means that $\alpha_{M_\varphi} = 0$.

Let $x \in N_0(M_\varphi) \cap M_\varphi^0$. By using Corollary 2 and property (b_1) we get

$$x \in N_0(M_\varphi^0) = \mathcal{M}(M_\varphi^0) = L_\infty.$$

This proves the right part of (13).

(ii) We must only prove the inclusion

$$M_\varphi \backslash M_\varphi^0 \subset N_0(M_\varphi).$$

Let $a \in M_\varphi \backslash M_\varphi^0, ||a||_{M_\varphi} = 1$ and $\psi(t) = \int_0^t a^*(s) \, ds$. It is well known that

$$\mathrm{dist}(a, M_\varphi^0) = \limsup_{t \to 0^+} \frac{1}{\varphi(t)} \int_0^t a^*(s) \, ds.$$

Therefore,

$$\gamma = \limsup_{t \to 0^+} \frac{\psi(t)}{\varphi(t)} > 0$$

and there exists a sequence $\{t_m\}$ tending to 0 such that

$$\lim_{m \to \infty} \frac{\psi(t_m)}{\varphi(t_m)} = \gamma.$$

Since

$$\lim_{m \to \infty} \frac{\varphi(t_m)}{2^n \varphi\left(\frac{t_m}{2^n}\right)} = 1$$

for every natural n, it follows that

$$\|a_{n,k}\|_{M_\varphi} \geqslant \limsup_{m \to \infty} \frac{\psi(t_m)}{2^n \varphi\left(\frac{t_m}{2^n}\right)} = \limsup_{m \to \infty} \frac{\psi(t_m)}{\varphi(t_m)} \lim_{m \to \infty} \frac{\varphi(t_m)}{2^n \varphi\left(\frac{t_m}{2^n}\right)} = \gamma$$

for every $1 \leqslant k \leqslant 2^n$, $n = 1, 2, \dots$.

Consider the subspaces $H_{n,k} = \text{closed span } [\{a_{n,i}\}_{i \neq k}]$. These subspaces are closed subspaces of M_φ and $a_{n,k} \notin H_{n,k}$. Thus, by the Hahn–Banach theorem, there are $b_{n,k} \in (M_\varphi)^*$ such that $b_{n,k}|_{H_{n,k}} = 0$, $b_{n,k}(a_{n,k}) = 1$ and $\|b_{n,k}\| = \frac{1}{\|a_{n,k}\|} \leqslant \frac{1}{\gamma}$. Then the operators

$$P_n x = \sum_{k=1}^{2^n} b_{n,k}(x) a_{n,k}$$

are projections from M_φ onto $Q_{a,n}$. Moreover, P_n are uniformly bounded since

$$\|P_n x\|_{M_\varphi} \leqslant \frac{1}{\gamma} \left\| \sum_{k=1}^{2^n} \|x\|_{M_\varphi} a_{n,k} \right\|_{M_\varphi} = \frac{1}{\gamma} \|x\|_{M_\varphi}.$$

Therefore, $a \in N_0(M_\varphi)$. The proof is complete. □

Example 2. There exists a non-separable r.i. space X such that $\mathscr{M}(X) \neq N_0(X)$.

Take $X = M_\varphi$ with $\varphi(t) = t \ln \frac{e}{t}$ on $(0, 1]$. Since $\alpha_{M_\varphi} = 0$ it follows that $\mathscr{M}(M_\varphi) = L_\infty$. The function $a(t) = \ln \frac{e}{t}$ for $t \in (0, 1]$ as unbounded is not in $\mathscr{M}(X)$ but it is in $M_\varphi \backslash M_\varphi^0$ and by Theorem 3(ii) it shows that $a \in N_0(X)$. Therefore, $N_0(X) \neq \mathscr{M}(X)$.

Corollary 6. *If $\varphi \in \mathscr{C}_0$ and $\limsup_{t \to 0^+} \frac{\varphi(2t)}{\varphi(t)} = 2$, then $\varphi' \in N_0(M_\varphi)$ and consequently $N_0(M_\varphi)$ is neither a linear space nor a lattice.*

Problem 1. *For $1 < p < \infty$ describe $N_0(L_{p,\infty})$.*

Note that $\mathscr{M}(L_{p,\infty}) = L_p$ and $N_0(L_{p,\infty}^0) = \mathscr{M}(L_{p,\infty}^0) = L_p$.

Theorem 4. *Let X be an r.i. space X on $[0,1]$. The following conditions are equivalent*:

(i) $\otimes : X \times X \to X(I \times I)$ *is bounded.*
(ii) $\mathscr{M}(X) = X$.

(iii) $X \in \mathcal{N}_0$, i.e., $N_0(X) = X$.

(iv) *There exists a constant $C > 0$ such that*

$$\left\| \sum_{k=1}^{2^n} c_{n,k} a_{n,k} \right\|_X \leqslant C \|a\|_X \left\| \sum_{k=1}^{2^n} c_{n,k} \chi_{\Delta_{n,k}} \right\|_X \tag{14}$$

for all $a \in X$ and all $c_{n,k} \in \mathbf{R}$, $k = 1, 2, \ldots, 2^n$, $n = 0, 1, 2, \ldots$.

Proof. Implication (i) $\Rightarrow$ (ii) follows by definition (ii) $\Rightarrow$ (iii) from Theorem 2 and (iv) $\Rightarrow$ (i) by the result in [A97, Theorem 1.14]. Therefore, it only remains to prove that (iii) implies (iv).

First, assume additionally that X is separable. If $X \in \mathcal{N}_0$, then, similarly as in the proof of Theorem 2, for any $a \in X$ there exist $C_1 > 0$ and $f \in V_0(X')$ such that $\int_0^1 f(t) a(t) \, dt = 1$ and

$$\left\| \sum_{k=1}^{2^n} c_{n,k} a_{n,k} \right\|_X \leqslant C_1 \|f\|_{L_1}^{-1} \left\| \sum_{k=1}^{2^n} c_{n,k} \chi_{\Delta_{n,k}} \right\|_X .$$

Let us introduce a new norm on X defined by

$$\|a\|_1 = \sup \left\{ \frac{\left\| \sum_{k=1}^{2^n} c_{n,k} a_{n,k} \right\|_X}{\left\| \sum_{k=1}^{2^n} c_{n,k} \chi_{\Delta_{n,k}} \right\|_X} : c_{n,k} \in \mathbf{R}, \ k = 1, 2, \ldots, 2^n, \ n = 0, 1, 2, \ldots \right\}.$$

Then $\|a\|_X \leqslant \|a\|_1$ and $\|a\|_1 < \infty$ for all $a \in X$. By the closed graph theorem we obtain that $\|a\|_1 \leqslant C \|a\|_X$ and (14) is proved.

Now, let X be a non-separable r.i. space. In the case $X = L_\infty$ both conditions (iii) and (iv) are fulfilled. Therefore, consider the case $X \neq L_\infty$. Then X^0 is a separable r.i. space. The canonical isometric imbedding $X^0 \subset X = X''$ gives that $X^0 \in \mathcal{N}_0$. Let $a \in V_0(X)$. The separability of X^0 implies

$$\left\| \sum_{k=1}^{2^n} c_{n,k} [a^{(m)}]_{n,k} \right\|_{X^0} \leqslant C \|a^{(m)}\|_{X^0} \left\| \sum_{k=1}^{2^n} c_{n,k} \chi_{\Delta_{n,k}} \right\|_{X^0}$$

$$= C \|a^{(m)}\|_{X^0} \left\| \sum_{k=1}^{2^n} c_{n,k} \chi_{\Delta_{n,k}} \right\|_X ,$$

where $a^{(m)}(t) = \min(a(t), m)$, $m = 1, 2, \ldots$.

Since $X = X''$ has the Fatou property and $[a^{(m)}]_{n,k} = [a_{n,k}]^{(m)}$ it follows that

$$\left\| \sum_{k=1}^{2^n} c_{n,k} a_{n,k} \right\|_X = \lim_{m \to \infty} \left\| \left[\sum_{k=1}^{2^n} c_{n,k} a_{n,k} \right]^{(m)} \right\|_X$$

$$\leqslant C\|a\|_X \left\| \sum_{k=1}^{2^n} c_{n,k} \chi_{\Delta_{n,k}} \right\|_X$$

and Theorem 4 is proved. $\quad\square$

Theorem 5. *Let X be an r.i. space X on $[0,1]$. Then $X \in \mathcal{N}_0$ if and only if $X'' \in \mathcal{N}_0$.*

Proof. The proof is similar to that of Theorem 4. The essential part is the proof of the estimate (14). We leave the details to the reader.

Theorem 6. *Let X be a separable r.i. space X on $[0,1]$. Then the following conditions are equivalent*:

(i) $a \in N_0(X)$.
(ii) *The operators $R_{n,a} : Q_{e,n} \to Q_{a,n}$ given by*

$$R_{n,a}\left(\sum_{k=1}^{2^n} c_{n,k} \chi_{\Delta_{n,k}} \right) = \sum_{k=1}^{2^n} c_{n,k} a_{n,k}$$

are uniformly bounded.
(iii) *The operators $R_{n,a}$ and their inverses are uniformly bounded.*

Proof. (i) $\Rightarrow$ (iii): Let $a \in N_0(X)$. Then $\|R_{n,a}\| \leqslant C$ for all $n = 0, 1, 2, \ldots$, by Theorem 2. Next, since $a \neq 0$ there exists $n_0 \in \mathbf{N}$ and $\varepsilon = \varepsilon(n_0) > 0$ such that $a(t) \geqslant u(t) = \varepsilon \chi_{(0,2^{-n_0})}(t)$. Therefore, for all $c_{n,k} \in \mathbf{R}$,

$$\left\| \sum_{k=1}^{2^n} c_{n,k} a_{n,k} \right\|_X \geqslant \left\| \sum_{k=1}^{2^n} c_{n,k} u_{n,k} \right\|_X = \varepsilon \left\| \sum_{k=1}^{2^n} c_{n,k} \chi_{((k-1)2^{-n}, (k-1+2^{n_0})2^{-n})} \right\|_X$$

$$= \varepsilon \left\| \sigma_{2^{-n_0}}\left(\sum_{k=1}^{2^n} c_{n,k} \chi_{\Delta_{n,k}} \right) \right\|_X \geqslant \varepsilon \|\sigma_{2^{n_0}}\|_{X \to X}^{-1} \left\| \sum_{k=1}^{2^n} c_{n,k} \chi_{\Delta_{n,k}} \right\|_X,$$

which shows that the inverses $(R_{n,a})^{-1}$ are uniformly bounded.

(ii) $\Rightarrow$ (i): If the operators $R_{n,a}$ are uniformly bounded, then we have estimate (14) or equivalently $a \in \mathcal{M}(X)$ and Theorem 2(i) gives that $a \in N_0(X)$. $\quad\square$

Now, we present a characterization of L_p spaces among all r.i. spaces on $[0,1]$.

Theorem 7. *Let X be an r.i. space X on $[0,1]$. The following conditions are equivalent*:

- (i) $X \in \mathcal{N}_0$ and $X' \in \mathcal{N}_0$.
- (ii) *There exists a constant $C > 0$ such that*

$$C^{-1} \left\| \sum_{k=1}^{2^n} c_{n,k} \chi_{\Delta_{n,k}} \right\|_X \leqslant \left\| \sum_{k=1}^{2^n} c_{n,k} a_{n,k} \right\|_X \leqslant C \left\| \sum_{k=1}^{2^n} c_{n,k} \chi_{\Delta_{n,k}} \right\|_X \tag{15}$$

  for all $a \in V_0(X)$ with $\|a\|_X = 1$ and all $c_{n,k} \in \mathbf{R}$ with $k = 1, 2, \ldots, 2^n$, $n = 0, 1, 2, \ldots$.
- (iii) *For any pair of functions (a, f) such that $a \in V_0(X)$, $f \in V_0(X')$ satisfying (6) the operators $P_{n,a,f}$ defined in (7) are uniformly bounded in X.*
- (iv) *The operator of the tensor product $\otimes$ is bounded from $X \times X$ into $X(I \times I)$ and from $X' \times X'$ into $X'(I \times I)$.*
- (v) *There exists a $p \in [1, \infty]$ such that $X = L_p$.*

Proof. (i) $\Rightarrow$ (iv). This follows from Theorem 4.

(iv) $\Rightarrow$ (ii): Let $a \in V_0(X), \|a\|_X = 1$. Assumption (iv) implies, by Theorem 4, that

$$\left\| \sum_{k=1}^{2^n} c_{n,k} a_{n,k} \right\|_X \leqslant C_1 \left\| \sum_{k=1}^{2^n} c_{n,k} \chi_{\Delta_{n,k}} \right\|_X$$

for some constant $C_1 > 0$.

Therefore it only remains to prove left estimate in (15). For arbitrary $b \in V_0(X')$ such that

$$\int_0^1 a(t)b(t)\, dt = 1 \quad \text{and} \quad \left\| \sum_{k=1}^{2^n} d_{n,k} b_{n,k} \right\|_{X'} \leqslant 1 \, (d_{n,k} \in \mathbf{R})$$

we obtain $\int_{\Delta_{n,k}} a_{n,k}(t) b_{n,k}(t)\, dt = 2^{-n}$ and

$$\left\| \sum_{k=1}^{2^n} c_{n,k} a_{n,k} \right\|_X \geqslant \int_0^1 \left(\sum_{k=1}^{2^n} c_{n,k} a_{n,k}(t) \right) \left(\sum_{k=1}^{2^n} d_{n,k} b_{n,k}(t) \right) dt = 2^{-n} \sum_{k=1}^{2^n} c_{n,k} d_{n,k}.$$

Since $\otimes$ is bounded from $X' \times X'$ into $X'(I \times I)$ it follows, again by Theorem 4 used to X', that

$$\left\| \sum_{k=1}^{2^n} d_{n,k} b_{n,k} \right\|_{X'} \leqslant C_2 \left\| \sum_{k=1}^{2^n} d_{n,k} \chi_{\Delta_{n,k}} \right\|_{X'}$$

for some constant $C_2 > 0$, from which we conclude that

$$\left\| \sum_{k=1}^{2^n} c_{n,k} a_{n,k} \right\|_X \geq C_2^{-1} \sup_d \int_0^1 \left(\sum_{k=1}^{2^n} c_{n,k} a_{n,k}(t) \right) \left(\sum_{k=1}^{2^n} d_{n,k} b_{n,k}(t) \right) dt$$

$$= 2^{-n} C_2^{-1} \sup_d \sum_{k=1}^{2^n} c_{n,k} d_{n,k},$$

where the supremum is taken over all $d = (d_{n,k})_{k=1}^{2^n}$ such that $\left\| \sum_{k=1}^{2^n} d_{n,k} \chi_{\Delta_{n,k}} \right\|_{X'} \leq 1$.

The operator $P_{n,e}$ defined as in (11) by

$$P_{n,e} x(t) = \sum_{k=1}^{2^n} \left(2^n \int_{\Delta_{n,k}} x(s)\, ds \right) \chi_{\Delta_{n,k}}(t) \quad (n = 1, 2, \ldots)$$

satisfies $\|P_{n,e}\|_{X' \to X'} \leq 1$. Therefore,

$$\left\| \sum_{k=1}^{2^n} c_{n,k} \chi_{\Delta_{n,k}} \right\|_X = \sup_{\|y\|_{X'} \leq 1} \int_0^1 \left(\sum_{k=1}^{2^n} c_{n,k} \chi_{\Delta_{n,k}}(t) \right) y(t)\, dt$$

$$= \sup_{\|y\|_{X'} \leq 1} \int_0^1 \left(\sum_{k=1}^{2^n} c_{n,k} \chi_{\Delta_{n,k}}(t) \right) P_{n,e} y(t)\, dt$$

$$\leq \sup_{\|P_{n,e} y\|_{X'} \leq 1} \int_0^1 \left(\sum_{k=1}^{2^n} c_{n,k} \chi_{\Delta_{n,k}}(t) \right) P_{n,e} y(t)\, dt$$

$$\leq 2^{-n} \sup_d \sum_{k=1}^{2^n} c_{n,k} d_{n,k}.$$

Hence

$$\left\| \sum_{k=1}^{2^n} c_{n,k} \chi_{\Delta_{n,k}} \right\|_X \leq C_2 \left\| \sum_{k=1}^{2^n} c_{n,k} a_{n,k} \right\|_X.$$

(ii) $\Rightarrow$ (v): By Krivine's theorem [K,LT, p. 141] for every r.i. space X there exists $p \in [1, \infty]$ with the following property:

for any $n = 0, 1, 2, \ldots$ and $\varepsilon > 0$ there exist disjoint and equimeasurable functions $v_k \in X$, $k = 1, 2, \ldots, 2^n$, such that

$$(1 - \varepsilon)\|c\|_p \leq \left\| \sum_{k=1}^{2^n} c_{n,k} v_k \right\|_X \leq (1 + \varepsilon)\|c\|_p \tag{16}$$

for any $c = (c_{n,k})_{k=1}^{2^n}$, where $\|c\|_p = \left(\sum_{k=1}^{2^n} |c_{n,k}|^p \right)^{1/p}$. Hence, in particular, it follows (with the notion $\frac{n}{\infty} = 0$) that

$$(1 - \varepsilon) 2^{n/p} \leq \left\| \sum_{k=1}^{2^n} v_k \right\|_X \leq (1 + \varepsilon) 2^{n/p}.$$

Let

$$a(t) = r^{-1} \left(\sum_{k=1}^{2^n} v_k \right)^* (t), \quad \text{where } r = \left\| \sum_{k=1}^{2^n} v_k \right\|_X.$$

Then $\|a\|_X = 1$ and $a_{n,k}$ are equimeasurable functions with $r^{-1} v_k$ for every $k = 1, 2, \ldots, 2^n$. Therefore,

$$\frac{1-\varepsilon}{1+\varepsilon} 2^{-n/p} \|c\|_p \leqslant \left\| \sum_{k=1}^{2^n} c_{n,k} a_{n,k} \right\|_X \leqslant \frac{1+\varepsilon}{1-\varepsilon} 2^{-n/p} \|c\|_p,$$

that is,

$$\frac{1-\varepsilon}{1+\varepsilon} \left\| \sum_{k=1}^{2^n} c_{n,k} \chi_{\Delta_{n,k}} \right\|_p \leqslant \left\| \sum_{k=1}^{2^n} c_{n,k} a_{n,k} \right\|_X \leqslant \frac{1+\varepsilon}{1-\varepsilon} \left\| \sum_{k=1}^{2^n} c_{n,k} \chi_{\Delta_{n,k}} \right\|_p.$$

Hence, by assumption (15), we have

$$C_\varepsilon^{-1} \left\| \sum_{k=1}^{2^n} c_{n,k} \chi_{\Delta_{n,k}} \right\|_p \leqslant \left\| \sum_{k=1}^{2^n} c_{n,k} \chi_{\Delta_{n,k}} \right\|_X \leqslant C_\varepsilon \left\| \sum_{k=1}^{2^n} c_{n,k} \chi_{\Delta_{n,k}} \right\|_p, \qquad (17)$$

where $C_\varepsilon = C(1+\varepsilon)/(1-\varepsilon)$.

Let $1 \leqslant p < \infty$. If X is a separable r.i. space, then (17) implies that $X = L_p$. In the case when $X = X''$ it is sufficient to consider r.i. spaces $X \neq L_\infty$. Then X^0 satisfies (15) and so $X^0 = L_p$. Hence, $X' = (X^0)' = L_{p'}$ and $X = X'' = (L_{p'})' = L_p$.

Let $p = \infty$. Suppose that there is a function $x \in X \backslash L_\infty$. Then from (17) we obtain

$$\|x\|_X \geqslant \left\| \sum_{k=1}^{2^n} x^*(k2^{-n}) \chi_{\Delta_{n,k}} \right\|_X \geqslant C_\varepsilon^{-1} \left\| \sum_{k=1}^{2^n} x^*(k2^{-n}) \chi_{\Delta_{n,k}} \right\|_\infty = C_\varepsilon^{-1} x^*(2^{-n}).$$

Since $x \notin L_\infty$ it follows that $\lim_{n \to \infty} x^*(2^{-n}) = \infty$. This contradiction shows that $X \subset L_\infty$. The reverse imbedding is always true.

(v) $\Rightarrow$ (iii): This follows from the estimate of the norm of natural projections in L_p space

$$\|P_{n,a,f}\|_{L_p \to L_p} \leqslant \|a\|_p \|f\|_{p'}.$$

(iii) $\Rightarrow$ (i): By definition of the operators $P_{n,a,f}$ we have that $X \in \mathcal{N}_0$. We want to show that also $X' \in \mathcal{N}_0$. For all $x \in X$ and $y \in X'$

$$\int_0^1 P_{n,a,f} x(t) y(t) \, dt = \sum_{k=1}^{2^n} 2^n \int_{\Delta_{n,k}} f_{n,k}(s) x(s) \, ds \int_{\Delta_{n,k}} a_{n,k}(t) y(t) \, dt = \int_0^1 P_{n,f,a} y(s) x(s) \, ds.$$

Therefore, the conjugate operator $(P_{n,a,f})^*$ to $P_{n,a,f}$ is $P_{n,f,a}$ on the space X'. Since X' is isometrically imbedded in X^* the last equality implies that the operators $P_{n,f,a}$ are uniformly bounded, and so $X' \in \mathcal{N}_0$. The proof is complete. $\quad \square$

4. Additional remarks and results

First we describe the difference between the cases on $[0,1]$ and $[0,\infty)$. Let $X[0,\infty)$ denote an r.i. space on $[0,\infty)$ and $X = \{x \in X[0,\infty) : x(t) = 0 \text{ for } t > 1\}$ the corresponding r.i. space on $[0,1]$. We use here also the notion $X[0,\infty) \in \mathcal{N}$ from the paper [HS99, p. 56] (cf. also our explanation after Definition 2). Let us present examples showing that no one of the following statements:

(i) $X[0,\infty) \in \mathcal{N}$,
(ii) $X \in \mathcal{N}_0$

implies the other one, in general.

Example 3. The Orlicz space $L_{\Phi_p}[0,\infty)$, where $\Phi_p(u) = e^{u^p} - 1$, $1 < p < \infty$, belongs to the class $\mathcal{N}$. On the other hand, the lower Boyd index $\alpha_{L_{\Phi_p}}$ of L_{Φ_p} on $[0,1]$ equals 0 and so $\mathcal{M}(L_{\Phi_p}) = L_\infty$. Therefore, by Theorem 4, $L_{\Phi_p} \notin \mathcal{N}_0$.

Example 4. Consider the function

$$\varphi(t) = \begin{cases} t^\alpha & \text{if } 0 \leqslant t \leqslant 1, \\ t^\alpha \ln^{-\beta}(t + e - 1) & \text{if } 1 \leqslant t < \infty, \end{cases}$$

where $0 < \beta \leqslant \alpha < 1$. Then φ is a quasi-concave function on $[0,\infty)$, i.e., $\varphi(t)$ is increasing on $[0,\infty)$ and $\varphi(t)/t$ is decreasing on $(0,\infty)$. Let $\tilde{\varphi}$ be the smallest concave majorant of φ. Then

$$\sup_{0 < t \leqslant 1, n \in \mathbf{N}} \frac{\tilde{\varphi}(nt)}{\tilde{\varphi}(n)\tilde{\varphi}(t)} = \infty.$$

In fact, for every $n = 1, 2, \ldots,$ we can choose $t \in [0,1]$ such that $nt < 1$. Then

$$\frac{\tilde{\varphi}(nt)}{\tilde{\varphi}(n)\tilde{\varphi}(t)} \geqslant \frac{1}{4} \frac{\varphi(nt)}{\varphi(n)\varphi(t)} = \ln^\beta(n + e - 1) \to \infty \quad \text{as } n \to \infty.$$

This implies that the Lorentz space $\Lambda_{\tilde{\varphi}}[0,\infty) \notin \mathcal{N}$ (see [HS99, Theorem 4.1]). At the same time for $\Lambda_{\tilde{\varphi}} = \Lambda_{t^\alpha} = L_{p,1}$ with $p = 1/\alpha$ on $[0,1]$ we have, by (3), that $\mathcal{M}(\Lambda_{\tilde{\varphi}}) = \mathcal{M}(L_{p,1}) = L_{p,1} = \Lambda_{\tilde{\varphi}}$, and, by Theorem 4, $\Lambda_{\tilde{\varphi}} \in \mathcal{N}_0$.

The reason of non-equivalences (i) and (ii) is coming from the fact that the dilation operator σ_t in r.i. spaces on $[0,\infty)$ does not satisfy an equation of the form

$$\|\sigma_t x\|_{X[0,\infty)} = f(t)\|x\|_{X[0,\infty)}, \quad \text{for } x \in X[0,\infty) \text{ and for all } t > 0.$$

If this equation is satisfied, then the function $f(t)$ is a power function $f(t) = t^\alpha$ for some $\alpha \in [0,1]$ and then the above statements (i) and (ii) are equivalent. This observation allows us to improve, for example, Theorem 4.2 from [HS99]: if

$1 < p < \infty$ and $1 \leqslant q \leqslant \infty$, then

$$N(L_{p,q}[0, \infty)) = L_{p,q} \Leftrightarrow 1 \leqslant q \leqslant p.$$

We can characterize $N(L_{p,q}[0, \infty))$ for $1 < p \leqslant q < \infty$.

Theorem 8. *If* $1 < p \leqslant q < \infty$, *then* $N(L_{p,q}[0, \infty)) = L_p$.

Proof. Let $a \in N(L_{p,q}[0, \infty))$. The spaces $L_{p,q}[0, \infty)$ are separable for $q < \infty$. Therefore, similarly as in the proof of Theorem 2, we can show that

$$\left\| \sum_{k=1}^{n} c_k a_k \right\|_{L_{p,q}[0,\infty)} \leqslant C \left\| \sum_{k=1}^{n} c_k \chi_{[k-1,k)} \right\|_{L_{p,q}[0,\infty)} \tag{18}$$

for all $c_k \in \mathbf{R}$, $k = 1, 2, \ldots, n$, $n = 1, 2, \ldots$. Since

$$\|\sigma_t x\|_{L_{p,q}[0,\infty)} = t^{1/p} \|x\|_{L_{p,q}[0,\infty)} \quad \text{for } x \in L_{p,q}[0, \infty) \text{ and all } t > 0, \tag{19}$$

it follows that

$$\left\| \sum_{k=1}^{n} c_k a_k^n \right\|_{L_{p,q}} \leqslant C \left\| \sum_{k=1}^{n} c_k \chi_{\left[\frac{k-1}{n}, \frac{k}{n}\right)} \right\|_{L_{p,q}},$$

and, by Theorem 1.14 in [A97] together with property (c), we obtain $a \in L_p$.

Conversely, if $a \in L_p$ then, by using property (c), Theorem 1.14 in [A97] and equality (19), we get (18) for all n of the form 2^m, $m = 1, 2, \ldots$. The space $L_{p,q}$ has the Fatou property, thus passing to the limit, we obtain

$$\left\| \sum_{k=1}^{\infty} c_k a_k \right\|_{L_{p,q}[0,\infty)} \leqslant \left\| \sum_{k=1}^{\infty} c_k \chi_{[k-1,k)} \right\|_{L_{p,q}[0,\infty)}.$$

Next, arguing as in the proof of Theorem 2 (see also [HS99, Theorem 2.3]) we obtain $a \in N(L_{p,q}[0, \infty))$. □

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