WAVEGUIDEFINITEELEMENTSAPPLIED
ON A CAR TYRE

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ABSTRACT

Structures acting as waveguides are quite common with examples being, construction beams, fluid filled pipes, rails and extruded aluminium profiles. Curved structures like car tyres and pipe-bends may also be considered as waveguides. Wave solutions in such structures may be found by a method called the Waveguide Finite Element Method or WFEM. This method uses a finite element approach on the cross-section of a waveguide to model the vibro–acoustic response as a set of linear coupled one dimensional wave equations.

In this thesis six novel waveguide finite elements are derived and validated. These elements are, straight and curved pre-stressed , orthotropic or anisotropic shell elements, straight and curved fluid elements, and straight and curved fluid-shell coupling elements.

Forced response and input power calculations for infinite and periodic waveguides are presented. The assembled waveguide models can also serve as input for the Super Spectral Finite Element Method, which enables forced response calculations for more complex boundaries. Furthermore, several properties of damped and undamped wave solutions are investigated.

Finally, a car tyre model, encompassing for the highly anisotropic material and the air cavity inside the tyre is set forth. A number of forced response calculations for this model are presented and compared with measurements with good agreement.

Key words

wave equation, wave solution, waveguide, finite element, spectral finite element, tyre noise, tyre vibration, input power, shells, pre–stress, fluid–shell coupling, axi–symmetric, two–and-half-dimensional
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PRESENTED WORK

The content of this thesis has partly been presented at,

- Nine workshops within the RATIN research project.
- Novem conference 2000, Lyon France, C-M Nilsson and S Finnveden, 'Input of vibration energy into waveguides using FE-formulations'.
- Internoise 2002, Dearborn, MI, USA, C-M Nilsson and S Finnveden, 'Tyre Vibration Analysis with Conical Waveguide Finite Elements'.

Furthermore, Chapter 7, 'Input power to waveguides calculated by a finite element method', which is made together with Dr. Svante Finnveden, will be submitted for publication to the 'Journal of Sound and Vibration'.

The plate strip elements in Chapter 2 are also used in the paper 'A spectral super element for modelling of plate vibrations: part 1, general theory.' by, F. Birgersson , S. Finnveden and C-M Nilsson, which is accepted for publication in the Journal of Sound and Vibration, subject to minor revision.
NOMENCLATURE

General remark

The following list of symbols is made to ease the reading of this thesis. The separate chapters contain explanations of symbols as they first appear in the text. Also, since Chapter 7 is mostly written by Dr. Svante Finnveden, the notation there differs from other chapters.

List of Symbols

∗ symbolize complex conjugate
T symbolize transpose
H T symbolize transpose and complex conjugate
δ symbolize variation
^ symbolize frequency domain.
′ symbolize time derivative.
′′ symbolize quantity per unit volume.
′′ symbolize quantity per unit area.
′′ symbolize quantity per unit length.
t time
ω angular frequency
$L_{s0}$ shell Lagrangian
$L_s$ shell Lagrangian without fluid coupling.
$L_{s\omega}$ shell Lagrangian without fluid coupling for frequency $\omega$.
$L_{f0}$ fluid Lagrangian.
$L_f$ fluid Lagrangian without shell coupling.
$U_s$ potential energy in shell.
$U_f$ potential energy in fluid.
$T_s$ kinetic energy in shell.

$T_f$ kinetic energy in fluid.

$\delta W_s$ virtual work on shell

$\delta W_{s,f}$ virtual work on shell from fluid

$\delta W_{s,ext}$ virtual work on shell from external forces

$\delta W_{s,\text{loss}}$ virtual work on shell from dissipative forces

$\delta W_f$ virtual work on fluid

$\delta W_{f,s}$ virtual work on fluid from shell

$\delta W_{f,ext}$ virtual work on fluid from external forces

$\delta W_{f,\text{loss}}$ virtual work on fluid from dissipative forces

$\delta B_c$ coupling term between fluid and shell

$p$ pressure

$\psi$ velocity potential in fluid

$c_f$ sound speed in unbounded fluid

$\rho_f$ density of fluid at equilibrium

$\rho_a$ acoustic change of density in fluid

$\rho_s$ density of shell

$m''$ shell mass per unit area

$f$ external forces on shell per unit area in time domain

$\hat{f}$ external forces on shell per unit area in frequency domain

$a_{kl}$ waveguide element matrices for shell corresponding to potential energy.

$m_2$ waveguide element matrix for shell corresponding to kinetic energy.

$b_{kl}$ waveguide element matrices for fluid corresponding to kinetic energy.

$n_2$ waveguide element matrix for fluid corresponding to potential energy.

$m_1$ waveguide element matrix for fluid–shell coupling.

$A_{kl}$ assembled matrix from $a_{kl}$ and/or $b_{kl}$

$M_2$ assembled matrix from $m_2$ and/or $n_2$

$M_1$ assembled coupling matrix
$E_x$ Young’s modulus in $x$-direction

$E_y$ Young’s modulus in $y$-direction

$\nu_x$ Poisson’s ratio $xy$

$\nu_y$ Poisson’s ratio $yx$

$\psi$ velocity potential

$\hat{\psi}_i$ velocity potential at node–line $i$

$\hat{\psi}$ $\begin{bmatrix} \psi_1 & \psi_2 & \psi_3 \end{bmatrix}^T$ velocity potential at element corners

$\xi, \eta, \zeta$ Co-ordinates for thin shell

$x, y, z$ Cartesian co-ordinate for straight waveguide

$s, \theta, z$ Conical co-ordinates

$\alpha$ Half cone angle

$u, v, w$ displacements of shell

$\xi, \eta, \zeta$ co–ordinates of arbitrary shell

$s, \theta, z$ conical co-ordinates

$x, R, \theta$ cylindrical co–ordinates

$\phi$ rotation about $x$ or $\theta$ -axis in shell

$u_s = \mathbf{u} \begin{bmatrix} u & v & w \end{bmatrix}^T$ displacement of shell

$u_f$ velocity of fluid.

$n$ normal out of fluid.

$Q_i$ generalized force

$q_i$ generalized co-ordinate

$\eta$ viscoelastic damping

$\eta_v$ viscous damping

$\mu$ scaling constant for fluid velocity potential

$[D]$ material stiffness matrix of shell

$D_{ij}$ entry of $[D]$

$\epsilon$ strain varying over shell thickness

$\epsilon$ strain at shell midsurface
\( \varepsilon_l \)  linear part of \( \varepsilon \)
\( \varepsilon_q \)  quadratic part of \( \varepsilon \)
\( \beta \)  linear functional defined from \( \varepsilon_q \)
\( \kappa \)  wavenumber
\( \gamma \)  Non–dimensional ‘polar’ wavenumber

\([M]\)  Mass matrix
\([G]\)  General damping matrix

\( L \)  with of shell element
\( a \)  half width of shell element
\( h \)  thickness of shell
\( R \)  Radius for curved elements

\( N_p \)  Test- and shape–function for in plane motion
\( N_b \)  Test- and shape–function for out of plane motion
\( N_f \)  Test- and shape–function for fluid

\( \vartheta \)  dummy variable in shape–functions for shells

\( \xi_1, \xi_2, \xi_3 \)  triangular coordinates

\( A \)  Cross section area of fluid element or area of shell
\( V \)  Volume of fluid or shell

\( \hat{a}, \hat{b}, \hat{c}, \hat{d} \)  Amplitudes of rigid body motion

\( R_0 \)  Radius of tyre
\( l \)  Length of lever creating torque excitation on tyre

\( \sigma \)  Stress in shell
\( \sigma_0 \)  Pre–stress in shell

\[ [N_s N_\theta N_{s\theta}] = h\sigma_0 \]

\( \Phi \)  Assembled degrees of freedom for shell
\( \Psi \)  Assembled degrees of freedom for fluid

\( \mathbf{W} = \begin{bmatrix} \Psi & \Phi \end{bmatrix}^T \)  Assembled degrees of freedom for fluid–shell system
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INTRODUCTION

1 Background

Acoustics is a branch of science concerning the study of phenomena leading to sound perception. Thus, the human perception of sound, having a frequency span of about 20 to 20 000 Hz plays an important role in acoustics. Also, since there are many causes leading to sound perception, acoustics is a very broad science, spanning such diverse disciplines as meteorology and psychology, [1].

To classify a study of acoustics, one may first consider the primary mechanism leading to an excitation of an acoustic system. Secondly, one may consider transmission in this system and, thirdly, the subsequent radiation from this system into some other system. This procedure can be repeated so that the radiation out of one system serves as the input into another and so forth until the sound finally reaches the human ear.

Due to the wide frequency span for human hearing, it is often found that at least one characteristic length of such a system is much longer than the characteristic length between two nodes in an oscillatory pattern in the system. The implications of this are twofold. First, numerical methods commonly used for low frequency analysis, such as the finite element method (FEM), become very costly in terms of algebraic manipulations. Secondly, and possibly more important, the physical interpretations of computational results from methods, such as FEM, are often quite difficult. These
implications sets acoustics apart from other branches of engineering science where the characteristic lengths of the system are of the same order as the oscillatory pattern.

To overcome the problems of dealing directly with oscillatory patterns, a number of alternative methods have been developed. One example of an alternative method that have found wide use is room acoustics, e.g. [2], where acoustic energy in air is considered. Another commonly used method is 'Statistical Energy Analysis' (SEA), e.g. [3]. Similar to room acoustics, SEA considers flow of acoustic energy but is expanded to include energy flow between multiple systems and structures. Other methods consider waves as a basis functions for the oscillatory pattern, see e.g. [4] and [5]. Such methods are best suited for large systems in which there are uniform properties in at least one direction.

In this thesis, systems where all properties are constant through one co-ordinate are considered. These systems are henceforth referred as waveguides. A definition of a waveguide found in a technical dictionary, [6], reads,

_Broadly: A device which, constrains or guides the propagation of electromagnetic waves along a path defined by the physical construction of the waveguide._

For the purpose of this thesis, the word 'electromagnetic' is replaced by 'mechanical'. Furthermore, all geometrical and material properties through the waveguide are required to be constant, and it should be noted, this also includes the curvature along the path of propagation.

In structural acoustics, according to the above definition, a waveguide may be considered as a generalised beam in which many different wave types can exist. Wave equations for beams have been used extensively and have found many applications in acoustics. Different beam theories yield wave equations for longitudinal waves,
torsional waves, shearing waves and flexural waves for Euler and Timoshenko beam theories. Wave equations for curved beams and plane fluid waves in pipes are other examples.

All of the wave equations mentioned above are, however, restricted by an upper frequency where the wavelength is comparable to at least one of the dimensions of the cross-section. For higher frequencies more complex shaped waves start to propagate. Consequently for higher frequencies there is a demand for more advanced wave equation models. For some special cases such as, waves in sandwich beams, [7], coupled elastic layers [8] and [9] and plate strips [10, Chapter IV-3.h.] and [9], analytical models have been derived. The analytical models suffer from a drawback in that they can only be used for simple geometries. Deriving and using 'high frequency' wave equations for more general geometries is the topic of this thesis. With respect to the methods developed in this thesis, this technique will be called the 'Waveguide Finite Element Method' (WFEM).

2 Waveguide Finite Element Methods

2.1 Waveguide finite elements

Consider a waveguide and a variational formulation for some physical field described with linear theory. Upon the recognition of constant physical properties along a single direction, separability of this field is possible. As an example, consider time harmonic displacements in three directions, \( u, v \) and \( w \), then,

\[
\begin{bmatrix}
  u(x, y, z, t) \\
  v(x, y, z, t) \\
  w(x, y, z, t)
\end{bmatrix}
= 
\begin{bmatrix}
  \xi(y, z) \cdot \tilde{u}(x) \\
  \psi(y, z) \cdot \tilde{v}(x) \\
  \zeta(y, z) \cdot \tilde{w}(x)
\end{bmatrix}
\cdot e^{i\omega t}
\]  

(1)

where, \( t \) and \( \omega \) are the time and the angular frequency respectively, \( x \) is the direction.
of propagation and $y$ and $z$ are co-ordinates of the cross section. The functions, $\xi$, $\psi$ and $\zeta$ define the displacement dependence with respect to the cross-section’s geometry, and $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$ are functions defining the $x$-dependence of the displacement.

The ansatz defined by equation (1) is subsequently used in the equations describing the motion and hence a two-dimensional finite element model is made over the cross section co-ordinates $y$ and $z$. One example is the straight waveguide assembled from plate strip elements, as seen in Figure 1.

![H-beam, modelled with waveguide FE](image)

Figure 1: H-beam, modelled with waveguide FE

Straight waveguide finite elements were, most likely first, formulated by Alaami, [11], in 1973. Alaami’s elements are formulated for isotropic solid materials. Use of these elements may be found in reference [12], where waves in railway rails are examined. Moreover, anisotropic elements is described in [13], where waves in anisotropic H-beams are considered. Twisted beams are examined in [14] and [15]. Plate elements
are described by Gavric in reference [16], where they are used for cylinders. Plate elements were also used by Orrenius and Finnveden for stiffened plate structures [17]. Prestressed orthotropic plates have been presented by the author in reference [18]. These elements have been utilized in [19] and [20], for a method denoted the 'Super Spectral Element Method' (SSEM).

Finite elements utilizing an axis of symmetry have been known at least since 1965, [21]. Published work using such elements include church bells, [22], and car tyres, [23]. These elements, sometimes referred to as axi-symmetric finite elements, are widely used both for dynamic and static problems, and can be found in commercial software, for instance ANSYS and ABAQUS.

In the derivations of axi-symmetric elements circumferential modes are assumed, see [24]. The resulting equations in [24] are similar to those for the strong form of the curved waveguide finite elements presented in this thesis. One characteristic that sets the curved waveguide finite elements of this thesis apart from conventional axi-symmetric finite elements is that a 'weak form', described in the following section, is explicitly presented here. This is important since this weak form implicitly contains information about the natural boundary conditions at the ends of the waveguide, and is thus essential for the formulation of spectral super elements.

Chapters 1 to 6 in this thesis are concerned with new waveguide finite elements as follows. Chapter 1 derives a variational formulation for fluid–shell coupling and shows some beneficial properties following from this particular formulation. Chapters 2 to 4 report on six new waveguide finite elements. These are, anisotropic, prestressed shell–elements for straight and curved waveguides, fluid–elements for straight and curved waveguides and coupling–elements for fluid–shell coupling for straight and curved waveguides. The derivation is made so that both a 'weak' and a 'strong' form of the equations of motion (with respect to the co-ordinate along the waveguide) for these elements are obtained. In Chapter 5 validations of the new elements are made. Dispersion relations for a fluid–filled pipe–bend has not been
found in the open literature, and are thus considered in Chapter 6.

After assembling waveguide finite elements, the wave equation for a straight shell structure with contained fluid is given by,

\[
\int \sum_{k=0}^{2} \sum_{l=0}^{2} \frac{\partial^k}{\partial x^k} \delta W^H A_{kl} \frac{\partial^l W}{\partial x^l} + i\omega \delta W^H M_1 W - \omega^2 \delta W^H M_2 W \, dx = \int \delta W^H F \, dx ,
\]

where the vector \( W(x) \) contains the degrees of freedom in the assembled fluid–shell system and the vector \( F \) describes the external forces acting on the shell. For the purpose of this thesis, equation (2) is denoted as the ‘weak form’ of the wave–equation.

The matrices \( A_{kl} \), in equation (2) originate from consideration of expressions for the potential and kinetic energies in the shell and fluid, respectively. The matrix \( M_2 \) originates from consideration of expressions for the potential and kinetic energies in the shell and fluid, respectively. The matrix \( M_1 \) originates from the coupling condition between the fluid and the shell, see Chapters 1 and 4. Dissipative forces, i.e. damping, can be included in this equation by allowing the matrices \( A_{kl} \) and \( M_2 \) to be complex valued.

By repeated integration by parts of equation (2), while neglecting the natural boundary terms, and subsequent usage of calculus of variation, the weak form is transformed into, what in this thesis is referred to as, the ‘strong form’ of the wave–equation,

\[
\left[ \sum_{j=0}^{4} K_j \frac{\partial^j}{\partial x^j} + i\omega M_1 - \omega^2 M_2 \right] W = F ,
\]

where,
\[ K_j = \sum_{k=0}^{2} \left( (-1)^{j-k} A_{(j-k)k} \right), \quad k \leq j \leq 4. \]  

For curved waveguides the differentiation with respect to \( x \) in equations (2) and (3) are replaced by differentiation with respect to \( \theta \), but the rest of the notations is kept.

Wave solutions to the strong form of the wave equation for straight waveguides are found by the wave ansatz,

\[ \mathbf{W}(x) = \hat{\mathbf{W}} e^{-i\kappa x} \]  

where, \( \hat{\mathbf{W}} \) defines the wave-shape and \( \kappa \) is the, possibly complex valued, wavenumber defining the wave propagation and decay. The wave solutions are given by using the ansatz of equation (5) in equation (3), yielding the twin valued eigenproblem,

\[ \left[ K(\kappa) + i\omega M_1 - \omega^2 M_2 \right] \hat{\mathbf{W}} = 0, \]  

where

\[ K(\kappa) = \sum_{j=0}^{4} K_j (-i\kappa)^j, \]  

which can be solved either for a given frequency, \( \omega \), or for a given wavenumber, \( \kappa \).

For curved waveguides, the wavenumbers \( \kappa \) are replaced with the 'polar' wavenumbers \( \gamma \).

The strong form of the wave equation given by equation (3) is used for all calculations in this thesis. It must however be emphasized that the weak form is very useful in many cases. Particularly the so called Super Spectral Elements are conveniently derived from using both the weak and the strong forms, see Section 2.2 in this introduction and references [20], [19] and [25]. To the authors knowledge previous work on waveguide finite elements has been derived exclusively for the strong form.
2.2 Forced response

Forced response solutions for waveguide finite element models can be handled in several different ways. In the following, three methods for forced responses are explained. Here, the first two methods use forces concentrated along the waveguide. These methods may however also include distributed forces, that can be described with a Fourier series or Fourier transform for the respective method.

Modal summation

Modes for undamped structures with rotational symmetry are given by the waves with real integer ‘polar’ wavenumbers, \( \gamma_p = 0, \pm 1, \pm 2, \pm 3, \ldots \), see Chapters 5, 6 and 8. In a modal summation the solution is written as,

\[
W(\theta) = \sum_{p=-\infty}^{+\infty} \sum_{m=1}^{N} a_{p,m} \hat{W}_{p,m} e^{-i\gamma_p \theta}
\]  
(8)

where \( \hat{W}_{p,m} \) are eigenvectors found from the equivalent of equation, (6) for curved waveguides and the amplitudes \( a_{p,m} \) are sought. To find the modal amplitudes \( a_{p,m} \), the orthogonality relation,

\[
\int_0^{2\pi} e^{-i\gamma_p \theta} e^{i\gamma_q \theta} d\theta = 2\pi \delta_{p,q}
\]  
(9)

can be used. For two cases orthogonality relations for the eigenvectors, \( \hat{W}_{p,m} \), can also be used. Firstly, when the coupling matrix, \( M_1 \) is zero and damping is proportional to \( K(\kappa) \) and to \( M_2 \), i.e. there are possibly two independent damping coefficients. Secondly, when \( M_1 \) is non-zero, for which the two damping coefficients must equal. For the second case, equation (6) is rewritten into an orthogonalisable eigenproblem as detailed in Chapter 1, but with damping included. Orthogonality between eigenvectors is applicable for systems with proportional damping only. As
a consequence the forced response calculations in Chapter 8, for a car tyre filled with air, do not employ this orthogonality between eigenvectors.

Orthogonality between eigenvectors is, however used in Chapter 7 where a straight waveguide model made from thin plate strip elements is derived. For this solution shear diaphragm boundary conditions are assumed at the ends of the waveguide so that no near–fields are present. The input power from this modal solution is subsequently averaged over both frequency and excitation position. This yields a closed form expression for calculating average input power into waveguide–FE models. The same approach used for calculating averaged input power into a plate is used in 'Structure borne sound', [10, Chapter IV 4]. There it is concluded that the result is identical to that for an infinite plate.

**Infinite waveguides**

The modal approach described above is useful for rotational symmetry and for waveguides where near–fields are negligible at the ends. For infinite waveguides a different approach based on the Fourier transform may be used. For the force,

\[ F = \hat{F} \delta(x) , \]  

the spatial Fourier transform, from the \( x \)–domain, to the wavenumber domain, i.e. the \( \kappa \)–domain, of equation (3) yields,

\[ \hat{W}(\kappa) = \left[ \sum_{j=0}^{4} K_j (\kappa)^j + i\omega M_1 - \omega^2 M_2 \right]^{-1} \hat{F} , \]  

where \( \hat{W}(\kappa) \) is the spatial Fourier transform of the response \( W(x) \). The response \( W(x) \), is found by the inverse Fourier transform here defined as,
\[ W(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{W}(\kappa) e^{-ikx} d\kappa. \] (12)

Equation (12) can be solved by residue calculus while replacing the integral limits \( \pm \infty \) with an integral curve in the complex plane. Two such curves are shown in Figure 2.

Figure 2: Paths of integration in complex plane

The integral curve for the upper half plane can only be used for \( x \leq 0 \). If \( x > 0 \) the integrand in equation (12) will not approach zero along the path \( \gamma_U \) in the upper half of the complex plane as \( R \to \infty \). Similarly the integral curve in the lower half plane can only be used for \( x \geq 0 \). Consequently there is one set of wave solutions for \( x \leq 0 \) and another for \( x \geq 0 \).

It may falsely be assumed that solutions for large positive \( x \) consist only of waves with positive real parts of the wavenumber, \( \kappa \), i.e. positive phase velocity, see e.g.
However, propagating wave solutions for $x > 0$ are given by propagating waves with positive group velocity and such waves may also exist for negative phase velocity. The proof of this behaviour is developed and exemplified in Chapter 7, where a structure built up by straight thin–walled plate strip elements is examined. These waves are however more common for curved structures, see e.g. dispersion relations developed in Chapters 5, 6 and 8.

Simple poles or poles for which eigenvectors are linearly independent are now assumed for the Fourier transformed solution of equation (11). The residues at the poles, $\kappa = \kappa_p$ are then calculated with equation (13), see for instance reference [27, Section II.2 Theorem 3] or [28].

$$\lim_{\kappa \to \kappa_p} (\kappa - \kappa_p) \hat{W}(\kappa) = \frac{\hat{W}_L^T \hat{F}}{\hat{W}_p L \left[ \frac{\partial (\sum K_i (-i \kappa)^r)}{\partial \kappa} \right]_{\kappa_p} \hat{W}_p R},$$

(13)

where $\hat{W}_L$ and $\hat{W}_R^T$ are the respective left and right eigenvectors for the eigenproblem (6) evaluated at the wavenumber $\kappa = \kappa_p$.

In Chapter 7, evaluation of input power using this approach is compared with an averaged modal summation, as described in the previous section. For waveguides with one degree of freedom on their cross-section this is also made in reference [26]. In the limit of zero damping, it is shown in Chapter 7, the two methods are identical. It is also shown that at the waves ‘cut–on’ frequencies the absence of damping yields infinite input power.

**Super spectral elements**

'Super Spectral Elements', (SSE), are derived by using wave solutions, given from the eigenproblem in equation (6), as test and shape functions in equation (2). The remaining integral over the length of the waveguide is made analytically, which makes the approach efficient, [19]. Furthermore, at the ends, the spectral elements
can be coupled to other spectral elements or to regular finite elements. Forces at the ends of the spectral elements can also be included as can distributed forces in the form of travelling pressure waves. However, SSEM is not treated further in this thesis. It is however an essential part of the motivation behind much of the presented work since the derivation of these elements start by using equations (2) to (6). References on SSEM are [20], [19] and [25]. For waveguides not described with WFEM, the corresponding method is referred to as the Spectral Finite Element Method, (SFEM), and is described in references, [29] and [30]. An equivalent to SFEM is the Dynamic Stiffness Method, (DSM), see [31], [32] and [33], which uses the strong form combined with natural boundary conditions to obtain the same equations as SFEM.

Summary of Waveguide Finite Element Methods

Figure 3 summarize the different methods related to Waveguide Finite Elements and how they can be obtained.
Figure 3: Relations between different theories. Thick borders indicate subjects treated in this thesis. Shaded boxes indicate methods to calculate forced response, and 'B.C.' represents boundary conditions.
3 Car tyre noise

Traffic has been a major environmental noise polluter for over half a century. Within the European Union and several other countries, traffic noise for new vehicles has been, and still is, regulated by the 70/157/EEC and ECE 51 regulations. These regulations are based on the ISO 362 testing procedures, originally written in the early 1960’s. ISO 362 stipulates full throttle acceleration over the test section, which is referred to as ‘pass by’ testing.

Since the first European regulations on traffic noise were implemented in 1970, the allowed A-weighted noise level has dropped by 8 dB(A), for passenger cars. During the same period traffic noise levels measured in-situ have hardly dropped at all, [34, pages 35 and 38]. This large discrepancy can not exclusively be explained with an increase in traffic. Consequently, the answer must be sought in the discrepancy between the noise level according to the regulations and the noise level emitted by vehicles in daily traffic. For higher speeds it has been found that tyres are the major contributors for traffic noise. Consequently regulations for car tyre noise has recently been implemented as the amendment 2001/43/EC to the 92/23/EEC directive for tyres, [35]. There, it is stipulated that tyre noise should be measured from a rolling vehicle, which is refereed to as ‘coast by’ testing. Furthermore the test stipulates, nearly new tyres mounted on a car chosen by the tyre manufacturer and a test–speed of 70 or 80 km/h, and with a load of $75 \pm 5\%$ of the maximum. Given that most tyre models are mounted on different cars, with different pressurization, different loads, and that car tyres both age and wear, it is possible that, future amendments to 92/23/EEC will not reduce car tyre noise in real traffic, even if stipulated noise levels are significantly lowered. Particularly, age and wear has been found to increase noise levels significantly, see [34, Chapter 10.8].

Also, vehicle interior noise due to tyres are becoming more important as other noise sources, such as engines, exhaust systems and gearboxes are better managed.
The increased concern for tyre noise has lead to a recent increase in tyre noise research. One example of this is the EU tyre noise research project RATIN, that has funded much of the work reported in this thesis.

Car tyre noise is due both to direct acoustic sources and to sources causing the tyre to vibrate. A study of vibration sources are given in reference, [34], these are:

- The impact when the tread changes direction at the road contact, see also [36] [37] and [38].
- The road roughness, see also [37] and [36].
- The excitation due to unevenness of the tyre, in particular the tread pattern, see also [39] and [40].
- The adhesion and frictional forces at the road contact causing stick–snap and stick–slip vibration respectively. This predominantly induces vibrations in the tread blocks transferring to the carcass of the tyre.

Direct acoustic sources include compression of air cavities in the tread pattern at the road contact, see for instance [41], and air turbulence around the tyre.

The relative importance of the different noise sources depends both on the road conditions and the tyre. Vibrational sources, however, dominate for rough roads and low frequencies. By considering vibrational sources exclusively, the tyre noise problem may be considered in three distinct stages: excitation, transmission and radiation.

Sound radiation from tyres has been treated in for instance [42], [43], [44], [45] and [46]. Worth noting with respect to radiation is the so called 'horn effect', e.g. [44], [45] and [46]. The 'horn effect' is an increase in transmission efficiency at the wedge between the road and the tyre close to the tyre–road contact. Thus, the horn effect
indicates that vibrations close to the tyre–road contact are, at least for external noise, of greater importance than those far from this contact area.

The vibration transmission problem, a model for which can be seen in Chapter 8, has been treated by several authors. Models include: simple orthotropic plate–strips, [47] and [9], assemblies of orthotropic plate strips [48], circular rings, [49], [50], [51] and [38], circular thin shells [52] and [53] beams, [54], coupled elastic layers [8] and finite element models [23] and [51]. References [23], [55], [56] and [52] also analyse the air cavities inside tyres, which is mainly interesting for structure borne noise into the vehicle passenger compartment. For practical applications it may be noticed that the flattening at the tyre–road contact divides the single air cavity resonance into two, see [56], and similar effects may also be attributed the tyre’s rotation.

The various models have different uses depending on for instance their frequency range or whether they are to be used for analysing structure–borne noise into passenger compartments or air–borne noise.

The structure borne noise is dominated by resonances for which rigid body motion of the rim is possible. There are two dominant modes for radial motion transmission to the hub. The first mode has a resonance frequency about 100 Hz and is seen in Figure 4. The second is the first air cavity mode which has a resonance frequency about 200-250 Hz. The first resonance of a wheel is found at around 300 Hz, so above this frequency rigid body motion of the wheel will not occur and a deformable wheel model could possibly be favour calculations concerning force transmission to the hub.

The air borne noise due to tyre vibrations, is important over a wide frequency range up to about 3 kHz. High frequency vibrations of the tread are virtually independent of the car tyre curvature. According to Larsson and Kropp, [8], this occurs above the first ring frequency, around 400 Hz. Furthermore, the local deformations in the
tread or tread–pattern becomes increasingly important above roughly 800 Hz. For this reason, a layered material model is developed in [8], this model is thus preferable for frequencies above, say 600 Hz.

The tyre model developed in Chapter 8 of this thesis aims at being applicable both for structure born noise and air borne noise from around 80 Hz up to around 800 Hz. Due to the nature of WFEM, the tyre model does not include any tread pattern. In the RATIN project a ‘slick’ tyre is considered and the excitation due to the tread pattern is included as a ‘roughness’ in the modelling of contact forces. The car tyre model in Chapter 8, includes: the curvature, the geometry of the cross–section, the air cavity inside the tyre, the pre–stress due to the inflation pressure, the anisotropic material properties, and the rigid body properties of the wheel, all modelled with the waveguide finite elements derived in Chapters 1 to 4.

An advanced model could also include: elastic properties of the wheel, rotation, coupling between wheel and vehicle, thick shell and solid elements. Elastic properties of the wheel have not been considered in the RATIN project. Rotation of the wheel causes an increased pre–stress due to centrifugal forces, Coriolis forces, and shifted wave–speeds due to the Doppler effect, see [53]. The increased pre–stress can be included in the modelled pre–stress due to the inflation. Coriolis forces have been shown to be of little importance for acoustic frequencies, [53]. The other developments listed above have been, at least partly, addressed by in the EU funded RATIN tyre research program.

The input data used for the model in Chapter 8 comes from an ABAQUS finite element model. This model use elements derived from solid and thick–shell theory rather than the thin–shell theory used for the waveguide finite elements derived in Chapter 2. As a result some modifications of the input data have been made. By comparing calculated results from tyre models with small changes of material data it was found that these tyre models are very sensitive to the flexural rigidity at the corner between the tread and the sidewall. The tyre damping is modelled as
frequency dependant and proportional to the structure’s stiffness matrices, whereas only a minor non-frequency dependent damping is attributed to the air.

The forced response from point forces are modelled with a modal expansion approach. Since non-proportional damping is used in the model there is no orthogonality relation between eigenvectors. The vector basis used for the expansion are the eigenvectors of the corresponding uncoupled problems. Since the structure and the air cavity inside a tyre are lightly coupled, a reduced number of eigenvectors may be used to enhance the speed of the numerical calculations.

The forced response is calculated for radial excitation and responses at: four positions on the tread, one at the hub and one for the air cavity pressure at the rim. Furthermore point mobility from tangential excitation has also been calculated. These calculations are compared to corresponding measurements.

Measurements for radial point mobility, transfer function to air pressure and transfer mobility to the tyre rim were made by the author and are reported in Chapter 8. The other measurements were made at Chalmers Institute of Technology, Gothenburg, Sweden, and are reported in [9]. The first wheel resonance in the tyre-wheel system occurs at 297 Hz, which is less than half of the lowest resonance frequency for the wheel without the mass-loading of the tyre. This resonance may have some effect on the air cavity resonance, but very little effect on the tyre vibrations at the tread. Also, it has been found that the response is fairly sensitive to small changes in the static air pressure that cause the larger part of the pre-stress in the tyre. Combining this with uncertainties of other parameters, it is concluded that it is very difficult to get truly deterministic models for car tyres in traffic conditions.

The proposed tyre model yields very good results in the frequency range between 80 Hz up to about 800 Hz. At higher frequencies thin shell elements yield large errors as shearing across the thickness of the tread becomes more important. Damping and partly some other material data are, however, based on curve fitting rather
Figure 4: Mode of tyre–wheel assembly about 100 Hz

Figure 5: Assemblage of a tyre model from conical shell strips
than physical modelling. For these reasons, derivations of thick shell elements and solid elements have been developed within the RATIN project. A more physically correct damping model is also to be developed in future tyre research projects.

4 Thesis Summary

Broadly, this thesis is an attempt to explore aspects of the method denoted here as the 'Waveguide Finite Element Method' (WFEM). Waveguide finite elements yield equations of motion for systems with wave-propagation along a single dimension in which the structure is uniform.

Conical and straight waveguide finite element shell strips, are derived in Chapter 2. These elements model the physical properties of anisotropic or orthotropic material and pre-stress. For the conical shell strip elements the derivation becomes quite cumbersome and is thus handled by a symbolic mathematical computer code. The inclusion of pre-stress into the shell theory requires non-linear displacement to strain relations. It is shown that, for any such relation found in the literature and applied to a circular shell, some of the shell’s rigid body motion is impaired.

In Chapter 3, straight and curved waveguide finite elements used for fluids for which the motion is governed by the Helmholtz equation are derived. For simplicity, these elements have triangular cross-sections and use linear interpolation functions.

In Chapter 4, elements used for coupling the fluid and shell elements, described in Chapters 2 and 3 respectively, are derived. This derivation is based on the variational formulation for full fluid-shell coupling shown in Chapter 1. This formulation yields a coupling term equivalent to that for gyroscopic forces. In Chapter 1, it is shown how such gyroscopic coupled systems can be rewritten into systems for which orthogonal eigenvectors exist for propagating waves. Since a 2D regular FE model is a special case of a waveguide finite element model with wavenumber equal to zero,
it is concluded that these orthogonality relations also exist for regular FE models based on the same variational formulation.

Wave-equations derived from the waveguide finite elements in this thesis may be written either in a weak form or in a strong form. Since, the weak form is not usually given by authors dealing with waveguide finite elements, the work reported here is original. From the weak form the strong form is derived by integration by parts and calculus of variation.

In Chapter 5, validations of the elements derived in earlier chapters are shown. The validations are made by comparing wavenumbers or resonance frequencies for waveguide finite element models with, either theoretical models or tabulated values found in the literature. The validations are made for simple systems such as an orthotropic and pre-stressed plate strip, toroids, straight and curved rectangular ducts, and a straight fluid filled pipe. In Chapter 6, effects on wave-propagation due to the inclusion of a full fluid-shell coupling in a fluid filled pipe bend model is examined. The results of this study are that the curvature stiffens the shell’s ‘breathing’ motion so that, compared to a straight pipe, the fluid experiences something close to rigid boundaries. Although, the mass-loading from the fluid on the shell still has a significant effect comparable with that for a straight pipe. Furthermore, the dynamically uncoupled waves that appear in a straight fluid-filled pipe are often coupled for the curved pipe, and as a result dispersion curves for different wave types generally do not cross.

Chapters 5 and 6 examine propagating waves only. Sets of only propagating waves can be used for some cases of forced response calculations. This is so, either for waveguides in which boundary conditions at the ends of the waveguide are such that no near-fields occur, or for periodic systems such as systems with rotational symmetry. The case of no near-field boundaries is utilized in Chapter 7. The other case with rotational symmetry is used for the forced response calculation of the car tyre model in Chapter 8. To this end, the waveguide finite element calculations
resemble those of the finite strip and axi-symmetric elements found both in many books on finite elements and in commercial software.

The waveguide finite elements presented in this thesis are novel with respect to the theories they use and physical properties they model. They also have wider applications than other waveguide finite elements found in the literature as well as finite-strip elements and axi-symmetric elements. This is so because the elements in this thesis are given in a weak form associated with the wave equation. This enables the waveguide finite elements (WFE) to derive super spectral elements (SSE) that couple to, and often may replace, regular finite elements, to greatly improve the numerically efficiency of FE models. This is discussed further in [19].

Another option for forced response calculations arises when infinite waveguides are considered. For such cases a spatial Fourier transform over the length of the waveguide is made so the solution in the wavenumber domain may be easily obtained. The inverse Fourier transform is then solved by complex integration and residue calculus. It should be noted that this method may be used also for axi-symmetric elements implemented in commercial FE software. Although, to the authors knowledge, this has never been attempted.

In Chapter 7, the input power into a waveguide from a point force is calculated by two different approaches. The first is based on a modal summation, whereas the second approach relies on the spatial Fourier transform discussed above. Although the latter expression is deemed to be the more physical of the two, it is also the more costly in terms of the computational effort required. In the limit of zero damping the expression derived by spatial Fourier transform equals the expression derived from modal summation. The derivations in Chapter 7 also considers waves for which the signs of the phase and the group velocities are different. It is shown that on the positive axis from a point of excitation the group velocity of any wave has to be positive, whereas the phase velocity may be negative. Finally, in Chapter 7, the input power into a stiffener used in a railway car is calculated and compared to an
in situ measurement. The agreement is good, considering that several simplifying assumptions are made.

In Chapter 8, a car tyre is modelled with the curved elements described in Chapters 2 to 4. Thus the tyre’s vibrations are described with thin walled shell elements encompassing pre-stress and anisotropic material, presented in Chapter 2. The wheel is also modelled with these elements, whereas the air cavity on the inside of the tyre is modelled with the elements in Chapter 3 and the coupling between fluid and shell elements is achieved through the elements described in Chapter 4.

The material data for the tyre are given from a regular FE model used by the tyre supplier (Goodyear). This data is slightly modified, so that calculated results of radial point mobility agree well with corresponding measured results. The main differences in material data compared to the regular FE model are as follows. First, material data describing the cross dependence between in-plane and flexural motion are discarded for the WFE model. This may be motivated by the differences in shell theory used for the different models. Thick shell theory, allowing for shear across the thickness, are used in the regular model, whereas thin shell theory is used for the waveguide shell elements. Secondly, the rigidity close to the corner between the tyre sidewall and the central section is lower in the WFE model.

Furthermore, the wheel is free but modelled as rigid for the frequencies of interest here. For the real tyre–wheel assembly, this is violated at frequencies above and slightly below the first wheel resonance, measured at 297 Hz. This wheel resonance frequency is less than half of that calculated by the wheel supplier for the wheel alone. However, since a rigid wheel is stipulated, the wheel model is made very rigid. This, very rigid, wheel model results in what is probably numerical cancellation at low frequencies and rigid body motions in the entire tyre–wheel system are impaired. This is, however, not a problem for the motion and frequency regime of interest.

Damping is modelled by one frequency dependent part, proportional to the structure’s stiffness and one frequency independent part proportional to the air density.
The magnitude of the structural damping was estimated so that calculations and measurements of the radial point mobility agree well. The magnitude of the air cavity damping was estimated by the half-value-bandwidth at resonances.

Dispersion relations and the corresponding, undamped, wave shapes are calculated. It is found that most features in the measured point mobility can be explained by considering the dispersion relations and the shapes of the undamped waves together with some knowledge of the tyre structure. As a consequence the tyre response may be subdivided into several different frequency regimes. These regimes are governed by the following motions respectively: rigid body motion, 'two degree of freedom' systems, beam like resonances, cross sectional bending waves and higher order waves.

The proposed model poorly estimates the first and the last of these regimes. The first regime is poorly estimated due to the stiffness associated with rigid body motion for pre-stressed shell models. The last regime is poorly estimated since the thin shell theory used here can not encompass different motion across the tread thickness.

Subsequently, forced response calculations from point loads on the tread are presented. These calculations are based on a modal summation approach, however due to the non-proportional damping (the difference in damping between the structure and the air), the orthogonality between eigenvectors is lost. For radial excitation centered on the tread, radial response at the excitation, radial responses at circumferential positions 100mm, 200mm, and 300mm away from the excitation, the wheel motion in the direction of excitation, and the pressure inside the air cavity were calculated. Furthermore, results for a tangential point mobility are also presented. The calculations are compared with corresponding measurements with favourable results.

The thesis may be summed in the following four sentences. Several curved and straight waveguide finite elements for systems with different properties are presented. A number of properties of the resulting wave-equations are shown. Several means to solve forced response problems for waveguide finite elements are proposed and
explored. A tyre model, devised from the presented waveguide finite elements, is compared to measurements yielding high quality results.

References


Appendix: Related topics and future developments

Axi–symmetric FE and wave solutions in tyres

The strong form of equation (3) can be represented with axi-symmetric finite elements that are available in commercial FE software. Equation (13) shows that it is possible to find solutions with decaying waves in such structures if they are considered as infinite in length. Since a radially excited car tyre may be considered infinite above roughly 500 Hz, commercial software for axi–symmetric finite elements and wave solutions for assumed infinite tyre models seem to be a promising combination. For lower frequencies an option might be to assume periodic excitation only for angles $-2\pi N < \theta < +2\pi N$, where the number $N$ is such that waves travelling more than $N$ times around the tyre are negligible due to damping. Although, these approaches may be costly in terms of the computations required, they may provide a good physical insight.

Notes on numerical issues

Wave solutions are obtained from WFEM models by solving the twin valued eigenproblem,

$$
\left[ \sum_{j=0}^{4} K_j (-i\kappa)^j + i\omega M_1 - \omega^2 M_2 \right] \hat{W} = 0
$$

(14)

where $\kappa$ is the wavenumber, $\omega$ is the angular frequency and $\hat{W}$ represents the wave–shape. Equation (14) is either solved for a known wavenumber $\kappa$, or for a known frequency, $\omega$. Unless $M_1 = 0$ and $\kappa$ is known, a polynomial eigenproblem must be solved. The most used approach to solve polynomial eigenproblems is to expand the system, [57]. One approach, for a known wavenumber, is shown in Chapter 1. For a known frequency a small complication arises since the $K_4$ matrix in equation (14)
is rank deficient and in theory infinite values of $\kappa$ are expected. This is examined further in references [16] and [20]. The expanded system is subsequently solved with a standard eigensolver method, see [57] for a review.

If there are $N$ degrees of freedom in the system, a known wavenumber, $\kappa$, will give rise to a $2N$-size system for a fluid–shell coupled system and an $N$-size system for an uncoupled system. For a known frequency the expanded system will be of size $4N$ or slightly less, if the methods in [16] or [20] is used. Then, even with efficient iterative solvers, the calculations can be quite time consuming.

In the eigenproblems here, wavenumbers are sought for a multitude of frequencies, but often only for a few number of waves. Good approximations of the wavenumbers are given from the wavenumbers calculated at the previous frequency. Furthermore an even better approximation of the wavenumber is obtained by Taylor expansion so that,

\begin{equation}
\kappa (\omega_0 + \Delta \omega) \approx \kappa (\omega_0) + \frac{\partial \kappa}{\partial \omega} \bigg|_{\omega_0} \Delta \omega
\end{equation}

where, $\omega_0$ is the previous frequency, $\omega_0 + \Delta \omega$ is the current frequency, $\kappa (\omega_0)$ is the previously calculated wavenumber and $\kappa (\omega_0 + \Delta \omega)$ is the sought wavenumber. The group velocity,

\begin{equation}
c_g = \frac{\partial \omega}{\partial \kappa},
\end{equation}

may be evaluated analytically by differentiating equation (6), with respect to $\kappa$ and subsequent left multiplication by the left eigenvector, $\hat{W}_L$, corresponding to equation (6). In the absence of fluid–shell coupling, i.e. with $M_1$ equal to zero, the result is written,

\begin{equation}
\frac{\partial \omega}{\partial \kappa} = \frac{\hat{W}_L \left[ \frac{\partial \kappa}{\partial \omega} \right] \hat{W}}{2\omega \hat{W}_L [M] \hat{W}},
\end{equation}
see also Chapter 7 and reference [58].

The start approximation in equation (15) can be used as a shift before applying an iterative eigensolver. Since the start approximation usually is very good, the convergence radius of the eigensolver may be small. Hence, eigensolvers working on the un-expanded problem might be useful. Such eigensolvers are found in references [59], [60] and [61]. Further investigation of this methodology is, however, outside the scope of this thesis.
1 Governing equation

Consider a fluid surrounded partly by a shell and partly by a rigid boundary as seen in Figure 1. From Petyt, [1, eq.1.38], Hamilton’s principle applied on this fluid is written as,

$$\delta L^f_0 = \int_{t_1}^{t_2} \delta (U_f - T_f) - \delta W_f \, dt = 0,$$

where $\delta$ symbolizes first variation. $L^f_0$ is the Lagrangian for the fluid. $U_f$ is the potential and $T_f$ is the kinetic energy in the fluid. $\delta W_f$ symbolizes the virtual work on the fluid. The values $t_1$ and $t_2$ symbolize start and end times, respectively. The corresponding equation for the shell can be written as,

$$\delta L^s_0 = \int_{t_1}^{t_2} \delta (U_s - T_s) - \delta W_s \, dt = 0.$$

This is equivalent to equation (1) except for the index 's', referring to the shell instead of 'f', referring to the fluid. Virtual work of three kinds are considered; virtual
work on the fluid-shell boundary, virtual work from external forces and virtual work corresponding to dissipative energy. The latter is here referred to as 'virtual loss'. Consequently, for the shell,

$$\delta W_s = \delta W_{s,f} + \delta W_{s,ext} + \delta W_{s,loss},$$  \hspace{1cm} (3)

where $\delta W_{s,f}$, $\delta W_{s,ext}$ and $\delta W_{s,loss}$ are the virtual work on the shell from the fluid, the external forces and the virtual loss, respectively. External forces are here confined to the shell. Thus,

$$\delta W_f = \delta W_{f,s} + \delta W_{f,loss}. \hspace{1cm} (4)$$

where $\delta W_{f,s}$ and $\delta W_{f,loss}$ are the virtual work on the fluid from the shell and the virtual loss respectively. Subracting equation (1) from equation (2) defines, $\delta L$, for the combined system as,
\[ \delta L = \delta L_{s0} - \delta L_{f0} = 0 , \]  

(5)

In principle any linear combination of \( \delta L_{s0} \) and \( \delta L_{f0} \) may be used to define \( \delta L \). The choice given by equation (5) has some very useful properties that are elaborated further in Section 4. Combining and rearranging equations (1)–(5) yields,

\[ \delta L = \delta L_s + \delta L_f + \delta B_c = 0 , \]  

(6)

where

\[ \delta L_s = \int_{t_1}^{t_2} \delta (U_s - T_s) - \delta W_{s,\text{loss}} - \delta W_{\text{ext}} \, dt , \]  

(7)

\[ \delta L_f = -\int_{t_1}^{t_2} \delta (U_f - T_f) - \delta W_{f,\text{loss}} \, dt \]  

(8)

and

\[ \delta B_c = \int_{t_1}^{t_2} \delta W_{f,s} - \delta W_{s,f} \, dt . \]  

(9)

The three functionals, \( \delta L_s \), \( \delta L_f \) and \( \delta B_c \) give rise to one waveguide finite element type each. Derivation of these elements are treated in Chapters 2, 3 and 4 respectively. Basic expressions for \( \delta L_s \) and \( \delta L_f \) may be found in the literature, [2] for the fluid, [3] and [4] for the shell, and repeated in Chapters 2 and 3. Details of the coupling term, \( \delta B_c \), are explored in the following. To begin, the virtual work on the shell from the fluid is given by,

\[ \delta W_{s,f} = \int_S \delta \mathbf{u}_s \cdot (p \mathbf{n}) \, dS , \]  

(10)
where \( \bullet \) refers to scalar product, \( S \) is the wetted surface of the fluid shell boundary, \( u_s \) is the displacement of the shell, \( p \) is the fluid pressure and \( n \) is the normal vector directed out of the fluid. This is indicated by the indices, \( s,f \). Correspondingly, the virtual work on the fluid from the shell is given by,

\[
\delta W_{f,s} = - \int_S \delta (p n) \bullet u_s \, dS.
\] (11)

Energy flow into the shell is defined to be positive by equation (10). The energy flow into the fluid is opposite to that into the shell, which explains the minus sign in equation (11). For the acoustic wave equation the fluid velocity is written in terms of a velocity potential, \( \psi \). Consequently,

\[
\frac{\partial u_f}{\partial t} = - \mu \nabla \psi,
\] (12)

where \( u_f \) is the fluid particle displacement and \( \mu \) is a non-dimensional scaling constant. The latter is used to improve numerical stability when there are large differences in the orders of magnitude for the parameters in the respective fluid and shell system. In terms of this velocity potential, the pressure is given by, [2],

\[
p = \rho_f \mu \frac{\partial \psi}{\partial t},
\] (13)

where \( \rho_f \) is the fluid density at equilibrium. Furthermore, the shell’s normal displacement into the fluid, \( w \), is defined by,

\[
w = - u_s \bullet n.
\] (14)

Hence, the coupling term, \( \delta B_c \) is written,
\[ \delta B_c = - \int_S \rho_f \mu \int_{t_1}^{t_2} \frac{\partial \delta \psi}{\partial t} w + \delta w \frac{\partial \psi}{\partial t} \, dt \, dS. \] (15)

Equation (15) is used in Chapter 4 for deriving fluid-shell coupling waveguide finite elements.

2 Waveguide finite elements

Throughout this thesis time-harmonic motion is assumed, thus the forthcoming equations in this chapter are expressed in the frequency domain rather than in the time domain. The Fourier transform, \( \hat{g}(\omega) \), of a function, \( g(t) \), is defined by,

\[ \hat{g}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(t) e^{-i\omega t} \, dt, \] (16)

where \( \omega \) is the angular frequency and \( t \) is the time.

Waveguide element formulations in the frequency domain are derived and detailed in Chapters 2, 3 and 4. From Chapter 2, a straight shell element is given by,

\[ \int \sum_{k=0}^{2} \sum_{l=0}^{2} \frac{\partial^k}{\partial x^k} \delta \hat{\phi}^H \frac{\partial^l}{\partial x^l} \hat{\phi} - \delta \hat{\phi}^H m_2 \hat{\phi} \, dx, \] (17)

where \( ^H \) is the complex conjugate transpose, i.e. \( ^H = ^*^T \),

\[ \hat{\phi} = \begin{bmatrix} \hat{u}_1 & \hat{u}_2 & \hat{v}_1 & \hat{v}_2 & \hat{w}_1 & \hat{w}_2 & \hat{\phi}_1 & \hat{\phi}_2 \end{bmatrix}^T \] are the displacements and rotations at the node-lines of the shell element as shown in Figure 2. The matrices, \( a_{kl} \) and \( m_2 \), are complex valued when damping is included but otherwise real valued.

Virtual work from external forces acting on the shell are described by the term,

\[ \int \delta \hat{\phi}^H \tilde{f} \, dx \] (18)
where the vector, \( \tilde{f} \), contains weighted external forces and torques per unit length corresponding to the displacement vector, \( \hat{\phi} \). From Chapter 3, a straight fluid element is given by,

\[
\int \partial \frac{\partial \hat{\psi}^H}{\partial x} b_{11} \partial \hat{\psi}^H + \delta \hat{\psi}^H b_{00} \hat{\psi}^H - \omega^2 \delta \hat{\psi}^H n_2 \hat{\psi}^H \, dx ,
\]

(19)

where \( b_{11}, b_{00} \) and \( n_2 \) are matrices describing the element. \( \hat{\psi} = [ \hat{\psi}_1 \, \hat{\psi}_2 \, \hat{\psi}_3 ]^T \). \( \hat{\psi}_i = \psi_i (x) \) is the value of \( \psi \) along node-line \( 'i' \), as seen in Figure 3. The matrices, \( b_{11}, b_{00} \) and \( n_2 \), are complex-valued when damping is included and real otherwise.

The fluid-shell coupling element for a wetted surface between a fluid and a shell element is given by,

\[
\int i\omega \left[ \begin{array}{cc} \delta \hat{\psi}^H & \delta \hat{\phi}^H \end{array} \right] m_1 \left[ \begin{array}{c} \hat{\psi} \\ \hat{\phi} \end{array} \right] \, dx ,
\]

(20)

where \( m_1 \) is a real valued antisymmetric coupling matrix.
Figure 3: Fluid element co-ordinates

3 Assembled systems

Straight but otherwise complex shaped waveguides are modelled by assembling several waveguide finite elements into larger systems of equations. These systems enable analysis of higher order waves. Assembling sets degrees of freedom of shared node-lines of adjacent elements equal. Practical considerations of assembling is treated in many books on finite elements, e.g. [5] and [6] and will not be discussed further.

After assembling all matrices, \( a_{ij}, b_{ij}, m_k, n_2 \) and adding all force vectors \( \hat{f} \) the fluid-shell system of equations is written,

\[
\int \sum_{k=0}^{2} \sum_{l=0}^{2} \frac{\partial^k \delta W^H}{\partial x^k} A_{ki} \frac{\partial^l W}{\partial x^l} + i \omega \delta W^H M_1 W - \omega^2 \delta W^H M_2 W \, dx = \int \delta W^H F \, dx ,
\]

where \( W = \begin{bmatrix} \hat{\Psi} & \hat{\Phi} \end{bmatrix}^T \), \( \hat{\Psi} \) denotes the degrees of freedom in the assembled fluid and \( \hat{\Phi} \) is the degrees of freedom in the assembled shell. Equation (21) is the weak
form of the waveguide finite element model. By repeated integration by parts and use of calculus of variation, see [7, p.166–169], the weak form is transformed into the strong differential form,

\[
\sum_{j=0}^{4} K_j \frac{\partial^j}{\partial x^j} + i\omega M_1 - \omega^2 M_2 \right] W = F, \quad (22)
\]

where \(K_j = \sum_{k+l=j} (-1)^k A_{kl}\).

Strictly, equation (22) should also include Neumann type natural boundary terms for the ends of the waveguide. However, in cases when Neumann boundary conditions are important, expression (21) may be employed. A method using both the strong and the weak form is the Spectral Super Element method described in [8] and [9]. In this thesis the strong form is used.

4 Properties of wave equation

In the absence of external forces, equation (22) has solutions \(W = \tilde{W} e^{-i\kappa x}\). \(\tilde{W}\) and \(\kappa\) may be determined by equation (23),

\[
\left[ \sum_{j=0}^{4} K_j (-i\kappa)^j + i\omega M_1 - \omega^2 M_2 \right] \tilde{W} = 0 \quad (23)
\]

Equation (23) is a twin parameter eigenproblem in \(\kappa\) and \(\omega\). The eigenvectors, \(\tilde{W}\) describe the cross sectional deformation of the wave whereas the wavenumbers, \(\kappa\), describe the wave propagation and decay. Solutions of equation (23) will either yield the wavenumbers for a known frequency, i.e. \(\kappa(\omega)\), or the frequencies for a known wavenumber, i.e. \(\omega(\kappa)\).

If there are \(N\) degrees of freedom in the model, there are \(4N\) solutions for \(\kappa(\omega)\), (including infinite valued \(\kappa\)), whereas there are \(2N\) solutions for the \(\omega(\kappa)\) problem. Thus the
problem $\kappa(\omega)$ is calculated at a numerically higher cost than $\omega(\kappa)$. The frequency, $\omega$, is real and the frequency range is usually known. Consequently, complex valued wavenumbers are calculated as $\kappa(\omega)$. Conversely, for real wavenumbers with a known range, $\omega(\kappa)$ is preferred since these calculations are made at a numerically lower cost.

Real wavenumbers correspond to propagating waves. Strictly, propagating waves occur for undamped systems only. For a real wavenumber and no damping the matrices in equation (23) have the following properties:

$$\left[ \sum_j K_j (-i\kappa) \right]$$

is Hermitian and positive definite or semi definite.

$iM_1$ is imaginary and Hermitian and

$M_2$ is real and positive definite.

These properties are a result of the choice for the virtual work term $\delta L$ in equation (5). The presence of only Hermitian matrices indicates that only conservative energies are present. Hence, the analysis of non-conservative work at the fluid shell boundary in Section 1, yields a conservative coupling when the total system is considered. Furthermore, all coupling between fluid and structural waves are governed by the terms in matrix, $M_1$.

When polynomial eigenproblems, such as equation (23) are solved in practice, they are most often expanded into linear systems, [10]. An expansion which uses the properties stated above is given in the following.

First, the matrix $M_2$ is Cholesky factorized. Hence, $M_2$ is written as $LL^H$, where $^H$ denotes complex conjugate and transpose. After multiplication by the non-singular matrix $L^{-1}$ from the left, equation (23) is written,
\[
[K' - \omega M'_{1} - \omega^{2}I] W' = 0,
\]  
(24)

where

\[W' = L^{H}W, \quad M'_{1} = -iL^{-1}M_{i}L^{-H} \quad \text{and} \quad K' = L^{-1} \left[ \sum_{m=0}^{4} K_m (-i\kappa_m) \right] L^{-H}\]

Now, substituting, \(V' = \omega W'\) and \(\lambda = -1/\omega\) gives,

\[
[A - \lambda B] U = 0 ,
\]

(25)

where

\[
A = \begin{bmatrix} M'_{1} & I \\ I & 0 \end{bmatrix} \quad B = \begin{bmatrix} K' & 0 \\ 0 & I \end{bmatrix} \quad U = \begin{bmatrix} W' \\ V' \end{bmatrix}
\]

(26)

Properties of the original system yields, \(A\) Hermitian and \(B\) positive definite or semi definite. For eigensystems with these properties a left eigenvector is always the conjugate of the right eigenvector and if \(B\) is positive definite, i.e. if rigid body motion is excluded, orthogonality relations for eigenvectors, \(U_n\) and eigenvalues, \(\lambda_n\) are given by,

\[
U_m^{H}AU_n = \delta_{m,n} \lambda_n \quad U_m^{H}BU_n = \delta_{m,n},
\]

(27)

where \(\delta_{m,n}\) is the Kronecker delta. These orthogonality relationships are due to the matrix properties in equation (23). Originally a result of the choice leading to equation (5). By setting \(\kappa = 0\) in equation (22) an ordinary 2D finite element model is obtained. Consequently, the fluid-shell theory and orthogonality relations derived in this chapter are also valid for ordinary finite element models. The orthogonality
relations in (27) can often be used for solving the forced problem of equation (22) with modal expansion. Modal expansions for a waveguide without fluid is described in Chapter 7.

5 Conclusions

Modelling of a fully coupled fluid–shell system is made by considering variational formulations both for the shell and for the fluid. Any (non–zero) linear combination of the two formulations yield a valid variational formulation for the coupled system. For harmonic vibration, the particular linear combination chosen here gives a Hermitian coupling matrix both for regular and for waveguide finite element models. Orthogonality relations for eigenvectors also follows from this formulation.

References


CHAPTER 2: WAVEGUIDE FINITE ELEMENTS FOR THIN WALLED, ANISOTROPIC, PRE–STRESSED SHELLS

1 Frequency domain Lagrangian

From Chapter 1, the variation of the shell Lagrangian, $\delta L_s$, in the time domain is defined by,

$$\delta L_s = \int_{t_1}^{t_2} \delta (U_s - T_s) - \delta W_{s,\text{loss}} - \delta W_{\text{ext}} \, dt,$$

(1)

where $\delta$ denotes first variation, $t_1$ and $t_2$ are the start and end times, $U_s$ and $T_s$ are the shells potential and kinetic energy, $\delta W_{s,\text{ext}}$ and $\delta W_{s,\text{loss}}$ are the virtual work from external and dissipative forces, respectively. In the absence of fluid–shell coupling the following identity holds,

$$\delta L_s = 0.$$  

(2)

All terms in equation (1) are here required to be bilinear or quadratic functionals of displacements. This restriction is necessary since linear differential equations
are sought. Furthermore, the system's state at \( t_1 \) and \( t_2 \) is irrelevant, given that harmonic motion over a long period of time is considered. Thus, \( t_2 \) and \( t_1 \) may tend to \( \pm \infty \) respectively without any loss of information.

Parseval’s identity for two real valued functions, \( f(t) \) and \( g(t) \) is given by,

\[
\int_{-\infty}^{+\infty} f(t) g(t) \, dt = \int_{-\infty}^{+\infty} \hat{f}(\omega)^* \hat{g}(\omega) \, d\omega ,
\]

where \( t \) is time, \( \omega \) is angular frequency, \( * \) denotes complex conjugate and \( \hat{\cdot} \) denotes the Fourier transform defined by,

\[
\hat{g}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(t) e^{-i\omega t} \, dt .
\]

Applying Parseval’s identity on equation (1) gives,

\[
\delta L_s(\omega) = \int_{-\infty}^{+\infty} \delta U_s(\omega) - \delta T_s(\omega) - \delta W_{s,\text{loss}}(\omega) - \delta W_{s,\text{ext}}(\omega) \, d\omega .
\]

Responses at different frequencies are independent when linear systems are considered. In analysis of a pre-stressed system, the state at \( \omega = 0 \) affects all other frequencies. Thus, strictly, the inclusion of static–stress yields a non–linear system. The system can still be considered linear if the static case is treated separately. Consequently, a variation formulation defined for each frequency is given by,

\[
\delta L_{s,\omega}(\omega) = \delta U_s(\omega) - \delta T_s(\omega) - \delta W_{s,\text{loss}}(\omega) - \delta W_{s,\text{ext}}(\omega) .
\]

In the following \( \delta L_{s,\omega} \) is referred to as the shells 'Lagrangian variation'. Each of the terms on the right hand side of equation (6) are treated independently in Section 2.
2 Terms in the shell’s Lagrangian variation

There are many theories for thin walled shells, see for instance [1] or [2] for reviews. In the following the Sander–Budiansky shell theory is used,[3].

2.1 Virtual work from external forces

External forces are separated into a static and a dynamic part. The virtual work, $\delta W_{s,\text{ext}}$ from external forces is written as,

$$\delta W_{s,\text{ext}} = \int_S \delta u^T f_{\text{stat}} + \delta u^T f_{\text{dyn}} dS,$$

where $^T$ denotes transpose, $S$ is the surface area of the shell, $\delta u$ is the virtual displacement of the shell, $f_{\text{stat}}$ and $f_{\text{dyn}}$ are static and dynamic parts of the external forces per unit area respectively.

Equation (7) is given in the time domain. Transformation of the time integral of equation (7) to the frequency domain is made as in Section 1. The result is written, for the static part, $\omega = 0$,

$$\delta W_{s,\text{ext}} (\omega = 0) = \int_S \delta \hat{u}^H \hat{f}_{\text{stat}} dV,$$

and for the dynamic part, $\omega \neq 0$,

$$\delta W_{s,\text{ext}} (\omega) = \int_S \delta \hat{u}^H \hat{f}_{\text{dyn}} dV,$$

where $^H$ denotes complex conjugate transpose.
2.2 Potential energy

Pre–stress yields an additional stiffening or weakening to a system. In this analysis the energy associated with the pre–stress’ change of stiffness, as well as, the energy associated with elasticity are derived simultaneously from the strain energy. Strain energy per unit volume is defined by, see for instance reference [4, equation 2.1],

$$U''' = \int \sigma^T d\epsilon,$$

(10)

Here, $\sigma$ is the stress and $\epsilon$ is the strain in the solid relative to the static equilibrium.

Consider a shell with co–ordinates, $\xi$, $\eta$ and $\zeta$ as seen in Figure 1,

![Figure 1: Shell co–ordinates](image)

then, $\sigma = \begin{bmatrix} \sigma_\xi & \sigma_\eta & \sigma_\zeta & \tau_{\xi\eta} & \tau_{\xi\zeta} & \tau_{\eta\zeta} \end{bmatrix}^T$ and $\epsilon = \begin{bmatrix} \epsilon_\xi & \epsilon_\eta & \epsilon_\zeta & \gamma_{\xi\eta} & \gamma_{\xi\zeta} & \gamma_{\eta\zeta} \end{bmatrix}^T$.

Now, since thin walled shells are considered, Love’s first approximation is adopted, see reference [1, page 6]. It is found that, $\gamma_{\xi\zeta} = \gamma_{\eta\zeta} = \sigma_\zeta = 0$. Thus, with regard to equation (10), it is sufficient to write,

$$\sigma = \begin{bmatrix} \sigma_\xi & \sigma_\eta & \tau_{\xi\eta} \end{bmatrix}^T \quad \text{and} \quad \epsilon = \begin{bmatrix} \epsilon_\xi & \epsilon_\eta & \gamma_{\xi\eta} \end{bmatrix}^T.$$
In the Sander–Budiansky, and several other, shell theories the thickness is assumed to be small compared to the radii of curvature of the shell. As a result, the strains dependence in the $\zeta$ coordinate are given by,

$$\epsilon = \epsilon + \zeta \kappa,$$

(12)

where $\epsilon$ is the strain at $\zeta = 0$ and $\kappa$ is the change of curvature at $\zeta = 0$. Both $\epsilon$ and $\kappa$ are functionals of the displacements, $u = [u \ v \ w]$. The exact expressions depend on the shell geometry.

Now, Hooke’s law for an anisotropic material is given by the relation,

$$\sigma = E \epsilon + \sigma_0,$$

(13)

where $\sigma_0$ is the pre–stress and $E$ is the material’s stiffness matrix, that may depend on $\zeta$. Due to reciprocity, $E$ is symmetric. Statically the shell is considered to be a membrane, therefore $\sigma_0$ is uniform over $\zeta$. Substituting equations (12) and (13) into equation (10) and integrating over the thickness gives the potential energy per unit area, $U_s''$, as,

$$U_s'' = \frac{1}{2} \left[ \begin{array}{c} \epsilon^T \\ \kappa^T \end{array} \right] [D] \left[ \begin{array}{c} \epsilon \\ \kappa \end{array} \right] + h \sigma_0^T \epsilon,$$

(14)

Within the shell, the radii of curvature are considered constant with respect to $\zeta$, consequently,

$$[D] = \int_{-\frac{b}{2}}^{\frac{b}{2}} \begin{bmatrix} E & \zeta E \\ \zeta E & \zeta^2 E \end{bmatrix} d\zeta.$$

(15)

If there is a neutral layer at $\zeta = 0$ the off diagonal blocks in $[D]$ are zero. In many cases no such neutral layer exist and $[D]$ must be treated as a full matrix. Calculation
of $[D]$ for complex layered anisotropic materials is treated in, for instance, reference [5].

The effect of pre–stress in equation (14) is given by, $h\sigma^T_0 \varepsilon$, where $\sigma^T_0$ is constant in time. If the components of $\varepsilon$ are linear functionals of the displacement, the variation of the energy associated with pre–stress is constant. Hence, dynamic effects of pre–stress require nonlinear displacement to strain relations. Expanding displacement to strain relations to include quadratic terms give,

$$\varepsilon = \varepsilon_l + \varepsilon_q,$$

where $\varepsilon_l$ is a linear functional of $u$ and $\varepsilon_q$ is a quadratic functional of $u$. Descriptions of nonlinear displacement to strain relations for Sander–Budiansky shell theory in complex geometries are given in references [2] and [3]. Nonlinear shell to displacement relations may give rise to strains also for rigid body modes, see Appendix D.

Using equation (16) in equation (14) and assuming small displacements, so that higher order terms of $u$ may be ignored, yields,

$$U_s'' = \frac{1}{2} \left[ \begin{array}{c} \varepsilon_l^T \\ \chi^T \end{array} \right] [D] \left[ \begin{array}{c} \varepsilon_l \\ \chi \end{array} \right] + \frac{1}{2} \beta^T [N] \beta + \varepsilon_l^T (h\sigma_0) ,$$

where $\beta$ and $N$ are defined such that,

$$\frac{1}{2} \beta^T [N] \beta = h\sigma_0^T \varepsilon_q .$$

With this definition, $\beta$ is required to be a linear functional in $u$. Taking the first variation of equation (17) and subsequently applying the transformation to the frequency domain similarly to Section 1 yields, for $\omega = 0$,
\[ \delta U_s (\omega = 0) = \int_S \delta U''_s (\omega = 0) \, dS = \int_S \delta \hat{\epsilon}_l^H (h\sigma_0) \, dS \]  \hspace{1cm} (19)

and, for \( \omega \neq 0 \)

\[
\delta U_s (\omega) = \int_S \delta U''_s (\omega) \, dS = \int_S \left[ \delta \hat{\epsilon}_l^H \delta \hat{\kappa}^H \right] [D] \begin{bmatrix} \hat{\epsilon}_l \\ \hat{\kappa} \end{bmatrix} + \delta \hat{\beta}^H [N] \hat{\beta} \, dS . \hspace{1cm} (20)
\]

### 2.3 Kinetic energy

The kinetic energy per unit area of the shell is defined by,

\[
T''_s = \frac{1}{2} \dot{\hat{u}}^T [M] \dot{\hat{u}} , \hspace{1cm} (21)
\]

where \( \dot{\hat{u}} \) is the velocity and \([M]\) is the mass matrix given by,

\[
[M] = \begin{bmatrix} m'' & 0 \\ 0 & m'' \end{bmatrix} , \hspace{1cm} (22)
\]

where, \( m'' \) is the mass per unit area of the shell.

Taking the first variation of equation (21) and subsequently applying the transformation to the frequency domain according to Section 1 yields,

\[
\delta T_s = \int_S \delta T''_s \, dS = \int_S \omega^2 \delta \hat{u}^H [M] \dot{\hat{u}} \, dS . \hspace{1cm} (23)
\]
2.4 Virtual work of dissipative forces

There are many material models that include damping, see for instance the review in reference [6]. There is little reason to restrict damping to any specific model here. Instead, a more general linear damping model outlined by Woodhouse [7] is considered.

In a linear system with generalized co-ordinates, \( q_j, j = 1, 2, \ldots, N \), the corresponding generalized dissipative internal forces, \( Q_j \) can be written, [7],

\[
Q_j = -\sum_{k=1}^{N} \int_{-\infty}^{t} g_{jk} (t - \tau) \dot{q}_k (\tau) d\tau
\]  

(24)

where \( g_{jk} \) are 'kernel' or 'relaxation' functions and \( \dot{q}_k \) is the velocity of the generalized co–ordinate \( q_k \).

For simplicity, the generalized co-ordinates, \( q_j \) are chosen as the variables used in \( \delta U \) and \( \delta T \). Considering the virtual work from the generalized forces and applying a transformation to the frequency domain according to Section 1 yields,

\[
-\delta W_{s,\text{loss}} = i\omega \int_V \left[ \begin{bmatrix} \delta \hat{\varepsilon}^H_l & \delta \hat{\kappa}^H_l \delta \hat{\varepsilon}_q^H & \delta \hat{\varepsilon}_q^H \delta \hat{\kappa}^H_q \end{bmatrix} \begin{bmatrix} \hat{\varepsilon}_l & \hat{\kappa}_l \hat{\varepsilon}_q \hat{\kappa}_q \hat{u} \end{bmatrix} \right] dV
\]  

(25)

where the matrix \( [G] \) is a function of \( \omega \). For the purpose of this thesis, only the diagonal blocks of \( [G] \) are considered. Thus \( [G] \) can be written as,

\[
\]  

(26)
It is required that $[G]$ is symmetric and positive (semi) definite for any set of physically realizable displacements. If there were an antisymmetric part of $[G]$, it would represent a gyroscopic coupling, see [7]. Since such coupling effects are not considered here, $[G]$ is required to be symmetric. The mathematical details leading to symmetry and positive (semi) definiteness requirements are further elaborated in Appendix A.

An important conclusion from equation (25) and (26) is that, instead of considering the dissipative virtual work as a separate term in $\delta L_{\text{dis}}$, the matrices $[D]$, $[N]$ and $[M]$ can be considered to have imaginary parts. This approach is used in the following.

### 3 Static equilibrium

The static case is treated separately in Section 2. For the static case, kinetic energy and losses are all zero. Consequently, by combining equations (2), (6), (8) and (19),

$$\int_S \delta \tilde{\epsilon}^H \left( h \sigma_0 \right) - \delta \tilde{u}^H f_{\text{stat}} \, dS = 0.$$  \hspace{1cm} (27)

The interpretation of equation (27) is that external virtual work must equal internal virtual work for $\omega = 0$. This relation is also known as 'the principle of virtual work', described in, for example, reference [4].

### 4 Orthotropic plate strip waveguide finite element

In the following a plate strip waveguide finite element is derived for orthotropic material with one main axis along the waveguide. The treatment of this element can be seen as a special case of the anisotropic conical element treated in Section 5.
4.1 Orthotropic plate strip

Here, a plate is considered a flat shell. Consequently the co-ordinates $\xi$, $\eta$ and $\zeta$ in Figure 1 are here replaced with the local Cartesian co-ordinates $x$, $y$ and $z$ respectively. Displacements and co-ordinates for a plate strip are seen in Figure 2. For this analysis, $x$ is chosen as the direction along the waveguide. The non-dimensional co-ordinate, $\vartheta$ is defined as,

$$\vartheta = \frac{y}{a},$$

(28)

Where $a$ is the half-width of the plate. Thus, $\vartheta = -1$ at node-line 1 and $\vartheta = +1$ at node-line 2. The displacements and the rotation about the $x-axis$ at node line $i = 1, 2$ are denoted $u_i$, $v_i$, $w_i$ and $\phi_i$ respectively.

Figure 2: Plate strip
4.1.1 Orthotropic material

'Orthotropic material' refers to a material with properties that are symmetric with respect to orthogonal planes of symmetry. Here, these planes are assumed to coincide with the cartesian co-ordinate system, \(x\ y\ z\). The stiffness matrix, \([D]\), as given by equation (15), is then written as,

\[
[D] = \frac{h}{1 - \nu_x \nu_y} \begin{bmatrix}
D_1 & \frac{h^2}{12} D_1 \\
D_1 & \frac{h^2}{12} D_1
\end{bmatrix},
\]  

(29)

where,

\[
[D_1] = \begin{bmatrix}
E_x & \nu_y E_x & 0 \\
\nu_x E_y & E_y & 0 \\
0 & 0 & (1 - \nu_x \nu_y) G_{xy}
\end{bmatrix}.
\]  

(30)

\(E_x\) and \(E_y\) are Young’s modulus and \(\nu_x\) and \(\nu_y\) are Poisson ratios in the respective directions and \(G_{xy}\) is the shear modulus in the \(x - y\) plane. Since \([D_1]\) is symmetric, \(\nu_y E_x = \nu_x E_y\).

4.1.2 Damping

From the discussion in Section 2.4, it is clear that the elastic stiffness matrix, \([D]\), the geometric stiffness matrix, \([N]\), and the mass matrix \([M]\) may have non-zero imaginary parts corresponding to dissipative forces. Furthermore, the imaginary parts of \([D]\), and \([N]\) must be positive definite or positive semi definite. Contrary, the imaginary part of \([M]\) must be at least negative semi definite, owing to the minus sign in front of the kinetic energy in equation (1). These restraints are met if the stiffness parameters, \(E_x\), \(E_y\), \(G_{xy}\), \(N_x\), \(N_y\) and \(N_{xy}\) are added with negative imaginary parts and the mass per unit area, \(m''\), is added with a positive imaginary part.
4.1.3 Displacement to strain relations

Displacement to strain relations for a thin walled plate are given in reference [1]. With the notation used here these relations are written,

\[ \varepsilon_l = \begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix}, \quad \varepsilon_q = \begin{bmatrix} \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \\ \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \\ \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \end{bmatrix} \quad \text{and} \quad \kappa = \begin{bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ -2 \frac{\partial^2 w}{\partial x \partial y} \end{bmatrix}. \quad (31) \]

4.1.4 Pre–stress energy

From equation (31) it is clear that the pre–stress energy, \( \frac{1}{2} \beta^T [N] \beta = h \sigma_0^T \varepsilon_q \) is given by,

\[ \frac{1}{2} \begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} \begin{bmatrix} N_x & N_{xy} \\ N_{xy} & N_y \end{bmatrix} \begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{bmatrix} \quad (32) \]

where,

\[ \begin{bmatrix} N_x & N_y & N_{xy} \end{bmatrix} = h \sigma_0^T \quad (33) \]

4.2 Waveguide finite element

The definition of a waveguide, as given in the introduction of this thesis, imply solutions that are separable with respect to the direction of wave propagation and to the cross–section. In the waveguide–FE method, subdivision into finite elements is made within the cross-section only.
4.2.1 Shape–functions

In this thesis trial–functions for the displacement, \( u \) and test–functions for the virtual displacements, \( \delta u \), are chosen equally. This procedure is here identical to what is known as the 'Galerkin method'. Both trial– and test–functions are henceforth referred to as 'shape–functions'.

The in–plane displacements, \( u \) and \( v \) require \( C^0 \) continuity only, since the strains including \( u \) and \( v \), i.e. \( \varepsilon_l \) in equation (31), include first order derivatives in \( y \) only. For simplicity, linear functions in \( y \) are chosen as shape–functions for \( u \) and \( v \). The out of plane displacement, \( w \), require \( C^1 \) continuity since second order derivatives are included in \( \kappa \). For simplicity, cubic functions in \( y \) are chosen as shape–functions for \( w \). Hence, in the frequency domain, displacements are given by,

\[
\hat{u} = \mathbf{N}_p^T(\vartheta) \hat{\mathbf{u}}(x), \quad \delta \hat{u}^* = \mathbf{\delta u}(x)^H \mathbf{N}_p(\vartheta)
\]

\[
\hat{v} = \mathbf{N}_p^T(\vartheta) \hat{\mathbf{v}}(x), \quad \delta \hat{v}^* = \mathbf{\delta v}(x)^H \mathbf{N}_p(\vartheta)
\]

\[
\hat{w} = \mathbf{N}_b^T(\vartheta) \hat{\mathbf{w}}(x), \quad \delta \hat{w}^* = \mathbf{\delta w}(x)^H \mathbf{N}_b(\vartheta)
\]

where

\[
\mathbf{N}_p(\vartheta) = \begin{bmatrix}
\frac{1}{2}(1-\vartheta) \\
\frac{1}{2}(1+\vartheta)
\end{bmatrix}
\quad \text{and} \quad
\mathbf{N}_b(\vartheta) = \begin{bmatrix}
\frac{1}{4}(2-3\vartheta+\vartheta^3) \\
\frac{a}{4}(1-\vartheta-\vartheta^2+\vartheta^3) \\
\frac{1}{4}(2+3\vartheta-\vartheta^3) \\
\frac{a}{4}(-1-\vartheta+\vartheta^2+\vartheta^3)
\end{bmatrix}
\]
The rationale for choosing the polynomials in equation (39), is that the displace-
ments and rotations at the node–lines are given directly by the entries of \( \hat{u}, \hat{v} \) and \( \hat{w} \), given in equation (37). The shape–functions in equation (39) are Hermitian polynomials that are often used for finite elements, see for instance [8].

4.2.2 Element description

The approximations given by equations (34) to (39) are substituted into the dis-
placement to strain relations (31) and subsequently into the Lagrangian variation (6). After expanding the matrices and sorting terms corresponding to the same derivatives with respect to the \( x \)-axis, the Lagrangian variation is written,

\[
\delta L_{xw} = \int 2 \sum_{k=0}^{2} \sum_{l=0}^{2} \frac{\partial^k}{\partial x^k} \frac{\partial^{H}}{\partial x^l} a_{kl} \frac{\partial^l}{\partial x^l} \phi^{H} \phi \, dx - \int \delta \phi^{H} \hat{f} \, dx, \tag{40}
\]

where, \( \hat{\phi} = [\hat{u}^T \ \hat{v}^T \ \hat{w}^T]^T \) are the displacements and rotations at the node–lines of the shell element, \( \hat{f} \) is the weighted force on the element node–lines, that is related to the real force, \( \hat{f} = [\hat{f}_x \ \hat{f}_y \ \hat{f}_z]^T \), by,

\[
\hat{f} = \int_{-a}^{+a} \begin{bmatrix} N_p \hat{f}_x \\ N_p \hat{f}_y \\ N_b \hat{f}_z \end{bmatrix} dy.
\tag{41}
\]

As an example, the matrix, \( a_{10} \) in equation (40) is given by,

\[
a_{10} = \begin{bmatrix}
0 & -\frac{hE_{xy}}{1-\nu_x\nu_y} [I_{pp01}] & 0 \\
-hG_{xy} [I_{pp01}] & 0 & 0 \\
0 & 0 & -h\sigma_{xy} [I_{bb01}] \\
\end{bmatrix},
\tag{42}
\]

where the matrices \([I_{pp01}]\) and \([I_{bb01}]\) are defined as.
\[
[I_{p01}] = \int_{-a}^{+a} N_p^T \frac{\partial N_p}{\partial y} dy \quad \text{and} \quad [I_{b01}] = \int_{-a}^{+a} N_b^T \frac{\partial N_b}{\partial y} dy. \quad (43)
\]

The other matrices \(a_{kl}\) in equation (40), written with the same notation, are detailed in Appendix B.

### 4.2.3 Evaluation of FE matrices

The matrices \([I_{p01}]\) and \([I_{b01}]\) and the similar matrices defined in Appendix B are evaluated analytically. This is possible since the entries in the integrands are products of polynomials. Another common approach is to use numerical integration such as Gauss quadrature.

### 5 Conical shell strip waveguide finite element

Anisotropic conical shell strip elements are derived similarly to orthotropic plate strip elements. The main difference is the added complexity due to the anisotropic material and the curved geometry. To manage this added complexity a derivation method suitable for computational analysis is developed.

#### 5.1 Anisotropic Conical Shell

Coordinates for a conical shell strip is introduced by Figure 3. Wave–propagation is considered in the \(\theta\)-direction. As for the plate strip in Section 4 the analysis is simplified by introducing a non–dimensional co-ordinate, \(\vartheta\). Here, the co–ordinate \(\vartheta\) is defined by,

\[
\vartheta = \frac{s - s_m}{L/2} \quad (44)
\]
where, $L$ is the length of the element in the $s$-direction and $s_m$ is the $s$ co-ordinate at the middle of the element.

### 5.1.1 Anisotropic material and damping

The real part of the material stiffness matrix, $[D]$, is a symmetric positive definite and, generally, full $6 \times 6$ matrix for an anisotropic thin shell. Damping is considered in the same manner as in Section 4.1.2. Consequently the imaginary parts of $[D]$ and $[N]$ are positive (semi) definite, whereas the imaginary part of $[M]$ is negative (semi) definite.

### 5.1.2 Displacement to strain relations

Displacement to strain relations in Sander–Budiansky theory for a general thin walled shell are found in reference [3]. These relations can be specified for any given thin shell geometry. In the following, displacement to strain relations for conical-coordinates are presented.

The linear strain to displacement relations for $z = 0$ are,

$$
\varepsilon_{l,s} = \frac{\partial u}{\partial s}
$$

$$
\varepsilon_{l,\theta} = \frac{1}{R(s)} \frac{\partial v}{\partial \theta} + \frac{u \sin(\alpha)}{R(s)} + \frac{w \cos(\alpha)}{R(s)}
$$

$$
\gamma_{l,s\theta} = \frac{1}{R(s)} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial s} - \frac{v \sin(\alpha)}{R(s)}
$$

The quadratic strain to displacement, $\varepsilon_{q,s}$, $\varepsilon_{q,\theta}$ and $\gamma_{q,s\theta}$ relations are given as,
\[
\varepsilon_{q,s} = \frac{1}{2} (\varepsilon_{l,s}^2 + \varepsilon_{l,s\theta}^2 + 2\varepsilon_{l,s\theta} \phi_z + \phi_s^2 + \phi_z^2)
\]

\[
\varepsilon_{q,\theta} = \frac{1}{2} (\varepsilon_{l,\theta}^2 + \varepsilon_{l,s\theta}^2 - 2\varepsilon_{l,s\theta} \phi_z + \phi_{\theta}^2 + \phi_z^2)
\]

\[
\gamma_{q,\theta} = \varepsilon_{l,z} (\varepsilon_{l,s} + \varepsilon_{l,\theta}) - (\varepsilon_{l,s} - \varepsilon_{l,\theta}) \phi_z + \phi_s \phi_{\theta}
\]

where, \( \varepsilon_{l,s\theta} = \frac{1}{2} \gamma_{l,s\theta} \), \( \phi_{\theta} \) is the rotation about the \( \theta \)-axis, \( -\phi_s \) is the rotation about the \( s \)-axis and \( -\phi_z \) is the rotation about the \( z \)-axis. For conical co-ordinates these rotations are,

\[
\phi_s = -\frac{\partial w}{\partial s},
\]

\[
\phi_{\theta} = -\frac{1}{R(s)} \frac{\partial w}{\partial \theta} + \frac{\cos(\alpha)}{R(s)} v,
\]

\[
\phi_z = \frac{1}{2} \left( \frac{1}{R(s)} \frac{\partial u}{\partial \theta} - \frac{\sin(\alpha)}{R(s)} v - \frac{\partial v}{\partial s} \right).
\]

The changes of curvature, \( \kappa_s \), \( \kappa_{\theta} \) and \( \kappa_{s\theta} \) are,

\[
\kappa_s = -\frac{\partial^2 w}{\partial s^2}
\]

\[
\kappa_{\theta} = -\frac{1}{R(s)^2} \frac{\partial^2 w}{\partial \theta^2} + \frac{\cos(\alpha)}{R(s)^2} \frac{\partial v}{\partial \theta} - \frac{\sin(\alpha)}{R(s)} \frac{\partial w}{\partial s}
\]

\[
\kappa_{s\theta} = -\frac{1}{R(s)} \frac{\partial^2 w}{\partial s \partial \theta} - \frac{3 \cos(\alpha)}{4} \frac{\sin(\alpha)}{R(s)^2} v + \frac{\sin(\alpha)}{R(s)} \frac{\partial w}{\partial \theta} + \left( 1 + \frac{\cos(\alpha)}{2} \right) \frac{\partial v}{\partial s}
\]

5.1.3 Potential and kinetic energy

From equation (20), the variation of the potential energy per unit area is written,

\[
\delta U_s''(\omega) = \left[ \delta \hat{\varepsilon}_H^I \ \delta \hat{\kappa}_H^I \right] \left[ \hat{\varepsilon}_I \ \hat{\kappa}_I \right] + \delta \tilde{\beta}_H^I \left[ \mathbf{N} \right] \tilde{\beta}
\]

(49)
where

\[
\begin{bmatrix}
\hat{\epsilon}_l^T & \hat{\zeta}_l^T
\end{bmatrix} = \begin{bmatrix}
\hat{\epsilon}_{l,s} & \hat{\epsilon}_{l,\theta} & \hat{\gamma}_{l,s\theta} & \hat{\kappa}_s & \hat{\kappa}_\theta & 2\hat{\kappa}_{s\theta}
\end{bmatrix}
\] (50)

and \([D]\) can be calculated from equation (15).

From equation (18) and (46) it is found that \(\hat{\beta}\) and \([N]\) can be written as,

\[
\hat{\beta} = \begin{bmatrix}
\hat{\epsilon}_{l,s} & \hat{\epsilon}_{l,\theta} & \hat{\gamma}_{l,s\theta} & \hat{\phi}_z & \hat{\phi}_s & \hat{\phi}_\theta
\end{bmatrix}^T
\] (51)

and

\[
[N] = \begin{bmatrix}
N_s & 0 & N_{s\theta} & -N_{s\theta} & 0 & 0 \\
0 & N_\theta & N_{s\theta} & N_{s\theta} & 0 & 0 \\
N_{s\theta} & N_{s\theta} & N_s + N_\theta & N_s - N_\theta & 0 & 0 \\
-N_{s\theta} & N_{s\theta} & N_s - N_{s\theta} & N_s + N_\theta & 0 & 0 \\
0 & 0 & 0 & 0 & N_s & N_{s\theta} \\
0 & 0 & 0 & 0 & N_{s\theta} & N_\theta
\end{bmatrix}
\] (52)

where,

\[
\begin{bmatrix}
N_s & N_\theta & N_{s\theta}
\end{bmatrix}^T = h\sigma_0.
\] (53)

The kinetic energy per unit area is given directly from equation (21).

### 5.2 Waveguide Finite Element

#### 5.2.1 Shape- and Test- functions

By inspection of equations (45) to (47) and (51) it is found that terms including the in–plane displacements \(\hat{u}\) and \(\hat{v}\) contain first order derivatives with respect to
s only. Consequently shape–functions for ˆ\(u\) and ˆ\(v\) require at least \(C^0\) continuity. Contrary to this, terms including the out–of–plane displacement ˆ\(w\) may contain second order derivatives and thus require at least \(C^1\) continuity. \(C^0\) continuity is satisfied with polynomials of degree equal to or higher than one. \(C^1\) continuity is satisfied with polynomials of degree equal to or higher than three. For simplicity, polynomials with degrees higher than three are not used here. Thus the shape function for ˆ\(w\) is chosen as a third degree polynomial. The shape functions chosen for the in–plane displacements ˆ\(u\) and ˆ\(v\) can then be chosen as first, second or third degree polynomials. Here second degree polynomials are chosen. This choice is supported by Ross, [9], who considered vibration and buckling problems with axi–symmetric thin walled shell finite elements. Results in [9] indicate that, for the same number of elements, the convergence with second degree polynomials for ˆ\(u\) and ˆ\(v\) is almost as good as with third degree polynomials and significantly better than first degree polynomials. Consequently, for the in plane displacements quadratic 'bubble' shape–functions with amplitude ˆ\(u_3\) and ˆ\(v_3\) are introduced. Hence,

\[
\begin{align*}
\hat{u} &= N_p^T(\vartheta) \hat{\mathbf{u}}(x), & \delta \hat{u}^* &= \delta \hat{\mathbf{u}}(x)^H N_p(\vartheta) \\
\hat{v} &= N_p^T(\vartheta) \hat{\mathbf{v}}(x), & \delta \hat{v}^* &= \delta \hat{\mathbf{v}}(x)^H N_p(\vartheta) \\
\hat{w} &= N_b^T(\vartheta) \hat{\mathbf{w}}(x), & \delta \hat{w}^* &= \delta \hat{\mathbf{w}}(x)^H N_b(\vartheta)
\end{align*}
\]

where

\[
\begin{align*}
\hat{\mathbf{u}} &= \begin{bmatrix} \hat{u}_1 & \hat{u}_2 & \hat{u}_3 \end{bmatrix}^T, \quad \hat{\mathbf{v}} &= \begin{bmatrix} \hat{v}_1 & \hat{v}_2 & \hat{v}_3 \end{bmatrix}^T, \quad \hat{\mathbf{w}} &= \begin{bmatrix} \hat{w}_1 & \hat{\phi}_1 & \hat{w}_2 & \hat{\phi}_2 \end{bmatrix}^T, \\
\delta \hat{\mathbf{u}} &= \begin{bmatrix} \delta \hat{u}_1 & \delta \hat{u}_2 & \delta \hat{u}_3 \end{bmatrix}^T, \quad \delta \hat{\mathbf{v}} &= \begin{bmatrix} \delta \hat{v}_1 & \delta \hat{v}_2 & \delta \hat{v}_3 \end{bmatrix}^T, \quad \delta \hat{\mathbf{w}} &= \begin{bmatrix} \delta \hat{w}_1 & \delta \phi_1 & \delta \hat{w}_2 & \delta \phi_2 \end{bmatrix}^T
\end{align*}
\]
\[ \mathbf{N}_p(\vartheta) = \begin{bmatrix} \frac{1}{2} (1 - \vartheta) \\ \frac{1}{2} (1 + \vartheta) \\ (1 - \vartheta^2) \end{bmatrix} \quad \text{and} \quad \mathbf{N}_b(\vartheta) = \frac{1}{4} \begin{bmatrix} (2 - 3\vartheta + \vartheta^3) \\ \frac{L}{2} (1 - \vartheta - \vartheta^2 + \vartheta^3) \\ (2 + 3\vartheta - \vartheta^3) \\ \frac{L}{2} (-1 - \vartheta + \vartheta^2 + \vartheta^3) \end{bmatrix}. \quad (59) \]

With the polynomials in equation (59), the in–plane displacements at the node–lines are given directly by the first two entries in \( \hat{u} \) and \( \hat{v} \). The displacements and rotations for the out–of–plane displacements are given by the entries in \( \hat{w} \). The polynomials in equation (59) are plotted in Figure 4.

### 5.2.2 Analytical derivation

After substituting the shape functions defined by equation (54) to (59) and relations (49) to (53) into equation (6), the derivation of the element wave equation is fairly straightforward. After sorting terms with respect to derivatives of \( \theta \), differentiation with respect to \( s \) and subsequent integration from \( s_1 \) to \( s_2 \), the desired result is written,

\[ \delta L_{sw} = \int \sum_{k=0}^{2} \sum_{l=0}^{2} \frac{\partial^k \partial^l H}{\partial \theta^k \partial \theta^l} \mathbf{a}_{kl} \frac{\partial^l}{\partial \theta^l} \hat{\phi} - \delta \hat{\phi} \mathbf{m}_2 \hat{\phi} d\theta - \int \delta \hat{\phi}^H \mathbf{f} d\theta, \quad (60) \]

where, \( \hat{\phi} = [\hat{u}^T \hat{v}^T \hat{w}^T]^T \). The problem with the derivation of equation (60) is that the expressions giving the variation of potential energy are quite involved. For this reason the derivation is made with codes written for the symbolic mathematical software package MAPLE. By virtue of symmetry, \( a_{10} = -a_{01}, a_{20} = a_{02} \) and \( a_{12} = -a_{21} \). Consequently, \( a_{01}, a_{02}, a_{21} \) are not calculated with these codes. The code corresponding to the pre–stress term is given in Appendix C together with some comments. The code corresponding to the elastic energy is quite similar. The output of the code described in Appendix C are sums of terms for each element matrix, \( \mathbf{a}_{00}, \mathbf{a}_{01}, \mathbf{a}_{11} \). One such term may be,
\[+N_t \cdot C_2(P_{va}, P_{w}, 1, 0, 2, A) \cdot C_2 \cdot S_a \ldots\]

where \( C_a = \cos(\alpha) \), \( S_a = \sin(\alpha) \), \( A = \begin{bmatrix} L/2 & \sin(\alpha) & R(s_m) \end{bmatrix} \), \( s_m \) is the \( s \)-coordinate at the center of the element, \( \alpha \), \( L \) and \( R(s) \) are explained by Figure 3. \( C_2(P_{va}, P_{w}, i, j, k, A) \) is notation for,

\[
C_2(P_{va}, P_{w}, i, j, k, A) = \int_{-L/2}^{+L/2} \frac{\partial^i N_p}{\partial s^i} \frac{\partial^j N_T^T}{\partial s^j} \frac{1}{R(s)^k} ds, \quad (61)
\]

Since \( N_p \) has 3 entries and \( N_b \) has 4, the output of this integral yield 12 values. The element matrices, \( a_{00}, a_{01} \) and \( a_{11} \) have 100 entries. \( P_{va} \) and \( P_w \) give information about the placement of the values calculated with equation (61) in the element matrices.

### 5.2.3 Evaluation of FE matrices

The integral in equation (61) is evaluated as follows. First \( s \) is replaced with the non-dimensional co-ordinate \( \vartheta \). Hence,

\[
d s = d s \overline{d \vartheta} \overline{d \vartheta} \quad (62)
\]

Since, \( \vartheta \) is a linear function in \( s \),

\[
\frac{\partial^i N_p}{\partial s^i} = \frac{\partial^i N_p}{\partial \vartheta} \left( \frac{d \vartheta}{d s} \right)^i \quad (63)
\]

and similarly for polynomial \( N_b \). Consequently, with the aid of equation (44),

\[
C_2(P_{va}, P_{w}, i, j, k, A) = \int_{-1}^{+1} \frac{\partial^i N_p}{\partial \vartheta^i} \frac{\partial^j N_T^T}{\partial \vartheta^j} \left( \frac{d \vartheta}{d \vartheta} \right)^{1-(i+j)} \left( R(\vartheta) \right)^{k-1} d \vartheta \quad (64)
\]
Unless \( R(s) \) is constant or \((k - 1) \leq 0\), the integrands above are not polynomials. As a result, equation (64) is evaluated numerically. For the results presented in this thesis, 4-point Gauss quadrature is used.

6 Conclusions

The first variation of the Lagrangian for a shell is taken from the time domain to the frequency domain. Subsequently, terms corresponding to potential energy, kinetic energy, external forces and damping are treated.

A waveguide finite element description for a thin walled plate strip with orthotropic material, prestress and damping is derived. The element description is stated in equation (40). The matrices associated with this element are detailed in Appendix B.

A waveguide finite element for a thin walled anisotropic pre-stressed conical shell strip is derived. The procedure for deriving this element is closely related to that of an orthotropic pre-stressed plate strip. As a result of the complexity corresponding to the conical geometry and the anisotropic material, the derivation is automated with a program code written in the symbolic mathematical software MAPLE.
Figure 3: Conical co-ordinates
Figure 4: Shape–functions. Upper left corner: \( N_{p1} \) and \( N_{p2} \). Upper right corner: \( N_{p3} \). Lower left corner: \( N_{b1} \) and \( N_{b3} \). Lower right corner: \( N_{b2} \) and \( N_{b4} \).
References


A Dissipative forces

Consider the definition of the generalized dissipative forces in Section 2.4. The virtual work from dissipative forces is written

\[ \delta W_{s,\text{loss}} = \sum_{j=1}^{N} \delta q_j Q_j = - \sum_{j=1}^{N} \delta q_j \sum_{k=1}^{N} \int_{-\infty}^{t} g_{jk} (t - \tau) \dot{q}_k (\tau) \, d\tau \]  

(65)

Now, if the virtual displacements \( \delta q_j \) are seen as taking place during time \( \delta t \), we have,

\[ \delta q_j = \frac{\partial q_j}{\partial t} \delta t \]  

(66)

Thus,

\[ \delta W_{s,\text{loss}} = - \sum_{j=1}^{N} \sum_{k=1}^{N} \dot{q}_j \int_{-\infty}^{t} g_{jk} (t - \tau) \dot{q}_k (\tau) \, d\tau \delta t = H (t) \delta t \]  

(67)

Since \( \delta W_{s,\text{loss}} \) is defined as virtual work on the system, a positive \( H \) is recognized as the rate of energy fed into the system from the forces \( Q_j \). Let \( E (t) \) denote the total energy in the system. In a linear system, \( H (t) \) is proportional to \( E (t) \), for all magnitudes of an otherwise specified state. Thus, \( h (t) \) is now defined by,

\[ H (t) = E (t) h (t) \]  

(68)

Energy conservation in the absence of external forces requires that,

\[ E (t + \delta t) = E (t) + E (t) h (t) \delta t \]  

(69)

as \( \delta t \to 0 \). Consequently,
\[ E(t) = E_0 \exp\left( \int_{t_0}^{t} h(t) \, dt \right), \quad (70) \]

where \( E_0 \) is the total energy at \( t = t_0 \).

Since \( E \) must decay in the absence of external forces, it is clear that, \( \int_{t_0}^{t} h(t) \, dt < 0 \) for any \( t_0 \) and \( t \). Since \( E(t) \) is positive, it follows that

\[ H = -\sum_{j=1}^{N} \sum_{k=1}^{N} \dot{q}_j \int_{-\infty}^{t} g_{jk}(t - \tau) \dot{q}_k(\tau) \, d\tau, \quad (71) \]

must be negative. In practical analysis undamped systems may be useful, thus \( H \) is here allowed to be zero as well. Transformation of \( H(t) \), to the frequency domain according to Section 1 yields,

\[ H(\omega) = -\frac{\omega^2}{2} \hat{q}^H [G_0(\omega)] \hat{q} \quad (72) \]

where \( G_0(\omega) \) contains the Fourier transforms of \( -g_{jk} \). \( H(\omega) \) may be seen as the mean rate of energy fed into the system for harmonic time dependence, see [7]. Consequently, \( H(\omega) \leq 0 \) and it follows that \( G_0(\omega) \) must be positive (semi) definite for all possible choices of \( \hat{q} \).

Now, let \( G_0(\omega) \) be decomposed in one symmetric and one antisymmetric part. Since contributions from the antisymmetric part cancel in pairs, only the symmetric part will contribute to \( H(\omega) \), whereas the antisymmetric part will contribute to gyroscopic forces. Since gyroscopic forces are not considered here the antisymmetric part is here set to zero.

The properties of \( G_0 \) also applies to \( G \), as defined in Section 2.4. Thus \( G \) must be symmetric and positive definite.
### B Matrices of plate strip element

Equation (40) gives the mathematical description of a plate strip element. The matrices $a_{00}$, $a_{01}$, $a_{11}$, $a_{02}$, $a_{22}$ and $m_2$ in equation (40) are given by,

$$a_{00} = \begin{bmatrix} hG_{xy}I_{pp11} & hE_y'I_{pp11} & \frac{h^3E_y}{12}I_{bb22} + N_yI_{bb11} \\ hE_y'I_{pp11} & hG_{xy}I_{pp11} & hG_{xy}I_{pp11} \\ \frac{h^3E_y}{12}I_{bb22} + N_yI_{bb11} & hG_{xy}I_{pp11} & \frac{h^3E_y}{12}I_{bb22} + N_yI_{bb11} \end{bmatrix}, \quad (73)$$

$$a_{01} = \begin{bmatrix} hG_{xy}I_{pp10} \\ hE_y'I_{pp10} \\ \frac{h^3E_y}{12}I_{bb10} + N_yI_{bb10} \end{bmatrix}, \quad (74)$$

$$a_{11} = \begin{bmatrix} hE_y'I_{pp00} \\ hG_{xy}I_{pp00} \\ \frac{4h^3E_y}{12}G_{xy}I_{bb11} + N_xI_{bb00} \end{bmatrix}, \quad (75)$$

$$a_{02} = \begin{bmatrix} \frac{E_y'h^3}{12}I_{bb20} \end{bmatrix}, \quad (76)$$

$$a_{22} = \begin{bmatrix} \frac{E_y'h^3}{12}I_{bb20} \end{bmatrix}, \quad (77)$$

and

$$m_2 = \begin{bmatrix} m''I_{pp00} \\ m''I_{pp00} \\ m''I_{bb00} \end{bmatrix}. \quad (78)$$
The block matrices, $I_{ppij}$ and $I_{bbij}$ are defined by,

$$I_{ppij} = \int_{-a}^{+a} \partial^i N_p^T \partial^j N_p \ dy \quad \text{and} \quad I_{bbij} = \int_{-a}^{+a} \partial^i N_b^T \partial^j N_b \ dy$$  \hspace{1cm} (79)

The matrices, $a_{21} = a_{12} = 0$. Due to symmetry, the matrices $a_{10}$ and $a_{20}$ are given by,

$$a_{10} = a_{01}^T \quad a_{20} = a_{02}^T$$  \hspace{1cm} (80)

### C  Program code for conical strip elements

Here a MAPLE program code used for deriving waveguide finite element matrices $a_{ij}$, is presented. The code presented here corresponds to pre-stress in a conical shell. The output of this program can, with minor changes be used in programming environments such as MATLAB. A corresponding code for the elastic energy also been developed and is quite similar to the code presented here.

**Program start.** (‘Budiansky’ refers to reference, [3].)

Prestress according to Budiansky

```maple
> restart; with('linalg');
```

The ‘dummy’ variables $A00R$, $A01R$ . . . must reset, if the program is to be used more than once.

```maple
> A00R:=0;
> A01R:=0;
> A02R:=0;
> A11R:=0;
```
> A12R:=0;
> A22R:=0;

The variation for the Lagrangian is written according to equations (49) and (51) to (53).

> L:=simplify(multiply(multiply(transpose([epsa,epta,epsta,pa,psa,pta]),
[[Ns,0,Nst,-Nst,0,0],[0,Nt,Nst,Nst,0,0],[Nst,Nst,Ns+Nt,Ns-Nt,0,0],
[-Nst,Nst,Ns-Nt,Ns+Nt,0,0],[0,0,0,0,Ns,Nst],[0,0,0,0,Nst,Nt]]),
[eps,ept,epst,p,ps,pt]));

Here, $\epsilon_{l,s} = \delta \hat{\epsilon}_{l,s}$, $\epsilon_{l,\theta} = \delta \hat{\epsilon}_{l,\theta}$, $\epsilon_{l,s\theta} = \delta \hat{\epsilon}_{l,s\theta}$, $\phi_{z} = \delta \hat{\phi}_{z}$, $\phi_{s} = \delta \hat{\phi}_{s}$, $\phi_{\theta} = \delta \hat{\phi}_{\theta}$

Now, let $B_{ua} = \hat{\delta}u$, $B_{u} = \hat{u}$, $B_{va} = \hat{\delta}v$, $B_{v} = \hat{v}$, $B_{wa} = \hat{\delta}w$ and $B_{w} = \hat{u}$.

Furthermore, let differentiation with respect to $s$ be denoted with 'dx'. Differentiation with respect to $\theta$ be denoted with 'dt'. If the differentiation is of a virtual displacement, this is denoted with 'a'. The order of the differentiation is denoted with 0, 1 or 2. For instance,

$$\frac{\partial \delta \hat{v}}{\partial s} = dta0*dx1*Bva.$$

From equations (45) and (47), the strains and rotations are now written,

> eps := dx1*dt0*Bu;
> ept := dx0*dt1*Bv/R+dx0*dt0*sin(alpha)*Bu/R+dx0*dt0*cos(alpha)*Bw/R;
> epst:=1/2*(dt0*dx1*Bv+dx0*dt1*Bu/R-sin(alpha)/R*dt0*dx0*Bv);
> ps := -dt0*dx1*Bw;
> pt := -dx0*dt1*Bw/R+dx0*dt0*Bv*cos(alpha)/R;
> p := -1/2*(dt1*dx0*Bu-R*dx0*dt0*sin(alpha)*Bv/R-dx1*dt0*Bv);
> epsa := dx1*dta0*Bua;
\[ epta := dxa0 \times dta1 \times Bva / R + dxa0 \times dta0 \times \sin(\alpha) \times Bua / R + dxa0 \times dta0 \times \cos(\alpha) \times Bwa / R; \]

\[ epsta := \frac{1}{2} \times (dta0 \times dxa1 \times Bva + dxa0 \times dt1 \times Bua / R - \sin(\alpha) / R \times dt0 \times dx0 \times Bva); \]

\[ psa := -dta0 \times dxa1 \times Bwa; \]

\[ pta := -dxa0 \times dta1 \times Bwa / R + dxa0 \times dta0 \times Bva \times \cos(\alpha) / R; \]

\[ pa := -\frac{1}{2} \times (dta1 \times dxa0 \times Bua / R - dxa0 \times dta0 \times \sin(\alpha) \times Bva / R - dxa1 \times dta0 \times Bva); \]

Differentiation with respect to \( s \) is made for the shape functions, that is replace \( \frac{\partial N_p}{\partial s} = dxa0 \times dx1 \) with \( N01 \) etc. Note that information of types of shape functions are not required in \( N01 \) since this information is contained in the displacements, \( Bua \), \( Bw \) etc.

Consequently, replace \( dxa0 \times dx0 \) in \( L \) with \( N00 \) etc.

\[ La := \text{diff(diff}(L, dxa0), dx0) \times N00 + \text{diff(diff}(L, dxa1), dx0) \times N10 + \text{diff(diff}(L, dxa0), dx1) \times N01 + \text{diff(diff}(L, dxa1), dx1) \times N11 + \text{diff(diff}(L, dxa2), dx0) \times N20 + \text{diff(diff}(L, dxa0), dx2) \times N02 + \text{diff(diff}(L, dxa2), dx1) \times N21 + \text{diff(diff}(L, dxa1), dx2) \times N12 + \text{diff(diff}(L, dxa2), dx2) \times N22 + \text{diff(diff}(L, dxa3), dx0); \]

Collect all terms \( dta0 \times dt0 \) into \( A00 \) and similarly for \( A10, A01, A11, A20, A02 \) and \( A12 \)

\[ A00 := \text{simplify(diff(diff}(La, dta0), dt0)); \]

\[ A10 := \text{simplify(diff(diff}(La, dta1), dt0)); \]

\[ A01 := \text{simplify(diff(diff}(La, dta0), dt1)); \]

\[ A11 := \text{simplify(diff(diff}(La, dta1), dt1)); \]

\[ A20 := \text{simplify(diff(diff}(La, dta2), dt0)); \]

\[ A02 := \text{simplify(diff(diff}(La, dta0), dt2)); \]

\[ A21 := \text{simplify(diff(diff}(La, dta2), dt1)); \]
> A12 := simplify(diff(diff(La,dta1),dt2));
> A22 := simplify(diff(diff(La,dta2),dt2));

Define the vectors,

> V := [Nt,Ns,Nst,alpha,Bua,Bva,Bwa,N00,N01,N11,N20,N02,N22,Bu,Bv,Bw];
> Rm := [Rp2,Rp1,Rm0,Rm1,Rm2,Rm3,Rm4,Rm5];

where Rp2 denotes multiplication with $R(s)^2$, Rm0 denotes multiplication with $R(s)^0 = 1$ and Rm2 denotes multiplication with $\frac{1}{R(s)^2}$, etc.

Now sort A00 so that N00*Bua*Bu is written as Bua*N00*Bu, i.e. with N00 in the middle etc.

> A00:=sort(A00,V);
> A01:=sort(A01,V);
> A02:=sort(A02,V);
> A11:=sort(A11,V);
> A12:=sort(A12,V);
> A22:=sort(A22,V);

Replace $R^1$ with Rp1 and $R^{-1}$ with Rm1 etc.

> for j from -5 to 2 do A00R:=coeff(A00,R,j)*Rm[3-j]+A00R; od;
> for j from -5 to 2 do A01R:=coeff(A01,R,j)*Rm[3-j]+A01R; od;
> for j from -5 to 2 do A02R:=coeff(A02,R,j)*Rm[3-j]+A02R; od;
> for j from -5 to 2 do A11R:=coeff(A11,R,j)*Rm[3-j]+A11R; od;
> for j from -5 to 2 do A12R:=coeff(A12,R,j)*Rm[3-j]+A12R; od;
> for j from -5 to 2 do A22R:=coeff(A22,R,j)*Rm[3-j]+A22R; od;

Define new V, needed later for sorting terms in A00R, A01R etc.
\[ V := \{\alpha, B_{ua}, B_{va}, B_{wa}, N_{00}, N_{01}, N_{10}, N_{11}, N_{20}, N_{22}, R_{m0}, R_{m1}, R_{m2}, R_{m3}, R_{m4}, B_{u}, B_{v}, B_{w}\}; \]

Sort so that \( R_{m2} \cdot B_{ua} \cdot N_{00} \cdot B_{u} \rightarrow B_{ua} \cdot N_{00} \cdot R_{m2} \cdot B_{u} \). Thus, let \( R_{m2} \) be connected with \( N_{00} \). This is needed since integration is made over \( s \) and \( R = R(s) \).

\[ A_{00} := \text{sort(simplify}(A_{00R}), V) \];
\[ A_{01} := \text{sort(simplify}(A_{01R}), V) \];
\[ A_{02} := \text{sort(simplify}(A_{02R}), V) \];
\[ A_{11} := \text{sort(simplify}(A_{11R}), V) \];
\[ A_{12} := \text{sort(simplify}(A_{12R}), V) \];
\[ A_{22} := \text{sort(simplify}(A_{22R}), V) \];

Define

\[ B := \{B_{u}, B_{v}, B_{w}\}; \]
\[ B_{a} := \{B_{ua}, B_{va}, B_{wa}\}; \]
\[ P := \{P_{u}, P_{v}, P_{w}\}; \]
\[ P_{a} := \{P_{ua}, P_{va}, P_{wa}\}; \]

Now, substitute \( B_{ua} \cdot N_{12} \cdot R_{m3} \cdot B_{v} \) with \( C_{2}(P_{ua}, P_{v}, 1, 2, 3) \) for all \( A_{00} \ A_{01} \ A_{10} \) etc. Here, \( C_{2}(P_{ua}, P_{v}, 1, 2, 3) \) defines an integral over \( s \).
A01:=simplify(sort(algsubs((Ba[m]*cat('N',i,j)*Rm[3-k]*B[n]) =
C2(Pa[m],P[n],i,j,-k,A),A01),V)); od; od; od; od; od;

> for m from 1 to 3 do for n from 1 to 3 do for i from 0 to 2 do
for j from 0 to 2 do for k from -5 to 2 do
A02:=simplify(sort(algsubs((Ba[m]*cat('N',i,j)*Rm[3-k]*B[n]) =
C2(Pa[m],P[n],i,j,-k,A),A02),V)); od; od; od; od; od;
> for m from 1 to 3 do for n from 1 to 3 do for i from 0 to 2 do
for j from 0 to 2 do for k from -5 to 2 do
A11:=expand(simplify(sort(algsubs((Ba[m]*cat('N',i,j)*Rm[3-k]*B[n]) =
C2(Pa[m],P[n],i,j,-k,A),A11),V))); od; od; od; od; od;
> for m from 1 to 3 do for n from 1 to 3 do for i from 0 to 2 do
for j from 0 to 2 do for k from -5 to 2 do
A12:=simplify(sort(algsubs((Ba[m]*cat('N',i,j)*Rm[3-k]*B[n]) =
C2(Pa[m],P[n],i,j,-k,A),A12),V)); od; od; od; od; od;
> for m from 1 to 3 do for n from 1 to 3 do for i from 0 to 2 do
for j from 0 to 2 do for k from -5 to 2 do
A22:=simplify(sort(algsubs((Ba[m]*cat('N',i,j)*Rm[3-k]*B[n]) =
C2(Pa[m],P[n],i,j,-k,A),A22),V)); od; od; od; od; od;

Finally, replace $\sin(\alpha)$ with $S_a$ and $\cos(\alpha)$ with $C_a$ and write the results.

> A00:=algsubs(cos(alpha)=Ca,algsubs(sin(alpha) = Sa, A00));

A00:=Nt*C2(Pua,Pu,0,0,2,A)-1/2*Nst*C2(Pva,Pu,0,0,2,A)*Ca^2+
1/2*Nt*C2(Pva,Pv,0,0,2,A)*Ca^2+Nt*C2(Pwa,Pw,0,0,2,A)*Ca^2-

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\[ \text{Nt} \cdot \text{C2}(\text{Pua}, \text{Pu}, 0, 0, 2, A) \cdot \text{Ca}^{-2} + \text{Ns} \cdot \text{C2}(\text{Pva}, \text{Pv}, 1, 1, 0, A) + \\
\text{Nst} \cdot \text{C2}(\text{Pua}, \text{Pu}, 0, 1, 1, A) \cdot \text{Sa} + \text{Nt} \cdot \text{C2}(\text{Pwa}, \text{Pu}, 0, 0, 2, A) \cdot \text{Sa} + \text{Ca} - \\
\text{Nst} \cdot \text{C2}(\text{Pua}, \text{Pv}, 0, 1, 1, A) \cdot \text{Sa} + 1/2 \cdot \text{Nst} \cdot \text{C2}(\text{Pva}, \text{Pv}, 0, 0, 2, A) + \text{Ns} \cdot \text{C2}(\text{Pua}, \text{Pu}, 1, 1, 0, A) + \\
1/2 \cdot \text{Nt} \cdot \text{C2}(\text{Pva}, \text{Pv}, 0, 0, 2, A) + \text{Ns} \cdot \text{C2}(\text{Pua}, \text{Pu}, 1, 1, 0, A) + \\
\text{Nst} \cdot \text{C2}(\text{Pva}, \text{Pv}, 0, 1, 1, A) \cdot \text{Ca} + 1/2 \cdot \text{Nst} \cdot \text{C2}(\text{Pva}, \text{Pv}, 0, 0, 2, A) + \text{Ns} \cdot \text{C2}(\text{Pwa}, \text{Pv}, 0, 0, 2, A) \cdot \text{Sa} + \text{Ca} + \\
\text{Nt} \cdot \text{C2}(\text{Pua}, \text{Pv}, 0, 0, 2, A) \cdot \text{Sa} + 1/2 \cdot \text{Nst} \cdot \text{C2}(\text{Pva}, \text{Pv}, 0, 0, 2, A) - \\
\text{Nt} \cdot \text{C2}(\text{Pua}, \text{Pv}, 0, 0, 2, A) \cdot \text{Sa} + \text{Nt} \cdot \text{C2}(\text{Pwa}, \text{Pu}, 0, 0, 2, A) + \text{Nst} \cdot \text{C2}(\text{Pua}, \text{Pv}, 0, 0, 2, A) - \\
\text{Nt} \cdot \text{C2}(\text{Pua}, \text{Pv}, 0, 0, 2, A) \cdot \text{Sa} + \text{Nst} \cdot \text{C2}(\text{Pva}, \text{Pv}, 0, 0, 2, A) \cdot \text{Sa} + \text{Ca} + \\
\text{Nt} \cdot \text{C2}(\text{Pua}, \text{Pv}, 0, 0, 2, A) \cdot \text{Sa} - \text{Nst} \cdot \text{C2}(\text{Pva}, \text{Pv}, 0, 1, 1, A) \cdot \text{Ca} \\
\]

\[ > \text{A01} := \text{algsubs}(\cos(\alpha) = \text{Ca}, \text{algsubs}(\sin(\alpha) = \text{Sa}, \text{A01})); \]

\[ \text{A01} := \text{Nst} \cdot \text{C2}(\text{Pva}, \text{Pv}, 1, 0, 1, A) - 1/2 \cdot \text{Nt} \cdot \text{C2}(\text{Pva}, \text{Pv}, 0, 0, 2, A) \cdot \text{Sa} + \text{Nst} \cdot \text{C2}(\text{Pwa}, \text{Pv}, 1, 0, 1, A) + 1/2 \cdot \text{Nst} \cdot \text{C2}(\text{Pva}, \text{Pv}, 0, 0, 2, A) \cdot \text{Sa} + \text{Nt} \cdot \text{C2}(\text{Pva}, \text{Pv}, 0, 0, 2, A) \cdot \text{Ca} + \text{Nst} \cdot \text{C2}(\text{Pua}, \text{Pu}, 1, 0, 1, A) - \text{Nt} \cdot \text{C2}(\text{Pva}, \text{Pv}, 0, 0, 2, A) \cdot \text{Sa} + \text{Ca} \]

\[ > \text{A02} := \text{algsubs}(\cos(\alpha) = \text{Ca}, \text{algsubs}(\sin(\alpha) = \text{Sa}, \text{A02})); \]

\[ \text{A02} := 0 \]

\[ > \text{A11} := \text{algsubs}(\cos(\alpha) = \text{Ca}, \text{algsubs}(\sin(\alpha) = \text{Sa}, \text{A11})); \]

\[ \text{A11} := 1/2 \cdot \text{Nt} \cdot \text{C2}(\text{Pua}, \text{Pu}, 0, 0, 2, A) + \text{Nt} \cdot \text{C2}(\text{Pva}, \text{Pv}, 0, 0, 2, A) + \text{Nt} \cdot \text{C2}(\text{Pwa}, \text{Pv}, 0, 0, 2, A) - 1/2 \cdot \text{Nst} \cdot \text{C2}(\text{Pua}, \text{Pu}, 0, 0, 2, A) + \text{Nst} \cdot \text{C2}(\text{Pva}, \text{Pv}, 0, 0, 2, A) \cdot \text{Sa} + \text{Ca} \]

\[ > \text{A12} := \text{algsubs}(\cos(\alpha) = \text{Ca}, \text{algsubs}(\sin(\alpha) = \text{Sa}, \text{A12})); \]

\[ \text{A12} := 0 \]
These are the desired results that can be used for numerical evaluation.

D Rigid body modes in prestressed cylinder

Not all rigid body motions are accurately modelled for pre–stressed shells. This is due to the nonlinear displacement–strain relations being nonzero for some rigid body motions. Although different shell theories restrict different rigid body modes, there is, to the authors knowledge, no theory that allows all rigid body motion. To study which theories apply to related rigid body motion the following analysis is made.

Consider a cylinder with two different coordinate systems as shown in Figure 5. The cylinder has radius 'r' and an unspecified width.

The cylinder, as all free bodies, has six rigid body modes. Due to symmetry of the axes $e_y$ and $e_z$, only four of these are required in this analysis. The four rigid body modes are illustrated in Figure 6.

A rigid body motion is most easily expressed in the coordinates $e_x, e_y$ and $e_z$ whereas strains calculations require displacements expressed in the shell coordinate system, $e_s, e_\theta$ and $e_r$. The relationship between these coordinates are given by,

\[
\begin{bmatrix}
    e_x \\
    e_y \\
    e_z
\end{bmatrix}
= \begin{bmatrix}
    1 & 0 & 0 \\
    0 & \cos(\theta) & \sin(\theta) \\
    0 & -\sin(\theta) & \cos(\theta)
\end{bmatrix}
\begin{bmatrix}
    e_s \\
    e_\theta \\
    e_r
\end{bmatrix}
\]

(81)
Figure 5: Coordinates for cylinder
While considering the displacements, $w$, to be small, they are found from,

$$w = w_0 + \varphi \times r,$$

(82)

where, $w_0$ is the displacement at the origin, i.e. the center of the cylinder, $\varphi$ is the rotation vector about the origin and $r = s e_s + r e_r$ is the position vector of a point on the cylinder.

**Mode A**

$$w_0 = \hat{a} e_x \quad \varphi = 0$$

(83)

where $\hat{a}$ is a constant. Consequently, the displacements are given by,

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \hat{a} & 0 & 0 \end{bmatrix}. \quad (84)$$
Mode B

\[ w_0 = \hat{b} e_z \quad \varphi = 0 \quad (85) \]

where \( \hat{b} \) is a constant. Consequently, the displacements are given by,

\[
\begin{bmatrix}
  u \\
v \\
w
\end{bmatrix} =
\begin{bmatrix}
  0 & -\hat{b} \sin(\theta) & \hat{b} \cos(\theta)
\end{bmatrix}
\quad .
\]

Mode C

\[ w_0 = 0 \quad \varphi = \hat{c} e_y \quad (87) \]

where \( \hat{c} \) is a constant. Consequently, the displacements are given by,

\[
\begin{bmatrix}
  u \\
v \\
w
\end{bmatrix} =
\begin{bmatrix}
  \hat{c} r \cos(\theta) & \hat{c} s \sin(\theta) & -\hat{c} s \cos(\theta)
\end{bmatrix}
\quad .
\]

Mode D

\[ w_0 = 0 \quad \varphi = \hat{d} e_x \quad (89) \]

where \( \hat{d} \) is a constant. Consequently, the displacements are given by,

\[
\begin{bmatrix}
  u \\
v \\
w
\end{bmatrix} =
\begin{bmatrix}
  0 & -\hat{d} r & 0
\end{bmatrix}
\quad .
\]

A few theories for nonlinear displacement to strains of shells of revolution are found in [2]. These are the Rotter–Jumikis theory, the Sander theory, the simplified Sander theory and the Donnell theory. The Stricklin nonlinear shell theory is used in [9]. The Sander–Budiansky theory used for the elements of this thesis can be found in [3]. Strains for the different rigid body motions are found upon inserting the
displacements for the rigid body modes into these theories. For all but the more complex Sander–Budiansky theory, nonzero strains are then directly evident from hand calculations. The rigid body modes permitted, i.e. with zero strain, and restrained, i.e. with nonzero strain, in the different theories are given by Table 1. The results for the Sander–Budiansky theory are given by considering strain energy in a cylindrical waveguide finite element undergoing rigid body motion.

<table>
<thead>
<tr>
<th>Theory</th>
<th>Mode A</th>
<th>Mode B</th>
<th>Mode C</th>
<th>Mode D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rotter–Jumikis</td>
<td>F</td>
<td>F</td>
<td>R</td>
<td>R</td>
</tr>
<tr>
<td>Sander</td>
<td>F</td>
<td>F</td>
<td>R</td>
<td>R</td>
</tr>
<tr>
<td>simplified Sander</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>R</td>
</tr>
<tr>
<td>Donnell</td>
<td>F</td>
<td>R</td>
<td>R</td>
<td>F</td>
</tr>
<tr>
<td>Stricklin</td>
<td>F</td>
<td>F</td>
<td>R</td>
<td>R</td>
</tr>
<tr>
<td>Sander–Budiansky</td>
<td>F</td>
<td>F</td>
<td>R</td>
<td>R</td>
</tr>
</tbody>
</table>

Table 1: Rigid body motion in pre–stressed cylinder; F=free rigid body motion, R=restrained rigid body motion
CHAPTER 3:
STRAIGHT AND CURVED WAVEGUIDE
FINITE ELEMENTS FOR FLUIDS

1 Frequency domain lagrangian

From Chapter 1, the variation of a fluid Lagrangian, $\delta L_f$, is defined by,

$$\delta L_f = -\int_{t_1}^{t_2} \delta \left( U_f - T_f \right) - \delta W_{f,\text{loss}} \, dt,$$

(1)

where $\delta$ denotes first variation, $t_1$ and $t_2$ are the start and end times, $U_f$ and $T_f$ are the fluid potential and kinetic energy and $\delta W_{f,\text{loss}}$ is the virtual work from dissipative forces. In the absence of fluid–shell coupling, e.g. for rigid walls,

$$\delta L_f = 0.$$ 

(2)

All terms in equation (1) are here required to be bilinear or quadratic functionals. This restriction is necessary since linear differential equations are sought. Furthermore, the system state at $t_1$ and $t_2$ is irrelevant, given that harmonic motion over a long period of time is considered. Thus, $t_2$ and $t_1$ may tend to $\pm \infty$ respectively without any loss of information.
Parseval’s identity for two real valued functions, \( f(t) \) and \( g(t) \) yields,

\[
\int_{-\infty}^{+\infty} f(t) g(t) \, dt = \int_{-\infty}^{+\infty} \hat{f}(\omega) \hat{g}(\omega) \, d\omega ,
\]

where \( t \) is time, \( \omega \) is angular frequency, \( * \) denotes complex conjugate and \( \hat{\ } \) denotes the Fourier transform defined by,

\[
\hat{g}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(t) e^{-i\omega t} \, dt .
\]

Applying Parseval’s identity on equation (1) gives,

\[
\delta L_f(\omega) = - \int_{-\infty}^{+\infty} \delta U_f(\omega) - \delta T_f(\omega) - \delta W_{f,\text{loss}}(\omega) \, d\omega .
\]

Calculated response at different frequencies are independent when linear systems are considered. Consequently, a variation formulation defined for each frequency is given by,

\[
\delta L_{f,\omega}(\omega) = \delta T_f(\omega) - \delta U_f(\omega) + \delta W_{f,\text{loss}}(\omega) .
\]

In the following \( \delta L_{f,\omega} \) is referred to as the fluid ‘Lagrangian variation’. Each of the terms on the right hand side of equation (6) are treated independently in Section 2.

2 The fluid Lagrangian variation

In this thesis the fluid is considered to be ‘ideal’ or close to ‘ideal’, i.e. the fluid has low viscosity and low heat conductivity.
2.1 Velocity potential

Acoustic pressure, $p$, the change of density due to acoustic pressure, $\rho_a$, and the fluid particle displacement, $\mathbf{u}_f$ in an ideal, undamped, fluid are related through the velocity potential $\psi$, such that, [1],

$$\frac{\partial \mathbf{u}_f}{\partial t} = -\mu \nabla \psi,$$

(7)

$$\rho_a = \mu \frac{\rho_f}{c_f^2} \frac{\partial \psi}{\partial t},$$

(8)

and

$$p = \rho_f \mu \frac{\partial \psi}{\partial t},$$

(9)

$\rho_f$ is the fluid density at equilibrium, $c_f$ is the sound velocity in an unbounded fluid and $\mu$ is a scaling constant introduced to enhance numerical stability in fluid–shell coupled systems.

2.2 Potential energy

From Temkin [1, Chapter 2.7], the acoustic potential energy per unit volume, $U''_f$ in an ideal fluid is given by,

$$U''_f = \frac{1}{2} \frac{c_f^2}{\rho_f \rho_a^2}$$

(10)

Combining equation (10) with equation (8), and taking the first variation of $U''_f$ and applying the transformation to the frequency domain according to Section 1 yields the expression,
\[ \delta U_f = \int \delta U_f'' dV = \omega^2 \int \mu^2 \frac{\rho_f}{c_f^2} \delta \hat{\psi}^* \hat{\psi} dV \]  

(11)

where \( V \) is the volume of the fluid and \( ^* \) denotes complex conjugate.

### 2.3 Kinetic energy

Also from \[1\], the kinetic acoustic energy, \( T_f'' \) per unit volume in an ideal fluid is given by,

\[ T_f'' = \frac{1}{2} \rho_f \left\| \frac{\partial u_f}{\partial t} \right\|^2, \quad (12) \]

where, \( \left\| \ldots \right\| \) symbolize Euclidean norm. Combining equation (12) with equation (7), taking the first variation of \( T_f'' \) and applying the transformation to the frequency domain according to Section 1 yields,

\[ \delta T_f = \int \delta T_f'' dV = \int \mu^2 \frac{\rho_f}{c_f^2} \nabla \delta \hat{\psi}^H \nabla \hat{\psi} dV, \quad (13) \]

where, \( ^H \) symbolize the conjugate transpose, i.e. \( ^*^T \).

### 2.4 Virtual work from dissipative forces

The virtual work from dissipative forces are treated similarly to those described in Chapter 2. Consequently, it is found that dissipative forces can be accounted for by adding imaginary parts to the coefficients in \( \delta U_f \) and \( \delta T_f \), i.e. to \( \mu^2 \frac{\rho_f}{c_f^2} \) and \( \mu^2 \rho_f \). As in Chapter 2, the signs of the complex parts are given from requirement of energy dissipation. Thus, the coefficients, \( \mu^2 \frac{\rho_f}{c_f^2} \) and \( \mu^2 \rho_f \) in equations (11) and (13) are replaced by,
\[ \mu^2 \rho_f (1 + i \eta_v) \quad \text{and} \quad \frac{\mu^2 \rho_f}{c_f^2} (1 - i \eta_e), \quad (14) \]

where \( \eta_v \geq 0 \) and \( \eta_e \geq 0 \) are, the generally frequency dependent, damping coefficients.

Note that, the existence of a velocity potential, \( \psi \), is valid since \( \mathbf{u}_f \) is irrotational in ideal conditions. In practice, dissipative forces in fluids are commonly due to shear viscosity. This viscosity may cause a rotational velocity field. Thus, an introduction of dissipative forces due to shear viscosity may violate the assumptions leading to equations (11) and (13). However, if shear forces are small compared to the acoustic pressure the assumption of an 'ideal fluid' is still valid in practice.

## 2.5 Lagrangian variational statement

The Lagrangian variation for the fluid is now given by combining equations (6), (11), (13) and (14). The result is simply stated here:

\[ \delta L_f = \mu^2 \int \rho_f (1 + i \eta_v) \nabla \delta \hat{\psi}^H \nabla \hat{\psi} - \omega^2 (1 - i \eta_e) \frac{\rho_f}{c_f^2} \delta \hat{\psi}^* \hat{\psi} \, dV \quad (15) \]

In the succeeding sections the coefficients, \( \rho_f \) and \( c_f \) are considered to be constant within each waveguide finite element. The scaling constant, \( \mu \), is required to be constant within the entire fluid.

## 3 Waveguide finite elements

### 3.1 Shape functions and cross section geometry

The waveguide finite element method yields wave equations along the waveguide. The dependence, with respect to the cross–section of a fluid element, is approxi-
mated with test- and shape-functions for the variation term and the velocity potential, respectively. Throughout this thesis approximating spaces for test- and shape- functions are set equal, this corresponds to Galerkin’s method, [2]. Here, the cross-sections of the fluid elements are triangular and the velocity potential approximation over an element’s cross-section is linear. Finite element methods with these properties are common, see for instance reference [2] or [3]. One advantage with this type of triangular elements is that they are simple and well proven. Furthermore, meshing software for these elements are readily available, for instance in the MATLAB pde-toolbox. The main drawback is the slow convergence to exact solutions due to a coarse approximation by the linear shape approximation space.

3.1.1 Triangle coordinates

The cross-section of a single element is seen in Figure 1. The triangle coordinates, $\xi_1, \xi_2$ and $\xi_3$ are defined as the ratios between the triangle sub areas, $A_1, A_2, A_3$ and the total area, $A$. Hence,

$$\xi_1 = \frac{A_1}{A}, \quad \xi_2 = \frac{A_2}{A} \quad \text{and} \quad \xi_3 = \frac{A_3}{A}.$$  \quad (16)

where, $A = A_1 + A_2 + A_3$.

3.1.2 Shape functions

Consider the straight fluid waveguide finite element in Figure 2. The values of  at the node-lines, 1, 2 and 3 are denoted by, $\hat{\psi}_1, \hat{\psi}_2$ and $\hat{\psi}_3$, respectively.

With linear interpolation, the value of $\hat{\psi}$ at any point within the element is given by,

$$\hat{\psi} = N_f^T (\xi_1, \xi_2, \xi_3) \hat{\Phi} (x).$$  \quad (17)
3.2 Straight element

For a straight waveguide, the gradient $\nabla \hat{\psi}$ is expressed in Cartesian coordinates. By using the shape–functions defined by equations (17) and (18) the approximations,\

$$
\int \nabla \delta \hat{\psi}^* \nabla \hat{\psi} dV = \int \int \delta \hat{\psi}^H \frac{\partial N_f}{\partial y} \frac{\partial N_f^T}{\partial y} \hat{\psi} + \delta \hat{\psi}^H \frac{\partial N_f}{\partial z} \frac{\partial N_f^T}{\partial z} \hat{\psi} dA dx \\
+ \int \int \frac{\partial \delta \hat{\psi}^H}{\partial x} N_f N_f^T \frac{\partial \hat{\psi}}{\partial x} dA dx
$$

(19)

and

where,

$$
N_f = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} \quad \text{and} \quad \hat{\psi}(x) = \begin{bmatrix} \hat{\psi}_1(x) \\ \hat{\psi}_2(x) \\ \hat{\psi}_3(x) \end{bmatrix}.
$$

(18)
Figure 2: Fluid element in Cartesian co-ordinates

\[ \int \delta \hat{\psi}^{*} \hat{\psi} dV = \int \int \delta \hat{\psi}^{H} N_i N_j^{T} \hat{\psi} dA dx \] \hspace{1cm} (20)

follow. Evaluation of these expressions is now developed. The spatial derivatives of the linear shape functions are constant and given by, [2],

\[ \left[ \frac{\partial N_i}{\partial z} \frac{\partial N_j}{\partial y} \right] = \frac{1}{2A} \begin{bmatrix} (y_2 - y_3) & (z_3 - z_2) \\ (y_3 - y_1) & (z_1 - z_3) \\ (y_1 - y_2) & (z_2 - z_1) \end{bmatrix} = B, \] \hspace{1cm} (21)

where \( z_i \) and \( y_i, i = 1, 2, 3, \) are the co-ordinates of node \( i \) and \( A \) is the triangle area of the cross-section. \( A \) is given by,
\[ A = (z_2 - z_1) (y_3 - y_1) - (z_3 - z_1) (y_2 - y_1). \]  

Consequently, since \( B \) is constant, the first area integral of equation (19) is evaluated as,

\[
\int \int \delta \tilde{\psi}^H \frac{\partial N_f}{\partial z} \frac{\partial N_f^T}{\partial z} \tilde{\psi} + \delta \tilde{\psi}^H \frac{\partial N_f}{\partial y} \frac{\partial N_f^T}{\partial y} \tilde{\psi} \, dA \, dx = A \int \delta \tilde{\psi}^H B B^T \tilde{\psi} \, dx. \quad (23)
\]

Evaluations of the second area integral of equation (19) and the area integral of equation (20) are equivalent. First, expansion of \( N_f N_f^T \) yields,

\[
N_f N_f^T = \begin{bmatrix}
\xi_1^2 & \xi_1 \xi_2 & \xi_1 \xi_3 \\
\xi_1 \xi_2 & \xi_2^2 & \xi_2 \xi_3 \\
\xi_1 \xi_3 & \xi_2 \xi_3 & \xi_3^2
\end{bmatrix} \quad (24)
\]

Integration of polynomials in triangular coordinates are given by, [2],

\[
\int \xi_k^l \xi_l^m \, dA = 2A \frac{k! \, l! \, m!}{(2 + k + l + m)!}. \quad (25)
\]

Thus, \( \int N_f N_f^T \, dA \) is evaluated analytically with equations, (22), (24) and (25). Hence, the mathematical description for the waveguide finite element is given by,

\[
\delta L_f = \int \delta \tilde{\psi}^H b_{00} \tilde{\psi} + \frac{\partial \delta \tilde{\psi}^H}{\partial x} b_{11} \frac{\partial \tilde{\psi}}{\partial x} - \omega^2 \delta \tilde{\psi}^H n_2 \tilde{\psi} \, dx \quad (26)
\]

where,

\[
b_{00} = (1 + i \eta_v) \rho_f \mu^2 (A B B^T), \quad (27)
\]

\[
b_{11} = (1 + i \eta_v) \rho_f \mu^2 \int N_f N_f^T \, dA, \quad (28)
\]

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\( \mathbf{n}_2 = (1 - i \eta_c) \frac{\rho f \mu^2}{c_f^2} \int \mathbf{N}_f \mathbf{N}^T_f \, dA . \)  

(29)

A validation example, including coupling to a shell, is given in Chapter 5.

### 3.3 Curved element

A curved fluid element is seen in Figure 3. For this element, cylindrical coordinates are used for expressing \( \nabla \delta \hat{\psi} \) and \( \nabla \hat{\psi} \) and \( dV \). The direction of propagation, \( x \), is replaced with \( \theta \) in equations (17) and (18). As a result the approximations in equation (30) and (31) follow:

\[
\int \nabla \delta \hat{\psi}^* \nabla \delta \hat{\psi} \, dV = \int \int \delta \hat{\psi}^H \left( \frac{\partial \mathbf{N}_f}{\partial x} \frac{\partial \mathbf{N}^T_f}{\partial x} + \frac{\partial \mathbf{N}_f}{\partial R} \frac{\partial \mathbf{N}^T_f}{\partial R} \right) \hat{\psi} \, R \, dA \, d\theta + \\
+ \int \int \frac{\partial \hat{\psi}^H}{\partial \theta} \left( \frac{1}{R^2} \mathbf{N}_f \mathbf{N}^T_f \right) \frac{\partial \hat{\psi}}{\partial \theta} \, R \, dA \, d\theta
\]

(30)

and

\[
\int \delta \hat{\psi}^* \hat{\psi} \, dV = \int \int \delta \hat{\psi}^H \mathbf{N}_f \mathbf{N}^T_f \hat{\psi} \, R \, dA \, d\theta ,
\]

(31)

where, \( dA = dR \, dx \). Evaluation is made numerically, since not all expressions in these equations are polynomials. The relation between triangular and cylindrical coordinates are given by,

\[
\begin{bmatrix}
1 \\
x \\
R
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 1 \\
x_1 & x_2 & x_3 \\
R_1 & R_2 & R_3
\end{bmatrix} \begin{bmatrix}
\xi_1 \\
\xi_2 \\
\xi_3
\end{bmatrix}
\]

(32)
Figure 3: Fluid element in cylinder co-ordinates

For a function, \( f(\xi_1, \xi_2, \xi_3) \), three-point internal Gauss quadrature over a triangle yields, [4],

\[
\frac{1}{A} \int f(\xi_1, \xi_2, \xi_3) \, dA \approx \frac{1}{3} f\left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right) + \frac{1}{3} f\left(\frac{1}{6}, \frac{2}{3}, \frac{1}{6}\right) + f\left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right), \tag{33}
\]

where, the triangle area, \( A \), is given by,

\[
A = (x_2 - x_1) (R_3 - R_1) - (x_3 - x_1) (R_2 - R_1) \tag{34}
\]

and \( x_i \) and \( R_i \) are the coordinates of node \( i \). Consequently, with the aid of equations
(32) to (34) the integrals,

\[
I_1 = \int \frac{\partial N_f}{\partial x} \frac{\partial N_f^T}{\partial x} R \, dA \\
I_2 = \int \frac{\partial N_f}{\partial R} \frac{\partial N_f^T}{\partial R} R \, dA \\
I_3 = \int \left( \frac{1}{R^2} N_f N_f^T \right) R \, dA \\
I_4 = \int N_f N_f^T R \, dA
\]

(35) to (38)

can be easily evaluated. Hence, the mathematical description for the curved waveguide finite element is given by,

\[
\delta L_f = \int \delta \hat{\psi}^H \hat{b}_{00} \hat{\psi} + \frac{\partial \delta \hat{\psi}^H}{\partial \theta} \hat{b}_{11} \frac{\partial \hat{\psi}}{\partial \theta} - \omega^2 \delta \hat{\psi}^H \hat{n}_2 \hat{\psi} \, d\theta ,
\]

(39)

where,

\[
\hat{b}_{00} = (1 + i\eta_e) \rho_f \mu^2 (I_1 + I_2) \\
\hat{b}_{11} = (1 + i\eta_e) \rho_f \mu^2 I_3 \\
\hat{n}_2 = (1 - i\eta_e) \frac{\rho_f \mu^2}{c_f^2} I_4
\]

(40) to (42)

A validation example of this element is also given in Chapter 5.
4 Conclusions

A variation of a Lagrangian, describing the acoustic velocity potential in an non–viscous fluid is derived. It is noted that the potential energy of the variation is associated with time derivatives whereas the kinetic energy is associated with spatial derivatives, which is opposite to solids or shells. Waveguide fluid elements for straight and curved geometries are derived. The element type used is a 2D triangular element with linear interpolation. The main reason for choosing this element type is the easily available meshing software for the element type.

References


CHAPTER 4:
STRAIGHT AND CURVED WAVEGUIDE
FINITE ELEMENTS FOR FLUID–SHELL COUPLING

1 Frequency domain coupling functional

From Chapter 1, a functional describing coupling between a fluid and a shell over a wetted surface $S$, is given by,

$$\delta B_c = -\int_S \rho_f \mu \int_{t_1}^{t_2} \partial \delta \psi \partial t w \, dt \, dS + \delta w \partial \psi \partial t \, dt \, dS,$$

(1)

where, $t$ denotes time, $\rho_f$ is the fluid density at equilibrium, $\mu$ is a positive scaling constant, $\delta$ denotes first variation, $w$ is the shell normal displacement into the fluid and $\psi$ is the fluids velocity potential defined by,

$$\frac{\partial u_f}{\partial t} = -\mu \nabla \psi,$$

(2)

where, $u_f$ is the fluid particle displacement.
The system state at $t_1$ and $t_2$ is irrelevant, given that harmonic motion over a long period of time is considered. Thus, $t_2$ and $t_1$ may tend to $\pm \infty$ respectively without any loss of information.

Parseval’s identity for two real valued functions, $f(t)$ and $g(t)$ yields,

$$\int_{-\infty}^{+\infty} f(t) \ g(t) \ dt = \int_{-\infty}^{+\infty} \hat{f}(\omega)^* \hat{g}(\omega) \ d\omega,$$

where $t$ is time, $\omega$ is angular frequency, * denotes complex conjugate and $\hat{}$ denotes the Fourier transform defined by,

$$\hat{g}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(t) \ e^{-i\omega t} \ dt.$$  

Letting $t_1$ and $t_2$ tend to $\pm \infty$ and applying Parseval’s identity to equation (1) gives,

$$\delta B_c(\omega) = i \omega \int_S \rho_f \mu \int_{-\infty}^{+\infty} \delta \hat{\psi}^* \hat{w} - \delta \hat{w}^* \hat{\psi} \ d\omega \ dS.$$  

Calculation of response at different frequencies are independent when linear systems are considered. Consequently, a variation formulation defined for each frequency is given by,

$$\delta B_c(\omega) = i \omega \mu \int_S \rho_f \left( \delta \hat{\psi}^* \hat{w} - \delta \hat{w}^* \hat{\psi} \right) \ dS,$$

where, $\mu$ is taken out of the integral since it is required to be constant throughout the fluid.
2 Fluid shell coupling elements

2.1 Node numbering

Equation (1) and consequently equation (5) are defined so that a positive value of the shell displacement, \( w \), corresponds to displacement into the fluid. This fact must be considered when fluid–shell coupling is introduced into the waveguide finite element model discussed in Chapter 1. Here, this problem is addressed with implementing requirements on the node numbering of the fluid and shell elements.

2.1.1 Shell element nodes

Plate and shell elements for Cartesian and conical co-ordinates, respectively, are described in Chapter 2. A plate element with local co-ordinate system are seen in Figure 1.

![Figure 1: Plate element](image)

The node numbering for this element is made so that the node associated with the lower value of \( y \) is denoted node 1, and the node associated with the higher value of \( y \) is denoted node 2.
2.1.2 Fluid element nodes

Fluid elements in Cartesian and cylindrical co-ordinates are described in Chapter 3. A straight element with Cartesian co-ordinate system is seen in Figure 2.

![Figure 2: Fluid element](image)

It should be noted that the numbering of the nodes are given counter-clockwise around the element cross-section. This is in agreement with the element is definition in Chapter 3 and with the meshing generator in MATLAB’s pde-toolbox and several other triangular elements, [1].

2.1.3 Coupling element node numbering

The coupling element is created to couple fluid and shell elements. Figure 3 shows two adjoining fluid and plate elements.

The co-ordinates for the two nodes ‘i’ and ‘l’ coincide. Similarly, nodes ‘j’ and ‘m’ also coincide. Node ‘k’ is an internal node in the fluid, not to be connected to a shell element. Upon requesting that the local z-coordinate of the shell element points
into the fluid, a positive out-of-plane displacement, $w$, corresponds to displacement into the fluid. The node numbering is then chosen such that,

Node '$i$' is the local node 1 or 2 of the fluid element.

Node '$l$' is the local node 1 of the plate element.

Node '$j$' is the local node 2 or 3 of the fluid element.

Node '$m$' is the local node 2 of the plate element.

With counter-clockwise node numbering, such as that given from the MATLAB’s pde-toolbox meshing generator, the requirements of the fluid element numbering is automatically satisfied for fluids enclosed by the shell if the local node numbering of the shell also is chosen counter-clockwise. Hence, equation (6) is then valid for fluids enclosed by a shell. For a fluid–shell coupling on the outside of a shell a clockwise node numbering of the shell elements is required, for counter clockwise numbering of the fluid elements.
2.2 Shape–functions

Trial and test–functions are chosen equal in the following and referred to as ‘shape–functions’. The shape–functions for the velocity potential, \( \hat{\psi} \) and the plate or shells out of plane displacement \( \hat{w} \) are described in Chapter 2 and 3 respectively.

2.2.1 Shape functions for plate

From Chapter 2, the out of plane displacement of the plate is denoted \( \hat{w} \). The element interpolation of \( \hat{w} \) and \( \delta \hat{w} \) is written,

\[
\hat{w} = N_b^T(\vartheta) \hat{w}(x) \quad \delta \hat{w}^* = \delta \hat{w}(x)^H N_b(\vartheta),
\]

(7)

where

\[
\hat{w} = \begin{bmatrix} \hat{w}_1 & \hat{\phi}_1 & \hat{w}_2 & \hat{\phi}_2 \end{bmatrix}^T, \quad \delta \hat{w} = \begin{bmatrix} \delta \hat{w}_1 & \delta \hat{\phi}_1 & \delta \hat{w}_2 & \delta \hat{\phi}_2 \end{bmatrix}^T,
\]

(8)

and

\[
N_b(\vartheta) = \begin{bmatrix} \frac{1}{4} (2 - 3\vartheta + \vartheta^3) \\ \frac{a}{4} (1 - \vartheta - \vartheta^2 + \vartheta^3) \\ \frac{1}{4} (2 + 3\vartheta - \vartheta^3) \\ \frac{a}{4} (-1 - \vartheta + \vartheta^2 + \vartheta^3) \end{bmatrix}.
\]

(9)

\( w_1, w_2, \theta_1 \) and \( \theta_2 \) are the displacements and rotations about the z-axis for the respective nodes.

2.2.2 Shape functions for fluid

From Chapter 3 it is found that the interpolation of \( \hat{\psi} \) on the wetted surface between nodes 1 and 2 is written,
\[
\hat{\psi} = \begin{bmatrix}
\xi_1 & \xi_2
\end{bmatrix}
\begin{bmatrix}
\hat{\psi}_1(z) \\
\hat{\psi}_2(z)
\end{bmatrix}
\]  \hspace{1cm} (10)

where, \(\hat{\psi}_1(z)\) is the value of \(\hat{\psi}\) at node 1 and \(\hat{\psi}_2(z)\) is the value of \(\hat{\psi}\) at node 2. The triangular co-ordinates \(\xi_1\) and \(\xi_2\) vary linearly between node 1 and node 2. At node 1,

\[
\begin{bmatrix}
\xi_1 & \xi_2
\end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}
\]  \hspace{1cm} (11)

and at node 2,

\[
\begin{bmatrix}
\xi_1 & \xi_2
\end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}
\]  \hspace{1cm} (12)

Thus, on the wetted surface, the relation between \(\xi_1\), \(\xi_2\) and \(\vartheta\) is,

\[
\xi_1 = \frac{1}{2} (1 - \vartheta) \hspace{0.5cm} \text{and} \hspace{0.5cm} \xi_2 = \frac{1}{2} (1 + \vartheta)
\]  \hspace{1cm} (13)

Consequently, the shape functions for \(\hat{\psi}\) along the wetted surface is the same as those for the linearly dependent in-plane displacements of a plate strip, i.e. \(\hat{u}\) and \(\hat{v}\), see Chapter 2. Hence, following the notations in Chapter 2, \(N_p\) is now defined as,

\[
N_p = \begin{bmatrix}
\xi_1 & \xi_2
\end{bmatrix}^T
\]  \hspace{1cm} (14)

\textbf{2.3 Straight coupling element}

In Cartesian co-ordinates a small wetted surface area element is, \(dS = dx \, dy\). Consequently, with the interpolations for \(\hat{\psi}\) and \(\hat{w}\) given above, the sought coupling
element is described by,

\[
\delta B_{f,\omega} = i\omega \int \left[ \delta \hat{\psi}^H \delta \hat{w}^H \right] m_1 \left[ \begin{array}{c} \hat{\psi} \\ \hat{w} \end{array} \right] dx
\]  
(15)

where,

\[
m_1 = \rho_f \mu \left[ \begin{array}{c} I_1 \\ -I_1^T \end{array} \right],
\]  
(16)

and

\[
I_1 = a \int_{-1}^{+1} N_p(\vartheta) N_b^T(\vartheta) d\vartheta,
\]  
(17)

This integral is evaluated analytically in Chapter 2.

### 2.4 Curved coupling element

For the conical coordinates, a small surface area element is \(dS = R(s) ds d\theta\), and thus,

\[
\delta B_{f,\omega} = i\omega \int \left[ \delta \hat{\Psi}^H \delta \hat{\mathbf{w}}^H \right] m_1 \left[ \begin{array}{c} \hat{\Psi} \\ \hat{\mathbf{w}} \end{array} \right] d\theta
\]  
(18)

where,

\[
m_1 = \rho_f \mu \left[ \begin{array}{c} I_2 \\ -I_2^T \end{array} \right],
\]  
(19)

and, as in Chapter 2,
\[ I_2 = \int (N_p N^T_b) R(s) \, ds \] (20)

is evaluated with 4–point Gauss quadrature.

3 Conclusions

Waveguide finite elements coupling fluid acoustic and shell motion in the frequency domain are presented. The first element is for coupling straight fluid and plate elements whereas the second is for curved elements. Here, correct node numbering is found to be critical, since the shells displacement must be directed into the fluid. Integration routines from the previously derived plate and shell elements are applicable to the coupling elements. The matrix describing the coupling is imaginary and antisymmetric. Consequently, it should be noted that the fluid shell coupling, mathematically, has the same form as gyroscopic forces in shells or solids.

References

CHAPTER 5:  
WAVEGUIDE FINITE ELEMENT VALIDATIONS

1 Dispersion relations

The waveguide finite elements described in Chapters 1 to 4 are used for assembling wave equations in a weak form. For a straight waveguide these are written,

$$\int \sum_{k=0}^{2} \sum_{l=0}^{2} \frac{\partial^k \delta W^H}{\partial x^k} A_{kl} \frac{\partial^l W}{\partial x^l} + i\omega \delta W^H M_1 W - \omega^2 \delta W^H M_2 W \, dx = \int \delta W^H F \, dx ,$$  \hspace{1cm} (1)

where $W = \left[ \Psi^T \Phi^T \right]^T$, $\Psi$ is a vector of the degrees of freedom in the assembled fluid and $\Phi$ is a vector of the degrees of freedom in the assembled shell. By repeated integration by parts and use of calculus of variation, see [1, p.166–169], the strong differential form follows. Upon disregarding natural type boundary terms this form is written:

$$\left[ \sum_{j=0}^{4} K_j \frac{\partial^j}{\partial x^j} + i\omega M_1 - \omega^2 M_2 \right] W = F ,$$  \hspace{1cm} (2)

where,
\[
K_j = \sum_{k=0}^{2} \left( (-1)^{j-k} A_{(j-k)k} \right), \quad k \leq j \leq 4 . \tag{3}
\]

In the absence of damping and external forces, propagating wave solutions given by,

\[
W = \hat{W} e^{-i\kappa x} ,
\tag{4}
\]

for real valued wavenumbers, \( \kappa \), may be obtained via the eigenvalue problem,

\[
\begin{bmatrix}
\sum_{j=0}^{4} K_j (-i\kappa)^j + i\omega M_1 - \omega^2 M_2 \\
\end{bmatrix} \hat{W} = 0
\tag{5}
\]

Equation (5) is a twin valued eigenproblem with solutions either for a known frequency, \( \omega \), or for a known wavenumber, \( \kappa \). In this chapter, \( \omega \) is found as a function of \( \kappa \). This is the easier problem of the two to solve. In the general case with fluid–shell coupling, \( M_1 \) is nonzero and equation (5) may be expanded according to Chapter 1 before solving the problem, see [2]. For reference, the eigenproblems are solved with the ‘eig’ or ‘polyeig’ commands in MATLAB 6.5. Relations between the wavenumbers and corresponding frequencies are called dispersion relations. Since dispersion relations are easily compared with each other they are well suited for comparing or validating different wave equations. Here, dispersion relations are used for all validations.

Analytically derived dispersion relations may be found for some simple geometries. Likewise, dispersion relations found in the literature are almost always given for relatively simple geometries. Consequently, in this chapter, waves in simply supported plates, acoustic waves in 2D curved ducts and waves in toroid shells and straight fluid filled pipes are studied.
2 Orthotropic prestressed plate

The orthotropic prestressed plate element derived in Chapter 2 is considered here. Validation for both in–plane and out–of–plane motion for the isotropic, non pre-stressed case is given in Section 7, where a fully coupled fluid filled pipe is modelled.

Out–of–plane motion of an orthotropic prestressed plate strip is modelled with an analytical model and with waveguide finite elements. Dispersion relations derived from the analytical model are compared with those calculated from the waveguide FE model.

2.1 Plate strip model

The plate strip, seen in Figure 1, is simply supported along the edges, $y = 0$ and $y = L$.

![Figure 1: The plate model assemblage](image)

The material parameters chosen for this model are given in Table 1, The parameters are such that both pre-stress and orthotropy have a significant effect on the dispersion relations.
Table 1: Parameters for orthotropic plate strip

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Width:</td>
<td>$L$ 0.2 m</td>
</tr>
<tr>
<td>Young’s modulus $x$–direction:</td>
<td>$E_x$ 200 GPa</td>
</tr>
<tr>
<td>Young’s modulus $y$–direction:</td>
<td>$E_y$ 50 GPa</td>
</tr>
<tr>
<td>Poisson’s ratio $x$–direction:</td>
<td>$\nu_x$ 0.3</td>
</tr>
<tr>
<td>Poisson’s ratio $y$–direction:</td>
<td>$\nu_y$ 0.075</td>
</tr>
<tr>
<td>Thickness:</td>
<td>$h$ 1 mm</td>
</tr>
<tr>
<td>Density:</td>
<td>$\rho$ 10 · 10$^3$ kg</td>
</tr>
<tr>
<td>Prestress:</td>
<td>$N_x, N_y$ 50 kN/m</td>
</tr>
</tbody>
</table>

2.2 Analytical model

The homogeneous equation of motion for the plate is given by,

$$B_x \frac{\partial^4 w}{\partial x^4} + 2B_{xy} \left( \frac{\partial^4 w}{\partial x^2 \partial y^2} \right) + B_y \frac{\partial^4 w}{\partial y^4} - N \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) - \omega^2 m'' w = 0 \quad (6)$$

where $w$ is the displacement, $N = N_x = N_y$ is the prestress per unit length, $B_x, B_y$ and $B_{xy}$ are flexural rigidities and $m''$ is the mass per unit area.

The relations between flexural rigidity and material parameters are found in reference [3].

$$B_x = \frac{E_x h^3}{12 (1 - \nu_x \nu_y)},$$

$$B_y = \frac{E_y h^3}{12 (1 - \nu_x \nu_y)},$$

$$2B_{xy} = B_x \nu_y + B_y \nu_x + 4 \left( \frac{G_{xy} h^3}{12} \right),$$

where $G_{xy}$ is the material shear modulus, here chosen as,

$$G_{xy} = \frac{E_x}{2 (1 + \nu_x)}. \quad (8)$$
Simply supported edges yield the boundary conditions,

\[ w = 0, \quad (9) \]

and

\[ \frac{\partial^2 w}{\partial y^2} = 0, \quad (10) \]

at \( y = 0 \) and \( y = L \).

With wave propagation in the positive x-direction, the solutions to equation (6) are given by,

\[ w(x, y) = e^{-i\kappa x} \sin\left(\frac{p\pi y}{L}\right) \quad \text{for} \quad \kappa \geq 0 \quad \text{and} \quad p = 1, 2, 3\ldots \quad (11) \]

Inserting these solutions into equation (6) and solving the resulting dispersion relation yields the frequencies \( \omega \) for any pair of wavenumbers \( \kappa \) and integers \( p \).

### 2.3 Waveguide FE solution of plate

The waveguide FE-model described in this paper is composed from assembling elements to a plate and imposing simply supported boundary conditions at the end nodes. As there is no fluid in this model, equation (5) is written as,

\[
\sum_{j=0}^{4} K_j (-i\kappa)^j - \omega^2 M_2 \hat{\Phi} = 0.
\]

### 2.4 Dispersion relations for plate strip

Dispersion relations for two, four, eight and sixteen elements have been calculated. The dispersion relations for the 'exact' model and the coarse two element model is seen in Figure 2. Two remarks may be worth mentioning with regard to this figure.
First, higher order waves, i.e. $p = 3$ and $p = 4$, will have a poorer convergence rate than the lower order modes since the cross-section modes are more intricate for these waves. Secondly, since the potential energy, represented by the stiffness matrices, $K_j$, involves spatial derivatives it should converge slower than the kinetic energy represented by the mass matrix, $M_2$. Thus, the matrices, $K_j$, govern the convergence rate and since these are generally too stiff, the frequencies calculated with the waveguide FE method will be higher than the exact value.

Figure 2: Dispersion relations for plate strip. From the left, branches correspond to $p = 1, 2, 3$ and 4. '•' solution from two element plate model; '○' solution from 'exact' model.
Table 2: Frequencies for ‘exact’ solution and waveguide FE models

<table>
<thead>
<tr>
<th>wavenumber: $\kappa$</th>
<th>'exact'</th>
<th>2 elements</th>
<th>4 elements</th>
<th>8 elements</th>
<th>16 elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>368.1270</td>
<td>373.6417</td>
<td>368.2949</td>
<td>368.1356</td>
<td>368.1275</td>
</tr>
<tr>
<td>4</td>
<td>372.4015</td>
<td>377.8736</td>
<td>372.5680</td>
<td>372.4101</td>
<td>372.4020</td>
</tr>
<tr>
<td>6</td>
<td>377.7179</td>
<td>383.1383</td>
<td>377.8825</td>
<td>377.7263</td>
<td>377.7183</td>
</tr>
<tr>
<td>8</td>
<td>385.1128</td>
<td>390.4640</td>
<td>385.2750</td>
<td>385.1211</td>
<td>385.1132</td>
</tr>
<tr>
<td>10</td>
<td>394.5430</td>
<td>399.8099</td>
<td>394.7023</td>
<td>394.5511</td>
<td>394.5434</td>
</tr>
</tbody>
</table>

Table 3: Relative errors of second cut on frequency compared to ‘exact’ solution

<table>
<thead>
<tr>
<th>Number of elements</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.014981</td>
<td>0.0004561</td>
<td>2.344e-005</td>
<td>1.331e-006</td>
<td>8.0895e-008</td>
<td>4.7874e-009</td>
</tr>
</tbody>
</table>

The error behaviour discussed previously is seen in Figure 2. For models with more than two elements the differences seen in Figure 2 are difficult to display. For these models, results may be tabulated instead of plotted. Results for $p = 2$ are found in Table 2.

By studying Table 2 it should be noticed that the differences between models with different number of elements decrease slightly for higher $\kappa$. The reason for this is that higher values of $\kappa$ relates to derivatives with respect to $x$ more than with respect to $y$. For instance, $K_4$ does not represent any derivative with respect to $y$ at all, see Chapter 2. It is easier to approximate cross sectional shapes with piecewise polynomials than it is to approximate derivatives of the same shapes with derivatives of the same piecewise polynomials. Consequently, for higher wavenumbers the approximated frequencies are more accurate.

Cut–on frequencies are the associated frequencies for which a wave in an undamped
system starts to propagate without decay. For the present model cut on occurs at wavenumbers $\kappa = 0$. Also for $\kappa = 0$, equation (12) turns into a standard 2D, for the plate 1D, FE-model. The convergence rate can then be estimated analytically if it is governed by flexural rigidity only, see reference [4]. The relative error between the 'exact' and 'waveguide FE' cut on frequency of the second wave using 2 up to 64 elements is given in Table 3. For more than 4 elements the error decreases by a factor ranging between 16.9 and 17.1 for each doubling of the number of elements. This is close to the theoretical value of $2^4 = 16$ given by Fried in [4].

3 Toroidal shell

In this section toroidal shells are studied. It is important to note that no analytical solution exist for conical shells or shells built up from conical shell strips. Furthermore, the conical shell elements in Chapter 2 uses Sander’s shell theory. Validation of these elements should then be preferably be made by comparison with examples also using the theory due to Sander. Here, validation of the conical element is made by studying an isotropic toroidal shell. In this section a non prestressed toroidal shell is considered whereas a prestressed toroidal shell is studied in Section 4. Resonances for non prestressed toroids was studied by Balderes and Armenekas who used a shell theory due to Reissner and Love, that was found to yield results close to that due to Sander, see reference [5]. In reference [5] the equations of motion were solved by assuming a Fourier series expansion over the cross-section resulting in a non–linear eigenproblem solved iteratively.

Solutions for undamped propagating waves in the $\theta$ direction are, with polar wavenumber $\gamma$,

$$\mathbf{W} = \hat{\mathbf{W}} e^{-i\gamma \theta}, \quad (13)$$
and the corresponding eigenproblem is given by,

\[
\left[ \sum_{j=0}^{4} K_j (-i\gamma)^j - \omega^2 M_2 \right] \Phi = 0.
\] (14)

For an axisymmetric structure integer values of \( \gamma \) correspond to resonance frequencies. Rigid body motion corresponds to two resonance frequencies equal to zero, for \( \gamma = 0 \), and two resonance frequencies equal to zero, for \( \gamma = 1 \). Other resonance frequencies for a toroidal shell are given by Balderes, [5]. For the purpose of this validation, the non-dimensional frequency,

\[
\alpha = \frac{\omega R}{\sqrt{\rho/E}},
\] (15)

is defined, where the radius \( R \) is seen in Figure 3 together with the minor radius, \( r \) and shell thickness, \( h \).

Consider the case for which, \( R/r = 10 \), \( h/R = 0.001 \) and \( \nu = 0.3 \). For this case, Figure 4 shows integer values of \( \gamma \) plotted against \( \alpha \) given from reference [5] and from a waveguide FE model with 16 elements around the cross-section circumference. Also in Figure 4, rigid body motions are indicated at the origin and at \( \gamma = 1, \alpha = 0 \) and at \( \gamma = 1, \alpha = 0 \). Table 4 tabulates the above mentioned values together with values obtained using 32 elements.

Table 5 shows corresponding values using parameter values \( R/r = 10 \), \( h/R = 0.002 \) and \( \nu = 0.3 \). Table 6 shows corresponding values for \( R/r = 5 \), \( h/R = 0.002 \) and \( \nu = 0.3 \). The eigenfrequencies for the 32 element models are close to converged and models using more elements will give practically identical results.

Comparison with rigid body motions yield good results. This implies that the eigenproblem is solved with high accuracy. Eigenfrequencies for the waveguide FE models most often decrease with more elements. This is expected from the same arguments as for the plate element in the previous Section. Also, frequencies for the waveguide
FE models are generally slightly lower than those given in reference,[5]. The convergence rate for the lowest frequency, $\alpha$ at $\gamma = 2$ for $R/r = 10$, $h/R = 0.001$ and $\nu = 0.3$ is found in Table 7.

Figure 3: Toroid with co-ordinates
Figure 4: Balderes (o) vs. Waveguide FE.(●), 16 elements
<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>Rigid body motion</th>
<th>Balderes [5]</th>
<th>Fine mesh</th>
<th>Coarse mesh</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
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</tr>
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<td>0.0499</td>
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<td></td>
</tr>
<tr>
<td>5</td>
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<td>0.2259</td>
<td>0.2274</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.2713</td>
<td>0.2682</td>
<td>0.2704</td>
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</tr>
</tbody>
</table>

Table 4: Frequency parameter, $\alpha$ for $R/r = 10$ and $h/r = 0.001$ Fine mesh uses 32 elements, coarse mesh uses 16 elements.
<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>Rigid body motion</th>
<th>Balderes [5]</th>
<th>Fine mesh</th>
<th>Coarse mesh</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>7.9802e-007i</td>
<td>5.3134e-007i</td>
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</tr>
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</table>

Table 5: Frequency parameter, $\alpha$ for $R/r = 10$ and $h/r = 0.002$ Fine mesh uses 32 elements, coarse mesh uses 16 elements.
<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>Rigid body motion</th>
<th>Balderes [5]</th>
<th>Fine mesh</th>
<th>Coarse mesh</th>
</tr>
</thead>
<tbody>
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<td>4.0588e-008i</td>
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</tr>
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</table>

Table 6: Frequency parameter, $\alpha$ for $R/r = 5$ and $h/r = 0.002$ Fine mesh uses 32 elements, coarse mesh uses 16 elements.

<table>
<thead>
<tr>
<th>Number of elements</th>
<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
<th>128</th>
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<td></td>
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<td>0.046338</td>
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</table>

Table 7: Convergence for lowest $\alpha$ at $\gamma = 2$, corresponding to Table 4
4 Prestressed toroid shell

A prestressed toroid shell is considered in this Section. Calculated estimates of resonance frequencies for such a shell are found in references Liepins, [6], and Plaut et.al., [7]. Both references use Sander–Budiansky theory. Both in [6] and [7] separation of variables proportional to \( cos(n\theta) \) and \( sin(n\theta) \) is made. The resulting differential equations with respect to \( \varphi \) are solved by applying a central difference method followed by iterations in [6], and with Galerkin’s method in [7].

As for the non prestressed case, rigid body motion for \( \gamma = 0 \) and \( \gamma = 1 \), corresponding to resonance frequencies equal to zero are also used for reference. Here, the non–dimensional frequency, \( \beta \),

\[
\beta = \omega \frac{R^2}{r} \sqrt{\frac{\rho}{E}}
\]

where, \( \rho \) is the density and \( E \) is Young’s modulus, is used for validation. Furthermore, from [6], the prestress is given by,

\[
N_\theta = \frac{pr}{2}, \quad N_\varphi = \frac{pr}{2} \left( \frac{2 + \frac{r}{R} \cos \varphi}{1 + \frac{r}{R} \cos \varphi} \right), \quad N_{\theta\varphi} = 0,
\]

where \( p \) is the internal pressure applied to the toroid. For all results presented here the following values are used,

\[
\eta = \frac{r}{R} = 0.3 \quad \text{and} \quad \zeta = \frac{pr}{Eh} = 0.002.
\]

Results for a 16 element model is plotted in Figure 5 together with the results of [6] and [7]. Similar calculations are also found in Table 8, where \( \beta^2 \) is tabulated, and, as in [6] and [7], given with two digit precision.

Convergence to \( \beta^2 \approx 0.04 \) at \( \gamma = 0 \) are given in Table 9.
Figure 5: Frequency parameters for prestressed toroid; (dot) waveguide FE; (o) Liepins; (□) Plaut et.al.
<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>Liepins</th>
<th>Plaut et al.</th>
<th>Waveguide–FE</th>
</tr>
</thead>
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<tr>
<td>0</td>
<td>0.39</td>
<td>0.40</td>
<td>0.38</td>
</tr>
<tr>
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<td>0.04</td>
<td>0.04</td>
<td>0.05</td>
</tr>
<tr>
<td>1</td>
<td>0.38</td>
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<td>0.05</td>
</tr>
<tr>
<td>2</td>
<td>0.04</td>
<td>0.04</td>
<td>0.04</td>
</tr>
</tbody>
</table>

Table 8: Squared frequency parameter: $\beta^2$. Waveguide FE with 16 elements

<table>
<thead>
<tr>
<th>Number of elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
</tr>
<tr>
<td>0.05656</td>
</tr>
</tbody>
</table>

Table 9: Squared frequency parameter: $\beta^2$ at $\gamma = 0$

The convergence given by Table 9 is monotonic, and the error in $\beta$ decrease with a factor about 4 for each element doubling from 16 elements to 128 elements. This may indicate that the convergence is not governed by flexural.

Note that only two rigid body motions are predicted accurately. This is not primarily a problem with the finite element formulation but rather a problem associated with the underlying shell theory, see Appendix D in Chapter 2. There, by studying a small cylindrical element described with different nonlinear shell theories, it is found that rigid body motions are impaired for any chosen theory.

A toroid in which $\zeta = 0.011$ and $\eta = 0.16$ is also considered in [7]. For this case, comparison with waveguide FE models yield poorer results. The convergence for the waveguide FE is still monotonic and there is no apparent reason to suspect numerical problems. One possible reason for this discrepancy might instead be that the waveguide finite elements are simply curved only. In situations where flexural
rigidity is dominating, i.e. $\zeta$ is lower, the prestressed elements are here considered accurate for practical applications. This is the case for the car tyre modelled in Chapter 8.

5 Straight duct: fluid only

A straight fluid filled duct with rigid walls, as seen in Figure 6, is studied.

Upon the assumptions $L_y \ll L_z$ and $L_y \ll \lambda$, where $\lambda$ is the wavelength for free propagation, the wave motion in the duct can be considered two dimensional. Then the $n^{th}$ order waves, ($n = 0, 1, 2, \ldots$), for the velocity potential, $\psi$, with frequency

$$\omega = c_f \sqrt{\left(\kappa^2 + \frac{n^2 \pi^2}{L_z^2}\right)}, \quad (19)$$

are given by,

$$\hat{\psi} = \tilde{\psi} \cos \left(\frac{n\pi y}{L_z}\right) e^{-i\kappa x}, \quad (20)$$
where, $c_f$ is the plane wave phase velocity and $\kappa$ is the wavenumber and $\tilde{\psi}$ is the complex amplitude of the wave’s velocity potential. By introducing the non dimensional frequency,

$$\alpha = \frac{\omega L_z}{c_f} \quad (21)$$

and the non dimensional wavenumber,

$$\gamma = \kappa L_z, \quad (22)$$

dispersion relations for all possible values of $c_f$ and $L_z$ are found from one set of dispersion relations only.

Two different meshes for fluid element models are given in Figure 7.

Figure 7: Two finite element meshes generated for duct problem; coarse mesh 72 elements; fine mesh 128 elements.

Dispersion relations for the coarser mesh against the analytical solutions are given in
Figure 8. The same conclusions for the convergence may be drawn from this case as for the plate strip in Section 2. Thus, for $\gamma = 0$, the waveguide–FE model turns into an ordinary 2D FE model. For higher wavenumbers, $\gamma$, the waveguide–FE model yields better results than for low wavenumbers, see Section 2.

Since the fluid elements are linear polynomials the rate of convergence is not as good as for the plate element in the previous section.

![Graph](image.png)

Figure 8: Frequency parameter $\alpha$ for a straight fluid filled duct; (\cdot) waveguide FE (coarse mesh); (o) Analytical, from the left branches correspond to $n = 0, 1, 2, 3, 4$.

Frequencies at the second cut on, i.e at $\alpha = 2\pi$, $\gamma = 0$, are tabulated in Table
Number of elements

<table>
<thead>
<tr>
<th>18</th>
<th>72</th>
<th>256</th>
<th>1078</th>
<th>exact</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.6731</td>
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<td>6.2998</td>
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<td>$2\pi \approx 6.2832$</td>
</tr>
</tbody>
</table>

Table 10: Asymptotic convergence for lowest $\alpha$ for $n = 2$

11 together with values for finer meshes. An increase in the number of elements by approximately a factor 4, means that the number of elements along $L_z$ increase by approximately a factor 2. Since linear elements are used the rate of convergence should then be approximately $2^2 = 4$, see [4]. This is close to the rate of convergence found from Table 10, which gives an error decrease of a factor 4.5 when the number of elements increase from 72 to 256 and 4.4 when they increase from 256 to 1078.

For $n = 0$, the analytical solutions are, $\alpha = \gamma$. Differences between $\alpha$ obtained from waveguide FE models and from the analytical solution are very small. For $\gamma = 0$ the absolute error is $7 \cdot 10^{-7}$ and for $\gamma = 1, 2, \ldots, 10$ the absolute error is less than $5 \cdot 10^{-13}$. These notable results are expected since $n = 0$ corresponds to a plane wave in the x-direction, which in principle can be modelled exactly with the piecewise linear polynomials used for the cross-section of the waveguide FE model.
<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>analytic</th>
<th>coarse</th>
<th>fine</th>
</tr>
</thead>
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<tr>
<td>0</td>
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</tr>
<tr>
<td>1</td>
<td>6.3623</td>
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<td>6.3787</td>
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<td>6.5938</td>
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</table>

Table 11: Calculated values of $\alpha$ for second cut on $(n = 2)$ wave in duct
6 Curved duct; fluid only

In this section a curved duct with rigid walls enclosing a fluid, as seen in Figure 9, is studied.

As for the straight duct it is assumed that $L_z \ll L_R$ and $L_z \ll \lambda$, where $\lambda$ is the wavelength for free propagation. Thus, the solutions are independent of $z$. Solutions for such 2D curved duct are given by Rostafinski, [8]. Accordingly, the fluid velocity potential, $\hat{\psi}$ for a propagating wave of order $n$, with polar wavenumber $\gamma$ at frequency $\omega$ is given by,

$$\psi = \hat{\psi} e^{-i\gamma \theta} F_\gamma (r) / \sin(\pi \gamma)$$

(23)

where, $\hat{\psi}$ is the complex amplitude and the function $F_\gamma (r)$ is defined as,
\[ F_\gamma (r) = J_\gamma (kr) J'_{-\gamma} (kR_1) - J'_{-\gamma} (kr) J_\gamma (kR_1) , \]  

(24)

where, \( k = \omega/c_0 \), \( J_\gamma (kr) \) are Bessel functions of the first kind with order \( \gamma \) and \( J'_{\gamma} (kr) \) is the derivative of \( J_\gamma (kr) \) with respect to \( kr \).

The relationship between wavenumbers \( \gamma \) and frequency parameters \( kR_1 \) are given by solutions of the equation,

\[ F'_\gamma (R_2) = 0 \]  

(25)

where, \( F'_\gamma \) is the derivative of \( F_\gamma \) with respect to \( r \), see [8]. Tabulated values for dispersion relations given by equation (25) are also found in [8].

For the waveguide FE model, the meshes in Figure 7 are used. Dispersion relations for \( R_2/R_1 = 2 \) for the coarse mesh waveguide FE model and the analytical model are seen in Figure 10. Tabulated values for the second order wave are given in Table 12.
Figure 10: Dispersion relation for $R_2/R_1 = 2$; (○) Rosafinski, [8]; (●) WFE.
<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>Rostafinski [8]</th>
<th>72 elements</th>
<th>128 elements</th>
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</tbody>
</table>

Table 12: Frequency parameter: $kR_1$
7 Fluid filled pipe

A straight fluid–filled pipe, as seen in Figure 11, is studied in this section.

Investigation of fluid filled pipes with full coupling between fluid and shell are described by Finnveden, [9]. Finnveden assumes trigonometric dependence with respect to $\varphi$, whereas high order polynomials are used for the radial dependence. This yields a very good convergence so that results from the software code described in [9] can be seen as highly accurate. In the following the code in [9], with a 6 degree polynomial for the fluids radial dependence, is used as reference.

The material parameters are given in Table 13. These values are chosen such that the fluid shell coupling significantly affects both the predominantly fluid waves as well as the predominantly structural waves.

Two different meshes are used for waveguide FE models. These are seen in Figure 12.
Table 13: Parameters used for fluid filled pipe models

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s modulus</td>
<td>$E$</td>
<td>210 GPa</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>$\nu$</td>
<td>0.3</td>
</tr>
<tr>
<td>Shell density</td>
<td>$\rho$</td>
<td>7800 kg/m³</td>
</tr>
<tr>
<td>Shell thickness</td>
<td>$h$</td>
<td>5 mm</td>
</tr>
<tr>
<td>Pipe radius</td>
<td>$r$</td>
<td>0.1 m</td>
</tr>
<tr>
<td>Fluid wave speed</td>
<td>$c_f$</td>
<td>1500 m/s</td>
</tr>
<tr>
<td>Fluid density</td>
<td>$\rho_f$</td>
<td>1000 kg/m³</td>
</tr>
</tbody>
</table>

The coarser mesh has 16 plate strip elements and 64 fluid elements. The finer mesh has 32 plate strip elements and 256 fluid elements. This corresponds to a total of 137 and 337 degrees of freedom respectively. Note that these meshes are not entirely symmetric with respect to the $y$–axis or the $z$–axis. As a result, the waveguide FE models yield two slightly separated wavenumbers for each wavenumber found in a perfectly symmetric pipe. The frequencies tabulated in Table 14, relate to the
average of each frequency pair. The dispersion relation is also plotted in Figure 13.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>Finnveden</th>
<th>coarse mesh</th>
<th>fine mesh</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>235.00</td>
<td>243.74</td>
<td>237.56</td>
</tr>
<tr>
<td>0</td>
<td>712.00</td>
<td>749.72</td>
<td>721.62</td>
</tr>
<tr>
<td>1</td>
<td>38.24</td>
<td>37.96</td>
<td>38.17</td>
</tr>
<tr>
<td>1</td>
<td>199.55</td>
<td>200.12</td>
<td>199.70</td>
</tr>
<tr>
<td>1</td>
<td>236.53</td>
<td>244.68</td>
<td>238.51</td>
</tr>
<tr>
<td>1</td>
<td>512.36</td>
<td>509.02</td>
<td>511.54</td>
</tr>
<tr>
<td>2</td>
<td>148.20</td>
<td>147.16</td>
<td>147.94</td>
</tr>
<tr>
<td>2</td>
<td>243.37</td>
<td>251.49</td>
<td>245.41</td>
</tr>
<tr>
<td>2</td>
<td>398.91</td>
<td>400.08</td>
<td>399.21</td>
</tr>
<tr>
<td>2</td>
<td>715.98</td>
<td>753.54</td>
<td>725.39</td>
</tr>
</tbody>
</table>

Table 14: Frequencies [Hz]

The frequencies written in *italic* in Table 14, corresponds to predominantly plane fluid cross-sectional waves. If no coupling was present, these waves would be completely plane. In Section 5 it is concluded that such plane waves are exceptionally well approximated, even with a coarse mesh. The wave shape in the fluid for the wave at $\kappa = 1$ at approximately 200 Hz from the coarse mesh model is seen in Figure 14. This wave shape approximately shows a half sine shape over the diameter. For the coarse mesh this half wavelength is approximated with 6 elements only. Simultaneously, the shell undergoes a simple breathing motion, approximated by 16 plate strip elements. This observation indicates a possible problem, since more fluid elements means more shell elements. Consequently, very large systems may be needed for resolving the velocity potential in the fluid. One solution to this problem is to use higher order polynomials for the fluid elements. This is however not within the scope of this thesis.
Figure 13: Dispersion relation for fluid filled pipe; (–) Finnveden; (·) Waveguide–FE coarse mesh
Figure 14: Mode in fluid for wave at $\kappa = 1$ and frequency $\approx 200\,Hz$
8 Validation for curved coupling element

No suitable validation for the curved coupling element has been found in the literature. However, in the limit of large curvatures, results for a curved fluid–filled pipe–bend yield dispersion relations very close to those for corresponding straight pipes. Furthermore, in Chapter 8, the curved coupling element is used to calculate the air response inside a car tyre. These calculations agree well with measurements. Consequently there is no apparent reason to doubt the validity of the curved coupling element.

Since pipe bends are common in practical applications there is arguably a need for studies of waves in such pipes. A preliminary study of some of these waves is given in Chapter 6.

9 Conclusions

Six different waveguide finite elements are derived in Chapters 1 to 4. To validate these elements, dispersion relations for analytic models and dispersion relations found in the literature are compared against models assembled from different elements. With the dispersion relations studied here, five of the six elements has been completely or partially validated. The exception is the curved fluid–shell coupling element, for which no suitable validation has been found. The only validation made for these elements are models of bent fluid filled pipes with large bend radii. These models yield results very close to those of corresponding straight pipes. Besides the curved coupling element, two problems concerning the curved shell elements have also been found. The first problem is with highly prestressed shells, for which resonance frequencies do not converge to those found in the literature. The second problem is that curved shells with general anisotropic material have not been found in the literature, and validations for such materials are consequently not made here.
References


CHAPTER 6:
WAVES IN FULLY COUPLED FLUID–FILLED PIPE BENDS

1 Dispersion relations for a fluid filled pipe bend

Fluid filled pipe bends and ducts are used in many applications. Studies of waves in pipe bends are given for example in [1], [2] and [3]. These studies does however not take account of full acoustic coupling between a compressible fluid and a surrounding shell. El-Raheb and Wagner, reference, [4] and [5], considers fully coupled pipe bends. Unfortunately their results are not in a suitable form for comparison here. In straight pipes full coupling is important in many practical situations, see for instance [6]. Dispersion relations for fluid–shell coupled curved pipes are arguably also important in many practical situations. Since dispersion relations for such pipes have not been found in the literature some results are presented in the following. For comparison, dispersion relations for a corresponding straight pipe are given. Associated dispersion relations from the uncoupled fluid and shell equations are also given.

A model for a curved pipe is made with the waveguide finite elements developed in Chapters 2 to 4 with the fine mesh from Figure 12 in Chapter 5. By assuming wave solutions, in the form:
\[ W(\theta) = \hat{W} e^{-i\gamma \theta} \]  

for free response, the eigenproblem,

\[
\left[ \sum_{j=0}^{4} K_j (-i\gamma)^j + i\omega M_1 - \omega^2 M_2 \right] \hat{W} = 0,
\]

follows. For a straight pipe the wave solutions,

\[ W(x) = \hat{W} e^{-ikx} \]

are assumed and the eigenproblem is given for \( \kappa \) instead of \( \gamma \). Upon approximating a curved waveguide with a straight, the distance travelled by a wave is approximated by, \( x \approx \theta R \), and he wavenumber is approximated by, \( \kappa \approx \gamma / R \), where \( R \) is the mean radius of the bend.

The pipe bend under study has properties typical of a water filled steel pipe as given in Table 1. The dimensions are given by Figure 3 in Chapter 5.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s modulus ( E )</td>
<td>210 GPa</td>
</tr>
<tr>
<td>Poisson’s ratio ( \nu )</td>
<td>0.3</td>
</tr>
<tr>
<td>Shell density ( \rho )</td>
<td>7800 kg/m³</td>
</tr>
<tr>
<td>Shell thickness ( h )</td>
<td>5 mm</td>
</tr>
<tr>
<td>Pipe radius ( r )</td>
<td>0.1 m</td>
</tr>
<tr>
<td>Fluid wave speed ( c_f )</td>
<td>1500 m/s</td>
</tr>
<tr>
<td>Fluid density ( \rho_f )</td>
<td>1000 kg/m³</td>
</tr>
<tr>
<td>Radius of bend ( R )</td>
<td>0.333 m</td>
</tr>
</tbody>
</table>

Table 1: Parameters used for pipe bend model

Except from having infinite bend curvature, the straight pipe used for comparison
has the same properties. The mesh used is the fine mesh seen in Figure 12 in Chapter 5.

Waves for uncoupled fluid and shell are found by setting the coupling matrix, $M_1$, to zero in equation (2).

## 2 Dispersion relations in a straight pipe

The dispersion relation for the fluid-shell coupled straight pipe is seen in Figure 1. The straight ‘curves’ at the bottom indicate the longitudinal and the shearing waves in the shell. The predominantly first flexural wave and the predominantly plane fluid wave are indicated with '□' and '○' respectively.

Dispersion relations for these two waves, together with the corresponding waves without fluid shell coupling included are seen in Figure 2. By comparing the coupled and the uncoupled waves it can be seen that the curve for the predominantly flexural wave is steeper for the coupled wave and thus the phase velocity is lower. Physically this can be explained as an increased mass loading from the fluid on the pipe. The phase velocity for the predominantly plane fluid wave also decreases when coupling is introduced. This effect can be explained by a decrease in the compressibility of the fluid when the walls are non–rigid.

Another interesting feature seen in Figure 1 are the dispersion curves crossing each other. This seems to indicate that the waves are orthogonal, since the crossing waves are described by higher order trigonometric functions about the circumference of the pipe.
Figure 1: Dispersion relation for straight fluid filled pipe. ‘□’: predominantly flexural wave; ‘○’: predominantly fluid wave
Figure 2: Dispersion relations for coupled and uncoupled waves in fluid filled pipe; ‘□’: predominantly flexural wave; ‘○’: predominantly plane fluid wave; Filled marks indicate uncoupled waves
3 Dispersion relations in a pipe bend

Dispersion relations for a pipe bend including fluid shell coupling is seen in Figure 3. Due to the curvature there is only one axis of symmetry of the cross-section of the pipe. Consequently the first flexural wave for the straight pipe is decomposed into two waves, one antisymmetric and one symmetric. The symmetric flexural wave and the predominantly fluid wave are indicated in Figure 3 by '□' and '○' respectively. As can be seen the dispersion relations for the symmetric and antisymmetric flexural waves are very close.

![Figure 3: Dispersion relations for coupled waves in fluid filled pipe bend. ’□’: predominantly flexural wave; ’○’: predominantly plane fluid wave.](image)

Distinct crossing of dispersion curves that was found for the straight pipe is, for some waves no longer present. As a consequence it is hard to define the predominantly...
fluid wave at frequencies above 890 Hz.

Dispersion curves for the symmetric flexural wave and the predominantly plane fluid wave are seen in Figure 4. By comparing Figure 4 with Figure 2 it is seen that the difference between coupled and uncoupled cases are less prominent for the fluid wave in the pipe bend than it is for the straight pipe. This is likely to be explained by a stiffening of the shell’s ‘breathing’ motion due to the bend curvature, which cause more ”rigid” boundaries for the fluid.

![Dispersion Curves](image)

Figure 4: Dispersion relations coupled and uncoupled waves in fluid filled pipe bend, □ predominantly flexural wave; ’○’ predominantly plane fluid wave; Filled marks indicate uncoupled waves.

Values for the symmetric flexural and the predominantly fluid waves are found in Table 2 and 3 respectively. The frequencies in these tables are given for the non–dimensional wavenumber $\gamma$ rather than $\gamma/R$. The values $\gamma = 0, 2, 4 \ldots$ corresponds
to resonance frequencies in a 90 degree pipe bend with shear diaphragm and pressure release boundary conditions.

<table>
<thead>
<tr>
<th>$\gamma$ [-]</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency [Hz]</td>
<td>348</td>
<td>1189</td>
<td>2158</td>
<td>2988</td>
<td>3683</td>
</tr>
</tbody>
</table>

Table 2: Frequencies for predominantly symmetric flexural wave in fluid filled pipe bend

<table>
<thead>
<tr>
<th>$\gamma$ [-]</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency [Hz]</td>
<td>1334</td>
<td>2684</td>
<td>3938</td>
<td>4940</td>
<td>5765</td>
</tr>
</tbody>
</table>

Table 3: Frequencies for predominantly plane fluid wave in fluid filled pipe bend

References


CHAPTER 7:
INPUT POWER TO WAVEGUIDES
CALCULATED BY A FINITE ELEMENT METHOD
Input power to waveguides calculated by a finite element method

Abstract

This paper considers the power injected into waveguide structures from forces that are concentrated along the waveguide but are otherwise arbitrary. The motion of the structure is described by a set of coupled, linear, one-dimensional wave equations. These equations are, in this work, devised by the waveguide finite element method, which is a versatile tool for describing wave motion in general structures that have uniform properties along one direction. Two separate procedures are derived. The first is based on spatial and frequency averaging of a modal solution. The second procedure is based on a spatial Fourier transform, which by nature assumes the waveguide to be infinite. It is shown that, in the limit of zero damping, the two methods are identical. Subsequently input power to a stiffener in a railway car structure is calculated and compared with an in situ measurement, showing a fair agreement.

1. Introduction

For the assessment of the severity of a structure’s vibration, it is useful to know not only the magnitude of the excitation forces but also the vibrational power they inject into the structure. This is particularly so for high frequency vibrations of structures described by Statistical Energy Analysis (SEA), since the input power is a critical input datum to such an analysis [1-3]. Moreover, it is an experience shared by many practitioners that if the criteria for the application of SEA are roughly met, see e.g., [4], and if the magnitude of input power is correct, even a crude SEA model will perform reasonably well. This is so as vibrational energy is a conserved quantity, in contrast to, e.g., vibration velocity or sound pressure.

The literature describes the evaluation of input power to common structural elements such as beams, plates and cylindrical shells and these formulations are made available in commercial SEA codes. For general structures, however, the evaluation must often rely on measurements or detailed numerical analysis, which might prove difficult and costly.

This paper is concerned with efficient procedures for the numerical evaluation of input power to general structures that have properties that are uniform along one direction. The formulation is based on the Waveguide Finite Element Method (WFEM). Previously this method has been used, e.g., for studies of vibration in rods with arbitrary cross section [5]; composite plates [6]; thin-walled beams [7]; railway
track [8]; rib-stiffened plates [9]; twisted beams [10]; anisotropic solids [11-12]; conical shells [13], car tyres [14], fluid-filled pipe bends, [15] and a wind tunnel [16]. Possibly, the methods for the evaluation of input power presented here will apply to any structure built up by the elements presented in the references [5-16].

A standard procedure for the evaluation of input power is described in detail for a point-excited plate in “Structure Borne Sound” [1, Chapter IV.4]. Thus, for a random location of the force and small damping, the frequency averaged input power is simply given by a numerical factor, the squared force magnitude, the plate mass and the modal density (number of modes per unit frequency). For structures that have dimensions that are large compared to the wavelength, the modal density is independent of the boundary conditions and the derived formula can be applied for any set of boundary conditions. In fact, the value derived in [1] equals the input power to an infinite plate.

The procedure in [1] is commonly used in SEA. However, the derivation considers a homogenous structure. For such structures, the ratio of the averaged square of the modal force to the averaged modal mass equals the ratio of squared force to total mass, which is not generally true. In practical applications of SEA, the expression for homogenous structures is, nevertheless, often used also for inhomogeneous structures. It is then assumed that the force intensity is proportional to the mass density [17].

Another source of error, when using the simple expression in [1], is peculiar to long structures that support a small number of wave types. Most often, as the frequency is increased, yet another wave type starts to propagate at its ‘cut-on’ frequency. The modal density increases largely at this frequency, as does the injected power. However, the mobility increases already below the cut-on frequency and, if the structure dissipates energy, so does the injected power, in contrast to the modal density. Thus, the proportionality between injected power and modal density does not apply. This phenomenon is illustrated in reference [18] for a fluid-filled pipe, where, for 1% damping, the simple formula give rise to errors of 3-5 dB in third octave bands just below the cut-on frequencies.

In an alternative formulation, Langley calculates the input power to infinitely long structures that supports wave motion described by one response variable [19]. The input power is expressed by the waves’ dispersion characteristics and it attributes dissipative effects.

In what follows, the modal procedure for the evaluation of input power in reference [1] and the wave based procedure in reference [19] are extended to apply also for structures that support many kinds of waves. The presented formulation
applies to waveguide structures that have uniform properties along one direction while it may have quite arbitrary, non-uniform, cross sections. The force is concentrated at a random location along the waveguide but is otherwise arbitrary. The equations of motion are a set of coupled one-dimensional wave equations. Here, these are devised by the waveguide FEM.

First, the derivation in reference [1] is applied for general finite length waveguide structures. The WFEM allows an efficient identification of the modes and natural frequencies for a convenient set of boundary conditions at the ends. Upon this basis the injected power is expressed by a modal summation and then frequency averaged as in [1]. The resulting expression is proportional to the modal density and the ratio of squared modal force to modal mass is identified.

In an alternative procedure, Langley’s result is expanded to apply for general waveguides. This procedure is numerically more costly than the one discussed above. However, it describes the increase in input power at frequencies below the cut-on.

The outline of the paper is as follows. Section 2 gives a brief overview of the waveguide FEM and demonstrates its application to a stiffener in a railway car structure. Section 3 develops the generalisation of the methods in reference [1] and [19] for waveguides. The methods are compared and it is shown that in the limit of zero damping they are identical. Finally, in Section 4, the developed procedure is demonstrated for the railway car structure and the results are compared with measurements, showing a fair agreement.

2. Waveguide Finite Elements

2.1 Variational statement

Hamilton’s principle states that the true motion of a structure minimises the time integral of the difference between strain and kinetic energy minus the work from external forces. Considering harmonic motion of the form $e^{-i\omega t}$, where $\omega$ is angular frequency and $t$ is time, and applying Parceval’s identity, the following functional is stationary for true motion

$$
H = \int \int \left( \varepsilon^T C \varepsilon - \omega^2 u^T \mathbf{m} u - F^T F u - u^T \mathbf{F} \right) dV d\omega,
$$

where upper indices * and T denote complex conjugate and vector transpose and $V$ is the structure’s domain, $\varepsilon$ is the strain, $u$ is the displacement and $F$ is the force. Moreover, $C$ is the rigidity matrix and $\mathbf{m}$ is the mass matrix, both possibly dependent on location. From reciprocity it follows that they are symmetric matrices. The mass matrix is positive definite and the rigidity matrix is positive definite or positive semi-
definite. For linear motion, the strain is a linear functional of the displacement and thus the functional (1) is a symmetric bi-linear functional of the displacement and its conjugate.

Now, Hamilton’s principle does not apply for non-conservative motion. A dodge proposed in reference [20], and used, e.g., in [21-22], is therefore applied in the following. Thus, the functional (1), of the displacement and the complex conjugate of the displacement, is replaced with the same symmetric bi-linear functional of the displacement and the complex conjugate of the displacement in an adjoint, mathematically designed, system. This adjoint system is similar to the system under investigation but has negative damping. Also, for linear motion, different frequencies are independent and, for simplicity, one frequency at a time is considered. The resulting functional, here denoted the Lagrangian, is given by,

\[
L = \int \left( \varepsilon^T C \varepsilon - \omega^2 u^s u^m - F^T u - u^s F \right) dV ,
\]

where the upper index, ‘\(a\)’, denotes the complex conjugate of the adjoint variable. In the Lagrangian, damping is attributed if the imaginary part of the rigidity matrix is negative (semi) definite. In the absence of damping, the adjoint system is identical to the system and the integral of the Lagrangian over frequency is identical to the functional (1).

The Lagrangian (2) is stationary for true motion, subject to the boundary conditions, and is taken as the basis for the finite element formulation in the next section.

### 2.2 Waveguide finite element formulation

Consider a structure that has uniform geometrical and material properties along one direction and define a coordinate system with the \(z\)-axis along the structure and the \(x\)- and \(y\)-axes in the plane of the cross section. The structure is sub divided into elements, which have finite dimensions on the cross section and extends along the \(z\)-axis. For each element, the rigidity and mass matrix are identified and the strain vector is defined as a function of the displacements. Thus, the Lagrangian is given by the sum of the sub Lagrangians defined for each element.

Within the elements the displacement’s dependence of the \(x\)- and \(y\)-coordinates is expressed with conventional, polynomial, finite element shape functions. Thus the displacement is given by

\[
u(x, y, z) = \Psi^T(x, y) v(z), \quad u^s(x, y, z) = \Psi^T(x, y) v^a(z),
\]

where the entries to the vector \(\Psi\) are FE shape functions, which are non-zero within one element only, and \(v\) and \(v^a\) contain the ‘nodal’ displacements, i.e., the
displacements along a line in the \( z \)-direction. The displacements (3) are inserted into the sub Lagrangians and the derivatives and integrals with respect to \( x \) and \( y \) are evaluated. Finally, the sub Lagrangians are assembled according to standard FE procedures, upon which it follows that

\[
L = \int \left( \sum_{n=0}^{2} \sum_{m=0}^{2} \left( \frac{\partial v^T}{\partial z^n} A_{n,m} \frac{\partial v^T}{\partial z^m} - \omega^2 v^T M v - v^T f - v^T f^* \right) \right) dz ,
\]

where \( A_{n,m} \) are generalised stiffness matrices, \( M \) is the mass matrix and \( f \) is a generalised force vector. From the properties of the elements’ mass matrices it follows that the mass matrix, \( M \), is symmetric and that its real part is positive definite. Also,

\[
A_{n,m} = A_{m,n}^T ,
\]

since the rigidity matrices are symmetric.

The literature describes various elements that may be used in the analysis presented above, see, e.g., [5-16]. The examples presented in the present work, uses the thin-walled element presented in [16]. It is equal to Gavric’s thin-walled element, except that the transformation to real valued matrices, given by [7, equation (5)], is not made.

### 2.3 Wave equation

From the calculus of variation applied on equation (4), see e.g. [20], follows the equation of motion

\[
\sum_n K_n \frac{\partial v}{\partial z^n} - \omega^2 M v = f .
\]

where

\[
K_n = \sum_{m=0}^{2} \left( (-1)^{n-m} A_{(n-m)} \right) , \quad m \leq n \leq 4 .
\]

Equation (6) constitutes a set of ordinary linear differential equations with constant coefficients. If there are no external forces, the solutions are exponential functions of the form of

\[
v = \phi e^{i \kappa z} ,
\]

and a twin-parameter eigenvalue problem in \( \kappa \) and \( \omega \) follows

\[
\left( K(\kappa) - \omega^2 M \right) \phi = 0 ,
\]

where
\[ K(\kappa) = \sum_n (i \kappa)^n K_n. \] (10)

Without dissipative losses and considering real \( \kappa \), the stiffness matrix \( K \) is Hermitian and positive semi definite, which follows from equation (5) and (7) and the properties of the rigidity matrix.

Equation (9) can be solved as a polynomial eigenvalue problem in the wavenumber \( \kappa \) for a given frequency, \( \omega \). Without dissipative losses, an eigenvalue may be real, in which case it describes a propagating wave. It can also be imaginary or complex, describing strictly decaying or oscillating and decaying, near field, solutions. With losses, the motion always decays away from the sources and there are no real valued wavenumbers. The latter is most easily demonstrated for a system that is proportionally damped, for which the wave solutions are found from the following eigenvalue problem

\[(1 - i\eta)K(\kappa)\phi - \omega^2 M\phi = 0,\] (11)

which is equally written

\[K(\kappa)\phi - \lambda M\phi = 0,\] (12)

\[\lambda = \omega^2/(1 - i\eta) \approx \omega^2 (1 + i\eta),\] (13)

where the last approximate equality is valid for small levels of damping. Now, suppose that \( \kappa \) is real, then \( K(\kappa) \) is Hermitian and since \( M \) is real, symmetric and positive definite, it follows that the eigenvalue \( \lambda \) is real. According to equation (13), however, \( \lambda \) is a complex number for a given frequency \( \omega \). Since this is clearly a contradiction, it is concluded that the wave numbers \( \kappa \) are complex for a damped system. This also shows that an evolution of the wavenumber with frequency, \( \kappa(\omega) \), is bound to either the upper or lower complex half plane.

Symmetry between waves travelling to the left and to the right indicates another important property of the eigenvalue problem (9). If \( \kappa_j \) is an eigenvalue to the equation for a given frequency \( \omega \), so is \(-\kappa_j\), since \( M \) is symmetric and

\[K(-\kappa_j) = K(\kappa_j)^T.\] (14)

which follows from equation (5) and (7). Thus, a wave component that propagates or decays in the positive \( z \)-direction has a left eigenvector, which equals the right eigenvector of a similar component in the negative \( z \)-direction.

For undamped systems, symmetry exists between wavenumbers in all four quadrants in the complex plane, see Fig. 1. These symmetries, which can be shown by
combinations of transposes and complex conjugates together with the Hermitian properties of the matrices in equation (9), are summarized in Table 1.

For studies of propagating waves in undamped structures, the wavenumbers takes real values only. Equation (9) can then, alternatively, be solved as a generalised linear eigenvalue problem in $\omega$ for a given $\kappa$. The linear problem is smaller than the polynomial and also it is on an efficient form, as the mass matrix is symmetric and positive definite and the stiffness matrix is Hermitian and positive semi-definite. Hence, numerical speed and stability is gained; typically, the calculation burden is reduced by a factor of 100. However, the choice between the polynomial eigenvalue problem in $\kappa$ and the linear eigenvalue problem in $\omega$ is a matter of objectives and both are used in the following analysis of input power to waveguide structures.

2.4 Example

To illustrate the discussion above, the beam structure in Fig. 2 is considered. This beam is attached to the base frame of the railway car section shown in Fig. 3. (The section is a full-scale experimental prototype built during the design of a new coach for the Stockholm metro some years ago.) The beam is made of steel and has a wall thickness of 4 mm. The mesh used for the waveguide FE model of the beam is shown in Fig. 4. This model neglects the holes in the beam shown in Fig. 2 and 3, which obviously is an approximation. Furthermore, simplifying boundary conditions are assumed at the connection between the beam and the corrugated plate. These boundary conditions are chosen based on the following reasoning.

First, the plate stiffness across the corrugation is very low and hence the axial motion of the beam is assumed to be free. Second, the rotation about the beam axis is assumed blocked, since the rotational rigidity of the open thin-walled beam is low and the flexural stiffness of the corrugated plate is in comparison large. Third, the in plane stiffness along the plate corrugation is high compared to the beam’s flexural and rotational stiffness and thus the motion along the corrugation is blocked. Fourth, the beam is there to increase the flexural stiffness of the structure, so it is reasonable to assume the motion in the vertical direction free. Thus, the following boundary conditions are assumed at the connection to the corrugated plate

$$\frac{\partial w}{\partial z} = \phi_z = u = \frac{\partial v}{\partial y} = 0,$$

where $u, v, w$ are the displacements in the $x, y, z$ directions and $\phi_z$ is the rotation about the $z$-axis. Obviously, these boundary conditions are somewhat simplistic but, as will be shown in Section 4, they result in a reasonably good estimate of the input power to the structure. Furthermore, as also shown in Section 4, the value of the
frequency band averaged input power is not critically dependent of the choice, except in a lower frequency regime.

Fig. 5 shows the dispersion characteristics for the beam. For reference, Fig. 5 shows also the wavenumbers for flexural waves in a 4 mm thick steel plate. In the beam, there are two waves propagating at low frequencies, since there are two degrees of freedom that are not restrained by the boundary conditions. At increasing frequencies, additional waves are “cut-on”, i.e., they start propagating. Normally, the dispersion curves do not cross. An exception is the axial, dilatational, wave, which is dynamically uncoupled from all other waves for the chosen boundary conditions. It is seen as a straight line through most of the other curves in Fig 5.

In Fig. 5, the dispersion characteristics are shown with dots, each defining a wave number and a frequency that constitutes and eigenvalue pair to equation (9). It is of interest to sort this collection of dots into branches that define different wave types. At lower frequencies, when there are only a few branches, this is not difficult, while at higher frequencies it becomes increasingly tricky. The problems are illustrated in Fig. 6 and 7. Fig. 6 shows the dispersion curves amplified around the near crossing of two branches and Fig. 7 shows the cross sectional mode shapes at frequencies just below and above the crossing. Based on the wave shapes in Fig. 7, it would be natural to sort the solutions B and C as belonging to the same branch; however, this would indicate that the dispersion curves cross, which is not the case. Also, the wave shapes for the solutions A and D are not similar. Thus, the proper sorting of the waves is not evident.

An interesting behaviour of the dispersion curves appears if the corners of the beam are modelled as sharp 90 degree bends. The branch that has a cut-on a little above 7.5 kHz is shown amplified for this case in Fig. 8 and it is seen that the lowest frequency for which the wave propagates is given by a finite wave number. At somewhat higher frequencies, the lower part of the branch shows wave numbers that decrease towards zero with increasing frequencies, while the upper part shows wave numbers that increase with frequency. It follows that for the lower part, the phase velocity and the group velocity are in opposite directions. Such waves have been observed previously, e.g., for cylindrical shells [26] and car tyres [14]. Fig. 9 illustrates the evolution of the complex wavenumber with frequency for a damped and an un-damped structure. The form of the observed evolution will have implications for the expressions for input power that are developed in the next section.

3. Input power to waveguide structures

Two methods are developed for the calculation of input power to waveguide structures in the following. The first is a modal approach, inspired by the calculation
of input power to a point loaded simply supported plate in reference [1, Chapter IV.4].
The second is a wave approach, expanding Langley’s result in reference [19] to
general waveguides.

3.1 The modal approach

Consider a waveguide of length $L$, located at: $z \in [0, L]$. It is excited by a
harmonic force that is concentrated at location $z_0$ but is otherwise arbitrary. Damping
is described as stiffness proportional and the equation of motion (6) takes the form of

$$
(1 - i \eta) \sum_n K_n \frac{\partial^n v}{\partial z^n} - \omega^2 M v = f_0 \delta(z - z_0).
$$

(16)

For structures that are long compared to the wavelength, the frequency averaged
input power does not largely depend on the precise boundary conditions at the ends.
Therefore, a ‘convenient’ set of boundary conditions is considered. This convenient
set is defined so that if a wave, of the form of equation (8), impinges on the boundary,
it is reflected but no other wave nor any near field components are generated. For
structures built-up by the solid element in reference [5] and the thin-walled shell
element in reference [16], this would be the shear-diaphragm boundary condition
where the motion in the plane of the cross section is blocked while the motion along
the waveguide is free. Similarly, if the structure contains a fluid, the convenient
boundary condition for this fluid would be the pressure release condition.

The solution to equation (16) is sought as a linear combination of the un-damped
modes, obeying the convenient boundary condition. To that end, we consider the
homogenous form of the equation for an un-damped waveguide. It was shown in
Section 2.3 that if $v = \varphi e^{i \kappa z}$ is a solution, so is $v = \varphi^* e^{-i \kappa z}$. Thus,

$$
v = \text{Re}(\varphi) \cos \kappa z + \text{Im}(\varphi) \sin \kappa z,
$$

(17)
is also a solution. The cross sectional modes $\varphi$ are normalised so that the real part
corresponds to the motion along the waveguide and the imaginary part to the motion
in the plane of the cross section. This is possible for structures built up by isotropic or
orthotropic elements, such as the solid element in reference [5] and the thin-walled
element in [7, 16]. This follows from arguments similar to those used for finite strip
elements and axi-symmetric finite elements, see for instance [23, Chapter 10.6 and
10.7]. Note, however, these arguments do not hold for general anisotropic media, as
also discussed in references [12, 23].

From the solution to the equations of motion (17), the proposed normalisation of
the cross sectional modes and the definition of the convenient boundary condition it
follows that the structure’s eigenmodes $\psi_p$ are given by
\[ \psi_{pr}(z) = \cos(\kappa z) \text{Re}(\varphi_{pr}) + \sin(\kappa z) \text{Im}(\varphi_{pr}), \]  

where

\[ \kappa = p \pi / L. \]  

The eigenfrequencies \( \omega_{pr} \) and the corresponding cross sectional mode shapes \( \varphi_{pr} \) are given by the linear eigenvalue problem that follows from the un-damped and homogenous version of equation (16) when the wavenumber, \( \kappa \), is given by equation (19). The eigenmodes obey the following orthogonality relations

\[ \int_0^L (\psi_{qs})^H M \psi_{pr} \, dz = \delta_{qr} \delta_{sr} m_{pr} L / 2 \]

\[ \int_0^L (\psi_{qs})^H K(\kappa_p) \psi_{pr} \, dz = \delta_{qr} \delta_{sr} \omega_{pr}^2 m_{pr} L / 2 \]

where upper index H denotes complex conjugate and transpose and

\[ m_{pr} = \varphi_{pr}^H M \varphi_{pr}. \]

Upon this basis, the power injected by the force is given by [1]

\[ P_{in} = \frac{1}{2} \text{Re} \left( \sum_p \sum_r 2 (-i \omega) \left| f_0^H \psi_{pr}(z_0) \right|^2 \right). \]

3.1.1 Statistical expectation of input power

In line with standard procedures in SEA [1], the frequency averaged value of the input power (22) is evaluated for a random location of the force,

\[ \bar{P}_{in} = \frac{1}{\Delta_\omega} \int_0^\omega \int_0^\omega P_{in} \, d\omega \, dz, \]

where \( \Delta_\omega = \omega_1 - \omega_0 \). It is assumed that the bandwidths of the resonances are small, so that most of the energy associated with a resonance, having its natural frequency within a frequency band, is contained in this band. Yet, the frequency band is so narrow that the cross sectional mode shape \( \varphi_{pr} \), for branch \( r \) of the dispersion relations, does not vary much for the modes within the band. Upon this basis, the integrals in equation (23) are evaluated as in reference [1]

\[ \bar{P}_{in} = \frac{1}{2} \sum_r \frac{\pi}{2} \left| f_0^H \varphi_{pr} \right|^2 N_r \frac{L m_{pr}}{\Delta_\omega} \]

where \( N_r \) is the number of resonances within the frequency band, associated with branch \( r \).
The cross sectional mode shape, $\varphi_{pr}$, is an implicit function of frequency, i.e., it is the mode shape for branch $r$ calculated for the wave number at frequency $\omega$. At lower frequencies, when the dispersion relations have only a few branches, it is not difficult to identify these branches and thus to find the cross sectional mode shape for a certain branch at a given frequency.

An alternative to equation (24) that avoids the identification of the branches, useful for waveguides that are not too long, is based on the “exact” mode count,

$$\bar{P}_{in} = \frac{\pi}{4} L \Delta_\omega \sum_r \sum_p \int_0^{\omega_{pr}} \left| \frac{1}{m_{pr}} \varphi_{pr} \right|^2 \delta(\omega_{pr} - \omega) d\omega. \quad (25)$$

Note, however, the apparent precision in this expression is illusory in most situations where SEA is useful. Instead, asymptotic expressions for the mode count are derived in the next section.

### 3.1.2 Mode count

It follows from equation (19) that the mode count $N_r$ is approximately given by,

$$N_r \approx \left( \kappa_r(\omega_r) - \kappa_r(\omega) \right) L/\pi, \quad (26)$$

provided that $\kappa_r$ is an increasing function of frequency. Sometimes, however, it may be otherwise, as discussed in Section 2.5.

The application of equation (26) to the dispersion characteristics shown in Fig. 8 requires that the region where the wavenumber is a multi-valued function of frequency is identified and that the upper and lower parts of the curve are handled separately. Moreover, the mode count associated with the lower part is the absolute value of the one given by the equation.

At higher frequencies, the dispersion relations may be intricate and it might be difficult to identify that two solutions belong to the same branch, which is required for the evaluation of equation (26). An alternative is then to estimate the mode count from the following asymptotic expression for the modal density [16]

$$\frac{N_r}{\Delta_\omega} = n_r(\omega) \approx \frac{L}{\pi} \left| \frac{\partial \kappa_r}{\partial \omega} \right| \approx \frac{L}{\pi} \frac{2 \omega m_{pr}^H}{\varphi_{pr}^H \kappa' \varphi_{pr}} \quad (27)$$

where $n_r$ is the modal density and

$$\kappa' = \frac{\partial \kappa_r}{\partial \kappa_r}. \quad (28)$$

This expression, in contrast to the one in reference [16], takes the absolute value of the denominator and therefore it is valid also when the wave number is a decreasing
function of frequency. The expression can be evaluated by considering one solution to the dispersion relations at a time. It is, however, singular at the cut-on frequencies, which is inferred from the following arguments.

At a cut-on frequency, \( \omega \neq 0 \), such that the wave number \( \kappa = 0 \), the eigenvector is given by

\[
K_0 \phi - \omega^2 M \phi = 0.
\]  

(29)

This eigenvector is real, since \( K_0 \) and \( M \) are real and symmetric. Consequently, the denominator in equation (27) is, for \( \kappa = 0 \),

\[
\phi^T (iK_i) \phi = \phi^T (-iK_i) \phi = 0,
\]  

(30)

since the transpose of the scalar \( \phi^T (iK_i) \phi \) equals itself and \( K_i \) is an anti symmetric matrix. A consequence of this result is that the group velocity is zero for \( \kappa = 0 \), see [16, equation (18)].

In conclusion, a waveguide is excited by a force concentrated at an axial location, as described by equation (16). The response is calculated for a set of convenient boundary conditions at the ends and is expressed as a modal sum. Upon this basis, the frequency averaged power injected by a force at a random axial location is given by equation (24). In this equation, the mode count may be evaluated, as in equation (25), by simply calculating the natural frequencies when the wavenumber is given by equation (19). It may also be evaluated by equation (26), provided that the regions where the dispersion relation for a branch is a multi valued function of frequency are identified. Finally, it can be estimated from the asymptotic modal density (27), in which case the injected power is given by

\[
\bar{P}_{\text{in}} = \sum_r \frac{\omega}{2} \left| \frac{H_0^H \phi_{pr}}{H_{pr}^H K_{pr} \phi_{pr}} \right|^2,
\]  

(31)

where the summation is taken over the branches of the dispersion characteristics. Unfortunately, this expression is singular at the cut-on frequencies.

### 3.2 The wave approach

In this section, the power injected by the force on the right hand side of equation (16) is calculated upon the assumption that the waveguide is infinitely long. This formulation is not singular at the cut-on frequencies, since damping is attributed.

First, a spatial Fourier transform is applied, and the equation becomes

\[
\mathbf{D}(\kappa) \mathbf{\tilde{u}}(\kappa) = \mathbf{\tilde{f}},
\]  

(32)

where
The input power to the structure is given by

\[ P_{in} = \frac{1}{2} \text{Re}\left( \int_{-\infty}^{0} -i\omega u(z) \, dz \right) \]

\[ = \frac{1}{4\pi} \text{Re}\left( f_0^H \lim_{z \to z_0} \int_{-\infty}^{0} -i\omega \tilde{u}(\kappa) e^{i\kappa z} \, dz \right) \]  

(35)

The integral appearing on the right hand side of this expression is evaluated by contour integration; the contour chosen here consists of the real axis together with a semi-circle enclosing the upper half plane. The poles and residues are identified by a theorem from reference [24], which is repeated for completeness.

**Theorem.** Let the elements of a square matrix \( D(\kappa) \) and a column vector \( \tilde{f}(\kappa) \) be analytic functions of \( \kappa \) in a neighbourhood of a simple zero \( \kappa_j \) of \( \det(D(\kappa)) \) and let \( \tilde{u}(\kappa) \) be the solution to equation (32). Then \( (\kappa - \kappa_j) \tilde{u}(\kappa) \) is analytic in a neighbourhood of \( \kappa_j \) and

\[ \lim_{\kappa \to \kappa_j} (\kappa - \kappa_j) \tilde{u}(\kappa) = \frac{\theta_j^T \tilde{f}(\kappa)}{\theta_j^T D'(\kappa_j) \phi_j} \phi_j \]  

(36)

where the non-zero vectors \( \theta_j \) and \( \phi_j \) are determined uniquely, apart from scalar factors, by

\[ D(\kappa_j) \phi_j = 0, \quad D(\kappa_j)^T \theta_j = 0 \]  

(37)

The assumptions of the theorem ensure that the denominator of the right hand side of equation (36) is non-zero. For an un-damped structure, however, there is a double pole at the cut-on, as seen in Fig. 9, and the expression is singular. Also, the restriction of a simple pole is violated for waveguides with cross sections having two axes of symmetry, for instance circular or quadratic waveguides. However, since the eigenvectors for such double poles are linearly independent, equation (36) still holds, see [25]. Thus, the poles required for the evaluation of the integral in equation (35) are identified and it follows that

\[ P_{in} = \frac{\omega}{2} \sum_{\text{Im}(\kappa_j < 0)} \text{Re}\left( \frac{(\theta_j^T f_0)(f_0^H \phi_j)}{\theta_j^T D'(\kappa_j) \phi_j} \right) \]  

(38)

where the poles \( \kappa_j \) and the left and right eigenvectors, \( \phi_j \) and \( \theta_j \), are given by the solutions to the polynomial eigenvalue problem, defined by equation (33) and (37) for
a given frequency $\omega$. This expression will predict a finite input power at the cut-on
frequencies, if damping is attributed. It is a generalisation of the result in reference
[19] to systems with multi dimensional dispersion characteristics and constitutes the
major result of the present study. It should be useful as the waveguide FEM is a
versatile tool that may be applied to virtually any structure that has constant properties
along one direction.

3.3 Input power to an infinite waveguide without damping

It is of interest to compare the wave expression for input power (38) to the modal
expression (24). To that end, equation (32) is considered for a structure without
damping. As shown in the Appendix, for such structures, waves associated with poles
that have nonzero imaginary parts do not receive any input power. Consequently, it is
only the poles on the real axis that need be included in the summation (38). Moreover,
it is only poles associated with the upper half plane that should be included. To
identify these, the wave number is considered for a lightly damped structure

$$\kappa = \kappa_0 + \frac{\partial \kappa}{\partial \eta}, \tag{39}$$

where $\kappa$ is a solution to equation (12) and $\kappa_0$ is the associated real solution found in
the limit of zero damping. The derivative above should also be evaluated in this limit.

To that end, the derivative of equation (12) with respect to $\eta$ is taken

$$\frac{\partial}{\partial \eta} \left( (K(\kappa) - \omega^2 (1 + i \eta) M) \phi \right)$$

$$= \left( K'(\kappa) \frac{\partial \kappa}{\partial \eta} - i \omega^2 M \right) \phi + \left( K(\kappa) - \omega^2 (1 + i \eta) M \right) \frac{\partial \phi}{\partial \eta} = 0 \tag{40}$$

This equation is pre multiplied with the left eigenvector, which for a propagating wave
in a system without damping is the complex conjugate of the right eigenvector. The
resulting expression is rearranged, and it follows that

$$\frac{\partial \kappa}{\partial \eta} = \frac{i \omega^2 \phi^H M \phi}{\phi^H K'(\kappa) \phi} = \frac{i \omega}{2 c_g}, \tag{41}$$

where the group velocity $c_g$ for an un-damped waveguide is given by [16]

$$c_g = \frac{\partial \omega}{\partial \kappa} = \frac{\phi^H K'(\kappa) \phi}{2 \omega \phi^H M \phi}. \tag{42}$$

It follows from equation (39) and (41) that waves having a positive group
velocity are associated with the upper complex plane and should be included in the
summation (38). Thus, the input power to an un-damped infinite waveguide is given by

$$P_{in} = \frac{\omega}{2} \sum_{\varepsilon_j > 0, \Im \kappa_j > 0} \frac{|\phi_j^H f_0|^2}{\phi_j^H K'(\kappa_j) \phi_j},$$  \hspace{1cm} (43)$$

where it is noted that the denominator is a real and positive quantity for each term.

In Section 2 it was seen that if $\kappa$ is a solution to the dispersion relations, so is $-\kappa$. From reasons of symmetry it follows that one of these solutions is associated with a positive and the other with a negative group velocity of the same magnitude. Thus, the summation in equation (43) is equally made for the positive real wave numbers, if the absolute value of the denominator is considered.

In conclusion, the expression (38) for the input power to an infinite waveguide equals equation (31). Thus, the asymptotic modal approach and the wave approach estimate the injected power equally, if the structure has no damping.

Finally, it is worth noting that, in a similar analysis, Langley draws the erroneous conclusion that it is the waves with a positive wavenumber that are associated with the upper complex half plane [19, p. 359]. Thus, an equation similar to equation (31) results, except that it will predict a negative input power for waves that have phase velocity and group velocity in different directions.

4 Input power to a railway car structure

To demonstrate the formulation derived above, the vibrational power injected by a point force on the beam in Fig. 2 was measured. Thus, a shaker excited a force in the vertical direction at the middle of the top flange, at the location $(x, y) = (0.028, 0.12)$, see Fig. 3. The force and the response of the beam were measured with a Brüel&Kjaer 8001 impedance head.

Fig. 10 shows the input power injected by a force of unit magnitude in narrow bands, while Fig. 11 shows the same result in third-octave bands. For reference, Fig. 11 shows also the SEA estimate of the input power to a 4 mm steel plate. This is the value that many SEA practitioners would choose for a high frequency estimate of the input power, in the lack of alternatives.

Damping is modelled with a hysteretic damping loss factor, $\eta = 0.02$. At lower frequencies, the beam’s length has a major influence on the results and the predicted value is greatly in error. From 50 Hz, however, the prediction is of the correct order of magnitude and for twenty three consecutive third-octaves, from 50 Hz to 8 kHz, the values predicted by equation (38) agree with the measured values, with an error that is less than 3.5 dB.
If the dispersion characteristics in Fig. 5 are compared to the calculated input power in Fig. 10, it is seen that the input power has maxima at the cut-on frequencies around 130 Hz, 550 Hz and 2.8 kHz. The cut-on around 1.4 kHz, however, is not quite evident in the input power, since this wave form has a nodal line at the cross sectional coordinate where the force acts. Comparing the measured and calculated values in Fig. 10, it appears as if the cut-on frequencies at 130 Hz and 550 Hz are over estimated, while the one at 2.8 kHz is under estimated.

Fig. 12 shows the influence of damping on the results. Thus, the results for a hysteretic damping of magnitude $\eta = 0.001$ are compared to those for $\eta = 0.1$. Much of the theoretical analysis above consider structures that are proportionally damped; equation (38), however, applies equally for structures that have an arbitrary distribution of damping. To illustrate the results for such a structure, Fig 12 shows also the results when the top flange has an added damping, $\eta_e = 0.2$, while the rest of the beam has a very low level of damping, $\eta_e = 0.001$. It appears as if, at the considered frequencies, the damping material is perhaps better used if it is evenly distributed.

The results presented above are calculated for a set of simple boundary conditions at the connection between the beam and the corrugated plate. The choice was based on an “educated guess” in Section 2.5 and it was claimed that it was not critical. This statement is substantiated in Fig. 13, showing the results for: a, the considered boundary condition, $u = \phi_z = 0$; b, blocked condition, $u = v = w = \phi_z = 0$; c, shear diaphragm condition, $u = w = 0$ and d, free condition, where it is understood that a non restricted degree of freedom is not impeded by any boundary force.

The blocked condition is special in that there are no waves propagating in the structure at lower frequencies and, consequently, the injected power is very low. Other than that, the injected power is quite indifferent to the chosen boundary condition.

5. Conclusions

This paper considers the power injected into waveguide structures from forces that are concentrated along the waveguide but are otherwise arbitrary. The derived formulation should be particularly useful for a Statistical Energy Analysis. The motion of the structure is described by a set of coupled, linear, one-dimensional wave equations. These equations are, in this work, devised by the waveguide FEM, which is a versatile tool for describing wave motion in general structures that have uniform properties along one direction. Possibly, the methods for the evaluation of input power presented here will apply to any structure built up by the elements presented in the references [5-16].
SEA may be conceived in a modal approach or a wave approach and in line with this there are in the literature a modal and a wave formulation for the evaluation of input power. The application of these formulations to structures described by the waveguide FEM is developed in this work.

First, the modal formulation in reference [1] is applied for general finite length waveguide structures. The WFEM allows an efficient identification of the modes and natural frequencies for a convenient set of boundary conditions at the ends. Upon this basis the injected power is expressed by a modal summation and then frequency averaged as in [1]. The resulting expression is proportional to the modal density and to the ratio of squared modal force to modal mass, which are explicitly identified.

Second, an infinite waveguide is considered and the derivation in reference [19] is expanded to apply also for multi-dimensional dispersion relations. The input power is given by an inverse Fourier transform, which is solved by residue calculus. The poles describe the damped wave solution and are given by a polynomial eigenvalue problem. This procedure is numerically more costly; however, it attributes damping effects and avoids some difficulties in the modal procedure.

The modal procedure requires the modal density, which may be identified by a direct mode count, for a structure obeying convenient boundary conditions. It can also be evaluated by the asymptotic expressions (26) and (27). In doing this, care must be taken whenever a wave has its group and phase velocity in different directions, in which case the wavenumber is a decreasing function of frequency. This fact seems to have been overlooked in the past.

Another complication appears for waveguides at the cut-on frequencies, where the asymptotic modal density becomes infinite. A good solution for the handling of this singularity has not appeared and it seems as the modal approach based on the asymptotic modal density is quite useless whenever a cut-on frequency is encountered.

A waveguide finite element model of a stiffener in a railway-car structure serves as a demonstration example of the method. Calculation of the input power from a point force is subsequently made using the wave approach. The results are compared with an in situ measurement made on a full scale model. The results agree within 3.5 dB for 23 consecutive third octave bands, from 50 Hz to 8 kHz. This level of accuracy is often sufficient and considering the ease by which the model is made, the presented procedures should be useful whenever the power input to waveguide structures is of interest.
Acknowledgement

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Appendix

For a structure without damping, the near field solutions cannot absorb any energy nor transport energy away from the excitation and, hence, there is no contribution to the input power (38) associated with imaginary or complex poles. For completeness, this is demonstrated in what follows.

For an imaginary pole, \( \text{Im}\{\kappa_j\} > 0 \) and \( \text{Re}\{\kappa_j\} = 0 \), \( \mathbf{K}'(\kappa_j) \) is real and \( \mathbf{K}'(\kappa_j) = \mathbf{D}'(\kappa_j) \) is imaginary. The eigenvectors, \( \varphi_j \) and \( \theta_j \), given by equation (37), are the column null-spaces of the real matrices \( \mathbf{D}(\kappa_j) \) and \( \mathbf{D}(\kappa_j)^\top \) respectively. Consequently, \( \varphi_j \) and \( \theta_j \) are real valued vectors. The input power,

\[
P_{in,j} = \frac{\omega}{2} \text{Re} \left( \varphi_j^\top \mathbf{f}_0 \mathbf{f}_0^\top \varphi_j \right)
\]

is thus zero, since the bracketed expression is imaginary.

Complex poles appear in quadruples, see Table 1. Of the four poles, \( \kappa_a = \kappa_j \) and \( \kappa_b = \kappa_j^* \), belongs to the upper half plane, if \( \text{Im}\{\kappa_j\} > 0 \) and \( \text{Re}\{\kappa_j\} > 0 \). Using the relations between the eigenvectors in Table 1, the combined input power, \( P_a + P_b \), from the poles, \( \kappa_j \) and \( -\kappa_j^* \), is thus given by

\[
P_a + P_b = \frac{\omega}{2} \text{Re} \left( \varphi_j^\top \mathbf{f}_0 \mathbf{f}_0^\top \varphi_j \right) + \frac{\omega}{2} \text{Re} \left( \theta_j^\top \mathbf{D}'(\kappa_j)^\top \varphi_j \right)
\]

Furthermore,

\[
\text{Re} \left( \varphi_j^\top \mathbf{D}'(-\kappa_j^*) \varphi_j^* \right) = \text{Re} \left( \theta_j^\top \mathbf{D}'(-\kappa_j^*) \varphi_j^* \right) = \text{Re} \left( \theta_j^\top \mathbf{D}'(\kappa_j) \varphi_j \right),
\]

and

\[
\mathbf{D}'(-\kappa_j^*) = -\mathbf{D}'(\kappa_j).
\]

It follows that

\[
P_a + P_b = P_a - P_a = 0.
\]
References


Fig. 1. Symmetry of complex wavenumbers for an un-damped structure.

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<td>d</td>
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<td>$\varphi_d = \varphi_a^*$</td>
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Table 1: Relations between eigenwavenumbers and eigenvectors in different quadrants for un-damped waveguides.
Fig. 2 Sketch of beam and corrugated plate.

Fig. 3 Sketch of the considered railway car section. The measurements are taken on the x’th beam from the near end.
Fig. 4. Waveguide FE mesh of beam. Circles indicate the node location.

Fig. 5. Dispersion characteristics. Dots, solutions to equation (9) for beam; dashed line, steel plate with 4 mm thickness.
Fig. 6. Magnified dispersion characteristics of beam

Fig 7. Wave forms corresponding to dispersion relations in Fig.6
Fig. 8. Dispersion characteristics for the beam modelled with sharp corners, amplified.

Fig 9. Evolution with frequency of complex wave number. Solid line, undamped structure; dashed line, damped structure.
Fig. 10. Input power to railway car structure. Solid line, measured; dashed line, calculated with equation (38).

Fig. 11. Input power to railway car structure. Solid line, measured; dashed line, calculated with equation (38); dotted line, SEA estimate for a 4 mm steel plate.
Fig. 12. Input power. Solid line, $\eta = 0.001$; dashed line, $\eta = 0.1$; dotted line, $\eta = 0.001$ except for top flange where $\eta = 0.2$.

Fig. 13. Calculated input power to railway car structure for different boundary conditions at the corrugated plate. Solid line, considered boundary conditions; dashed line, blocked condition; dotted line, shear diaphragm condition; dash-dot line, free condition.
1 Tyre Design

The investigated tyre in this chapter is a 'slick', i.e. a tyre without tread pattern or grooves. The tyre was especially made by Goodyear for the EC funded research project RATIN, although with properties typical of a production tyre.

A tyre is made of a variety of materials, such as different types of rubber, fabric and steel. These materials are used in various forms in different regions of the tyre structure. As a consequence, a tyre may be divided into three types of sections, depending on the structural properties of each section. These sections are here referred to as the central area and the upper and lower sidewall areas as seen in Figure 1.

The ply is a layer of fabric that is embedded in rubber. At the lower sidewalls the ply encloses a volume filled with both steel wires and hard rubber materials, this makes the lower sidewall areas quite rigid. On the contrary, the upper sidewall areas are quite flexible, since the ply layer there is simple and there is no steel in there.

The tread is an about 13 mm thick rubber layer. The breakers of the central area
Figure 1: Tyre design and notation.
are made with steel wires in the circumferential direction of the tyre. This makes
the central area rigid with respect to flexural waves in the circumferential direction
but fairly flexible with respect to motion in the $x - R$ plane. Furthermore, due to
the rubber in the tread, the latter motion is highly damped.

2 Waveguide FE-model

The waveguide-FE model of the tyre and wheel are made with thin anisotropic
prestressed conical shell elements described in Chapter 2 of this thesis. The air
cavity is modelled with the fluid elements described in Chapter 3 whereas the fluid–
shell coupling is made with elements described in Chapter 4. To establish a full
waveguide–FE model, a mesh and a variety of input data are needed.

2.1 Mesh

Co–ordinates for the tyre geometry are provided by Goodyear, and the geometry
for the air cavity’s ’wetted’ rim surface was measured with a ruler. The mesh used
for the tyre air cavity was made with the MATLAB pde–toolbox. This ’toolbox’
gives meshes for 2D triangular elements, and is thus suitable for the fluid elements
in Chapter 3. Geometry data for the mesh boundary, which is suitable for the shell
elements, is also given from this ’toolbox’. The mesh used here is given in Figure 2.

2.2 Parameters

2.2.1 Tyre data

Material data used for a regular FE–model of the tyre was provided by Goodyear.
This data also included values of prestress corresponding to an inflation pressure of
2 bar over atmospheric pressure. It is stressed here that using the material data
Figure 2: Tyre mesh; Thin lines symbolize fluid element boundaries; Thick lines symbolize shell elements.
provided by Goodyear directly yields poor results compared to both measurements and an original model. There are several possible reasons for this.

Firstly
The regular FE–model uses elements based on thick shell theory rather than the thin shell theory used for the waveguide finite elements. Material data for thick and thin shell elements should generally differ since the thick shells can also encompass shear across the thickness. As seen in Chapter 2, equation (14), the material stiffness matrix is given as $6 \times 6$ matrices, $D$. $6 \times 6$ matrices are also provided by Goodyear. For thin shell theory, the $3 \times 3$ off–diagonal blocks in the $D$ matrices defines the (layered) materials coupling between out–of–plane and in–plane motion. However, for thick shell theory, the off diagonal blocks also affects the shearing across the thickness. It has been found that by excluding the off–diagonal blocks, calculations made with the waveguide–FE model are improved.

Secondly
The two breakers seen in Figure 1 are each made of rubber enclosing a layer of steel wires running in the circumferential direction. When the diagonal blocks in the $D$ matrices are excluded, there is no compensation for the different neutral layers for the flexural rigidities with respect to the $x$ and $\theta$ directions. Here, half the flexural rigidity in the $\theta$ direction is used in the waveguide–FE model. This is the same result as given by moving a flexural reference layer from the bottom of an ideal sandwich beam to the middle, neutral, layer.
Thirdly

The tread is the thick rubber layer on top of the breakers in Figure 1. The tread is modelled by adding both a mass and a material rigidity matrix to the suitable shell elements. The additional mass was provided by Goodyear, whereas the rigidity is found by approximating the tread as an isotropic shell. In reference [1], the shear modulus of the tread on a truck tyre is reported as \( G = 1.03 \text{ MPa} \). The shear modulus for the tread of the investigated tyre is expected to be roughly the same. Here, \( G = 1.15 \text{ MPa} \) is chosen to optimize calculations of the magnitude of the radial point mobility compared to measurements. The tread is measured to be about 13mm thick and Poisson’s ratio for rubber is assumed to be 0.5.

Fourthly

The regular FE-model uses more shell elements on the cross-section than the waveguide FE-model. Numerically, this should not give rise to a less accurate model, since the shape-functions of the regular elements are of lower order than those of the waveguide elements. Larger elements do however give rise to a less detailed geometrical model. This means that when the material data changes rapidly from one element to the next much care must be taken when replacing these elements with a single waveguide element. Here, all waveguide elements in the upper sidewall regions are chosen to have the material properties of the regular element in these regions that have the weakest \( x - R \) flexural rigidity. The parts of the central area close to the sidewalls also have rapidly changing material parameters. The material data for waveguide elements are set equal to one of the regular elements in the same region, (with the off-diagonal blocks in \( D \) removed).

The exact quantities for the third and fourth modifications stated above together with a frequency dependent damping have been chosen to optimize calculations of the magnitude of the radial point mobility compared to measurements, see Figure 17. This ‘optimization’ is solely made by visual inspection, i.e. the calculated curve should look the same as the measured curves. It was found that the calculated
results are sensitive to changes of parameters for the sidewalls.

The flexural rigidities in the $\theta$ and $x - R$ planes for both the waveguide-FE model and the original regular FE-model can be seen in Figures 4 and 5. A magnification of Figure 5 at the upper sidewall is shown in Figure 6. The tyre damping is frequency dependent and proportional to the material stiffness. The frequency dependence of the damping is seen in Figure 3.

![Figure 3: Frequency dependent damping. The roman numbers, I, II and III indicate frequency regimes with different types of motion](image)

### 2.2.2 Wheel

The wheel on which the tyre is mounted is made of aluminium and chosen to be as rigid as possible. Thus, at least for low frequencies, the wheel can be considered as a rigid body. The weight of the rim is 8.9 kg. The data given for the moment of inertia and center of gravity are not used here, since they were found to be in error. The rim is modelled with the same elements as the tyre, i.e. with thin-walled shell elements. The shape of the rim’s surface on the air cavity was measured with a
Figure 4: Flexural rigidity in the $\theta$ direction. Solid lines indicate values used for the waveguide FE model. Dots indicate values of the original regular FE model. Dashed lines shows values at the rim.
Figure 5: Flexural rigidity in the $x - R$ plane. Solid lines indicate values used for the waveguide FE model. Dots indicate values of the original regular FE model. Dashed lines shows values at the rim, which are larger than seen here.
Figure 6: Magnification of the corner of Figure 5
ruler. The mass distribution is such that 7.6 kg of the total 8.9 kg is placed on the
disc corresponding to the hub and spokes. The material data is chosen to give a
very stiff wheel so that the wheel is ensured to be rigid at the frequencies considered
here.

2.2.3 Air

The air inside the tyre is inflated to 2.0 bar over the normal atmospheric pressure
and measurements were made indoors at about 20°C. At these conditions, fluid
air speed is about 343 m/s, whereas the air density is about 3.63 kg/m³. The,
frequency independent damping, \( \eta_f \), in the air cavity is chosen to approximately
give the same half value bandwidths of the calculated fluid resonance peaks as those
of the measured peaks. Here, \( \eta_f = 0.003 \), is used.

3 Free response

3.1 Wave solutions

Truly propagating waves are found for undamped systems only. The waves are given
by solving the twin valued eigenproblem,

\[
\left[ \sum_{j=0}^{4} K_j (-i\gamma) + i \omega M_1 - \omega^2 M_2 \right] \hat{W}(\theta) = 0,
\]

(1)

for real valued \( \gamma \). The vectors \( \hat{W} \) give the wave shape corresponding to a pair,
\( \omega \) and \( \gamma \). A dispersion relation shows the dependence between frequency, \( \omega \), and
polar wavenumber, \( \gamma \). The dispersion relation for the tyre model is seen in Figure
7. Different wave shapes, 'A', 'B' and 'C', corresponding to '□' indicated points in
Figure 7, are given in Figures 8 to 10. The rings in Figure 7 show the dispersion
relation for the uncoupled fluid system.
Figure 7: Dispersion relation; Dots: coupled system, Rings: Uncoupled fluid, Squares: Displayed wave types, Diamond: wheel rigid body motion
Figure 8: Shape of wave 'A' at wavenumber, $\kappa$, and frequency, $\omega$, marked in Figure 7. Grey line depict original shape.
Figure 9: Shape of wave 'B' at wavenumber, $\kappa$, and frequency, $\omega$, marked in Figure 7. Grey line depict original shape.
Figure 10: Shape of wave 'C' at wavenumber, $\kappa$, and frequency, $\omega$, marked in Figure 7. Grey line depict original shape.
3.2 Interpretation of dispersion relation

The system solution is required to be periodic in $\theta$, so that $\hat{W}(\theta) = \hat{W}(\theta + 2\pi)$. Since the homogeneous solutions, $e^{-i\gamma \theta}$ are periodic only if $\gamma = 0, \pm 1, \pm 2, \ldots, \pm \infty$, it is concluded that these polar wavenumbers corresponds to resonance frequencies for the undamped system. Especially $\gamma = 0, \pm 1$, corresponds to wavenumbers where rigid body motion of the wheel is possible. Due to the pre–stress rigid body motions for the tyre–wheel assembly are restrained, see Chapter 2 Appendix D. These restraints has been found to become more severe when the flexural rigidity of the wheel increases. This can be seen in Figure 7 where there is no point close to zero frequency for $\gamma = 1$. The forced response calculations presented in Section 6 are, however, hardly affected by this phenomena.

One interesting rigid body motion of the wheel is indicated with '$\Diamond$' in Figure 7. This mode includes large shearing of the sidewalls, as seen in Figure 11. This may be expected to yield high damping about 100 Hz, since shearing of rubber most often results in high damping characteristics. Thus, since both the tread and the wheel behaves as rigid bodies, this mode can be seen as a 2 degree of freedom, (2 DOF), system with two masses coupled by a spring a damper.

As seen in Figure 9, wave 'B' has a node at the middle of the central area, thus this wave will hardly be excited for any force positioned there. Wave 'A', deforms the central tyre area mostly in the $\theta$ direction where the rigidity is dominated by the steel wires of the breakers. Consequently wave 'A' is not expected to be highly damped. On the other hand, wave 'C' also deforms the central tyre area in the $x - R$ plane. Thus, wave 'C' is expected to have high damping since the flexural rigidity for this motion largely depends on the rubber in the tread. At the cut on of the branch for wave 'C' the point mobility is expected to rise since the modal density, i.e the number of resonances in a frequency band, is high.

In conclusion, there are three frequency regimes of interest. Frequencies between,
Figure 11: Motion of tyre–wheel assembly about 100 Hz
roughly, 50 to 100 Hz are dominated by '2 DOF' systems, with the wheel and the tread acting as masses and the tyre sidewalls as springs and dampers. Frequencies from about 100 Hz up to about 300 Hz are dominated by the branches including waves 'A' and 'B', that hardly deforms the cross-section of the central area of the tyre. Frequencies above 350 Hz are dominated by higher order waves that largely deform the cross-section of the central area. These three frequency regimes are indicated in Figure 3, that also show the frequency dependence of the damping. A fourth frequency regime starts when shearing across the tread’s thickness become important, e.g. [2] and [3]. Since this shearing can not be modelled with thin shell elements, this frequency regime is not considered here.

The coupled and uncoupled dispersion relations for the air cavity in Figure 7 are quite close. This is also true for the structure’s dispersion relations and indicates that also wave types, such as those seen in Figures 8 to 10, and forced responses on the structure will differ very little between coupled and uncoupled systems.

The trend of the measured radial point mobility in Figure 16 may be expected from the above discussion. At about 100 Hz there is a wide peak which may be caused by the '2 DOF' system in Figure 11. Between 100 Hz and 300 Hz, a number of resonances corresponding to resonances for 'Wave A' appear. Since 'Wave B' has a node line at the excitation position its effect on the measurement is small. At 350 Hz 'Wave C' cuts on, which due to its high modal density and high damping, only appears as one wide peak.
4 Forced response

4.1 Equations of motion

In this thesis the wave equation for a curved waveguide–FE model including shell, fluid and fluid-shell coupling in curved coordinates may be written, as,

\[ \sum_{j=0}^{4} K_j \frac{\partial^j}{\partial \theta^j} + i \omega M_1 - \omega^2 M_2 \] \( W(\theta) = F(\theta) \),

(2)

where, the vector \( W(\theta) \) contains parameters describing the cross-sectional displacements and air-cavity pressure. \( F(\theta) \) is a force vector corresponding to \( W(\theta) \), see also Chapter 1.

The air, inside the tyre–wheel enclosure, has much less damping than the tyre structure. It is thus unsatisfactory to use damping directly proportional to the matrices in equation (2). As a result orthogonality relations such as that shown in Chapter 1 can not be used here.

4.2 Response calculations

By numbering fluid nodes with low numbers and subsequently the shell nodes with higher numbers, the matrices of equation (2) are partitioned as,

\[ K_j = \begin{bmatrix} K_{f,j} & 0 \\ 0 & K_{s,j} \end{bmatrix}, \quad M_1 = \begin{bmatrix} 0 & M_{1,sf} \\ -M_{1,sf}^T & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} M_{f,2} & 0 \\ 0 & M_{s,2} \end{bmatrix}, \]

(3)

where 0 denotes an appropriately sized matrix of zeros.

The homogeneous, undamped and uncoupled systems corresponding to equation (2) are then written,
for the fluid and

\[
\sum_{j=0}^{4} K_{f,j} \frac{\partial^j}{\partial \theta^j} - \omega^2 M_{f,2} \right] \Phi(\theta) = 0, \quad (4)
\]

for the shell. For periodic systems with periodicity \(2\pi\), equations (4) and (5) have solutions,

\[
\Phi(\theta) = \hat{\Phi}_{p,n} e^{-in\theta} \quad \text{for} \quad \omega = \omega_{p,n} \quad (6)
\]

and

\[
\Psi(\theta) = \hat{\Psi}_{q,n} e^{-in\theta} \quad \text{for} \quad \omega = \omega_{q,n} \quad (7)
\]

where \(n = 0, \pm 1, \pm 2... \pm \infty\), \(p = 1, 2, ..., P\), \(q = 1, 2, ..Q\), and \(P\) and \(Q\) are the number of degrees of freedom in the respective system. The frequencies, \(\omega_{q,n}\) and \(\omega_{p,n}\) and the vectors \(\hat{\Phi}_{p,n}\) and \(\hat{\Psi}_{q,n}\) are found by solving the eigenproblems arising from inserting equations (6) and (7) into the respective equations (4) and (5).

The solution \(W(\theta)\) of equation (2) must be periodic due to the nature of the geometry. Thus, the solutions given in equations (4) and (5) must span the solution space of \(W(\theta)\), so that,

\[
W(\theta) = \sum_{n=-\infty}^{+\infty} \left[ \begin{array}{cc} \Phi_n & 0 \\ 0 & \Psi_n \end{array} \right] \left[ \begin{array}{c} a_n \\ b_n \end{array} \right] e^{-in\theta}, \quad (8)
\]

where the matrices, \(\Phi_n\) and \(\Psi_n\) are given by,
\[
[\Phi_n] = [\Phi_{1,n} \Phi_{2,n} \Phi_{3,n} \ldots \Phi_{P,n}]
\]  
(9)

and

\[
[\Psi_n] = [\Psi_{1,n} \Psi_{2,n} \Psi_{3,n} \ldots \Psi_{Q,n}].
\]  
(10)

Expanding the force vector of equation (2) as an exponential Fourier series gives,

\[
F(\theta) = \sum_{-\infty}^{\infty} \begin{bmatrix} \hat{F}_{f,n} \\ \hat{F}_{s,n} \end{bmatrix} e^{-in\theta}
\]  
(11)

Now, inserting equation (8) into equation (2), multiplying by, \(e^{i\omega \theta}\) and integrating over \(\theta : 0 \rightarrow 2\pi\) yields,

\[
\begin{bmatrix}
K_f(n) - \omega^2 M_{2,f} & i\omega M_{1,sf} \\
-i\omega M_{1,bs}^T & K_s(n) - \omega^2 M_{2,s}
\end{bmatrix}
\begin{bmatrix}
\Phi_n & 0 \\
0 & \Psi_n
\end{bmatrix}
\begin{bmatrix}
a_n \\ b_n
\end{bmatrix}
= \begin{bmatrix}
\hat{F}_{f,n} \\ \hat{F}_{s,n}
\end{bmatrix},
\]  
(12)

for any \(n = 0, \pm 1, \pm 2, \ldots, \pm \infty\), where,

\[
K_f(n) = \sum K_{f,j} (-i n)^j \quad \text{and} \quad K_s(n) = \sum K_{s,j} (-i n)^j.
\]  
(13)

Subsequently, multiplication from the left with, \([\Psi_n \ \Phi_n]\) results in,

\[
\begin{bmatrix}
[\Phi_n]^T [K_f(n) - \omega^2 M_{2,f}] [\Phi_n] & [\Phi_n]^T [i\omega M_{1,sf}] [\Psi_n] \\
[\Psi_n]^T [-i\omega M_{1,bs}^T] [\Phi_n] & [\Psi_n]^T [K_s(n) - \omega^2 M_{2,s}] [\Psi_n]
\end{bmatrix}
\begin{bmatrix}
a_n \\ b_n
\end{bmatrix}
= \begin{bmatrix}
\hat{F}_{f,n} \\ \hat{F}_{s,n}
\end{bmatrix}
\]  
(14)

This linear set of equations is solved for each frequency, \(\omega\) and each wavenumber \(n\) of interest. Back substitution into equation (8) gives the forced solution \(W(\theta, \omega)\).
4.2.1 Torque excitation

Torque excitation in the θ direction is made exactly with the method derived in the previous section, since at each node–line there has one degree of freedom for rotation in the θ direction. For moment excitation in the x – R-plane there is no such degree of freedom and results from the following analysis are employed. Consider the response from a unit force,

\[ F(\theta) = \hat{F} \delta(\theta). \]  

(15)

From the previous section it is found that response from this force may be written,

\[ W_f(\theta) = \| \hat{F} \| \sum_{n=-\infty}^{+\infty} \sum_{r=R}^{c_{k,n}} \hat{W}_{k,n} e^{-in\theta}, \]  

(16)

where \( \| \cdot \| \) denotes Euclidean norm, \( c_{k,n} \) are scalars and the vectors \( \hat{W}_{k,n} \) form a basis for the solution for every \( n \). Thus for the basis in the previous section,

\[ \hat{W}_{k,n} = \begin{bmatrix} \hat{\Phi}_{k,n} \\ 0 \end{bmatrix} \text{ for } 1 \leq k \leq P \]  

(17)

and

\[ \hat{W}_{k,n} = \begin{bmatrix} 0 \\ \hat{\Psi}_{k-P,n} \end{bmatrix} \text{ for } P + 1 \leq k \leq P + Q. \]  

(18)

A torque is now defined by two point forces so that,

\[ \hat{T}(\theta) = \lim_{\theta_0 \to 0} \hat{F} (\delta(\theta - \theta_0) - \delta(\theta + \theta_0)) R\theta_0, \]  

(19)

where, \( 2R\theta_0 \) is the separation distance between the two forces.
The solution for the response from this torque is,

\[
W_t(\theta) = \lim_{\theta_0 \to 0} \| \hat{F} \| R\theta_0 (W_f(\theta - \theta_0) - W_f(\theta + \theta_0)) \tag{20}
\]

Substituting equation (16) into (20) and subsequently letting \( \theta_0 \) tend to zero, while keeping \( \hat{T} \) constant yields,

\[
W_t(\theta) = \| \hat{T} \| \sum_{n=-\infty}^{+\infty} \sum_r i^n c_{k,n} \hat{W}_{k,n} e^{-in\theta}, \tag{21}
\]

### 4.3 Practical considerations

All wavenumbers \( n = 0, \pm 1, \pm 2, \ldots, \pm \infty \) in equations (8) and (21) can obviously not be used. The series in \( n \) must be truncated at some coefficient \( n = \pm N \).

By considering Figure 7, it can be seen that the dispersion relations for the predominantly fluid waves for the coupled and the uncoupled system differ little. The same is true for the structural subsystem and for the corresponding wave shapes. Thus, the fluid and structure are weakly coupled. For uncoupled systems the response is dominated by resonances close to the excitation frequency. Consequently, in practice, the solution of the uncoupled system requires only a few of the original cross sectional mode-shapes. For weakly coupled systems this is also true and so \( P \) and \( Q \) in equations (6) and (7) are replaced by \( P \leq \overline{P} \) and \( Q \leq \overline{Q} \).

The model used here has, \( P = 168 \) and \( Q = 354 \) whereas the results given in the following uses \( \overline{P} = 10 \) and \( \overline{Q} = 50 \). Furthermore a value \( N = 150 \) is chosen. The forced response results are then practically indistinguishable from results with larger \( \overline{P}, \overline{Q} \) and \( N \). For comparison, calculated point mobility with \( \overline{P} = 2, \overline{Q} = 10, N = 150 \) is shown in Figure 12 together with results for \( \overline{P} = 10, \overline{Q} = 50, N = 30 \).
Figure 12: Magnitude of calculated radial point mobility at tread centre; Solid line; \( P = 10, \ O = 50, \ N = 150 \). Dotted line; \( P = 10, \ O = 50, \ N = 30 \). Dashed line; \( P = 2, \ O = 10, \ N = 150 \).
5 Measurements

Measurements of three quantities were made by the author. These are the point mobility from a radial force positioned at the middle of the tread and transfer mobilities to the air–cavity pressure and to the rim motion from the same excitation. The measurement setup is seen in Figure 13. The measurements were made with a swept–sine analysis. This gave a very good coherence and an extra fine resolution around the lightly damped fluid resonance peaks. In addition, measurements made by Andersson [2] on a tyre from the same batch are also used here. The measurements used here include radial point mobility on the center of the tread and transfer mobilities to points with 100 mm, 200 mm and 300 mm circumferential separation from the same excitation. Furthermore, point mobility from a tangential circumferential force at the center of the tread is also used here.

The equipment used for the authors measurements is listed in Table 1. According to Andersson, [2], the shape and size of the excitation area is of large importance and a square plate 15 mm x 15 mm is recommended as the smallest sized plate that avoids local deformation in the tread. The mounting of the impedance head in the author’s measurement is seen in Figure 14.

Radial point mobility on the center of the tread was measured both by the author and by Andersson, [2]. The measured magnitude of the point mobility is shown in Figure 16. The differences are acceptably small, but not likely to be explained by original differences in the individual tyres, since measurements at Goodyear have shown these to be much smaller. Also great care has been taken to ensure the same measurement conditions. One possible reason for the discrepancy is the ageing effect of the tyre, since more than two years passed between the measurements. An ageing will make the rubber harder and more rigid and thus lower the point mobility. At higher frequencies this corresponds with the behaviour seen in Figure 16.
### Table 1: Parameters for orthotropic plate strip

<table>
<thead>
<tr>
<th>Device</th>
<th>Type</th>
<th>Serial number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Impedance head:</td>
<td>B&amp;K 8001</td>
<td>10109</td>
</tr>
<tr>
<td>Accelerometer</td>
<td>B&amp;K 4393V</td>
<td>1929298</td>
</tr>
<tr>
<td>Hydrophone</td>
<td>B&amp;K 8103</td>
<td>1442849</td>
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<td>Analyzer</td>
<td>Tektronix</td>
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<tr>
<td>Shaker</td>
<td>LDS V203</td>
<td>45104/17</td>
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<td>Shaker amplifier</td>
<td>Zachry D250</td>
<td></td>
</tr>
<tr>
<td>Amplifier force</td>
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<td>1799669</td>
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<td>Amplifier tread acc.</td>
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<tr>
<td>Amplifier rim acc.</td>
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<tr>
<td>Amplifier pressure</td>
<td>B&amp;K 2635</td>
<td>1447225</td>
</tr>
</tbody>
</table>

Figure 13: Measurement setup.
Figure 14: Impedance head mounting.
Figure 15: Measurement setup. The white cable connects to the hydrophone measuring the internal air cavity pressure. The black cable connects to the accelerometer that measures the motion of the wheel.
Figure 16: Magnitude of point mobility measurements; Grey: ref [2], Black: author’s
6 Results

6.1 Point mobility radial direction

The calculated point mobility from a radial force acting on the center of the tread is compared to the measurement. The absolute value, the phase and the real part are seen in Figures 17 to 19.

Figure 17: Magnitude of point mobility measurement and calculation; Grey: ref [2], Black: calculation

The small peaks at around 225 Hz and 450 Hz corresponds to the first and second air cavity resonance. These are observed both in the measurement and the calculation data. The phase discrepancy at higher frequencies is probably an effect of local,
tread, flexibility in the measurements.

Figure 18: Phase of point mobility measurement and calculation; Grey: ref [2], Black: calculation

6.2 Transfer mobility radial excitation

The magnitude of the transfer mobility from a radial point force to response locations 100 mm, 200 mm and 300 mm separated in the circumferential direction are seen in Figures 20, 21 and 22 respectively. The measured mobilities are indicated with grey lines whereas calculated are indicated with black lines.
Figure 19: Real part of mobility measurement and calculation; Grey: ref [2], Black: calculation
Figure 20: Magnitude of transfer mobility, 100 mm separation in $\theta$ direction. Measurement and calculation; Grey: [2], Black: calculation
Figure 21: Magnitude of transfer mobility, 200 mm separation in $\theta$ direction. Measurement and calculation; Grey: [2], Black: calculation
Figure 22: Magnitude of transfer mobility, 300 mm separation in $\theta$ direction. Measurement and calculation; Grey: [2], Black: calculation
6.3 Point mobility tangential excitation

Tangential excitation of a tyre is due to tangential forces at the tread–road interaction. The tread is about 13 mm thick and seen in Figure 1. By adding this thickness to the distance to the middle of the breakers, and the distance above the surface at which the excitation acts, the tangential force can be seen as acting on a lever with length \( l \). The tread rotation due to this torque is,

\[
\left( \frac{l}{R_0} \right) \frac{\partial W_t(\theta)}{\partial \theta},
\]

(22)

where \( l \) is the length of the lever and \( R_0 \) is the radius of the tyre. For the point mobility the pure tangential part will not give rise to a rotation. Likewise, the torque will not give rise to any outward displacement at the excitation. Thus, the point mobility of the tangential force acting at radius \( R = R_0 + l \) is given as a sum of the tangential displacement at \( R_0 \) and the rotation at \( R_0 \) times \( \frac{l}{R_0} \). The length, \( l \), of the lever is here chosen to be \( l = 18 \) mm, which is a realistic value for the tyre under consideration. The magnitude of the measured and calculated tangential point mobilities are seen in Figure 24 whereas the real parts are seen in Figure 25. The real parts are in closer agreement. This is likely to be explained with the local, reactive, deformation of the tread. The calculated real parts of the two separate contributions are shown in Figure 26. As seen, the in–plane tangential excitation is dominating at lower frequencies, whereas the rotation dominates the higher frequencies. The calculated peak at 65 Hz is due to a, '2 DOF' mode including rigid body motion of the wheel as seen in Figure 23. This mode also exist for the real tyre–wheel assembly.
Figure 23: Motion for resonance calculated at 65 Hz.
Figure 24: Magnitude of calculated and measured magnitude of tangential point mobility, Grey: [2], Black: Calculated
Figure 25: Real parts of calculated and measured tangential point mobility, Grey: [2], Black: Calculated
Figure 26: Real parts of calculated tangential point mobility contributions, Solid: In plane contribution, Dashed: Rotational contribution
6.4 Air pressure response

The transfer function from the radial point excitation to the air at about 81 degrees separation is considered. The measurement shows a peak at 297 Hz which has been found to correspond to the first rim resonance. The rim alone is reported to have its first resonance at about 800 Hz. The discrepancy between the measured and reported resonances is probably due to the mass loading of the tyre on the rim. Since the rim is not rigid, differences at frequencies in-between fluid resonances above the first rim resonance are possibly due to the rim motion.

![Graph showing air pressure response](image)

Figure 27: Magnitude of transfer function to air cavity, measurement and calculation; Grey: measured, Black: calculation
6.5 Wheel acceleration response

The transfer function to the wheel is measured and calculated. Here the transfer function is related to the acceleration, since this relates more naturally to the force on the wheel. The discrepancies, at about 100 Hz may be explained with decreased rotational motion around the $\theta$–axis in the calculations. There are two reasons for this, either the response position coincides with the rotation center or the motion is close to rotation–free. By considering several response positions on the wheel it is found that the latter explanation accounts for the larger part of the discrepancy.

In reality a rotation–free motion is obtained when the net force on the wheel acts through the mass center. Since the data for the wheels moments of inertia and center of gravity are approximates only, better wheel data may improve the calculations.

Another possible reason is the restriction of this rigid body rotation due to pre–stress as discussed in Appendix D of Chapter 2. In the calculations here a very high flexural rigidity in the wheel is used to prevent any rim resonances below 1600 Hz. This high flexural rigidity impedes some rigid body motions of the tyre–wheel model.
Figure 28: Magnitude of transfer function to rim acceleration, measurement and calculation; Grey: measured, Black: calculation. The measured acceleration at the fluid resonance peak at 225 Hz is about 1 dB higher than the calculated
7 Conclusions

A car tyre is modelled with waveguide finite elements. The model includes the tyre, the wheel and the air cavity enclosed by the tyre-wheel assemblage. Input data used for a conventional FE model is modified to improve response calculations. It is observed that calculated responses are particularly sensitive to differences in the stiffness at the upper side–walls.

Calculations of point mobility and transfer mobilities of a freely suspended tyre/wheel are compared to measurements. The results indicate good agreement in a frequency regions from about 100 Hz up to about 600 Hz.

Calculations up to about 100 Hz are less satisfying, since rigid body motion of the wheel is poorly estimated. This is partly explained with poor wheel data and partly by the fact that rigid body motion is restricted with the elements used, see Appendix D of Chapter 2. This is not a problem for a fixed wheel and probably a minor problem for a wheel mounted on a car suspension.

Some of the discrepancies at frequencies above 600 Hz is explained with local flexibility of, and shearing across, the tread thickness. It is difficult to include the effects of this shearing in the thin shell theory used here and it probably sets the upper frequency limit for the, thin shell, elements used here.

The, unwanted, effects of local flexibility of the tread on the measurements may be overcome by measurements on the inside of the tyre. The tread deformation on ’normal patterned’ tyres might partly be overcome by including locally reacting distributed spring mass systems to the thin shell theory, see [4].

Tyre damping is modelled proportional to the structural stiffness matrices and is frequency dependent. Future improvements should include damping proportional to different sub–elements of the tyre such as the tread and the sidewalls.
References


