On the mitigation of shock waves by exciting vibrational modes in water droplets

Michael Försth, Raúl Ochoterena

Fire Technology
SP Report 2013:52
On the mitigation of shock waves by exciting vibrational modes in water droplets

Michael Försth, Raúl Ochoterena
Abstract

On the mitigation of shock waves by exciting vibrational modes in water droplets

The vibrational energy of water droplets has been investigated. Viscosity was neglected and only the lowest order symmetrical vibration, the quadrupole mode with \( m=0 \), was considered. An approximate solution was given by Rayleigh in 1879:

\[
\omega_{vlb} = \alpha^2 \gamma r_0^2 \quad \text{where} \quad \alpha \text{ is the relative vibration amplitude, } \gamma \text{ is the surface tension, and } r_0 \text{ is the droplet radius at equilibrium, that is the radius of the spherical droplet. It was found that this solution agrees with the exact solution within a few percent for moderate values of the relative amplitude } \alpha.
\]

The rationale of this study was to investigate whether dissipation of energy from a shock wave, e.g. from an explosion, into vibrational energy of droplets is a mechanism that needs to be considered in order to accurately model explosion mitigation using water mists. Traditionally droplet vibrations are not considered in such models whereas other mechanisms such as translational energy, heating, evaporation, and droplet breakup are considered. A useful parameter for quantifying the relative importance of vibrations as compared to linear acceleration of droplets is the ratio between vibrational and translational kinetic energy. This ratio is given by \( 3\alpha^2 \gamma / (\rho \pi r_0 v_{trans}^2) \) where \( v_{trans} \) is the translational velocity to which the droplet has been accelerated by the shock front. This ratio was found to be negligible for a severe explosion scenario where information on \( v_{trans} \) was available in the literature. For other cases vibrations might be more significant but detailed information about specific scenarios is needed to obtain detailed results.

More work is needed in order to fully understand the effect of viscosity and higher order vibrations. Furthermore, a model for the excitation of droplet vibrations by shock waves needs to be developed for modelling. Finally, there is a lack of experimental data on droplet size distribution for water mist interacting with a shock wave.

Key words: explosion, shock waves, water mist, droplet, vibration, spherical harmonics.

SP Sveriges Tekniska Forskningsinstitut
SP Technical Research Institute of Sweden

SP Report 2013:52
ISBN 978-91-87461-40-8
ISSN 0284-5172
Borås 2013
Contents

Abstract 3

Contents 4

Preface 5

1 Introduction 6

2 Gap analysis 9

3 Vibrations of inviscid droplets 11
   3.1 Energy comparisons 18
   3.2 Viscosity and dissipation 19

4 Discussion and conclusions 21

Appendix A: Calculations of integrals using expansions in spherical harmonics 22

Appendix B: Expansion of $r^5$ in spherical harmonics 27

Appendix C: The Rayleigh solution 52

References 54
Preface

This work was partly sponsored by Lars och Astrid Albergers stiftelse which support is gratefully acknowledged.
1 Introduction

 Explosion mitigation is typically achieved by venting through designed weakened areas in the enclosure boundary. In some instances this is not possible for examples due to extreme and fast overpressures or physical limitations on the size and placement of such vents [1]. An attractive option for explosion mitigation is water spray systems since these require relatively low maintenance and are often already installed for fire suppression. This also make such systems relatively cost-effective.

 One of the highest reported efficiencies for water mist was reported by Buzukov [2] who used auxiliary explosions to create a water mist acting as a curtain to mitigate open explosions from a concentrated charge. A 90% reduction in overpressure was observed in some cases.

 The nature of explosions differs depending on the distribution of the charge. For distributed charges of combustible gas or dust the situation is somewhat different as compared to concentrated charges, such as TNT. For concentrated charges the blast consists of a shock front followed by a reaction front (that is a non-combusted charge that is ejected from the initial position). The reactivity of the shock front depends on the oxygen content of the explosive. Mitigation depends on the interaction of the water mist both with the shock front and the reaction front [3]. In military applications the concentrated charge is the typical threat; whereas in, for example, the offshore industry the charge is normally distributed within an enclosure.

 The quasi-static pressure is a commonly used quantity to define the nature of the explosion, which describes the long-term pressure buildup in an enclosure. Unlike the overpressure, \( p \), the quasi-static pressure is less dependent on the observer’s position relative to the charge. The quasi-static pressure is defined as [3]:

 \[
 p_{qs} = \frac{I(t_1, t_2)}{t_2 - t_1} 
 \]

 (1)

 Where \( I \) is the impulse per area for a time interval defined as

 \[
 I(t_1, t_2) = \int_{t_1}^{t_2} p \, dt 
 \]

 (2)

 Thomas [4] envisioned gas explosion mitigation by water spray as the following process:

 atomization of larger droplets due to an explosion wave \( \rightarrow \) flame quenching due to evaporation and cooling

 He also pointed out that spray suppression systems can lead to explosion enhancement due to turbulence generated by the spray. Lane [1, 5] noted that transient flow can be more efficient in inducing droplet breakup than steady flow. Pilch and Erdman [6] found that a critical Weber number of 12 can describe droplet fragmentation in shock studies. This value was subsequently found to hold for steady as well as transient flows [1, 7, 8].
Many of these studies were, however, performed on relatively large droplets, on the order of mm, and it is not clear how valid the value is for very small droplets.

Experiments on polydispersed sprays [1, 8] have shown that the breakup of large droplets, with diameters on the orders of mm, can occur within 10 ms after exposure to a velocity transient. It was also noted that this timescale is relatively short compared to the relevant timescales for large scale gas explosions.

Large-scale experiments simulating an offshore platform configuration show that the mitigation efficiency increases with increased explosion severity [1]. At the same time it has been reported that explosion mitigation using water mist in large complex systems can show a lower performance than in simple configurations [3]. This indicates that accurate detailed models are required to fully understand, predict and design such mitigation systems based on water mist.

Thomas [1, 4] presented critical number densities for inhibiting laminar methane-air flames, assuming mono-dispersed mists. The results are reproduced in Table 1 with an additional column for surface density. Thomas also noted that the critical mass concentration for water vapour is around a factor of 4-5 times higher than for the most effective droplet diameters, that is droplets can be much more efficient that water vapour.

It is interesting to note that the surface density, that is the amount of droplet surface per unit of water volume, does not vary more than a factor of approximately two within the 10 – 100 µm droplet range.

van Wingerden [9] found that droplets below 20 µm and above 200 µm are most effective. The small droplets range can probably be explained by rapid heating and evaporation for such droplets while the large droplets range indicates that other mechanisms, such as droplet breakup for example, play an important role. In a study by Lentati [10] an optimal drop diameter of 15 µm was proposed, determined based on the lowest extinction strain rate.

The reduction in burning velocity in methane-air flames was theoretically found to be mainly due to heat transfer, and that the effect of water vapour evaporation was less important [4]. The effect of the addition of KCl to the water mist on the flame speed was investigated by Thomas [4]. There was an effect but it was relatively small.

Schwer and Kailasanath [3] presented simulations of the interaction between a blast from 2.12 kg of TNT with water mist in open space and in enclosures. Only droplets smaller than 50 µm were considered and the breakup mechanism, emphasized by Thomas [1], was not considered. The simulations show that in open explosions the reaction front is relatively far behind the shock front and do not contribute to the overpressure at the shock front. Very close to the charge, the distance between the fronts is smaller however. The water mist was most efficient in reducing overpressure peaks, particularly in the first tens
of milliseconds. For longer times the overpressure, that is the quasi-static pressure, becomes less affected by the mist.

For enclosures, as opposed to open explosions, the situation is more complex. When the shock front is reflected from the walls, interactions occur with the reaction front which can increase the mixing between excess explosive and air, thereby speeding up the reactions.

With water mist it was found that the mist was to a large extent pushed by the shock front and therefore was ahead of the reaction zone and did therefore not significantly suppress the reactions with the exception of the initial stages. Physically this was explained by the higher inertia of the accelerated droplets as compared to the gases. As the gases contract after the over expansion the droplets tend to continue outwards.

A detailed study of the mitigation of the shock front showed that evaporation has a relatively small effect. The explanation for this was twofold. Firstly, the vaporization at the shock front is limited since the temperature is not very high and since the time scale is very short. Secondly, the phase change from liquid to gaseous water increases the gas density at the front, adding gas pressure which partially cancels the effect of decreased temperature. The authors suggested that the momentum extraction is the most important mechanism for mitigation of the shock front (but not the reaction front). The effect of convective heating of the droplets was not specifically discussed.

Studies of the location of the mist were also performed and these indicated that the location is much less important than the total amount of water between the explosion and the observation point. This suggests that a diffuse spray could be as effective as a concentrated spray near the charge. It was also noted that droplet size played a minor role compared to the total amount of water. Interestingly, for high mass loadings of water the overpressure increased close to the charge. This effect was not very large however.

Ananth et al. [11] performed a numerical analysis between water mist and a charge equivalent of 23 kg TNT in a 3.5 m radius spherical chamber. They found that droplet fragmentation is an important part of the mitigation mechanism and that most of the water mist is completely evaporated before the shock front hits the wall. The droplets were accelerated to velocities on the order of 500 ms$^{-1}$.

This report continues with a brief summary in Section 2 with challenges, especially regarding the level of understanding, concerning water mists for explosion mitigation. Section 3 contains the core part of the report which is the theory for vibrational modes of water droplets and the amount of energy that can be absorbed by such modes. Section 4 contains discussion and conclusions with directions for future work, experimental as well as theoretical. Appendices A and B contain the mathematical formalism used to analytically solve integrals in Section 3. Finally Appendix C contains a derivation of the vibrational energy using the approximative solution by Rayleigh [12].

The delimitations of this work are:
- Only the lowest order, quadrupole, symmetric vibrational mode is considered.
- The theory is only strictly valid for small amplitudes of the vibrations.
- Viscosity is neglected.
- The kinetic energy of a vibrating mode is calculated but no attention is given to how and whether such a mode can be excited by the shock front.
- The fact that internal vibration can only absorb energy, and not momentum, from the shock front is not discussed further.
2 Gap analysis

The relative importance of the different mitigation mechanisms of water mist, and the interaction between these mechanisms, is poorly understood. The level of understanding needs to be improved in order to appropriately design explosion mitigation systems using water mist. In this section some of the outstanding issues are pointed out.

Large-scale experiments simulating an offshore platform configuration show that the mitigation efficiency increases with increased explosion severity [1]. At the same time it has been reported that explosion mitigation using water mist in large complex systems can show a lower performance than in simple configurations [3]. Acceleration of gas flows in explosions depends on local congestion and confinements. This indicates that accurate detailed models are required in order to fully understand, predict and design such mitigation systems based on water mist.

Schwer and Kailasanath [3] found that, especially for small droplets and high mass loadings, the overpressure increased close (50 cm) to a concentrated charge. This mechanism needs to be understood in order to avoid exacerbating the situation with water mist. The smallest diameter considered was 15 µm. What if the water mist system creates even smaller droplets, or if smaller droplets are created by breakup? This has not been considered by the authors.

A too fine mist could potentially be vented out from the enclosure to be protected before the combustion front arrives [1]. This is of particular interest for blasts with combustion fronts (such as gas- and TNT explosions for example) since one desirable mitigation mechanism for such blasts is flame quenching.

Schwer and Kailasanath [3] suggested that the water mist fails to suppress the reaction zone due to the water mist being pre-dispersed in the enclosure. They proposed that if the water is continuously pumped into the explosion area it would be likely to penetrate into and suppress the reaction zone. This assumption needs to be experimentally validated before designing mitigation systems. This question is also closely linked to the filling capacity of the mist systems in different enclosures [13], how early must the system be activated in order to be fully effective in mitigating the blast?

The effect of turbulence induced by the spray should be further studied since studies suggest that the burning velocity increases by around three times with sprays [1].

Understanding of the interaction between the explosion wave and the mist is impeded by the lack of information concerning the actual size distribution of the mist residue – after fragmentation by the wave itself. Such measurements are complicated by high numbers densities and short durations [1]. Advanced in-situ measurements are required in order to obtain the required knowledge on the actual size distribution that interacts with the shock front and possibly with the reaction front.

A critical Weber number of 12 has been found to describe fragmentation [1, 6-8, 14]. Many of these studies were however performed on relatively large droplets, on the order of mm, and it is not clear how valid the value is for very small droplets. Schwer and Kailasanath [3] specifically pointed out that there is a lack of information on breakup for droplets smaller than 100 µm. Knowledge on the breakup mechanism is required for several reasons:
1. Small droplets appear to increase the overpressure close to the explosive, at least for large mass loadings [3].
2. Knowledge of the final droplets sizes is important for understanding of transport of the mist (Will it escape the enclosure? To what degree will it penetrate the reaction zone?)
3. Knowledge of the breakup and final droplets sizes is important for the understanding of kinetic extraction of energy from the blast.
4. Knowledge of the final droplets sizes is important for understanding of the heat extraction of energy from the blast (evaporation and convective heating).

Knowledge of the explosion and combustion characteristics for explosives other than TNT is required since many other explosives are used for military applications. This knowledge is required basically for two reasons:

1. Understanding the interaction between the shock front and the mist.
2. TNT is oxygen deficient which means that the reaction front strongly contributes to the pressure buildup in an enclosure. This suggests that the positive results (experiments [14] as well as simulations [3]) from water mist mitigation of TNT-explosions in enclosures might be overly optimistic if other explosives, which have a less pronounced reaction front, are considered.

The effect of surfactants might be positive from a droplet size perspective since the surface tension is reduced, resulting in higher Weber numbers and smaller droplets. On the other hand, foaming [1] might influence the mitigation negatively though by for example reduced the distribution of water, or other unforeseeable effects. Also, if there is a lower diameter limit that should not be undercut (due to findings of increased overpressure near the blast using very small droplets as discussed above) surfactants could become a problem.

Finally, only breakup, heating, evaporation, and momentum extraction of the droplets has been considered, and not internal vibrations of the droplets excited by the shock front. Such vibration are the topic of this report. A better understanding of internal vibrations could also lead to an increased level of understanding regarding the detailed droplet breakup mechanism.
3 Vibrations of inviscid droplets

The oscillation angular frequency of an inviscid droplet vibrating at small amplitudes is given by Landau and Lifshitz [15], p. 247, as

\[ \omega^2 = \frac{\gamma}{\rho r_0^3} l(l - 1)(l + 2) \]  

(3)

where

- \( \gamma \) is the surface tension, \( \gamma = 72.75 \cdot 10^{-3} \) J m\(^{-2} \) for water at 293 K.
- \( \rho \) is the density, \( \rho = 998 \) kg m\(^{-3} \) for water at 293 K.
- \( r_0 \) is the droplet equilibrium radius

\( l=0 \) corresponds to the breathing mode, that is a spherically symmetrical pulsation of the droplet. This is not possible for an incompressible fluid and therefore this mode vanishes. \( l=1 \) corresponds to a translatory motion of the entire droplet which is not of interest for this analysis where the vibrational energy is investigated. The first non-vanishing mode is therefore \( l=2 \), the quadrupole mode. It will be assumed that no higher order vibrations are excited.

The shape of a vibrating droplet can be expressed as [16]:

\[ r(\theta, \varphi, t) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} R_{lm}(t) Y_{lm}(\theta, \varphi) \]  

(4)

where the expansion coefficients \( Y_{lm}(\theta, \varphi) \) are spherical harmonics, see [17], pp 670-672 and 683-696.

As mentioned above we will consider a droplet with an equilibrium radius \( r_0 \) which is vibrating in the quadrupole mode, \( l=2 \). Therefore the translation mode \( l=1 \) and higher order modes with \( l>2 \) vanish, i.e.:

\[ r(\theta, \varphi, t) = R_{00}Y_{00} + \sum_{m=-2}^{2} R_{2m}(t) Y_{2m}(\theta, \varphi) \]  

(5)

\( m=0 \) corresponds to vibrations that are symmetrical about an axis while \( m\neq0 \) corresponds to non-symmetric vibrations, with a dependence in the azimuthal angle \( \varphi \). In this report we will for simplicity assume symmetric vibrations with \( m=0 \). This gives

\[ r(\theta, t) = R_{00}Y_{00} + R_{20}(t)Y_{20}(\theta) \]  

(6)

Denoting the vibrational amplitude by \( A=\alpha r_0 \), where \( \alpha \) is the relative amplitude, and expressing the time dependence as \( \sin(\omega t + \epsilon) \) we can write the radius as:

\[ r(\theta, t) = r_0 + \alpha r_0 \sin(\omega t + \epsilon)Y_{20} \]  

(7)

The spherical harmonics are given by[18]:
\[ Y_{00} = \frac{1}{2\sqrt{\pi}} \] (8)

and

\[ Y_{20}(\theta) = \frac{1}{4 \sqrt{\pi}} \cdot (3\cos^2(\theta) - 1) \] (9)

By comparing Eqs. (6) and (7) we identify:

\[ R_{00} = \frac{r_0}{Y_{00}} = 2\sqrt{\pi} r_0 \] (10)

and

\[ R_{20} = A \sin(\omega t + \varepsilon) = \alpha r_0 \sin(\omega t + \varepsilon) \] (11)

The shape of a body vibrating according to Eq. (7), with \( \varepsilon=0.2 \) and \( \varepsilon=1 \), is shown in Figure 1 and Figure 2, respectively. In the figures the equilibrium, spherical, shape for \( \omega t + \varepsilon=0 \) is shown to the left. The most elongated shape, for \( \omega t + \varepsilon=\pi/2 \), is shown in the middle, and the most oblate shape, for \( \omega t + \varepsilon=-\pi/2 \), is shown to the right.

Figure 1. Time evolution of a body described by Eq. (7) with \( \varepsilon=0.2 \). The phase \( \omega t + \varepsilon \) assumes the values 0, \( \pi/2 \), and \( -\pi/2 \) from left to right.

Figure 2. Time evolution of a body described by Eq. (7) with \( \varepsilon=1 \). The phase \( \omega t + \varepsilon \) assumes the values 0, \( \pi/2 \), and \( -\pi/2 \) from left to right.
The density of kinetic energy is
\[ \rho_{\text{kin}} = \frac{1}{2} \rho_m |v|^2 \]  
(12)

The velocity is given by the gradient of the velocity potential according to [19], page 17:
\[ \mathbf{v} = \nabla \phi \]  
(13)

and consequently the total kinetic vibrational energy is
\[
\begin{align*}
\mathcal{W}_{\text{kin}} &= \int \rho_{\text{kin}} dV \\
&= \int \frac{1}{2} \rho_m |\nabla \phi|^2 dV \\
&= \int \frac{1}{2} \rho_m |\nabla \phi|^2 r^2 \sin(\theta) \, dr \, d\theta \, d\varphi
\end{align*}
\]  
(14)

The velocity potential for a vibrating sphere with \(l=2\) is given by Lamb [19], page 474, as:
\[
\phi = -\frac{\alpha r}{n r_0} \alpha r_0 Y_{20} \cos(\omega t + \epsilon)
\]
\[
\phi = -\frac{\omega r^2}{8} \sqrt{\frac{5}{\pi}} (3 \cos^2(\theta) - 1) \cos(\omega t + \epsilon)
\]  
(15)

The gradient operator in spherical coordinates is given by
\[
\nabla = \hat{r} \frac{\partial}{\partial r} + \frac{1}{r} \hat{\phi} \frac{\partial}{\partial \phi} + \frac{1}{r \sin(\theta)} \hat{\theta} \frac{\partial}{\partial \theta}
\]  
(16)

and therefore
\[
\begin{align*}
\mathbf{v} &= \nabla \phi \\
&= -\left( \frac{\alpha \omega r^2 - 1}{4} \sqrt{\frac{5}{\pi}} (3 \cos^2(\theta) - 1) \hat{r} \\
&\quad - \frac{\alpha \omega r^2 - 1}{2} \sqrt{\frac{5}{\pi}} 6 \cos(\theta) \sin(\theta) \hat{\phi} \right) \cos(\omega t + \epsilon) \\
&\quad + \epsilon \frac{\alpha \omega r}{4} \sqrt{\frac{5}{\pi}} (3 \cos(\theta) \sin(\theta) \hat{\theta} \\
&\quad - (3 \cos^2(\theta) - 1) \hat{r})
\end{align*}
\]  
(17)
Equation (14) becomes

\[
\omega_{\text{kin}} = \frac{\alpha^2 \omega^2 \rho_m}{32} \frac{5 \cos^2(\omega t + \epsilon)}{\pi} \int \left(3 \cos(\theta) \sin(\theta) \right) d\theta \\
- \left(3\cos^2(\theta) - 1\right) d\phi
\]

which can be expressed as

\[
\omega_{\text{kin}} = \frac{5\alpha^2 \omega^2 \rho_m \cos^2(\omega t + \epsilon)}{32\pi} \int_0^{2\pi} \int_0^\pi r^4 (9 \cos^2(\theta) \sin^2(\theta)) \\
+ 9 \cos^4(\theta) \sin(\theta) - 6\cos^2(\theta) \sin(\theta) + \sin(\theta)) dr d\theta d\phi
\]

The integration of \( r \) can be performed analytically:

\[
\omega_{\text{kin}} = \frac{\alpha^2 \omega^2 \rho_m \cos^2(\omega t + \epsilon)}{32\pi} \int_0^{2\pi} \int_0^\pi r^5(\theta, \varphi)(9 \cos^2(\theta) \sin^2(\theta)) \\
+ 9 \cos^4(\theta) - 6\cos^2(\theta) + 1) \sin(\theta) d\theta d\varphi
\]

with \( r \) given by Eq. (6) or (7).

For the quadrupole mode, \( l=2 \), Eq. (3) gives the square of the angular frequency as:

\[
\omega^2 = \frac{8\gamma}{\rho r_0^3}
\]

which, inserted in Eq. (20), gives:

\[
\omega_{\text{kin}} = \frac{\alpha^2 \gamma \cos^2(\omega t + \epsilon)}{4\pi r_0^3} \int_0^{2\pi} \int_0^\pi r^5(\theta, \varphi)(9 \cos^2(\theta) \sin^2(\theta)) \\
+ 9 \cos^4(\theta) - 6\cos^2(\theta) + 1) \sin(\theta) d\theta d\varphi
\]

The integration in Eq. (22) is feasible but relatively cumbersome due to the factor \( r^5 \). Baxansky and Kiryati [16] presented a method based on expansion of the integrand factors into spherical harmonics, and the orthogonality of spherical harmonics. Details of the solution are given in Appendices A and B. The solution is
\[ w_{\text{kin}} = 2\alpha^2 \gamma \, r_0^2 \cos^2(\omega t + \epsilon) \left( 1 + \frac{1}{2} \sqrt{\frac{5}{\pi}} \alpha \sin(\omega t + \epsilon) + \frac{45}{14\pi} \alpha^2 \sin^2(\omega t + \epsilon) 
+ \frac{25}{28} \sqrt{\frac{5}{\pi^3}} \alpha^3 \sin^3(\omega t + \epsilon) 
+ \frac{1325}{1232\pi^2} \alpha^4 \sin^4(\omega t + \epsilon) 
+ \frac{375}{4576} \sqrt{\frac{5}{\pi^5}} \alpha^5 \sin^5(\omega t + \epsilon) \right) \] (23)

For small values of \( \alpha \sin(\omega t + \epsilon) \) this is close to the solution by Rayleigh [12], p. 95, see Appendix C:

\[ w_{\text{kin, Rayleigh}} = 2\alpha^2 \gamma \, r_0^2 \cos^2(\omega t + \epsilon) \] (24)

Interestingly the solution by Rayleigh is obtained by approximating \( \alpha^0 \alpha^3 \) with \( \alpha^0 \) in Eq. (22).

The total energy of the vibrating inviscid droplet is

\[ w_{\text{tot}} = w_{\text{kin}} + w_{\text{pot}} \] (25)

where \( w_{\text{pot}} \) is the potential energy. The total energy is constant, that is not time-dependent. Since we are free to define the potential energy within an arbitrary constant term we can define \( w_{\text{pot}} \) as zero when \( w_{\text{kin}} \) reaches its maximum. In this way \( w_{\text{tot}} \) equals the maximum, over time, value of \( w_{\text{kin}} \).

\[ w_{\text{tot}} = \max(w_{\text{kin}}) \] (26)
Figure 3 shows the maximum value of $w_{\text{kin}}$, calculated using Matlab, in Eq. (23) divided by $2\alpha^2 \gamma r_0^2$. It is seen that $w_{\text{tot}}$ differs from $2\alpha^2 \gamma r_0^2$ by only a few percent up to $\alpha=0.5$. It should also be noted that the theory presented here is strictly valid only for small values of $\alpha$. In the analysis that will follow, we will use the approximate solution

$$w_{\text{tot}} = w_{\text{vib}} = 2\alpha^2 \gamma r_0^2$$

(27)

The vibrational frequencies for the quadrupole mode, $l=2$, is shown in Figure 4. Note that the frequency in this figure is $f = \text{vibrations per second}$, $f=\omega/(2\pi)$, where $\omega$ is given by Eq. (21).
The velocity of the surface is obtained by taking the derivative of Eq. (7)

$$v(\theta, \varphi, t) = \alpha \omega r_0 \cos(\omega t + \epsilon)Y_{20}$$

(28)

The maximum surface velocity is obtained when the \( \cos(\omega t + \epsilon) \) factor is one and along the polar axis, \( \theta=0 \) or \( \theta=\pi \).

$$v_{m,\text{ax}} = \alpha \omega r_0 \frac{1}{2} \sqrt{\frac{5}{\pi}} = \alpha \frac{8\gamma}{\rho r_0^3} r_0 \frac{1}{2} \sqrt{\frac{5}{\pi}} = \alpha \frac{10\gamma}{\pi \rho r_0}$$

(29)

The maximum surface velocity for the quadrupole mode, \( l=2 \), with \( \alpha=0.2 \), is shown in Figure 5.
3.1 Energy comparisons

In order to obtain a first estimate of the potential efficiency of the vibrational mechanism for explosion mitigation, it is relevant to compare the vibrational energy with the translational energy obtained by a droplet that has absorbed momentum from the shock wave. The translational kinetic energy of a droplet that has been accelerated to the speed $v_{\text{transl}}$ is

$$w_{\text{transl}} = \frac{1}{2} m v_{\text{transl}}^2 = \frac{1}{2} \rho \frac{4}{3} \pi r_0^3 v_{\text{transl}}^2 = \frac{2}{3} \rho \pi r_0^3 v_{\text{transl}}^2$$  \hspace{1cm} (30)

The ratio between vibrational and translational energy becomes, from Eqs. (29) and (30):

$$\frac{w_{\text{tot}}}{w_{\text{transl}}} = \frac{2 \alpha^2 y r_0^2}{\frac{2}{3} \rho \pi r_0^3 v_{\text{transl}}^2} = \frac{3 \alpha^2 y}{\rho \pi r_0 v_{\text{transl}}^2}$$  \hspace{1cm} (31)

This is illustrated in Figure 6 for values of $v_{\text{transl}}$ corresponding to realistic droplet velocities for explosion mitigation [11], 500 ms$^{-1}$, and for a much lower velocity, 10 ms$^{-1}$, for comparison.
It is seen in Figure 6 that for \( v_{\text{transl}} = 500 \text{ ms}^{-1} \) the vibrational energy is very modest as compared to the translational energy, even for a large vibrational relative amplitude \( \alpha = 1 \). As an example, a very small droplet with radius \( r_0 = 1 \mu\text{m} \) has a vibrational energy which corresponds to \( 3 \cdot 10^{-4} \) of the translational energy of the same droplet traveling at a speed of \( 500 \text{ ms}^{-1} \).

### 3.2 Viscosity and dissipation

The vibrations will eventually be relaxed to heat due to the viscous friction in the droplets [19], p. 640. This effect will not be considered here but it is an easy task to calculate what the temperature increase of the droplets will be as a result of transformation of vibrational kinetic energy into heat. This increase is given by:

\[
\Delta T = \frac{W_{\text{heat}}}{c_p m}.
\]

where

- \( c_p \) is the specific heat capacity of water at constant pressure, \( c_p = 4.18 \cdot 10^3 \text{ Jkg}^{-1}\text{K}^{-1} \) [20]. \( c_p \) is essentially independent of temperature in the relevant temperature range 293-393 K.

- \( m \) is the mass of the droplet.
If all vibrational energy $w_{vib}$ is dissipated into heat, the temperature increase is given by Eqs. (27) and (32):

$$\Delta T = \frac{w_{heat}}{c_p m} = \frac{3}{4c_p \rho \pi r_0^3} \cdot 2a^2 \gamma r_0^2 = \frac{3a^2 \gamma}{2c_p \rho \pi r_0}.$$  

(33)

The temperature increase is shown in Figure 7, and it can be seen that the droplet heating due to dissipation of vibrational energy is negligible for all foreseeable purposes.

Figure 7 Temperature increase when all vibrational energy has been dissipated into heat for $\alpha=0.2$. 


4 Discussion and conclusions

It has been found that quadrupole vibrational energy is negligible compared to the translational energy for a droplet traveling at 500 ms\(^{-1}\). Such a speed is relevant for water mist near explosions of a TNT equivalent of 23 kg in a 3.5 radius spherical chamber. This is a relatively severe explosion scenario and for smaller charges, larger volumes, volumes with partial venting, and gas phase explosions, it is reasonable to expect lower translational velocities of the droplets. This would increase the relative importance of the vibrational mechanism. Detailed information about the explosion scenario is therefore necessary in order to assess the relative importance of droplet vibrations. For very low velocities, 10 ms\(^{-1}\), the vibrational energy becomes comparable to the translational energy.

In this study only the quadrupole vibrational mode has been considered. Excitation of multiple higher order modes would increase the vibrational energy. Further theoretical studies are required and especially the excitation mechanism must explored. If there is no or little coupling between the shock wave and the vibrational modes, the importance of the theory in Section 0 becomes marginal for explosion mitigation.

In-situ experiments are required in order to obtain information of typical droplet size distributions at the shock front. Similar experiments are also necessary to validate theories on vibrational excitation mechanisms.
Appendix A: Calculations of integrals using expansions in spherical harmonics

The vibrational kinetic energy is given by Eq. (22):

$$w_{\text{kin}} = \frac{\alpha^2 \gamma \cos^2(\omega t + \epsilon)}{4\pi r_0^3} \int_0^{2\pi} \int_0^\pi r^5(\theta, \varphi) (9 \cos^2(\theta) \sin^2(\theta)$$

$$+ 9 \cos(\theta) - 6 \cos^2(\theta) + 1) \sin(\theta) d\theta d\varphi$$

We need to calculate the integrals

$$9 \int_0^{2\pi} \int_0^\pi r^5(\theta, \varphi) \cos^2(\theta) \sin^2(\theta) \sin(\theta) d\theta d\varphi$$ \hspace{1cm} (34)

$$9 \int_0^{2\pi} \int_0^\pi r^5(\theta, \varphi) \cos^4(\theta) \sin(\theta) d\theta d\varphi$$ \hspace{1cm} (35)

$$6 \int_0^{2\pi} \int_0^\pi r^5(\theta, \varphi) \cos^2(\theta) \sin(\theta) d\theta d\varphi$$ \hspace{1cm} (36)

$$\int_0^{2\pi} \int_0^\pi r^5(\theta, \varphi) \sin(\theta) d\theta d\varphi$$ \hspace{1cm} (37)

with $r$ given by Eq. (6) or (7).

This can be achieved by direct but very cumbersome integration. Symbolic software could also be used but the experience, using Mathcad, is that the integrals are too complex and the solution must be worked out in several steps. Therefore we will perform a direct analytic integration based on the expansion of the integrand into spherical harmonics. This is described by Baxansky and Kiriyati [16] and a very brief explanation is given here. This approach requires relatively much work but a great advantage is that the results can be re-used in future work when for example other vibrational modes are considered.

Consider the integral

$$\int_0^{2\pi} \int_0^\pi f(\theta, \varphi) g^*(\theta, \varphi) \sin(\theta) d\theta d\varphi$$ \hspace{1cm} (38)

where the asterisk $^*$ denotes the complex conjugate. Assume that the functions $f(\theta, \varphi)$ and $g(\theta, \varphi)$ can be expanded in terms of spherical harmonics:

$$f(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} P_{lm} Y_{lm}(\theta, \varphi)$$ \hspace{1cm} (39)

$$g(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} B_{lm} Y_{lm}(\theta, \varphi)$$ \hspace{1cm} (40)
The base set of spherical harmonics, $Y_{lm}(\theta, \varphi)$, is orthonormal:

$$\int_0^{2\pi} \int_0^\pi Y_{l'm'}(\theta, \varphi) Y_{lm}^*(\theta, \varphi) \sin(\theta) \, d\theta \, d\varphi = \delta_{ll'}\delta_{mm'}$$  \hspace{1cm} (41)

This means that the integral (38) can be written as:

$$\int_0^{2\pi} \int_0^\pi f(\theta, \varphi) g^*(\theta, \varphi) \sin(\theta) \, d\theta \, d\varphi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} P_{lm} B_{lm}^*$$  \hspace{1cm} (42)

The integral (37) can be calculated using Eq. (42) with

$$f(\theta, \varphi) = r^5(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} P_{lm} Y_{lm}(\theta, \varphi)$$  \hspace{1cm} (43)

$$g(\theta, \varphi) = 1 = B_{00} Y_{00} = 2\sqrt{\pi} Y_{00}$$  \hspace{1cm} (44)

Since there is only one non-zero $B_{lm}$ coefficient, namely $B_{00}$, integral (37) can be written as

$$\int_0^{2\pi} \int_0^\pi r^5(\theta, \varphi) \cdot 1 \cdot \sin(\theta) \, d\theta \, d\varphi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} P_{lm} B_{lm}^* = P_{00} B_{00}$$  \hspace{1cm} (45)

where the last equality comes from the fact that $B_{00}$ is real-valued.

So, integral (37) becomes

$$2\sqrt{\pi} P_{00}$$  \hspace{1cm} (46)

In order to calculate all integrals (34) - (37) the expansion coefficients $P_{lm}$ for $r^5$ are required. These coefficients are derived in Appendix B.

In order to evaluate integral (36) we need to express $\cos^2(\theta)$ in terms of $Y_{lm}$. We have:

$$Y_{20}(\theta, \varphi) = \frac{1}{4} \sqrt{\frac{5}{\pi}} \left(3\cos^2(\theta) - 1\right)$$  \hspace{1cm} (47)

That is

$$\cos^2(\theta) = \frac{1}{3} \left(4 \sqrt{\frac{\pi}{5}} Y_{20} + 1\right)$$  \hspace{1cm} (48)

Which, using Eq. (44) gives

$$\cos^2(\theta) = \frac{1}{3} \left(4 \sqrt{\frac{\pi}{5}} Y_{20} + 2\sqrt{\pi} Y_{00}\right) = \frac{2\sqrt{\pi}}{3} Y_{00} + \frac{4}{3} \sqrt{\frac{\pi}{5}} Y_{20}$$  \hspace{1cm} (49)
Integral (36) can now be evaluated as:

\[
6 \int_0^{2\pi} \int_0^{\pi} r^5(\theta, \varphi) \cos^2(\theta) \sin(\theta) \, d\theta d\varphi = 6 \sum_{l=0}^{\infty} \sum_{m=-l}^{l} P_{lm} B_{lm} = 6 \left( \frac{2\sqrt{\pi} P_{00}}{3} + \frac{4}{3} \sqrt{\frac{\pi}{5}} P_{20} \right)
\]

\[= 4\sqrt{\pi} P_{00} + 8 \sqrt{\frac{\pi}{5}} P_{20} \tag{50}\]

In order to evaluate integral (35) we need to express \( \cos^4(\theta) \) in terms of \( Y_{lm} \). We have [18]:

\[
Y_{40}(\theta, \varphi) = \frac{3}{16\sqrt{\pi}} \cdot (35 \cos^4(\theta) - 30 \cos^2(\theta) + 3)
\]

\[= \frac{3}{16\sqrt{\pi}} \cdot \left( 35 \cos^4(\theta) - 30 \left( \frac{2\sqrt{\pi}}{3} Y_{00} + \frac{4}{3} \sqrt{\frac{\pi}{5}} Y_{20} \right) + 6\sqrt{\pi} Y_{00} \right) \tag{51}\]

where the last equality comes from Eqs. (44) and (49). This gives:

\[
\cos^4(\theta) = \frac{2\sqrt{\pi}}{5} Y_{00} + \frac{8}{7} \sqrt{\frac{\pi}{5}} Y_{20} + \frac{16\sqrt{\pi}}{105} Y_{40} \tag{52}\]

Integral (35) can now be evaluated as

\[
9 \int_0^{2\pi} \int_0^{\pi} r^5(\theta, \varphi) \cos^4(\theta) \sin(\theta) \, d\theta d\varphi = 9 \sum_{l=0}^{\infty} \sum_{m=-l}^{l} P_{lm} B_{lm} \]

\[= 9 \left( \frac{9}{35} \left( \frac{9}{2\sqrt{\pi}} P_{00} + 40 \sqrt{\frac{\pi}{5}} P_{20} + \frac{16\sqrt{\pi}}{3} P_{40} \right) \right)
\]

\[= \frac{18\sqrt{\pi}}{5} P_{00} + \frac{72}{7} \sqrt{\frac{\pi}{5}} P_{20} + \frac{48\sqrt{\pi}}{35} P_{40} \tag{53}\]

In order to evaluate integral (34) we need to express \( \cos^2(\theta) \sin^2(\theta) \) in terms of \( Y_{lm} \). We start by rewriting

\[
\cos^2(\theta) \sin^2(\theta) = \cos^2(\theta) \left(1 - \cos^2(\theta)\right) = \cos^2(\theta) - \cos^4(\theta) \tag{54}\]

Using Eqs. (49) and (52) we obtain:
\[
\cos^2(\theta)\sin^2(\theta) = \cos^2(\theta) - \cos^4(\theta) \\
= \frac{2\sqrt{\pi}}{3} Y_{00} + \frac{4}{3} \sqrt{5} Y_{20} \\
- \left( \frac{2\sqrt{\pi}}{5} Y_{00} + \frac{8}{7} \sqrt{5} Y_{20} + \frac{16\sqrt{\pi}}{105} Y_{40} \right) \\
= \frac{4\sqrt{\pi}}{15} Y_{00} + \frac{4}{21} \sqrt{5} Y_{20} - \frac{16\sqrt{\pi}}{105} Y_{40} 
\]

(55)

We can now evaluate integral (34) as:

\[
9 \int_0^{2\pi} \int_0^\pi r^5(\theta, \varphi) \cos^2(\theta)\sin^2(\theta) \sin(\theta) \, d\theta \, d\varphi \\
= 9 \sum_{l=0}^\infty \sum_{m=-l}^l P_{lm} B_{lm} \\
= 9 \left( \frac{4\sqrt{\pi}}{15} P_{00} + \frac{4}{21} \sqrt{5} P_{20} - \frac{16\sqrt{\pi}}{105} P_{40} \right) \\
= \frac{36\sqrt{\pi}}{15} P_{00} + \frac{36}{21} \sqrt{5} P_{20} - \frac{48\sqrt{\pi}}{35} P_{40} 
\]

(56)

Equation (22) can now be evaluated using Eqs. (34)-(37), (46), (50), (53), and (56):

\[
w_{\text{kin}} = \frac{\alpha^2 \omega^2 \rho_m \cos^2(\omega t + \epsilon)}{32\pi} \left\{ \left( \frac{36\sqrt{\pi}}{15} P_{00} + \frac{36}{21} \sqrt{5} P_{20} \\
- \frac{48\sqrt{\pi}}{35} P_{40} \right) \\
+ \left( \frac{18\sqrt{\pi}}{5} P_{00} + \frac{72}{7} \sqrt{5} P_{20} + \frac{48\sqrt{\pi}}{35} P_{40} \right) \\
- \left( 2\sqrt{\pi} P_{00} + 8\sqrt{5} P_{20} \right) + \left( 2\sqrt{\pi} P_{00} \right) \right\} \\
= \frac{\alpha^2 \omega^2 \rho_m \cos^2(\omega t + \epsilon)}{32\pi} \left\{ 4\sqrt{\pi} P_{00} + 4 \sqrt{5} P_{20} \right\} 
\]

(57)

By iterative resubstitution of \( P_{00} \) and \( P_{20} \) in Appendix B we obtain the result presented in expression (23):
\[ w_{\text{kin}} = 2\alpha^2 \gamma r_0^2 \cos^2(\omega t + \epsilon) \left( 1 \right. \\
+ \frac{1}{2} \sqrt{\frac{5}{\pi}} \alpha \sin(\omega t + \epsilon) + \frac{45}{14\pi} \alpha^2 \sin^2(\omega t + \epsilon) \\
+ \frac{25}{28} \sqrt{\frac{5}{\pi^3}} \alpha^3 \sin^3(\omega t + \epsilon) \\
+ \frac{1325}{1232\pi^2} \alpha^4 \sin^4(\omega t + \epsilon) \\
+ \frac{375}{4576} \sqrt{\frac{5}{\pi^5}} \alpha^5 \sin^5(\omega t + \epsilon) \left. \right) \]
Appendix B: Expansion of $r^5$ in spherical harmonics

In order to obtain the coefficients $P_{lm}$ which describes $r^5$ in terms of spherical harmonics,

$$r^5(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} P_{lm} Y_{lm}(\theta, \varphi), \quad (58)$$

we perform an iterative process where firstly the coefficients $S_{lm}$ describing $r^2$ are calculated:

$$r(\theta, \varphi) \cdot r(\theta, \varphi) = r^2(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} S_{lm} Y_{lm}(\theta, \varphi), \quad (59)$$

Thereafter the coefficients $Q_{lm}$ describing $r^4$:

$$r^2(\theta, \varphi) \cdot r^2(\theta, \varphi) = r^4(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Q_{lm} Y_{lm}(\theta, \varphi), \quad (60)$$

And finally the coefficients $P_{lm}$:

$$r(\theta, \varphi) \cdot r^4(\theta, \varphi) = r^5(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} P_{lm} Y_{lm}(\theta, \varphi), \quad (61)$$

Theory for this is briefly explained here. Given two functions

$$f(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} Y_{lm}(\theta, \varphi) \quad (62)$$

$$g(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} B_{lm} Y_{lm}(\theta, \varphi) \quad (63)$$

The product $f(\theta, \varphi)g^*(\theta, \varphi)$ can be expanded with the coefficients $C_{lm}$

$$f(\theta, \varphi)g^*(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C_{lm} Y_{lm}(\theta, \varphi) \quad (64)$$
The expressions in brackets are the Clebsch-Gordan coefficients \([21]\), p. 1025. These can be evaluated numerically using for example the online calculator by WolframAlpha \([22]\).

From Eqs. (59) and (65) we have

\[
S_{lm} = \sum_{l_1=0}^{\infty} \sum_{m_1=-l_1}^{l_1} \sum_{l_2=0}^{\infty} \sum_{m_2=-l_2}^{l_2} R_{l_1 m_1} R_{l_2 m_1 - m} (-1)^{m_1 - m} \frac{(2l_1 + 1)(2l_2 + 1)}{4\pi(2l + 1)} \langle l_1, l_2; 0, 0 \mid l, 0 \rangle \langle l_1, l_2; m_1, m - m_1 \mid l, m \rangle
\]  

(66)

where \(R_{lm}\) are given by expressions (10) and (11).

The outer summation only takes on the values \(l=0\) and \(l=2\) (see expressions (10) and (11)) which gives

\[
S_{lm} = \sum_{m_1=0}^{\infty} \sum_{l_2=0}^{2} R_{0 m_1} R_{l_2 m_1 - m} (-1)^{m_1 - m} \frac{(2 \cdot 0 + 1)(2l_2 + 1)}{4\pi(2l + 1)} \langle 0, l_2; 0, 0 \mid l, 0 \rangle \langle 0, l_2; m_1, m - m_1 \mid l, m \rangle
\]  

(67)

\[
+ \sum_{m_1=-2}^{2} \sum_{l_2=0}^{\infty} R_{2 m_1} R_{l_2 m_1 - m} (-1)^{m_1 - m} \frac{(2 \cdot 2 + 1)(2l_2 + 1)}{4\pi(2l + 1)} \langle 2, l_2; 0, 0 \mid l, 0 \rangle \langle 2, l_2; m_1, m - m_1 \mid l, m \rangle
\]

Now the outer summations only take on the values \(m_1\) (see expressions (10) and (11)) which gives (only the \(m_1\) index in the first of the second pair of summations is changed below)

\[
S_{lm} = \sum_{m_1=0}^{\infty} \sum_{l_2=0}^{2} R_{0 m_1} R_{l_2 m_1 - m} (-1)^{m_1 - m} \frac{(2 \cdot 0 + 1)(2l_2 + 1)}{4\pi(2l + 1)} \langle 0, l_2; 0, 0 \mid l, 0 \rangle \langle 0, l_2; m_1, m - m_1 \mid l, m \rangle
\]  

(68)

\[
+ \sum_{m_1=0}^{\infty} \sum_{l_2=0}^{2} R_{2 m_1} R_{l_2 m_1 - m} (-1)^{m_1 - m} \frac{(2 \cdot 2 + 1)(2l_2 + 1)}{4\pi(2l + 1)} \langle 2, l_2; 0, 0 \mid l, 0 \rangle \langle 2, l_2; m_1, m - m_1 \mid l, m \rangle
\]
\[
S_{lm} = \sum_{l_2=0}^{\infty} R_{00} R_{l_2-m} (-1)^{-m} \sqrt{\frac{(2 \cdot 0 + 1)(2l_2 + 1)}{4\pi (2l + 1)}} \\
\cdot \langle 0, l_2; 0, 0 \mid l, 0 \rangle \langle 0, l_2; 0, m \mid l, m \rangle \\
+ \sum_{l_2=0}^{\infty} R_{20} R_{l_2-m} (-1)^{-m} \sqrt{\frac{(2 \cdot 2 + 1)(2l_2 + 1)}{4\pi (2l + 1)}} \\
\cdot \langle 2, l_2; 0, 0 \mid l, 0 \rangle \langle 2, l_2; 0, m \mid l, m \rangle
\]

(69)

We have [21]:

\[
\langle 0, l_2; 0, m_2 \mid l, m \rangle = \delta_{ll_2} \delta_{mm_2}
\]

(70)

So the first summation becomes

\[
\sum_{l_2=0}^{\infty} R_{00} R_{l_2-m} (-1)^{-m} \sqrt{\frac{(2 \cdot 0 + 1)(2l_2 + 1)}{4\pi (2l + 1)}} \cdot \delta_{ll_2} \delta_{00} \delta_{ll_2} \delta_{mm}
\]

(71)

\[
= R_{00} R_{l-m} (-1)^{-m} \sqrt{\frac{(2 \cdot 0 + 1)(2l + 1)}{4\pi (2l + 1)}}
\]

The second summation becomes

\[
\sum_{l_2=0}^{\infty} R_{20} R_{l_2-m} (-1)^{-m} \sqrt{\frac{(2 \cdot 2 + 1)(2l_2 + 1)}{4\pi (2l + 1)}} \\
\cdot \langle 2, l_2; 0, 0 \mid l, 0 \rangle \langle 2, l_2; 0, m \mid l, m \rangle
\]

(72)

The summation only takes on the values \(l_2=0\) and \(l_2=2\) (see expressions (10) and (11)) which gives

\[
R_{20} \left( R_{0-m} (-1)^{-m} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 0 + 1)}{4\pi (2l + 1)}} \cdot \langle 2, 0; 0, 0 \mid l, 0 \rangle \langle 2, 0; 0, m \mid l, m \rangle \\
+ R_{2-m} (-1)^{-m} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 2 + 1)}{4\pi (2l + 1)}} \\
\cdot \langle 2, 2; 0, 0 \mid l, 0 \rangle \langle 2, 2; 0, m \mid l, m \rangle \right)
\]

So finally we have
$$S_{tm} = R_{00}R_{l-m}(-1)^{-m} \sqrt{\frac{(2 \cdot 0 + 1)(2l + 1)}{4\pi(2l + 1)}}$$

$$+ R_{20}\left( R_{0-m}(-1)^{-m} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 0 + 1)}{4\pi(2l + 1)}} \right) \cdot \langle 2,0;0,0|l,0\rangle\langle 2,0;0,m|l,m\rangle$$

$$+ R_{-2-m}(-1)^{-m} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 2 + 1)}{4\pi(2l + 1)}} \cdot \langle 2,2;0,0|l,0\rangle\langle 2,2;0,m|l,m\rangle$$

(73)

Since only $R_{00}$ and $R_{20}$ are the only non-zero $R_{lm}$ we obtain

$$S_{l0} = R_{00}R_{l0}(-1)^{-0} \sqrt{\frac{(2 \cdot 0 + 1)(2l + 1)}{4\pi(2l + 1)}}$$

$$+ R_{20}\left( R_{00}(-1)^{-0} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 0 + 1)}{4\pi(2l + 1)}} \right) \cdot \langle 2,0;0,0|l,0\rangle\langle 2,0;0,0|l,0\rangle$$

$$+ R_{20}(-1)^{-0} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 2 + 1)}{4\pi(2l + 1)}} \cdot \langle 2,2;0,0|l,0\rangle\langle 2,2;0,0|l,0\rangle$$

(74)

Which, according to Eq. (70) gives

$$S_{l0} = R_{00}R_{l0}(-1)^{-0} \sqrt{\frac{(2 \cdot 0 + 1)(2l + 1)}{4\pi(2l + 1)}}$$

$$+ R_{20}\left( R_{00} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 0 + 1)}{4\pi(2l + 1)}} \cdot \delta_{2l}\delta_{2l} \right) \cdot \langle 2,2;0,0|l,0\rangle\langle 2,2;0,0|l,0\rangle$$

$$+ R_{20} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 2 + 1)}{4\pi(2l + 1)}} \cdot \langle 2,2;0,0|l,0\rangle\langle 2,2;0,0|l,0\rangle$$

(75)

And the non-zero $S_{l0}$ are
\[ S_{00} = R_{00} R_{00} \sqrt{\frac{(2 \cdot 0 + 1)(2 \cdot 0 + 1)}{4\pi(2 \cdot 0 + 1)}} + R_{20} R_{20} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 2 + 1)}{4\pi(2 \cdot 2 + 1)}} \cdot \frac{1}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} \]  
\[ = \frac{R_{00}^2}{2\sqrt{\pi}} + \frac{R_{20}^2}{2\sqrt{\pi}} \]  
\[ S_{10} = R_{20} \left( R_{20} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 2 + 1)}{4\pi(2 \cdot 2 + 1)}} \right) \cdot (2,2;0,0|1,0)(2,2;0,0|1,0) = 0 \]  
\[ S_{20} = R_{00} R_{20} \sqrt{\frac{(2 \cdot 0 + 1)(2 \cdot 2 + 1)}{4\pi(2 \cdot 2 + 1)}} + R_{20} \left( R_{00} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 0 + 1)}{4\pi(2 \cdot 2 + 1)}} \right) + R_{20} \left( R_{20} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 2 + 1)}{4\pi(2 \cdot 2 + 1)}} \right) \cdot \left( -\frac{\sqrt{14}}{7} \right)^2 \]  
\[ = \frac{R_{00} R_{20}}{\sqrt{\pi}} + \frac{R_{20}^2}{7\sqrt{\pi}} \]  
\[ S_{30} = R_{20} R_{20} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 2 + 1)}{4\pi(2 \cdot 3 + 1)}} \cdot (2,2;0,0|3,0)(2,2;0,0|3,0) = 0 \]  
\[ S_{40} = R_{20} R_{20} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 2 + 1)}{4\pi(2 \cdot 4 + 1)}} \cdot (2,2;0,0|4,0)(2,2;0,0|4,0) \]  
\[ = \frac{R_{20}^2}{6\sqrt{\pi}} \left( \frac{5}{35} \right)^2 \]  
\[ = \frac{3R_{20}^2}{7\sqrt{\pi}} \]
Using Eqs. (60) and (65) we now go on with:

\[
Q_{lm} = \sum_{l_1=0}^{\infty} \sum_{m_1=-l_1}^{l_1} \sum_{l_2=0}^{\infty} S_{l_1 m_1} S_{l_2 m_1-m} (-1)^{m_1-m} \sqrt{(2l_1 + 1)(2l_2 + 1)} \frac{(2l_1 + 1)(2l_2 + 1)}{4\pi(2l + 1)}
\]  

The outer summation only takes on the values \(l_j=0, l_j=2, \) and \(l_j=4\) (see expressions (76) to (80)) which gives

\[
Q_{lm} = \sum_{m_1=-2}^{2} \sum_{l_2=0}^{\infty} S_{lm} S_{l_2 m_1-m} (-1)^{m_1-m} \sqrt{(2 \cdot 0 + 1)(2l_2 + 1)} \frac{(2 \cdot 0 + 1)(2l_2 + 1)}{4\pi(2l + 1)}
\]  

\[
\cdot \langle 0, l_2; 0,0|l, 0 \rangle \langle 0, l_2; m_1, m - m_1 |l, m \rangle
\]  

\[
+ \sum_{m_1=-2}^{2} \sum_{l_2=0}^{\infty} S_{2m_1} S_{l_2 m_1-m} (-1)^{m_1-m} \sqrt{(2 \cdot 2 + 1)(2l_2 + 1)} \frac{(2 \cdot 2 + 1)(2l_2 + 1)}{4\pi(2l + 1)}
\]  

\[
\cdot \langle 2, l_2; 0,0|l, 0 \rangle \langle 2, l_2; m_1, m - m_1 |l, m \rangle
\]  

\[
+ \sum_{m_1=-4}^{4} \sum_{l_2=0}^{\infty} S_{4m_1} S_{l_2 m_1-m} (-1)^{m_1-m} \sqrt{(2 \cdot 4 + 1)(2l_2 + 1)} \frac{(2 \cdot 4 + 1)(2l_2 + 1)}{4\pi(2l + 1)}
\]  

\[
\cdot \langle 4, l_2; 0,0|l, 0 \rangle \langle 4, l_2; m_1, m - m_1 |l, m \rangle
\]  

Non-zero \(S_{lm}\) exists only for \(m_j=0\) which gives

\[
Q_{lm} = \sum_{l_2=0}^{\infty} S_{00} S_{l_2-m} (-1)^{-m} \sqrt{(2 \cdot 0 + 1)(2l_2 + 1)} \frac{(2 \cdot 0 + 1)(2l_2 + 1)}{4\pi(2l + 1)}
\]  

\[
\cdot \langle 0, l_2; 0,0|l, 0 \rangle \langle 0, l_2; 0, m |l, m \rangle
\]  

\[
+ \sum_{l_2=0}^{\infty} S_{20} S_{l_2-m} (-1)^{-m} \sqrt{(2 \cdot 2 + 1)(2l_2 + 1)} \frac{(2 \cdot 2 + 1)(2l_2 + 1)}{4\pi(2l + 1)}
\]  

\[
\cdot \langle 2, l_2; 0,0|l, 0 \rangle \langle 2, l_2; 0, m |l, m \rangle
\]  

\[
+ \sum_{l_2=0}^{\infty} S_{40} S_{l_2-m} (-1)^{-m} \sqrt{(2 \cdot 4 + 1)(2l_2 + 1)} \frac{(2 \cdot 4 + 1)(2l_2 + 1)}{4\pi(2l + 1)}
\]  

\[
\cdot \langle 4, l_2; 0,0|l, 0 \rangle \langle 4, l_2; 0, m |l, m \rangle
\]  

The first summation becomes
\[
\sum_{l_2=0}^{\infty} S_{00} S_{l_2-m} (-1)^{-m} \frac{(2 \cdot 0 + 1)(2l_2 + 1)}{4\pi(2l + 1)} \cdot \langle 0, l_2; 0, 0 | l, 0 \rangle \langle 0, l_2; 0, m | l, m \rangle \\
= \sum_{l_2=0}^{\infty} S_{00} S_{l_2-m} (-1)^{-m} \frac{(2 \cdot 0 + 1)(2l_2 + 1)}{4\pi(2l + 1)} \cdot \delta_{l_2} \delta_{l_2} = S_{00} S_{l_2-m} (-1)^{-m} \frac{(2 \cdot 0 + 1)(2l_2 + 1)}{4\pi(2l + 1)}
\]

The second summation becomes

\[
\sum_{l_2=0}^{\infty} S_{20} S_{l_2-m} (-1)^{-m} \frac{(2 \cdot 2 + 1)(2l_2 + 1)}{4\pi(2l + 1)} \cdot \langle 2, l_2; 0, 0 | l, 0 \rangle \langle 2, l_2; 0, m | l, m \rangle \\
= S_{20} \left( S_{0-m} (-1)^{-m} \frac{(2 \cdot 2 + 1)(2 \cdot 0 + 1)}{4\pi(2l + 1)} \cdot \langle 2, 0; 0, 0 | l, 0 \rangle \langle 2, 0; 0, m | l, m \rangle \\
+ S_{2-m} (-1)^{-m} \frac{(2 \cdot 2 + 1)(2 \cdot 2 + 1)}{4\pi(2l + 1)} \cdot \langle 2, 2; 0, 0 | l, 0 \rangle \langle 2, 2; 0, m | l, m \rangle \\
+ S_{4-m} (-1)^{-m} \frac{(2 \cdot 2 + 1)(2 \cdot 4 + 1)}{4\pi(2l + 1)} \cdot \langle 2, 4; 0, 0 | l, 0 \rangle \langle 2, 4; 0, m | l, m \rangle \right)
\]

The third summation becomes
\[
\sum_{l_2=0}^{\infty} S_{40} S_{l_2-m} (-1)^{-m} \sqrt{\frac{(2 \cdot 4 + 1)(2l_2 + 1)}{4\pi(2l + 1)}} \\
\cdot \langle 4, l_2; 0,0 | l, 0 \rangle \langle 4, l_2; 0, m | l, m \rangle \\
= S_{40} (S_{0-m} (-1)^{-m} \sqrt{\frac{(2 \cdot 4 + 1)(2 \cdot 0 + 1)}{4\pi(2l + 1)}}) \\
\cdot \langle 4,0; 0,0 | l, 0 \rangle \langle 4,0; 0, m | l, m \rangle \\
+ S_{2-m} (-1)^{-m} \sqrt{\frac{(2 \cdot 4 + 1)(2 \cdot 2 + 1)}{4\pi(2l + 1)}} \\
\cdot \langle 4,2; 0,0 | l, 0 \rangle \langle 4,2; 0, m | l, m \rangle \\
+ S_{4-m} (-1)^{-m} \sqrt{\frac{(2 \cdot 4 + 1)(2 \cdot 4 + 1)}{4\pi(2l + 1)}} \\
\cdot \langle 4,4; 0,0 | l, 0 \rangle \langle 4,4; 0, m | l, m \rangle \\
\]  

(86)

Since non-zero \(Q_m\) exists for \(m=0\) we obtain by summing (84) (85) (86) and setting \(m=0\):
\[ Q_{l0} = s_{00} s_{l0} \left( \frac{2 \cdot 0 + 1)(2l + 1)}{4\pi(2l + 1)} \right) \]

\[ + s_{20} \left( \frac{(2 \cdot 2 + 1)(2 \cdot 0 + 1)}{4\pi(2l + 1)} \right) \cdot \langle 2,0;0,0|l,0 \rangle \langle 2,0;0,0|l,0 \rangle \]

\[ + s_{20} \left( \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 2 + 1)}{4\pi(2l + 1)}} \right) \cdot \langle 2,2;0,0|l,0 \rangle \langle 2,2;0,0|l,0 \rangle \]

\[ + s_{40} \left( \frac{(2 \cdot 2 + 1)(2 \cdot 4 + 1)}{4\pi(2l + 1)} \right) \cdot \langle 2,4;0,0|l,0 \rangle \langle 2,4;0,0|l,0 \rangle \]

\[ + s_{40} \left( \frac{(2 \cdot 4 + 1)(2 \cdot 0 + 1)}{4\pi(2l + 1)} \right) \cdot \langle 4,0;0,0|l,0 \rangle \langle 4,0;0,0|l,0 \rangle \]

\[ + s_{20} \left( \sqrt{\frac{(2 \cdot 4 + 1)(2 \cdot 2 + 1)}{4\pi(2l + 1)}} \right) \cdot \langle 4,2;0,0|l,0 \rangle \langle 4,2;0,0|l,0 \rangle \]

\[ + s_{40} \left( -1 \right)^{0} \left( \frac{(2 \cdot 4 + 1)(2 \cdot 4 + 1)}{4\pi(2l + 1)} \right) \cdot \langle 4,4;0,0|l,0 \rangle \langle 4,4;0,0|l,0 \rangle \]

(87)

And the non-zero \( Q_{l0} \) are
\[ Q_{00} = S_{00} S_{00} \sqrt{\frac{(2 \cdot 0 + 1)(2 \cdot 0 + 1)}{4\pi (2 \cdot 0 + 1)}} \]
\[ + S_{20} \left( S_{00} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 0 + 1)}{4\pi (2 \cdot 0 + 1)}} \cdot 0^2 \right) \]
\[ + S_{20} \left( \frac{(2 \cdot 2 + 1)(2 \cdot 2 + 1)}{4\pi (2 \cdot 0 + 1)} \cdot \left( \frac{1}{\sqrt{5}} \right)^2 \right) \]
\[ + S_{40} \left( \frac{(2 \cdot 2 + 1)(2 \cdot 4 + 1)}{4\pi (2 \cdot 0 + 1)} \cdot 0^2 \right) \]
\[ + S_{40} \left( S_{00} \sqrt{\frac{(2 \cdot 4 + 1)(2 \cdot 0 + 1)}{4\pi (2 \cdot 0 + 1)}} \cdot 0^2 \right) \]
\[ + S_{20} \left( \frac{(2 \cdot 4 + 1)(2 \cdot 2 + 1)}{4\pi (2 \cdot 0 + 1)} \cdot 0^2 \right) \]
\[ + S_{40} \left( \frac{(2 \cdot 4 + 1)(2 \cdot 4 + 1)}{4\pi (2 \cdot 0 + 1)} \cdot \left( \frac{1}{\sqrt{3}} \right)^2 \right) \]
\[ = \frac{S_{00}^2}{2\sqrt{\pi}} + \frac{S_{20}^2}{2\sqrt{\pi}} + \frac{S_{40}^2}{2\sqrt{\pi}} \]  

where the last step came from Eq. (77).

\[ Q_{10} = S_{00} S_{10} \sqrt{\frac{(2 \cdot 0 + 1)(2 \cdot 1 + 1)}{4\pi (2 \cdot 1 + 1)}} \]
\[ + S_{20} \left( S_{00} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 1 + 1)}{4\pi (2 \cdot 1 + 1)}} \cdot 0^2 \right) \]
\[ + S_{20} \left( \frac{(2 \cdot 2 + 1)(2 \cdot 2 + 1)}{4\pi (2l + 1)} \cdot 0^2 \right) \]
\[ + S_{40} \left( \frac{(2 \cdot 2 + 1)(2 \cdot 4 + 1)}{4\pi (2l + 1)} \cdot 0^2 \right) \]
\[ + S_{40} \left( S_{00} \sqrt{\frac{(2 \cdot 4 + 1)(2 \cdot 0 + 1)}{4\pi (2l + 1)}} \cdot 0^2 \right) \]
\[ + S_{40} \left( \frac{(2 \cdot 4 + 1)(2 \cdot 2 + 1)}{4\pi (2l + 1)} \cdot 0^2 \right) \]
\[ + S_{20} \left( \frac{(2 \cdot 4 + 1)(2 \cdot 4 + 1)}{4\pi (2l + 1)} \cdot 0^2 \right) \]
\[ + S_{40} (-1)^0 \left( \frac{(2 \cdot 4 + 1)(2 \cdot 4 + 1)}{4\pi (2l + 1)} \cdot 0^2 \right) \]
\[ = \frac{S_{00} S_{10}}{2\sqrt{\pi}} = 0 \]
\[ Q_{20} = S_{00}S_{20} \sqrt{\frac{(2 \cdot 0 + 1)(2 \cdot 2 + 1)}{4\pi(2 \cdot 2 + 1)}} \]

\[ + S_{20} \left( S_{00} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 0 + 1)}{4\pi(2 \cdot 2 + 1)}} \cdot 1^2 \right) \]

\[ + S_{20} \left( \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 2 + 1)}{4\pi(2 \cdot 2 + 1)}} \cdot \left( \frac{2}{\sqrt{7}} \right)^2 \right) \]

\[ + S_{40} \left( \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 4 + 1)}{4\pi(2 \cdot 2 + 1)}} \cdot \left( \frac{2}{\sqrt{7}} \right)^2 \right) \]

\[ + S_{40} \left( S_{00} \sqrt{\frac{(2 \cdot 4 + 1)(2 \cdot 0 + 1)}{4\pi(2 \cdot 2 + 1)}} \cdot 0^2 \right) \]

\[ + S_{20} \left( \sqrt{\frac{(2 \cdot 4 + 1)(2 \cdot 2 + 1)}{4\pi(2 \cdot 2 + 1)}} \cdot \left( \frac{2}{\sqrt{7}} \right)^2 \right) \]

\[ + S_{40} \left( \sqrt{\frac{(2 \cdot 4 + 1)(2 \cdot 4 + 1)}{4\pi(2 \cdot 2 + 1)}} \cdot \left( \frac{10}{3\sqrt{77}} \right)^2 \right) \]

\[ = \frac{S_{00}S_{20}}{2\sqrt{\pi}} + \frac{S_{20}}{2\sqrt{\pi}} \left( \frac{S_{00}}{2\sqrt{\pi}} + \frac{\sqrt{5}S_{20}}{7\sqrt{\pi}} + \frac{3S_{40}}{7\sqrt{\pi}} \right) \]

\[ + \frac{S_{40}}{7\sqrt{\pi}} \left( \frac{3S_{20}}{7\sqrt{\pi}} + \frac{10\sqrt{5}S_{40}}{77\sqrt{\pi}} \right) \]

\[ = \frac{S_{00}S_{20}}{\sqrt{\pi}} + \frac{\sqrt{5}S_{20}}{7\sqrt{\pi}} + \frac{6S_{20}S_{40}}{7\sqrt{\pi}} + \frac{10\sqrt{5}S_{40}}{77\sqrt{\pi}} \]
$$Q_{30} = S_{00} S_{30} \frac{(2 \cdot 0 + 1)(2 \cdot 3 + 1)}{4\pi(2 \cdot 3 + 1)}$$

$$+ S_{20} \left( S_{00} \frac{(2 \cdot 2 + 1)(2 \cdot 0 + 1)}{4\pi(2 \cdot 3 + 1)} \cdot 0^2 \right)$$

$$+ S_{20} \frac{(2 \cdot 2 + 1)(2 \cdot 2 + 1)}{4\pi(2 \cdot 3 + 1)} \cdot 0^2$$

$$+ S_{40} \left( S_{00} \frac{(2 \cdot 2 + 1)(2 \cdot 4 + 1)}{4\pi(2 \cdot 3 + 1)} \cdot 0^2 \right)$$

$$+ S_{40} \left( S_{00} \frac{(2 \cdot 4 + 1)(2 \cdot 0 + 1)}{4\pi(2 \cdot 3 + 1)} \cdot 0^2 \right)$$

$$+ S_{20} \left( S_{40} \frac{(2 \cdot 2 + 1)(2 \cdot 4 + 1)}{4\pi(2 \cdot 3 + 1)} \cdot 0^2 \right)$$

$$+ S_{40} \left( S_{40} \frac{(2 \cdot 4 + 1)(2 \cdot 4 + 1)}{4\pi(2 \cdot 3 + 1)} \cdot 0^2 \right) = 0$$

since $S_{30}=0$. 

(91)
\[ Q_{40} = S_{00}S_{40} \sqrt{\frac{(2 \cdot 0 + 1)(2 \cdot 4 + 1)}{4\pi(2 \cdot 4 + 1)}} \]

\[ + S_{20} \left( S_{00} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 0 + 1)}{4\pi(2 \cdot 4 + 1)}} \cdot 0^2 \right) \]

\[ + S_{20} \left( S_{00} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 2 + 1)}{4\pi(2 \cdot 4 + 1)}} \cdot \left( \frac{3}{\sqrt{35}} \right)^2 \right) \]

\[ + S_{40} \left( S_{00} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 4 + 1)}{4\pi(2 \cdot 4 + 1)}} \cdot \left( -2 \frac{5}{\sqrt{77}} \right)^2 \right) \]

\[ + S_{40} \left( S_{00} \sqrt{\frac{(2 \cdot 4 + 1)(2 \cdot 0 + 1)}{4\pi(2 \cdot 4 + 1)}} \cdot 1^2 \right) \]

\[ + S_{40} \left( S_{00} \sqrt{\frac{(2 \cdot 4 + 1)(2 \cdot 2 + 1)}{4\pi(2 \cdot 4 + 1)}} \cdot \left( -2 \frac{5}{\sqrt{77}} \right)^2 \right) \]

\[ + S_{40} \left( S_{00} \sqrt{\frac{(2 \cdot 4 + 1)(2 \cdot 4 + 1)}{4\pi(2 \cdot 4 + 1)}} \cdot \left( \frac{9\sqrt{2002}}{1001} \right)^2 \right) \]

\[ = S_{00}S_{20} \frac{3S_{20}^2}{2\sqrt{\pi}} + \frac{10\sqrt{5}S_{20}S_{40}}{7\sqrt{\pi}} + \frac{S_{00}S_{40}}{2\sqrt{\pi}} \]

\[ + \frac{10\sqrt{5}S_{20}S_{40}}{77\sqrt{\pi}} + \frac{243S_{40}^2}{1001\sqrt{\pi}} \]

\[ = \frac{3S_{20}^2}{7\sqrt{\pi}} + \frac{20\sqrt{5}S_{20}S_{40}}{77\sqrt{\pi}} + \frac{S_{00}S_{40}}{\sqrt{\pi}} + \frac{243S_{40}^2}{1001\sqrt{\pi}} \]
\[ Q_{50} = S_{50} \frac{(2 \cdot 0 + 1)(2 \cdot 5 + 1)}{4\pi(2 \cdot 5 + 1)} \]

\[ + S_{20} \left( S_{00} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 0 + 1)}{4\pi(2 \cdot 5 + 1)}} \cdot 0^2 \right) \]

\[ + S_{20} \left( S_{00} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 2 + 1)}{4\pi(2 \cdot 5 + 1)}} \cdot 0^2 \right) \]

\[ + S_{40} \left( S_{00} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 0 + 1)}{4\pi(2 \cdot 5 + 1)}} \cdot 0^2 \right) = 0 \]

since \( S_{50} = 0 \).
\[ Q_{60} = S_{00} s_{60} \sqrt{\frac{(2 \cdot 0 + 1)(2 \cdot 6 + 1)}{4\pi(2 \cdot 6 + 1)}} \]

\[ + S_{20} \left( S_{00} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 0 + 1)}{4\pi(2 \cdot 6 + 1)}} \cdot 0^2 \right) \]

\[ + S_{20} \left( \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 2 + 1)}{4\pi(2 \cdot 6 + 1)}} \cdot 0^2 \right) \]

\[ + S_{40} \left( \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 4 + 1)}{4\pi(2 \cdot 6 + 1)}} \cdot \left( \frac{5}{\sqrt{11}} \right)^2 \right) \]

\[ + S_{40} \left( S_{00} \sqrt{\frac{(2 \cdot 4 + 1)(2 \cdot 0 + 1)}{4\pi(2 \cdot 6 + 1)}} \cdot 0^2 \right) \]

\[ + S_{20} \left( \sqrt{\frac{(2 \cdot 4 + 1)(2 \cdot 2 + 1)}{4\pi(2 \cdot 6 + 1)}} \cdot \left( \frac{5}{\sqrt{11}} \right)^2 \right) \]

\[ + S_{40} \left( \sqrt{\frac{(2 \cdot 4 + 1)(2 \cdot 4 + 1)}{4\pi(2 \cdot 6 + 1)}} \cdot \left( \frac{2}{3\sqrt{11}} \right)^2 \right) \]

\[ = \frac{15\sqrt{5} S_{20} S_{40}}{22\sqrt{13\pi}} + \frac{15\sqrt{5} S_{20} S_{40}}{22\sqrt{13\pi}} + \frac{10S_{40}^2}{11\sqrt{13\pi}} \]

\[ = \frac{15\sqrt{5} S_{20} S_{40}}{11\sqrt{13\pi}} + \frac{10S_{40}^2}{11\sqrt{13\pi}} \]
\[ Q_{80} = S_{00} S_{80} \left\{ \frac{(2 \cdot 0 + 1)(2 \cdot 8 + 1)}{4\pi(2 \cdot 8 + 1)} \right\}^{0^2} \]
\[ + S_{20} \left\{ \frac{(2 \cdot 2 + 1)(2 \cdot 0 + 1)}{4\pi(2 \cdot 8 + 1)} \right\}^{0^2} \]
\[ + S_{40} \left\{ \frac{(2 \cdot 2 + 1)(2 \cdot 4 + 1)}{4\pi(2 \cdot 8 + 1)} \right\}^{0^2} \]
\[ + S_{40} \left\{ \frac{(2 \cdot 4 + 1)(2 \cdot 0 + 1)}{4\pi(2 \cdot 8 + 1)} \right\}^{0^2} \]
\[ + S_{40} \left\{ \frac{(2 \cdot 4 + 1)(2 \cdot 4 + 1)}{4\pi(2 \cdot 8 + 1)} \right\} \left( \frac{\sqrt{7}}{3 \sqrt{143}} \right)^2 \]
\[ = \frac{245 \cdot S_{40}^2}{143 \sqrt{17\pi}} \]
The first summation becomes

\[
P_{lm} = \sum_{l_2=0}^{\infty} R_{00} Q_{l_2-m} (-1)^{-m} \sqrt{\frac{(2 \cdot 0 + 1)(2l_2 + 1)}{4\pi(2l + 1)}} \cdot \langle 0, l_2; 0, 0 \mid l, 0\rangle \langle 0, l_2; 0, m \mid l, m\rangle \\
+ \sum_{l_2=0}^{\infty} R_{20} Q_{l_2-m} (-1)^{-m} \sqrt{\frac{(2 \cdot 2 + 1)(2l_2 + 1)}{4\pi(2l + 1)}} \cdot \langle 2, l_2; 0, 0 \mid l, 0\rangle \langle 2, l_2; 0, m \mid l, m\rangle
\]

(98)

The second summation becomes (see expressions (88)-(95))

\[
\sum_{l_2=0}^{\infty} R_{20} Q_{l_2-m} (-1)^{-m} \sqrt{\frac{(2 \cdot 2 + 1)(2l_2 + 1)}{4\pi(2l + 1)}} \cdot \langle 2, l_2; 0, 0 \mid l, 0\rangle \langle 2, l_2; 0, m \mid l, m\rangle \\
= R_{20} \left[ \sum_{l_2=0}^{\infty} Q_{m-0} (-1)^{-m} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 0 + 1)}{4\pi(2l + 1)}} \cdot \langle 2, 0; 0, 0 \mid l, 0\rangle \langle 2, 0; m \mid l, m\rangle \\
+ Q_{2-m} (-1)^{-m} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 4 + 1)}{4\pi(2l + 1)}} \cdot \langle 2, 2; 0, 0 \mid l, 0\rangle \langle 2, 0; m \mid l, m\rangle \\
+ Q_{4-m} (-1)^{-m} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 6 + 1)}{4\pi(2l + 1)}} \cdot \langle 2, 4; 0, 0 \mid l, 0\rangle \langle 2, 2; m \mid l, m\rangle \\
+ Q_{6-m} (-1)^{-m} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 8 + 1)}{4\pi(2l + 1)}} \cdot \langle 2, 6; 0, 0 \mid l, 0\rangle \langle 2, 4; m \mid l, m\rangle \\
+ Q_{8-m} (-1)^{-m} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 10 + 1)}{4\pi(2l + 1)}} \cdot \langle 2, 8; 0, 0 \mid l, 0\rangle \langle 2, 6; m \mid l, m\rangle \right) \]

(100)
Since non-zero $Q_{lm}$ exists for $m=0$ we obtain by summing (99) and (100) and setting $m=0$;

\[
P_{l0} = R_{00} Q_{l0} \sqrt{\frac{(2 \cdot 0 + 1)(2l + 1)}{4\pi(2l + 1)}} \\
+ R_{20} \left( Q_{00} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 0 + 1)}{4\pi(2l + 1)}} \\
\cdot \langle 0,0|l,0\rangle (2,0;0,m|l,m) \right) \\
+ Q_{20} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 2 + 1)}{4\pi(2l + 1)}} \\
\cdot \langle 2,0|0,l,0\rangle (2,2;0,m|l,m) \\
+ Q_{40} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 4 + 1)}{4\pi(2l + 1)}} \\
\cdot \langle 2,2|0,0,l,0\rangle (2,4;0,m|l,m) \\
+ Q_{60} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 6 + 1)}{4\pi(2l + 1)}} \\
\cdot \langle 2,4|0,0,0,l,0\rangle (2,6;0,m|l,m) \\
+ Q_{80} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 8 + 1)}{4\pi(2l + 1)}} \\
\cdot \langle 2,6|0,0,0,0,l,0\rangle (2,8;0,m|l,m) \\
\right)
\]

(101)

and the non-zero $P_{jo}$ are
\[ P_{00} = R_{00} Q_{00} \sqrt{\frac{(2 \cdot 0 + 1)(2 \cdot 0 + 1)}{4\pi(2 \cdot 0 + 1)}} + R_{20} \left( Q_{00} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 0 + 1)}{4\pi(2 \cdot 0 + 1)}} \cdot 0^2 \right. \\
+ Q_{20} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 2 + 1)}{4\pi(2 \cdot 0 + 1)}} \cdot \left( \frac{1}{\sqrt{5}} \right)^2 \\
+ Q_{40} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 4 + 1)}{4\pi(2 \cdot 0 + 1)}} \cdot 0^2 \\
+ Q_{60} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 6 + 1)}{4\pi(2 \cdot 0 + 1)}} \cdot 0^2 \\
+ Q_{80} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 8 + 1)}{4\pi(2 \cdot 0 + 1)}} \cdot 0^2 \left) \right. \\
= \frac{R_{00} Q_{00}}{2\sqrt{\pi}} + \frac{R_{20} Q_{20}}{2\sqrt{\pi}} \right) \]

\[ P_{10} = R_{00} Q_{10} \sqrt{\frac{(2 \cdot 0 + 1)(2 \cdot 1 + 1)}{4\pi(2 \cdot 1 + 1)}} + R_{20} \left( Q_{00} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 0 + 1)}{4\pi(2 \cdot 1 + 1)}} \cdot 0^2 + Q_{20} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 2 + 1)}{4\pi(2 \cdot 1 + 1)}} \cdot 0^2 \right. \\
+ Q_{40} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 4 + 1)}{4\pi(2 \cdot 1 + 1)}} \cdot 0^2 \left. \\
+ Q_{60} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 6 + 1)}{4\pi(2 \cdot 1 + 1)}} \cdot 0^2 \\
+ Q_{80} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 8 + 1)}{4\pi(2 \cdot 1 + 1)}} \cdot 0^2 \right) = 0 \]

since \( Q_{10} = 0 \).
\[ P_{20} = R_{00} Q_{20} \sqrt{\frac{(2 \cdot 0 + 1)(2 \cdot 2 + 1)}{4\pi(2 \cdot 2 + 1)}} \]
\[ + R_{20} \left( Q_{00} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 0 + 1)}{4\pi(2 \cdot 2 + 1)}} \cdot 1^2 + Q_{20} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 2 + 1)}{4\pi(2 \cdot 2 + 1)}} \cdot \left( -\frac{2}{\sqrt{7}} \right)^2 \right) \]
\[ + Q_{40} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 4 + 1)}{4\pi(2 \cdot 2 + 1)}} \cdot \left( \frac{2}{\sqrt{7}} \right)^2 \]
\[ + Q_{60} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 6 + 1)}{4\pi(2 \cdot 2 + 1)}} \cdot 0^2 \]
\[ + Q_{80} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 8 + 1)}{4\pi(2 \cdot 2 + 1)}} \cdot 0^2 \right) \]
\[ = \frac{R_{00} Q_{20}}{2\sqrt{\pi}} + \frac{R_{20} Q_{00}}{2\sqrt{\pi}} + \frac{\sqrt{5}R_{20} Q_{20}}{7\sqrt{\pi}} \]
\[ + \frac{3R_{20} Q_{40}}{7\sqrt{\pi}} \]
\( P_{30} = R_{00} Q_{30} \sqrt{\frac{(2 \cdot 0 + 1) (2 \cdot 3 + 1)}{4\pi (2 \cdot 3 + 1)}} \)

\[ + R_{20} \left( Q_{00} \sqrt{\frac{(2 \cdot 2 + 1) (2 \cdot 0 + 1)}{4\pi (2 \cdot 3 + 1)}} \right) \]

\[ \cdot 0^2 + Q_{20} \sqrt{\frac{(2 \cdot 2 + 1) (2 \cdot 2 + 1)}{4\pi (2 \cdot 3 + 1)}} \]

\[ \cdot 0^2 + Q_{40} \sqrt{\frac{(2 \cdot 2 + 1) (2 \cdot 4 + 1)}{4\pi (2 \cdot 3 + 1)}} \]

\[ \cdot 0^2 + Q_{60} \sqrt{\frac{(2 \cdot 2 + 1) (2 \cdot 6 + 1)}{4\pi (2 \cdot 3 + 1)}} \]

\[ \cdot 0^2 + Q_{80} \sqrt{\frac{(2 \cdot 2 + 1) (2 \cdot 8 + 1)}{4\pi (2 \cdot 3 + 1)}} \]

\[ \cdot 0^2 \right) = 0 \]

Since \( Q_{30} = 0 \).
\[ P_{40} = R_{00} Q_{40} \sqrt{\frac{(2 \cdot 0 + 1)(2 \cdot 4 + 1)}{4\pi(2 \cdot 4 + 1)}} \]
\[ + R_{20} \left( Q_{00} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 0 + 1)}{4\pi(2 \cdot 4 + 1)}} \cdot 0^2 \right) \]
\[ + Q_{20} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 2 + 1)}{4\pi(2 \cdot 4 + 1)}} \cdot \frac{(3\sqrt{2})}{\sqrt{35}} \]
\[ + Q_{40} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 4 + 1)}{4\pi(2 \cdot 4 + 1)}} \cdot 0 \]
\[ \cdot \left( -\frac{2\sqrt{5}}{\sqrt{77}} \right)^2 \]
\[ + Q_{60} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 6 + 1)}{4\pi(2 \cdot 4 + 1)}} \cdot 0 \]
\[ \cdot \frac{(3\sqrt{5})}{\sqrt{143}} \]
\[ + Q_{80} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 8 + 1)}{4\pi(2 \cdot 4 + 1)}} \cdot 0 \]
\[ = \frac{R_{00} Q_{40}}{2\sqrt{\pi}} + \frac{3R_{20} Q_{20}}{7\sqrt{\pi}} + \frac{10\sqrt{5}R_{20} Q_{40}}{77\sqrt{\pi}} \]
\[ + \frac{15\sqrt{5}R_{20} Q_{60}}{22\sqrt{13}\sqrt{\pi}} \]

\[ P_{50} = R_{00} Q_{50} \sqrt{\frac{(2 \cdot 0 + 1)(2 \cdot 5 + 1)}{4\pi(2 \cdot 5 + 1)}} \]
\[ + R_{20} \left( Q_{00} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 0 + 1)}{4\pi(2 \cdot 5 + 1)}} \cdot 0^2 \right) \]
\[ + Q_{20} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 2 + 1)}{4\pi(2 \cdot 5 + 1)}} \cdot 0^2 \]
\[ + Q_{40} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 4 + 1)}{4\pi(2 \cdot 5 + 1)}} \cdot 0^2 \]
\[ + Q_{60} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 6 + 1)}{4\pi(2 \cdot 5 + 1)}} \cdot 0^2 \]
\[ + Q_{80} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 8 + 1)}{4\pi(2 \cdot 5 + 1)}} \cdot 0^2 \]
\[ = 0 \]
Since $Q_{50} = 0$.

\[ P_{60} = R_{00} Q_{60} \sqrt{\frac{(2 \cdot 0 + 1)(2 \cdot 6 + 1)}{4\pi(2 \cdot 6 + 1)}} \]

\[ + R_{20} \left( Q_{00} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 0 + 1)}{4\pi(2 \cdot 6 + 1)}} \cdot 0^2 \right) \]

\[ + Q_{20} \left( \frac{(2 \cdot 2 + 1)(2 \cdot 2 + 1)}{4\pi(2 \cdot 6 + 1)} \cdot 0^2 \right) \]

\[ + Q_{40} \left( \sqrt{\frac{(2 \cdot 4 + 1)^2}{4\pi(2 \cdot 6 + 1)}} \right) \left( \frac{5}{11} \right)^2 \]

\[ + Q_{60} \left( \sqrt{\frac{(2 \cdot 6 + 1)^2}{4\pi(2 \cdot 6 + 1)}} \right) \left( -\frac{14}{55} \right)^2 \]

\[ + Q_{80} \left( \sqrt{\frac{(2 \cdot 8 + 1)^2}{4\pi(2 \cdot 6 + 1)}} \right) \left( \frac{2\sqrt{7}}{\sqrt{85}} \right)^2 \]

\[ = \frac{R_{00} Q_{60}}{2\sqrt{\pi}} + \frac{15\sqrt{5} R_{20} Q_{40}}{22\sqrt{13}\sqrt{\pi}} + \frac{7R_{20} Q_{60}}{11\sqrt{5}\sqrt{\pi}} + \frac{14R_{20} Q_{80}}{\sqrt{1105}\sqrt{\pi}} \]

\[ P_{70} = R_{00} Q_{70} \sqrt{\frac{(2 \cdot 0 + 1)(2 \cdot 7 + 1)}{4\pi(2 \cdot 7 + 1)}} \]

\[ + R_{20} \left( Q_{00} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 0 + 1)}{4\pi(2 \cdot 7 + 1)}} \cdot 0^2 \right) \]

\[ + Q_{20} \left( \frac{(2 \cdot 2 + 1)(2 \cdot 2 + 1)}{4\pi(2 \cdot 7 + 1)} \cdot 0^2 \right) \]

\[ + Q_{40} \left( \sqrt{\frac{(2 \cdot 4 + 1)^2}{4\pi(2 \cdot 7 + 1)}} \right) \]

\[ + Q_{60} \left( \sqrt{\frac{(2 \cdot 6 + 1)^2}{4\pi(2 \cdot 7 + 1)}} \right) \]

\[ + Q_{80} \left( \sqrt{\frac{(2 \cdot 8 + 1)^2}{4\pi(2 \cdot 7 + 1)}} \right) = 0 \]
Since $Q_{70}=0$.

\[
P_{80} = R_{00}Q_{80} \sqrt{\frac{(2 \cdot 0 + 1)(2 \cdot 8 + 1)}{4\pi(2 \cdot 8 + 1)}}
\]
\[+ R_{20} \left( Q_{00} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 0 + 1)}{4\pi(2 \cdot 8 + 1)}} \cdot 0^2 \right)
\]
\[+ Q_{20} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 2 + 1)}{4\pi(2 \cdot 8 + 1)}} \cdot 0^2
\]
\[+ Q_{40} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 4 + 1)}{4\pi(2 \cdot 8 + 1)}} \cdot 0^2
\]
\[+ Q_{60} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 6 + 1)}{4\pi(2 \cdot 8 + 1)}} \cdot \left( \frac{2\sqrt{7}}{\sqrt{65}} \right)^2
\]
\[+ Q_{80} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 8 + 1)}{4\pi(2 \cdot 8 + 1)}} \cdot \left( \frac{-2\sqrt{6}}{\sqrt{95}} \right)^2
\]
\[= \frac{R_{00}Q_{80}}{2\sqrt{\pi}} + \frac{14R_{20}Q_{60}}{\sqrt{1105\sqrt{\pi}}} + \frac{12R_{20}Q_{80}}{19\sqrt{5\sqrt{\pi}}}
\]

Since $Q_{90}=0$.

\[
P_{90} = R_{00}Q_{90} \sqrt{\frac{(2 \cdot 0 + 1)(2 \cdot 9 + 1)}{4\pi(2 \cdot 9 + 1)}}
\]
\[+ R_{20} \left( Q_{00} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 0 + 1)}{4\pi(2 \cdot 9 + 1)}} \cdot 0^2 \right)
\]
\[+ Q_{20} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 2 + 1)}{4\pi(2 \cdot 9 + 1)}} \cdot 0^2
\]
\[+ Q_{40} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 4 + 1)}{4\pi(2 \cdot 9 + 1)}} \cdot 0^2
\]
\[+ Q_{60} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 6 + 1)}{4\pi(2 \cdot 9 + 1)}} \cdot 0^2
\]
\[+ Q_{80} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 8 + 1)}{4\pi(2 \cdot 9 + 1)}} \cdot 0^2 \right) = 0
\]
\[ P_{10,0} = R_{00} Q_{10,0} \sqrt{\frac{(2 \cdot 0 + 1)(2 \cdot 10 + 1)}{4\pi(2 \cdot 10 + 1)}} + R_{20} \left( Q_{00} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 0 + 1)}{4\pi(2 \cdot 10 + 1)}} \cdot 0^2 \right) + Q_{20} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 2 + 1)}{4\pi(2 \cdot 10 + 1)}} \cdot 0^2 + Q_{40} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 4 + 1)}{4\pi(2 \cdot 10 + 1)}} \cdot 0^2 + Q_{60} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 6 + 1)}{4\pi(2 \cdot 10 + 1)}} \cdot 0^2 + Q_{80} \sqrt{\frac{(2 \cdot 2 + 1)(2 \cdot 8 + 1)}{4\pi(2 \cdot 10 + 1)}} \cdot \left( \frac{3\sqrt{15}}{\sqrt{323}} \right)^2 \]

\[ = \frac{45\sqrt{15}R_{20} Q_{80}}{38\sqrt{115\pi}} \]

since \( Q_{10,0} = 0 \).
Appendix C: The Rayleigh solution

According to Rayleigh [12] the kinetic energy is given by

\[ w_{\text{kin}} = 2\pi r_0^3 \sum_{n} \frac{1}{n(2n + 1)} \left( \frac{\mathrm{d}a_n}{\mathrm{d}t} \right)^2 \]  

(112)

where the \( a_n \) describe the radius as

\[ r = a_0 + a_1 P_1(\cos(\theta)) + a_2 P_2(\cos(\theta)) + \cdots \]  

(113)

\( P_1(\cos(\theta)) \) are Legendre polynomials. It is seen that Rayleigh also only considers symmetric vibrations with \( m=0 \), that is no \( \phi \)-dependence. We have [23]

\[ P_2(\cos(\theta)) = \frac{1}{2} (3\cos^2(\theta) - 1) \]  

(114)

Comparing expressions (7), (9), and (113) we identify

\[ a_2 \frac{1}{2} (3\cos^2(\theta) - 1) = \alpha r_0 \sin(\omega t + \epsilon) \frac{1}{4} \sqrt{\frac{5}{\pi}} (3\cos^2(\theta) - 1) \]  

(115)

that is

\[ a_2 = \alpha r_0 \sqrt{\frac{5}{4\pi}} \sin(\omega t + \epsilon) \]  

(116)

with

\[ \frac{\mathrm{d}a_2}{\mathrm{d}t} = \omega \alpha r_0 \sqrt{\frac{5}{4\pi}} \cos(\omega t + \epsilon) \]  

(117)

Eqs. (112) and (116) gives:

\[ w_{\text{kin}} = 2\pi r_0^3 \frac{1}{2(2 \cdot 2 + 1)} \omega^2 \alpha^2 r_0^2 \frac{5}{4\pi} \cos^2(\omega t + \epsilon) \]  

(118)

\[ = \frac{\rho \omega^2 \alpha^2 r_0^5}{4} \cos^2(\omega t + \epsilon) \]

Inserting the expression (21) for \( \omega^2 \) gives

\[ w_{\text{kin}} = 2 \gamma \alpha^2 r_0^2 \cos(\omega t + \epsilon) \]  

(119)
The potential energy is given by Rayleigh as for \( l=2 \) is according to Rayleigh

\[
\omega_{\text{pot}} = 2\pi \gamma \sum (n-1)(n+2) \frac{\alpha_n^2}{(2n+1)}
\]  

(120)

Of the potential energy of the equilibrium sphere is ignored and only the oscillation part, \( \alpha_2 \), is considered we obtain

\[
\omega_{\text{pot}} = 2\pi \gamma \frac{(2-1)(2+2)}{2 \cdot 2 + 1} \alpha_2^2
\]  

(121)

Or, using Eq.(116)

\[
\omega_{\text{pot}} = \frac{8}{5} \pi \gamma \alpha^2 r_0^2 \frac{5}{4\pi} \sin^2(\omega t + \epsilon) = 2\gamma \alpha^2 r_0^2 \sin^2(\omega t + \epsilon)
\]  

(122)

We get the total energy as

\[
\omega_{\text{tot}} = \omega_{\text{pot}} + \omega_{\text{kin}} = 2\gamma \alpha^2 r_0^2 (\sin^2(\omega t + \epsilon) + \cos^2(\omega t + \epsilon))
\]  

(123)
References

SP Technical Research Institute of Sweden
Our work is concentrated on innovation and the development of value-adding technology. Using Sweden's most extensive and advanced resources for technical evaluation, measurement technology, research and development, we make an important contribution to the competitiveness and sustainable development of industry. Research is carried out in close conjunction with universities and institutes of technology, to the benefit of a customer base of about 10000 organisations, ranging from start-up companies developing new technologies or new ideas to international groups.