Mean Field Games for Jump Non-Linear Markov Process
MEAN FIELD GAMES FOR JUMP NON-LINEAR MARKOV PROCESS

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Abstract

The mean-field game theory is the study of strategic decision making in very large populations of weakly interacting individuals. Mean-field games have been an active area of research in the last decade due to its increased significance in many scientific fields. The foundations of mean-field theory go back to the theory of statistical and quantum physics. One may describe mean-field games as a type of stochastic differential game for which the interaction between the players is of mean-field type, i.e. the players are coupled via their empirical measure. It was proposed by Lasry and Lions and independently by Huang, Malhame, and Caines. Since then, the mean-field games have become a rapidly growing area of research and has been studied by many researchers. However, most of these studies were dedicated to diffusion-type games. The main purpose of this thesis is to extend the theory of mean-field games to jump case in both discrete and continuous state space. Jump processes are a very important tool in many areas of applications. Specifically, when modeling abrupt events appearing in real life. For instance, financial modeling (option pricing and risk management), networks (electricity and Banks) and statistics (for modeling and analyzing spatial data). The thesis consists of two papers and one technical report which will be submitted soon:

In the first publication, we study the mean-field game in a finite state space where the dynamics of the indistinguishable agents is governed by a controlled continuous time Markov chain. We have studied the control problem for a representative agent in the linear quadratic setting. A dynamic programming approach has been used to drive the Hamilton Jacobi Bellman equation, consequently, the optimal strategy has been achieved. The main result is to show that the individual optimal strategies for the mean-field game system represent 1/N-Nash equilibrium for the approximating system of N agents.

As a second article, we generalize the previous results to agents driven by a non-linear pure jump Markov processes in Euclidean space. Mathematically, this means working with linear operators in Banach spaces adapted to the integro-differential operators of jump type and with non-linear partial differential equations instead of working with linear transformations in Euclidean spaces as in the first work. As a by-product, a generalization for the Koopman operator has been presented. In this setting, we studied the control problem in a more general sense, i.e. the cost function is not necessarily of linear quadratic form. We showed that the resulting unique optimal control is of Lipschitz type. Furthermore, a fixed point argument is presented in order to construct the approximate Nash Equilibrium. In addition, we show that the rate of convergence will be of special order as a result of utilizing a non-linear pure jump Markov process.

In a third paper, we develop our approach to treat a more realistic case from a modelling perspective. In this step, we assume that all players are subject to an additional common noise of Brownian type. We especially study the well-posedness and the regularity for a jump version of the stochastic kinetic equation. Finally, we show that the solution of the master equation, which is a type of second order partial differential equation in the space of probability measures, provides an approximate Nash Equilibrium. This paper, unfortunately, has not been completely finished and it is still in preprint form. Hence, we have decided not to enclose it in the thesis. However, an outlook about the paper will be included.

Keywords: Mean-field games, Dynamic Programing, Non-linear continuous time Markov chains, Non-linear Markov pure jump processes, Koopman Dynamics, McKean-Vlasov equation, Epsilon--Nash equilibrium.
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Växjö, September 2016
Rani
Preface

This thesis consists of an introduction two papers, Papers I–II and a technical report about the third paper. The introduction part provides mathematical definitions and tools which have been used in this thesis. Introducing the Mean-field game Theory on which Papers I–III are based. It ends with a short summary of the results of the included papers.

Papers included in the thesis


III. Outlook of the paper. "Jump Mean-Field Games disturbed with common Noise". 2016 (preprint).
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I An Epsilon-Nash equilibrium for non-linear Markov games of mean-field-type on finite spaces.

II An Approximate Nash Equilibrium for Pure Jump Markov Games of Mean-field-type on Continuous State Space.

III Outlook for the paper Jump Mean-Field Games disturbed with common Noise.
1 Introduction

In this section, we present the main mathematical tools which are used in the thesis. In Chapter one, we present the theory of Markov processes. Chapter two is dedicated to some results form the theory of ordinary differential equations. Optimal Control theory is introduced in Chapter 3 in both the diffusion case and then in the jump case. In Chapter 4 we display an introduction to Differential games theory. Finally, Chapter 5 is an introduction to mean-field game theory.

1.1 Continuous Time Markov Processes

Consider a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), where \(\Omega\) is called the sample space, \(\mathcal{F}\) is the \(\sigma\)-algebra and \(\mathbb{P}\) is a probability measure on \(\mathcal{F}\).

1.1.1 Markov Chains

Let \((X, \beta(X))\) be a measurable space, \(p_n(x, dy)\) transition probabilities on \((X, \beta(X))\), and \(\mathcal{F}_n\) for \((n = 0, 1, 2, \ldots)\) a filtration on \((\Omega, \mathcal{F}, \mathbb{P})\).

Definition 1.1. A stochastic process \((X_n)_{n=0,1,2,\ldots}\) on \((\Omega, \mathcal{F}, \mathbb{P})\) is called \((\mathcal{F}_n)\)-Markov chain with transition probabilities \(p_n\) if and only if

1. \(X_n\) is \(\mathcal{F}_n\) measurable \(\forall n \geq 0\)
2. \(\mathbb{P}[X_{n+1} \in B | \mathcal{F}_n] = p_{n+1}(X_n, B)\) \(\mathbb{P}\)-a.s. \(\forall n \geq 0, B \in \beta(S)\).

Theorem 1.2. Given a measurable space \((X, \beta(X))\) with distribution \(\mu\), and transition probabilities \(p_n\), there exist a Markov chain on the space and it’s distribution \(\mathbb{P}_\mu\) is unique, see [22].

1.1.2 Weak Convergence

Let \(X\) be a polish space and let

\[ C_b(X) := \{ f : f : X \to R \text{ bounded continuous}\} \]

be the space of bounded continuous real-valued functions on \(X\). We equip \(C_b(X)\) with supremum norm

\[ \|f\| := \sup_{x \in X} |f(x)|. \]

With this norm, \(C_b(X)\) is a Banach space. Moreover let us define the space

\[ \mathcal{P} := \{ \mu : \mu \text{ probability measure on}(X, \beta(X))\} \]

is the space of all probability measures on \(X\). We equip \(\mathcal{P}(X)\) with the topology of weak convergence. We say that a sequence of measures \(\mu_n \in \mathcal{P}(X)\) converges weakly to a limit \(\mu \in \mathcal{P}(X)\), denoted as \(\mu_n \Rightarrow \mu\) if

\[ \int f d\mu_n \to \int f d\mu, \quad n \to \infty \]
Proposition 1.3. (Prokhorov) A subset \( K \) of \( \mathcal{P}(X) \) is relatively compact in \( \mathcal{P}(X) \) if and only if it is tight, i.e,
\[
\forall \epsilon > 0 \quad \exists X_\epsilon \quad \text{compact set of } X \quad \text{with} \quad \mu(X \setminus X_\epsilon) \leq \epsilon \quad \forall \mu \in K.
\]

For the proof of this proposition and for more details see [10].

There are several ways to metricize the topology of weak convergence, at least on some subsets of \( \mathcal{P}(X) \). Let us denote by \( d \) the distance on \( X \) and, for \( p \in [0, \infty) \), by \( \mathcal{P}_p(X) \) the set of probability measures \( \mu \) such that
\[
\int_X d^p(x, y) \, d\mu(y) \leq \infty, \quad \text{for all point } x \in X
\]
The Monge-Kantorovich distance on \( \mathcal{P}_p(X) \) is given by
\[
d_p(\mu, \xi) = \inf_{\gamma \in \Pi(\mu, \xi)} \left[ \int_{X^2} d(x, y)^p \, d\gamma(x, y) \right]^{1/p}
\]
where \( \Pi(\mu, \xi) \) is the set of Borel probability measures on \( \mathbb{R}^d \) such that \( \gamma(A \times \mathbb{R}^d) = \mu(A) \) and \( \gamma(\mathbb{R}^d \times A) = \xi(A) \) for any set \( A \subset \mathbb{R}^d \). The proof that \( d \) constitute a metric and to show the existence of an optimal measure in (1.1) we refer to [6].

Theorem 1.4. (Kantorovich-Rubinstein Theorem) For any \( \mu, \xi \in \mathcal{P}_1(X) \),
\[
d_1(\mu, \xi) = \sup_{f \in C_{Lip}(X)} \left\{ \int_X f(x) \, d\mu(x) - \int_X f(x) \, d\xi(x) \right\}
\]
where \( C_{Lip}(X) \) is the set of Lipschitz continuous functions.

For a proof of this Theorem see [6].

1.1.3 Markov Processes

Let \( t \in \mathbb{R}^+ \), \( X \) be a Polish space (complete, separable, metric space), \( \beta(X) \) the Borel \( \sigma \)-algebra, \( p_{s,t}(x, dy) \) transition probabilities (Markov kernels) on \( (X, \beta(X)) \), \( 0 \leq s \leq t \leq \infty \), and \( (\mathcal{F}_t)_{t \geq 0} \) a filtration on \( (\Omega, \mathcal{F}, \mathbb{P}) \).

Definition 1.5. A stochastic process \( (X_t)_{t \geq 0} \) on the state space \( X \) is called an \( (\mathcal{F}_t) \)-Markov process with transition probabilities \( p_{s,t} \) if and only if

1. \( X_t \) is \( \mathcal{F}_t \) measurable \( \forall t \geq 0 \),
2. \( \mathbb{P}[X_t \in B | \mathcal{F}_s] = p_{s,t}(X_s, B) \) \( \mathbb{P} \)-a.s. \( \forall 0 \leq s \leq t, B \in \beta(X) \).

Definition 1.6. Let \( I \) be a finite set. A Q-Matrix on \( I \) is a matrix \( Q = (q_{ij} : i, j \in I) \) satisfying the following conditions

- \( 0 \leq -q_{ii} \leq \infty \) for all \( i \)
- \( q_{ij} \geq 0 \) for all \( i \neq j \)
• \( \sum_{j \in I} q_{ij} = 0 \) for all \( i \)

Thus in each row of \( Q \) we can choose the off-diagonal entries to be any non-negative real numbers, subject only to the constraint that the off-diagonal row sum is finite:

\[
q_i = \sum_{j \neq i} q_{ij} \leq \infty.
\]

The diagonal entry \( q_{ii} \) is then \(-q_i\), making the total row sum zero.

For more details see [19].

Definition 1.7. \( PC(\mathbb{R}^+, X) := \{ X : [0, \infty) \rightarrow X \} \) is constant on the interval \( [t, t+\epsilon) \).

Definition 1.8. A Markov process \( (X_t)_{t \geq 0} \) on \( (\Omega, \mathcal{F}, \mathbb{P}) \) is called a pure jump process or continuous time Markov chain if and only if \( (t \rightarrow X_t) \in PC(\mathbb{R}^+, X) \mathbb{P}\)-a.s.

1.1.4 The construction of time-inhomogeneous jump processes:

Let \( q_t : X \times \beta(X) \rightarrow [0, \infty) \) be a kernel of positive measure, i.e. \( x \rightarrow q_t(x, A) \) is measurable and \( A \mapsto q_t(x, A) \) is a positive measure. Our aim in this section is to construct a pure jump process with instantaneous jump rates \( q_t(x, dy) \). Let \( \lambda_t(x) := q_t(x, X\{x\}) \) be the total rate of jumping away from \( x \). And assume that \( \lambda_t(x) < \infty \) \( \forall x \in X \), no instantaneous jumps.

Now suppose that \((Y_n, J_n, h_n)_{n \in \mathbb{N}}\) is a Markov chain with \( J_n \) jumping times, and \( h_n \) holding times. Then \( J_n = \sum_{i=1}^{n} h_i \in [0, \infty) \) with jump times \( \{ J_n : n \in \mathbb{N} \} \).

Suppose that with respect to \( \mathbb{P}_{(t_0, \mu)} \),

\[
J_0 := t_0, \quad Y \sim \mu
\]

\[
\mathbb{P}_{(t_0, \mu)}[J_1 > t|Y_0] := e^{-\int_{t_0}^{t} \lambda_s(Y_0)ds}
\]

for all \( t \geq t_0 \), then \((Y_{n-1}, J_{n})_{n \in \mathbb{N}}\) is a time-homogeneous Markov chain on \( X \times [0, \infty) \) with transition law

\[
\mathbb{P}_{(t_0, \mu)}[Y_n \in dy, J_{n+1} > t|Y_0, J_1, ..., Y_{n-1}, J_n] = \pi_{J_n}(Y_{n-1}, dy) \cdot e^{-\int_{J_n}^{t} \lambda_s(Y)ds}
\]

i.e.

\[
\mathbb{P}_{(t_0, \mu)}[Y_n \in A, J_{n+1} > t|Y_0, J_1, ..., Y_{n-1}, J_n] = \int_A \pi_{J_n}(Y_{n-1}, dy) \cdot e^{-\int_{J_n}^{t} \lambda_s(Y)ds}
\]

\( \mathbb{P}\)-a.s. for all \( A \in \beta(X), t \geq 0 \)
For $Y_n \in \mathbb{X}$, $t_n \in [0, \infty)$ strictly increasing define

$$X := \Phi((t_n, Y_n)_{n=0,1,2,...}) \in PC([t_0, \infty), \mathbb{X} \cup \{\Delta\})$$

by

$$X_t := \begin{cases} Y_n & \text{for } t_n \leq t < t_{n+1}, n \geq 0 \\ \Delta & \text{for } t \geq \sup t_n. \end{cases}$$

Let

$$(X_t)_{t \geq t_0} := \Phi((J_n, Y_n)_{n \geq 0})$$

$$F_t^X := \sigma(X_s | s \in [t_0, t]), \ t \geq t_0.$$ 

**Theorem 1.9.** Under $P_{(t_0,\mu)}$, $(X_t)_{t \geq t_0}$ is a Markov jump process with initial distribution $X_{t_0} \sim \mu$

**Theorem 1.10.** 1. The transition probabilities

$$p_{s,t}(x,B) = P_{(s,x)}[X_t \in B] \quad (0 \leq s \leq t, x \in \mathbb{X}, B \in \beta(\mathbb{X}))$$

satisfying the Chapman-Kolmogorov equations for each $0 \leq t \leq s < \infty, \ x \in \mathbb{X}, A \in \beta(\mathbb{X})$

$$p_{t,s}(x,A) = \int_{\mathbb{R}^d} p_{t,s}(y,A)p_{s,t}(x,dy). \quad (1.2)$$

2. If $t \mapsto \lambda_t(x)$ is continuous for all $x \in \mathbb{X}$, then

$$(p_{s,s+h})f(x) = (1 - \lambda_s(x) \cdot h)f(x) + h \cdot (q_s f)(x) + o(h) \quad (1.3)$$

holds for all $s \geq 0, x \in \mathbb{X}$ and bounded functions $f : \mathbb{R} \to \mathbb{R}$ such that $t \mapsto (q_t f)(x)$ is continuous.

For a proof of the previous two theorems see [9].

We have showed in Theorem 1.9 the existence of a Markov process first and then obtained the Chapman-Kolmogorov equations for the transition probabilities in Theorem 1.10. There is a partial converse to this, which we will now develop. First we need a definition. Let $\{p_{t,s}; 0 \leq t \leq s < \infty\}$ be a family of mappings from $\mathbb{X} \times \beta(\mathbb{X}) \to [0,1]$. We say that they are a normal transition family if, for each $0 \leq t \leq s < \infty$:

1. the maps $x \mapsto p_{t,s}(x,A)$ are measurable for each $A \in \beta(\mathbb{X})$;
2. $p_{t,s}(x,A)$ is a probability measure on $\beta(\mathbb{X})$ for each $x \in \mathbb{X}$;
3. The Chapman-Kolmogorov equation (1.2) are satisfied.
Proposition 1.11. If \( \{p_{t,r}(x,A) := (T_{t,r}X_A)(x) := \int_A f(y)p(t,x,r,dy), 0 \leq t \leq r < \infty \} \) is a normal transition family and \( \mu \) is a fixed probability measure on the measurable space \( (\mathcal{X}, \beta(\mathcal{X})) \), then there exists a probability space \( (\Omega, \mathcal{F}, \mathbb{P}_\mu) \), a filtration \( (\mathcal{F}_t, t \geq 0) \) and a Markov process \( (X_t, t \geq 0) \) on that space such that:

- \( \mathbb{P}[X(r) \in A \mid X(t) = x] = p_{t,r}(x,A) \) for each \( 0 \leq t \leq r, x \in \mathcal{X}, A \in \beta(\mathcal{X}) \).
- \( X(0) \) has law \( \mu \).

For the proof see [2].

Proposition 1.12. Let \( X_t \) be a pure Markov jump process, then

\[
\tau_1 = \inf t \geq 0, \quad X_t \neq X_0
\]

is an \( \mathcal{F}_t \) stopping time.

Theorem 1.13. Under \( \mathbb{P}_x, \tau_1 \) and \( X_{\tau_1} \) are independent and there is a \( \beta(\mathcal{X}) \) measurable function \( \lambda(x) \) on \( \mathcal{X} \), different to \( \lambda \) above, such that

\[
\mathbb{P}_x[\tau_1 > t] = e^{-\lambda(x)t}.
\]

For a proof for the above proposition and Theorem see [22].

1.1.5 Forward and Backward Equations

Definition 1.14. The infinitesimal generator \( A_t \) of a Markov jump process at time \( t \) is the rate of change of the average for a function \( f : \mathcal{X} \to \mathbb{R} \) of the process.

\[
A_t f(x) = \lim_{h \to 0} \frac{E_x f(X_h) - f(x)}{h}, \quad f \in D_{A_t}.
\]

\[
E_x f(X_h) = E_x[f(X_h)\mid \tau_1 > h]\mathbb{P}_x[\tau_1 > h] + E_x[f(X_h)\mid \tau_1 < h]\mathbb{P}_x[\tau_1 < h]
\]

\[
E_x f(X_h) = f(x)e^{-\lambda(x)h} + \int_{\mathcal{X}} f(y)q_t(x,dy)(1 - e^{-\lambda(x)h}) + O(h)
\]

\[
E_x f(X_h) - f(x) = (1 - e^{-\lambda(x)h})\left(\int_{\mathcal{X}} f(y)q_t(x,dy) - f(x)\right) + O(h)
\]

Thus

\[
A_t f(x) = \lambda(x)\int_{\mathcal{X}} (f(y) - f(x))q_t(x,dy)
\]

In the case of a countable state space, the infinitesimal generator (or intensity matrix, kernel) has the following form:

\[
E_x f(X_h) = \sum_{y \neq x} f(y)q_t(x,y)h + f(x)(1 - \sum_{y \neq x} q_t(x,y)h) + O(h)
\]

\[
E_x f(X_h) - f(x) = \sum_{y \neq x} (f(y) - f(x))q_t(x,y)h + O(h)
\]

\[
A_t f(x) = \sum_{y \in \mathcal{X}} (f(y) - f(x))q_t(x,y)
\]
Theorem 1.15. (Kolmogorov’s backward equation) 
If \( t \to q_t(x,.) \) is continuous in total variation norm for all \( x \in \mathbb{X} \), then the transition kernels \( p_{s,t} \) of the Markov jump process constructed above are the minimal solutions of the backward equation

\[
- \frac{\partial}{\partial s} (p_{s,t}f)(x) = -(A_s p_{s,t}f)(x)
\]

for all bounded functions \( f : \mathbb{X} \to \mathbb{R} \), \( 0 \leq s \leq t \), with terminal condition \( (p_{t,t}f)(x) = f(x) \).

Remark 1.16. 
1. \( A_t \) is a linear operator on functions \( f : \mathbb{X} \to \mathbb{R} \).
2. (1.4) describes the backward evolution of the expectation values \( \mathbb{E}_{(s,x)}[f(X_t)] \) respectively the probabilities \( \mathbb{P}_{(s,x)}[X_t \in B] \) when varying the starting times \( s \).
3. In a discrete state space, (1.4) reduces to

\[
- \frac{\partial}{\partial s} p_{s,t}(x,z) = \sum_{y \in \mathbb{X}} A_s(x,y)p_{s,t}(y,z), \quad p_{t,t}(x,z) = \delta_{xz}
\]

a system of ordinary differential equations.

For \( \mathbb{X} \) being finite,

\[
p_{s,t} = \exp \left( \int_s^t A_r dr \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \int_s^t A_r dr \right)^n
\]

is the unique solution.
If \( \mathbb{X} \) is infinite, the solution is not necessarily unique (hence the process is not unique).

Theorem 1.17. (Kolmogorov’s forward equation) 
The forward equation

\[
\frac{d}{dt}(p_{s,t}f)(x) = (p_{s,t}A_t f)(x), \quad (p_{s,s}f)(x) = f(x)
\]

holds for all \( 0 \leq s \leq t \), \( x \in \mathbb{X} \) and all bounded functions \( f : \mathbb{X} \to \mathbb{R} \) such that \( t \to (q_t f)(x) \) and \( t \to \lambda_t(x) \) are continuous for all \( x \).

Corollary 1.18. (Fokker-Planck equation) Under the assumptions in the Theorem,

\[
\frac{d}{dt}(\mu_t, f) = (\mu_t, A_t f)
\]

for all \( t \geq s \) and bounded functions \( f : \mathbb{X} \to \mathbb{R} \) such that \( t \to \lambda_t \) are pointwise continuous. One sometimes writes

\[
\frac{d}{dt}\mu_t = \partial_t^* \mu_t
\]

A proof of the above Theorems may be found in [9].
1.1.6 Markov Evolution and Propagators

We start first with definition of the propagators. For a set $S$, a family of mappings $U^{t,r}$ from $S$ to itself, parametrized by the pairs of numbers $r \leq t$ (respectively $t \leq r$) from a given finite interval is called a (forward) propagator (respectively a backward propagator) is $S$, if $U^{t,t}$ is the identity operator in $S$ for all $t$ and the following chain rule, or propagator equation, holds for $r \leq s \leq t$ (respectively for $t \leq s \leq r$):

$$U^{t,s}U^{s,r} = U^{t,r}.$$

A backward propagator $U^{t,r}$ of bounded linear operator on Banach space $B$ is called strongly continuous if the operator $U^{t,r}$ depend strongly continuously on $t$ and $r$.

Suppose $U^{t,r}$ is a strongly continuous backward propagator of bounded linear operator on a Banach space with a common invariant domain $D$. Let $L_t$, $t \geq 0$ be a family of bounded operators $D \to B$ depending continuously on $t$. Let us say that the family $L_t$ generates $U^{t,r}$ on $D$ if, for any $f \in D$, the equations

$$\frac{d}{ds}U^{t,s}f = U^{t,s}L_s f, \quad \frac{d}{ds}U^{s,r}f = -L_s U^{s,r}f, \quad 0 \leq t \leq s \leq r,$$

where the derivatives exist in the Banach topology of $B$.

One needs to estimate the difference between two propagators when the difference between their generators is available. Which lead to the following Theorem.

**Proposition 1.19.** Let $D$ and $B$, $D \subset B$ be two Banach spaces equipped with a continuous inclusion and let $L_i$, $i = 1, 2$, $t \geq 0$, be two families of bounded linear operators, which are continuous in time $t$. Assume moreover, that $U^{t,r}_i$ are two propagators in $B$ generated by $L_i$, $i = 1, 2$, respectively, i.e. satisfying

$$\frac{d}{ds}U^{t,s}_i f = U^{t,s}_i L_s f_i, \quad \frac{d}{ds}U^{s,r}_i f = -L_s U^{s,r}_i f, \quad t \leq s \leq r,$$

for any $f \in D$, which satisfy $\|U^{t,r}_i\|_B \leq c_1, i = 1, 2$. Moreover, let $D$ be invariant under $U^{t,s}_i$ and $\|U^{t,s}_i\|_D \leq c_2$. Then we have

i) $U^{t,r}_2 - U^{t,r}_1 = \int_t^r U^{t,s}_2 (L^2_s - L^1_s) U^{s,r}_1 ds$ \hspace{1cm} (1.8)

ii) $\|U^{t,r}_2 - U^{t,r}_1\|_{B(0,1) \to R^k} \leq c_2^2 (r - t) \sup_{t \leq s \leq r} \|L^2_s - L^1_s\|_{B_1(0) \to R^k}$

**Proof.** By (1.8), We have that

$$U^{t,r}_2 - U^{t,r}_1 = U^{t,s}_2 U^{s,r}_1 \bigg|_{s=t} = \int_t^r \frac{d}{ds}(U^{t,s}_2 U^{s,r}_1)ds$$

$$= \int_t^r U^{t,s}_2 L^2_s U^{s,r}_1 - U^{t,s}_2 L^1_s U^{s,r}_1 ds$$

$$= \int_t^r U^{t,s}_2 (L^2_s - L^1_s) U^{s,r}_1 ds$$
Which implies the proof.

With each Markov process $X_t$ we associate a family of operators $(T_{s,t}, \ 0 \leq t \leq s < \infty)$ on $L^\infty(X)$ by prescription

$$(T_{t,s}f)(x) = E(f(X_t)|X_s = x)$$

for each $f \in L^\infty(X), \ x \in X$. We recall that $I$ is the identity operator, $If = f$ for each $f \in L^\infty(X)$.

**Theorem 1.20.**  
1. $T_{t,s}$ is a linear operator on $L^\infty(X)$ for each $0 \leq t \leq s < \infty$.
2. $T_{s,s} = I$ for each $s \geq 0$.
3. $T_{r,t}T_{t,s} = T_{r,s}$ whenever $0 \leq r \leq t \leq s < \infty$.
4. $f \geq 0 \Rightarrow T_{t,s}f \geq 0$ for all $0 \leq t \leq s < \infty, \ f \in L^\infty(X)$.
5. $T_{t,s}$ is a contraction, i.e. $\|T_{t,s}\| \leq 1$ for each $0 \leq t \leq s < \infty$.
6. $T_{t,s}(1) = 1$ for all $t \geq 0$.

Any family satisfying (1) to (6) of Theorem (1.20) is called Markov evolution or Markov propagator. It is obvious from previous that the following hold

$$p_{t,s}(x,A) = (T_{t,s}1_A)(x) = P[X_s \in A|X_t = x].$$

By properties of conditional probability each $p_{t,s}(x,.)$ is a probability measure, where $p_{t,s}(x,.)$ is the transition probabilities we define them above and we have the following

$$(T_{t,s}f)(x) = \int_{\mathbb{R}^d} f(y)p_{t,s}(x,dy) \quad (1.9)$$

**Definition 1.21.** A Markov propagator is said to be strongly continuous if for each $0 \leq t \leq s < \infty, \ f \in L^\infty(X), \ x \in X$.

$$\lim_{t \to 0}\|T_t f - f\| = 0 \text{ for all } f \in C_0(\mathbb{R}^d). \quad (1.10)$$

Now let us define

$$D_A = \left\{ \psi \in B; \exists \phi_\psi \in B \text{ such that } \lim_{t \to 0}\left\| \frac{T_{s,t}\psi - \psi}{t} - \phi_\psi \right\| = 0 \right\}$$

and let us define the operator $A$ in $B$ by the prescription

$$A\psi = \phi_\psi$$

Then $A$ is the generator of the propagator $T_{s,t}$.

**Theorem 1.22.**
1. $D_A$ is dense in $B$.

2. $T_tD_A \subseteq D_A$ for each $t \geq 0$.

3. $T_tA\psi = A T_t \psi$ for each $t \geq 0, \psi \in D_A$.

**Theorem 1.23.** $A$ is closed.

**Definition 1.24.** The following are the basic definitions related to the generators of Markov processes. One says that an operator $A$ in $C(R^d)$ defined on a domain $D_A$

- is conditionally positive, if $Af(x) \geq 0$ for any $f \in D_A$ s.t. $f(x) = 0 = \min_y f(y)$;
- satisfies the positive maximum principle (PMP), if $Af(x) \leq 0$ for any $f \in D_A$ s.t. $f(x) = \max_y f(y) \geq 0$;
- is dissipative if $\|((\lambda - A)f\| \geq \lambda \|f\|$ for $\lambda > 0, f \in D_A$.

**Theorem 1.25.** Let $A$ be a generator of a Feller semigroup $\Phi_t$. Then

- $A$ is conditionally positive,
- satisfies the PMP on $D_A$.

If moreover $A$ is local and $D_A$ contains $C_c^\infty$, then it is locally conditionally positive and satisfies the local PMP on $C_c^\infty$. Where $C_c$ is the space of continuous functions with compact support.

**Corollary 1.26.** Let $B$ be a Banach space and let $A : B \rightarrow B$ be a bounded dissipative linear operator. Then $A$ generates a strongly continuous contraction semigroup $(T_t)_{t \geq 0}$ on $B$, which is given by

$$T_t f = e^{At} f := \sum_{n=0}^{\infty} \frac{1}{n!} (At)^n f \quad (t \geq 0). \tag{1.11}$$

**Proposition 1.27.** Let $X$ be a locally compact metric space and $A$ be a bounded conditionally positive operator from $C_\infty(X)$ to $B(X)$. Then there exists a bounded transition kernel $\nu(x,dy)$ in $X$ with $\nu(x,\{x\}) = 0$ for all $x$, and a function $a(x) \in B(X)$ such that

$$Af(x) = \int_X f(z) \nu(x, dz) - a(x)f(x) \tag{1.12}$$

Conversely, if $A$ is of this form, then it is a bounded conditionally positive operator $C(X) \rightarrow B(X)$. Where $C_\infty(X)$ is the space of continuous functions that vanish at infinity.
Theorem 1.28. Let $\nu(x, dy)$ be a weakly continuous uniformly bounded transition kernel in a complete metric space $X$ such that $\nu(x, \{x\}) = 0$ and $a \in C(X)$. Then operator (3.35) has $C(X)$ as its domain and generates a strongly continuous semigroup $T_t$ in $C(X)$ that preserves positivity and is given by transition kernels $p_t(x, dy)$:

$$T_t f(x) = \int p_t(x, dy) f(y).$$  \hspace{1cm} (1.13)

In particular, if $a(x) = \|\nu(x, \cdot)\|$, then $T_t 1 = 1$ and $T_t$ is the semigroup of a Markov process that we shall call a pure jump or jump-type Markov process.

Theorem 1.29. Let $\nu(x, dy)$ be a weakly continuous uniformly bounded transition kernel in a metric space $X$ such that $\nu(x, \{x\}) = 0$. Let $a(x) = \nu(x, X)$. Define the following process $X^\tau_x$. Starting at a point $x$, the process remains there for a random $a(x)$-exponential time $\tau$, i.e., this time is distributed according to $P(\tau > t) = \exp\left[-ta(x)\right]$, and then jumps to a point $y \in X$ distributed according to the probability law $\nu(x, .)/a(x)$. Then the procedure is repeated.

Remark 1.30. If $X$ in Theorem 1.28 is locally compact and a bounded $\nu$ (depending weakly continuous on $x$) is such that $\lim_{x \to \infty} \int_K \nu(x, dy) = 0$ for any compact set $K$, then $L$ of form (1.12) preserves the space $C_\infty(X)$ and hence generates a Feller semigroup.

In order to deal with one of the most important class of Markov processes we will make the following definition

**Definition 1.31.** A strongly continuous propagator of positive linear contractions on $C_\infty(X)$ is called a Feller Propagator. In other words, A Markov process in a locally compact metric space $X$ is called a Feller process if its Markov propagator reduced to $C_\infty(X)$ is a Feller propagator, i.e., it preserves $C_\infty(X)$ and it is strongly continuous there.

Theorem 1.32. Every Lévy process is a Feller process.

Theorem 1.33. If $\Phi$ is a Feller propagator, then the dual propagator $\Phi^*$ on $\mathcal{M} (X)$ is a positivity-preserving propagator of contractions depending continuously on $t$ and $s$. Where $\mathcal{M} (X)$ is the set of bounded signed Borel measures on $X$.

The proof of the theorems and propositions above can be found in [2, 10, 14] and [15].

**1.1.7 Law of Large Numbers for Empirical Measures**

Let $\mathcal{P}(X)$ denote the space of probability vectors on $X$. We are given a family of jump rates matrices or pure jump Markov operator $A_t(x, y, p)$ indexed by $p \in \mathcal{P}(X)$, and assume for simplicity that the map $p \to A_t(p)$ is Lipschitz continuous. Given the states $X_1^N(t), \ldots, X_N^N(t)$ of $N$ particles at any time $t$, the empirical measure has the form

$$\mu^N_t := \frac{1}{N} \sum_{i=1}^N \delta_{X_i^N(t)}, \quad t \geq 0.$$  \hspace{1cm} (1.14)

where $\delta_x$ is Dirac measure at $x$, and $x \in X$ where $X$ is the state space.
Theorem 1.34. Suppose that $A_t(x, y, .)$ is Lipschitz continuous for all $x, y \in X$, and assume that $\mu_0^N$ convergence in probability to $q \in \mathcal{P}(X)$ as $N$ tends to infinity. Then $(\mu_0^N)_{N \in \mathbb{N}}$ converge uniformly on compact time interval in probability to $\mu_t$, where $\mu_t$ is the unique solution to equation (1.5) with $\mu_0 = q$.

For more details see [8] and [20].

1.2 Initial Value Problems for Ordinary Differential Equations

Let $E$ be a complete normed space (Banach space) and $U$ an open subset of $E$. Assume that $J \subset \mathbb{R}$ is an open interval containing 0 (for the sake of simplification) then we define

$$f : J \times U \longrightarrow E.$$ 

By an integral curve with initial condition $x_0$ we mean a mapping

$$X : J \longrightarrow U$$

such that $X$ is differentiable, $X(0) = x_0$ and

$$\dot{X} = f(t, X(t))$$

for all $t \in J$. We define a local flow for $f$ at $x_0$ to be a mapping

$$X : J \times U_0 \longrightarrow U$$

where $U_0$ is an open subset of $U$ containing $x_0$ such that for each $x \in U_0$ the map

$$t \mapsto X_x(t) = X(t,x)$$

is an integral curve for $f$ with initial condition $x$.

Theorem 1.35. Let $J$ be an open interval containing 0. Let $U$ be open in $E$. Let $x_0 \in U$. Let $0 < a < 1$ such that the closed ball $B_{2a}$ is contained in $U$. Let

$$f : J \times U \longrightarrow E$$

be a continuous map, bounded by a constant $C > 0$, and satisfying a Lipschitz condition on $U$ with Lipschitz constant $K$ uniformly with respect to $J$. If $b < a/C$ and $b < 1/K$ then there exists a unique flow

$$X : J_b \times B_a(x_0) \longrightarrow U.$$

If $f$ is of class $C^p$, then so is each integral curve $X_x$. 
By definition of the integral curve and with new notation $D_1 := \frac{\partial}{\partial t}$, we have
\[ D_1 X(t, x) = f(t, X(t, x)). \]

We now investigate regularity in the initial value for the flow.

**Theorem 1.36.** Let $J$ be an open interval containing 0, and let $U$ be open in $E$. Let
\[ f : J \times U \rightarrow E. \]
be a $C^p$ map with $p \geq 1$ (possibly $p = \infty$), and let $x_0 \in U$. There exists a unique local flow for $f$ at $x_0$. We can select an open subinterval $J_0$ of $J$ containing 0 such that the unique local flow
\[ X : J_0 \times U_0 \rightarrow U. \]
is of class $C^p$, and such that $D_2 X(t, x) := \frac{\partial}{\partial x} X(t, x)$ satisfies the differential equation
\[ D_1 D_2 X(t, x) = D_2 f(t, X(t, x)) D_2 X(t, x). \]
on $J_0 \times U_0$ with initial condition $D_2 X(0, x)$.

**Theorem 1.37.** Let $J$ be an open interval containing 0, and let $U$ be open in $E$. Let
\[ f : J \times U \rightarrow E. \]
be a continuous map in $t$ and a $C^1$ map with $p \geq 1$ on $U$ and let $x_0 \in U$. Then there exists a unique local flow for $f$ at $x_0$ such that the integral curve $X(t, x)$ is of class $C^1$ in both $t$ and $x$.

The proof of the above theorems can be found in [18].

### 1.2.1 Continuous Dependence on Parameter and Differentiability

Let as before $J$ denote an open interval in $R$ and $U$ be an open subset in the Banach space $E$. Moreover, let $\Lambda$ denote a metric space then we are going to study the parameter-dependent initial value problem
\[ \dot{X} = f(t, x, \lambda), \quad X(\tau) = \xi \]

**Theorem 1.38.** Assume that $\Lambda$ is as above and that $M := U \times \Lambda$, and that $f$ is a lipschitz continuous in $x$ and in the parameter $\lambda$. Then there exist a solution flow which is lipschitz continuous uniformly in all variables.

Let us now turn to recall results about the differentiability of the solution flow. We now assume that the function $f$ is differentiable in $x$ and the parameter $\lambda$. If we differential the equation
\[ \dot{X} = f(t, x, \lambda) \]
with respect to the initial data $\xi$ we get the following linear equation
\[ \dot{z} = D_2 f(t, X(t, \tau, \xi, \lambda)) \]
\[ z(\tau) = I \]
Theorem 1.39. Let the function \( f \) be of the type \( C^1 \) in the variable \( x \) and the parameter \( \lambda \) then the solution flow \( X(\tau, t, \xi, \lambda) \) is of type \( C^1 \) in all variables and \( \frac{\partial X}{\partial \xi} \) is a solution of the linearized initial value problem

\[
\dot{z} = D_2 f(t, X(t, \tau, \xi, \lambda))
\]

\[
z(\tau) = I.
\]

Theorem 1.40. Let the function \( f \) be depending continuously on \( t \) and of type \( C^m \) in \( x \in U \) and \( \lambda \in \Lambda \). Then the solution flow \( X(\tau, t, \xi, \lambda) \) is of type \( C^m \) in \( \tau, \xi \) and \( x \) and the parameter \( \lambda \).

The proof of the above theorems can be found in [18] and [1].

1.2.2 Linearization of Ordinary Differential Equations

Definition 1.41. Let \( \beta(t, s), \quad 0 \leq t \leq s \leq T \) be a family of non-singular transformation in a Banach space \( \mathbb{X} \) with Borel \( \sigma \)-algebra \( \mathcal{F} \) and measure \( \mu \). Recall that \( \beta(t, s) \) is non-singular if and only if for every \( A \in \mathcal{F} \) such that \( \mu(A) = 0 \), \( \mu(\beta^{-1}(t, s)A) = 0 \). Further let \( F \in L^\infty(\mathbb{X}) \). Then the family operators \( \Phi^{t,s} : L^\infty(\mathbb{X}) \to L^\infty(\mathbb{X}) \) defined by

\[
(\Phi^{t,s}F)(x) = F(\beta(s, t)(x)) \quad 0 \leq t \leq s \leq T \quad (1.15)
\]

is called the Koopman propagator with respect to \( \beta(t, s) \). Traditionally the Koopman propagator is called Koopman operator.

It can be shown, see [17], that the Koopman operator has the following properties:

- \( \Phi^{t,s} \) is a linear operator.
- \( \Phi^{t,s} \) is a contraction on \( L^\infty(\mathbb{X}) \), i.e. \( \|\Phi^{t,s}F\|_{L^\infty(\mathbb{X})} < \|F\|_{L^\infty(\mathbb{X})} \) \( \forall F \in L^\infty(\mathbb{X}) \).

Being a contraction the Koopman operator is bounded. It is also straightforward to show that \( \Phi^{t,s} \) is a time inhomogeneous propagator see [17].

1.3 Optimal Control

Throughout this section we suppose that \( (\Omega, \mathcal{F}, \mathbb{P}) \) is a probability space with filtration \( \{\mathcal{F}_t, t \geq 0\} \) satisfying the usual conditions.

Definition 1.42. (Control Processes)

Given a subset \( U \) of \( \mathbb{R}^m \), we denote by \( U_0 \) the set of all measurable processes \( u = \{u_t, t \geq 0\} \) valued in \( U \). The elements of \( U_0 \) are called a control processes.

In most cases it is natural to require that the control processes \( u \) is adapted to the \( X \) process. Then we define the control process \( u \) by

\[
u_t = u(t, X_t)
\]

such a function called feedback control law.
1.3.1 Optimal Control for Diffusions

**Definition 1.43. (Controlled Diffusion Processes)**

Let \( b : (t, x, u) \in \mathbb{R}_+ \times \mathbb{X} \times U \to b(t, x, u) \in \mathbb{X} \) and \( \sigma : (t, x, u) \in \mathbb{R}_+ \times \mathbb{X} \times U \to \sigma(t, x, u) \in \mathbb{X}^d \) be two functions. For a given point \( x_0 \in \mathbb{X} \) we will consider the following controlled stochastic differential equation

\[
dX_t = b(s, X_s, u(s, X_s)) \, dt + \sigma(s, X_s, u(s, X_s)) \, dB_t,
\]

\( X_t = x, \quad (1.16) \)

where \( B = \{B_t, t \geq 0\} \) is a Brownian motion valued in \( \mathbb{X}^d \) defined on \( (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}) \).

In most concrete cases we also have to satisfy some control constraints, and we model this by taking as given a fixed subset \( U \subseteq \mathbb{R}^m \) and requiring that \( u_t \in U \) for each \( t \). We can now define the class of admissible control laws.

**Definition 1.44.** A control law \( u \) is called admissible if

- \( u(t, x) \in U \) for all \( t \in \mathbb{R}_+ \) and all \( x \in \mathbb{X} \).

- For any given initial point \((t, x)\) the SDE (1.16) has a unique solution.

The class of admissible control laws is denoted by \( \mathbb{U} \subseteq \mathbb{U}_0 \).

**Definition 1.45.** The control problem is defined as the problem to maximize

\[
E_{t,x} \left[ \int_{t}^{T} J(s, X_s^u, u_s) \, ds + G(X_T^u) \right],
\]

\( (1.17) \)

given the dynamics (1.16) and the constraints

\[
u(s, y) \in U, \quad \forall (s, y) \in [t, T] \times \mathbb{X}. \quad (1.18)\]

**Definition 1.46.** The value function

\[
V : [0, T] \times \mathbb{X} \to \mathbb{R}
\]

is defined such that

\[
V(t, x) = \sup_{u \in \mathbb{U}} E_{t,x} \left[ \int_{t}^{T} J(s, X_s^u, u_s) \, ds + G(X_T^u) \right]
\]

given the dynamics (1.16).

**Assumption** We assume the following.
1. There exists an optimal control law $\hat{u}$.

2. The optimal value function $V$ is regular in the sense that $V \in C^{1,2}$.

3. A number of limiting procedures in the following arguments can be justified.

**Theorem 1.47.** *(Hamilton Jacobi Bellman equation)* Under the Assumption, the following hold:

1. $V$ satisfies the Hamilton Jacobi Bellman equation

\[
\frac{\partial V}{\partial t}(t,x) + \sup_{u \in U} \{J(t,x,u) + A^u V(t,x)\} = 0, \quad \forall (t,x) \in (0,T) \times \mathbb{X}
\]

\[V(t,x) = G(x), \quad \forall x \in \mathbb{X}.
\]

2. For each $(t,x) \in [0,T] \times \mathbb{X}$ the supremum for the HJB equation above is attained at $u = \hat{u}(t,x)$.

**Theorem 1.48.** *(Verification theorem)* Suppose that we have two functions $H(t,x)$ and $g(t,x)$, such that

- $H$ is sufficiently integrable (see Remark 19.3.4 below), and solves the HJB equation

\[
\frac{\partial H}{\partial t}(t,x) + \sup_{u \in U} \{J(t,x,u) + A^u V(t,x)\} = 0, \quad \forall (t,x) \in (0,T) \times \mathbb{X}
\]

\[H(t,x) = G(x), \quad \forall x \in \mathbb{X}.
\]

- The function $g$ is an admissible control law.

- For each fixed $(t,x)$, the supremum in the expression

\[
\sup_{u \in U} \{J(t,x,u) + A^u V(t,x)\}
\]

is attained by the choice $u = g(t,x)$.

Then the following hold:

1. The optimal value function $V$ to the control problem is given by

\[V(t,x) = H(t,x)\]

2. There exists an optimal control law $\hat{u}$, and in fact $\hat{u}(t,x) = g(t,x)$.

The proof of the above theorems can be found in [5] and [12].
1.3.2 Optimal Control for Markov Jump Processes

In this subsection we shall describe the principle of dynamic programing for a finite state Markov Jump Process $X_t, \quad t \in [0, T]$ and the corresponding HJB equation for finding the optimal control strategy. We start by recalling the pure jump Markov process on a locally compact space $X$ specified by the integral generator of the form

$$A_t f(x) = \int_X (f(y) - f(x))\nu(t, x, dy)$$  \hspace{1cm} (1.19)

with bounded kernel $\nu$.

**Definition 1.49. (Controlled Jump Processes)**

Assuming that we can control the jumps of this process, i.e. the measure $\nu$ depends on the control $u_t$, then we consider the following controlled dynamics

$$\frac{d}{dt}(\mu_t, f) = (\mu_t, A_t[u_t] f)$$ \hspace{1cm} (1.20)

Suppose that the value function $V : [0, T] \times X \times \mathbb{R}^k \rightarrow \mathbb{R}$ starting at time $t$ and position $x$ is defined as:

$$V(t, x, \mu_s) := \sup_{u \in U} \mathbb{E}_t \left[ \int_t^T J(s, X_s, \mu_s, u_s) ds + V_T(X_T, \mu_T) \right]$$

where $\mu \in \mathcal{P}({\mathbb{R}}^k)$ the set of probability measure.

In this subsection and for simplicity we will fix all parameters but $u$ and drop them.

**Definition** Let $V(t, x)$ be a real valued function on $[0, T] \times X$. We define a linear operator $\mathcal{L}$ by

$$\mathcal{L} = \lim_{h \rightarrow 0^+} \frac{E_{t,x} V(t+h, X(t+h)) - V(t,x)}{h}$$  \hspace{1cm} (1.21)

provided the limit exist for each $x \in X$ and each $t$.

Let $D(\mathcal{L})$ be the domain of operator $\mathcal{L}$ and moreover assume that the following hold for each $V \in D(\mathcal{L})$

- $V$, $\frac{\partial V}{\partial t}$ and $\mathcal{L}V$ are continuous on $[0, T] \times X$
- $E_{t,x} |V(t, X(s))| \leq \infty$, $E_{t,x} \int_t^s |\mathcal{L}V(r, X(r))| dr \leq \infty \quad \forall t \leq s \in [0, T]$
- (Dynkin formula) For $t \leq s$,

$$E_{t,x} V(t, X(s)) - V(t,x) = E_{t,x} \int_t^s \mathcal{L}V(r, X(r)) dr$$

**Proposition 1.50.** Let $V(t, x)$ be of class $C^1([0, T] \times X)$. Then $V(t, x) \in D(\mathcal{L})$ and

$$\mathcal{L}V(t, x) = \frac{\partial V}{\partial t}(t, x) + A_t[u]V(t, x).$$

Where $A_t[u]$ is the generator of the jump nonlinear time-inhomogeneous Markov process.
A proof for the above Proposition can be found in [12].

The dynamic programming equation is derived from the following procedures. If we take a constant control \( u(s) = u \) for \( t \leq s \leq T \), then
\[
V(t, x) \leq \mathbb{E}_x \left[ \int_t^T J(s, X_s, u_s) ds + V^T(X_T) \right].
\]
If we deduct \( V(t, x) \) from both sides, divide by \( h \), let \( h \to 0 \) and use Dynkin’s formula we get that:
\[
\lim_{h \to 0} h^{-1} \mathbb{E}_x \left[ \int_t^T J(s, X_s, u_s) ds \right] = J(t, x, u)
\]
\[
\lim_{h \to 0} h^{-1} \mathbb{E}_x \left[ V(t, x) - V^T(T, X_T) \right] = 0
\]
\[
\lim_{h \to 0} h^{-1} \mathbb{E}_x \left[ \int_t^T \mathcal{L}[t, u] V(s, X_s) ds \right] = \mathcal{L}[t, u] V(t, x)
\]
Therefore, for all \( u \in U \) we have:
\[
0 \geq \mathcal{L}[t, u] V(t, x) + J(t, x, u).
\]
If \( \hat{u} \) is the optimal control strategy, we will have:
\[
V(t, x) := \mathbb{E}_x \left[ \int_t^T J(s, X_s, \hat{u}_s) ds + V^T(T, X_T) \right]
\]
\[
0 = \mathcal{L}[t, \hat{u}] V(t, x) + J(t, x, \hat{u}),
\]
which leads to the dynamic programming equation
\[
0 = \sup_{u \in U} \left[ \mathcal{L}[t, u] V(t, x) + J(t, x, u) \right].
\]
if \( x_t \) is a controlled Markov chain with finite state space and jump rates \( \nu(t, x, y, u) \) using the above proposition. The HJB equation becomes the following system of ordinary differential equations, see [12]
\[
\frac{\partial V}{\partial t} + \max_u [J(t, x, u) + \mathbf{A}[t, u] V] = 0.
\] (1.22)

Next we present a well-posedness result and a verification Theorem for the HJB equation in the space of bounded continuous functions \( C_b(\mathbb{X}) \) for the case of non homogenous pure jump Markov process. let us first formulate the assumptions that we need

1. The set \( U \) is compact.
2. The kernel \( \nu(t, x, dy) \) is a Feller transition kernel.
3. The cost function \( J(t, x, u) \in C_b(X) \).

4. The terminal value function \( G(X_T) \in C_b(X) \).

**Theorem 1.51.** Under the above assumptions there exists a unique solution \( v \) to the HJB equation \((1.22)\)

**Theorem 1.52.** Under the above assumptions the unique solution \( v \) to the HJB equation \((1.22)\) coincide with the value function \( V(t, x) \). Moreover, there exists an optimal control \( u \), which is given by any function satisfying

\[
A[t, u(t, x)]v(t, x) + J(t, x, u(t, x)) = \sup_{u \in U} A[t, u]v(t, x) + J(t, x, u)
\]

For the proof of the above Theorems see [3].

The optimal control \( u^* \) in the above Theorem may be discontinuous. If the maximum has achieved at only one point for every \((t, x)\) then it follow from the compactness of \( U \) that the optimal control \( u^* \) is continuous. Sufficient additional conditions for such a unique maximum is the strict concavity of the function \( \Theta(t, x, u) = A[t, u]V(t, x) + J(t, x, u) \) and the convexity of \( U \). Under somewhat stronger assumptions we show next a Lipschitz continuity of the optimal control \( u^* \).

**Lemma 1.53.** Let the assumptions above be fulfilled. And Let the function \( \Theta(t, x, \cdot) \) satisfy the following

1. \( \Theta(t, x, \cdot) \) is \( C^2 \) for each \((t, x) \in [0, T] \times X \).
2. \( \Theta(t, \cdot, u) \) satisfies on \( X \) a Lipschitz condition, uniformly with respect to \( t, u \).
3. The absolute value of the Eigenvalues of the matrices \( \Theta_{uu} \) are bounded above by \( \gamma > 0 \).

**Proof.** It is clear that the assumptions above on the function \( \Theta(t, x, u) \) are inherited by the cost function \( J(t, x, u) \) and the Jump coefficient \( \nu(t, x, u) \). We will divide the proof to two steps.

Assumption 3 means that the cost function \( J(t, x, \cdot) \) is a strictly concave function, hence it has a unique maximum on the compact set \( U \). Given \( s \in [0, T], x_1, x_2 \in X \) let

\[
u_1 = u^*(t, x_1), \quad u_2 = u^*(t, x_2)
\]

From 1, 3 and Taylor formula we have,

\[
J(t, x_1, u_2) - J(t, x_1, u_1) \geq J_u(s, x_1, u_1)(u_2 - u_1) - \frac{|\Theta_{uu}|}{2} |u_2 - u_1|^2
\]

\[
\geq J_u(s, x_1, u_1)(u_2 - u_1) - \frac{\gamma}{2} |u_2 - u_1|^2
\]

due to the fact that the cost function reach a maximum at \( u_1 \) on the convex set \( U \), the first term on the right hand side of the above inequality vanish. By applying the integral form of the mean value Theorem on the left hand side we have that

\[
\int_0^1 J_u(P_1(\lambda))(u_2 - u_1) d\lambda \geq \frac{\lambda}{2} |u_2 - u_1|^2,
\]

\[(1.23)\]
where $P_1(\lambda) = (t, x_1, u_1 + \lambda(u_2 - u_1))$. Likewise, if we exchange $x_1$ by $x_2$ we get,

$$-\int_0^1 J_u(P_2(\lambda))(u_2 - u_1)d\lambda \geq \frac{\lambda}{2}|u_2 - u_1|^2,$$

where $P_2(\lambda) = (t, x_2, u_1 + \lambda(u_2 - u_1))$. By adding (1.23) and (1.24) together we get

$$\int_0^1 [J_u(P_1(\lambda)) - J_u(P_2(\lambda))](u_2 - u_1)d\lambda \geq \gamma |u_2 - u_1|^2$$

By Cauchy’s inequality

$$\int_0^1 [J_u(P_1(\lambda)) - J_u(P_2(\lambda))]d\lambda \geq \gamma |u_2 - u_1|.$$

From assumption B6, and $|P_1(\lambda) - P_2(\lambda)| = |x_1 - x_2|

$$|J_u(P_1(\lambda)) - J_u(P_2(\lambda))| \leq C |x_1 - x_2|,$$

where $C$ is the Lipschitz constant for $J_u(t,.,u)$. Hence, we have that

$$C|x_2 - x_1| \geq \gamma |u_2 - u_1|$$

$$|u^*_A(t, x_2) - u^*_A(t, x_1)| \leq \frac{C}{\gamma} |x_2 - x_1|.$$

Repeating the same line arguments in the jump coefficient $\nu(t,x,u)$ guarantee the claimed regularity property.

For more details and discussions on the continuity, existence and uniqueness of the optimal control see [11].

1.4 Differential Game Theory

A non-cooperative game with an arbitrary, finite number of players $A; B; C; \cdots$ in normal form can be described by the sets $S_A, S_B, S_C, \cdots$ of possible strategies of these players and by their payoff functions

$$\Pi_A(s_A; s_B; s_C; \cdots), \Pi_B(s_A; s_B; s_C; \cdots), \Pi_C(s_A; s_B; s_C; \cdots); \cdots.$$

These functions specify the payoffs of the players $A; B; C; \cdots$ for an arbitrary profile $(s_A; s_B; s_C; \cdots)$, where a profile (or a situation) is any collection of strategies $s_A$ from $S_A$, $s_B$ from $S_B$, $s_C$ from $S_C$, etc.

Strictly and weakly dominated strategies, dominant strategies and Nash equilibria are defined in the same way as for two players. In particular, a situation $(s_A^*; s_B^*; s_C^*; \cdots)$ is called a Nash equilibrium, if none of the players can win by deviating from this situation, or, in other words, if the strategy $s_A^*$ is the best reply
to the collection of the strategies $s_A^*, s_B^*, s_C^*, \ldots$, the strategy $s_B^*$ is the best reply to the collection of the strategies $s_A^*, s_B^*, s_C^*, \ldots$. In formal language this means that
\[
\Pi_A(s_A^*; s_B^*; s_C^*; \ldots) \geq \Pi_A(s_A^*; s_B^'; s_C^*; \ldots)
\]
for all $s_A$ from $S_A$, \[\Pi_B(s_A^*; s_B^*; s_C^*; \ldots) \geq \Pi_A(s_A^*; s_B^'; s_C^*; \ldots)\]
for all $s_B$ from $S_B$, etc.

Differential games or continuous-time infinite dynamic games study a class of decision problems, under which the evolution of the state is described by a differential equation and the players act throughout a time interval.

In particular, in the general $n$-person differential game, Player $i$ seeks to:
\[
\max_{u_i} \int_{t_0}^T J_i[s, x(s), u_1(s), u_2(s), \ldots, u_n(s)]ds + G_i(x(T)), \quad \text{for } i \in \{1, 2, \ldots, n\}
\]
subject to the dynamics
\[
\dot{x}(s) = f[s, x(s), u_1(s), u_2(s), \ldots, u_n(s)], \quad x(t_0) = x_0,
\]
where $x(s) \in \mathbb{R}^m$ denotes the state variables of game, and $u_i \in U_i$ is the control of Player $i$, for $i \in N$.

The functions $f[s, x, u_1, u_2, \ldots, u_n]$, $J_i[s, \cdot, u_1, u_2, \ldots, u_n]$ and $G_i(\cdot)$, for $i \in N$, and $s \in [t_0, T]$ are differentiable functions. A set-valued function $\eta_i(\cdot)$ defined for each $i \in N$ as
\[
\eta_i(s) = \{x(t), \quad t_0 \leq t \leq c_i^s\}, \quad t_0 \leq c_i^s \leq s,
\]
where $c_i^s$ is non decreasing in $s$, and $\eta_i(s)$ determines the state information gained and recalled by Player $i$ at time $s \in [t_0, T]$. Specification of $\eta_i(\cdot)$ (in fact, $c_i^s$ in this formulation) characterizes the information structure of Player $i$ and the collection (over $i \in N$) of these information structures is the information structure of the game.

**Definition 1.54.** A set of strategies $\{v_1^*(s), v_2^*(s), \ldots, v_n^*(s)\}$ is said to constitute a non-cooperative Nash equilibrium solution for the $n$-person differential game, if the following inequalities are satisfied for all $v_i(s) \in U_i$, $i \in N$:
\[
\int_{t_0}^T J_i[s, x^*(s), v_1^*(s), v_2^*(s), \ldots, v_{i-1}^*(s), v_i^*(s), v_{i+1}^*(s), \ldots, v_n^*(s)]ds + G^i(x^*(T)) \geq
\int_{t_0}^T J_i[s, x[i](s), v_1^*(s), v_2^*(s), \ldots, v_{i-1}^*(s), v_i(s), v_{i+1}^*(s), \ldots, v_n^*(s)]ds + G^i(x[i](T))
\]
where on the time interval $[t_0, T]$
\[
\dot{x}[i](s) = f[s, x[i](s), v_1^*(s), v_2^*(s), \ldots, v_{i-1}^*(s), v_i(s), v_{i+1}^*(s), \ldots, v_n^*(s)], \quad x[i](t_0) = x_0.
\]

The set of strategies $\{v_1^*(s), v_2^*(s), \ldots, v_n^*(s)\}$ is known as a Nash equilibrium of the game.
Definition 1.55. Open-loop Nash Equilibria If the players choose to commit their strategies from the outset, the players information structure can be seen as an open-loop pattern in which \( \eta'(s) = x_0, s \in [t_0, T] \). Their strategies become functions of the initial state \( x_0 \) and time \( s \), and can be expressed as \( \{u_i(s) = \phi_i(s, x_0), \text{for } i \in N \} \).

Definition 1.56. Closed-loop Nash Equilibria Under the memoryless perfect state information, the players information structures follow the pattern \( \eta'(s) = \{x(s), t_0 \leq t \leq s\} \). The players strategies become functions of the initial state \( x_0 \), current state \( x(s) \) and current time \( s \), and can be expressed as \( \{u_i(s) = \phi_i(s, x, x_0), \text{for } i \in N \} \).

Definition 1.57. Feedback Nash Equilibria To eliminate information nonuniqueness in the derivation of Nash equilibria, one can constrain the Nash solution further by requiring it to satisfy the feedback Nash equilibrium property. In particular, the players information structures follow either a closed-loop perfect state (CLPS) pattern in which \( \eta'(s) = \{x(s), t_0 \leq t \leq s\} \) or a memoryless perfect state (MPS) pattern in which \( \eta'(s) = \{x_0, x(s)\} \). Moreover, we require the following feedback Nash equilibrium condition to be satisfied.

Definition 1.58. For the \( n \)-person differential game with MPS or CLPS information, an \( n \)-tuple of strategies \( \{u_i^*(s) = \phi_i^*(s, x) \in U^i, \text{for } i \in N \} \) constitutes a feedback Nash equilibrium solution if there exist functionals \( V^i(t, x) \) defined on \([t_0, T] \times \mathbb{R}^m \) and satisfying the following relations for each \( i \in N \):

\[
\begin{align*}
V^i(T, x) &= G^i(x), \\
V^i(t, x) &= \int_t^T J^i[s, x^*(s), \phi_1^*(s, \eta), \phi_2^*(s, \eta), \ldots, \phi_n^*(s, \eta)] ds + G^i(x^*(T)) \\
&\quad + \int_t^T J^i[s, x^{[1]}(s), \phi_1^*(s, \eta), \phi_2^*(s, \eta), \ldots, \phi_n^*(s, \eta)] ds + q^i(x^{[1]}(T)),
\end{align*}
\]

\( \forall \phi_i(\cdot, \cdot) \in \Gamma^i, x \in \mathbb{R}^n \), where on the interval \([t_0, T] \),

\[
\begin{align*}
x^{[1]}(s) &= f[s, x^{[1]}(s), \phi_1^*(s, \eta), \phi_2^*(s, \eta), \ldots, \phi_n^*(s, \eta)], \\
&\quad \ldots, \phi_{i-1}^*(s, \eta), \phi_i(s, \eta), \phi_{i+1}^*(s, \eta), \ldots, \phi_n^*(s, \eta)], \\
x^*[t] &= x(s) = \{x(s), x_0\} \text{ or } \{x(\tau), \tau \leq s\},
\end{align*}
\]

and \( \eta \) stands for either the data set \( \{x(s), x_0\} \) or \( \{x(\tau), \tau \leq s\} \), depending on whether the information pattern is MPS or CLPS.

Theorem 1.59. An \( n \)-tuple of strategies \( \{u_i^*(s) = \phi_i^*(t, x) \in U^i, \text{for } i \in N \} \) provides a feedback Nash equilibrium solution to the game if there exist continuously differentiable functions \( V^i(t, x) : [t_0, T] \times \mathbb{R}^m \rightarrow \mathbb{R}, i \in N \), satisfying the following set of partial differential equations:
\[-V^i(t, x) = \max_{u_i} \{ J_i[t, x, \phi^1_1(s, x), \phi^1_2(s, x), \cdots, \phi^1_{i-1}(s, x), u_i(s, x), \phi^1_{i+1}(s, x), \cdots, \phi^1_n(s, x)] \\
+ V_x^i(t, x) f[t, x, \phi^1_1(s, x), \phi^1_2(s, x), \cdots, \phi^1_{i-1}(s, x), u_i(s, x), \phi^1_{i+1}(s, x), \cdots, \phi^1_n(s, x)] \} \]

\[V(T, x) = G^i(x), \quad i \in \mathbb{N}.\]

The proof may be found in [7] or [16].

1.5 Mean-Field Games

The aim of this section is to present in a simplified framework some of the ideas developed in the Mean Field Games area. The Mean Field Game theory in diffusion case is well studied in literature see [6], [13] and [4], we will leave the jump case to the papers after the introduction. It is not our intention to give a full picture of this fast growing area, but we will try to provide an approach as self content as possible.

The typical model for Mean Field Games (MFG) is the following system

\[i) \quad - \partial_t V - \nu \Delta V + H(x, \mu, D\mu) = 0 \]
\[ii) \quad \partial_t \mu - \nu \Delta \mu - \nabla(D_p H(x, \mu, D\mu)) = 0 \]

\[\mu(0) = \mu_0, \quad V(x, T) = G(x, \mu(T)), \tag{1.25}\]

where in the above system, \( \nu \) is a nonnegative parameter, \( V \) is the value function \( V : [0, T] \times \mathbb{X} \to \mathbb{R} \), \( \mu \) is the distribution function, and \( H \) is the Hamiltonian. The first equation has to be understood backward in time and the second one is forward in time. There are two crucial structure conditions for this system: the first one is the convexity of \( H = H(x, \mu, D\mu) \) with respect to the last variable. This condition implies that the first equation (a Hamilton-Bellman-Jacobi equation) is associated with an optimal control problem. This first equation shall be the value function associated with a typical small player. The second structure condition is that \( \mu_0 \) (and therefore \( \mu(t) \)) is (the density of) a probability measure.

The heuristic interpretation of this system is the following. An average agent controls the stochastic differential equation:

\[dX_t = a_t dt + \sqrt{2\nu} B_t \tag{1.26}\]

where \((B_t)\) is a standard Brownian motion. She aims at minimizing the quantity

\[E \left[ \int_0^T \frac{1}{2} J(s, X_s, \mu(s), V_s) ds + G(X_T, \mu(T)) \right]. \tag{1.27}\]

Note that in this cost the evolution of the measure \( \mu_s \) enters as a parameter. The value function of our average player is then given by (1.25(i)). Her optimal control
is, at least heuristically, given in feedback form by \( a^*(x; t) = -D_p H(x, \mu, DV) \). Now, if all agents argue in this way, their repartition will move with a velocity which is due, on the one hand, to the diffusion, and, one the other hand, on the drift term \(-D_p H(x, \mu, DV)\). This leads to the Kolmogorov equation (1.25(ii)).

The Mean Field Game theory developed so far has been focused on two main issues: first investigate equations of the form (1) and give an interpretation (in economics for instance) of such systems. Second analyze differential games with a finite but large number of players and link their limiting behavior as the number of players goes to infinity and equation (1).

1.5.1 Symmetric functions of many variables

Let \( X \) be a compact metric space and \( v_N : \mathbb{X}^N \rightarrow \mathbb{R} \) be a symmetric function. Let assume the following

1. There is some \( C > 0 \) such that
   \[
   \|v_N\|_{L^{\infty}(X)} \leq 0
   \]

2. There is a constant \( w \) independent of \( N \) such that
   \[
   |v_N(X) - v_N(Y)| \leq w(d_1(\mu^N_X, \mu^N_Y)), \quad \forall Y, X \in \mathbb{X}^N,
   \]
   where \( \mu^N_X = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} \) and \( \mu^N_Y = \sum_{i=1}^{N} \delta_{y_i} \).

**Theorem 1.60.** If the \( v_N \) are symmetric and satisfy the assumptions (1) and (2) above, then there is a subsequence \((v_{n_k})\) of \((v_N)\) and a continuous map \( U : \mathcal{P}(X) \rightarrow \mathbb{R} \) such that

\[
\lim_{k \rightarrow \infty} \sup_{X \in \mathbb{X}^N} |v_{n_k}(X) - U(\mu^{n_k}_X)| = 0
\]

For a proof of the above Theorem see [6].

1.5.2 Mean-Field Equation

\[
\begin{align*}
& i) \quad -\partial_t V + \frac{1}{2} |DV(x, t)|^2 = F(x, \mu(t)) \\
& ii) \quad \partial_t \mu - \nabla(DV(x, t) \mu(x, t)) = 0 \quad \mu(0) = \mu_0, \quad V(x, T) = G(x, \mu(T))
\end{align*}
\]

Our aim is to prove the existence of classical solutions for this system and give interpretation in term of game with finitely many players. Let us briefly recall the heuristic interpretation of this system: the map \( V \) is the value function of a typical agent who controls her velocity \( a(t) \) and has to minimize her cost

\[
\int_0^T \left( \frac{1}{2} |a(t)|^2 + F(x(t), \mu(t)) \right) dt + G(x(T), \mu(T))
\]
where \( x(t) = x_0 + \int_0^t a(s) ds \). Her only knowledge on the overall world is the distribution of the one-all agent, represented by the density \( \mu(t) \) of some probability measure. Then her feedback strategy i.e., the way she ideally controls at each time and at each point her velocity is given by \( a(x, t) = -DV(x, t) \). Now if all agents argue in this way, the density \( \mu(x, t) \) of their distribution \( \mu(t) \) over the space will evolve in time with the equation of conservation law (1.28 (ii)).

We then have to prove the existence and uniqueness of a fixed point. Starting from a given initial distribution (which is the agents anticipation of the overall players dynamics) \( \mu \). From this initial distribution each player uses a backward reasoning achieved by the Hamilton-Jacobi- Bellman equation to obtain his optimal strategy \( u \). These optimal strategies can be plugged into the forward Kolmogorov equation to know the actual dynamics of the overall community implied by individual behavior, which is the distribution \( \mu^* \). Finally the rational expectation hypothesis implies that there are coherent between the anticipated initial distribution \( \mu \) and the resulted one \( \mu^* \). This forward/backward procedure is the essence of the Mean Field Game theory in continuous time.

Let assume that all measures have a finite first order moment. Let \( \mathcal{P}_1(\mathbb{R}) \) be the set of such Borel probability measures \( \mu \) on \( \mathbb{R}^d \) such that \( \int_{\mathbb{R}^d} |x| d\mu(x) < \infty \). The set \( \mathcal{P}_1(\mathbb{R}) \) can be endowed with the following distances:

\[
d_1(\mu, \xi) = \inf_{\gamma \in \Pi(\mu, \xi)} \left[ \int |x - y| d\gamma(x, y) \right]
\]

where \( \Pi(\mu, \xi) \) is the set of Borel probability measures on \( \mathbb{R}^{2d} \) such that \( \gamma(A \times \mathbb{R}^d) = \mu(A) \) and \( \gamma(\mathbb{R}^d \times A) = \xi(A) \) for any set \( A \subset \mathbb{R}^d \). We assume that \( J : X \times \mathcal{P}_1(\mathbb{R}) \to \mathbb{R} \) and \( G : X \times \mathcal{P}_1(\mathbb{R}) \to \mathbb{R} \) are satisfying the following assumptions

1. \( J \) and \( G \) are uniformly bounded by \( C_0 \) over \( \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}) \)

2. \( J \) and \( G \) are Lipschitz continuous i.e. for all \( (x_1, \mu_1), (x_2, \mu_2) \in \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}) \) we have

\[
|F(x_1, \mu_1) - F(x_2, \mu_2)| \leq C_0 \left[ |x_1 - x_2| + d(\mu_1, \mu_2) \right]
\]

and

\[
|G(x_1, \mu_1) - G(x_2, \mu_2)| \leq C_0 \left[ |x_1 - x_2| + d(\mu_1, \mu_2) \right]
\]

3. Finally we suppose that \( \mu_0 \) is absolutely continuous, with a density still denoted \( \mu_0 \) which is Hölder continuous that satisfies \( \int_{\mathbb{R}^d} |x|^2 \mu_0(x) dx \).

A pair \( (V, \mu) \) is a classical solution to (1.28) if \( V, \mu : \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}) \to \mathbb{R} \) are continuous, of class \( C^2 \) in space and \( C^1 \) in time and \( (V, \mu) \) satisfies (1.28) in the classical sense.

The main result of this section is the following existence result:

**Theorem 1.61.** Under the above assumptions, there is at least one solution to (1.28)
Let us assume that, besides assumptions given at the beginning of the section, the following conditions hold:

\[ \int_X (F(x, \mu_1) - F(x, \mu_2))d(\mu_1 - \mu_2)(x) \geq 0 \quad \forall \mu_1, \mu_2 \in \mathcal{P}_1(\mathbb{R}), \mu_1 \neq \mu_2 \]

and

\[ \int_X (G(x, \mu_1) - G(x, \mu_2))d(\mu_1 - \mu_2)(x) \geq 0 \quad \forall \mu_1, \mu_2 \in \mathcal{P}_1(\mathbb{R}) \]

**Theorem 1.62.** Under the above conditions, there is a unique classical solution to the mean field equation (1.28)

**Remark 1.63.** The case that we treated above for \( \nu = 1 \) in equation (1.26) called the first order mean field equation. For \( \nu = 0 \) a second order equation appear and a solutions in viscosity and distributional sense to the Mean Field Game arise, for more details see [6].

### 1.5.3 Application to games with finitely many players

Let us assume that \((V, \mu)\) is a solution for the mean field equation (1.28) and let us investigate the optimal strategy for a representative player who counts the density \( \mu \) of the other players as given. She faces the following minimization problem

\[
\inf_a J(a), \quad J(a) = E \left[ \int_0^T \frac{1}{2} |a|^2 + F(X_s, \mu_s)ds + G(X_T, \mu_T) \right].
\]

where \( X_t = X_0 + \int_0^t a_s ds + \sqrt{2}B_s \), \( X_0 \) is a fixed random initial condition with law \( \mu_0 \) and the control \( a \) is adapted to some filtration \((\mathcal{F}_t)\). We assume that \( B_t \) is an \( d \)-dimensional Brownian motion adapted to the filtration \((\mathcal{F}_t)\) and that \( X_0 \) and \( (B_t) \) are independent. We claim that the feedback strategy \( \bar{a}(t, X_t) := -D_x V(t, x) \) is optimal for this optimal stochastic control problem.

**Lemma 1.64.** Let \((\bar{X}_t)\) be the solution of the stochastic differential equation

\[
d\bar{X}_t = \bar{a}(t, \bar{X}_t)dt + \sqrt{2}dB_t
\]

\[ \bar{X}_0 = X_0 \]

and \( \bar{a}(t) = \bar{a}(t, X_t) \). Then

\[
\inf_a J(a) = J(\bar{a}) = \int_{\mathbb{R}^N} V(0, x)d\mu_0(x).
\]

We will now look at differential game with \( N \) players. In this game a player \( i(i = 1, ..., N) \) is controlling through her control \( a' \) a dynamic of the form

\[
dX^i_t = a'_i dt + \sqrt{2}dB^i_t
\]
where $B_t^i$ is a $d$-dimensional Brownian motion. The initial condition $X_0^i$ for this system is also random and has for law $\mu_0$. We assume that the all $X_0^i$ and all the Brownian motions $B_t^i$, $i(i = 1, ..., N)$ are independent. However player $i$ can choose his control $a^i$ adapted to the filtration $(\mathcal{F}_t = \sigma(X_{0}^{i}, B^{i}_{s}, s \leq t, j = 1, ..., N))$. Her payoff is then given by

$$J^N_i(a^1, ...a^N) = E \left[ \int_0^T \frac{1}{2}|a_t^i|^2 + F \left( X_t^i, \frac{1}{N-1} \sum_{i \neq j} \delta X_t^j \right) ds + G \left( X_T^i, \frac{1}{N-1} \sum_{i \neq j} \delta X_T^j \right) \right]$$

Our aim is to explain that the strategy given by the Mean Field Game is suitable for this problem. More precisely, let $(u, m)$ be one classical solution to (1.28) and let us set $\bar{a}(t, x) = -D_x(t, x)$. With the closed loop strategy $\bar{a}$ one can associate the open-loop control $\hat{a}^i$ obtained by solving the SDE

$$dX_t^i = \bar{a}(t, X_t^i)dt + \sqrt{2}dB_t^i$$

with random initial condition $X_0^i$ and setting $\hat{a}^i_t = \bar{a}(t, X_t^i)$. Note that this control is just adapted to the filtration $(\mathcal{F}_t = \sigma(X_{0}^{i}, B^{i}_{s}, s \leq t))$ and not to the full filtration $(\mathcal{G}_t)$ defined above.

**Theorem 1.65.** For any $\epsilon > 0$ there is some $N_0$ such that, if $N \geq N_0$, then the symmetric strategy $(\bar{a}^1, ...\bar{a}^N)$ is an $\epsilon$-Nash equilibrium in the game $J^N_1, ..., J^N_N$.

Mathematically

$$J^N_i(\bar{a}^1, ..., \bar{a}^N) \leq J^N_i(\hat{a}^j_{\hat{a}^{i}}, a) + \epsilon$$

for any control $a$ adapted to the filtration $(\mathcal{F}_t)$ and any $i \in \{1, ..., N\}$.

For a proof of the Theorem see [6].

**References**


2 Summary of included papers

In this chapter we present a summary of the results of the thesis. Let us adopt the notations from the included papers and suppose that the assumptions in these papers hold.

**Paper I: A 1/n Nash equilibrium for non-linear Markov games of mean-field-type on finite state space**

In this paper, we extend the method of mean-field games approximation to finite state space settings. In particular, the time-inhomogeneous semi group and the controlled continuous time Markov chain are introduced. We set up the limiting dynamics, the kinetic equation. Moreover, we derive explicitly the generator of the Koopman propagator

$$A(t, x, u)F(x) = \sum_{i=1}^{k} \left( \frac{\partial F}{\partial x_i} a^*_i(t, x)ux_i \right),$$

where $a^*_i(t, x)$ corresponds to row $i$ of the matrix valued function $A^*[t, x]$, $u$ is the control parameter, and $F$ is a functional on the space of finite measures. The control problem is discussed in the linear quadratic framework. Smooth dependence of the solutions of the HJB equation on the functional vector valued parameter $y$ is presented. Furthermore, the limit when the number of players approach infinity is investigated. In addition, the bounds for error estimation are derived, namely

$$\left| (\psi_{N,\gamma}^0 F)(x^N_0) - (\phi_{\gamma}^0 F)(x_0) \right| \leq \frac{C(T)}{N} \left( T \| F \|_{C^2(M)} + k_1 \right).$$

Finally, a tagged player deviating from the optimal strategy is introduced and the approximate Nash equilibrium is established. We present the final result as follow:

**Theorem 2.1.** Let $\{A(t, j, y, u) \mid t \geq 0, j \in \mathcal{X}, y \in M, u \in \mathcal{U}\}$ be the family of jump type operators given in (2.1) and $x$ be the solution to equation (3.2). Assume the following

i) The kernel $\nu(t, j, y, u)$ satisfies the Hypotheses $A$ and $B$;

ii) The time-dependent Hamiltonian $H_t$ is of the form (4.8);

iii) The terminal function $V_T$ is in $C^1_\infty(\mathcal{X} \times \mathbb{R}^k)$.

iv) The initial conditions $x^N_0$ of an $N$ players game converge to $x_0$ in $\mathbb{R}^k$ in a way that (5.5) is satisfied and (6.5) holds.

Then the strategy profile $u = \Gamma(t, x^N_0, \alpha(0, t, x^N_0))$, defined via HJB (4.4) and (4.8) is an $\epsilon$-Nash equilibrium in a $N$ players game, with

$$\epsilon = \frac{C(T)}{N} \left( \| J \|_{C(\mathcal{U})} + \| J^* \|_2 + \| V^T \|_{C^2(\mathcal{X} \times \mathbb{R}^k)} + 1 \right).$$

**Paper II: An Approximate Nash Equilibrium for Pure Jump Markov Games of Mean-field-type on Continuous State Space**

In this paper, we generalize our results of the first paper to the case of a continuous state space. In contrast to the first paper, the operator $A$, which is contained
in the kinetic equation, is the time-dependent integral operator. The generator of the Koopman propagator in this case takes the form
\[ A[t, \mu]F(\mu) = \int_{\mathbb{R}^d} A[t, \mu_x](\delta_{\mu_x} F) \mu_t(dx), \]
where \( \delta_{\mu_x} F \) is the variational derivative of the functional \( F \) on the space of finite measures. The control problem for a representative player is studied in the general sense. Moreover, under specific assumptions on the space \( U \) and both the cost function \( J \) and the integral operator \( A \), the feedback regularity feature of the unique optimal control is derived. An argument leading to the consistency condition and existence of the fixed point in the space of flows of probability measure in the weak topology is presented. Over and above, a weak form of the law of large number with rigorous estimate for the error term is displayed
\[ \left| \left( \psi_{N, t}^{0} \right)(\mu_N^n) - \left( \phi_{\gamma}^{0, t} \right)(\mu_0) \right| \leq \frac{C(T)}{N} \left( T \| F \|_{C^2(\mathcal{A})} + k_1 \right) \]
with a constant \( C(T) \) independent on \( \gamma \). Finally, after introducing the tagged player and preforming the optimization, a special order of convergence to the mean-field limit is constructed and the approximate Nash equilibrium is established with
\[ \epsilon = \left( \frac{1}{N^{1/(2+d)}} + \epsilon(N) \right) \left( \| J \|_{C^2([0,T] \times U, \mathcal{C}^{0,2}(\mathbb{R}^d \times \mathcal{A}))} + \| V_T \|_{C^{2,2}(\mathbb{R}^d \times \mathcal{A})} + 1 \right). \]