On a class of commutative algebras associated to graphs

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Abstract

In 2004 Alexander Postnikov and Boris Shapiro introduced a class of commutative algebras for non-directed graphs. There are two main types of such algebras, algebras of the first type count spanning trees and algebras of the second type count spanning forests. These algebras have a number of interesting properties including an explicit formula for their Hilbert series.

In this thesis we mainly work with the second type of algebras, we discover more properties of the original algebra and construct a few generalizations. In particular we prove that the algebra counting forests depends only on graphical matroid of the graph and converse. Furthermore, its "K-theoretic" filtration reconstructs the whole graph. We introduce $t$-labelled algebras of a graph, their Hilbert series contains complete information about the Tutte polynomial of the initial graph.

Finally we introduce similar algebras for hypergraphs. To do this, we define spanning forests and trees of a hypergraph and the corresponding "hypergraphical" matroid.
List of Papers

The following papers are included in this thesis.

**PAPER I:** On Postnikov-Shapiro Algebras and their generalizations

**PAPER II:** "K-theoretic" analog of Postnikov-Shapiro algebra distinguishes graphs
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1. Introduction

The main object of study in this thesis is square-free commutative algebras \( C_G \) and \( C^T_G \) associated to an arbitrary graph \( G \) and as well their generalizations. These algebras were defined by Alexander Postnikov and Boris Shapiro in [9] and have a number of interesting properties. In particular, their total dimension is the number of spanning forests/trees; moreover there are explicit formulas for their Hilbert series which are specializations of the Tutte polynomial of \( G \).

This topic was started in an earlier paper [10], where the algebra \( C_A \) was introduced for a vector configuration \( A \); we present this definition in subsection 1.3.

Below we always assume that all graphs are undirected, without loops, but might have multiple edges. We also fix a field \( \mathbb{K} \) of zero characteristic, for example \( \mathbb{R} \) or \( \mathbb{C} \). For a graph \( G \), by \( X \subset G \) we understand a subset of edges of \( G \) and by \(|X|\) we denote the number of edges of this subset.

\( F \subset G \) is called a spanning forest if it has no cycles, and a spanning tree if it is a spanning forest of size \(|F| = v(G) - 1\), where \( v(G) \) is the number of vertices of \( G \).

In this thesis we use a number of classical graph-theoretical notions such as the Tutte polynomial of a graph and the graphical matroid.

Algebras \( C_G \) and \( C^T_G \) count spanning forests and trees resp. To describe their Hilbert series, we need to recall definitions of the Tutte polynomial of \( G \) and of an external activity. Fix an arbitrary linear order of edges of \( G \). Now, given a spanning forest \( F \) in \( G \) and an edge \( e \in G \setminus F \) in its complement, we say that \( e \) is externally active for \( F \), if there exists a cycle \( C \) in \( F \cup e \) such that \( e \) is minimal in \( C \) with respect to the chosen linear order. The total number \( \text{act}(F) \) of externally active edges is called the external activity of \( F \). For a spanning tree \( T \), an edge \( e \in T \) is called internally active for \( T \) if there is a cut \( C \) (i.e., \( G \setminus C \) is disconnected) such that \( C \cap T = \{e\} \) and \( e \) is the minimal edge in \( C \). The total number of internally active edges is called the internal activity of \( T \).

The original definition of the Tutte polynomial \( T_G \) for a connected graph \( G \) is

\[
T_G(x,y) := \sum_{i,j} t_{ij} x^i y^j,
\]

where \( t_{ij} \) denotes the number of spanning trees of internal activity \( i \) and external activity \( j \). For a disconnected graph \( G \), its Tutte polynomial is the product
of the Tutte polynomials corresponding to its connected components.

Although the external/internal activity of a given forest/tree in $G$ depends on the choice of a linear ordering of edges, it is well known that the total number of forests/trees with a given size and external/internal activity is independent of this ordering. In other words, the Tutte polynomial depends only on $G$. There are other definitions of the Tutte polynomial: such as deletion-contraction recurrence, Whitney rank generating function etc. (See more information about this polynomial in any classic graph theory book, for example, in Tutte’s book [12] or in [4].)

Algebras counting spanning forests store substantial information about the underlying graph. Namely, they are in 1-1-correspondence with graphical matroids of graphs, see Theorems 1.4. A matroid $M$ is a pair $(E, I)$, where $E$ is a ground set and $I$ is a family of independent subsets of $E$. Here a ground set is finite and independent subsets have the following properties:

- The empty set is independent;
- Every subset of an independent set is independent;
- If $A$ and $B$ are two independent sets and $|A| > |B|$, then there exists an element $e$ in $A$ such that $B \cup \{e\}$ is also independent.

For a graph $G$, the ground set of its graphical matroid is the set of all edges of $G$ and independent subsets are spanning forests of $G$. We will also use vector matroids, where a ground set consists of vectors in $\mathbb{K}^n$ and independent subsets are just linearly independent subsets of these vectors. (See more information about matroids and their Tutte polynomials in [8], [4].)

Almost all algebras below have two types of definitions. The first definition as a subalgebra of an algebra and the second definition as a quotient algebra of the polynomial ring, see for example Theorems 1.2 and 1.6. The second definition is more interesting, however it is easier to work with the first definition. For example, it is faster to calculate the Hilbert series on computer or by hands using the first definition. Generalizations of algebras of second type (here denoted by $\mathcal{B}$) see in papers [2] (inside denoted by $\mathcal{C}$) and [5]. We focus here more on algebras of first type (here denoted by $\mathcal{C}$) associated to graphs and to hypergraphs.

Our thesis has the following structure.

In Section 1 we present definitions and properties of algebras $\mathcal{C}_G$, $\mathcal{C}_T^G$ and $\mathcal{C}_V$, see subsections 1.1, 1.2 and 1.3 resp. Section 1 is based on papers [2] and [10] and contains two new statements, which show that algebras $\mathcal{C}_G$ and $\mathcal{C}_T^G$ correspond to the graphical matroid and the bridge-free matroid resp. Theorem 1.4 and Proposition 1.8 were proven in [1].
In Sections 2-4 we present new generalizations of the above algebras, which were introduced in papers [I] and [II]. Namely, in Section 2 we define the 'K-theoretic' filtration of algebras, which distinguishes graphs in case of spanning forests. In Section 3 we present what happens if we change the definition of a square-free algebra \( \Phi_G \) by a cubic-free etc. In Section 4 we define an algebra counting spanning forests for a hypergraph. For this, we also need to define spanning trees/forests for a hypergraph, the hypergraphical matroid and the corresponding Tutte polynomial.

1.1 Algebras \( \mathcal{C}_G \) and \( \mathcal{B}_G \)

Let \( G \) be a graph without loops on the vertex set \( \{0,...,n\} \). Let \( \Phi_G \) be the graded commutative algebra over \( \mathbb{K} \) generated by the variables \( \phi_e, e \in G \), with the defining relations:

\[
(\phi_e)^2 = 0, \quad \text{for any edge } e \in G.
\]

Let \( \mathcal{C}_G \) be the subalgebra of \( \Phi_G \) generated by the elements

\[
X_i = \sum_{e \in G} c_{i,e} \phi_e,
\]

for \( i = 1,...,n \), where

\[
c_{i,e} = \begin{cases} 
1 & \text{if } e = (i,j), i < j; \\
-1 & \text{if } e = (i,j), i > j; \\
0 & \text{otherwise.} 
\end{cases}
\]

We call \( \mathcal{C}_G \) the spanning forests counting algebra of \( G \).

**Example 1.1** Consider the complete graph \( K_3 \) on vertices \( \{0,1,2\} \) and with edges \( \{a,b,c\} \), see Fig. 1.1.

Elements corresponding to the vertices are

\[
X_0 = \phi_a + \phi_c, \\
X_1 = -\phi_a + \phi_b, \\
X_2 = -\phi_b - \phi_c.
\]

Note that the sum of elements corresponding to all vertices vanishes. Therefore algebras generated by elements \( X_i, i \in \{1,...,n\} \) or by elements \( X_i, i \in \{0,...,n\} \) coincide.

Our algebra has a natural graded structure and its graded components are
Figure 1.1: $K_3$

- $\mathbb{K} = \text{span}\{1\}, \ dim = 1$;
- $\text{span}\{\phi_a - \phi_b, \ \phi_a + \phi_c\}, \ dim = 2$;
- $\text{span}\{\phi_a\phi_b, \ \phi_b\phi_c, \ \phi_a\phi_c\}, \ dim = 3$;
- $\text{span}\{\phi_a\phi_b\phi_c\}, \ dim = 1$.

Its Hilbert series is $1 + 2t + 3t^2 + t^3$, and its total dimension (as linear space over $\mathbb{K}$) is 7, which is exactly the number of spanning forests of $K_3$.

It is clear that for a disconnected graph $G$, the algebra $\mathcal{C}_G$ is the Cartesian product of algebras corresponding to the connected components of $G$. Furthermore, the following stronger statement holds. Define a block of $G$ as its biconnected component. In other words, a block is a maximal subgraph (induced by a subset of vertices) such that it remains connected whenever a vertex is removed.

**Proposition 1.1** Given a graph $G$, the algebra $\mathcal{C}_G$ is the Cartesian product of algebras corresponding to all blocks of $G$.

Its Hilbert series and the set of defining relations were calculated in [10]. Namely, let $J_G$ be the ideal in $\mathbb{K}[x_1, \ldots, x_n]$ generated by the polynomials

$$p_I = \left( \sum_{i \in I} x_i \right)^{D_I+1},$$

(1.2)

where $I$ ranges over all nonempty subsets in $\{1, \ldots, n\}$ and $D_I$ is the total number of edges between $I$ and the complementary set $\bar{I}$. Set $\mathcal{B}_G := \mathbb{K}[x_1, \ldots, x_n]/J_G$.

The algebra $\mathcal{B}_G$ is independent of the choice of the vertex 0, because if we consider the same definition including the vertex 0, we have an ideal in $\mathbb{K}[x_0, \ldots, x_n]$ generated by $p_I$ for $I \subset \{0, 1, \ldots, n\}$. In particular, we will have
the generator \( p_{\{0,1,\ldots,n\}} = x_0 + \ldots + x_n \), i.e., we can assume that \( x_0 = -(x_1 + \ldots + x_n) \).

It is clear that, for the algebra \( \mathcal{C}_G \), relations from the ideal \( J_G \) hold. Indeed

\[
\sum_{i \in I} X_i = \sum \pm \phi_e,
\]

where the latter sum is taken over all edges which have exactly one end in \( I \), i.e., their number equals \( D_I \). The sum raised to power \( D_I + 1 \) vanishes. So to show the isomorphism, it is enough to show that algebras \( \mathcal{C}_G \) and \( \mathcal{B}_G \) have the same dimension.

**Example 1.2** For the graph \( K_3 \), the ideal is

\[
J = \langle x_1^3, x_2^3, (x_1 + x_2)^3 \rangle.
\]

Then by definition, the algebra \( \mathcal{B}_\Delta \) is given by:

\[
\mathcal{B}_\Delta = \mathbb{K}[x_1, x_2]/J = \mathbb{K}[x_1, x_2]/\langle x_1^3, x_2^3, (x_1 + x_2)^3 \rangle.
\]

It is easy to see that its Hilbert series equals

\[
HS(t) = 1 + 2t + 3t^2 + t^3.
\]

For the case of \( K_3 \), we know that \( \dim(\mathcal{B}_\Delta) = \dim(\mathcal{C}_\Delta) = 7 \). Therefore \( \mathcal{C}_\Delta \) and \( \mathcal{B}_\Delta \) are isomorphic, because we always have a natural surjective homomorphism from \( \mathcal{B} \) to \( \mathcal{C} \).

For any graph, this isomorphism was proven in [9] (in fact it is a corollary of results of paper [10]).

**Theorem 1.2** For any simple graph \( G \), algebras \( \mathcal{B}_G \) and \( \mathcal{C}_G \) are isomorphic. The total dimension of these algebras (as vector spaces over \( \mathbb{K} \)) is equal to the number of spanning forests in \( G \). The dimension of the \( k \)-th graded component of these algebras equals the number of spanning forests \( F \) in \( G \) with external activity \( |G| - |F| - k \).

**Corollary 1.3** Given a graph \( G \), the Hilbert series of the algebra \( \mathcal{C}_G \) is given by

\[
HS_{\mathcal{C}_G}(t) = T_G \left( 1 + t, \frac{1}{t} \right) \cdot t^{|G| - v(G) + c(G)},
\]

where \( c(G) \) is the number of connected components of \( G \).
Example 1.3 For the graph $K_3$, its Tutte polynomial is
\[ T_\Delta(x,y) = x^2 + x + y. \]

After substitution we get
\[ T_\Delta\left(1 + t, \frac{1}{t}\right) \cdot t^{3-3+1} = \left(\frac{1}{t} + 2 + 3t + t^2\right) \cdot t = 1 + 2t + 3t^2 + t^3, \]
which is exactly the Hilbert series of $\mathcal{C}_\Delta$ and $\mathcal{B}_\Delta$.

Let $a < b < c$ be a linear order of edges. Then the spanning forests of $K_3$ have the following activities:

- $\text{act}(\emptyset) = 0$, then $|G| - |F| - \text{act}(F) = 3$;
- $\text{act}(\{a\}) = 0$, then $|G| - |F| - \text{act}(F) = 2$;
- $\text{act}(\{b\}) = 0$, then $|G| - |F| - \text{act}(F) = 2$;
- $\text{act}(\{c\}) = 0$, then $|G| - |F| - \text{act}(F) = 2$;
- $\text{act}(\{a, b\}) = 0$, then $|G| - |F| - \text{act}(F) = 1$;
- $\text{act}(\{a, c\}) = 0$, then $|G| - |F| - \text{act}(F) = 1$;
- $\text{act}(\{b, c\}) = 1$, then $|G| - |F| - \text{act}(F) = 0$.

In other words, 0 occurs once, 1 occurs 2 times, 2 occurs 3 times and 3 occurs once, which gives the sequence $(1, 2, 3, 1)$. It is again exactly the Hilbert series of $\mathcal{C}_\Delta$ (or $\mathcal{B}_\Delta$).

The following important properties of algebras counting spanning forests were found by the author in paper [I].

**Theorem 1.4** Given two graphs $G_1$ and $G_2$, algebras $\mathcal{C}_{G_1}$ and $\mathcal{C}_{G_2}$ are isomorphic if and only if their graphical matroids are isomorphic. (The algebraic isomorphism can be thought of either as graded or as non-graded, the statement holds in both cases.)

Theorem 1.4 means that algebras $\mathcal{C}_G$ and $\mathcal{B}_G$ remember a lot of information about $G$. For example, if $G$ is 3-connected (i.e. it remains connected after deletion of any 2 vertices), then the algebra remembers the whole graph. However if $G$ is not 3-connected it is not always true. In fact, it is true up to 2-isomorphisms of graphs, see the definition in [13] and example of such graphs below. In addition, Theorem 1.4 implies that if $\mathcal{C}_{G_1}$ and $\mathcal{C}_{G_2}$ are isomorphic as non-graded algebras, then they are also isomorphic as graded algebras.
Consider the graphs $G_1$ and $G_2$, see Fig. 1.2. It is clear that they have isomorphic graphical matroids. It means that algebras $B_{G_1}$ and $B_{G_2}$ (in $C_{G_1}$ and $C_{G_2}$) should be isomorphic by Theorem 1.4. Let us check it.

Let $\mathbb{K}[x_1, x_2, x_3]$ be the polynomial ring for $G_1$. Then the ideal $\mathcal{I}_{G_1}$ is given by
\[
\mathcal{I}_{G_1} = \langle x_1^4, x_2, x_3, (x_1 + x_2)^5, (x_2 + x_3)^3, (x_1 + x_3)^7, (x_1 + x_2 + x_3)^4 \rangle = \langle x_1^4, x_2, x_3, (x_1 + x_2)^5, (x_2 + x_3)^3, (x_1 + x_2 + x_3)^4 \rangle,
\]
because $(x_1 + x_3)^7 \in \langle x_1^4, x_2^3 \rangle$.

Let $\mathbb{K}[\tilde{x}_1, \tilde{x}_2, \tilde{x}_3]$ be the polynomial ring for $G_2$. Then the ideal $\mathcal{I}_{G_2}$ is given by
\[
\mathcal{I}_{G_2} = \langle \tilde{x}_1^5, \tilde{x}_2^4, \tilde{x}_3^2, (\tilde{x}_1 + \tilde{x}_2)^4, (\tilde{x}_2 + \tilde{x}_3)^4, (\tilde{x}_1 + \tilde{x}_3)^7, (\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3)^4 \rangle = \langle \tilde{x}_1^5, \tilde{x}_2^4, \tilde{x}_3^2, (\tilde{x}_1 + \tilde{x}_2)^4, (\tilde{x}_2 + \tilde{x}_3)^4, (\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3)^4 \rangle,
\]
because $(\tilde{x}_1 + \tilde{x}_3)^7 \in \langle \tilde{x}_1^5, \tilde{x}_2^3 \rangle$.

Consider the ring isomorphism $\psi : \mathbb{K}[x_1, x_2, x_3] \to \mathbb{K}[\tilde{x}_1, \tilde{x}_2, \tilde{x}_3]$, defined by:

\begin{itemize}
  \item $\psi(x_1) = -\tilde{x}_1 - \tilde{x}_2$;
  \item $\psi(x_2) = \tilde{x}_2$;
  \item $\psi(x_3) = -\tilde{x}_2 - \tilde{x}_3$.
\end{itemize}

Let us see how $\psi$ acts on the ideal $\mathcal{I}_{G_1}$
\[
\psi(\mathcal{I}_{G_1}) = \langle (-\tilde{x}_1 - \tilde{x}_2)^4, \tilde{x}_2^4, (\tilde{x}_2 - \tilde{x}_3)^4, (\tilde{x}_1)^5, (\tilde{x}_3)^3, (-\tilde{x}_1 - \tilde{x}_3)^4 \rangle = \mathcal{I}_{G_2}.
\]

Then we get
\[
\psi(B_{G_1}) = \psi(\mathbb{K}[x_1, x_2, x_3]/\mathcal{I}_{G_1}) = \mathbb{K}[\tilde{x}_1, \tilde{x}_2, \tilde{x}_3]/\mathcal{I}_{G_2} = B_{G_2},
\]
hence algebras $B_{G_1}$ and $B_{G_2}$ are isomorphic.
1.2 Algebras $\mathcal{C}_G^T$ and $\mathcal{B}_G^T$

It is not surprising that to construct algebras counting spanning trees we need to add some relations to $\Phi_G$ corresponding to cuts of $G$. However, Theorem 1.6 which was proven by A. Postnikov and B. Shapiro is not a corollary of paper [10]. Nevertheless algebras counting spanning trees have many similar properties. At first we define $\mathcal{C}_G^T$; the structure of this subsection coincides with the structure of subsection 1.1.

Let $G$ be a graph without loops on the vertex set $\{0, \ldots, n\}$. Let $\Phi_G^T$ be the graded commutative algebra over $\mathbb{K}$ generated by the variables $\phi_e, e \in G$, with the defining relations:

$$(\phi_e)^2 = 0, \quad \text{for any edge } e \in G,$$

$$\prod_{e \in C} (\phi_e) = 0, \quad \text{for any cut } C \subset G.$$

Let $\mathcal{C}_G^T$ be the subalgebra of $\Phi_G^T$ generated by the elements

$$X_i = \sum_{e \in G} c_{i,e}\phi_e,$$

for $i = 1, \ldots, n$, where $c_{i,e}$ as in 1.1.

We call $\mathcal{C}_G^T$ the spanning trees counting algebra of $G$.

**Example 1.5** Consider the complete graph $K_3$ on vertices $\{0,1,2\}$ and edges $\{a,b,c\}$, see Fig. 1.1. Elements corresponding to vertices are

$$X_0 = \phi_a + \phi_c,$$

$$X_1 = -\phi_a + \phi_b,$$

$$X_2 = -\phi_b - \phi_c.$$

Our algebra has natural graded structure and its graded components are

- $\mathbb{K} = \text{span}\{1\}, \dim = 1$;
- $\text{span}\{\phi_a - \phi_b, \phi_a + \phi_c\}, \dim = 2$.

The Hilbert series is $1 + 2t$, and its total dimension is 3, which is exactly the number of spanning trees of $K_3$. 

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It is clear that for a disconnected graph $G$, the algebra $\Phi^T_G$ is zero, because empty set of edges is a cut. So $C^T_G$ is non-trivial only when $G$ is connected. In this case we have an analog of Proposition 1.1. Define a 2-edge connected component of $G$ as a maximal subgraph (induced by a subset of vertices) such that it remains connected whenever an edge is removed.

**Proposition 1.5** For a connected graph $G$, the algebra $C^T_G$ is the Cartesian product of algebras corresponding to all 2-edge connected components of $G$.

Its Hilbert series and the set of defining relations were calculated in [9]. Namely, let $J^T_G$ be the ideal in $\mathbb{K}[x_1, \ldots, x_n]$ generated by the polynomials

$$p^T_I = \left(\sum_{i \in I} x_i\right)^{D_I}, \quad \text{(1.3)}$$

where $I$ ranges over all nonempty subsets in $\{1, \ldots, n\}$ and $D_I$ is the total number of edges between $I$ and the complementary set $\overline{I}$.

Set $B^T_G := \mathbb{K}[x_1, \ldots, x_n]/J^T_G$.

The algebra $B^T_G$ is independent of the choice of the vertex 0, because we again can consider the same definition with the vertex 0. However now we need the relation $x_0 + \ldots + x_n = 0$ instead $p_{\{0, \ldots, n\}} = (x_0 + \ldots + x_n)^0$.

It is clear that for the algebra $C^T_G$, relations from the ideal $J^T_G$ hold. Indeed,

$$\sum_{i \in I} X_i = \sum \pm \phi_e,$$

where the latter sum is taken over all edges which have exactly one end in $I$, i.e., their number is $D_I$. Since it is a cut, in power $D_I$ it should be zero. So to show the isomorphism, it is again enough to prove that algebras $C^T_G$ and $B^T_G$ have the same dimension.

**Example 1.6** For the graph $K_3$, the ideal is

$$J^T = <x_1^2, x_2^2, (x_1 + x_2)^2>.$$  

Then $B^T_\Delta$ is given by:

$$B^T_\Delta = \mathbb{K}[x_1, x_2]/J^T = \mathbb{K}[x_1, x_2]/<x_1^2, x_2^2, (x_1 + x_2)^2>.$$  

It is easy to see that its Hilbert series is

$$HS(t) = 1 + 2t.$$  

For the case of $K_3$, we know that $\dim(B^T_\Delta) = \dim(C^T_\Delta) = 3$, implying that they are isomorphic, because we have a natural surjective homomorphism from $B^T_G$ to $C^T_G$.  

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The following theorem shows it for a general graph; it is the main result of paper [9].

**Theorem 1.6** For any simple graph $G$, algebras $B^T_G$ and $C^T_G$ are isomorphic. The total dimension of these algebras (as vector spaces over $\mathbb{K}$) is equal to the number of spanning trees in $G$. The dimension of the $k$-th graded components of these algebras equals the number of trees $T$ in $G$ with external activity $|G| - (v(G) - 1) - k$.

**Corollary 1.7** Given a graph $G$, the Hilbert series of the algebra $C^T_G$ is given by

$$HS_{C^T_G}(t) = T_G \left( 1, \frac{1}{t} \right) \cdot t^{|G| - v(G) + c(G)},$$

where $c(G)$ is the number of connected components of $G$.

**Example 1.7** For the graph $K_3$, its Tutte polynomial is

$$T_\Delta(x, y) = x^2 + x + y.$$  

After substitution we get

$$T_\Delta \left( 1, \frac{1}{t} \right) \cdot t^{3-3+1} = \left( \frac{1}{t} + 2 \right) \cdot t = 1 + 2t,$$

which is exactly the Hilbert series of $C^T_\Delta$ and $B^T_\Delta$.

Let $a < b < c$ be a linear order of edges. Then the spanning trees have the following activities.

- $act(\{a, b\}) = 0$, then $|G| - 2 - act(T) = 1$;
- $act(\{a, c\}) = 0$, then $|G| - 2 - act(T) = 1$;
- $act(\{b, c\}) = 1$, then $|G| - 2 - act(T) = 0$.

The following important property of algebras counting spanning trees was found by the author in [I]. Edge $e$ is called a bridge if graph $G - e$ has more connected components than $G$. Define bridge-free matroid of graph $G$ as graphical matroid of graph $G'$, where $G'$ is $G$ after deleting of all its bridges. The algebra is independent of bridges, because any bridge $e$ is cut and so we have the relation $\phi_e = 0$.

**Proposition 1.8** Given two graphs $G_1$ and $G_2$ with a bridge-free matroids, then algebras $C^T_{G_1}$ and $C^T_{G_2}$ ($B^T_{G_1}$ and $B^T_{G_2}$) are isomorphic as graded algebras.
We think that the converse is also true.

**Conjecture 1.1** Given two graphs $G_1$ and $G_2$, algebras $C^T_{G_1}$ and $C^T_{G_2}$ are isomorphic if and only if bridge-free matroids corresponding to $G_1$ and $G_2$ are isomorphic.

**Example 1.8** Consider the graphs $G_1$ and $G_2$, see Fig. 1.2. It is clear that they have isomorphic matroids. It means that by Theorem 1.8 they should be isomorphic. Let us check it.

Let $\mathbb{K}[x_1, x_2, x_3]$ be the polynomial ring for $G_1$, then the ideal $\mathcal{J}_{G_1}$ is given by

$$\mathcal{J}_{G_1} = <x_1^3, x_2^3, (x_1 + x_2)^4, (x_2 + x_3)^2, (x_1 + x_3)^6, (x_1 + x_2 + x_3)^3> = <x_1^3, x_2^3, x_3^3, (x_1 + x_2)^4, (x_2 + x_3)^2, (x_1 + x_2 + x_3)^3>,$$

because $(x_1 + x_3)^6 \in <x_1^3, x_2^3>$.

Let $\mathbb{K}[\tilde{x}_1, \tilde{x}_2, \tilde{x}_3]$ be the polynomial ring for $G_2$, then the ideal $\mathcal{J}_{G_2}$ is given by

$$\mathcal{J}_{G_2} = <\tilde{x}_1^4, \tilde{x}_2^3, (\tilde{x}_1 + \tilde{x}_2)^3, (\tilde{x}_2 + \tilde{x}_3)^3, (\tilde{x}_1 + \tilde{x}_3)^6, (\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3)^3> = <\tilde{x}_1^4, \tilde{x}_2^3, \tilde{x}_3^2, (\tilde{x}_1 + \tilde{x}_2)^3, (\tilde{x}_2 + \tilde{x}_3)^3, (\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3)^3>,$$

because $(\tilde{x}_1 + \tilde{x}_3)^6 \in <\tilde{x}_1^4, \tilde{x}_2^3>$.

Consider the ring isomorphism $\psi : \mathbb{K}[x_1, x_2, x_3] \rightarrow \mathbb{K}[\tilde{x}_1, \tilde{x}_2, \tilde{x}_3]$, defined by:

- $\psi(x_1) = -\tilde{x}_1 - \tilde{x}_2$;
- $\psi(x_2) = \tilde{x}_2$;
- $\psi(x_3) = -\tilde{x}_2 - \tilde{x}_3$.

Let us see how $\psi$ acts on the ideal $\mathcal{J}_{G_1}$.

$$\psi(\mathcal{J}_{G_1}) = <(-\tilde{x}_1 - \tilde{x}_2)^3, \tilde{x}_2^3, (-\tilde{x}_2 - \tilde{x}_3)^3, (-\tilde{x}_1)^4, (-\tilde{x}_3)^2, (-\tilde{x}_1 - \tilde{x}_3)^3> = \mathcal{J}_{G_2}.$$

Then we get

$$\psi(\mathcal{B}_{G_1}) = \psi(\mathbb{K}[x_1, x_2, x_3]/\mathcal{J}_{G_1}) = \mathbb{K}[\tilde{x}_1, \tilde{x}_2, \tilde{x}_3]/\mathcal{J}_{G_2} = \mathcal{B}_{G_2},$$

hence algebras $\mathcal{B}_{G_1}$ and $\mathcal{B}_{G_2}$ are isomorphic.
1.3 Algebras corresponding to vector configurations

The following algebra was introduced by A. Postnikov, B. Shapiro and M. Shapiro in [10].

Given a finite set \( A = \{a_1, \ldots, a_m\} \) of vectors in \( \mathbb{K}^n \), let \( \Phi_A \) be the commutative algebra over \( \mathbb{K} \) generated by \( \{\phi_i : i \in [1,m]\} \) with relations \( \phi_i^2 = 0 \), for each \( i \in [1,m] \).

For \( i = 1, \ldots, n \), set \( X_i = \sum_{k \in [1,m]} a_k, i \phi_k \). Denote by \( \mathcal{C}_A \) the subalgebra of \( \Phi_A \) generated by \( X_1, \ldots, X_n \). Its Hilbert series was calculated in paper [10].

**Theorem 1.9** The dimension of the algebra \( \mathcal{C}_A \) is equal to the number of independent subsets in \( V \). Moreover, the dimension of the \( k \)-th graded component is equal to the number of independent subsets \( S \subset A \) such that \( k = m - |S| - \text{act}(S) \).

Here \( \text{act}(S) \) is the external activity of subset \( S \). Its definition is the same as for the activity of spanning forests, i.e. the number of elements \( e \) such that \( S \cup e \) is not independent and \( e \) is the minimal element among elements with nonzero coefficients in this linear dependence. Theorem 1.9 means that the Hilbert series of \( \mathcal{C}_A \) is also a specialization of the Tutte polynomial of the corresponding vector matroid. It is easy to understand this if one uses the definition of the Tutte polynomial via Whitney rank generating function.

**Corollary 1.10** Given a vector configuration \( A \) in \( \mathbb{K}^n \), the Hilbert series of the algebra \( \mathcal{C}_A \) is

\[
HS_{\mathcal{C}_A}(t) = T_A \left( 1 + t, \frac{1}{t} \right) \cdot t^{|A| - \text{rk}(A)},
\]

where \( T_A \) is the Tutte polynomial corresponding to \( A \), \( |A| \) is the number of vectors in the configuration and \( \text{rk}(A) \) is the dimension of the linear span of these vectors.

The algebra \( \mathcal{C}_G \) counting forests of \( G \) is a particular case of algebras corresponding to vector configurations. Indeed, let \( G \) be a graph on \( n \) vertices with edges \( e_1 = (v_{b_1}, v_{c_1}), \ldots, e_m = (v_{b_m}, v_{c_m}) \). Consider the vector configuration \( A = \{a_1, \ldots, a_m\} \), where

\[
a_i = (0, \ldots, 0, 1, 0, \ldots, 0, -1, 0, \ldots, 0).
\]

By definitions algebras \( \mathcal{C}_G \) and \( \mathcal{C}_A \) coincide. Their matroids (graphical and vector) are also isomorphic, and, then the Tutte polynomials are the same.
There is again a definition as a quotient algebra (type $\mathcal{B}$), see information about this definition in [10].

One would like to get results similar to Theorem 1.4 for vector configurations, however it is impossible. Indeed, since the set of different vector configurations depends on continuous parameters, there are uncountably many non-isomorphic algebras. At the same time, the number of matroids is countable and, furthermore, this number is finite for a fixed number of vectors. It means that there are many different vector configurations with the same corresponding matroid. In other words, it is in principle impossible to reconstruct a vector configuration and its algebra from the corresponding vector matroid.
2. “K-theoretic” filtration

(based on paper [II])

This section deals with a "K-theoretic" filtered algebra \( \mathcal{K}_G \), which was suggested by A. Kirillov and B. Shapiro. The section is based on paper [II], which was written by the author jointly with B. Shapiro. Here we study a filtered "K-theoretical" analog of algebras \( \mathcal{C}_G \) and \( \mathcal{C}_T^G \).

Generators of the algebra \( \mathcal{K}_G \) (and \( \mathcal{K}_T^G \)) are non-homogeneous. We can consider its filtration \( \mathcal{K} = F_0 \subset F_1 \subset \ldots \mathcal{K}_G \), where \( F_k \) is the span of \( F_{k-1} \) and all products of \( k \)-tuples of generators (not necessarily distinct). The Hilbert series of the filtered algebra is just the Hilbert series of its associated graded algebra, in other words,

\[
HS_{\mathcal{K}_G} = 1 + (\dim(F_1) - 1)t + (\dim(F_2) - \dim(F_1))t^2 + \ldots.
\]

The algebra \( \mathcal{K}_G \) remembers the whole graph, see Theorem 2.4 and its dimension is the number of spanning forests. However, at the moment we do not know the combinatorial meaning of its Hilbert series. In this case it is not a specialization of the Tutte polynomial of \( G \), see example 2.2. There is a similar problem for \( \mathcal{K}_T^G \). Besides that other filtered algebras were introduced, see their definitions and problems in paper [II].

The structure of this section is as follows: Subsection 2.1 is about the algebra \( \mathcal{K}_G \) counting spanning forests and Subsection 2.2 is about the algebra \( \mathcal{K}_T^G \) counting spanning trees.

2.1 Algebras \( \mathcal{K}_G \) and \( \mathcal{D}_G \)

In the notation of subsection 1.1, our next object of consideration is the filtered subalgebra \( \mathcal{K}_G \subset \Phi_G \) defined by the generators:

\[
Y_i = \exp(X_i) = \prod_{e \in G} (1 + c_{i,e} \phi_e), \quad i = 0, \ldots, n.
\]
Notice that the set of generators $Y_i - 1, i \in \{0, 1, \ldots, n\}$ gives the same filtered structure of $Y_i$ and it is easy to work with them, because they are nilpotent. In this case we have one more generator than for $C_G$, since $(Y_0 - 1) + \ldots + (Y_n - 1)$ is not always zero.

Since $Y_i$ is obtained by exponentiation of $X_i$, we call $K_G$ the “K-theoretic” analog of $C_G$. Furthermore, because $Y_j - 1$ is nilpotent and $\ln(1 + (Y_j - 1)) = X_j$, then algebras $K_G$ and $C_G$ coincide as non-filtered subalgebras of $\Phi_G$, however we interested in the filtered structure of $K_G$.

Example 2.1 Consider the complete graph $K_3$ on vertices $\{0, 1, 2\}$ and edges $\{a, b, c\}$, see Fig 1.1. Generators of $K_D$ are

$$Y_0 - 1 = e^{X_0} - 1 = \phi_a + \phi_c + \phi_a \phi_c;$$
$$Y_1 - 1 = e^{X_1} - 1 = -\phi_a + \phi_b - \phi_a \phi_b;$$
$$Y_2 - 1 = e^{X_2} - 1 = -\phi_b - \phi_c + \phi_b \phi_c.$$

Then the filtered structure is

- $F_0 = \text{span}\{1\}, \dim = 1$;
- $F_1 = \text{span}\{1, Y_0, Y_1, Y_2\}, \dim(F_1) - \dim(F_0) = 3$;
- $F_2 = \text{span}\{1, Y_0, Y_1, Y_2, Y_0^2, Y_1^2, Y_0 Y_1, \ldots\}, \dim(F_2) - \dim(F_1) = 3$.

It is clear that $X_i = (Y_i - 1) - \frac{(Y_i - 1)^2}{2}$ in this example, therefore $K_D$ and $C_D$ coincide as subalgebras of $\Phi_G$.

Similarly to the algebra $C_G$, this filtered algebra is the Cartesian product of filtered algebras corresponding to connected components, however a stronger statement does not hold.

Proposition 2.1 Given a graph $G$, then filtered algebra $K_G$ is the Cartesian product of filtered algebras corresponding to all connected components of $G$.

Similarly to algebras $C_G$ and $C^T_G$, we can present $K_G$ as a quotient algebra. Define the ideal $J_G$ in $\mathbb{K}[y_0, y_1, \ldots, y_n]$ as generated by the polynomials

$$q_I = \left( \prod_{i \in I} y_i - 1 \right)^{D_I + 1}, \quad (2.1)$$

where $I$ ranges over all nonempty subsets in $\{0, 1, \ldots, n\}$. Set $D_G := \mathbb{K}[y_0, \ldots, y_n]/J_G$. 

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Theorem 2.2 For any graph $G$, algebras $B_G$, $C_G$, $D_G$ and $K_G$ are isomorphic as (non-filtered) algebras. Their total dimension is equal to the number of spanning forests of $G$.

Since we have natural surjective homomorphism $D_G \rightarrow K_G$, then the following stronger statement holds.

Theorem 2.3 For any graph $G$, algebras $D_G$ and $K_G$ are isomorphic as filtered algebras.

Algebras $D_G$ and $K_G$ considered as non-filtered algebras remember the graphical matroid of $G$ and only it. However, as filtered algebras they contain complete information about $G$.

Theorem 2.4 Given two graphs $G_1$ and $G_2$ without isolated vertices, $K_{G_1}$ and $K_{G_2}$ are isomorphic as filtered algebras if and only if $G_1$ and $G_2$ are isomorphic.

Example 2.2 Consider the two graphs $G_1$ and $G_2$ presented in Fig. [1.2] We know that algebras $C_{G_1}$ and $C_{G_2}$ are isomorphic as graded algebras. However by Theorem 2.4 filtered algebras $K_{G_1}$ and $K_{G_2}$ should distinguish graphs. Furthermore, in this case algebras $K_{G_1}$ and $K_{G_2}$ have different Hilbert series, namely,

$$HS_{K_{G_1}}(t) = 1 + 4t + 10t^2 + 14t^3 + 3t^4,$$

$$HS_{K_{G_2}}(t) = 1 + 4t + 10t^2 + 15t^3 + 2t^4.$$ 

This means that they are not isomorphic as filtered algebras. It also means that, in general, it is impossible to calculate the Hilbert series of $K_G$ from the Tutte polynomials of $G$, because in the above case $G_1$ and $G_2$ have the same Tutte polynomial.

2.2 Algebras $K_T^G$ and $D_T^G$

In notation of subsection 1.2, our next object is the filtered subalgebra $K_T^G \subset \Phi_T^G$ defined by the generators:

$$Y_i = \exp(X_i) = \prod_{e \in G} (1 + c_{i,e}\Phi_e), \quad i = 0, \ldots, n.$$
Here we usually have one more generator than in the case of algebras $G_T$.

As above in subsection 2.1, $K^T_G$ and $C^T_G$ coincide as subalgebras of $Φ^T_G$, however we are interested in the filtered structure of $K^T_G$.

**Example 2.3** Consider the complete graph $K_3$ on vertices $\{0, 1, 2\}$ and edges $\{a, b, c\}$, see Fig 1.1. Generators of $K^T_G$ are

\[
Y_0 - 1 = e^{X_0} - 1 = φ_a + φ_c;
\]
\[
Y_1 - 1 = e^{X_1} - 1 = -φ_a + φ_b;
\]
\[
Y_2 - 1 = e^{X_2} - 1 = -φ_b - φ_c.
\]

Then the filtered structure is

- $F_0 = \text{span}\{1\}$, $\text{dim} = 1$
- $F_1 = \text{span}\{1, Y_0, Y_1, Y_2\}$, $\text{dim}(F_1) - \text{dim}(F_0) = 2$

Then for $K_3$, we have that $G^T_\Delta$ and $K^T_\Delta$ are isomorphic as filtered algebras (filtered structure of $G_\Delta$ coincides with its graded structure). In general $G^T_G$ and $K^T_G$ are not isomorphic as filtered algebras.

Similarly to $G^T_G$, the algebra $K^T_G$ makes sense only for connected graphs. Next statement is similar to Proposition 1.5.

**Proposition 2.5** Given a connected graph $G$, the filtered algebra $K^T_G$ is the Cartesian product of the filtered algebras corresponding to all 2-edge connected components of $G$.

As it happens for algebras $G_G$ and $G^T_G$, we can present this algebra as a quotient algebra. Define the ideal $J^T_G$ in $K[y_0, y_1, \ldots, y_n]$ as generated by the polynomials

\[
q_I^T = \left( \prod_{i \in I} y_{i} - 1 \right)^{D_I},
\]

where $I$ ranges over all nonempty subsets in $\{0, 1, \ldots, n\}$. Set $D^T_G := K[y_0, \ldots, y_n]/J^T_G$.

**Theorem 2.6** For any graph $G$, algebras $B^T_G$, $G^T_G$, $D^T_G$ and $K^T_G$ are isomorphic as (non-filtered) algebras. Their total dimension is equal to the number of spanning trees of $G$.

Since we have natural surjective homomorphism $D^T_G \to K^T_G$, then the following stronger statement holds.
Theorem 2.7 For any graph G, algebras $D_T^G$ and $K_T^G$ are isomorphic as filtered algebras.

Similarly to the case of $C_T^G$, we get the following one side statement and the problem. Define the $\Delta$-subgraph of $G$ as the subgraph obtained from $G$ after removal of all its bridges and produced isolated vertices.

Proposition 2.8 Given connected graphs $G_1$ and $G_2$ with isomorphic $\Delta$-subgraphs $\hat{G}_1$, algebras $K_T^{G_1}$ and $K_T^{G_2}$ are isomorphic as filtered algebras.

Problem 2.1 Is it true that if $K_T^{G_1}$ and $K_T^{G_2}$ are isomorphic, then their $\Delta$-subgraphs are also isomorphic? If not, then what is a criterium for graphs?

Example 2.4 Consider the two graphs $G_1$ and $G_2$ presented in Fig. [1.2]. We know that algebras $C_T^{G_1}$ and $C_T^{G_2}$ are isomorphic as graded algebras, however in this case filtered algebras $K_T^{G_1}$ and $K_T^{G_2}$ distinguish graphs. Furthermore in this case algebras $K_T^{G_1}$ and $K_T^{G_2}$ have different Hilbert series, namely,

$$HS_{K_T^{G_1}}(t) = 1 + 5t + 3t^2,$$

$$HS_{K_T^{G_2}}(t) = 1 + 6t + 2t^2.$$ 

This means that they are not isomorphic as filtered algebras. It also means that it is impossible to calculate the Hilbert series from their Tutte polynomials, because they have the same Tutte polynomial.
3. Algebra counting \(t\)-labelled spanning forests/trees

(based on paper \([I]\))

In this section we substitute the square-free algebra \(\Phi_G\) for the \((t+1)\)-free algebra \(\Phi^F_G\). In the case of trees we additionally change relations corresponding to cuts. Subalgebras of \(\Phi^F_G\) and \(\Phi^T_G\) of type \(\mathcal{C}\) have similar properties to \(\mathcal{C}_G\) and \(\mathcal{C}_T^G\). They enumerate the so-called \(t\)-labelled spanning forests and trees resp.

Consider a finite labelling set \([1,2,\ldots,t]\) containing \(t\) different labels; each label being a number from 1 to \(t\). A spanning forest/tree of \(G\) with a label from \([1,2,\ldots,t]\) on each edge is called a \(t\)-labelled spanning forest/tree. The weight of a \(t\)-labelled spanning forest \(F\), denoted by \(\omega(F)\), is the sum of the labels of all its edges.

The structure of this section is as follows: Subsection 3.1 is the about algebra \(\mathcal{C}^F_G\) counting \(t\)-labelled spanning forests and Subsection 3.2 is the about algebra \(\mathcal{C}^T_G\) counting \(t\)-labelled spanning trees.

Algebras \(\mathcal{C}^F_G\) and \(\mathcal{C}^T_G\) are isomorphic to \(\mathcal{B}^F_G\) and \(\mathcal{B}^T_G\) resp., which are a particular case of algebras from paper \([2]\) (there denoted by \(\mathcal{C}\) instead of \(\mathcal{B}\)). In fact, they consider algebras where each edge \(e\) replaced to \(a_e\) its clones and their Hilbert series are specializations of the multivariate Tutte polynomial of a graph (see definition in \([11]\)). Also the Hilbert series of \(\mathcal{B}^F_G\) and \(\mathcal{B}^T_G\) were calculated in \([7]\).

3.1 Algebras \(\mathcal{C}^F_G\) and \(\mathcal{B}^F_G\)

Let \(G\) be a graph without loops on the vertex set \([0,\ldots,n]\). Let \(t > 0\) be a positive integer.

Let \(\Phi^F_G\) be the commutative algebra over \(\mathbb{K}\) generated by \(\{\phi_e : e \in E(G)\}\) satisfying the relations

\[(\phi_e)^{t+1} = 0, \quad \text{for any edge } e \in E(G).\]
Let $C^F_G$ be the subalgebra of $\Phi^F_G$ generated by the elements

$$X_i = \sum_{e \in G} c_{i,e} \phi_e,$$

for $i = 1, \ldots, n$, where $c_{i,e}$ as in 1.1.

Similarly to the case of $C_G$, we have the following property.

**Proposition 3.1** Given a graph $G$, the algebra $C^F_G$ is the Cartesian product of algebras corresponding to all blocks of $G$.

We can again present $C^F_G$ as a quotient algebra of a polynomial ring. Let $J^F_G$ be the ideal in the ring $\mathbb{K}[x_1, \ldots, x_n]$ generated by

$$p^F_I = \left( \sum_{i \in I} x_i \right)^{t \cdot D_I + 1},$$

where $I$ ranges over all nonempty subsets of vertices and $D_I$ is the total number of edges between vertices in $I$ and in the complementary set $\bar{I}$.

Set $B^F_G := \mathbb{K}[x_1, \ldots, x_n] / J^F_G$.

In fact, these algebras are not new objects. Indeed, algebra $B^F_G$ is isomorphic to algebra $B_{\hat{G}}$, where $\hat{G}$ is the $t$-strength copy of $G$, i.e., the graph $\hat{G}$ is constructed from $G$ by replacing every edge by its $t$ clones. The same is true for algebras $C_G$ and $C_{\hat{G}}$, so we can generalize Theorem 1.2 for the case $t > 1$.

**Theorem 3.2** For any graph $G$ and a positive integer $t$, algebras $B^F_G$, $C^F_G$, $B_{\hat{G}}$ and $C_{\hat{G}}$ are isomorphic, where $\hat{G}$ is the $t$-strength copy of $G$. Their total dimension over $\mathbb{K}$ is equal to the number of $t$-labelled spanning forests in $G$.

Graphical matroids of $G$ and of $\hat{G}$ uniquely restore each other. Since by Theorem 1.4 we get that, for any positive $t > 0$, algebras $B^F_G$ and $B_G$ contain the same information about $G$.

**Corollary 3.3** Given two graphs $G_1$, $G_2$ and positive integer $t > 0$, algebras $C^F_{G_1}$ and $C^F_{G_2}$ are isomorphic if and only if their graphical matroids are isomorphic.

Given two graphs $G_1$, $G_2$ and positive integers $t_1, t_2 > 0$, algebras $C^F_{G_1}$ and $C^F_{G_2}$ are isomorphic if and only if $C^F_{G_1}$ and $C^F_{G_2}$ are isomorphic.

(The algebraic isomorphism can be thought of either as graded or as non-graded; the statement holds in both cases.)
Also we can get that the Hilbert series of \( C_{FG} \) is a specialization of the Tutte polynomial of \( G \).

**Corollary 3.4** Given a graph \( G \), the Hilbert series of algebra \( C_{FG} \) is given by

\[
HS_{C_{FG}}(z) = T_{G} \left( \frac{z^{t+1} - 1}{z^{t+1} - z}, \frac{1}{z} \right) \cdot z^{t(|G| - v(G) + c(G))} \cdot \left( \frac{1 - z^{t}}{1 - z} \right)^{v(G) - c(G)} ,
\]

where \( c(G) \) is the number of connected components of \( G \).

Furthermore, in this case we have one more interesting property: it is possible to reconstruct the Tutte polynomial of \( G \) from the Hilbert series of \( B_{FG} \) for large \( t \), for an algorithm see the proof of the following theorem in paper [I].

**Theorem 3.5** For any positive integer \( t \geq n \), it is possible to restore the Tutte polynomial of any connected graph \( G \) on \( n \) vertices knowing only the dimensions of each graded component of the algebra \( B_{FG} \).

### 3.2 Algebras \( C_{TG} \) and \( B_{TG} \)

Let \( G \) be a graph without loops on the vertex set \( \{0, \ldots, n\} \). Let \( t > 0 \) be a positive integer.

Let \( \Phi_{G}^{T} \) be the commutative algebra over \( \mathbb{K} \) generated by \( \{ \phi_e : e \in E(G) \} \) satisfying the relations:

\[
(\phi_e)^{t+1} = 0, \quad \text{for any edge} \ e \in G;
\]

\[
\left( \prod_{e \in H} \phi_e \right)^{t} = 0, \quad \text{for any non-slim subgraph} \ H \subset G.
\]

Let \( C_{TG} \) be the subalgebra of \( \Phi_{G}^{T} \) generated by the elements

\[
X_i = \sum_{e \in G} c_{i,e} \phi_e,
\]

for \( i = 1, \ldots, n \), where \( c_{i,e} \) as in [1.1].

Similarly to the previous case, we have the following proposition.
Proposition 3.6  Given a connected graph $G$ and positive $t > 0$, the algebra $C_T^G$ is the Cartesian product of algebras corresponding to all blocks of $G$.

However we can not get a stronger statement as in Proposition 1.5; it is not true for $t > 1$.

We can also present $C_T^G$ as a quotient algebra of a polynomial ring. Let $J_T^G$ be the ideal in the ring $\mathbb{K}[x_1, \ldots, x_n]$ generated by

$$p_I^T = \left( \sum_{i \in I} x_i \right)^t D_I,$$

where $I$ ranges over all nonempty subsets of vertices and $D_I$ is the total number of edges between vertices in $I$ and in the complementary set $\bar{I}$.

Set $B_T^G := \mathbb{K}[x_1, \ldots, x_n]/J_T^G$.

It is again clear that the algebra $B_T^G$ is isomorphic to the algebra $B_T^G$, where $\widehat{G}$ is the $t$-strenght copy of $G$. The similar fact is true for algebras $C_T^G$ and $C_T^{\widehat{G}}$, so we can generalize Theorem 1.6 for case $t > 1$.

Theorem 3.7 For any graph $G$ and a positive integer $t$, algebras $B_T^G$, $C_T^G$, $B_T^{\widehat{G}}$ and $C_T^{\widehat{G}}$ are isomorphic, where $\widehat{G}$ is the $t$-strength copy of $G$. Their total dimension over $\mathbb{K}$ is equal to the number of $t$-labelled spanning trees in $G$.

As a consequence we can get that the Hilbert series of $C_T^G$ is a specialization of the Tutte polynomial of $G$.

Corollary 3.8 Given a graph $G$, the Hilbert series of the algebra $C_T^G$ is given by

$$HS_{C_T^G}(z) = T_G \left( \frac{z - 1}{z^t + 1 - z^{-t}} \right) \cdot z^t \cdot \left( \frac{1 - z^t}{1 - z} \right)^{v(G) - c(G)},$$

where $c(G)$ is the number of connected components of $G$.

However, in this case it is impossible to restore the Tutte polynomial from the Hilbert series of $HS_{B_T^G}$. The reason is that all spanning trees have the same size and therefore $HS_{B_T^G}$ has the same information that $HS_{B_T^G}$.

For $t > 1$, graph $\widehat{G}$ has no bridges, since its bridge-free matroid and its graphical matroid coincide. We get the following corollary of Proposition 1.8.

Proposition 3.9  Given two graphs $G_1$, $G_2$ with isomorphic graphical matroid and $t > 0$, algebras $C_T^{G_1}$ and $C_T^{G_2}$ are isomorphic as graded algebra.
In fact, it is possible to prove the converse of Proposition 3.9 by using the same idea as in the proof of Theorem 1.4 with one upgrade. For the element $X_i$ corresponding to the vertex $i$, $d(X_i)$ is not the degree of the vertex $i$, however, for vertices $i$ and $j$, $\left\lfloor \frac{d(X_i+X_j)}{d(X_i)+d(X_j)+1} \right\rfloor$ is the number of edges between $i$ and $j$. We do not present the proof here and nor in paper [I], because we think that the most interesting problem is Conjecture 1.1 and the converse of Proposition 3.9 is just its corollary.
4. Family of algebras for hypergraphs

(based on paper [I])

In this section we present a family of algebras corresponding to a hypergraph. Almost all algebras from this family (generic algebras) have the same Hilbert series and this generic Hilbert series counts spanning forests of this hypergraph. We present two equivalent definitions of the hypergraphical matroid and spanning forests/trees. One definition is more algebraic and another one is more combinatorial. There are other definitions of spanning trees and forests of a hypergraph, see for example paper [I] or book [3] about hypergraphs. Our definition has not good analog of Matrix-tree theorem (see [6]), however we also define the Tutte polynomial of a hypergraph, whose points $T(1,1)$ and $T(2,1)$ are exactly the numbers of spanning trees and forests resp. At first we define the family of algebras.

Given a hypergraph $H$ on $n$ vertices, let us associate commuting variables $\phi_e, e \in H$ to all edges of $H$.

Set $\Phi_H$ to be the algebra generated by $\{\phi_e : e \in H\}$ with relations $\phi_e^2 = 0$, for any $e \in H$.

Define $C = \{c_{i,e} \in \mathbb{K} : i \in [1,n], e \in H\}$ as a set of parameters of $H$, for any edge $e \in H$, $c_{i,e} = 0$ for vertices non-incident to $e$, and $\sum_{i=1}^n c_{i,e} = 0$.

For $i = 1, \ldots, n$, set

$$X_i = \sum_{e \in H} c_{i,e} \phi_e,$$

Denote by $\mathcal{C}_{H(C)}$ the subalgebra of $\Phi_H$ generated by $X_1, \ldots, X_n$, and denote by $\hat{\mathcal{C}}_H$ the family of such subalgebras.

Then algebras generalize the algebra $\mathcal{C}_G$ of a graph.

**Proposition 4.1** For a usual graph $G$, a generic algebra from $\hat{\mathcal{C}}_G$ is isomorphic to $\mathcal{C}_G$.

**Example 4.1** Consider the graph $K_3$. Then any set of parameters of $K_3$ is represented as:
We can change variables \( \phi_{e_1}, \phi_{e_2} \) and \( \phi_{e_3} \) to \( p\phi'_{e_1}, q\phi'_{e_2} \) and \( r\phi'_{e_3} \) resp. If \( p, q \) and \( r \) are non-vanishing, then the algebra is isomorphic to \( \mathbb{C}G \).

We define a hypergraphical matroid using the definition of an independent set of edges of a hypergraph. Let \( H \) be a hypergraph on \( n \) vertices. A set \( F \) of edges is called independent if there is a set of parameters \( C \) of \( H \), such that vectors corresponding to edges from \( F \) are linearly independent. In other words, \( F \) is independent if, for a generic set of parameters of \( H \), vectors are linearly independent. Define the hypergraphical matroid of \( H \) as the matroid with ground set \( E(H) \).

**Example 4.2** Let \( H \) be the hypergraph with vertices set \( V = \{v_1, v_2, v_3\} \) and edges \( E = \{(v_1, v_2), (v_2, v_3), (v_1, v_3), (v_1, v_2, v_3)\} \), see Fig. 4.1.

**Figure 4.1**

Vectors corresponding to edges are

- \( e_1 : (x, -x, 0) = x \cdot (1, -1, 0) \);
- \( e_2 : (0, y, -y) = y \cdot (0, 1, -1) \);
- \( e_3 : (-z, 0, z) = z \cdot (-1, 0, 1) \);
- \( e_4 : (p, q, -p - q) = p \cdot (1, 0, -1) + q \cdot (0, 1, -1) \),

where \( x, y, z, p \) and \( q \) are generic real numbers. It is easy to see that any three edges are linearly dependent and any two are independent.

There is a combinatorial definition of an independent set of edges. At first we need to define a cycle of \( H \). Subset of edges \( C \subset E \) is called a cycle if
• \(|C| = \bigcup_{e \in C} e|\)

• There is no subset \(|C' \subset C|\), such that, for \(C'\), the first property holds.

Definitions of dependence and of cycle are similar.

**Theorem 4.2** A subset of edges \(X \subset E\) is dependent if and only if there is a cycle \(C \subset X\).

A set of edges \(F\) is called a **spanning forest** if \(F\) has no cycles, in other words, \(F\) is spanning forest if and only if \(F\) is an independent set (by Theorem 4.2). Furthermore, all maximal spanning forests have the same size. A set of edges \(T\) is called a **spanning tree** if it is a spanning forest and \(T\) has exactly \(v(H) - 1\) edges.

The Hilbert series of algebras in \(\mathcal{C}_H(C)\) are also counting spanning forests of \(H\).

**Theorem 4.3** For a hypergraph \(H\), generic algebras from \(\hat{\mathcal{C}}_H\) have the same Hilbert series. The dimension of the \(k\)-th graded component of a generic algebra equals the number of spanning forests \(F\) in \(H\) with external activity \(\big|H\big| - |F| - k\).

We define the Tutte polynomial of \(H\) as the Tutte polynomial of the corresponding hypergraphical matroid. By above results we know

• \(T_H(2,1)\) is the number of spanning forests;

• \(T_H(1,1)\) is the number of maximal spanning forests. In fact, \(T_H(1,1)\) is the number of spanning trees if \(H\) has at least one spanning tree.

By Theorem 4.3 we get that a generic Hilbert series is a specialization of the Tutte polynomial of \(H\).

**Corollary 4.4** Given a hypergraph \(H\) and its generic set of parameters \(C\), the Hilbert series of the algebra \(\mathcal{C}_{H(C)}\) is given by

\[
HS_{\mathcal{C}_{H(C)}}(t) = T_G\left(1 + t, \frac{1}{t}\right) \cdot t^{|H| - rk_H},
\]

where \(rk_H\) is the size of a maximal spanning forest of \(H\).

**Example 4.3** Consider the hypergraph \(H\), see Fig. 4.1. It is clear that any three edges form a cycle of \(H\), so \(H\) has \(\binom{4}{2} + \binom{4}{1} + \binom{4}{0} = 11\) spanning forests and \(\binom{4}{2} = 6\) of them are spanning trees.

The Tutte polynomial of the corresponding hypergraphical matroid is \(x^2 + y^2 + 2x + 2y\), then \(T_H(1,1) = 6\) and \(T_H(2,1) = 11\). So we get that \(T_H(1,1)\) and \(T_H(2,1)\) are the number of spanning trees and forests of \(H\) resp.
There is another definition of spanning forests/trees $H$, which again shows that it is a generalization of spanning forests/trees of a usual graph.

**Theorem 4.5** A subset of edges $X \subset E$ is a forest (tree) if and only if there is a map from edges to pairs: $e_k \rightarrow (i, j)$, where $v_i, v_j \in e_k$, such that these pairs form a spanning forest (tree) in the complete graph $K_n$.

**Problem 4.1** The main problem is to construct a family $\mathcal{C}_T^H$ of algebras, which counts spanning trees of $H$.

By paper [10], for hypergraph $H$ and set of parameters $C$, we can present $\mathcal{C}_{H(C)}$ as a quotient algebra, i.e., as $\mathcal{B}_{H(C)}$. We can consider the algebra $\mathcal{B}_{H(C)}^T$, which is obtained from $\mathcal{B}_{H(C)}$ by changing the powers of the generators of the ideal (writing always one less). By paper [2] algebra $\mathcal{B}_{H(C)}^T$ should count spanning trees of $H$. However at this moment, we cannot present $\Phi_T^H$ such that its generic subalgebra $\mathcal{C}_{H(C)}^T$ counts spanning trees of $H$.

Probably, we need to add to $\Phi_H$ relations corresponding to cuts, where a cut is a subset of edges such that without it $H$ has no a spanning tree. However, we need to prove it and if we want to do something similar to the proof of Theorem 1.6, then we need to define $H$-parking functions for a hypergraph.
5. Notations

Our main notations are

- \(|G|, |H|, |F|\) and \(|T|\) are the number of edges
- \(v(G)\) and \(v(H)\) are the number of vertices
- \(\mathbb{K}\) is the fixed field of zero characteristic
- \(D_I\) is the total number of edges between vertex set \(I\) and its complementary
- \(T_G\) is the Tutte polynomial of \(G\)
- \(H_{S_A}\) is the Hilbert series of algebra \(A\)

List of algebras.

- \(C_G\) and \(B_G\) are the algebras counting spanning forests of \(G\)
- \(C_G^T\) and \(B_G^T\) are the algebras counting spanning trees of \(G\)
- \(C_A\) is the algebra corresponding to vector configuration \(A\)
- \(C_G^{F_t}\) and \(B_G^{F_t}\) are the algebras counting \(t\)-labelled spanning forests
- \(C_G^{T_t}\) and \(B_G^{T_t}\) are the algebras counting \(t\)-labelled spanning trees of \(G\)
- \(\hat{C}_H\) is the family of algebras associated to hypergraph \(H\)
- \(C_{H(C)}\) is the member of \(\hat{C}_H\) corresponding to set of parameters \(C\)
- \(\mathcal{K}_G\) and \(\mathcal{D}_G\) are "K-theoretic" algebras counting spanning forests
- \(\mathcal{K}_G^T\) and \(\mathcal{D}_G^T\) are "K-theoretic" algebras counting spanning trees
## Differences in the notation of Licentiate, Papers I and II

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Bibliography


